In [1]:

```
import sympy
from sympy import Matrix
import numpy as np
```

In [2]:

```
%matplotlib notebook
import matplotlib.pyplot as plt
from mpl_toolkits.mplot3d import axes3d
```

Mathematics for Machine Learning

Session 02: Inner product, solving systems of linear equations

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The inner product of vectors

- the **inner product** of two vectors is a scalar

$$\begin{aligned}\mathbf{x}\cdot\mathbf{y} &\doteq x_1y_1 + x_2y_2 + \cdots + x_ny_n\\ &= \sum_i x_iy_i\end{aligned}$$

(Sometimes the inner product is written $\langle\mathbf{x},\mathbf{y}\rangle$.)

- the inner product is commutative

$$\mathbf{x}\cdot\mathbf{y} = \mathbf{y}\cdot\mathbf{x}$$

- Furthermore, the inner product is **linear** in both arguments

$$\begin{aligned}(a\mathbf{x})\cdot(b\mathbf{y}) &= ab(\mathbf{x}\cdot\mathbf{y})\\ \mathbf{x}\cdot(\mathbf{y}+\mathbf{z}) &= \mathbf{x}\cdot\mathbf{y}+\mathbf{x}\cdot\mathbf{z}\\ (\mathbf{x}+\mathbf{y})\cdot\mathbf{z} &= \mathbf{x}\cdot\mathbf{z}+\mathbf{y}\cdot\mathbf{z}\end{aligned}$$

norm of a vector

The **norm** (=length) of a vector is defined as

$$\begin{aligned}\|\mathbf{x}\| &\doteq \sqrt{\mathbf{x}\cdot\mathbf{x}}\\ &= \sqrt{\sum_i x_i^2}\end{aligned}$$

properties of the norm

properties of the norm

- for all vectors $\mathbf{x}, \mathbf{y}$ and scalars a:

$$\begin{aligned} \|\mathbf{x}\| &\geq 0 \\ \|\mathbf{x}\| &= 0 \text{ if and only if } \mathbf{x} = \mathbf{0} \; (\forall i.\, x_i = 0) \\ \|a\mathbf{x}\| &= |a|\|\mathbf{x}\| \\ \|\mathbf{x} + \mathbf{y}\| &\leq \|\mathbf{x}\| + \|\mathbf{y}\| \end{aligned}$$

unit vectors

A **unit vector** is a vector of length 1.

Examples:

In [3]:
```
Matrix([0.6, 0.8])
```

Out[3]:

$$\begin{bmatrix} 0.6 \\ 0.8 \end{bmatrix}$$

In [4]:
```
from sympy import Rational
Matrix([
    Rational(1,3),
    Rational(2,3),
    Rational(2,3)
])
```

Out[4]:

$$\begin{bmatrix} \frac{1}{3} \\ \frac{2}{3} \\ \frac{2}{3} \end{bmatrix}$$

In [5]:
```
Matrix([
    sympy.Rational(1,2),
    sympy.Rational(1,2),
    sympy.Rational(1,2),
    sympy.Rational(1,2)
])
```

Out[5]:

$$\begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}$$

In [6]:
```
x, y, z, t = sympy.symbols('x y z t')
Matrix([sympy.sin(x), sympy.cos(x)])
```

Out[6]:

$$\begin{bmatrix} \sin(x) \\ \cos(x) \end{bmatrix}$$

In [7]:
```
Matrix([0, 0, 0, 1, 0])
```

Out[7]: $\begin{bmatrix}0\\0\\0\\1\\0\end{bmatrix}$

We can always shrink or stretch a vector into a unit vector of the same direction by dividing it by its norm.

$$\frac{\mathbf{u}}{\|\mathbf{u}\|} \text{ is always a unit vector, } \text{provided } \mathbf{u} \neq \mathbf{0}.$$

angle between vectors

For unit vectors, the dot product has a simple geometric interpretation:

