Coronavirus and a look at the logistic curve

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We take a look at the logistic curve as a simple model of infection spreading. This is a standard tool in population modeling, and has the form

$$N(t) = \frac{C}{1 + \mathrm{e}^{-\lambda(t - t_{\max})}}$$

for the number of infected cases as a function of time. The derivative is

$$N'(t) = \frac{\lambda C \mathrm{e}^{-\lambda(t - t_{\max})}}{(1 + \mathrm{e}^{-\lambda(t - t_{\max})})^2},$$

which takes its maximal value

$$N'(t_{\max}) = \frac{\lambda C}{4} \tag{1}$$

at $t = t_{\max}$, the time of the maximal infection rate. The initial phase of the curve is well approximated by the exponential growth $\hat{N}(t) = C\mathrm{e}^{\lambda(t - t_{\max})}$.

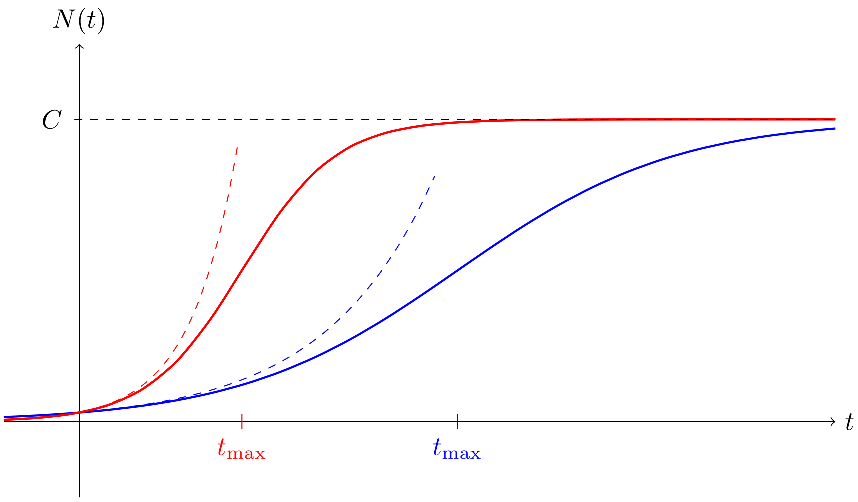

Figure 1: Number of cases with a lower and a higher infection rate λ. The curves start from the same initial value at time $t = 0$, are steepest at $t = t_{\max}$, and converge to the constant C. The dashed line is the approximating exponential growth $\hat{N}(t)$ for the early development of the curve.

Unchecked growth. According to various sources in the media, Coronavirus infections doubled roughly 2.5 days in the early stages of the outbreak before state intervention. Let us see what that means for our λ. From now on, we will measure time in days, hence growth rates λ in 1/days.

$$\begin{aligned}
\hat{N}(t + 2.5) &= 2\hat{N}(t) \\
C\mathrm{e}^{\lambda(t + 2.5 - t_{\max})} &= 2C\mathrm{e}^{\lambda(t - t_{\max})} \\
\mathrm{e}^{2.5\lambda} &= 2 \\
\lambda &= \frac{\ln 2}{2.5} \simeq 0.28.
\end{aligned}$$

Experts say around 80% of the population gets infected before the virus dies out (herd immunity). Focusing our attention to the UK, that means 80% of around 67 million, roughly $C = 54\,000\,000$ for the asymptotic value of the spread. To find out where we are at the moment, we could take an estimation of around $E = 50\,000$ cases and find, using the approximation $\hat{N}$,

$$E = \hat{N}(0) = C \cdot \mathrm{e}^{-0.28 t_{\max}} \tag{2}$$

which gives $t_{\max} = \frac{1}{0.28} \ln\left(\frac{C}{E}\right) \simeq 25$ days. This suggests that without any intervention, in 25 days the UK reaches peak infection rate with

$$N'(t_{\max}) = \frac{\lambda C}{4} = \frac{0.28 \cdot 54\,000\,000}{4} \simeq 3\,800\,000$$

new cases per day. If 1% of these cases need intensive care for two weeks, this would require $3\,800\,000 \cdot 0.01 \cdot 14 \simeq 530\,000$ intensive care unit beds (ICU's). The UK currently has a bit over 4 000 of these. (Since 14 days is comparable to the characteristic time scale of 25 days, we are loosing a bit of accuracy here but that shouldn't change the order of these numbers.)

Flatten the curve. Let us now try to look at a manageable way to acquire herd immunity. Being optimistic, assume capacity can be more than doubled to $I := 10\,000$ ICU's, and set the maximal infection rate accordingly. Again, if 1% requires ICU's for two weeks, that allows for $I/0.01/14$ peak infection rate per day. By (1), that limits the permissible rate as

$$\begin{aligned} \frac{I}{0.14} &= \frac{\lambda C}{4} \\ \lambda &= \frac{200 I}{7C} \end{aligned}$$

which means, in the early phases, doubling cases every t_{doubl} days, where

$$\begin{aligned} \hat{N}(t + t_{\mathrm{doubl}}) &= 2\hat{N}(t) \\ C\mathrm{e}^{\lambda(t + t_{\mathrm{doubl}} - t_{\max})} &= 2C\mathrm{e}^{\lambda(t - t_{\max})} \\ \mathrm{e}^{\lambda t_{\mathrm{doubl}}} &= 2 \\ t_{\mathrm{doubl}} &= \frac{\ln 2}{\lambda} = \frac{7 \ln 2 \cdot C}{200 I}. \end{aligned}$$

If drastic measures could suddenly create such a low value of λ, that would set $t_{\max}$, similar to (2), at

$$t_{\max} = \frac{1}{\lambda} \ln\left(\frac{C}{E}\right) = \frac{7C}{200I} \ln\left(\frac{C}{E}\right).$$

Plugging in the numbers $I = 10\,000$, $C = 54\,000\,000$ and $E = 50\,000$, we get $\lambda \simeq 0.0053$, $t_{\mathrm{doubl}} \simeq 130$, and $t_{\max} \simeq 1300$ days. Compare this with the values for unchecked growth $\lambda \simeq 0.28$, $t_{\mathrm{doubl}} \simeq 2.5$ and $t_{\max} \simeq 25$ days. There is a factor of over 50 between these numbers. In fact this is exactly the factor of ICU's needed for unchecked growth (cca 530 000) vs. the number available (assumed at 10 000).

These calculations suggest that *to get herd immunity in a controlled fashion with (more than doubling) the current resources we have to essentially stop* **any** *further spreading of the virus, and keep such drastic measures up for 3.5 years. And we have to act very quickly.*