SIR with Vital Dynamics

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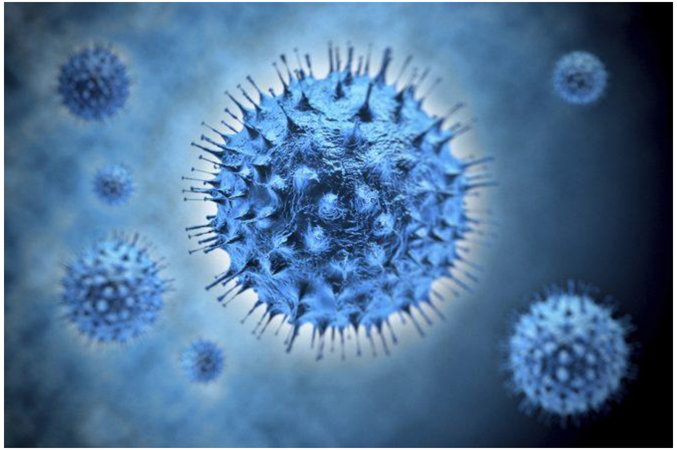
Background

The SIR model with vital dynamics is usually used for modeling endemic diseases. A disease is called **endemic** if it persists in a population. Thus, due to the long time period involved, a model for an endemic disease must include births and natural deaths.

- Malaria in many areas of Africa
- Chickenpox in the UK



Assumptions

- The population is sufficiently large and homogeneously mixing.
- Besides the disease, births and natural deaths also affect the change in the size of the three classes.
- Births and natural deaths occur at equal rates to keep the population size constant.
- All newborns are susceptible.
- Individuals recover from the disease with permanent immunity.

SIR Model

$$S'(t) = -\lambda SI$$

$$I'(t) = \lambda SI - \gamma I$$

$$R(t) = 1 - S(t) - I(t)$$

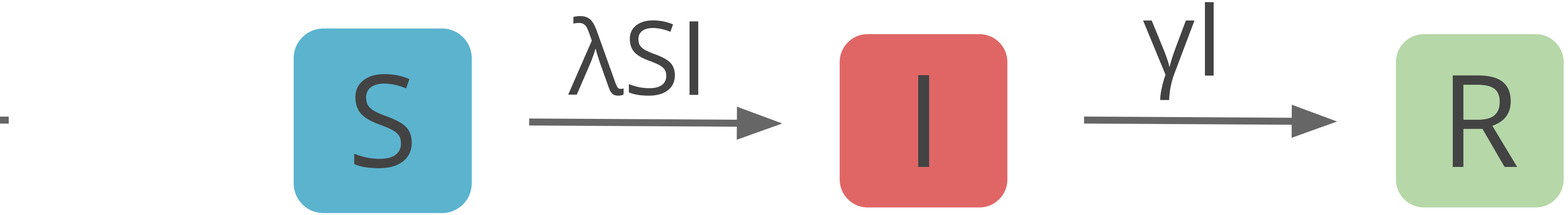
Notation

- $S(t) + I(t) + R(t) = 1$
- the daily death removal rate μ
- the daily recovery removal rate γ
- the daily contact rate λ
- the contact number σ