If

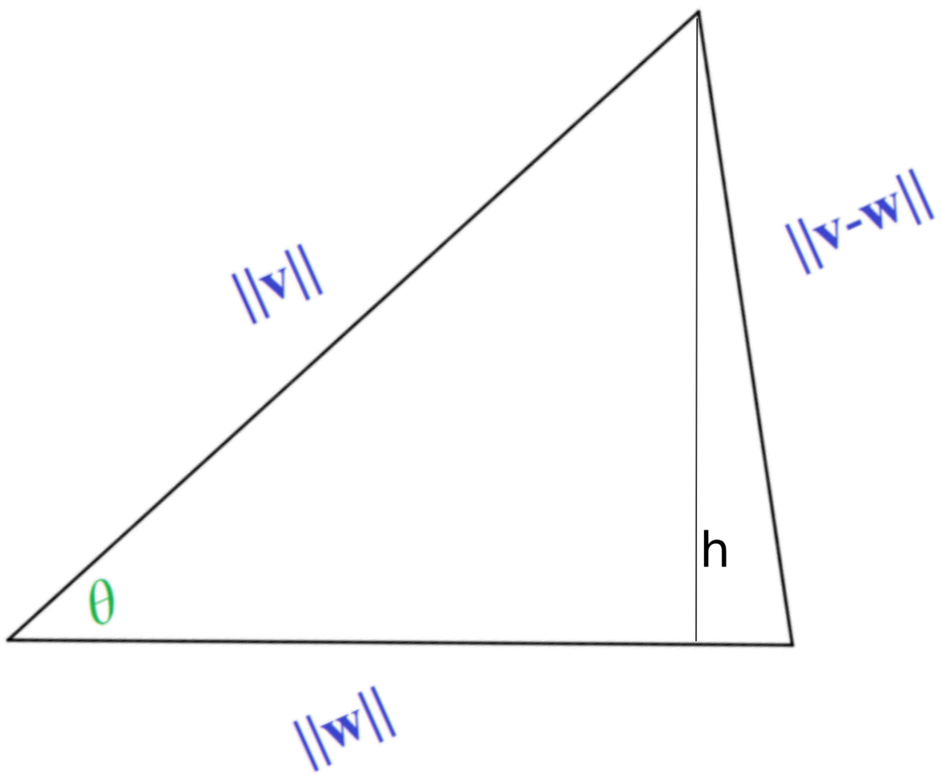
$$\|\mathbf{u}\| = \|\mathbf{v}\| = 1,$$

then

$$\mathbf{u} \cdot \mathbf{v} = \cos\theta,$$

where θ is the angle between $\mathbf{u}$ and $\mathbf{v}$.

Proof: https://proofwiki.org/wiki/Cosine_Formula_for_Dot_Product



$$\begin{aligned} h^2 &= \|\mathbf{v}\|^2 - \|\mathbf{v}\|^2\cos^2\theta \\ &= \|\mathbf{v}-\mathbf{w}\|^2 - (\|\mathbf{w}\| - \|\mathbf{v}\|\cos\theta)^2 \end{aligned}$$

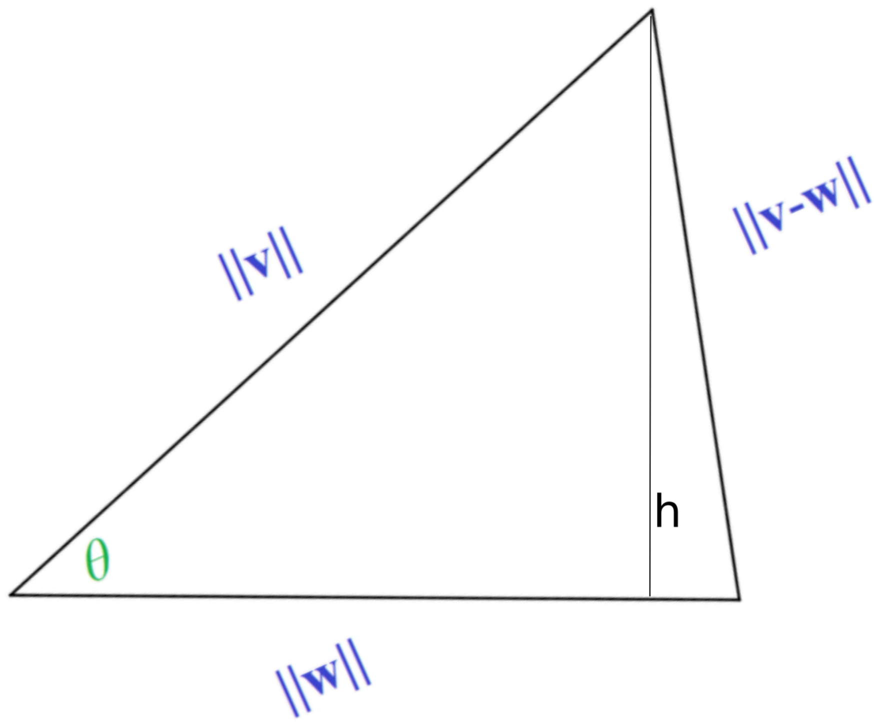
$$
\begin{aligned}
\|\mathbf{v}\|^2 - \|\mathbf{v}\|^2 \cos^2\theta &= \|\mathbf{v}-\mathbf{w}\|^2 - (\|\mathbf{w}\| - \|\mathbf{v}\|\cos\theta)^2 \\
&= \|\mathbf{v}\|^2 - 2\mathbf{v}\cdot\mathbf{w} + \|\mathbf{w}\|^2 - \|\mathbf{w}\|^2 + 2\|\mathbf{v}\|\|\mathbf{w}\|\cos\theta - \|\mathbf{v}\| \\
0 &= -2\mathbf{v}\cdot\mathbf{w} + 2\|\mathbf{v}\|\|\mathbf{w}\|\cos\theta \\
\cos\theta &= \frac{\mathbf{v}\cdot\mathbf{w}}{\|\mathbf{v}\|\|\mathbf{w}\|}
\end{aligned}
$$

