ADAPTIVE GREEDY REJECTION SAMPLING

GERGELY FLAMICH AND LUCAS THEIS

26/06/2023

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1. IN COLLABORATION WITH

Lucas Theis

2. BACKGROUND

2.1. CHANNEL SIMULATION

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When common randomness S available (SFRL):

$$H[X \mid S] \leq I[X;Y] + \log(I[X;Y] + 1) + 4.$$

2.2. LOSSY COMPRESSION

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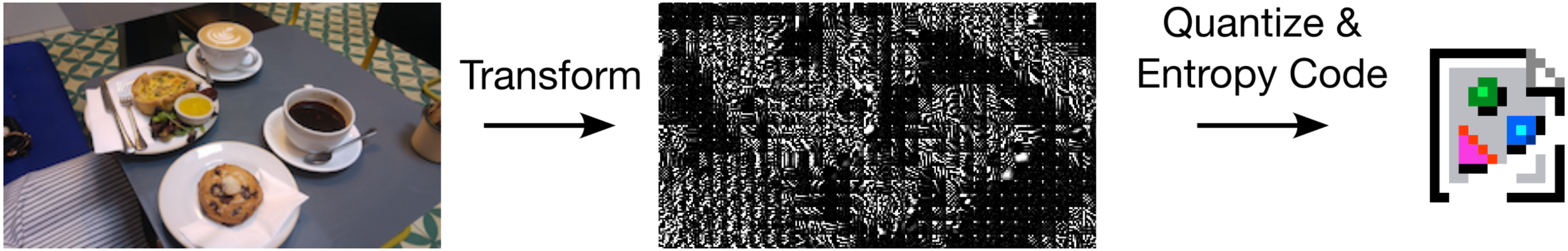

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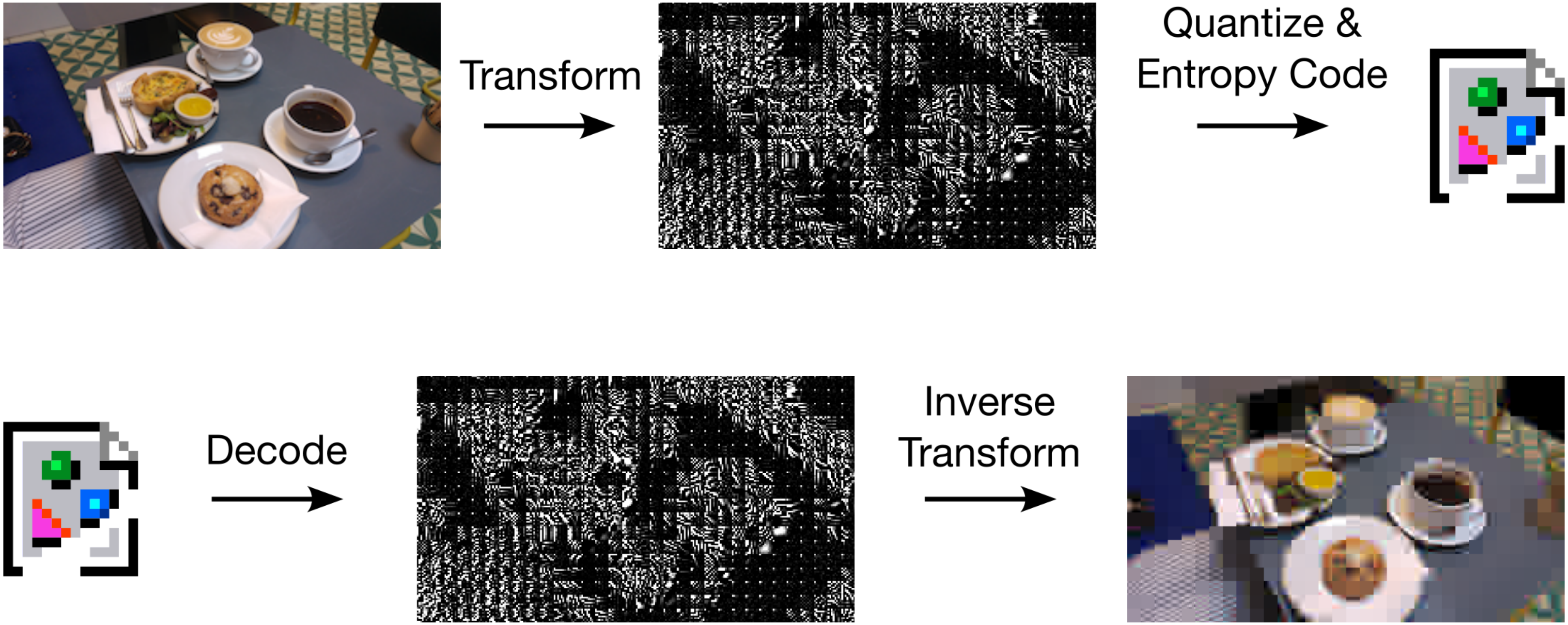

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We extend it to Borel probability measures over Polish spaces.