$$\sigma = \frac{\lambda}{\mu+\gamma}$$

- the replacement number σS

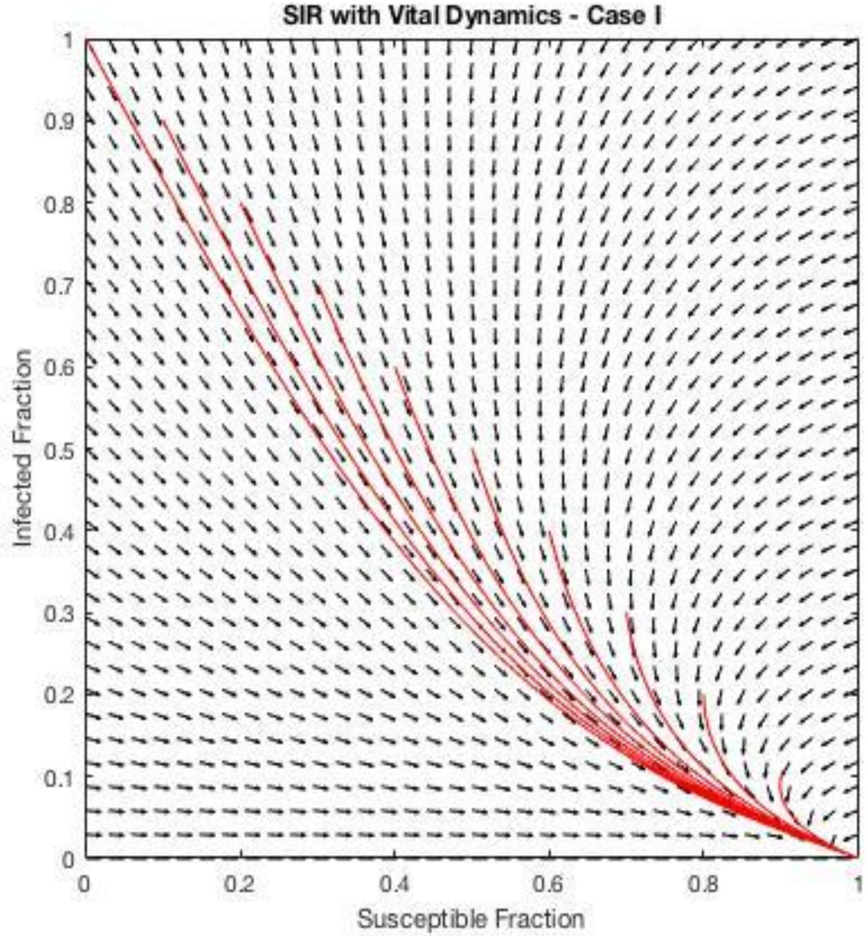
SIR Model



Two Distinct Asymptotic Behaviors

$$\frac{dS}{dt} = -0.25SI + 0.2 - 0.2S$$
$$\frac{dI}{dt} = 0.25SI - 0.15I - 0.2I$$

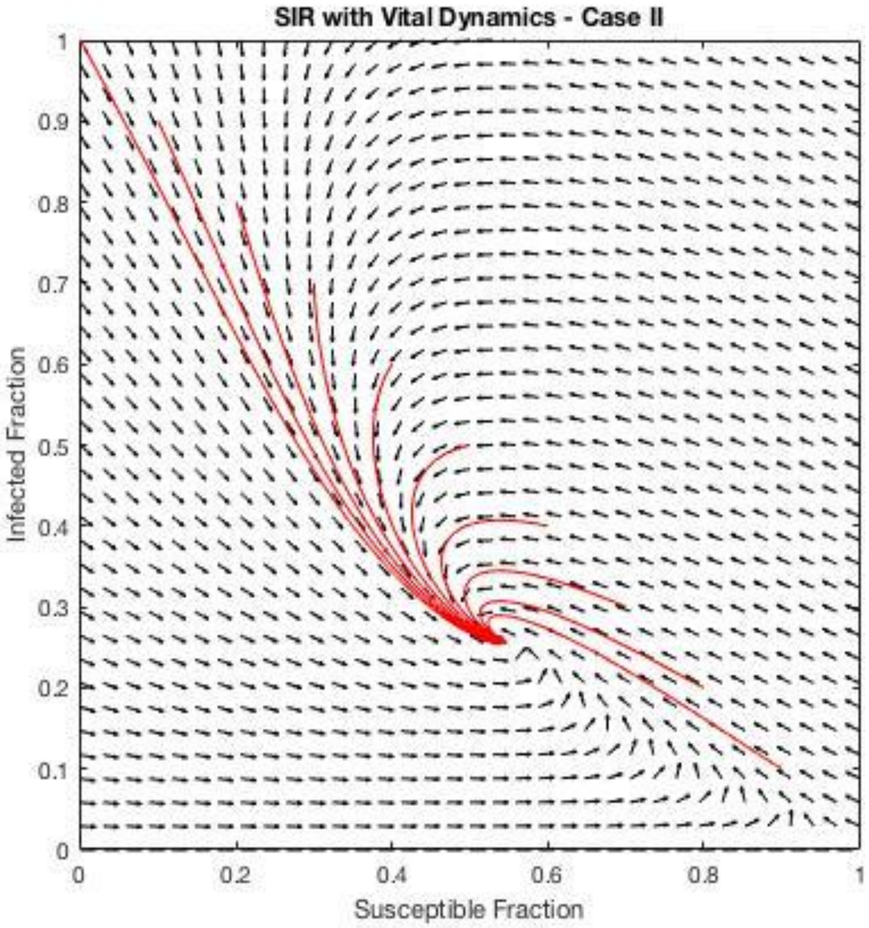
- μ = 0.2, λ = 0.25, γ = 0.15
- σ = 0.25/(0.2+0.15) = 0.714 < 1
- **(S_e, I_e) = (1, 0)**
- The disease dies out



