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RELATIVISTIC LIMITATIONS ON RUNAWAY ELECTRONS

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ABSTRACT. The non-relativistic theory of a plasma in an electric field E predicts that there will always be runaway electrons, although their number will be exponentially small for fields less than the Dreicer field E_D. However, when $E/E_D \sim kT/m_ec^2$, the ratio of the electron thermal energy to the rest mass energy, relativistic effects become important. After comparing earlier non-relativistic calculations we extend the approach of Kruskal and Bernstein to take account of relativistic effects and also to investigate the influence of impurities. It is found that below the critical electric field $E_R = E_D\,(kT/m_ec^2)$ absolutely no runaways are generated. In addition, the number of runaway electrons produced by electric fields in excess of E_R is calculated and we find significant modifications to the non-relativistic estimates when $(E_D/E)^2\,(kT/m_ec^2) > 1$.

1. INTRODUCTION

In a non-relativistic theory, application of an electric field to a fully ionized plasma gives rise to runaway electrons. This phenomenon results from the decrease in the Coulomb collision frequency with increasing energy, so that, whatever the value of the electric field, above some critical energy the electrons will be continuously accelerated. This defines a critical speed v_c by balancing the electric field E against the frictional drag of Coulomb collisions which have a frequency

$$\nu(v) = \frac{4\pi\, e^4\, n\, \ln\lambda}{m^2\, v^3} \tag{1}$$

where n is the electron density, m the electron mass, e the electron charge, ln λ the Coulomb logarithm and v the electron speed. Thus we obtain

$$v_c{}^2 = \frac{4\pi\, e^3\, n\, \ln\lambda}{m\, E} \tag{2}$$

We may also define a critical electric field E_D, the Dreicer field, as that field for which $v_c^2 = kT/m$, i.e. $v_c = v_{Th}$ where $v_{Th} = \sqrt{(kT/m)}$ is the thermal speed of electrons at a temperature T. Thus, when $E > E_D$ where

$$E_D = \frac{4\pi\, e^3\, n\, \ln\lambda}{m\, v_{Th}{}^2} \tag{3}$$

thermal electrons will run away. When $E < E_D$, $v_c > v_{Th}$ and only the tail of the Maxwellian distribution runs away, leading to an exponentially small production rate of runaways.

Several authors [1-6] have considered the problem of determining the number of runaways when $E \ll E_D$. The earliest attempt was by Dreicer [1] who divided velocity space into a collisional region, where the distribution function was almost spherically symmetric and a runaway region outside the spherical surface $v = v_c$, which he considered to be empty as a result of rapid depletion by the electric field. The Fokker-Planck equation for the electrons was solved numerically as an initial-value problem following the application of the electric field to a Maxwellian plasma, with the condition that the distribution function f vanish at the transition surface $v = v_c$ between the two regions. The decay rate of f represented the diffusion of particles into the runaway region, i.e. the production rate of runaways.

This simplified picture of velocity space was improved by Gurevich [2] who realized that at higher velocities near $v = v_c$ the distribution function f would depart from Maxwellian and acquire a directional character concentrated near $\mu = 1$ (where μ is the cosine of the angle between $\vec{E}$ and the velocity $\vec{v}$). He analysed the region $v \lesssim v_c$ for $\mu \sim 1$ as an expansion in $(1-\mu)$, writing

$$f = \exp\,\{\varphi_0(v) + (1-\mu)\,\varphi_1(v) + (1-\mu)^2\,\varphi_2(v) + \dots\} \tag{4}$$

This form of f was substituted into the Fokker-Planck equation and the resulting equations for φ_0, φ_1, etc. solved as an expansion in powers of $(E/E_D)^{1/2}$ with φ_1 arbitrarily set equal to zero in leading order. The resulting f was matched onto the Maxwellian solution valid for $v \sim v_{Th}$ and a runaway rate

$$S_G = \frac{2}{\sqrt{\pi}}\, n\, \nu\,(v_{Th}) \left(\frac{E}{E_D}\right)^{\frac{1}{2}} \exp\left\{-\frac{E_D}{4E} - \left(\frac{2E_D}{E}\right)^{\frac{1}{2}}\right\} \tag{5}$$

was deduced. The solution for f, however, shows a singular behaviour for $v \to v_c$ which, as observed by Lebedev [3], is due to the assumption that $\varphi_1(v) = 0$ in leading order. Furthermore, the solution is only valid in a region $v \lesssim v_c$ and cannot properly be connected onto the Maxwellian solution when $v \sim v_{Th}$ without considering an intermediate region $v_{Th} < v < v_c$, because of certain logarithmic terms in $\varphi_0(v)$. Finally, it is necessary to consider the region $v \gg v_c$ in order to determine the runaway rate correctly, and this was absent from Gurevich's solution.

The first and last of these deficiencies were rectified in the treatment of Lebedev [3] who considered a boundary layer about $v = v_c$ and expanded φ_0, φ_1, etc. in powers of $(E/E_D)^{1/3}$. Lebedev did not make Gurevich's assumption that $\varphi_1(v)$ was zero in leading order but did make it for $\varphi_2(v)$! The solution obtained matched onto Gurevich's solution for $v < v_c$ and also matched onto a solution valid for $v > v_c$ in the true runaway region. Thus Lebedev could compute a true runaway rate but still had the remaining deficiency inherent in Gurevich's treatment, namely neglect of the intermediate region $v_{Th} < v < v_c$. The result differs from Gurevich's by a pre-exponential factor:

$$S_L = 0.36\ n\ \nu(v_{Th}) \left(\frac{E}{E_D}\right)^{-\frac{1}{4}} \times \exp\left\{-\frac{E_D}{4E} - \left(\frac{2\ E_D}{E}\right)^{\frac{1}{2}}\right\} \tag{6}$$

The most consistent and sophisticated treatment of this problem has been given by Kruskal and Bernstein [4] who found it necessary to consider five distinct regions of velocity space with appropriate matching between them. As a result their treatment did not contain the errors in the other theories. Thus, they treated the intermediate region between thermal velocities and Gurevich's solution and considered the boundary layer at v_c more carefully. A comparison with Lebedev shows that he was in error in setting $\varphi_2(v) = 0$ in leading order in $(E/E_D)^{1/3}$. Their result again differs in the pre-exponential factor, although Lebedev's two errors almost cancel,

$$S_{KB} = c\ n\ \nu(v_{Th}) \left(\frac{E}{E_D}\right)^{-\frac{3}{8}} \times \exp\left\{-\frac{E_D}{4E} - \left(\frac{2\ E_D}{E}\right)^{\frac{1}{2}}\right\} \tag{7}$$

where c is an unknown constant of the analysis. Thus, although Kruskal and Bernstein [4] were the first authors to determine the pre-exponential factor correctly, the dominant exponential term is unchanged from the original result of Gurevich [2]!