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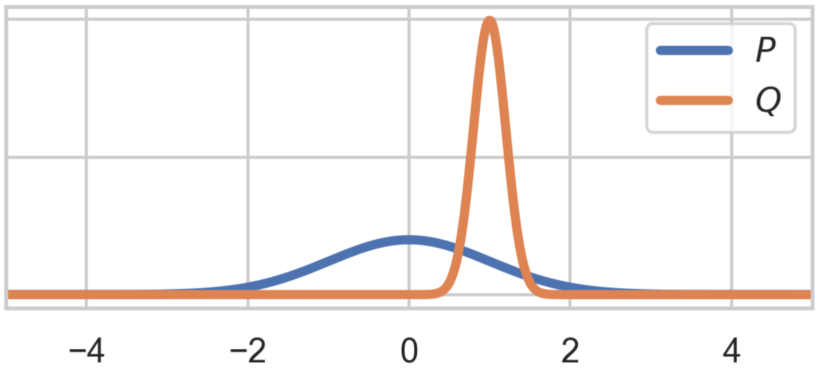

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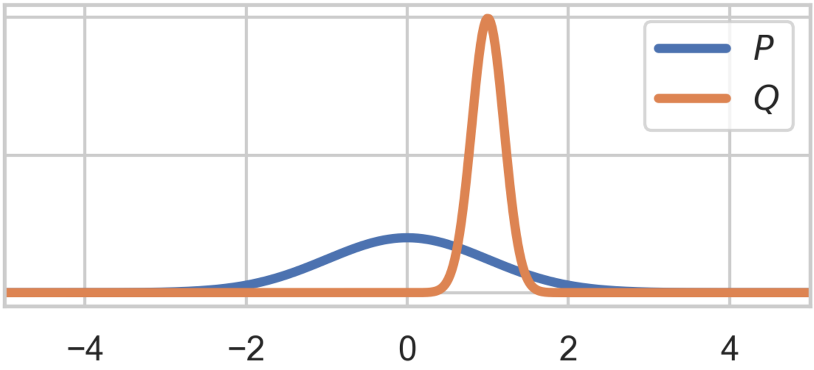

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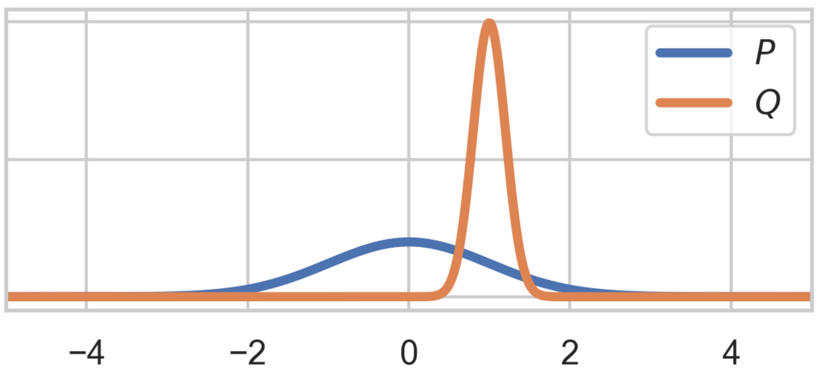

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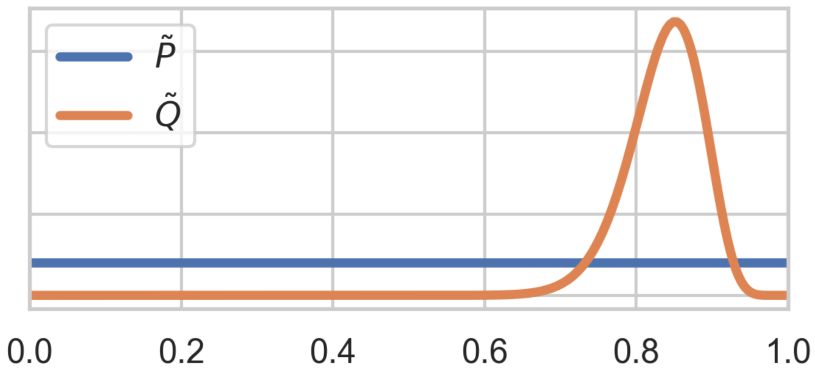

