Spacecraft Dynamics and Control

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Lecture 10: Rendezvous and Targeting - Lambert's Problem

Introduction

In this Lecture, you will learn:

Introduction to Lambert's Problem

- The Rendezvous Problem
- The Targeting Problem
  - Fixed-Time interception

Solution to Lambert's Problem

- Focus as a function of semi-major axis, a
- Time-of-Flight as a function of semi-major axis, a
  - Fixed-Time interception
- Calculating Δv.

Numerical Problem: Suppose we are in an equatorial parking orbit or radius r. Given a target with position $\vec{r}$ and velocity $\vec{v}$, calculate the Δv required to intercept the target before it reaches the surface of the earth.

Problems we Have Solved

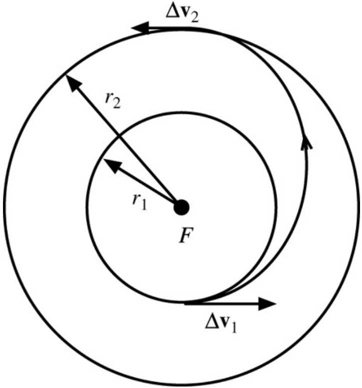
How to change our orbit to a desired one.

- Raise orbit
- Inclination change

Rendezvous?

- OK, when mission is not time-sensitive.
- Must use phasing...

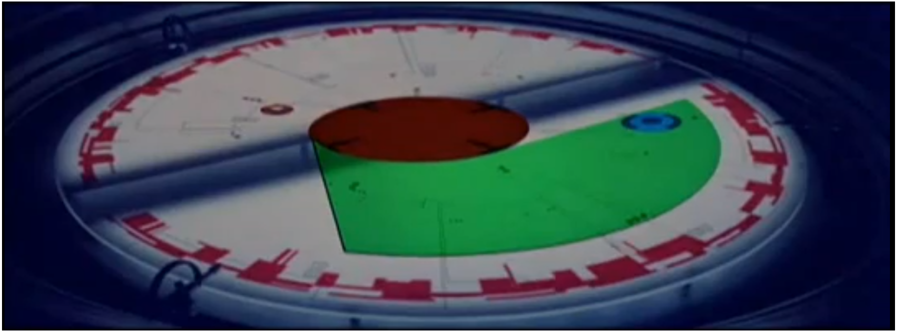
Let θ be the angle between the position vectors of the two craft at the beginning of the Hohman transfer.



We need $\theta = \pi - n_o \cdot T_{hohman} = \pi - 2\pi \frac{T_{hohman}}{T_{outer}}$, where n_o is the mean motion of the outer orbit and T_{hohman} is the period of the Hohman transfer.

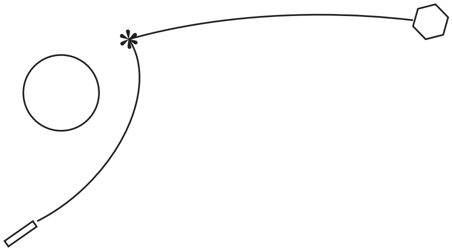
The Problem with phasing

Problem: We have to wait.



Remember what happened to the Death star?

Asteroid Interception



Suppose that:

- Our time to intercept is limited.
- The target trajectory is known.

Problem: Design an orbit starting from $\vec{r}_0$ which intersects the orbit of the asteroid at the same time as the asteroid.

- *Before* the asteroid intersects the earth (when $r(t) = 6378$)

Missile Defense

Problem: ICBM's have re-entry speeds in excess of 8km/s (Mach 26).

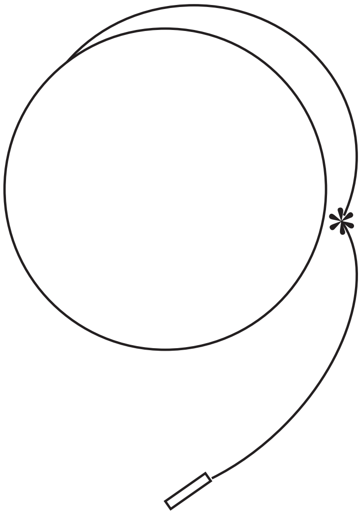
- Patriot missiles can achieve max of Mach 5.