This is how the cosine is defined in analytical geometry. (Note that this only holds if $\mathbf{u} \neq \mathbf{0}, \mathbf{v} \neq \mathbf{0}$.)

Since the cosine is always ≤ 1, it follows **("Schwarz Inequality")**:

$$\mathbf{u}\cdot\mathbf{v} \leq \|\mathbf{u}\|\|\mathbf{v}\|$$

triangle inequality



$$-1 \leq \cos\theta \leq 1$$

Therefore

$$
\begin{aligned}
\|\mathbf{v}-\mathbf{w}\| &= \sqrt{(\mathbf{v}-\mathbf{w})\cdot(\mathbf{v}-\mathbf{w})} && (1) \\
&= \sqrt{\mathbf{v}\cdot\mathbf{v} - 2\mathbf{v}\cdot\mathbf{w} + \mathbf{w}\cdot\mathbf{w}} && (2) \\
&= \sqrt{\|\mathbf{v}\|^2 - 2\|\mathbf{v}\|\|\mathbf{w}\|\cos\theta + \|\mathbf{w}\|^2} && (3) \\
&\leq \sqrt{\|\mathbf{v}\|^2 + 2\|\mathbf{v}\|\|\mathbf{w}\| + \|\mathbf{w}\|^2} && (4) \\
&\leq \|\mathbf{v}\| + \|\mathbf{w}\| && (5)
\end{aligned}
$$

orthogonal vectors

orthogonal vectors

The inner product of two vectors can be 0. Examples

$$\begin{bmatrix}3\\4\end{bmatrix},\begin{bmatrix}-8\\6\end{bmatrix}\quad\begin{bmatrix}5\\10\end{bmatrix},\begin{bmatrix}6\\-3\end{bmatrix}\quad\begin{bmatrix}0\\1\end{bmatrix},\begin{bmatrix}1\\0\end{bmatrix}\quad\begin{bmatrix}1\\2\\3\end{bmatrix},\begin{bmatrix}1\\1\\-1\end{bmatrix}$$

Since $\cos\theta = 0$ if $\theta \in \{90^\circ, -90^\circ\}$, vectors with a 0 inner product are **orthogonal** (perpendicular).

Matrices

A $m \times n$ matrix is a sequence of m row-vectors, each of length n or, equivalently, a sequence of n column vectors, each of length m.

Example of a 3×2 matrix:

$$A = \begin{bmatrix}1 & 2\\3 & 4\\5 & 6\end{bmatrix}$$

Individual cells are referred to with two indices (first: row, second: column), often using the lowercase version of the name of the matrix.

$$\begin{aligned}a_{1,1} &= 1\\a_{3,2} &= 6\\&\vdots\end{aligned}$$

The **transpose** of a matrix (written with T as an exponent) is the result of flipping rows and columns.

$$A^T = \begin{bmatrix}1 & 3 & 5\\2 & 4 & 6\end{bmatrix}$$

Obviously, the transpose of a $m \times n$ matrix is an $n \times m$ matrix.