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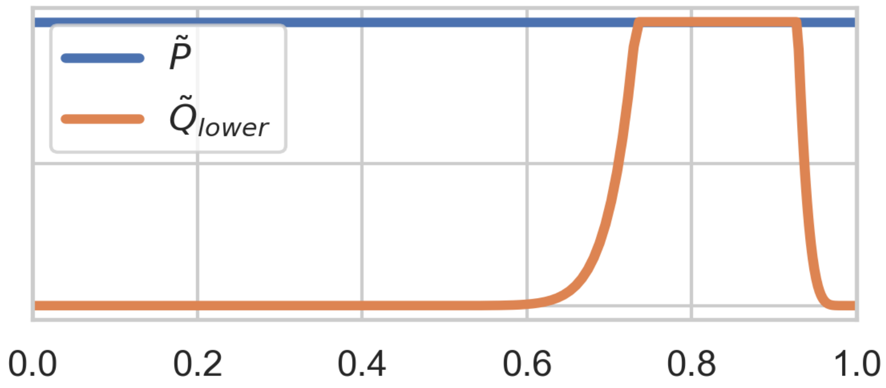

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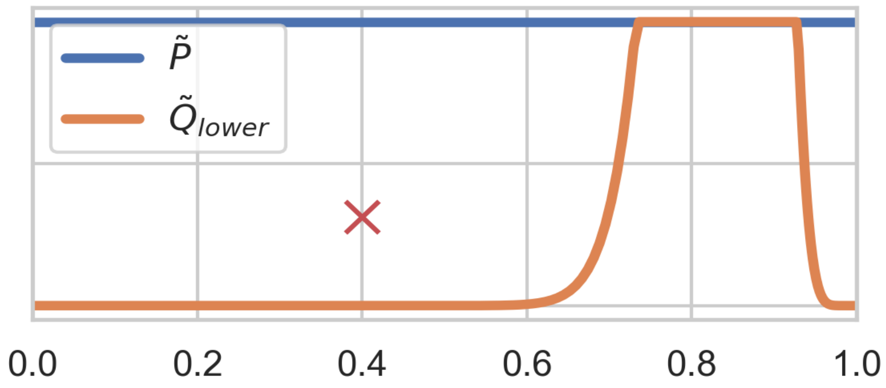

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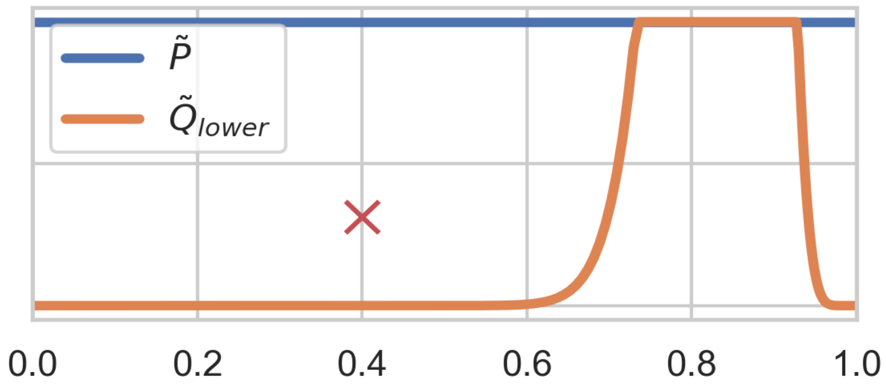

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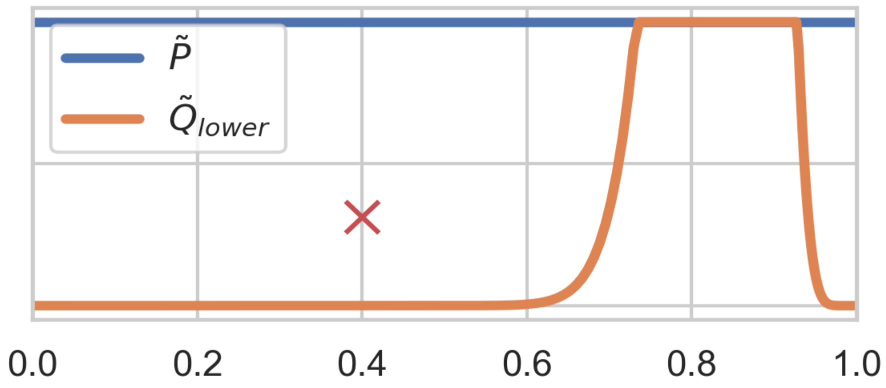

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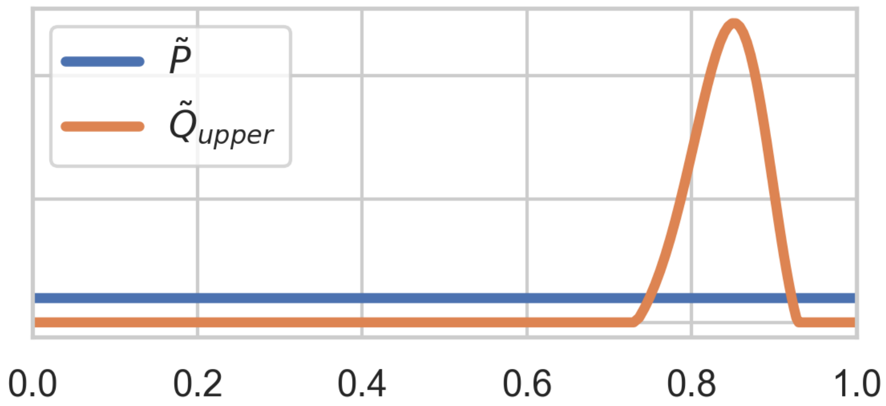

