Calculus

Gergo Nador

June 19, 2024

Contents

1 Functions

1.1 Definition of a Function

A function is a relationship between two variables, typically denoted as x and y, where the value of y depends on the value of x. This relationship can be expressed as:

$$y = f(x) \tag{1}$$

In this expression, f represents the function that maps each value of x to a corresponding value of y.

1.1.1 Function Representations

- Table
- Formula
- Description
- Graph

1.2 Domain and Range

Domain: The set of all possible input values (often referred to as the independent variable) for which a function is defined.

Range: The set of all possible output values (often referred to as the dependent variable) that a function can produce.

1.3 The Vertical Line Test

A function f can only have one value for each x in its domain.

1.4 Piecewise-Defined Functions

Piecewise-defined function: A function defined by multiple sub-functions, each of which applies to a different interval of the domain.

$$f(x) = \begin{cases} -x & \text{if } x < 0 \\ x^2 & \text{if } 0 \leq x \leq 1 \\ 1 & \text{if } x > 1 \end{cases} \tag{2}$$

1.5 Monotonic Functions

1.5.1 Increasing Functions

Increasing function: A function f is said to be increasing on an interval if for any two numbers x_1 and x_2 in the interval, $x_1 < x_2$ implies $f(x_1) < f(x_2)$.

1.5.2 Decreasing Functions

Decreasing function: A function f is said to be decreasing on an interval if for any two numbers x_1 and x_2 in the interval, $x_1 < x_2$ implies $f(x_1) > f(x_2)$.

1.6 Parity of Functions

1.6.1 Even Functions

Even function: A function f is said to be even if for all x in the domain of f, $f(x) = f(-x)$.

1.6.2 Odd Functions

Odd function: A function f is said to be odd if for all x in the domain of f, $f(x) = -f(-x)$.

1.7 Common Functions

Linear function: A function of the form $f(x) = ax + b$, where a and b are constants.

Polynomial function: A function of the form $f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$, where $a_0, a_1, \ldots, a_n$ are constants and n is a non-negative integer.

Trigonometric function: Functions such as $\sin x$, $\cos x$, $\tan x$, which relate the angles of a triangle to the lengths of its sides.

Exponential function: A function of the form $f(x) = e^x$, 2^x, 10^x, where the base is a constant and the exponent is the variable.

Logarithmic function: A function of the form $f(x) = \ln x$, $\log_2 x$, where ln denotes the natural logarithm and $\log_2$ denotes the logarithm to the base 2.

Transcendental functions: Functions that are not algebraic, such as trigonometric, exponential, and logarithmic functions.

1.8 Composite Functions

Composite function: A function that is formed when one function is applied to the result of another function. It is denoted as

$$(f \circ g)(x) = f(g(x)) \tag{3}$$

For example, if $f(x) = \sin x$ and $g(x) = x^2 - 1$, then:

$$(f \circ g)(x) = \sin(x^2 - 1) \tag{4}$$

The domain of the composite function $(f \circ g)(x)$ is the set of all x such that x is in the domain of g and $g(x)$ is in the domain of f.

2 Limits

Limit: The value that a function $f(x)$ approaches as x approaches a particular point. It is denoted as

$$\lim_{x \to c} f(x) \tag{5}$$

2.1 Example of a Limit

Given the function $f(x) = \frac{x^2-1}{x-1}$, we want to find the limit as x approaches 1.

To simplify $f(x)$:

$$f(x) = \frac{x^2 - 1}{x - 1} = \frac{(x-1)(x+1)}{x-1} = x + 1 \tag{6}$$

As x approaches 1:

$$\lim_{x \to 1} \frac{x^2 - 1}{x - 1} = \lim_{x \to 1}(x + 1) = 2 \tag{7}$$

Therefore, the limit of $f(x)$ as x approaches 1 is 2.

2.2 What is the Limit?

Let's explore different types of limits using examples and graphs.

- For the function $f(x) = \frac{1}{x}$:

$$\lim_{x \to \infty} f(x) = \lim_{x \to \infty} \frac{1}{x} = 0 \tag{8}$$

As x approaches infinity, the value of $f(x)$ approaches 0.

- For the function $g(x)$ defined as:

$$g(x) = \begin{cases} 0 & \text{if } x < 1 \\ 1 & \text{if } x \geq 1 \end{cases} \tag{9}$$

$$\lim_{x \to 1} g(x) \text{ does not exist} \tag{10}$$

The limit does not exist because the left-hand limit and the right-hand limit are not equal. However, $g(1) = 1$.

2.3 One-Sided Limits

One-sided limit: The limit of a function as the input approaches a specified value from one side (either from the left or from the right).

2.3.1 Right-hand Limit

The right-hand limit of a function $f(x)$ as x approaches c from the right (denoted as $x \to c^+$) is the value that $f(x)$ approaches as x approaches c from values greater than c. It is written as:

$$\lim_{x \to c^+} f(x) \tag{11}$$

2.3.2 Left-hand Limit

The left-hand limit of a function $f(x)$ as x approaches c from the left (denoted as $x \to c^-$) is the value that $f(x)$ approaches as x approaches c from values less than c. It is written as:

$$\lim_{x \to c^-} f(x) \tag{12}$$

2.3.3 Example of One-Sided Limits

Consider the function:

$$f(x) = \frac{x}{|x|} \tag{13}$$

This function can be expressed as:

$$f(x) = \begin{cases} 1 & \text{if } x > 0 \\ -1 & \text{if } x < 0 \\ \text{undefined} & \text{if } x = 0 \end{cases} \tag{14}$$

Examining the one-sided limits as x approaches 0:

$$\lim_{x \to 0^+} f(x) = 1 \quad \text{(right-hand limit)} \tag{15}$$

$$\lim_{x \to 0^-} f(x) = -1 \quad \text{(left-hand limit)} \tag{16}$$

Since the right-hand limit and the left-hand limit are not equal, the two-sided limit does not exist at $x = 0$.

2.3.4 Formal Definition of a Two-Sided Limit

A function $f(x)$ has a limit L as x approaches c if and only if both the right-hand limit and the left-hand limit exist and are equal to L. Formally, this can be written as:

$$\lim_{x \to c} f(x) = L \quad \Leftrightarrow \quad \lim_{x \to c^-} f(x) = \lim_{x \to c^+} f(x) = L \tag{17}$$

2.4 The Sandwich Theorem

Sandwich Theorem (Squeeze Theorem): If $f(x) \leq g(x) \leq h(x)$ for all x in some interval around c (except possibly at c itself), and if

$$\lim_{x \to c} f(x) = \lim_{x \to c} h(x) = L, \tag{18}$$

then

$$\lim_{x \to c} g(x) = L. \tag{19}$$

2.4.1 Example of the Sandwich Theorem

Consider the function $\sin x$ and the limits as x approaches 0. We want to find $\lim_{x \to 0} \sin x$.

From trigonometric properties, we know that:

$$-|x| \leq \sin x \leq |x| \quad \text{for all } x. \tag{20}$$

Taking the limit as x approaches 0 for the bounding functions $-|x|$ and $|x|$:

$$\lim_{x \to 0} -|x| = \lim_{x \to 0} |x| = 0. \tag{21}$$

By the Sandwich Theorem, since $\sin x$ is squeezed between $-|x|$ and $|x|$, we have:

$$\lim_{x \to 0} \sin x = 0. \tag{22}$$

This result is visually evident when considering the unit circle, where $\sin x = \frac{\text{opposite}}{\text{hypotenuse}}$. As x approaches 0, the opposite side of the triangle approaches 0, leading to $\sin x$ approaching 0.

3 Continuity

Continuity: A function $f(x)$ is said to be continuous at a point c if the following three conditions are met:

1. $f(c)$ is defined.
2. $\lim_{x \to c} f(x)$ exists.
3. $\lim_{x \to c} f(x) = f(c)$.

3.1 Intuitive Explanation of Continuity

Intuitively, a function is continuous if its graph can be drawn without lifting the pen from the paper. This means there are no jumps, breaks, or holes in the graph of the function.

3.2 Types of Discontinuities

Discontinuities in functions can be categorized into several types. The primary types are removable discontinuities and jump discontinuities.

Removable Discontinuity: A discontinuity at $x = c$ is called removable if $\lim_{x\to c} f(x)$ exists, but $f(c)$ is either not defined or not equal to the limit. This type of discontinuity can be "removed" by redefining $f(c)$ to be equal to the limit. For example:

$$\lim_{x\to 2} f(x) \text{ exists, but } f(2) \neq \lim_{x\to 2} f(x). \tag{23}$$

Jump Discontinuity: A discontinuity at $x = c$ is called a jump discontinuity if the left-hand limit and the right-hand limit exist but are not equal:

$$\lim_{x\to c^-} f(x) \neq \lim_{x\to c^+} f(x). \tag{24}$$

3.3 Continuous Functions

Continuous Functions: Functions that are continuous at every single point in their domain. This means that for each point c in the domain of f, the following conditions hold:

1. $f(c)$ is defined.
2. $\lim_{x\to c} f(x)$ exists.
3. $f(c) = \lim_{x\to c} f(x)$.

If these conditions are met for every point in the domain, the function is considered continuous on that domain. Otherwise, the function is said to be discontinuous.

Continuity ensures that the function behaves predictably without any sudden changes in value. In mathematical terms, this means there are no breaks, jumps, or holes in the graph of the function.

4 Derivatives

4.1 The Derivative at a Point

The derivative of function f at point x_0 is

$$f'(x_0) = \lim_{\Delta x\to 0} \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x} \tag{25}$$

- The slope of the tangent line to the curve $f(x)$ at $x = x_0$; the slope of the graph of $f(x)$ at $x = x_0$.
- The slope of the graph of $f(x)$ at $x = x_0$; the rate of change of $f(x)$ with respect to x.
- The rate of change of $f(x)$ with respect to x; provided that the limits exist.

4.2 The Derivative as a Function

The derivative of function f is

$$f'(x) = \lim_{\Delta x\to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} \tag{26}$$

- $f'(x)$ is a new function of x; it represents the rate of change of $f(x)$ with respect to x.
- If f' exists at every point in the domain of f; then f is **differentiable**.