Matrix operations

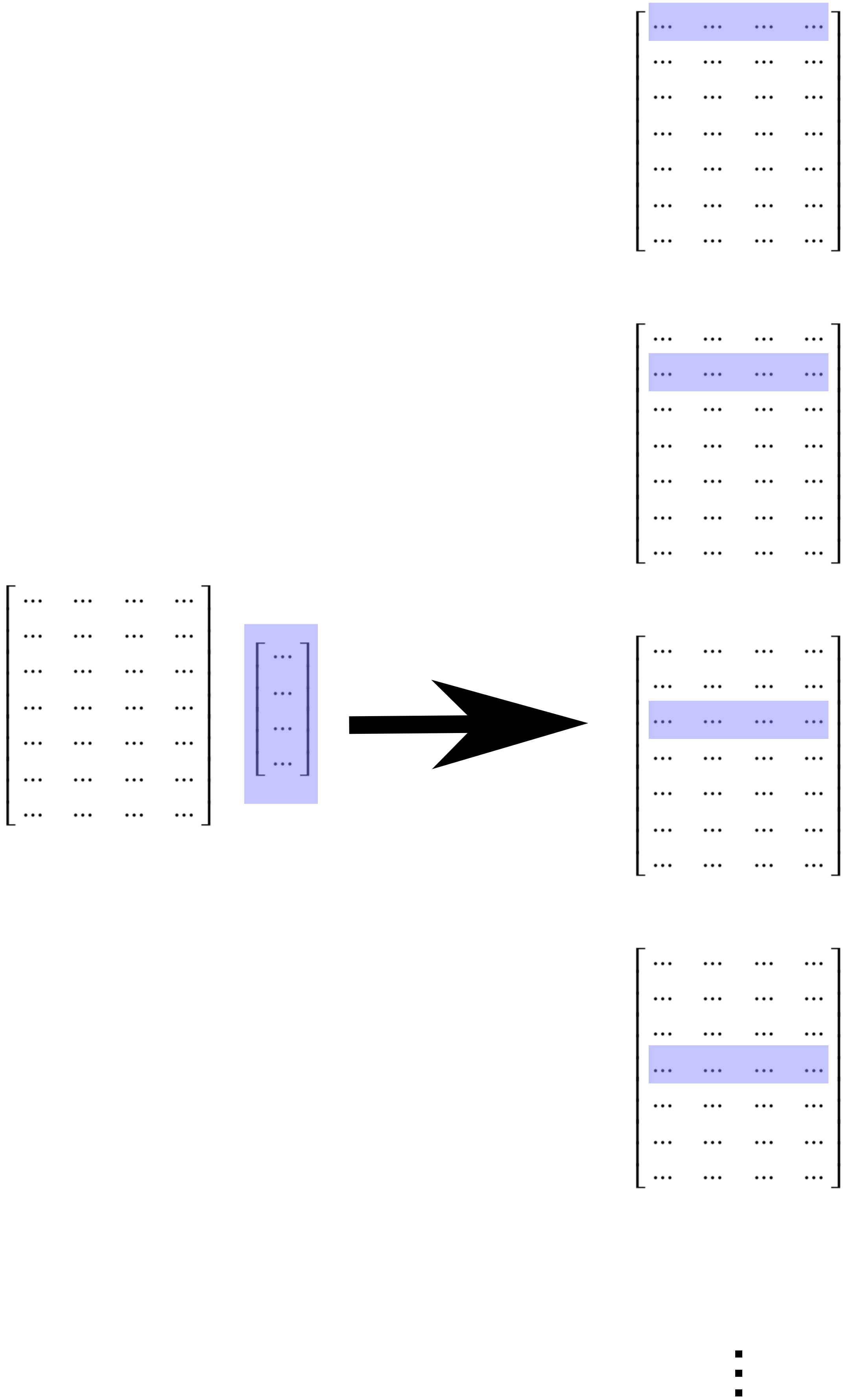
Applying a matrix to a vector

A $m \times n$ matrix can be seen as a function from $\mathbb{R}^n$ into $\mathbb{R}^m$. (Note that the number of columns reflects the input size and the number of rows the output size.)