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Transmit K using a Zeta coding distribution:

$$\zeta(k \mid \alpha) \propto k^{-\alpha}$$

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When $Q \leftarrow P_{X|Y}$ and $P \leftarrow P_X$, average over Y:

$$H[K] \leq I[X;Y] + \log(I[X;Y] + 1) + 4$$

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$$\mathbb{E}[K+1] = 2^{D_\infty[Q \,\|\, P]}$$

where

$$D_\infty[Q \,\|\, P] = \log\left(\operatorname{ess\,sup}_{x \in \Omega}\left\{\frac{dQ}{dP}(x)\right\}\right)$$

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Now, we find:

$$\mathbb{E}[K+1] = \infty$$

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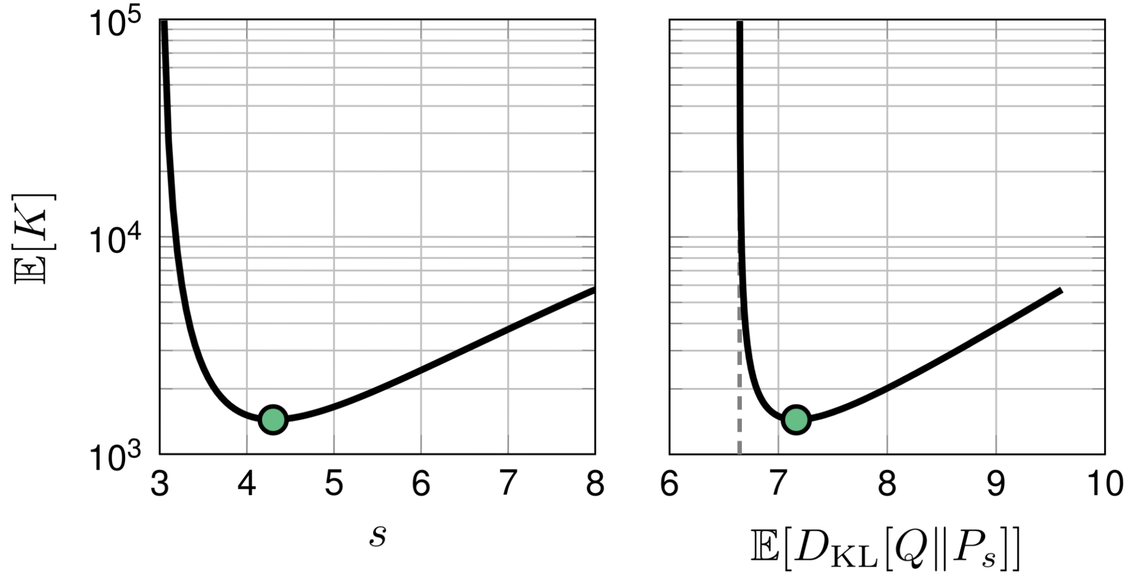

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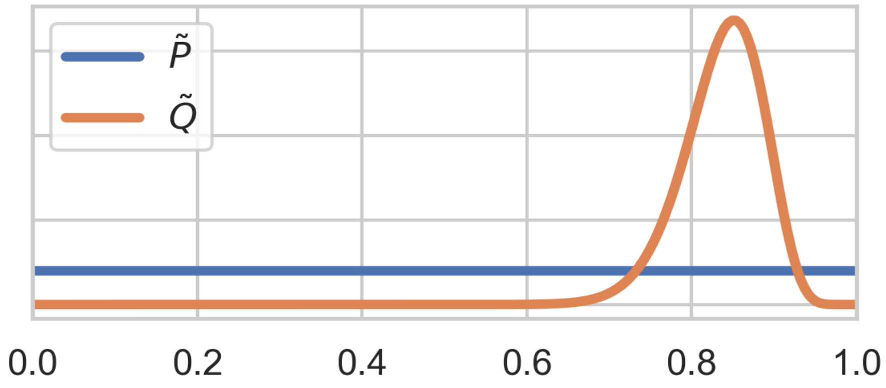