4.3 Derivatives of Common Functions

Constant function:

$$f(x) = c \tag{27}$$
$$f'(x) = 0 \tag{28}$$

Polynomial function:

$$f(x) = x^n \tag{29}$$
$$f'(x) = nx^{n-1} \tag{30}$$

Exponential function:

$$f(x) = e^x \tag{31}$$
$$f'(x) = e^x \tag{32}$$

Logarithmic function:

$$f(x) = \ln x \tag{33}$$
$$f'(x) = \frac{1}{x} \tag{34}$$

Trigonometric functions:

$$f(x) = \sin x \tag{35}$$
$$f'(x) = \cos x \tag{36}$$
$$f(x) = \cos x \tag{37}$$
$$f'(x) = -\sin x \tag{38}$$
$$f(x) = \tan x \tag{39}$$
$$f'(x) = \frac{1}{\cos^2 x} \tag{40}$$

4.4 Differentiation

Differentiation: Differentiation is the process of calculating the derivative.

$$f'(x) = \frac{d}{dx} f(x) \tag{41}$$

A function is differentiable at a point if its derivative exists at that point. If the function has a corner or a discontinuity at the point, it is not differentiable there.

A corner is a point where the graph of the function has a sharp turn, causing the derivative to be undefined. A discontinuity is a point where the function is not continuous, leading to the derivative not existing at that point.

4.5 Differentiation Rules

$$\frac{d}{dx}(f+g) = \frac{df}{dx} + \frac{dg}{dx} \quad \text{[sum rule]} \tag{42}$$

$$\frac{d}{dx}(cf) = c\frac{df}{dx} \quad \text{[constant multiple rule]} \tag{43}$$

$$\frac{d}{dx}(fg) = f\frac{dg}{dx} + \frac{df}{dx}g \quad \text{[product rule]} \tag{44}$$

$$\frac{d}{dx}f(g(x)) = f'(g(x))g'(x) \quad \text{[chain rule]} \tag{45}$$

4.6 L'Hôpital's Rule

L'Hôpital's Rule:

L'Hôpital's Rule is used to evaluate limits of indeterminate forms such as $\frac{0}{0}$ or $\frac{\infty}{\infty}$.
If $\lim_{x\to c} f(x) = 0$ and $\lim_{x\to c} g(x) = 0$ or $\lim_{x\to c} f(x) = \pm\infty$ and $\lim_{x\to c} g(x) = \pm\infty$, then

$$\lim_{x\to c} \frac{f(x)}{g(x)} = \lim_{x\to c} \frac{f'(x)}{g'(x)} \tag{46}$$

provided that the limit on the right-hand side exists.
L'Hôpital's Rule can also be applied to one-sided limits and limits at infinity.

4.6.1 Solving Indeterminate Forms with L'Hôpital's Rule

L'Hôpital's Rule can be used to resolve the following indeterminate forms:

$$\frac{0}{0} \qquad \frac{\infty}{\infty} \qquad 0\cdot\infty \qquad \infty-\infty \qquad 0^0 \qquad \infty^0 \qquad 1^\infty$$

To apply L'Hôpital's Rule, follow these steps:

1. **Verify the Indeterminate Form**: Ensure that the limit results in one of the indeterminate forms mentioned above.
2. **Differentiate the Numerator and Denominator**: Compute the derivatives of the numerator and the denominator of the function separately.
3. **Take the Limit**: Evaluate the limit of the new function formed by the derivatives. If the result is still an indeterminate form, you may need to apply L'Hôpital's Rule multiple times.
4. **Apply the Rule**: Use L'Hôpital's Rule iteratively until the limit can be determined.

4.6.2 Examples

1. Form $\frac{0}{0}$ or $\frac{\infty}{\infty}$:

$$\lim_{x\to c} \frac{f(x)}{g(x)} = \lim_{x\to c} \frac{f'(x)}{g'(x)} \tag{47}$$

2. Form $0\cdot\infty$:

$$\lim_{x\to c} f(x)\cdot g(x) = \lim_{x\to c} \frac{f(x)}{\frac{1}{g(x)}} \quad \text{or} \quad \lim_{x\to c} \frac{g(x)}{\frac{1}{f(x)}} \tag{48}$$

Convert the product to a fraction and then apply L'Hôpital's Rule.

3. Form $\infty-\infty$:

$$\lim_{x\to c}(f(x)-g(x)) \tag{49}$$

Combine the terms into a single fraction and then apply L'Hôpital's Rule.

4. Forms 0^0, ∞^0, 1^∞:

$$\lim_{x \to c} f(x)^{g(x)} \tag{50}$$

Take the natural logarithm to transform the exponentiation into a product, then apply L'Hôpital's Rule.

5 Optimization

5.1 Extreme Values

Extreme values of a function are the points where the function reaches its minimum or maximum values. These points can be either local or absolute.

To find the extreme values of a function $f(x)$, follow these steps:

1. **Find the derivative**: Compute the derivative of the function, $\frac{df}{dx}$.
2. **Set the derivative to zero**: Solve the equation $\frac{df}{dx} = 0$ to find the critical points. These points are where the slope of the tangent line is zero, indicating potential maxima or minima.
3. **Determine the nature of the critical points**: Use the second derivative test or the first derivative test to classify the critical points as local maxima, local minima, or points of inflection.

The slope of the function, represented by its derivative, is zero at the extreme values.

$$\frac{df}{dx} = 0 \tag{51}$$

- **Absolute Maximum**: The highest point over the entire domain of the function.
- **Absolute Minimum**: The lowest point over the entire domain of the function.
- **Local Maximum**: The highest point within a small interval around the point.
- **Local Minimum**: The lowest point within a small interval around the point.

5.1.1 Other Candidates for Extreme Values

Extreme values can also occur at:

- **Interior points where $f'(x)$ is undefined**: For example, the function $f(x) = |x|$ has a corner at $x = 0$, where the derivative is undefined. This point can be an extreme value.
- **Endpoints of the domain of f**: For example, the function $f(x) = \sqrt{x}$ has an endpoint at $x = 0$, which can be an extreme value.

5.2 Critical Points

A critical point is an interior point c where $f'(c)$ is zero or undefined.

$$f'(c) = 0 \quad \text{or} \quad f'(c) \text{ is undefined} \tag{52}$$

Critical points are potential candidates for local maxima or minima.

5.3 First and Second Derivatives

The first and second derivatives of a function provide important information about the function's extreme values. The first derivative test and the second derivative test are used to classify critical points as local minima or maxima.

	$\frac{df}{dx}$	$\frac{d^2f}{dx^2}$
Minimum	0	> 0
Maximum	0	< 0

- **Minimum**: At a minimum point, the first derivative $\frac{df}{dx} = 0$ and the second derivative $\frac{d^2f}{dx^2} > 0$.
- **Maximum**: At a maximum point, the first derivative $\frac{df}{dx} = 0$ and the second derivative $\frac{d^2f}{dx^2} < 0$.

The first derivative test can indicate whether a critical point is a local minimum, maximum, or neither. The second derivative test provides further confirmation by examining the concavity of the function at the critical points.

5.4 Newton's Method (or the Newton-Raphson Method)

Newton's Method finds approximate solutions to equations of the form $f(x) = 0$, known as "roots".

5.4.1 Procedure for Newton's Method

To apply Newton's Method, follow these steps:

1. **Initial Approximation**: Start with an initial guess x_0 for a root of the equation $f(x) = 0$.
2. **Tangent Line Approximation**: Use the tangent line at x_0 to approximate the function. The equation of the tangent line at $x = x_0$ is:
$$y = f(x_0) + f'(x_0)(x - x_0) \tag{53}$$
3. **Find the x-intercept**: The x-intercept of the tangent line (where $y = 0$) provides a new approximation x_1:
$$0 = f(x_0) + f'(x_0)(x_1 - x_0) \implies x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} \tag{54}$$
4. **Iterate**: Repeat the process using x_1 to find x_2, and so on. The general formula for the $(n+1)$-th approximation is:
$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \tag{55}$$

This iterative process continues until the desired accuracy is achieved.

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \tag{56}$$

5.4.2 Estimating $\sqrt{2}$

To estimate $\sqrt{2}$ using Newton's Method, we start by noting that $\sqrt{2}$ is a root of the function $f(x) = x^2 - 2$. The derivative of this function is $f'(x) = 2x$.

Starting with an initial guess $x_0 = 1$, we apply the Newton's Method formula:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \tag{57}$$

For $f(x) = x^2 - 2$ and $f'(x) = 2x$, the iterations are as follows:

n	x_n	$f(x_n)$	$f'(x_n)$	x_{n+1}
0	1	-1	2	$1 - \frac{-1}{2} = 1.5$
1	1.5	0.25	3	$1.5 - \frac{0.25}{3} = 1.416\overline{6}$
2	$1.416\overline{6}$	0.006944	2.8333	$1.416\overline{6} - \frac{0.006944}{2.8333} = 1.414216$
3	1.414216	≈ 0	≈ 2.828	≈ 1.414216

After three iterations, we find that $x_3 \approx 1.414216$, which is an approximation of $\sqrt{2}$.

5.4.3 Newton's Method for Optimization

Say we don't want to find roots, but critical points. Then Newton's Method can be modified:

To find critical points of $f(x)$, where $f'(x) = 0$, we use the following iterative formula:

$$x_{n+1} = x_n - \frac{f'(x_n)}{f''(x_n)} \tag{58}$$

This method uses the first and second derivatives of the function to converge to a critical point, which can be a local minimum, local maximum, or a saddle point.

5.4.4 A Note of Caution

When using Newton's Method, be aware of the following potential issues:

- **May diverge**: Newton's Method can fail to converge to a solution if the initial guess is not close enough to the actual root or if the function behaves poorly.
- **May find a different solution than the one you want**: The method may converge to a different root than expected, especially if there are multiple roots or if the function has inflection points.

5.5 Antiderivatives

A function F is an antiderivative of f if $F'(x) = f(x)$ for all x.

5.5.1 Finding Antiderivatives

Examples of finding antiderivatives:

1. For $f(x) = 2x$:
$$F(x) = x^2 + C \tag{59}$$
2. For $g(x) = \cos x$:
$$G(x) = \sin x + C \tag{60}$$
3. For $h(x) = x^n$:
$$H(x) = \frac{1}{n+1}x^{n+1} + C \tag{61}$$

Where C is the constant of integration.

5.6 Indefinite Integrals

The indefinite integral represents the collection of all antiderivatives of a function f. It is denoted as:

$$F(x) = \int f(x)\,dx \tag{62}$$

For example:

$$\int (x^2 - 2x + 5)\,dx = \frac{1}{3}x^3 - x^2 + 5x + C \tag{63}$$

Where C is the constant of integration.