General definition

$$\begin{bmatrix}\sum_{1\leq i\leq n} a_{1,i}x_i\\\sum_{1\leq i\leq n} a_{2,i}x_i\end{bmatrix}\quad\begin{bmatrix}A_{1,\_}\cdot\mathbf{x}\\A_{2,\_}\cdot\mathbf{x}\end{bmatrix}$$

$$A\mathbf{x} = \begin{bmatrix} \vdots \\ \sum_{1 \leq i \leq n} a_{m,i} x_i \end{bmatrix} = \begin{bmatrix} \vdots \\ A_{m,\_} \cdot \mathbf{x} \end{bmatrix}$$



Example

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Column picture

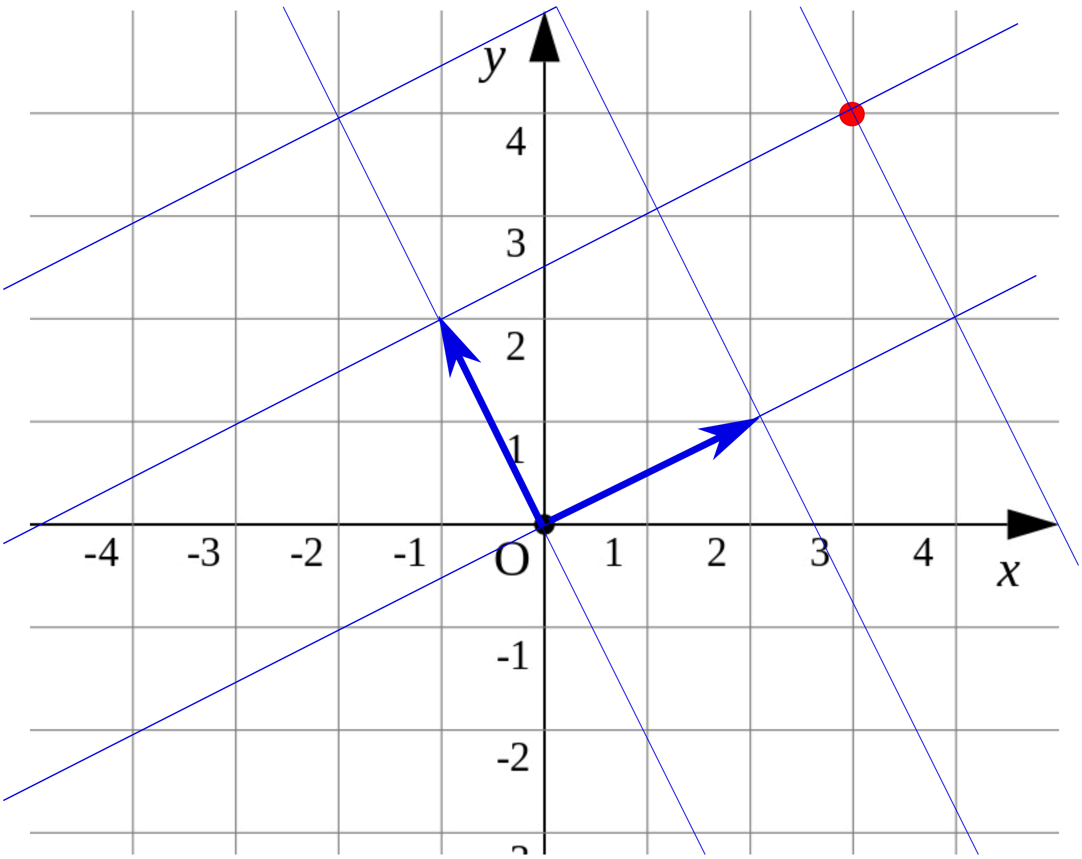
So far we focused on rows. Equivalently, this can be conceived as a column operation:

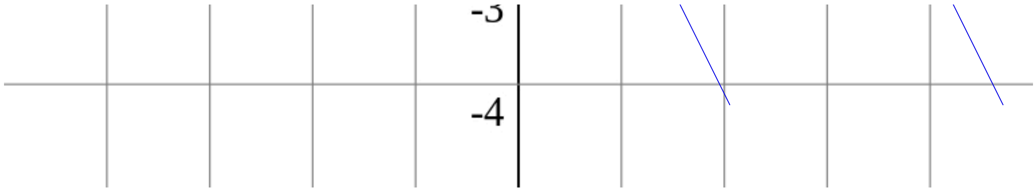
$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \end{bmatrix} = -1 \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix} + 1 \begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