Objective: Intercept ballistic trajectory before missile re-entry

- *Before* the missile intersects the atmosphere
- When $r(t) = 6378km + \cong 200km$

Complications:

- Plane changes may be required.
- The required time-to-intercept may be small.
  - Hohman transfer is not possible



The Targeting Problem

Step 1: Determine the orbit of the target

Step 1 can be accomplished one of two ways:

Method 1:

1. Given $\vec{r}(t_0)$ and $\vec{v}(t_0)$, find $a, e, i, \omega_p, \Omega$ and $f(t_0)$
   - we have covered this approach in Lecture 6.
2. Unfortunately, it is difficult to measure $\vec{v}$

Method 2:

1. Given two observations $\vec{r}(t_1)$ and $\vec{r}(t_2)$, find $a, e, i, \omega_p, \Omega$ and $f(t_0)$.
   - Alternatively, find $\vec{v}(t_1)$ and $\vec{v}(t_2)$
2. This is referred to as Lambert's problem (the topic of this lecture)

Note: This is a *boundary-value* problem:

- We know some states at two points.
- In contrast to the *initial value* problem, where we know all states at the initial time.
- Unlike initial-value problems, boundary-value problems cannot always be solved.

Carl Friedrich Gauss (1777-1855)

The Problem of orbit determination was originally solved by C. F. Gauss

Mathematician First

- Astronomer Second

Boring/Conservative.

Considered by many as one of the greatest mathematicians

- Professor of Astronomy in Göttingen
- He was a mathematician's mathematician.

Discovered

- Gaussian Distributions
- Gauss' Law (collaboration with Weber)
- Non-Euclidean Geometry (maybe)
- Least Squares (maybe)

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- Gauss' most lasting contribution may be in simplification/distillation of existing ideas.
- Actually, Lambert solved Lambert's problem. Gauss solved the 3 observation problem where we don't have range, only declination and right ascension. We won't actually cover the solution to this problem. It involves error minimization. However, the story of Gauss and Piazzi is much more interesting.
- Least-squares is also claimed by Legendre.
- Non-Euclidean geometries discovered in 1829 by Bolyai. Problem of parallel lines. No hard evidence to support Gauss; claim.

Discovery and Rediscovery of Ceres

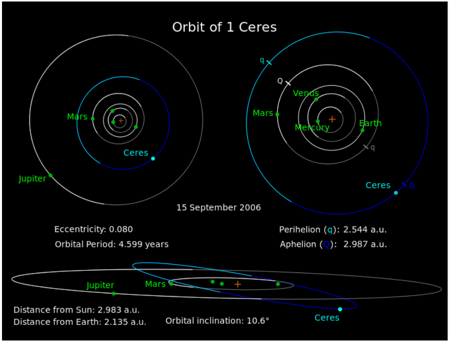
The pseudo-planet Ceres was discovered by G. Piazzi

- Observed 12 times between Jan. 1 and Feb. 11, 1801
- Planet was then lost.

Complication:

- Observation was only declination and right-ascension.
- Observations were only spread over 1% of the orbit.
  - No ranging info.
- For this case, three observations are needed.

C. F. Gauss solved the orbit determination problem and correctly predicted the location.

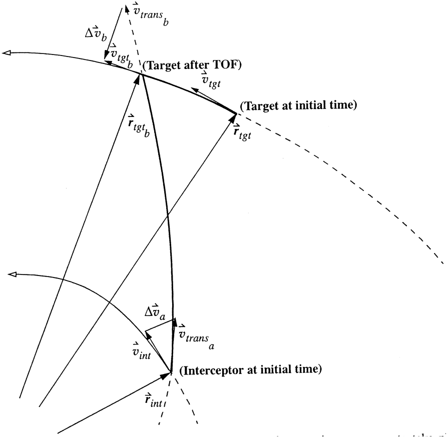


- Planet was re-found on Dec 31, 1801 in the correct location.