It is worth remarking in parenthesis that Kruskal and Bernstein [5] also considered a related problem — the runaway rate in a Lorentz electron gas. The absence of electron-electron collisions means that the electron distribution function f is only forced to be isotropic and not Maxwellian in the collision dominated region. This has the consequence that the runaway rate is no longer exponentially small in E/E_D. However, the form of f in the far runaway region $v \gg v_c$ does resemble that in the real physical problem with electron-electron collisions.

Finally, we mention a numerical solution [6] of the electron Fokker-Planck equation which agreed with the analytic theory of Kruskal and Bernstein and served to determine an unknown constant of integration in their theory which appears as the constant of proportionality c in the expression (7) for S_{KB}, best agreement being found with $c = 0.35$.

It should be emphasized that these discussions of the pre-exponential factor are essentially of mathematical rather than practical significance. In the first place, the differences obtained by the various authors are not large, and, secondly, the approximations used for the collision term can introduce comparable errors. In particular, as pointed out by Gurevich [2], a more correct approach to the runaway problem would replace the Coulomb logarithm $\ln \lambda$ by $\ln(\lambda\ mv_c^2/2\ kT_e)$ for the velocities of interest. This alone leads to a correction factor $\sim (E_D/E)^{-E_D/4E \ln \lambda}$ to the runaway flux S which can be more significant than the factor $(E_D/E)^{-3/8}$. Analogous considerations apply to the relativistic treatment considered below.

All the above references assumed $E/E_D \gg kT/mc^2$ so that $v_c \ll c$, the velocity of light, and relativistic effects could be ignored when considering velocities in the neighbourhood of v_c and beyond. The condition above is likely to be violated in the coming generation of tokamak experiments. This reason, as well as applications in basic and cosmic plasma physics provides the motivation for the treatment in this paper of the relativistic runaway problem, when $E/E_D \sim kT/mc^2$. Apart from calculating the number of runaways in this situation we shall also be interested in the explicit form of the electron distribution function f in the region $v \sim v_c$. This will have significance for calculations of synchrotron radiation and bremsstrahlung radiation from a plasma with runaway electrons. Synchrotron radiation from a high-temperature fusion reactor [7] depends on the behaviour of f near $v = c$ and will thus be strongly affected by runaways when $v_c \sim c$.

Consideration of relativistic effects on runaways can be expected to yield qualitative differences rather than minor modifications to the non-relativistic theory outlined above because of the significantly different velocity dependence of coulomb collisions when $v \sim c$. It is more convenient to discuss relativistic scattering in terms of the momentum p, rather than the velocity, of the incident particle. At high energies, the dominant cause of momentum loss by electrons in Coulomb scattering is due to loss of energy by the scattered electron to the recoil of target electrons (this is true of both relativistic and non-relativistic theories). This momentum loss rate can be written

$$\frac{dp}{dt} \sim \frac{dp}{d\mathcal{E}}\frac{d\mathcal{E}}{dt} \sim \frac{1}{v} \cdot \int_{\Delta} n\, v\, \Delta\, d\sigma(\Delta) \tag{8}$$

where the energy $\mathcal{E} = \sqrt{(p^2 c^2 + m^2 c^4)}$ is the energy of the incident electron, v is its velocity, n the density of target electrons and $d\sigma(\Delta)$ is the Coulomb cross-section for scattering with loss of energy Δ to the recoil electrons. Small-angle scattering $\theta \ll \pi/2$ is most important in Coulomb scattering and, for this, conservation of relativistic energy and momentum yields $\Delta \sim p^2\theta^2/m \ll \mathcal{E}$. Thus the recoil motion of the target can be ignored in calculating the scattering angle as a function of impact parameter ρ. This may be estimated [8] by observing that the incident particle is subjected to a transverse force $(e^2/r^2)\cdot(\rho/r)$ where $r = \sqrt{(\rho^2 + v^2t^2)}$ is the distance between the electrons as a function of time t. As a consequence

$$\theta \sim \frac{1}{p}\, e^2 \int_{-\infty}^{+\infty} \frac{\rho dt}{(\rho^2 + v^2t^2)^{3/2}} \sim \frac{e^2}{pv\rho} \tag{9}$$

Using $\Delta \sim p^2\theta^2/m$ we may write

$$d\sigma(\Delta) \sim \rho d\rho \sim \frac{e^4}{mv^2}\frac{d\Delta}{\Delta^2} \tag{10}$$

and conclude

$$\frac{dp}{dt} \sim \frac{ne^4}{mv^2}\int \frac{d\Delta}{\Delta} \sim \frac{ne^4}{mv^2}\ln\lambda \tag{11}$$

As momentum p increases, $v = pc/\sqrt{(p^2 + m^2c^2)} \rightarrow c$ and dp/dt no longer decreases with p, unlike the non-relativistic case, where $dp/dt \sim p^{-2}$ as follows from Eq. (1). Thus Eq. (11) implies that for electric fields such that $E \lesssim ne^3 \ln\lambda/mc^2$ no electrons will run away!

In Section 2, we derive a Fokker-Planck collision term, suitable for the scattering of relativistic electrons, which will allow us to discuss the above phenomenon in detail. Section 3 is devoted to a solution of the resulting kinetic equation for the electron distribution function by adapting the asymptotic techniques of Kruskal and Bernstein [4] to the relativistic problem. The detailed analysis allows us to define a critical electric field E_R

$$E_R = \frac{4\pi\, ne^3 \ln\lambda}{mc^2} \tag{12}$$

below which no electrons run away, and to obtain an expression for the production rate of runaways when $E > E_R$, thus giving a relativistic generalization of expression (7). We find significant modifications to the non-relativistic result when $E/E_D \sim (kT/mc^2)^{1/2}$ which situation already holds in existing tokamaks.

Our main concern has been the role of relativistic effects on runaway electrons. However, we also include the effect of ions with charge $Z \neq 1$, although we limit our discussion to the case $ZE/E_D \ll 1$. This situation was briefly referred to, in a footnote, by Gurevich [2], and, in a later paper, Gurevich and Zhivlyuk [9] treated the case of arbitrary ZE/E_D to exponential accuracy. The numerical treatment of Kulsrud et al. [6] also considered this latter situation. We obtain below an analytic formula for this effect which can be compared with this earlier work in the non-relativistic limit. The presence of impurities introduces an effective value of Z, $Z_{eff} = \Sigma n_\alpha Z_\alpha^2/\Sigma n_\alpha Z_\alpha$, where α is summed over all ions present in the plasma, and our formulae enable us to investigate the influence of impurities on the runaway rate.