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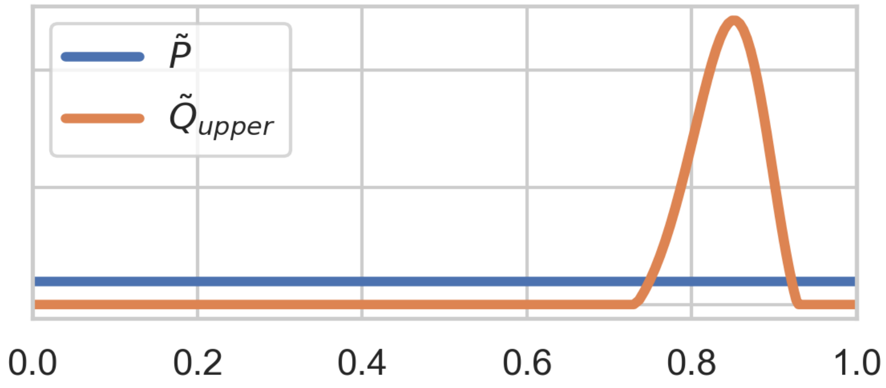

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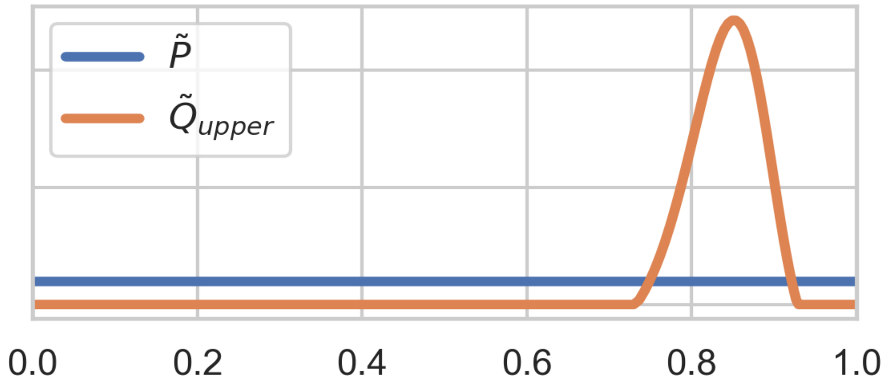

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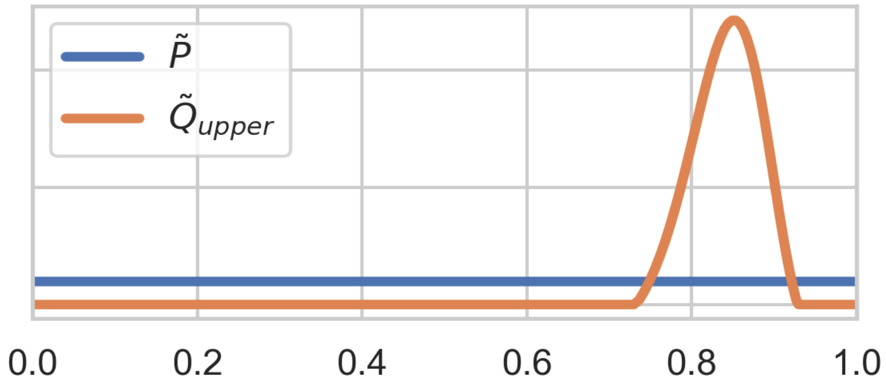

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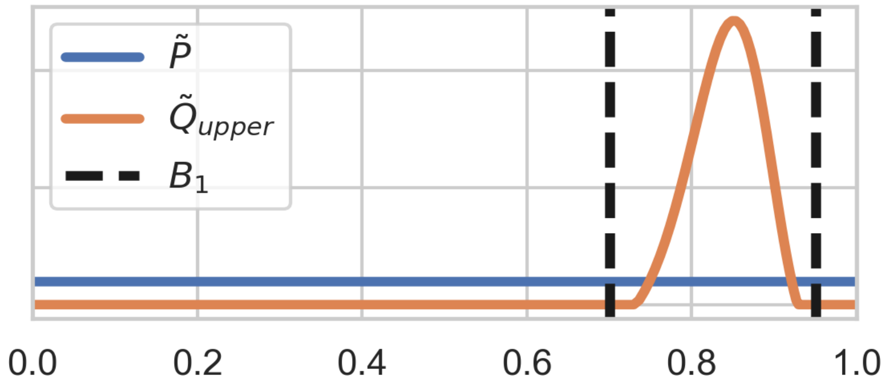