Two Distinct Asymptotic Behaviors

$$\frac{dS}{dt} = -0.65SI + 0.2 - 0.2S$$
$$\frac{dI}{dt} = 0.65SI - 0.15I - 0.2I$$

- μ = 0.2, λ = 0.65, γ = 0.15
- σ = 0.65/(0.2+0.15) = 1.857 > 1
- **(S_e, I_e) = ?**
- The disease remains endemic
- The disease occurs in cycles of outbreaks



The contact number is the threshold quantity

$$\sigma = \frac{\lambda}{\mu+\gamma}$$ → **the replacement number** σS

Case I: $\sigma \leq 1 \Rightarrow \lambda \leq \gamma + \mu$

Case II: $\sigma > 1 \Rightarrow \lambda > \gamma + \mu$

$\sigma S > 1$, $I(t)$ increases to a peak, then decreases to I_e

$\sigma S < 1$, $I(t)$ decreases then rebounces, eventually stabilizes at I_e

The endemic equilibrium point in case II: **$(S_e, I_e) = (1/\sigma, \mu(\sigma-1)/\lambda)$**

The replacement number **$\sigma S = 1$** at the endemic equilibrium point.

Linearization near the equilibrium points

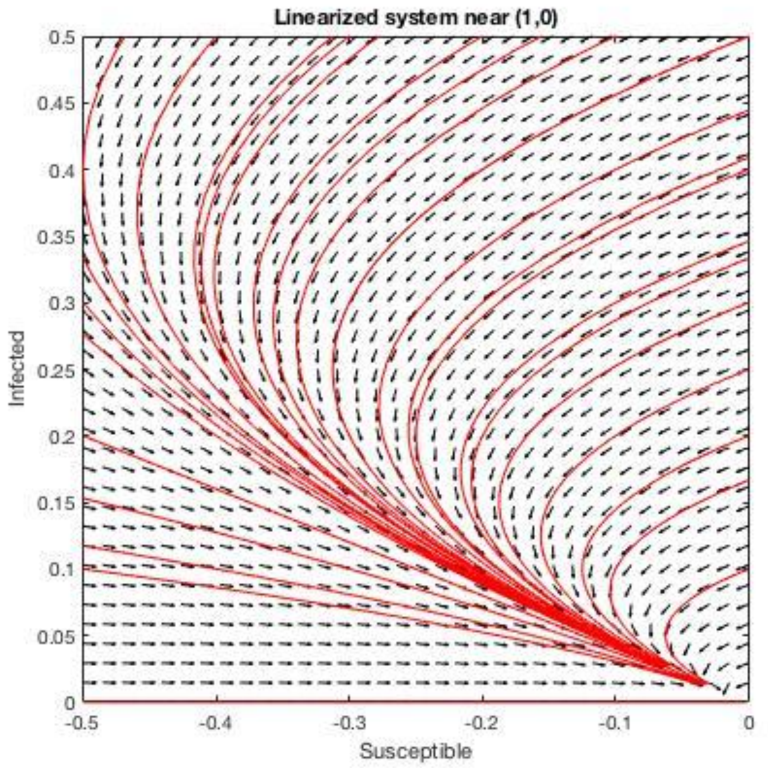
- Case I - linearization near (1, 0)