When computing $A\mathbf{x}$, each column of A can be seen as the axis of some (possibly skewed or degenerate) **coordinate system**. Applying A to $\mathbf{x}$ means:

- $\mathbf{x}$ is a vector in the coordinate system defined by the columns of A
- $A\mathbf{x}$ is the translation of $\mathbf{x}$ into the "objective" coordinate system.

$$A = \begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix}$$
$$\mathbf{x} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$
$$A\mathbf{x} = 2 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + 1 \begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$$



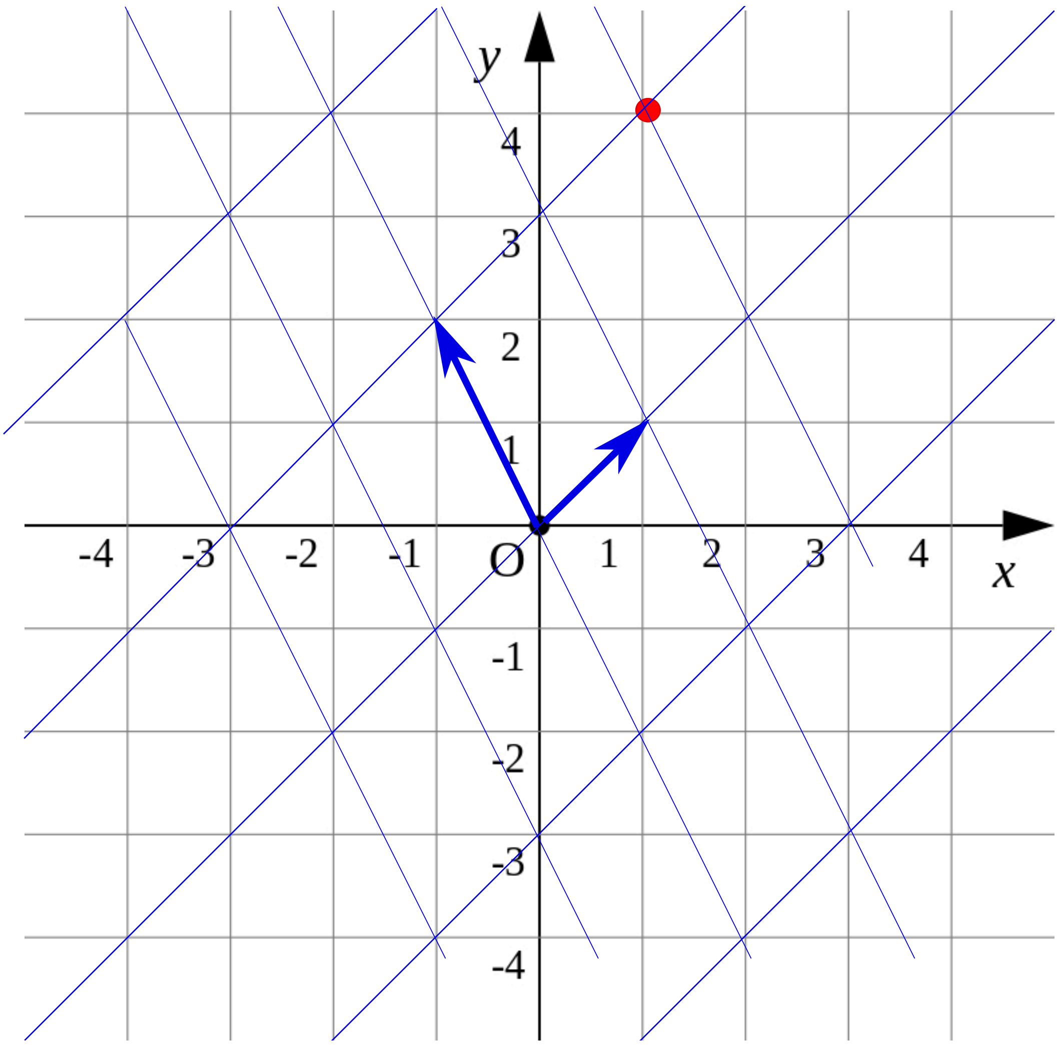


The columns of A need not be perpendicular.