2. FOKKER-PLANCK EQUATION FOR RELATIVISTIC ELECTRONS

In non-relativistic scattering, the Landau form of the collision operator is appropriate in a fully ionized gas. For species 'a' scattering on species 'b' we write the collision operator as [10]

$$C_{ab}(f_a(p)) = n_b \frac{\partial}{\partial p_i} \times \int Q_{ij}^{ab}(\vec{v} - \vec{v}')\left(\frac{\partial f_a}{\partial p_j} f_b - \frac{\partial f_b}{\partial p_j'} f_a\right) d^3p' \tag{13}$$

with

$$Q_{ij}^{ab}(\vec{v} - \vec{v}') = 2\, e_a^2\, e_b^2 \times \int_K \frac{k_i\, k_j}{k^4}\delta[(\vec{k}.\vec{v}) - (\vec{k}.\vec{v}')]\, d^3k \tag{14}$$

with the d^3k integration extending over the region $r_d^{-1} < |k| < r_{min}^{-1}$ which we label K, where r_{min} is the distance of closest approach in Coulomb scattering and r_d is the Debye length, e_a and e_b are the charges of species a and b and n_b is the density of species b. We consider the distribution functions to be functions of momentum p rather than velocity v, since this will be more natural in discussing relativistic scattering. As is well known Q_{ij}^{ab} can be evaluated to give

$$Q_{ij}^{ab} = \frac{2\pi\, e_a^2\, e_b^2}{|\vec{v} - \vec{v}'|^3}\{(\vec{v} - \vec{v}')^2\delta_{ij} - (v - v')_i\,(v - v')_j\}\ln\lambda \tag{15}$$

where $\ln\lambda$ is the Coulomb logarithm.

In discussing the scattering of high-energy electrons we may consider $v' \ll v$ and expand Q_{ij}^{ab} in v'/v. Terms representing slowing down and energy diffusion appear in order $(v'/v)^2$ relative to the dominant angle-scattering terms. Since we shall consider $v \sim c$, for consistency we require a form of the collision operator relativistically correct to order $(v'/c)^2$. A relativistic expression for C_{ab} has been given by Klimontovich [10] and can be written in the form

$$C_{ab} = n_b \frac{\partial}{\partial p_i}\left\{ \int Q_{ij}^{ab}(\vec{v} - \vec{v}') \times \left(\frac{\partial f_a}{\partial p_j} f_b - \frac{\partial f_b}{\partial p_j'} f_a \right) d^3 p' + \int R_{ij}^{ab} f_b(p') \frac{\partial f_a}{\partial p_j} d^3 p' - \int S_{ij}^{ab} \frac{\partial f_b}{\partial p_j'} f_a \, d^3 p' \right\} \tag{16}$$

where R_{ij}^{ab} and S_{ij}^{ab} are relativistic modifications arising from transverse fluctuating electric fields and are given by

$$R_{ij}^{ab}(\vec{v},\vec{v}') = 2 e_a^2 e_b^2 \times \int_K \frac{d^3 k}{k^4} \frac{(\vec{k}\cdot\vec{v})^2 \left((\vec{k} \times \vec{v}') \times \vec{k}\right)_j}{\left|(\vec{k}\cdot\vec{v})^2 - c^2 k^2\right|^2} \times \left((\vec{k} \times \vec{v}') \times \vec{k}\right)_i \delta(\vec{k}\cdot\vec{v} - \vec{k}\cdot\vec{v}') \tag{17}$$

$$S_{ij}^{ab}(\vec{v},\vec{v}') = e_a^2 e_b^2 \times \int_K \frac{d^3 k}{k^4} \frac{(\vec{k}\cdot\vec{v})^2 \left((\vec{k} \times \vec{v}) \times \vec{k}\right)_i \left((\vec{k} \times \vec{v}') \times \vec{k}\right)_j}{\left|(\vec{k}\cdot\vec{v})^2 - c^2 k^2\right|^2} \times \delta(\vec{k}\cdot\vec{v} - \vec{k}\cdot\vec{v}') \tag{18}$$

We adopt a cylindrical set of axes in momentum space, i.e. p, μ, φ where p is the magnitude of the momentum $\vec{p}$, μ is the cosine of the angle between $\vec{p}$ and the electric field $\vec{E}$ and φ is the azimuthal angle about the electric field direction. The distribution of field particles f_b is taken to be Maxwellian, correct to second order in the relativistic parameter $kT_b/m_b c^2$:

$$f_b(v') = \left(\frac{m_b}{2\pi k T_b}\right)^{3/2} \times \left\{1 + \frac{3}{8}\frac{k T_b}{m_b c^2}\left[15 - \left(\frac{m_b v'^2}{k T_b}\right)^2\right]\right\} \exp\left(-\frac{m_b v'^2}{2k T_b}\right) \tag{19}$$

Evaluating expression (16) with the aid of expressions (15), (17), (18) and (19) to second order in the relativistic parameter we obtain

$$C_{ab}(f_a) = \beta_a \frac{\partial^2 f_a}{\partial p^2} + \frac{\gamma_a}{p^2}\frac{\partial}{\partial \mu}(1-\mu^2)\frac{\partial f_a}{\partial \mu} + \frac{\gamma_a}{p^2(1-\mu^2)}\frac{\partial^2 f_a}{\partial \varphi^2} + \left(\frac{1}{p^2}\frac{\partial}{\partial p}(\beta_a p^2) - \lambda_a p\right)\frac{\partial f_a}{\partial p} - \frac{1}{p^2}\frac{\partial}{\partial p}(\lambda_a p^3) f_a \tag{20}$$

where

$$\beta_a = \sum_b \frac{2\pi e_a^2 e_b^2 n_b \ln\lambda}{v^3}\frac{2kT_b}{m_b} \times \left[1 + \frac{kT_b}{m_b c^2}\left(\frac{v^2}{c^2} - \frac{15}{2}\right)\right] \tag{21}$$

$$\gamma_a = \sum_b \frac{2\pi e_a^2 e_b^2 n_b \ln\lambda}{v}\left[1 - \frac{kT_b}{m_b v^2}\right] \tag{22}$$

$$\lambda_a = -\sum_b \frac{4\pi e_a^2 e_b^2 n_b \ln\lambda}{p v^2 m_b} \times \left[1 + \frac{kT_b}{m_b c^2}\left(\frac{v^2}{c^2} - \frac{15}{2}\right)\right] \tag{23}$$