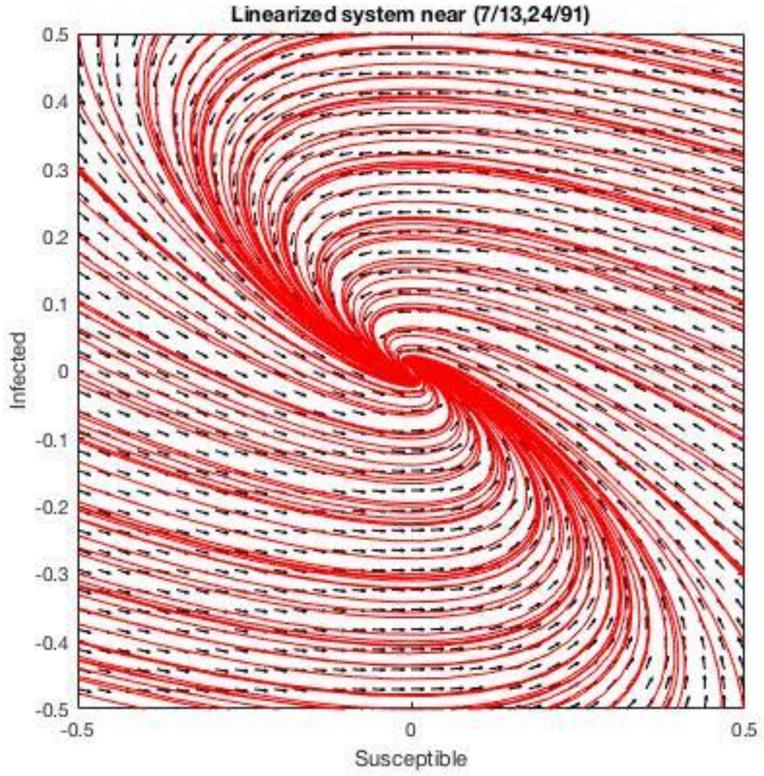
$$J(1,0) = \begin{bmatrix} -0.2 & -0.25 \\ 0 & -0.1 \end{bmatrix}$$

- Case II - linearization near (7/13, 24/91)

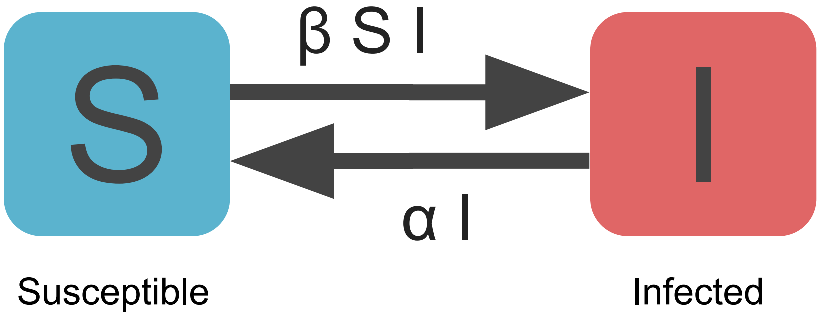
$$J(\tfrac{7}{13}, \tfrac{24}{91}) = \begin{bmatrix} -\frac{13}{35} & -\frac{7}{20} \\ \frac{12}{70} & 0 \end{bmatrix}$$