5.7 The Definite Integral

The definite integral of a function $f(x)$ over the interval $[a, b]$ is defined as the limit of a Riemann sum as the number of subintervals approaches infinity:

$$S_n = \sum_{k=1}^{n} f\left(a + k\frac{b-a}{n}\right) \cdot \frac{b-a}{n} \tag{64}$$

The definite integral is given by:

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} S_n \tag{65}$$

Alternatively, using the Fundamental Theorem of Calculus, the definite integral can be evaluated as:

$$\int_a^b f(x)\,dx = F(b) - F(a) = [F(x)]_a^b \tag{66}$$

where $F(x)$ is an antiderivative of $f(x)$.

6 Integral Techniques

6.1 Integration by Substitution

Integration by substitution is the opposite of the chain rule. It is used to simplify integrals by making a substitution that turns the integral into an easier form.

$$\int f(g(x))g'(x)\,dx = \int f(u)\,du \tag{67}$$

Where $u = g(x)$ and $du = g'(x)\,dx$.

6.1.1 The Procedure

1. Let $u = g(x)$.
2. Calculate $du = \frac{du}{dx}\,dx$.
3. Rewrite the integral in terms of u.
4. Integrate with respect to u.
5. Replace u by $g(x)$.

For example, if your integral looks like:

$$\int f(g(x))g'(x)\,dx \tag{68}$$

Then rename $g(x) = u$, and the integral becomes:

$$\int f(u)\,du \tag{69}$$

6.1.2 And with a Definite Integral

When applying integration by substitution to a definite integral, the limits of integration change according to the substitution. Specifically, if $u = g(x)$, then the limits of integration a and b change to $g(a)$ and $g(b)$ respectively.

$$\int_a^b f(g(x))g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,du \tag{70}$$

6.1.3 Example

Find $\int \cos(7x + 5)\,dx$.

1. Define $u = 7x + 5$.
2. Then $du = \frac{du}{dx}dx = 7 \implies du = 7\,dx$.

   This means we can solve:

$$\begin{aligned}
\int \cos(7x+5)\,dx &= \frac{1}{7}\int \cos(7x+5)\cdot 7\,dx & (71)\\
&= \frac{1}{7}\int \cos(u)\,du & (72)\\
&= \frac{1}{7}\sin(u) + C & (73)\\
&= \frac{1}{7}\sin(7x+5) + C & (74)
\end{aligned}$$

6.1.4 Another Example

Evaluate $\int (2x + 4)^5\,dx$.

1. $u = 2x + 4$.
2. $du = \frac{du}{dx}\,dx = 2\,dx$.

$$\begin{aligned}
\int (2x+4)^5\,dx &= \frac{1}{2}\int (2x+4)^5 \cdot 2\,dx & (75)\\
&= \frac{1}{2}\int u^5\,du & (76)\\
&= \frac{1}{2}\cdot\frac{1}{6}u^6 + C & (77)\\
&= \frac{1}{12}u^6 + C & (78)\\
&= \frac{1}{12}(2x+4)^6 + C & (79)
\end{aligned}$$

6.2 Integration by Parts

Integration by parts is the opposite of the product rule in differentiation. It is used to integrate products of functions and is given by the formula:

$$\int f(x)g'(x)\,dx = f(x)g(x) - \int g(x)f'(x)\,dx \tag{80}$$

Where $f(x)$ is a function you like to differentiate, and $g(x)$ is a function you like to integrate.

This method is often used when the integrand is a product of two functions, and one of them can be easily differentiated while the other can be easily integrated.

6.2.1 Example

Evaluate $\int x \cos x \, dx$.

Using the integration by parts formula:

$$\int f(x)g'(x)\,dx = f(x)g(x) - \int f'(x)g(x)\,dx \tag{81}$$

Let $f(x) = x$ and $g'(x) = \cos x$. Then, $f'(x) = 1$ and $g(x) = \sin x$.

Applying the integration by parts formula:

$$\begin{aligned}
\int x \cos x \, dx &= x \sin x - \int 1 \cdot \sin x \, dx && (82)\\
&= x \sin x - \int \sin x \, dx && (83)\\
&= x \sin x - (-\cos x) + C && (84)\\
&= x \sin x + \cos x + C && (85)
\end{aligned}$$

To verify, check the derivative of the result:

$$\begin{aligned}
\frac{d}{dx}(x \sin x + \cos x + C) &= x \cos x + \sin x + (-\sin x) && (86)\\
&= x \cos x && (87)
\end{aligned}$$

6.2.2 The Integral of $\ln(x)$

Evaluate $\int \ln(x) \, dx$.

Using the integration by parts formula:

$$\int f(x)g'(x)\,dx = f(x)g(x) - \int f'(x)g(x)\,dx \tag{88}$$

Let $f(x) = \ln(x)$ and $g'(x) = 1$. Then, $f'(x) = \frac{1}{x}$ and $g(x) = x$.

Applying the integration by parts formula:

$$\begin{aligned}
\int \ln(x)\,dx &= \ln(x) \cdot x - \int \frac{1}{x} \cdot x \, dx && (89)\\
&= x \ln(x) - \int dx && (90)\\
&= x \ln(x) - x + C && (91)
\end{aligned}$$

To verify, check the derivative of the result:

$$\begin{aligned}
\frac{d}{dx}\left(x \ln(x) - x + C\right) &= \ln(x) + 1 - 1 && (92)\\
&= \ln(x) && (93)
\end{aligned}$$

6.3 Integrals of Odd Functions

An **odd function** is defined as a function $f(x)$ such that:

$$f(x) = -f(-x) \tag{94}$$

For odd functions, the definite integral over a symmetric interval $[-a, a]$ is zero. Specifically:

$$\int_{-a}^{a} f_{\text{odd}}(x)\,dx = 0 \tag{95}$$

6.3.1 How to Determine if a Function is Odd

To determine if a function is odd, you can use the following characteristics:

Even Functions	**Odd Functions**
$\lvert x \rvert$	x
x^n, n even	x^n, n odd
$\cos x$	$\sin x$
e^{-x^2}	$\tan x$

Properties of operations on even (e) and odd (o) functions:

Addition	**Multiplication**	**Composition**	**Differentiation**
$e + e = e$	$e \cdot e = e$	$e \circ e = e$	$e' = o$
$o + o = o$	$o \cdot o = e$	$o \circ o = o$	$o' = e$
$e + o = \varnothing$	$e \cdot o = o$	$e \circ o = e$	

6.4 Integrals of Even Functions

For an even function, the definite integral over a symmetric interval $[-a, a]$ can be simplified as follows:

$$\int_{-a}^{a} f_{\text{even}}(x)\, dx = 2 \int_{0}^{a} f_{\text{even}}(x)\, dx \tag{96}$$

6.5 Trigonometric Substitution

Trigonometric substitution is a technique used to evaluate integrals involving expressions of the form $\sqrt{a^2 - x^2}$, $\sqrt{a^2 + x^2}$, and $\sqrt{x^2 - a^2}$. This method uses trigonometric identities to simplify the integrand.

6.5.1 Example: Integrating with $\sqrt{a^2 - x^2}$

Consider the integral:

$$\int \frac{1}{\sqrt{a^2 - x^2}}\, dx \tag{97}$$

We use the substitution $x = a \sin\theta$. This implies $dx = a\cos\theta\, d\theta$. From the triangle:

$$\sin\theta = \frac{x}{a} \implies x = a\sin\theta \tag{98}$$

Thus, the differential dx becomes:

$$dx = a\cos\theta\, d\theta \tag{99}$$

Substituting these into the integral, we get:

$$\int \frac{1}{\sqrt{a^2 - x^2}}\, dx = \int \frac{a\cos\theta}{\sqrt{a^2 - a^2\sin^2\theta}}\, d\theta \tag{100}$$

Simplifying the integrand using the Pythagorean identity $\sin^2\theta + \cos^2\theta = 1$:

$$\sqrt{a^2 - a^2\sin^2\theta} = a\sqrt{1 - \sin^2\theta} = a\cos\theta \tag{101}$$

The integral becomes:

$$\int \frac{a\cos\theta}{a\cos\theta}\, d\theta = \int d\theta = \theta + C \tag{102}$$

Converting back to x using $\theta = \sin^{-1}\left(\frac{x}{a}\right)$, we get:

$$\int \frac{1}{\sqrt{a^2 - x^2}}\, dx = \sin^{-1}\left(\frac{x}{a}\right) + C \tag{103}$$

6.5.2 Reference Triangles for Trigonometric Substitution

To aid in understanding, reference triangles are used for each type of trigonometric substitution:

1. For $\sqrt{a^2 - x^2}$, use $x = a \sin\theta$:

$$\text{Triangle:} \quad \sqrt{a^2 - x^2} = a \cos\theta \tag{104}$$

2. For $\sqrt{a^2 + x^2}$, use $x = a \tan\theta$:

$$\text{Triangle:} \quad \sqrt{a^2 + x^2} = a \sec\theta \tag{105}$$

3. For $\sqrt{x^2 - a^2}$, use $x = a \sec\theta$:

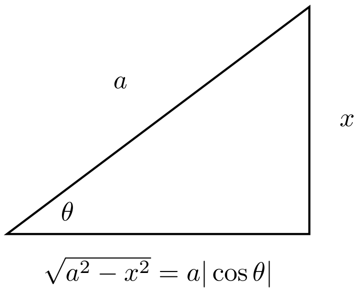
$$\text{Triangle:} \quad \sqrt{x^2 - a^2} = a \tan\theta \tag{106}$$

Each substitution simplifies the integral by using trigonometric identities, converting a complex algebraic integrand into a more manageable trigonometric form.