Once we have found the orbit of the target, we can determine where the target will be at the desired time of impact, t_f.

Procedure:

- The difference $t_f - t_0$ is the Time of Flight (TOF)
- Calculate $M(t_f) = M(t_0) + n(t_f - t_0)$
- Use $M(t_f)$ to find $E(t_f)$.
- Use $E(t_f)$ to find $f(t_f)$.
- Use $f(t_f)$ to find $\vec{r}(t_f)$.



The Targeting Problem

Step 3: Find the Intercept Trajectory

For a given

- Initial Position, $\vec{r}_1$
- Final Position, $\vec{r}_2$
- Time of Flight, TOF

the transfer orbit is uniquely determined.

Challenge: Find that orbit!!!

Difficulties:

- Where is the second focus?
- May require initial plane-change.
- May use LOTS of fuel.

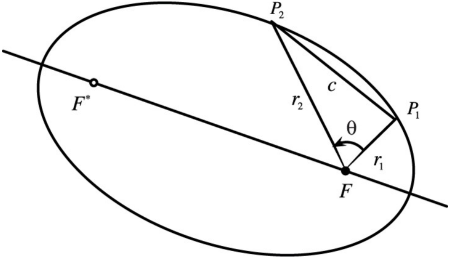


Figure: For given P_1 and P_2 and TOF, the transfer ellipse is uniquely determined.

On the Plus Side:

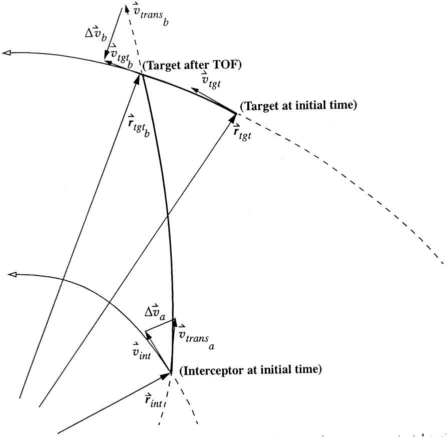
- We know the change in true anomaly, Δf...
- For this geometry, TOF only depends on a.

The Targeting Problem

Step 4: Calculate the Δv

Once we have found the transfer orbit,

- Calculate $\vec{v}_{tr}(t_0)$ of the transfer orbit.
- Calculate our current velocity, $\vec{v}(t_0)$
  - If a ground-launch, use rotation of the earth.
  - If in orbit, use orbital elements.
- Calculate $\Delta v = \vec{v}_{tr}(t_0) - \vec{v}(t_0)$



Finding the transfer orbit (The Hard Part)

Semi-major axis, a and Focus

The difficult part is to determine a.

- TOF only depends on a

The value of a dictates the location of the focus.

Focal Circle 1: For given $\vec{r}_0$ and a,

- The set of potential locations for the second focus is a circle.
  - Let $r_0 = \|\vec{r}_0\|$ be the distance to the known focus.
  - Then $r_{f1} = 2a - r_0$ is the distance from the spacecraft to the unknown focus.
- The unknown focus lies on a circle of radius $r_{f1} = 2a - r_0$ around our current position.

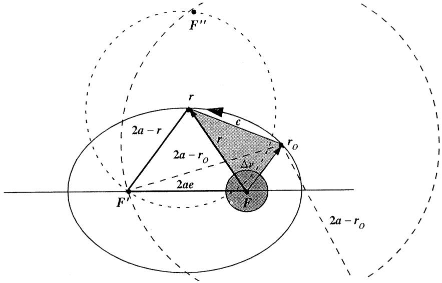


Figure: Potential Locations of Second Focus

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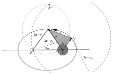


Figure: Potential Locations of Second Focus

- That TOF only depends on a was Lambert's conjecture! Only provide rigorously the year before he died (Tuberculosis?).

Finding Semi-major axis, a and Focus

Focal Circle 2: The same geometry holds for the final position vector, $\vec{r}$.