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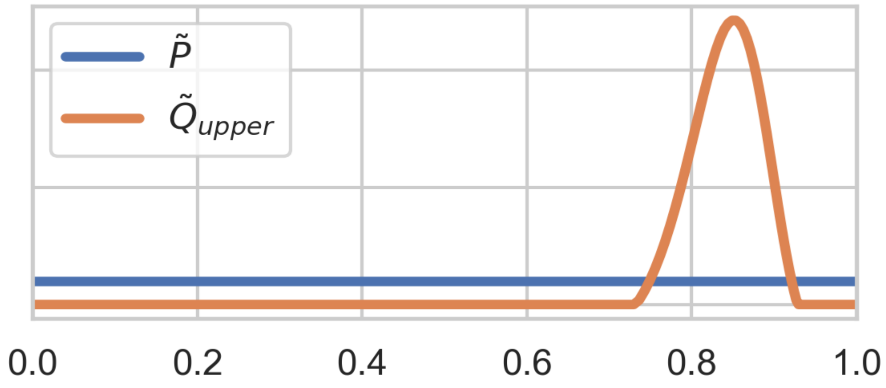

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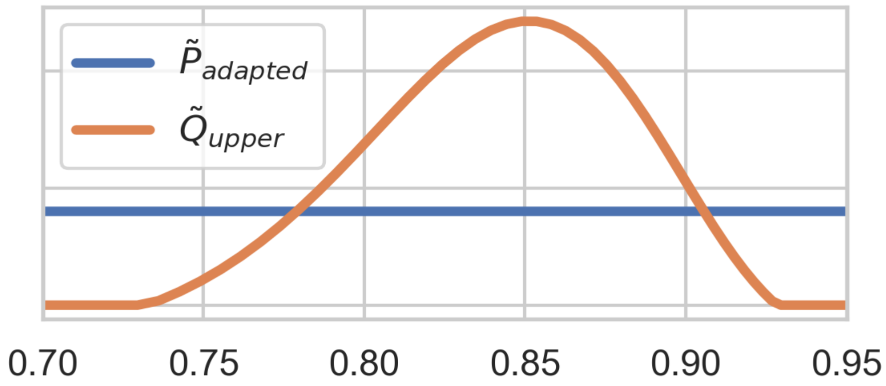

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Theorem:

$$H[K] \leq C + \log(C+1) + 3.63,$$

where $C = I[X;Y] + \mathbb{E}[\log P(B_{m_K})]$.

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$$\lfloor c + U \rceil - U \stackrel{d}{=} c + U'$$

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If $c \in [1/2, M - 3/2)$, then $\lfloor c + U \rceil \in [0 : M - 1]$.

$$\frac{\lfloor c + U \rceil - U}{M} \sim \mathrm{Unif}\left(\frac{c}{M} - \frac{1}{2M}, \frac{c}{M} + \frac{1}{2M}\right).$$

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DQ + Bits-back: Bits-back Quantization (BBQ)

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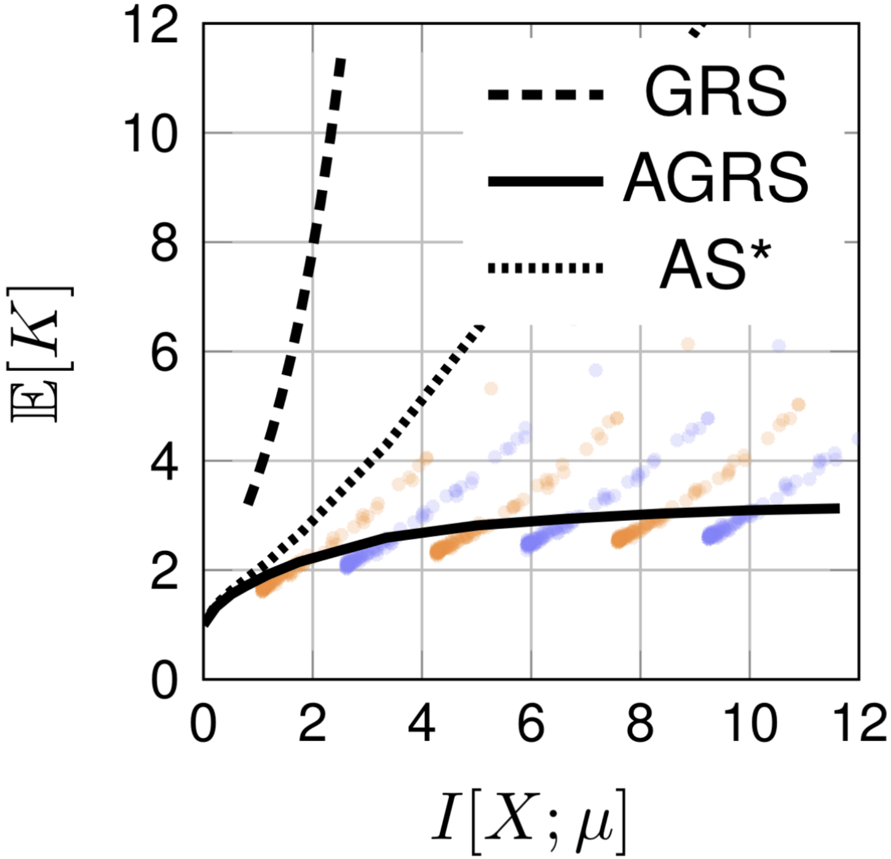