6.5.3 Reference Triangles for Trigonometric Substitution

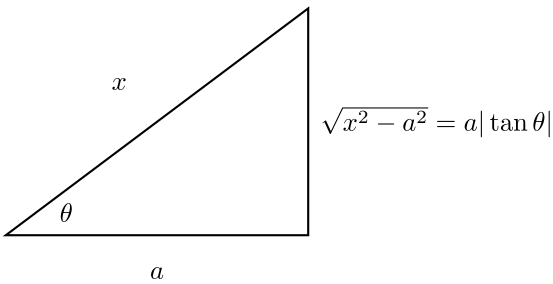
To aid in understanding, reference triangles are used for each type of trigonometric substitution:

For $\sqrt{a^2 - x^2}$, use $x = a \sin\theta$:



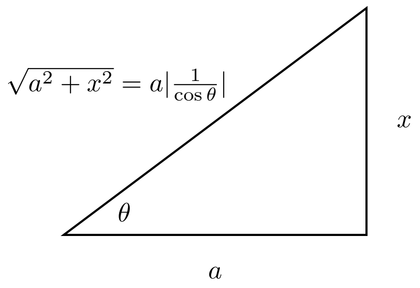
$$\sqrt{a^2 - x^2} = a \cos\theta \tag{107}$$
$$x = a \sin\theta \tag{108}$$
$$dx = a \cos\theta d\theta \tag{109}$$

For $\sqrt{x^2 - a^2}$, use $x = a\frac{1}{\cos\theta}$:



$$\sqrt{x^2 - a^2} = a|\tan\theta| \tag{110}$$
$$x = a\frac{1}{\cos\theta} \tag{111}$$
$$dx = a\frac{\tan\theta}{\cos\theta} d\theta \tag{112}$$

For $\sqrt{a^2 + x^2}$, use $x = a|\tan\theta|$:



$$\sqrt{a^2+x^2} = a|\frac{1}{\cos\theta}| \tag{113}$$
$$x = a|\tan\theta| \tag{114}$$
$$dx = a\frac{1}{\cos^2\theta}d\theta \tag{115}$$

6.6 Numerical Integration

Numerical integration is a method used to approximate the area under a curve when an exact integral is difficult or impossible to calculate analytically. One common method for numerical integration is the **Trapezoidal Rule**.

6.6.1 The Trapezoidal Rule

The trapezoidal rule approximates the area under a curve by dividing it into a series of trapezoids and summing their areas. The formula for the trapezoidal rule is:

$$T = \frac{1}{2}(y_0+y_1)\Delta x + \frac{1}{2}(y_1+y_2)\Delta x + \cdots + \frac{1}{2}(y_{n-1}+y_n)\Delta x \tag{116}$$
$$= \frac{\Delta x}{2}\left(y_0 + 2y_1 + 2y_2 + \cdots + 2y_{n-1} + y_n\right) \tag{117}$$

The area of each trapezoid is calculated by taking the average of the two heights and multiplying by the width:

$$\text{Area} = \frac{1}{2}(y_1+y_2)\Delta x \tag{118}$$

6.6.2 Simpson's Rule

Simpson's rule is a method for numerical integration that approximates the integral of a function by using parabolic segments. It provides a more accurate approximation than the trapezoidal rule by fitting a parabola through each pair of subintervals.

The formula for Simpson's rule is:

$$\int_a^b f(x)\,dx \approx \frac{\Delta x}{3}\left(y_0 + 4y_1 + 2y_2 + 4y_3 + \cdots + 2y_{n-2} + 4y_{n-1} + y_n\right) \tag{119}$$

where $\Delta x = \frac{b-a}{n}$ and n is even.

7 Improper Integrals

7.1 Improper Integrals of Type I

Improper integrals of type I occur when one or both limits of integration are infinite. These integrals are evaluated using limits.

Consider the integral $\int_a^\infty f(x)\,dx$:

$$\int_a^\infty f(x)\,dx = \lim_{b\to\infty}\int_a^b f(x)\,dx \tag{120}$$

Similarly, for the integral $\int_{-\infty}^b f(x)\,dx$:

$$\int_{-\infty}^b f(x)\,dx = \lim_{a\to-\infty}\int_a^b f(x)\,dx \tag{121}$$

For integrals over the entire real line:

$$\int_{-\infty}^\infty f(x)\,dx = \int_{-\infty}^c f(x)\,dx + \int_c^\infty f(x)\,dx, \quad c \text{ any real number} \tag{122}$$

- If the limit exists and is finite, the improper integral **converges**.
- Otherwise, the improper integral **diverges**.

7.1.1 A Diverging Improper Integral

Consider the integral $\int_0^\infty x^2\,dx$:

$$\int_0^\infty x^2\,dx = \lim_{b\to\infty}\int_0^b x^2\,dx \tag{123}$$

$$= \lim_{b\to\infty}\left[\frac{x^3}{3}\right]_0^b \tag{124}$$

$$= \lim_{b\to\infty}\left(\frac{b^3}{3} - \frac{0^3}{3}\right) \tag{125}$$

$$= \lim_{b\to\infty}\frac{b^3}{3} \tag{126}$$

$$= \infty \tag{127}$$

Since the limit is infinite, the integral diverges.

7.1.2 Simplified Notation

Alternatively, the integral can be expressed directly using the improper limit:

$$\int_0^\infty x^2\,dx = \left[\frac{1}{3}x^3\right]_0^\infty \tag{128}$$

$$= \frac{1}{3}\left(\infty^3 - 0^3\right) \tag{129}$$

$$= \frac{1}{3}\cdot\infty^3 \tag{130}$$

$$= \infty \tag{131}$$

Thus, the integral diverges.

7.1.3 Improper Integral of $\frac{1}{x}$

Consider the integral $\int_1^\infty \frac{1}{x}\,dx$:

$$\int_1^\infty \frac{1}{x}\,dx = \lim_{b\to\infty}\int_1^b \frac{1}{x}\,dx \tag{132}$$

$$= \lim_{b\to\infty}\left[\ln x\right]_1^b \tag{133}$$

$$= \lim_{b\to\infty}(\ln b - \ln 1) \tag{134}$$

$$= \lim_{b\to\infty}\ln b \tag{135}$$

$$= \infty \tag{136}$$

Since $\ln(\infty)$ is infinite, the integral diverges.

7.1.4 General Case

In fact, the integral $\int_1^\infty \frac{1}{x^p}\,dx$ converges or diverges depending on the value of p:

- $\int_1^\infty \frac{1}{x^p}\,dx$ converges to $\frac{1}{p-1}$ for $p > 1$
- $\int_1^\infty \frac{1}{x^p}\,dx$ diverges for $p \leq 1$

This is because:

$$\begin{aligned}
\int_1^\infty \frac{1}{x^p}\,dx &= \lim_{b\to\infty} \int_1^b \frac{1}{x^p}\,dx && (137)\\
&= \lim_{b\to\infty} \left[\frac{x^{1-p}}{1-p}\right]_1^b && (138)\\
&= \lim_{b\to\infty} \left(\frac{b^{1-p}}{1-p} - \frac{1^{1-p}}{1-p}\right) && (139)
\end{aligned}$$

For $p > 1$:

$$\begin{aligned}
\lim_{b\to\infty} \frac{b^{1-p}}{1-p} &= 0 && (140)\\
\int_1^\infty \frac{1}{x^p}\,dx &= \frac{1}{p-1} && (141)
\end{aligned}$$

For $p \leq 1$:

$$\lim_{b\to\infty} \frac{b^{1-p}}{1-p} \text{ does not converge to a finite value} \qquad (142)$$

Hence, the integral diverges.

7.2 Improper Integrals of Type II

An improper integral of type II is an integral of a function over an interval where the function value approaches infinity within the interval. This can occur when the integrand has a **vertical asymptote** at some point within the interval of integration.

7.2.1 Example 1

Consider the integral $\int_a^1 \frac{1}{x}\,dx$:

$$\begin{aligned}
\int_a^1 \frac{1}{x}\,dx &= [\ln x]_a^1 && (143)\\
&= \ln 1 - \ln a && (144)\\
&= -\ln a && (145)
\end{aligned}$$

Taking the limit as a approaches 0 from the positive side:

$$\begin{aligned}
\lim_{a\to 0^+} \int_a^1 \frac{1}{x}\,dx &= -\lim_{a\to 0^+} \ln a && (146)\\
&= -(-\infty) && (147)\\
&= \infty && (148)
\end{aligned}$$

Since the result is infinite, the integral diverges.

7.2.2 Example 2

Consider the integral $\int_a^1 \frac{1}{\sqrt{x}}\,dx$:

$$\int_a^1 \frac{1}{\sqrt{x}}\,dx = \left[2\sqrt{x}\right]_a^1 \tag{149}$$
$$= 2\sqrt{1} - 2\sqrt{a} \tag{150}$$
$$= 2 - 2\sqrt{a} \tag{151}$$

Taking the limit as a approaches 0 from the positive side:

$$\lim_{a\to 0^+} \int_a^1 \frac{1}{\sqrt{x}}\,dx = 2 - 2\sqrt{0} \tag{152}$$
$$= 2 \tag{153}$$

Since the result is finite, the integral converges.

7.3 The Direct Comparison Test

The Direct Comparison Test is used to determine the convergence or divergence of improper integrals by comparing them to a known integral.

Let $0 \le f(x) \le g(x)$ for all $x \ge a$. Then:

1. If $\int_a^\infty g(x)\,dx$ converges, then $\int_a^\infty f(x)\,dx$ also converges.
2. If $\int_a^\infty f(x)\,dx$ diverges, then $\int_a^\infty g(x)\,dx$ also diverges.

7.3.1 Example 1

Consider the integral $\int_1^\infty e^{-x^2}\,dx$. Note that e^{-x^2} has no elementary antiderivative. However, for $x \ge 1$, $e^{-x^2} \le e^{-x}$.

Using the Direct Comparison Test, we compare $\int_1^\infty e^{-x^2}\,dx$ with $\int_1^\infty e^{-x}\,dx$:

$$\int_1^\infty e^{-x}\,dx = \left[-e^{-x}\right]_1^\infty \tag{154}$$
$$= \lim_{b\to\infty}\left(-e^{-b} + e^{-1}\right) \tag{155}$$
$$= 0 + \frac{1}{e} \tag{156}$$
$$\approx 0.368 \tag{157}$$

Since $\int_1^\infty e^{-x}\,dx = \frac{1}{e}$, which is finite, we conclude that:

$$\int_1^\infty e^{-x^2}\,dx \le \int_1^\infty e^{-x}\,dx = \frac{1}{e} \tag{158}$$
$$\therefore \int_1^\infty e^{-x^2}\,dx \text{ converges} \tag{159}$$

7.3.2 Example 2

Consider the integral $\int_1^\infty \frac{\sin^2 x}{x^2}\,dx$. We know that $0 \le \sin^2 x \le 1$ for all x.