- The set of potential locations for the second focus is a circle.
  - Let $r_2 = \|\vec{r}\|$ be the distance of the final position to the known focus.
  - Then $r_{f2} = 2a - r_2$ is the distance from the final position to the unknown focus.
- The unknown focus lies on a circle of radius r_{f2} around the target position.

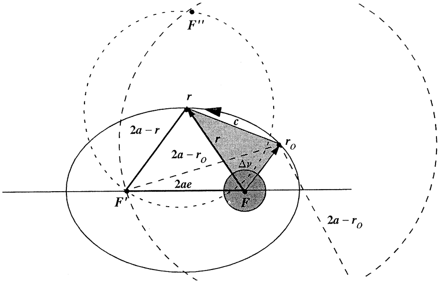


Figure: Potential Locations of Second Focus using Final Destination

Finding Semi-major axis, a and Focus

Geometry: By intersecting the two circles about $\vec{r}_0$ and $\vec{r}$,

- We find two potential locations for the focus (F' and F'')

We can discard the more distant focus.

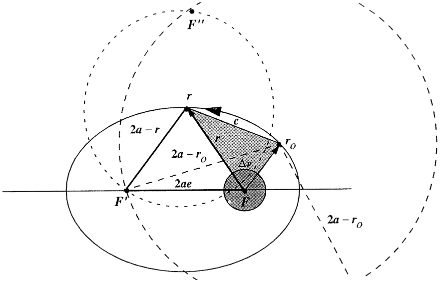


Figure: Potential Locations of Second Focus for a given choice of a

Note the focal locations vary continuously as we change a.

- As a increases, both circles get proportionally bigger.
- This changes the vacant focal locations.

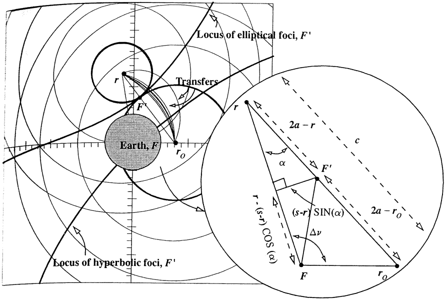
The minimum energy transfer

Note: As we vary a, the set of foci points F' and F'' form a *hyperbola*.

Conclusions

- There is a minimum-energy transfer corresponding to minimum a.
  - $a_{\min} = \frac{r_0+r+c}{4}$ where $c = \|\vec{r}_1 - \vec{r}_2\|$.
  - Focus lies on the line between $\vec{r}_0$ and $\vec{r}$.
  - But we don't want a *minimum energy* transfer!
  - Still, its nice to have a lower bound.
  - This is NOT the Hohman transfer.

Question: How to use TOF to find a?



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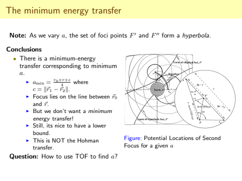

- Minimum Energy means the orbit has minimum energy as per $E = -\frac{\mu}{2a}$. The Δv required is not necessarily minimized.
- That means you probably don't want to use this transfer.
- $c = \|\vec{r}_1 - \vec{r}_2\|$ is the *chord*.
- At minimum energy orbit, $F' = F''$! Δt the long way is the same as Δt the short way.

Finding Semi-major axis, a and Focus

From Geometry, we already know

- $\theta = \Delta f = f(t_f) - f(t_0)$ - change in true anomaly.
- $\Delta t = TOF$

Can we use these to find a?

Kepler's Equation:

$$\sqrt{\frac{\mu}{a^3}}\Delta t = E(t_f) - E(t_0) - e(\sin E(t_f) - \sin E(t_0))$$

Problem: We know Δf, but not ΔE!

- E depends on e as well as f and a
- We don't know e

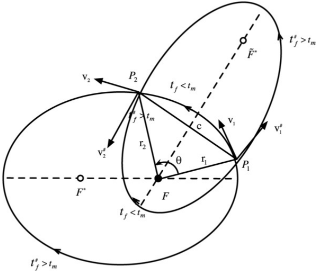

Figure: Geometry of the Problem

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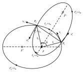

Figure: Geometry of the Problem

- LHS of Kepler's equation is $\Delta M(t) = n\Delta t$

Finding Semi-major axis, a and Focus

Solution

First: Calculate some lengths

- $c = \|\vec{r}_1 - \vec{r}_2\|$ is the *chord*.
- $s = \frac{c+r_1+r_2}{2}$ is the *semi-perimeter*.
  - NOT semiparameter.