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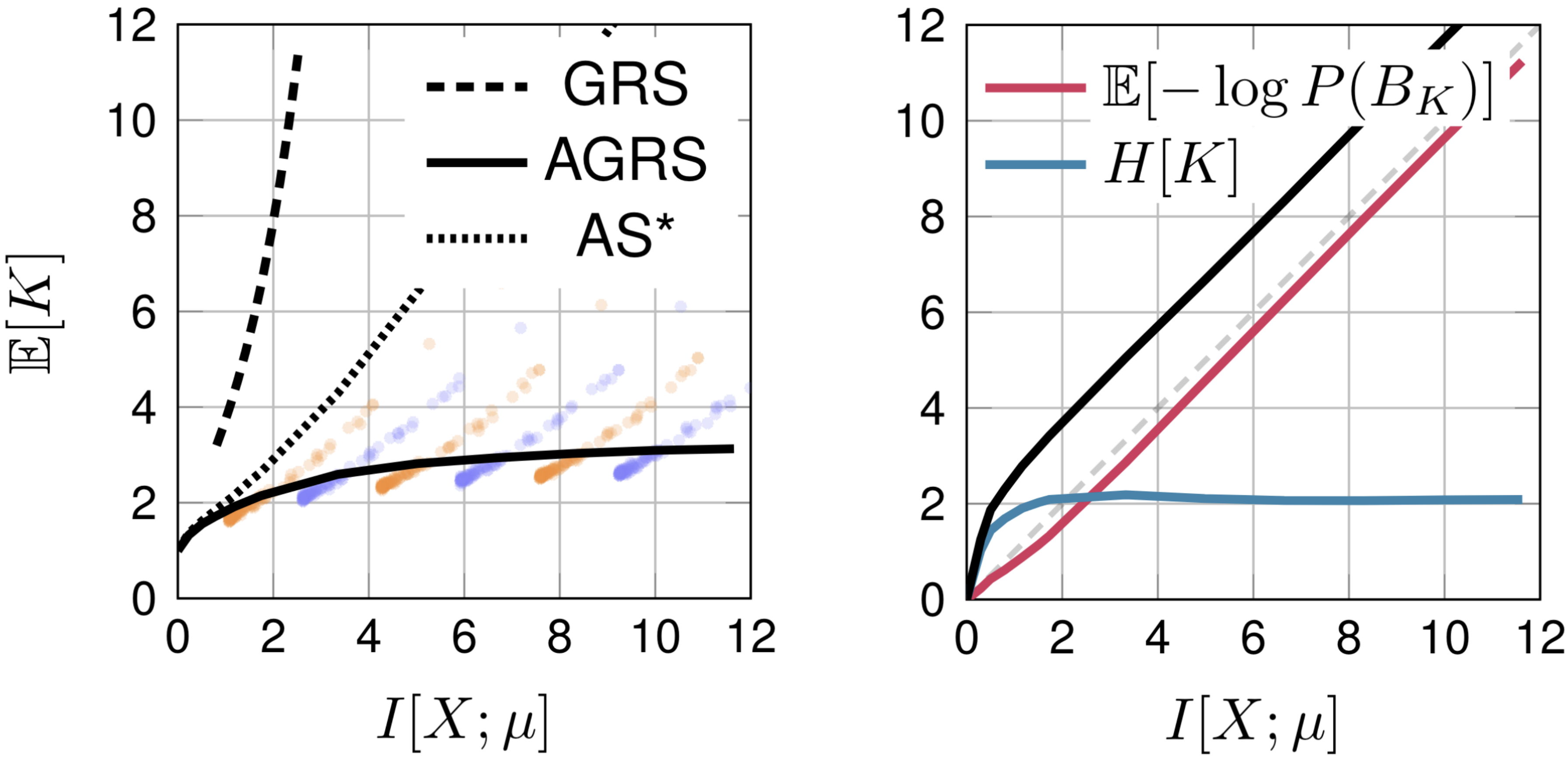