Thus,

$$0 \le \frac{\sin^2 x}{x^2} \le \frac{1}{x^2} \tag{160}$$

Using the Direct Comparison Test, we compare $\int_1^\infty \frac{\sin^2 x}{x^2}\,dx$ with $\int_1^\infty \frac{1}{x^2}\,dx$:

$$
\begin{aligned}
\int_1^\infty \frac{1}{x^2}\,dx &= \left[-\frac{1}{x}\right]_1^\infty && (161)\\
&= \lim_{b\to\infty}\left(-\frac{1}{b}+1\right) && (162)\\
&= 0+1 && (163)\\
&= 1 && (164)
\end{aligned}
$$

Since $\int_1^\infty \frac{1}{x^2}\,dx = 1$, which is finite, we conclude that:

$$
\begin{aligned}
\int_1^\infty \frac{\sin^2 x}{x^2}\,dx &\le \int_1^\infty \frac{1}{x^2}\,dx = 1 && (165)\\
\therefore \int_1^\infty \frac{\sin^2 x}{x^2}\,dx &\text{ converges} && (166)
\end{aligned}
$$

7.4 The Limit Comparison Test

The Limit Comparison Test (LCT) is used to determine the convergence or divergence of improper integrals by comparing them to a known integral with similar asymptotic behavior.

Find a function $g(x)$ that behaves asymptotically like $f(x)$. If

$$
\lim_{x\to\infty}\frac{f(x)}{g(x)} = C \quad \text{where} \quad 0 < C < \infty \tag{167}
$$

then

$$
\int_a^\infty f(x)\,dx \quad \text{and} \quad \int_a^\infty g(x)\,dx \quad \text{either both converge or both diverge.} \tag{168}
$$

7.4.1 Workflow

1. Find a function $g(x)$ that behaves asymptotically like $f(x)$ (Big-O notation).
2. Calculate

$$
C = \lim_{x\to\infty}\frac{f(x)}{g(x)} \tag{169}
$$

   If $0 < C < \infty$, you can proceed.
3. Check by some other means whether $\int_a^\infty g(x)\,dx$ converges or diverges.
4. By the LCT, $\int_a^\infty f(x)\,dx$ converges (or diverges) if $\int_a^\infty g(x)\,dx$ does.

7.4.2 Example 1

Consider the integral $\int_1^\infty \frac{x^2-2x-3}{x^3+\cos x}\,dx$. For large x,

$$
\frac{x^2-2x-3}{x^3+\cos x} \approx \frac{x^2}{x^3} = \frac{1}{x}. \tag{170}
$$

Using the Limit Comparison Test (LCT):

$$\lim_{x\to\infty} \frac{\frac{x^2-2x-3}{x^3+\cos x}}{\frac{1}{x}} = \lim_{x\to\infty} \frac{x(x^2-2x-3)}{x^3+\cos x} \tag{171}$$
$$= \lim_{x\to\infty} \frac{x^3-2x^2-3x}{x^3+\cos x} \tag{172}$$
$$= \lim_{x\to\infty} \frac{x^3(1-\frac{2}{x}-\frac{3}{x^2})}{x^3(1+\frac{\cos x}{x^3})} \tag{173}$$
$$= \lim_{x\to\infty} \frac{1-\frac{2}{x}-\frac{3}{x^2}}{1+\frac{\cos x}{x^3}} \tag{174}$$
$$= \frac{1-0-0}{1+0} \tag{175}$$
$$= 1. \tag{176}$$

Since $0 < 1 < \infty$, both integrals either converge or diverge together. As $\int_1^\infty \frac{1}{x}\,dx$ diverges, so does $\int_1^\infty \frac{x^2-2x-3}{x^3+\cos x}\,dx$.

7.4.3 Example 2

Consider the integral $\int_1^\infty \frac{e^x+x^2}{e^{2x}-x^5}\,dx$. For large x,

$$\frac{e^x+x^2}{e^{2x}-x^5} \approx \frac{e^x}{e^{2x}} = e^{-x}. \tag{177}$$

Using the Limit Comparison Test (LCT):

$$\lim_{x\to\infty} \frac{\frac{e^x+x^2}{e^{2x}-x^5}}{e^{-x}} = \lim_{x\to\infty} \frac{e^x+x^2}{e^{2x}-x^5}\cdot e^x \tag{178}$$
$$= \lim_{x\to\infty} \frac{e^x(e^x)+x^2e^x}{e^{2x}-x^5} \tag{179}$$
$$= \lim_{x\to\infty} \frac{e^{2x}+x^2e^x}{e^{2x}-x^5} \tag{180}$$
$$= \lim_{x\to\infty} \frac{e^{2x}(1+\frac{x^2e^{-x}}{e^{2x}})}{e^{2x}(1-\frac{x^5}{e^{2x}})} \tag{181}$$
$$= \lim_{x\to\infty} \frac{1+\frac{x^2}{e^x}}{1-\frac{x^5}{e^{2x}}} \tag{182}$$
$$= \frac{1+0}{1-0} \tag{183}$$
$$= 1. \tag{184}$$

Since $0 < 1 < \infty$, both integrals either converge or diverge together. As $\int_1^\infty e^{-x}\,dx$ converges, so does $\int_1^\infty \frac{e^x+x^2}{e^{2x}-x^5}\,dx$.

8 Infinite Series

8.1 Sequences

A sequence is a list of numbers. For example,

$$\{a_n\} = \{2, 4, 6, 8, \ldots, 2n, \ldots\} \tag{185}$$

The n-th term of the sequence is given by $a_n = 2n$. An infinite sequence can be represented as:

$$\{a_n\} = \{2n\}_{n=1}^{\infty} \tag{186}$$

8.1.1 Convergence and Divergence

A sequence is said to converge if the numbers in the sequence approach a certain value as n tends to infinity. Conversely, a sequence is said to diverge if the numbers do not approach a specific value.

Examples:

1. The sequence

$$\left\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \ldots, \frac{1}{n}, \ldots\right\} \tag{187}$$

converges to 0.

2. The sequence

$$\left\{1, \frac{2}{2}, \frac{3}{3}, \frac{4}{4}, \frac{5}{5}, \ldots, 1 - \frac{1}{n}, \ldots\right\} \tag{188}$$

converges to 1.

3. The sequence

$$\left\{\sqrt{1}, \sqrt{2}, \sqrt{3}, \sqrt{4}, \sqrt{5}, \ldots, \sqrt{n}, \ldots\right\} \tag{189}$$

diverges.

8.1.2 Sequences Defined by Recursion

A sequence defined by recursion is one where the n-th term depends on one or more of the previous terms.

Fibonacci Numbers :

The Fibonacci sequence is defined by the recursive formula:

$$f_n = f_{n-1} + f_{n-2} \quad \text{with} \quad f_1 = f_2 = 1 \tag{190}$$

This generates the sequence:

$$\{1, 1, 2, 3, 5, 8, \ldots\} \tag{191}$$

Divide-and-Conquer Algorithms For divide-and-conquer algorithms, such as MergeSort, the time complexity can be expressed as a recurrence relation:

$$t_n = 2t_{\frac{n}{2}} + n \quad \text{with some initial condition} \quad t_1 = \text{something} \tag{192}$$

8.1.3 Infinite Series

An infinite series is the sum of the terms of an infinite sequence. For example,

$$a_1 + a_2 + a_3 + a_4 + \ldots + a_n + \ldots \tag{193}$$

can be written using summation notation as:

$$\sum_{k=1}^{\infty} a_k \tag{194}$$

We define the partial sum S_n as:

$$S_n = \sum_{k=1}^{n} a_k = a_1 + a_2 + \ldots + a_n \tag{195}$$

8.2 Geometric Series

A geometric series is a series of the form:

$$a + ar + ar^2 + ar^3 + \ldots + ar^{n-1} + \ldots \tag{196}$$

This can be written using summation notation as:

$$\sum_{n=1}^{\infty} ar^{n-1} \tag{197}$$

8.2.1 Case I: $r = 1$

For the geometric series where the common ratio $r = 1$,

$$S_n = a + a \cdot 1 + a \cdot 1^2 + a \cdot 1^3 + \ldots = a \cdot n \tag{198}$$

Taking the limit as n approaches infinity:

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} (a \cdot n) = \infty \quad \text{(diverges)} \tag{199}$$

8.2.2 Case II: $r = -1$

For the geometric series where the common ratio $r = -1$,

$$S_n = a + a(-1) + a(-1)^2 + a(-1)^3 + \ldots = a - a + a - a + a - a + \ldots \tag{200}$$

The series alternates between 0 and a, and thus:

$$\text{(diverges)} \tag{201}$$

8.2.3 Case III: $|r| \neq 1$

For the geometric series where the common ratio $|r| \neq 1$,

$$S_n = a + ar + ar^2 + ar^3 + \ldots + ar^{n-1} \tag{202}$$
$$rS_n = ar + ar^2 + ar^3 + \ldots + ar^n \tag{203}$$
$$S_n - rS_n = a - ar^n \tag{204}$$
$$S_n(1-r) = a(1-r^n) \tag{205}$$
$$S_n = \frac{a(1-r^n)}{1-r} \tag{206}$$

Taking the limit as n approaches infinity:

$$\text{If } |r| < 1: \quad \lim_{n\to\infty} r^n = 0 \quad \text{so} \quad S_n \to \frac{a}{1-r} \quad \text{(converges)} \tag{207}$$
$$\text{If } |r| > 1: \quad \lim_{n\to\infty} r^n = \infty \quad \text{so the series diverges} \tag{208}$$

8.3 The nth Term Test

Given a series $a_1 + a_2 + a_3 + \cdots + a_n + \cdots$:

If $\sum_{n=1}^{\infty} a_n$ converges, then $a_n \to 0$.

So, $\sum_{n=1}^{\infty} a_n$ diverges if $\lim_{n\to\infty} a_n$ fails to exist or is $\neq 0$.

$$\sum_{n=1}^{\infty} n^2 \text{ diverges because } n^2 \to \infty$$

Caution: Many series diverge even though $a_n \to 0$.

$$\sum_{n=1}^{\infty} \frac{1}{n} \text{ diverges}$$

8.4 The Integral Test

Suppose $a_n = f(n)$ where f is continuous, positive, and decreasing.

Then, $\sum_{n=N}^{\infty} a_n$ and $\int_N^{\infty} f(x)dx$ either both converge or both diverge.