Then we get **Lambert's Equation:**

$$\Delta t = \sqrt{\frac{a^3}{\mu}}\left(\alpha - \beta - (\sin\alpha - \sin\beta)\right)$$

where

$$\sin\left[\frac{\alpha}{2}\right] = \sqrt{\frac{s}{2a}}, \qquad \sin\left[\frac{\beta}{2}\right] = \sqrt{\frac{s-c}{2a}}$$

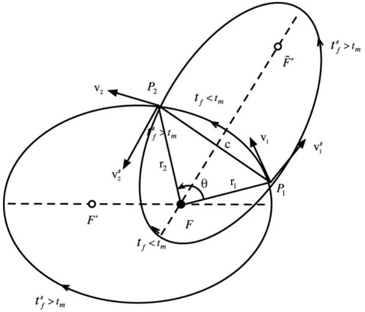

Figure: Geometry of the Problem

Conclusion: We can express TOF, solely as a function of a.

- albeit through a complicated function.
- But we are given TOF and need to FIND a

Finding Semi-major axis, a and Focus

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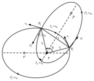

Figure: Geometry of the Problem

Conclusion: We can express TOF, solely as a function of a.

- albeit through a complicated function.
- But we are given TOF and need to FIND a

- The semi-perimeter is half the perimeter of the triangle shown in the figure.

There are several ways to solve Lambert's Equation

- Newton Iteration
  - More Complicated than Kepler's Equation
- Series Expansion
  - Probably the easiest...
- Bisection
  - Relatively Slow, but easy to understand
  - Only works for *monotone* functions.

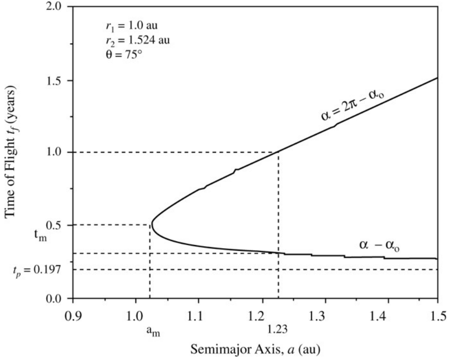

Figure: Geometry of the Problem

Solving Lambert's Equation
Bisection

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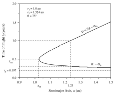


Figure: Geometry of the Problem

- In the figure, a_m is the minimum energy transfer orbit. (Recall NOT minimum Δv).
- t_m is the transfer time (TOF) obtained by plugging a_m into Lambert's equation.
- t_p is the flight time of a parabolic orbit (corresponding to $a = \infty$)
- The function is *monotone* in the interval $TOF \in [t_p, t_m]$
- The other branch of the plot ($TOF > t_m$) corresponds to use of the distant focus F''.

Solving Lambert's Equation via Bisection

Define $g(a) = \sqrt{\frac{a^3}{\mu}}\left(\alpha - \beta - (\sin\alpha - \sin\beta)\right)$.

Root-Finding Problem:

Find a :

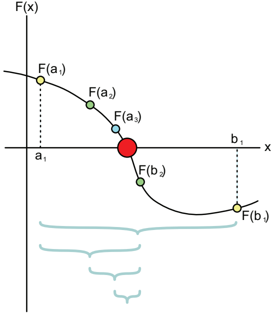
such that $g(a) = \Delta t$

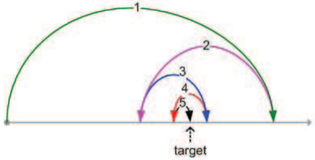
Bisection Algorithm:

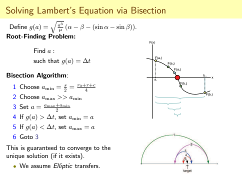
1. Choose $a_{\min} = \frac{s}{2} = \frac{r_0+r+c}{4}$
2. Choose $a_{\max} >> a_{\min}$
3. Set $a = \frac{a_{\max}+a_{\min}}{2}$
4. If $g(a) > \Delta t$, set $a_{\min} = a$
5. If $g(a) < \Delta t$, set $a_{\max} = a$
6. Goto 3

This is guaranteed to converge to the unique solution (if it exists).