with

$$v = \frac{pc}{\sqrt{p^2 + m^2c^2}} \tag{24}$$

In the presence of an electric field, the electron distribution function f satisfies the equation

$$e E_j \frac{\partial f}{\partial p_j} = \sum_{b=e,i} C_{eb}(f) \tag{25}$$

if we consider a quasi-steady state in which Ohmic heating and the exponentially small depletion due to runaways are neglected. Denoting the electron mass and temperature by m and T, respectively, this equation becomes

$$eE\left(\mu\frac{\partial f}{\partial p} + \frac{1-\mu^2}{p}\frac{\partial f}{\partial \mu}\right) = 2\pi e^4 n \ln\lambda \left\{\frac{\sqrt{p^2+m^2c^2}}{p^3 c} \times \left(1 + Z - \frac{kT}{mc^2}\frac{(p^2+m^2c^2)}{p^2}\right)\frac{\partial}{\partial \mu}(1-\mu^2)\frac{\partial f}{\partial \mu} + \frac{2}{mp^2c^2}\left[p^2 + m^2c^2 - \frac{kT}{2mc^2}\right.\right.$$

$$\times\left(\frac{2mc}{p^2}(m^2c^2-2p^2)(p^2+m^2c^2)^{\frac{1}{2}}\right.$$

$$\left.\left.+15m^2c^2+13p^2\right)\right]\frac{\partial f}{\partial p}$$

$$+\frac{2kT}{mc^2}\;\frac{(p^2+m^2c^2)^{3/2}}{p^3c}\cdot$$

$$\times\left[1-\frac{kT}{2mc^2}\left(\frac{13p^2+15m^2c^2}{p^2+m^2c^2}\right)\right]\frac{\partial^2 f}{\partial p^2}+\left.\frac{4f}{mc^2p}\right\} \qquad (26)$$

where we have assumed that $m_e/m_i \ll kT/m_e c^2$, f is independent of the phase angle φ and have introduced Z, the effective charge of the plasma ions.

In this equation, we shall consider relativistic velocities and it is convenient to normalize p so that $p = mcq$. Further, we consider the small quantity $\epsilon = kT/mc^2$ as an expansion parameter. The electric field E may be expressed in terms of the Dreicer field $E_D = 4\pi e^3 n \ln\lambda/kT$ and the quantity E/E_D is taken to be of order ϵ so that relativistic effects may compete with the tendency to runaway. Thus the quantity α

$$\alpha = \frac{E}{E_D\,\epsilon} = \frac{E\,mc^2}{4\pi e^3 n\,\ell n\lambda} \qquad (27)$$

is considered of order unity. The equation now takes the form

$$\alpha\left(\mu\frac{\partial f}{\partial q}+\frac{1-\mu^2}{q}\frac{\partial f}{\partial \mu}\right)=\frac{\sqrt{q^2+1}}{2q^3}$$

$$\times\left(1+Z-\frac{\epsilon(q^2+1)}{q^2}\right)\frac{\partial}{\partial\mu}(1-\mu^2)\frac{\partial f}{\partial\mu}$$

$$+\frac{1}{q^2}\left[q^2+1-\frac{\epsilon}{2}\left(\frac{2(1-2q^2)\sqrt{1+q^2}}{q^2}+15+13q^2\right)\right]\frac{\partial f}{\partial q}$$

$$+\frac{\epsilon(q^2+1)^{3/2}}{q^3}\left[1-\frac{\epsilon}{2}\left(\frac{13q^2+15}{q^2+1}\right)\right]\frac{\partial^2 f}{\partial q^2}+\frac{2f}{q} \qquad (28)$$

and in the next section we discuss its solution.

3. SOLUTION OF THE FOKKER-PLANCK EQUATION

The Fokker-Planck equation (28) closely resembles that solved by Kruskal and Bernstein [4], differing only through relativistic effects and allowance for $Z \neq 1$. They solved the equation by asymptotic techniques using the small parameter E/E_D. Although we introduce kT/mc^2 as the basic small parameter we order $E/E_D \sim kT/mc^2$, so that the two problems are essentially equivalent. Kruskal and Bernstein have described the asymptotic treatment but because of the brevity of their exposition it would seem desirable to expand their arguments here in the relativistic case.

Following Ref. [4] we divide momentum space into a number of distinct regions beginning with the non-relativistic region $q^2 < \epsilon^{1/2}$, region I.

Region I

For small q, $q^2 < \epsilon^{1/2}$ we can expand

$$f = f^{(0)} + \epsilon\, f^{(1)} + \ldots \qquad (29)$$

where $f^{(0)}$ is the non-relativistic Maxwellian $\exp(-(q^2/2\epsilon) + C_I)$ with C_I a normalization constant and $f^{(1)} = \alpha E_D f_s$ where f_s is the Spitzer solution [11]. When $q^2 \sim \epsilon^{1/2}$ this expansion fails and we must reconsider the solution of Eq. (28).

Region II

In region II, $q^2 \sim \epsilon^{1/2}$ and a new variable $u \sim \epsilon^{-1/4} q$ is introduced. Since the Maxwellian $\exp(-u^2/2\epsilon^{1/2})$ possesses no series expansion in powers of $\epsilon^{1/2}$ we must consider $F = \ln f$ writing

$$F = \epsilon^{-\frac{1}{2}} F^{(0)} + F^{(1)} + \epsilon^{\frac{1}{2}} F^{(2)} \ldots \qquad + C_{II} \qquad (30)$$

where the constant C_{II} is determined by matching onto the solution in region I. Substituting this expansion into Eq. (28) and solving order by order, we find in leading order $\partial F^{(0)}/\partial\mu = 0$, while in $\epsilon^{-1/2}$ order we obtain $F^{(0)} = u^2/2$, the non-relativistic Maxwellian. It is convenient to separate out the next-order correction to the relativistic Maxwellian, $\exp(-\sqrt{(1+q^2)/\epsilon})$, in $F^{(1)}$. Thus, $F^{(1)} = u^4/8 + F'$ where, from ϵ^0 order,

$$\frac{(1+Z)}{2}\left\{(1-\mu^2)\left[\left(\frac{\partial F'}{\partial\mu}\right)^2+\frac{\partial^2 F'}{\partial\mu^2}\right]\right.$$

$$\left.-2\mu\frac{\partial F'}{\partial u}\right\} - u\frac{\partial F'}{\partial u} + \alpha\mu u^4 = 0 \qquad (31)$$

which can be rewritten as a linear equation for $h = \exp F'$

$$\frac{(1+Z)}{2}\frac{\partial}{\partial\mu}(1-\mu^2)\frac{\partial h}{\partial\mu} - u\frac{\partial h}{\partial u} + \alpha\mu u^4 h = 0 \qquad (32)$$