8.4.1 Example

Consider the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$.

To determine whether it converges or diverges, we use the function $f(x) = \frac{1}{x^2}$.

$$\sum_{n=1}^{\infty} \frac{1}{n^2} < 1 + \int_1^{\infty} \frac{1}{x^2} dx = 1 + \left[-\frac{1}{x}\right]_1^{\infty} = 1 + (0+1) = 2 \tag{209}$$

Thus, $\sum_{n=1}^{\infty} \frac{1}{n^2} < 2$, so it converges.

8.5 The Relationship Between Infinite Series and Improper Integrals

$$\sum_{n=1}^{\infty} a_n \quad \longleftrightarrow \quad \int_1^{\infty} f(x)\,dx \tag{210}$$

Sequence $\longleftrightarrow$ Integrand
Series $\longleftrightarrow$ Integral
Discrete $\longleftrightarrow$ Continuous

8.6 Remainders

Consider a convergent series $S = \sum a_n$.

Define the remainder R_n:

$$R_n = S - S_n = a_{n+1} + a_{n+2} + a_{n+3} + \cdots \tag{211}$$

where S is the full sum, and S_n is the partial sum until n terms.

To estimate R_n, we use the concept of integral approximation.

$$R_n \approx \sum (\text{rectangle areas}) \tag{212}$$

$$\int_n^{\infty} f(x)\,dx \geq R_n \geq \int_{n+1}^{\infty} f(x)\,dx \tag{213}$$

8.6.1 Example

For a convergent series $\sum_{n=1}^{\infty} \frac{1}{n^2}$, we can estimate the remainder R_n as follows:

$$\int_{n+1}^{\infty} f(x)\,dx \leq R_n \leq \int_n^{\infty} f(x)\,dx$$

Estimate $\sum_{n=1}^{\infty} \frac{1}{n^2}$:

$$S_{10} = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \cdots + \frac{1}{100} = 1.54977\ldots$$

$$\int_z^\infty \frac{1}{x^2}\,dx = \left[-\frac{1}{x}\right]_z^\infty = \frac{1}{z} \implies \begin{cases} \text{For } z=10: \int_{10}^\infty \frac{1}{x^2}\,dx = \frac{1}{10} \\ \text{For } z=11: \int_{11}^\infty \frac{1}{x^2}\,dx = \frac{1}{11} \end{cases}$$

$$\frac{1}{11} \leq R_{10} \leq \frac{1}{10}$$

Thus,

$$S_{10} + \frac{1}{11} \leq S \leq S_{10} + \frac{1}{10}$$

$$1.64068\ldots \leq S \leq 1.64977\ldots$$

Taking the midpoint,

$$S \approx 1.6452\,(\pm 0.005)$$

9 Special Series

9.1 Power Series

The idea is to create an infinite series, not of numbers, but of powers of x.

$$\sum_{n=0}^{\infty} C_n x^n = C_0 + C_1 x + C_2 x^2 + C_3 x^3 + \cdots \tag{214}$$

This series is often referred to as an infinite polynomial.

9.1.1 Geometric Power Series

Consider the series where all coefficients are 1:

$$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots \tag{215}$$

For $|x| < 1$, this series converges to

$$\frac{1}{1-x} \tag{216}$$

Thus, we have:

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots \tag{217}$$

Rather than using $\frac{1}{1-x}$ to calculate $1+x+x^2+x^3+\cdots$, we consider the partial sums of $1+x+x^2+x^3+\cdots$ as approximations to $\frac{1}{1-x}$.

9.2 Series Representations

Assume that $f(x)$ can be written as:

$$f(x) = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \cdots \tag{218}$$

Then, the derivatives are:

$$f'(x) = a_1 + 2a_2x + 3a_3x^2 + 4a_4x^3 + \cdots \tag{219}$$
$$f''(x) = 2a_2 + 2 \cdot 3a_3x + 3 \cdot 4a_4x^2 + \cdots \tag{220}$$
$$f'''(x) = 2 \cdot 3a_3 + 2 \cdot 3 \cdot 4a_4x + \cdots \tag{221}$$

Considering the function and its derivatives evaluated at $x = 0$:

$$f(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + \cdots \qquad \Rightarrow f(0) = a_0 \tag{222}$$
$$f'(x) = a_1 + 2a_2x + 3a_3x^2 + 4a_4x^3 + \cdots \qquad \Rightarrow f'(0) = a_1 \tag{223}$$
$$f''(x) = 2a_2 + 2 \cdot 3a_3x + 3 \cdot 4a_4x^2 + \cdots \qquad \Rightarrow f''(0) = 2 \cdot a_2 \tag{224}$$
$$f'''(x) = 2 \cdot 3a_3 + 2 \cdot 3 \cdot 4a_4x + \cdots \qquad \Rightarrow f'''(0) = 2 \cdot 3a_3 \tag{225}$$

In general, for the n-th derivative:

$$f^{(n)}(0) = n! \cdot a_n \tag{226}$$

Thus, the coefficients a_n can be expressed as:

$$a_n = \frac{f^{(n)}(0)}{n!} \tag{227}$$

If a series representation exists, these are the coefficients.

9.3 Maclaurin Series

The Maclaurin series of $f(x)$ is given by:

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!}x^k = f(0) + f'(0)x + \frac{f''(0)}{2}x^2 + \cdots + \frac{f^{(n)}(0)}{n!}x^n + \cdots \tag{228}$$

9.3.1 Example

Find the Maclaurin series of e^x:

$$f(x) = e^x \tag{229}$$
$$f^{(n)}(x) = e^x \quad \forall n \tag{230}$$
$$\text{so} \quad f^{(n)}(0) = e^0 = 1 \tag{231}$$

The Maclaurin series is:

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!}x^k = \frac{f(0)}{0!} + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots \tag{232}$$
$$= 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \cdots = \sum_{k=0}^{\infty} \frac{x^k}{k!} \tag{233}$$

9.4 Taylor Series

The Maclaurin series is a special case of the Taylor series generated by f at $x = a$:

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots \tag{234}$$

- The Maclaurin series approximates f near 0.
- The Taylor series approximates f near a.

9.5 Taylor Polynomials

To create a Taylor polynomial, we terminate a Taylor series at n, resulting in a Taylor polynomial of order n.

For $f(x) = e^x$, the third-order Taylor polynomial $P_3(x)$ is:

$$P_3(x) = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 \tag{235}$$

9.5.1 Cosine

For $f(x) = \cos x$:

$$f'(x) = -\sin x \tag{236}$$
$$f''(x) = -\cos x \tag{237}$$
$$f'''(x) = \sin x \tag{238}$$
$$f^{(4)}(x) = \cos x \tag{239}$$

For even derivatives:

$$f^{(2k)}(x) = (-1)^k \cos x \tag{240}$$
$$f^{(2k)}(0) = (-1)^k \tag{241}$$

For odd derivatives:

$$f^{(2k+1)}(x) = (-1)^k \sin x \tag{242}$$
$$f^{(2k+1)}(0) = 0 \tag{243}$$

Thus, the series alternates as follows: $1, 0, -1, 0, 1, 0, -1, 0, \cdots$.

The series for $\cos x$ is:

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots \tag{244}$$
$$= \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{(2k)!} \tag{245}$$

9.5.2 Sine and Cosine

The series for $\cos x$ and $\sin x$ are:

$$\cos x = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{(2k)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots \tag{246}$$
$$\sin x = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots \tag{247}$$

9.6 Taylor Polynomials - Applications

9.6.1 Application No. 1: Solving Weird Equations

To solve the equation $\cos x = x$:

$$
\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots \approx x \tag{248}
$$

$$
1 - \frac{x^2}{2} \approx x \tag{249}
$$

We solve the quadratic equation:

$$
\frac{1}{2}x^2 + x - 1 = 0 \tag{250}
$$

Using the quadratic formula:

$$
x = \frac{-1 \pm \sqrt{1^2 - 4\left(\frac{-1}{2}\right)}}{2\left(\frac{1}{2}\right)} = \frac{-1 \pm \sqrt{3}}{1} \tag{251}
$$

$$
x = -1 \pm \sqrt{3} \tag{252}
$$

$$
x \approx 0.732 \quad \text{(since } x = -2.732 \text{ is not valid)} \tag{253}
$$

So, $x \approx 0.739$ is a pretty good approximation.

9.6.2 Application No. 2: Evaluating Nonelementary Integrals

Given the series for e^x:

$$
e^x = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \cdots = \sum_{k=0}^{\infty} \frac{x^k}{k!} \tag{254}
$$

The series for e^{-x^2}:

$$
e^{-x^2} = \sum_{k=0}^{\infty} \frac{(-x^2)^k}{k!} = 1 - x^2 + \frac{1}{2}x^4 - \frac{1}{6}x^6 + \cdots \tag{255}
$$

The integral of e^{-x^2}:

$$
\int e^{-x^2}\,dx = x - \frac{1}{3}x^3 + \frac{1}{10}x^5 - \frac{1}{42}x^7 + \cdots \tag{256}
$$

Evaluating the integral from 0 to 1:

$$
\int_0^1 e^{-x^2}\,dx \approx 1 - \frac{1}{3} + \frac{1}{10} - \frac{1}{42} = 0.742 \tag{257}
$$

$$
\text{compare with } 0.747\cdots \tag{258}
$$

9.6.3 Application No. 3: Euler's Identity

Given $i = \sqrt{-1}$:

$$
i^2 = -1 \tag{259}
$$

$$
i^3 = i^2 \cdot i = -i \tag{260}
$$

$$
i^4 = i^2 \cdot i^2 = 1 \tag{261}
$$

$$
i^5 = i^4 \cdot i = i \tag{262}
$$

The exponential form $e^{i\theta}$:

$$e^{i\theta} = 1 + \frac{i\theta}{1!} + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^3}{3!} + \frac{(i\theta)^4}{4!} + \frac{(i\theta)^5}{5!} + \frac{(i\theta)^6}{6!} + \cdots \tag{263}$$

$$= 1 + \frac{i\theta}{1!} - \frac{\theta^2}{2!} - \frac{i\theta^3}{3!} + \frac{\theta^4}{4!} + \frac{i\theta^5}{5!} - \frac{\theta^6}{6!} + \cdots \tag{264}$$

$$= \left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \cdots\right) + i\left(\frac{\theta}{1!} - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \cdots\right) \tag{265}$$

$$= \cos\theta + i\sin\theta \tag{266}$$

Euler's identity is given by:

$$e^{i\theta} = \cos\theta + i\sin\theta \tag{267}$$

$$e^{i\pi} = \cos\pi + i\sin\pi = -1 + i\cdot 0 \tag{268}$$

$$e^{i\pi} + 1 = 0 \tag{269}$$

10 Partial Derivatives

10.1 Functions of Several Variables

1. $f(x) = 3x^2 - x$
2. $f(x, y) = yx^2 + e^{-y}$
3. $f(x_1, x_2, x_3, \ldots, x_n)$