- We assume *Elliptic* transfers.







- By Elliptic solutions, we mean that we assume that the transfer orbit is elliptic.
- Parabolic solutions are possible, but not covered by Lambert's equations.
- We must check to make sure the solution is not parabolic before starting.

Make Sure a Solution Exists!!

- First calculate the minimum TOF.
- This corresponds to a parabolic trajectory

$$\Delta t_p = \frac{\sqrt{2}}{3}\sqrt{\frac{s^3}{\mu}}\left(1 - \left(\frac{s-c}{s}\right)^{\frac{3}{2}}\right)$$

  - Can get there even faster by using a hyperbolic approach (Not Covered).
- We should also calculate the Maximum TOF

$$\Delta t_{\max} = \sqrt{\frac{a_{\min}^3}{\mu}}\left(\alpha_{\max} - \beta_{\max} - (\sin\alpha_{\max} - \sin\beta_{\max})\right)$$

  where

$$\sin\left[\frac{\alpha_{\max}}{2}\right] = \sqrt{\frac{s}{2a_{\min}}}, \qquad \sin\left[\frac{\beta_{\max}}{2}\right] = \sqrt{\frac{s-c}{2a_{\min}}}$$

- One can exceed this by going the long way around (Not Covered Here)

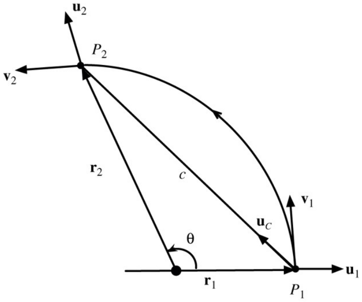
Calculating $\vec{v}(t_0)$ and $\vec{v}(t_f)$

Once we have a, calculating $\vec{v}$ is not difficult.

$$\vec{v}(t_0) = (B + A)\vec{u}_c + (B - A)\vec{u}_1, \qquad \vec{v}(t_f) = (B + A)\vec{u}_c - (B - A)\vec{u}_2$$

where

$$A = \sqrt{\frac{\mu}{4a}} \cot\left(\frac{\alpha}{2}\right), \qquad B = \sqrt{\frac{\mu}{4a}} \cot\left(\frac{\beta}{2}\right)$$



and the unit vectors

- $\vec{u}_1$ and $\vec{u}_2$ point to positions 1 and 2.

$$\vec{u}_1 = \frac{\vec{r}(t_0)}{r_1}, \qquad \vec{u}_2 = \frac{\vec{r}(t_f)}{r_2}$$

- $\vec{u}_c$ points from position 1 to 2.

$$\vec{u}_c = \frac{\vec{r}(t_f) - \vec{r}(t_0)}{c}$$

Calculating Δv

Once we have found the transfer orbit,

- Calculate $\vec{v}_{tr}(t_0)$ of the transfer orbit.
- Calculate our current velocity, $\vec{v}(t_0)$
  - If a ground-launch, use rotation of the earth.
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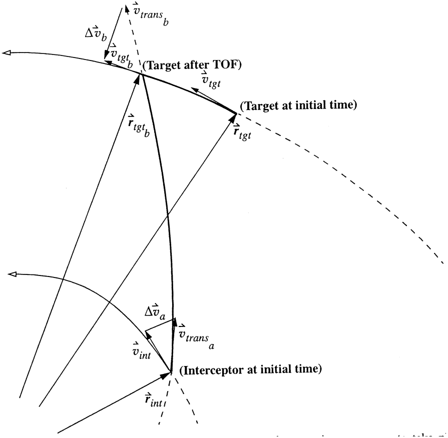
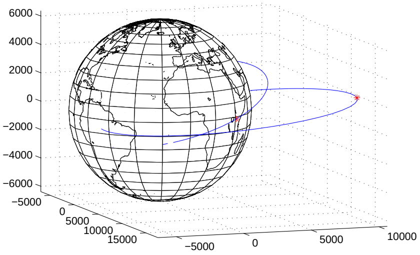