Linearization near the equilibrium points





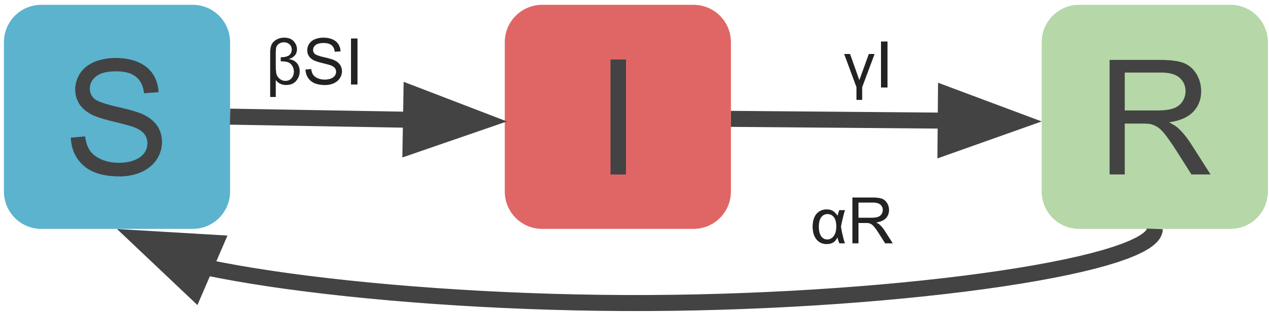
SIS Model



$$\frac{dS}{dt} = \alpha I - \beta S I$$

$$\frac{dI}{dt} = -\alpha I + \beta S I$$

SIRS Model without Vital Dynamics

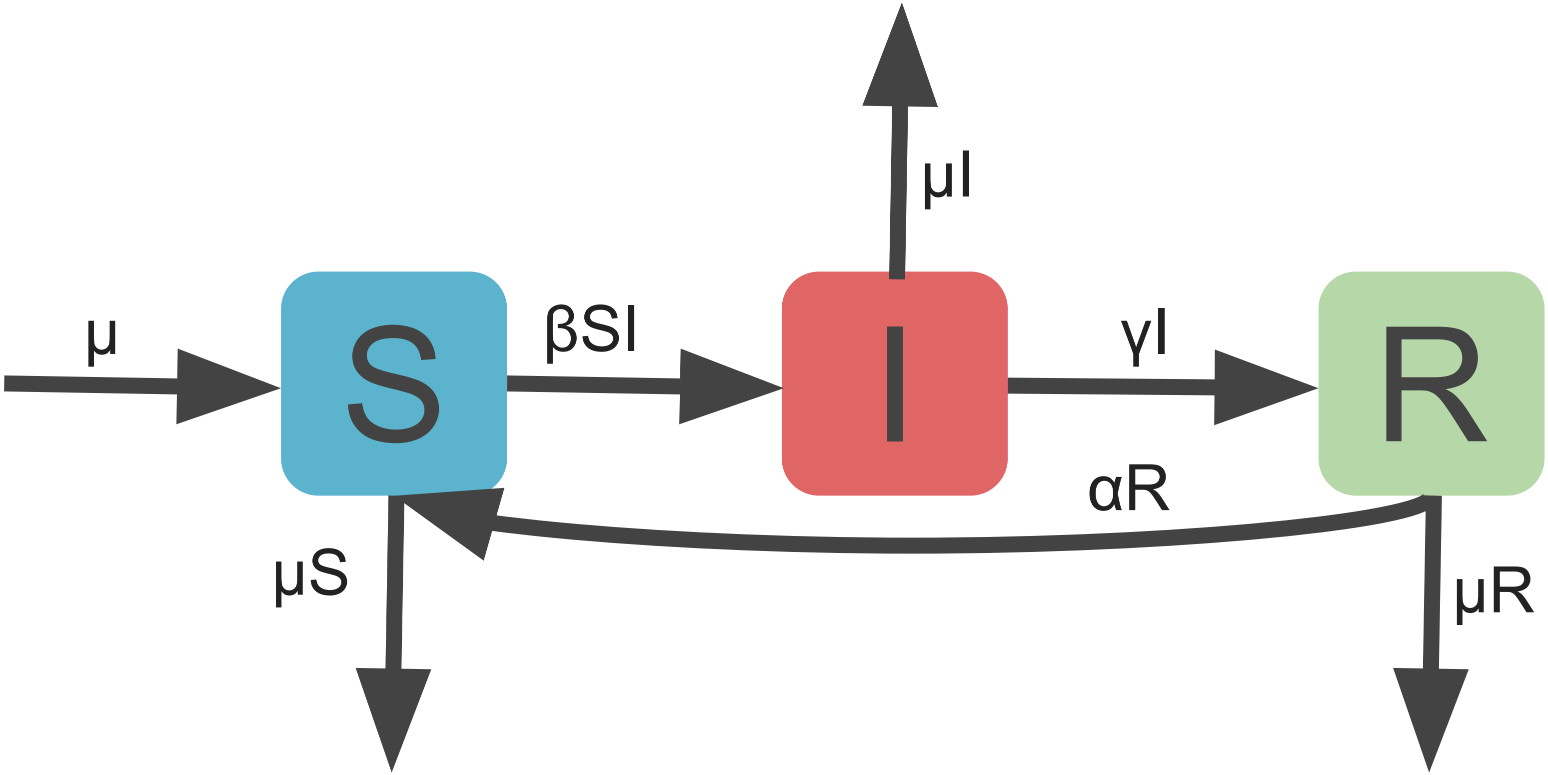


$$\frac{dS}{dt} = -\beta SI + \alpha R$$

$$\frac{dI}{dt} = \beta SI - \gamma I$$

$$\frac{dR}{dt} = \beta I - \alpha R$$

SIRS Model with Vital Dynamics