10.2 Partial Derivatives

Partial derivatives involve taking the derivative of a function with respect to one variable while treating all other variables as constants.

$$\frac{\partial f}{\partial x} = f_x \tag{270}$$

$$\frac{\partial f}{\partial y} = f_y \tag{271}$$

10.2.1 Second-Order Derivatives

Given the function:

$$f(x, y) = \sin(xy) + y^2 \tag{272}$$

The first-order partial derivatives are:

$$\frac{\partial f}{\partial x} = y\cos(xy) \tag{273}$$

$$\frac{\partial f}{\partial y} = x\cos(xy) + 2y \tag{274}$$

The second-order partial derivatives are:

$$\frac{\partial^2 f}{\partial x^2} = -y^2\sin(xy) \tag{275}$$

$$\frac{\partial^2 f}{\partial y^2} = -x^2\sin(xy) + 2 \tag{276}$$

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(y\cos(xy)\right) = y(-x\sin(xy)) = -xy\sin(xy) + \cos(xy) \tag{277}$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(x\cos(xy) + 2y\right) = x(-y\sin(xy)) = -xy\sin(xy) + \cos(xy) \tag{278}$$

10.2.2 Mixed Derivative Theorem

The Mixed Derivative Theorem states that if the second-order mixed partial derivatives of a function are continuous, then they are equal. Mathematically, this is expressed as:

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x} \tag{279}$$

10.3 The Gradient Vector

The gradient vector of a function f with respect to its variables is a vector that contains all its partial derivatives. For a function $f(x_1, x_2, x_3, \ldots, x_n)$, the gradient vector is given by:

$$\nabla f = \left(\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \ldots, \frac{\partial f}{\partial x_n} \right)^\top \tag{280}$$

For a function $f(x, y)$, the gradient vector is:

$$\nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right)^\top \tag{281}$$

10.3.1 The Gradient Vector at a Point

Consider the function:

$$f(x, y) = -(x^2 + y^2) \tag{282}$$

The gradient vector is given by:

$$\nabla f = \begin{bmatrix} -2x \\ -2y \end{bmatrix} \tag{283}$$

At the point $(2, 2)$, the gradient vector is:

$$\nabla f \Big|_{(2,2)} = \begin{bmatrix} 4 \\ 4 \end{bmatrix} \tag{284}$$

The gradient vector has the following properties:

1. It is perpendicular to contour lines.
2. It points in the direction of the steepest slope, i.e., towards the maximum.
3. Its length is proportional to the steepness.

10.4 Critical Points

In 1D, a critical point occurs where the derivative is zero:

$$\frac{df}{dx} = 0 \tag{285}$$

In nD, a critical point occurs where the gradient vector is zero:

$$\nabla f = \vec{0} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \tag{286}$$

This means that all partial derivatives are equal to zero.

10.4.1 Saddle Points

A saddle point is a point on the surface of the graph of a function where the slopes (derivatives) in orthogonal directions are zero, but it is not a local extremum (minimum or maximum). Saddle points are critical points that are neither local maxima nor local minima.

10.4.2 The Second Derivative Test

Consider a critical point (a, b) where $f_x(a, b) = 0$ and $f_y(a, b) = 0$. To classify the nature of this critical point, we use the second derivative test with the discriminant D, defined as:

$$D = f_{xx} f_{yy} - (f_{xy})^2 \tag{287}$$

The critical point (a, b) is classified as:

1. A local maximum if $f_{xx} < 0$ and $D > 0$
2. A local minimum if $f_{xx} > 0$ and $D > 0$
3. A saddle point if $D < 0$
4. Inconclusive if $D = 0$

10.4.3 Gradient Descent

Gradient descent is a numerical procedure for finding the minima of a function. The key idea is to iteratively move in the direction opposite to the gradient of the function at the current point, as the gradient points in the direction of the steepest ascent.

The procedure can be described as follows:

- The gradient vector ∇f points upwards.
- Therefore, to find the minimum, move in the direction of $-\nabla f$.

Mathematically, this is represented by the update rule:

$$\begin{bmatrix} x_{n+1} \\ y_{n+1} \end{bmatrix} = \begin{bmatrix} x_n \\ y_n \end{bmatrix} - \eta \nabla f(x_n, y_n) \tag{288}$$

where η is the learning rate or step size, which is typically set by trial and error.

10.4.4 Example

Find the minimum point on the function $f(x, y) = x^2 + y^2 + x - y$ using the gradient descent method with a learning rate $\eta = 0.3$ and starting at $r_0 = (x_0, y_0) = (1, 1)$. Stop when both partial derivatives are less than 0.01.

The update rule is:

$$r_{n+1} = \begin{bmatrix} x_{n+1} \\ y_{n+1} \end{bmatrix} = \begin{bmatrix} x_n \\ y_n \end{bmatrix} - \eta \nabla f\big|_{(x_n, y_n)} \tag{289}$$

The gradient is:

$$\nabla f = \begin{bmatrix} 2x + 1 \\ 2y - 1 \end{bmatrix} \tag{290}$$

Thus, the update rule becomes:

$$\begin{bmatrix} x_{n+1} \\ y_{n+1} \end{bmatrix} = \begin{bmatrix} x_n \\ y_n \end{bmatrix} - 0.3 \begin{bmatrix} 2x + 1 \\ 2y - 1 \end{bmatrix} \tag{291}$$

n	r_n	$\nabla f\big\|_{r_n}$	$-\eta\nabla f\big\|_{r_n}$
0	$\begin{bmatrix} 1 \\ 1 \end{bmatrix}$	$\begin{bmatrix} 3 \\ 1 \end{bmatrix}$	$\begin{bmatrix} -0.9 \\ -0.3 \end{bmatrix}$
1	$\begin{bmatrix} 0.1 \\ 0.7 \end{bmatrix}$	$\begin{bmatrix} 1.2 \\ 0.4 \end{bmatrix}$	$\begin{bmatrix} -0.36 \\ -0.12 \end{bmatrix}$
2	$\begin{bmatrix} -0.26 \\ 0.58 \end{bmatrix}$	$\begin{bmatrix} 0.48 \\ 0.16 \end{bmatrix}$	$\begin{bmatrix} -0.144 \\ -0.048 \end{bmatrix}$
3	$\begin{bmatrix} -0.404 \\ 0.532 \end{bmatrix}$	$\begin{bmatrix} 0.192 \\ 0.064 \end{bmatrix}$	$\begin{bmatrix} -0.058 \\ -0.019 \end{bmatrix}$
4	$\begin{bmatrix} -0.462 \\ 0.513 \end{bmatrix}$	$\begin{bmatrix} 0.077 \\ 0.026 \end{bmatrix}$	$\begin{bmatrix} -0.023 \\ -0.008 \end{bmatrix}$
5	$\begin{bmatrix} -0.485 \\ 0.505 \end{bmatrix}$	$\begin{bmatrix} 0.031 \\ 0.010 \end{bmatrix}$	$\begin{bmatrix} -0.009 \\ -0.003 \end{bmatrix}$
6	$\begin{bmatrix} -0.494 \\ 0.502 \end{bmatrix}$	$\begin{bmatrix} 0.012 \\ 0.004 \end{bmatrix}$	$\begin{bmatrix} -0.004 \\ -0.001 \end{bmatrix}$
7	$\begin{bmatrix} -0.498 \\ 0.501 \end{bmatrix}$	$\begin{bmatrix} 0.005 \\ 0.001 \end{bmatrix}$	$\begin{bmatrix} -0.0015 \\ -0.0003 \end{bmatrix}$

10.4.5 The Importance of the Learning Rate

Choosing the correct learning rate η is crucial for the efficiency of gradient descent:

- Too small: The algorithm will be slow, taking small steps.
- Just right: The algorithm converges efficiently to the minimum.
- Too big: The algorithm may overshoot the minimum or diverge.

11 Multiple Integrals

11.1 Double Integrals

A double integral extends the concept of a single integral to functions of two variables. While a single integral represents the area under a curve, a double integral represents the volume under a surface.

The single integral:

$$\int_{x_1}^{x_2} f(x)\,dx \tag{292}$$

represents the area under the curve $f(x)$.

The double integral:

$$\iint_D f(x,y)\,dx\,dy \tag{293}$$

represents the volume under the surface $f(x,y)$ over the region D.

11.1.1 Fubini's Theorem

Fubini's Theorem states that if $f(x, y)$ is continuous on a rectangular region $[x_1, x_2] \times [y_1, y_2]$, then the order of integration does not matter. Mathematically, this is expressed as:

$$\iint_D f(x,y)\,dx\,dy = \int_{y_1}^{y_2} \int_{x_1}^{x_2} f(x,y)\,dx\,dy = \int_{x_1}^{x_2} \int_{y_1}^{y_2} f(x,y)\,dy\,dx \tag{294}$$

11.2 Double Integrals Over General Regions

So far, we have considered integration over a rectangular region. Now, we extend this to general regions.