This parabolic equation (32) can, in principle, be solved with 'initial' values in v obtained from matching onto the Spitzer solution as $u \to 0$ (i.e. if we define $C_{II} = C_I$, $h(0,\mu) = 1$). In fact, a numerical solution appears to be necessary but an analytic solution can be obtained for large u from Eq. (31). Solving Eq. (31) as a descending series in u^2, and requiring a finite solution at $\mu = 1$ in each order (which results in the appearance of a term in $\ln u$) we find

$$F' = \frac{\alpha u^4}{4} + \sqrt{\frac{2\alpha}{1+Z}}\, u^2\left(2\sqrt{1+\mu} - \frac{9+Z}{2\sqrt{2}}\right)$$
$$+ \frac{5+Z}{4}\ln u + \frac{1}{4}\ell n(1+\mu) + \frac{1}{1+Z}\int_\mu^1 \frac{d\mu}{1-\mu^2}$$
$$\times\left\{4 + \mu(5+Z) - (9+Z)\sqrt{\frac{1+\mu}{2}}\right\} + b + O(u^{-2}) \tag{33}$$

where b is an unknown constant of O(1) which can only be obtained by matching the full solution of Eq. (31) onto the solution in region I, and into which the difference $C_{II} - C_I$ can be absorbed. (In the notation of Ref. [4], $a = b - 3/8 \ln\alpha$ when $Z = 1$.)

The expansion (30) for F fails when $u^2 \sim \epsilon^{-1/2}$, i.e. $q^2 \sim O(1)$, and the solution of the Fokker-Planck equation (28) must again be reconsidered.

Region III

In region III $q^2 \sim O(1)$ and we have the fully relativistic situation. The appropriate expansion for F has the form

$$F = \epsilon^{-1} F^{(0)} + \epsilon^{-\frac{1}{2}} F^{(1)} + F^{(2)} + \ldots \qquad + C_{III} \tag{34}$$

where the coefficients $F^{(n)}$ should not be confused with those in expansion (30). Substitution in Eq. (28) yields $\partial F^{(0)}/\partial\mu = 0$ in ϵ^{-2} order which also trivially satisfies $\epsilon^{-3/2}$ order. In ϵ^{-1} order, we find

$$\alpha\mu\frac{\partial F^{(0)}}{\partial q} = \frac{\sqrt{1+q^2}}{q^3}\left(\frac{1+Z}{2}\right)\left(\frac{\partial F^{(1)}}{\partial\mu}\right)^2(1-\mu^2)$$
$$+ \frac{(q^2+1)^{3/2}}{q^3}\left(\frac{\partial F^{(0)}}{\partial q}\right)^2 + \frac{(q^2+1)}{q^2}\frac{\partial F^{(0)}}{\partial q} \tag{35}$$

Annihilating the $\partial F^{(1)}/\partial\mu$ term by considering this equation at the point $\mu = +1$ (at $\mu = -1$ there is a boundary layer) an equation for $\partial F^{(0)}q/\partial q$ is obtained, with solution

$$F^{(0)} = \alpha\left(\sqrt{q^2+1} + \frac{1}{\sqrt{q^2+1}}\right) - \sqrt{q^2+1} \tag{36}$$

where the first part modifies the relativistic Maxwellian $F^{(0)} = -\sqrt{q^2+1}$. An equation for $\partial F^{(1)}/\partial\mu$ is obtained by subtracting Eq. (35) evaluated at $\mu = 1$, from itself to give

$$F^{(1)} = 2\sqrt{\frac{2\alpha}{1+Z}}\,\frac{q^2}{q^2+1}$$
$$\times\left(1 - (\alpha-1)q^2\right)^{\frac{1}{2}}\sqrt{1+\mu} + \bar{F}^{(1)}(q) \tag{37}$$

where $\bar{F}^{(1)}$ must be determined from the next-order equation. Thus the $\epsilon^{-1/2}$-order equation, evaluated at $\mu = 1$ provides an equation for $\partial F^{(1)}/\partial q$ ($\mu = 1$) which can be integrated and combined with Eq. (37) to give

$$F^{(1)} = 2(\sqrt{1+\mu} - \sqrt{2})$$
$$\times\sqrt{\frac{2\alpha}{(1+Z)}}\,\frac{q^2}{q^2+1}\left(1-(\alpha-1)q^2\right)^{\frac{1}{2}}$$
$$- \frac{1}{2}\sqrt{\frac{\alpha(1+Z)}{\alpha-1}}\sin^{-1}\left[1 - \frac{2}{\alpha} + 2q^2\left(1-\frac{1}{\alpha}\right)\right] \tag{38}$$

provided $\alpha > 1$. We discuss the case $\alpha < 1$ below. The $\epsilon^{-1/2}$ order equation can also be used to obtain $\partial F^{(2)}/\partial\mu$ which can be integrated, introducing an arbitrary function $\bar{F}^{(2)}(q)$, which in turn may be obtained from the ϵ^0 order equation evaluated at $\mu = 1$. After a great deal of algebra we obtain

$$F^{(2)} = \frac{1}{4}\left(\frac{1+Z}{2}\right)\left\{\ln\left(\frac{\sqrt{1+q^2}-1}{\sqrt{1+q^2}+1}\right)\right.$$
$$+ \frac{(\alpha-2)\sqrt{\alpha}}{(\alpha-1)^{3/2}}\ln\left(\frac{\sqrt{\alpha}+\sqrt{\alpha-1}\sqrt{1+q^2}}{\sqrt{\alpha}-\sqrt{\alpha-1}\sqrt{1+q^2}}\right)$$
$$\left.+ \frac{2\alpha\sqrt{1+q^2}}{(\alpha-1)[1-(\alpha-1)q^2]}\right\} + \ln\frac{q}{\sqrt{1+q^2}}$$
$$+ \alpha\left[\frac{13}{2}\sqrt{1+q^2} + \frac{11}{2}\frac{1}{\sqrt{1+q^2}} + \frac{1}{3(1+q^2)^{3/2}}\right]$$
$$- \frac{1}{4}\frac{\alpha+1}{\alpha-1}\ln[1-(\alpha-1)q^2] + \frac{\alpha q^2}{(1+Z)}\frac{(\mu-1)}{\sqrt{1+q^2}}$$
$$+ \frac{1}{4}\ln(1+\mu) - \int_\mu^1 \frac{d\mu}{1-\mu^2}\left\{\mu\right.$$
$$+ \frac{2}{(1+Z)}\frac{[\alpha(\mu-2)q^2 + 1 + q^2]}{[1-(\alpha-1)q^2]}$$
$$\times\left[[(1+\mu) - \sqrt{2(1+\mu)}]\right.$$
$$\left.\left.\times\frac{[(q^2+1)(q^2+2) - \alpha q^2(q^2+3)]}{(q^2+1)^{3/2}} - \frac{\sqrt{1+\mu}}{2\sqrt{2}}\right]\right\} \tag{39}$$