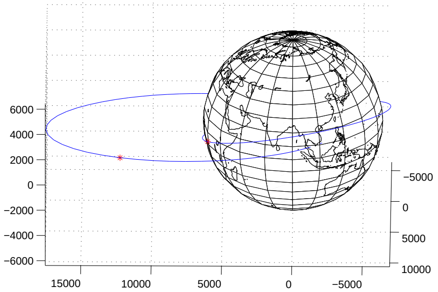
Numerical Example of Missile Targeting

Problem: Suppose that Brasil launches an ICBM at Bangkok, Thailand.

- We have an interceptor in the air with position and velocity
$$\vec{r} = \begin{bmatrix}6045 & 3490 & 0\end{bmatrix} km \qquad \vec{v} = \begin{bmatrix}-2.457 & 6.618 & 2.533\end{bmatrix} km/s.$$
- We have tracked the missile at $r_t = \begin{bmatrix}12214.839 & 10249.467 & 2000\end{bmatrix} km$ heading $\vec{v} = \begin{bmatrix}-3.448 & .924 & 0\end{bmatrix} km/s$.

Question: Determine the Δv required to intercept the missile before re-entry, which occurs in 30 minutes.





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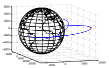
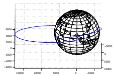




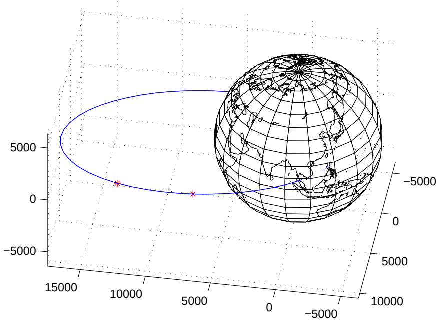
- The figure shows both the path of the ICBM and the current (temporary) orbit of the interceptor.
- The $*$ indicates the current positions of the ICBM and interceptor in their respective orbits.

Numerical Example of Missile Targeting

The first step is to determine the position of the ICBM in $t + 30$min.

Recall: To propagate an orbit in time:

1. Use $\vec{r}_t$ and $\vec{v}_t$ to find the orbital elements, including $M(t_0)$.
2. Propagate Mean anomaly $M(t_f) = M(t_0) + n\Delta t$ where $\Delta t = 1800s$.
3. Use $M(t_f)$ to find true anomaly, $f(t_f)$.
   - Requires iteration to solve Kepler's Equation.
4. Use the orbital elements, including $f(t_f)$ to find $\vec{r}(t_f)$



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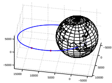

- This figure shows the position of the ICBM at the initial point and the desired point of interception.

Numerical Example of Missile Targeting

The next step is to determine whether an intercept orbit is feasible using TOF=30min.

Geometry of the Problem:

$$r_1 = \|\vec{r}\| = 6,980km, \quad r_2 = \|\vec{r}_t(t_f)\| = 12,282km,$$

$$c = \|\vec{r} - \vec{r}_t(t_f)\| = 7,080km, \quad s = \frac{c + r_1 + r_2}{2} = 13,171km$$

Minimum Flight Time: Using the formula, the minimum (parabolic) flight time is

$$t_{\min} = t_p = \frac{\sqrt{2}}{3}\sqrt{\frac{s^3}{\mu}}\left(1 - \left(\frac{s-c}{s}\right)^{\frac{3}{2}}\right) = 18.2min$$

Thus we have more than enough time.

Maximum Flight Time: Geometry yields a minimum semi-major axis of

$$a_{\min} = \frac{s}{2} = 6,586km$$

Plugging this into Lambert's equation yields a maximum flight time of $t_{\max} = 37.3$min.