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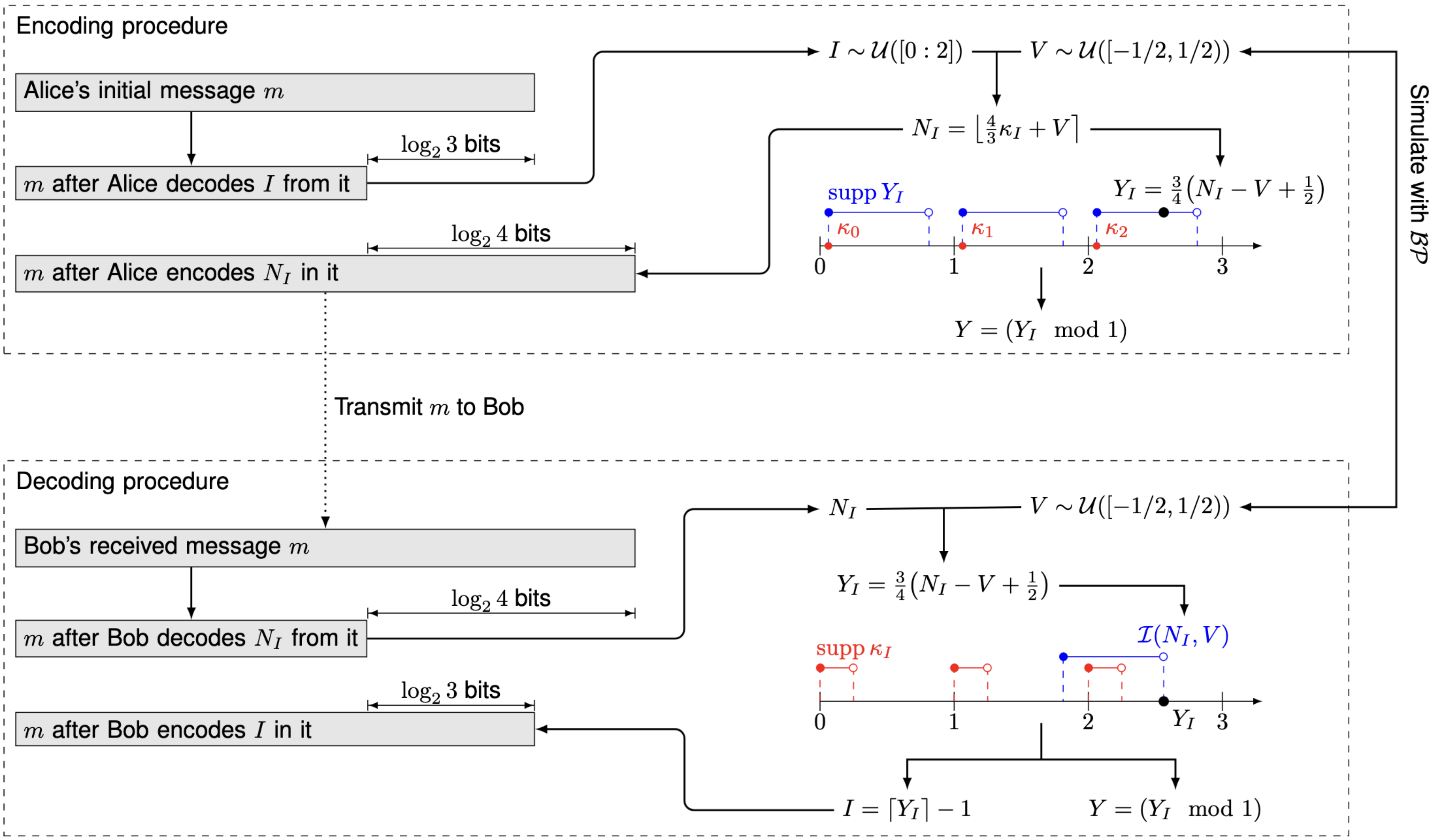
3. Specialized algorithms for Gaussians?

6. REFERENCES

- P. Harsha, R. Jain, D. McAllester, and J. Radhakrishnan, “The communication complexity of correlation,” IEEE Transactions on Information Theory, vol. 56, no. 1, pp. 438–449, 2010.
- C. T. Li and A. El Gamal, “Strong functional representation lemma and applications to coding theorems,” IEEE Transactions on Information Theory, vol. 64, no. 11, pp. 6967–6978, 2018.
- F. “Greedy Poisson Rejection Sampling,” arXiv preprint arXiv:2305.15313, 2023.

7. BBQ



Encoding procedure
Alice's initial message m
$\log_2 3$ bits
m after Alice decodes I from it
$\log_2 4$ bits
m after Alice encodes N_I in it
$I \sim \mathcal{U}([0:2])$
$V \sim \mathcal{U}([-1/2, 1/2))$
$N_I = \lfloor \frac{4}{3}\kappa_I + V \rceil$
$Y_I = \frac{3}{4}(N_I - V + \frac{1}{2})$
supp Y_I
κ_0
κ_1
κ_2
0 1 2 3
$Y = (Y_I \mod 1)$
Simulate with $\mathcal{BP}$
Transmit m to Bob
Decoding procedure
Bob's received message m
$\log_2 4$ bits
m after Bob decodes N_I from it
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m after Bob encodes I in it
N_I
$V \sim \mathcal{U}([-1/2, 1/2))$
$Y_I = \frac{3}{4}(N_I - V + \frac{1}{2})$
supp κ_I
$\mathcal{I}(N_I, V)$
0 1 2 Y_I 3
$I = \lceil Y_I \rceil - 1$
$Y = (Y_I \mod 1)$