The constant C_{III} is obtained by matching this solution in region III for $q^2 \ll 1$ onto the solution in region II as $u \to \infty$ with the result

$$C_{III} = b - \frac{(2\alpha - 1)}{\epsilon} + \frac{1}{2}\sqrt{\frac{(1+Z)\alpha}{\epsilon(\alpha - 1)}} \times \sin^{-1}\left(1 - \frac{2}{\alpha}\right) - \frac{37\alpha}{3} - \frac{(5+Z)\ln\epsilon}{16} - \frac{(1+Z)}{4}\frac{\alpha}{(\alpha-1)} + \frac{(1+Z)}{4}\ln 2 - \frac{(1+Z)}{4}\frac{(\alpha - 2)\sqrt{\alpha}}{(\alpha-1)^{3/2}}\ln(\sqrt{\alpha} + \sqrt{\alpha - 1}) \tag{40}$$

Note that we recover the results of Ref. [4] by taking the non-relativistic limit $q^2 \ll 1$, $\alpha \gg 1$ but retaining αq^2 finite.

It is clear that a singularity exists as $q \to q_c = 1/\sqrt{\alpha - 1}$ and we must investigate a boundary layer around $q = q_c$ which we label region IV. However, before discussing this we observe that as $\alpha \to 1$, $q_c \to \infty$ so that region III extends to infinity and the runaway region $q > q_c$ disappears. In addition, when $\alpha \leq 1$, Eq. (36) for $F^{(0)}$ still holds but different solutions for $F^{(1)}$ and $F^{(2)}$ are required. Thus

$$F^{(1)} = 2(\sqrt{1+\mu} - \sqrt{2})\sqrt{\frac{2\alpha}{1+Z}\frac{q^2}{q^2+1}} \times (1 + (1-\alpha)q^2)^{\frac{1}{2}} - \frac{1}{2}\sqrt{\frac{(1+Z)\alpha}{1-\alpha}}\cosh^{-1}\left[\frac{2}{\alpha} - 1 + 2q^2\left(\frac{1}{\alpha} - 1\right)\right] \tag{41}$$

while $F^{(2)}$ possesses no singularity in q, i.e. for $\alpha \leq 1$ region III extends to infinity and no runaway region exists. The solution in this case will be used in a later paper to discuss synchrotron radiation from a plasma in an electric field.

Returning to the runaway case $\alpha > 1$ we consider the singularity at $q = q_c$.

Region IV

Examination of the source of the singularity in Eq. (35) shows that it can be removed by the introduction of the new variable x where

$$q = q_c(1 + \epsilon^{\frac{1}{3}} x)$$

and the expansion

$$F = \frac{F^{(0)}}{\epsilon^{\frac{1}{3}}} + F^{(1)} + \epsilon^{\frac{1}{3}} F^{(2)} + \ldots \qquad + C_{IV} \tag{42}$$

Expanding Eq. (35) about q_c and inserting the expansion (42) we find in $\epsilon^{-2/3}$ order

$$\left(\frac{1+Z}{2}\right)\sqrt{\frac{\alpha - 1}{\alpha}}(1+\mu)\left(\frac{\partial F^{(0)}}{\partial \mu}\right)^2 + \frac{\partial F^{(0)}}{\partial x} = 0 \tag{43}$$

This non-linear partial differential equation can be solved in terms of its boundary values on $\mu = 1$ by the method of characteristics. These boundary values are obtained by considering subregion IV' defined by $\mu = 1 - \epsilon^{1/3}\nu$, $\nu > 0$ and finite, and again expanding

$$F = \frac{F^{(0)}}{\epsilon^{\frac{1}{3}}} + F^{(1)} + \epsilon^{\frac{1}{3}} F^{(2)} + \ldots \qquad + C_{IV'} \tag{44}$$

Using the boundary condition that $F^{(n)}$ be finite as $\nu \to 0$ we find in ϵ^{-1} order

$$\frac{\partial F^{(0)}}{\partial \nu} = 0 \tag{45}$$

while $\epsilon^{-2/3}$ order is trivially satisfied, and in $\epsilon^{-1/3}$ order

$$(1+Z)\left\{\nu\left[\frac{\partial^2 F^{(1)}}{\partial \nu^2} + \left(\frac{\partial F^{(1)}}{\partial \nu}\right)^2\right] + \frac{\partial F^{(1)}}{\partial \nu}\right\} + \alpha\left(\frac{\partial F^{(0)}}{\partial x}\right)^2 = \sqrt{\frac{\alpha}{\alpha - 1}}\left\{-\nu + 2\left(1 - \frac{1}{\alpha}\right)\right\}\frac{\partial F^{(0)}}{\partial x} \tag{46}$$

This has a solution

$$F^{(1)} = P(x) + \nu Q(x) \tag{47}$$

where by equating powers of ν we find

$$\frac{\partial F^{(0)}}{\partial x} = -\sqrt{\frac{\alpha - 1}{\alpha}}(1+Z)Q^2 \tag{48}$$

and

$$Q^3 + \frac{2x}{\alpha(1+Z)}Q + \frac{1}{(1+Z)(\alpha - 1)} = 0 \tag{49}$$

which can be integrated parametrically to give

$$F^{(0)} = \sqrt{\alpha(\alpha - 1)}(1+Z)^2\frac{Q^4}{4} - \frac{1+Z}{2}\sqrt{\frac{\alpha}{\alpha - 1}}Q \tag{50}$$

Similarly, in ϵ^0 order, we find an equation for $F^{(2)}$ satisfied by the form

$$F^{(2)} = A(x) + B(x)\,\nu + C(x)\,\nu^2 \tag{51}$$

where equating powers of ν leads to an equation for $\partial P/\partial x$. Integrating this parametrically we find

$$P = -\frac{(1+Z)^3}{8}\sqrt{\alpha(\alpha-1)}\,(\alpha-2)\,Q^6 - \frac{(1+Z)^2}{4}\sqrt{\frac{\alpha}{\alpha-1}}\left(\alpha-\frac{7}{3}\right)Q^3 - \frac{1}{2}\ln\left[2\alpha(1+Z)\,Q^3 - \frac{\alpha}{\alpha-1}\right] + \left(\frac{\alpha-2}{\alpha-1}\right)\left[1-\frac{(1+Z)}{4}\sqrt{\frac{\alpha}{\alpha-1}}\right]\ln Q \tag{52}$$

Lebedev's error in considering φ_2 to be zero in leading order becomes clear at this point. It corresponds to setting $C = 0$ and ignoring the condition imposed by equating powers of ν^2. This leads to different logarithmic terms in P which are reflected through the matching conditions in the pre-exponential factors in the runaway rates. Indeed, Eq. (49) for Q and the logarithmic terms in Eq. (52) should correspond to Eq. (7) for ψ and (6) for φ_1, respectively, in Lebedev [3], as $\alpha \to \infty$ and $Z = 1$; the differences arise from the comments above.