Numerical Example of Missile Targeting

What remains is to solve Lambert's equation:

$$\Delta t = \sqrt{\frac{a^3}{\mu}} \left(\alpha - \beta - (\sin\alpha - \sin\beta)\right)$$

where

$$\sin\left[\frac{\alpha}{2}\right] = \sqrt{\frac{s}{2a}}, \qquad \sin\left[\frac{\beta}{2}\right] = \sqrt{\frac{s-c}{2a}}$$

Initialize our search parameters using $a \in [a_l, a_h] = [a_{\min}, 2s]$.

1. $a_1 = \frac{a_l + a_h}{2} = 8,232$ - $TOF = 21.14$min - too low, decrease a
   1.1 Set $a_h = a_1$
2. $a_2 = \frac{a_l + a_h}{2} = 7,409$ - $TOF = 24$min - too low, decrease a
   2.1 Set $a_h = a_2$
3. $a_3 = \frac{a_l + a_h}{2} = 6,997$ - $TOF = 26.76$min - too low, decrease a
   3.1 Set $a_h = a_3$
4. $\cdots$

K. $a_k = \frac{a_l + a_h}{2} = 6,744$ - $TOF = 29.99$
   K.1 Close Enough!

Numerical Example of Missile Targeting

Now we need to calculate Δv.

$$\vec{v}(t_0) = (B+A)\vec{u}_c + (B-A)\vec{u}_1, \qquad \vec{v}(t_f) = (B+A)\vec{u}_c - (B-A)\vec{u}_2$$

where

$$A = \sqrt{\frac{\mu}{4a}}\cot\left(\frac{\alpha}{2}\right) = .597, \qquad B = \sqrt{\frac{\mu}{4a}}\cot\left(\frac{\beta}{2}\right) = 4.2363$$

and the unit vectors

$$\vec{u}_1 = \begin{bmatrix} .866 \\ .5 \\ 0 \end{bmatrix}, \quad \vec{u}_2 = \begin{bmatrix} .52 \\ .8414 \\ .1451 \end{bmatrix}, \quad \vec{u}_c = \begin{bmatrix} .0493 \\ .9666 \\ .2516 \end{bmatrix}$$

which yields

$$\vec{v}_t(t_0) = \begin{bmatrix} 3.3901 & 6.4913 & 1.2163 \end{bmatrix} km/s$$

Calculating Δv

$$\Delta v = \vec{v}_t(t_0) - \vec{v} = \begin{bmatrix} 5.847 & -.1267 & -1.3167 \end{bmatrix} km/s$$

For a total impulse of 6km/s.

Numerical Example of Missile Targeting

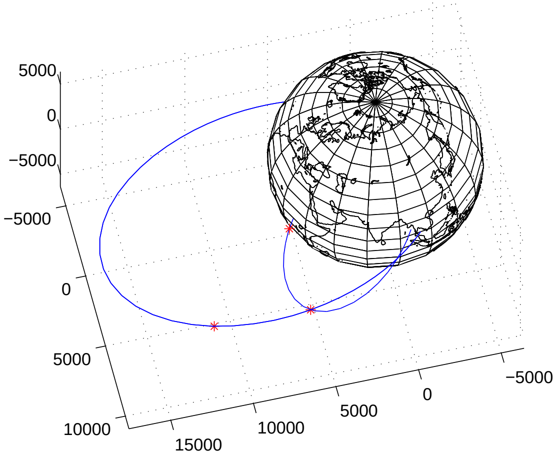

Figure: Intercept Trajectory

Numerical Example of Missile Targeting

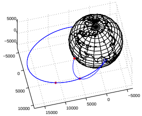

Figure: Intercept Trajectory

- This figure shows the ICBM and the path of the intercept trajectory.

Summary

This Lecture you have learned:

Introduction to Lambert's Problem

- The Rendezvous Problem
- The Targeting Problem
  - Fixed-Time interception

Solution to Lambert's Problem

- Focus as a function of semi-major axis, a
- Time-of-Flight as a function of semi-major axis, a
  - Fixed-Time interception
- Calculating Δv.

Next Lecture: Rocketry.