Consider the function $f(x, y) = x^2 + y$ over the region bounded by $y = x$ and $y = x^2$:

$$V = \int_0^1 \int_{x^2}^{x} (x^2 + y)\,dy\,dx \tag{295}$$

Evaluating the integral:

$$\begin{aligned}
V &= \int_0^1 \left[x^2 y + \frac{1}{2}y^2\right]_{y=x^2}^{y=x} dx &(296)\\
&= \int_0^1 \left(x^2 x + \frac{1}{2}x^2 - \left(x^2 x^2 + \frac{1}{2}(x^2)^2\right)\right) dx &(297)\\
&= \int_0^1 \left(x^3 + \frac{1}{2}x^2 - x^4 - \frac{1}{2}x^4\right) dx &(298)\\
&= \int_0^1 \left(x^3 + \frac{1}{2}x^2 - \frac{3}{2}x^4\right) dx &(299)\\
&= \left[\frac{1}{4}x^4 + \frac{1}{6}x^3 - \frac{3}{10}x^5\right]_0^1 &(300)\\
&= \frac{1}{4} + \frac{1}{6} - \frac{3}{10} = \frac{7}{60} \approx 0.12 &(301)
\end{aligned}$$

11.2.1 Estimating π

To estimate π using double integrals, consider the volume of a cylinder with radius 1 and height π.

$$\begin{aligned}
\iint_D 1\,dy\,dx &= \int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} 1\,dy\,dx &(302)\\
&= \int_{-1}^{1} [y]_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\,dx &(303)\\
&= \int_{-1}^{1} 2\sqrt{1-x^2}\,dx &(304)
\end{aligned}$$

Using trigonometric substitution:

$$\int_{-1}^{1} 2\sqrt{1-x^2}\,dx = \int_{\sin^{-1}(-1)}^{\sin^{-1}(1)} 2\cos^2(\theta)\,d\theta = \pi \tag{305}$$

Thus, the volume of this cylinder is π, confirming our estimation.

11.3 Monte Carlo Integration

Monte Carlo integration is a numerical method for estimating the value of an integral using random sampling. It is particularly useful for high-dimensional integrals.

For a 1D integral, the method involves the following steps:

1. Select n random points x_i in the interval $[a, b]$.
2. Calculate the average value of the function at these points:

$$\langle f \rangle_n = \frac{1}{n} \sum_{i=1}^{n} f(x_i)$$

3. Estimate the integral as:

$$I_n = (b - a)\langle f \rangle_n$$

The integral is given by:

$$I = \langle f \rangle (b - a)$$

11.3.1 Going to Higher Dimensions

In higher dimensions, Monte Carlo integration remains efficient. The error of Monte Carlo integration decreases as:

$$\frac{1}{\sqrt{n}}$$

In contrast, the error of Simpson's rule decreases as:

$$\frac{1}{n^{4/d}}$$

where d is the dimensionality of the integral. For dimensions $d > 8$, Monte Carlo integration generally outperforms other methods.

11.3.2 Procedure for Monte Carlo Integration in 1 Dimension

If you want to estimate the integral

$$\int_{x_1}^{x_2} f(x)\, dx$$

1. Generate n random points x_i in the interval $[x_1, x_2]$ and calculate $f(x_i)$.
2. Take the average of all $f(x_i)$ values to get $\langle f \rangle_n$:

$$\langle f \rangle_n = \frac{1}{n} \sum_{i=1}^{n} f(x_i)$$

3. Multiply by the length of the integration interval to get:

$$I_n = \langle f \rangle_n (x_2 - x_1)$$

11.3.3 Procedure for Monte Carlo Integration in 3 Dimensions

If you want to estimate the integral

$$\int_{x_1}^{x_2} \int_{y_1}^{y_2} \int_{z_1}^{z_2} f(x, y, z)\, dz\, dy\, dx$$

1. Generate n random points (x_i, y_i, z_i) in the region $[x_1, x_2] \times [y_1, y_2] \times [z_1, z_2]$ and calculate $f(x_i, y_i, z_i)$.
2. Take the average of all $f(x_i, y_i, z_i)$ values to get $\langle f \rangle_n$:

$$\langle f \rangle_n = \frac{1}{n} \sum_{i=1}^{n} f(x_i, y_i, z_i)$$

3. Multiply by the volume of the integration region to get:

$$I_n = \langle f \rangle_n (x_2 - x_1)(y_2 - y_1)(z_2 - z_1)$$

12 Differential Equations

12.1 Differential Equations

Given the differential equation

$$\frac{dy}{dx} \tag{306}$$

we seek to determine the function y.

12.2 A Simple Differential Equation

Consider the differential equation

$$\frac{dy}{dx} = x^2 \tag{307}$$

This can be rewritten as

$$dy = x^2 \, dx \tag{308}$$

Integrating both sides, we obtain

$$\int dy = \int x^2 \, dx \tag{309}$$

which yields the solution

$$y = \frac{x^3}{3} + C \tag{310}$$

where C is the constant of integration.

12.3 A Less Simple Differential Equation

Consider the differential equation

$$\frac{dy}{dx} = 6xy \tag{311}$$

Separating variables, we get

$$\frac{dy}{y} = 6x \, dx \tag{312}$$

Integrating both sides, we have

$$\int \frac{1}{y} \, dy = \int 6x \, dx \tag{313}$$

which results in

$$\ln |y| = 3x^2 + C \tag{314}$$

Exponentiating both sides, we obtain

$$|y| = e^{3x^2+C} \tag{315}$$

Simplifying, the general solution is

$$y = Ce^{3x^2} \tag{316}$$

where C is a constant.

12.4 Separation of Variables

The method of separation of variables is a general technique for solving separable differential equations. Given a differential equation of the form

$$\frac{dy}{dx} = f(x, y) \tag{317}$$

we can rearrange it to isolate the y-terms on one side and the x-terms on the other side, resulting in

$$\frac{dy}{g(y)} = h(x)\, dx \tag{318}$$

where $g(y)$ and $h(x)$ are functions of y and x, respectively. Then, we integrate both sides:

$$\int \frac{1}{g(y)}\, dy = \int h(x)\, dx \tag{319}$$

This will yield a solution for y in terms of x.

12.4.1 Example of a Separable Equation

Consider the differential equation

$$\frac{dy}{dx} = g(x) \cdot f(y) \tag{320}$$

By separating the variables, we get

$$\frac{dy}{f(y)} = g(x)\, dx \tag{321}$$

Integrating both sides, we obtain

$$\int \frac{1}{f(y)}\, dy = \int g(x)\, dx \tag{322}$$

which can then be solved to find the function y.

12.5 Initial Value Problems

So far, all solutions we have found have been families of solutions. An initial value problem (IVP) seeks a specific solution by providing an initial condition.

12.5.1 Example of an Initial Value Problem

Solve the differential equation

$$\frac{dy}{dx} = \frac{x}{y} \tag{323}$$

such that $y(0) = 5$.

From the previous section, we have the general solution:

$$y = \pm\sqrt{x^2 + C} \tag{324}$$

Applying the initial condition $y(0) = 5$:

$$5 = \sqrt{C} \implies C = 25 \tag{325}$$

Thus, the specific solution is:

$$y = \sqrt{x^2 + 25} \tag{326}$$

12.6 Mixing Problems

Mixing problems involve determining the concentration of a substance, such as salt, in a solution over time.

12.6.1 General Concept

Consider a tank containing a mixture (e.g., salt water). Salt water at a different concentration flows into the tank, and the mixture flows out. We aim to find the concentration of salt as a function of time.

12.6.2 Examples of Mixing Problems

1. Injection of medicine into the bloodstream.
2. Pollution in a lake.
3. Chemical reactions.

12.6.3 Solving Mixing Problems

To solve mixing problems, follow these steps:

1. Read the problem carefully.
2. Identify the rates of flow in and flow out.
3. Write the differential equation
$$\frac{dy}{dt} = \text{flow in} - \text{flow out} \tag{327}$$
4. Solve the differential equation and isolate y.
5. Use the initial condition to eliminate the integration constant.

12.6.4 Example: A Salt Tank

Consider a tank that contains 20 kg of salt dissolved in 5000 L of water. Brine containing 0.03 kg of salt per liter enters the tank at a rate of 25 L/min. The solution is thoroughly mixed and drains from the tank at the same rate. We aim to determine the amount of salt remaining in the tank after half an hour.

Let y be the amount of salt in the tank at time t (in minutes). The rate of change of the amount of salt in the tank is given by the differential equation:

$$\frac{dy}{dt} = (\text{rate of salt in}) - (\text{rate of salt out}) \tag{328}$$

The rate of salt entering the tank is:

$$\text{rate of salt in} = 0.03\,\frac{\text{kg}}{\text{L}} \times 25\,\frac{\text{L}}{\text{min}} = 0.75\,\frac{\text{kg}}{\text{min}} \tag{329}$$

The rate of salt leaving the tank is:

$$\text{rate of salt out} = \frac{y(t)}{5000\,\text{L}} \times 25\,\frac{\text{L}}{\text{min}} = \frac{y(t)}{200}\,\frac{\text{kg}}{\text{min}} \tag{330}$$

Thus, the differential equation becomes:

$$\frac{dy}{dt} = 0.75 - \frac{y}{200} \tag{331}$$

To solve this differential equation, we separate the variables:

$$\frac{dy}{0.75 - \frac{y}{200}} = dt \tag{332}$$

Rewriting the equation, we get:

$$\frac{dy}{150-y} = \frac{dt}{200} \tag{333}$$

Integrating both sides, we obtain:

$$-\ln|150-y| = \frac{t}{200} + C \tag{334}$$

Exponentiating both sides, we have:

$$150 - y = Ce^{-\frac{t}{200}} \tag{335}$$

Solving for y, we get:

$$y = 150 - Ce^{-\frac{t}{200}} \tag{336}$$

Using the initial condition $y(0) = 20$:

$$20 = 150 - C \implies C = 130 \tag{337}$$

Thus, the specific solution is:

$$y = 150 - 130e^{-\frac{t}{200}} \tag{338}$$

After half an hour (30 minutes), the amount of salt remaining in the tank is:

$$y(30) = 150 - 130e^{-\frac{30}{200}} \approx 38.1\,\text{kg} \tag{339}$$

12.7 Euler's Method

Many differential equations are either impossible or tedious to solve analytically. Therefore, we can solve them numerically.

The main idea is that we know $\frac{dy}{dx}$, so we can always calculate it.

12.7.1 Steps for Euler's Method

Consider the differential equation

$$\frac{dy}{dx} = f(x, y) \tag{340}$$

with the initial condition $y(x_0) = c$.

To solve this numerically using Euler's Method, follow these steps:

1. Decide on a step size Δx.
2. Let $x_1 = x_0 + \Delta x$.
3. Calculate y_1 using the first-order Taylor expansion:

$$y_1 = y_0 + \left(\left. \frac{dy}{dx} \right|_{x_0} \right) \Delta x = y_0 + f(x_0, y_0)\Delta x \tag{341}$$

4. Continue this process to find y_{n+1} from y_n:

$$y_{n+1} = y_n + f(x_n, y_n)\Delta x \tag{342}$$