Matching the limit of F obtained in subregion IV' as $x \to -\infty$ onto the solution in region III as $q \to q_c$ and $\mu \to 1$ we find we must take the root of Eq. (49) for which $Q \to -\infty$ as $\kappa \to -\infty$ which allows us to calculate C_{IV}'. For brevity we omit O(1) quantities to obtain

$$C_{IV}' = \frac{2}{\epsilon}\left(\sqrt{\alpha(\alpha-1)} - \alpha + \frac{1}{2}\right) - \frac{1}{2}\sqrt{\frac{\alpha(1+Z)}{\epsilon(\alpha-1)}}\left[\frac{\pi}{2} - \sin^{-1}\left(1-\frac{2}{\alpha}\right)\right] - \frac{\ln\epsilon}{48(\alpha-1)}\left\{\alpha(3Z+19) - 11 - 3Z + 2(1+Z)(\alpha-2)\sqrt{\frac{\alpha}{\alpha-1}}\right\} \tag{53}$$

Returning to the full region IV, we find that the characteristic curves for Eq. (43) can be parameterized in terms of the same quantity Q appearing in Eq. (49) which now satisfies the more general equation

$$Q^3 + \frac{2xQ}{\alpha(1+Z)} + \frac{4}{(1+Z)^2}\sqrt{\frac{\alpha}{\alpha-1}} \times \left[\sqrt{\frac{1+\mu}{2}} - 1 + \frac{(1+Z)}{4}\sqrt{\frac{\alpha}{\alpha-1}}\right] = 0 \tag{54}$$

away from $\mu = 1$. $F^{(0)}$ can be expressed in terms of Q as

$$F^{(0)} = \frac{3\sqrt{\alpha(\alpha-1)}(1+Z)^2 Q^4}{4} + (1+Z)\sqrt{\frac{\alpha-1}{\alpha}}\,xQ^2 \tag{55}$$

which coincides with the solution (50) on $\mu = 1$ using Eq. (49). Continuing the expansion of Eq. (35) in powers of $\epsilon^{1/3}$, we find in $\epsilon^{1/3}$ order

$$\frac{(1+Z)}{2}\left[(1-\mu^2)\left(\frac{\partial^2F^{(0)}}{\partial\mu^2} + 2\frac{\partial F^{(0)}}{\partial\mu}\frac{\partial F^{(1)}}{\partial\mu}\right) - 2\mu\frac{\partial F^{(0)}}{\partial\mu}\right] + \alpha\left(\frac{\partial F^{(0)}}{\partial x}\right)^2 + \sqrt{\frac{\alpha}{\alpha-1}}\left[(1-\mu)\frac{\partial F^{(1)}}{\partial x} + x\frac{(\alpha+1)}{\alpha}\frac{\partial F^{(0)}}{\partial x} - \frac{(3\alpha-1)}{\alpha}x\mu\frac{\partial F^{(0)}}{\partial x} - (1-\mu^2)\frac{\partial F^{(0)}}{\partial\mu}\right] = 0 \tag{56}$$

Changing variables from x and μ to x and Q defined through Eq. (54) this equation may be directly integrated using the boundary condition at $\mu = 1$, namely that $F^{(1)}$ coincides with Eq. (47). The result is rather complicated and since we do not need to make use of it in calculating the runaway rate we shall not quote it here, but we observe that

$$C_{IV} = C_{IV}' \tag{57}$$

As discussed in Ref. [4], the characteristics of Eq. (43) cover parts of the region $-\infty < x < \infty$, $-1 < \mu < 1$ doubly while some subregions are inaccessible to the characteristics. Further scalings are needed to penetrate these inaccessible subregions of region IV but this is not necessary for our present purpose of determining the runaway rate.

As $x \to +\infty$, we solve Eq. (54) for Q (taking the solution $Q \to 0$) and hence for F. Since we find F is only a logarithmic function of x and therefore of q in this limit we may simply expand $f = \exp(F)$ itself in powers of ϵ in the region $q > q_c$.

Region V

In region V, $q > q_c$, it is no longer necessary to expand $F = \ln f$ as in regions II, III and IV for we can revert to expanding f itself, as in region I:

$$f = f^{(0)} + \epsilon f^{(1)} + \dots \quad (58)$$

We assume that region V covers all $q > q_c$ which can only be justified by continuing through the inaccessible subregions of region IV. Substituting expansion (58) into the Fokker-Planck equation (28), we find

$$\left(\alpha\mu - \frac{(q^2+1)}{q^2}\right)\frac{\partial f^{(0)}}{\partial q} + \left[\alpha(1-\mu^2) + \frac{(1+Z)\sqrt{q^2+1}}{q^2}\right]\frac{1}{q}\frac{\partial f^{(0)}}{\partial \mu} - \frac{2f^{(0)}}{q} = \frac{(1+Z)}{2}\frac{\sqrt{q^2+1}}{q^3}(1-\mu^2)\frac{\partial^2 f^{(0)}}{\partial \mu^2} \quad (59)$$

As discussed by Kruskal and Bernstein [4] this is a two-way parabolic equation, the direction in q in which the solution can be 'advanced' depending on the sign of $\alpha\mu - (q^2+1)/q^2$. Thus for the region $\alpha\mu - (q^2+1)/q^2 < 0$ the solution is determined, in principle, by the boundary conditions at $q \to \infty$ (namely that f vanishes at $q \to \infty$, since there are no sources of runaways there), while for the region $\alpha\mu - (q^2+1)/q^2 > 0$ a specification of the source strength at the singularity $q = q_c$, $\mu = 1$ of this equation is necessary. This source strength, given by matching onto the solution in region IV, is interpreted as a source of runaways from lower velocities.