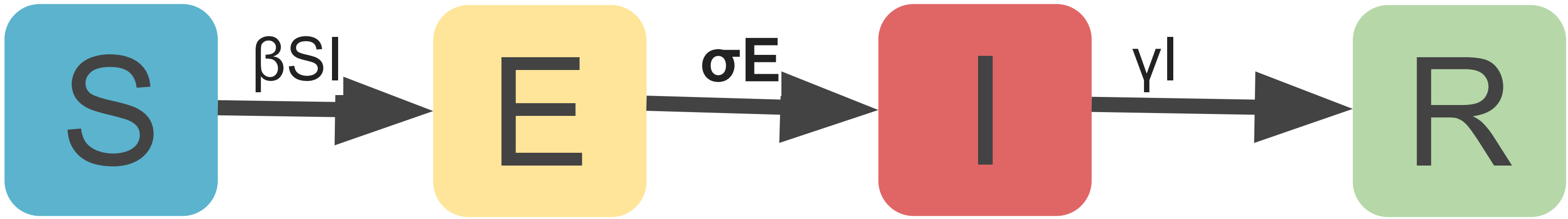
SIRS Model with Vital Dynamics

$$\frac{dS}{dt} = \mu - \beta SI + \alpha R - \mu S$$

$$\frac{dI}{dt} = \beta SI - \gamma I - \mu I$$

$$\frac{dR}{dt} = \gamma I - \alpha R - \mu R$$

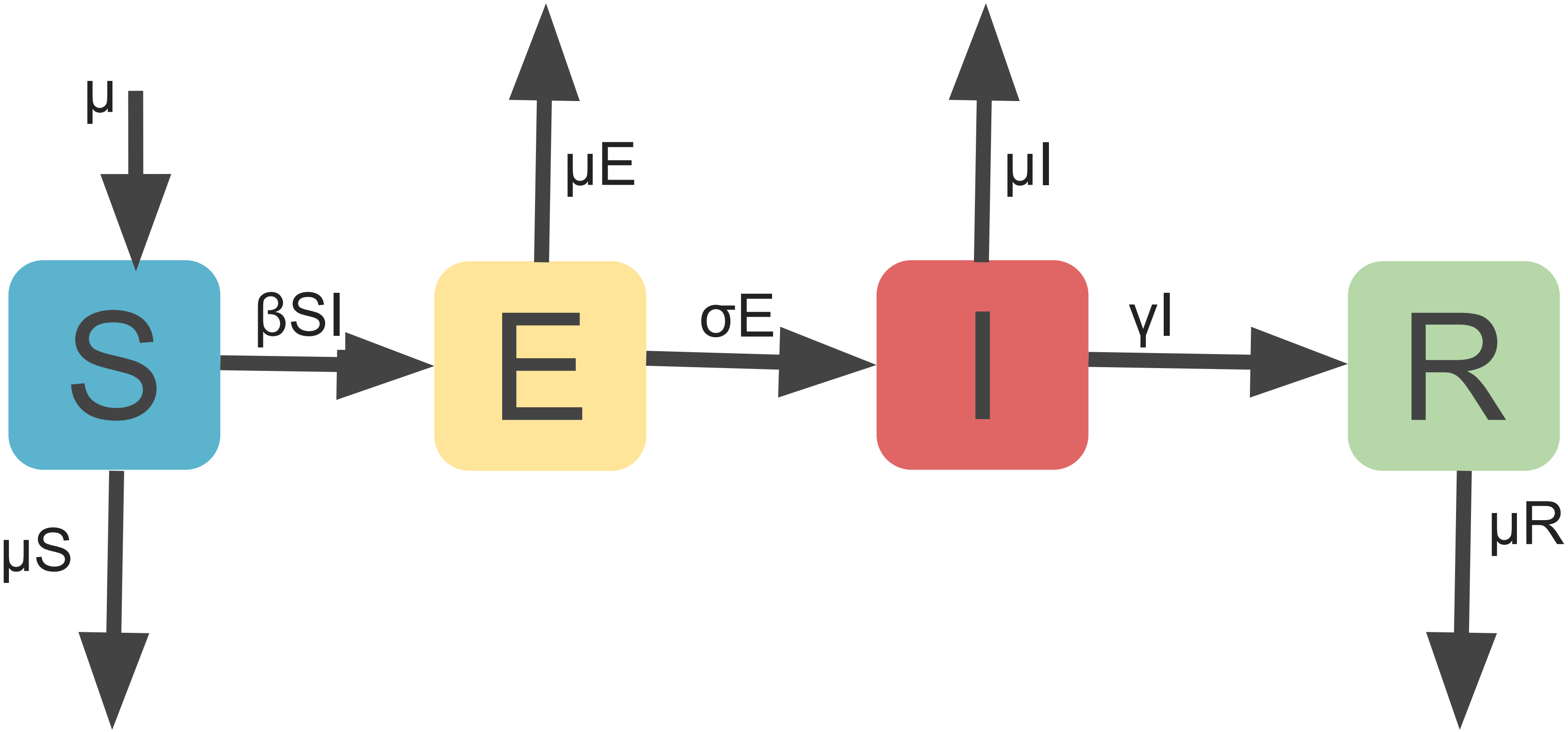
SEIR Model



$$\frac{dS}{dt} = -\beta SI \qquad \frac{dI}{dt} = \sigma E - \gamma I$$

$$\frac{dE}{dt} = \beta SI - \sigma E \qquad \frac{dR}{dt} = \gamma I$$

SEIR model with dynamics



SEIR Model with Dynamics

$$\frac{dS}{dt} = \mu - \mu S - \beta SI$$

$$\frac{dE}{dt} = \beta SI - \mu E - \sigma E$$

$$\frac{dI}{dt} = \sigma E - \gamma I - \mu I$$

$$\frac{dR}{dt} = \gamma I - \mu R$$

Sources

Hethcote H.W. (1989) Three Basic Epidemiological Models. In: Levin S.A., Hallam T.G., Gross L.J. (eds) Applied Mathematical Ecology. Biomathematics, vol 18. Springer, Berlin, Heidelberg