As in region II we can only obtain an analytic solution to Eq. (59) for large q. Changing variables from q and μ to cylindrical co-ordinates $q_\perp$ and $q_\parallel$ where $q = \sqrt{(q_\perp^2 + q_\parallel^2)}$, $\mu = q_\parallel/\sqrt{(q_\perp^2 + q_\parallel^2)}$, we seek a beam-like solution (i.e. peaked near $\mu = 1$ so that $q_\perp \ll q_\parallel$). With this approximation we obtain

$$f^{(0)} = A\,\frac{1}{q_\parallel}\exp\left\{-\frac{(\alpha+1)\,q_\perp^2}{2(1+Z)\,q_\parallel}\right\} \quad (60)$$

where A is a constant related to the source strength at $q = q_c$ as discussed in Ref. [4].

The runaway flux can be directly computed from Eq. (60). Calculating the limit of F in region IV' as $x \to +\infty$ i.e. $q > q_c$ we can calculate A to O(1). This then leads to a runaway rate

$$S_R \propto \epsilon^{-h(\alpha,Z)}\exp\left\{\frac{2}{\epsilon}\left(\sqrt{\alpha(\alpha-1)} - \alpha + \frac{1}{2}\right) - \frac{1}{2}\sqrt{\frac{\alpha(1+Z)}{\epsilon(\alpha-1)}}\left[\frac{\pi}{2} - \sin^{-1}\left(1-\frac{2}{\alpha}\right)\right]\right\} \quad (61)$$

where

$$h(\alpha,Z) = \frac{1}{16(\alpha-1)}\left\{\alpha(Z+1) - Z + 7 + 2\sqrt{\frac{\alpha}{\alpha-1}}(1+Z)(\alpha-2)\right\} \quad (62)$$

For comparison with earlier results it is convenient to express this result in terms of E/E_D, giving a runaway accumulation rate

$$S_R = C_R\, n\nu(v_{Th})\left(\frac{E}{E_D}\right)^{-h(\alpha,Z)} \times \exp\left\{-\frac{\lambda(\alpha)}{4}\frac{E_D}{E} - \left(\frac{2E_D}{E}\right)^{\frac{1}{2}}\gamma(\alpha,Z)\right\} \quad (63)$$

where $C_R(\alpha, Z)$ is an unknown constant of order unity in ϵ and

$$\lambda(\alpha) = 8\alpha\left(\alpha - \frac{1}{2} - \sqrt{\alpha(\alpha-1)}\right)$$

$$\gamma(\alpha,Z) = \sqrt{\frac{(1+Z)\,\alpha^2}{8(\alpha-1)}}\left[\frac{\pi}{2} - \sin^{-1}\left(1-\frac{2}{\alpha}\right)\right] \quad (64)$$

4. CONCLUSIONS

In Section 3, we have derived expressions for the electron distribution function in the presence of a weak electric field covering many regions of velocity space. These calculations include relativistic effects and allow for the effects of ions with $Z \neq 1$. The main result of this analysis is that the critical velocity at which electrons can run away is modified by relativistic effects. In terms of the normalized momentum q this occurs at $q = q_c = 1/\sqrt{(\alpha-1)}$. Clearly, if $\alpha = (E/E_D)\cdot(mc^2/kT) \to 1$ this critical momentum recedes to infinity, i.e. there are no runaway electrons produced if

$$E < \frac{E_D kT}{mc^2} = E_R \equiv \frac{4\pi n e^3 \ln\lambda}{mc^2} \quad (65)$$

(for $n \sim 5\times10^{19}$ m^{-3}, $E_R \sim 5\times10^{-2}$ V/m).

In this case, the distribution function at large values of q is given by Eqs (34), (36), (40) and (41). This distribution function contains a larger number of particles at higher energies than a Maxwellian but also has a characteristic width $\sim\sqrt{(E/E_D)}$ about $\mu = 1$, although even at $\mu \to -1$ there are more particles than in the Maxwellian.

For electric fields $E > E_R$, runaway electrons are generated in the region of momentum space given by

$$p > mcq_c = \frac{mc}{\left(\frac{E}{E_D}\frac{mc^2}{kT} - 1\right)^{\frac{1}{2}}}$$

For large p the distribution function has a beam-like nature with a Gaussian distribution of perpendicular momenta, whose width increases in proportion to the parallel momentum $q_\parallel$ as given by Eq. (60). This width exceeds that of the non-relativistic calculation [4] (namely $\ln q_\parallel$) because of the increased scattering effect at relativistic energies (see the Lorentz term in Eq. (28)).

In the situation, $E > E_R$, the runaway production rate is given by Eq. (63). Clearly, as $E \to E_R$, i.e. as $\alpha \to 1$, $S_R \to 0$. Thus the runaway result is strongly influenced when $E/E_D \gtrsim kT/mc^2$, but even when the condition $E/E_D \gg kT/mc^2$ is satisfied, relativistic corrections of order $\exp\{-(E_D/E)^2 kT/mc^2\}$ appear. Expanding the functions $\lambda(\alpha)$ and $\gamma(\alpha)$ for large α in Eq. (63) shows that the dominant relativistic corrections are given by

$$S_R \sim S_{NR} \exp\left\{-\frac{kT}{mc^2}\left[\frac{1}{8}\left(\frac{E_D}{E}\right)^2 + \frac{2}{3}\left(\frac{E_D}{E}\right)^{3/2}(1+Z)^{\frac{1}{2}}\right]\right\} \tag{66}$$

where the non-relativistic limit S_{NR} is given by

$$S_{NR} = C\, n\nu(v_{Th})\left(\frac{E}{E_D}\right)^{-3/16(Z+1)} \times \exp\left\{-\frac{E_D}{4E} - \left((1+Z)\frac{E_D}{E}\right)^{\frac{1}{2}}\right\} \tag{67}$$

TABLE I. THE EFFECT OF Z ON THE RUNAWAY RATE S. $(E/E_D \equiv 0.10)$

Z	$\frac{S(Z)}{S(1)}$ (Ref.[6])	$\frac{S(Z)}{S(1)}$ (Eq.(67))
2	0.582	0.564
3	0.374	0.372

Thus even in the supposedly non-relativistic limit, $\alpha \gg 1$, the exponential correction factor in Eq.(66) may be significant and S_{NR} may give an overestimate of the runaway flux.

The Z-dependence of the exponential term in Eq. (67) agrees with that found by Gurevich [2] who did not, however, calculate the Z-dependence in the pre-exponential factor. A comparison with numerical results obtained by Kulsrud et al. [6] is shown in Table I where the ratio $S(Z)/S(Z=1)$ is listed for $Z = 2$ and 3. A comparison with the $Z = 10$ result obtained in that reference would be inappropriate since $E_D/E = 10$ was taken in Ref. [6] and $Z = E_D/E$ violates the ordering in our analysis.

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