$$A = \begin{bmatrix} 1 & -1 \\ 1 & 2 \end{bmatrix} \tag{6}$$

$$\mathbf{x} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \tag{7}$$

$$A\mathbf{x} = 2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + 1 \begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \end{bmatrix} \tag{8}$$



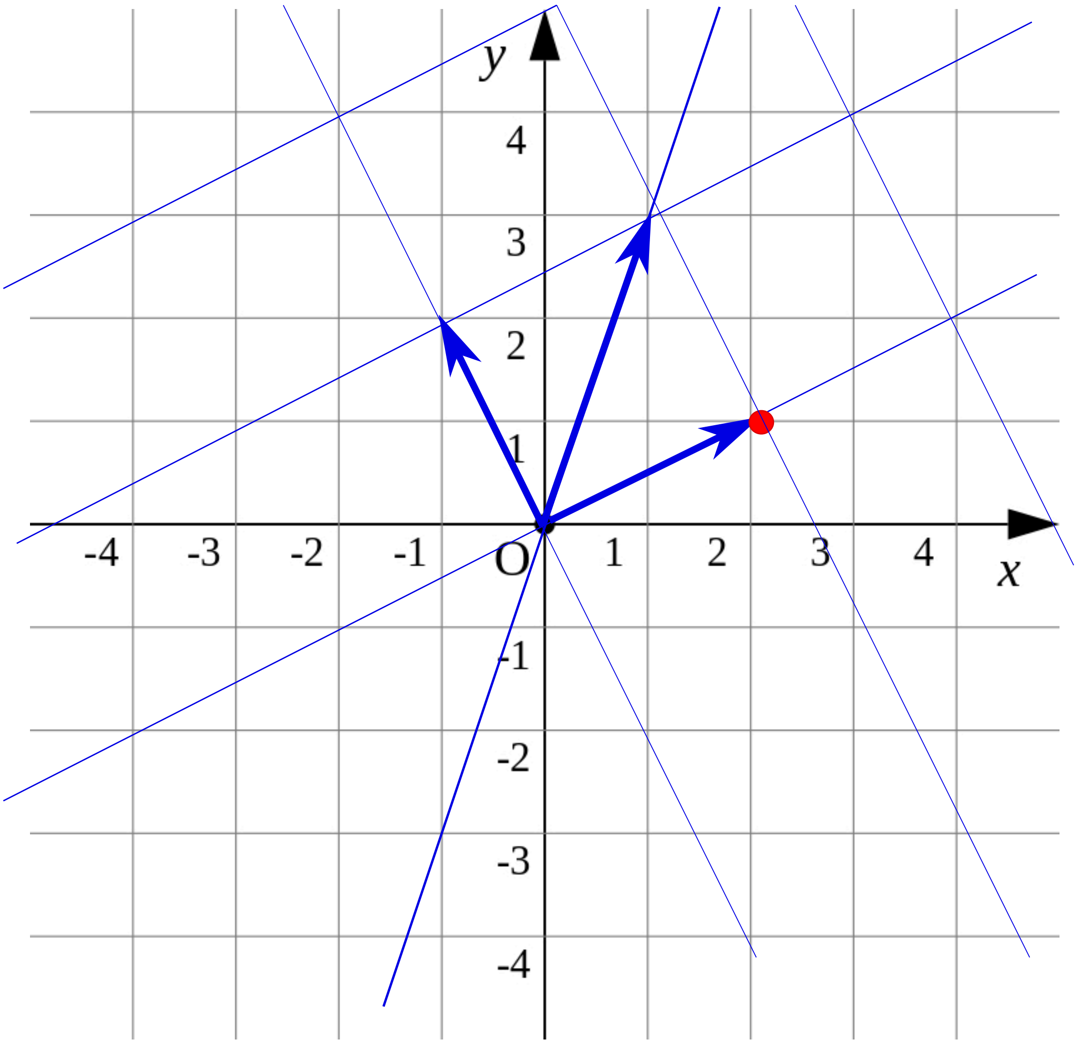
We can also have degenerate cases where

the columns of A are not independent.

$$A = \begin{bmatrix} 2 & -1 & 1 \\ 1 & 2 & 3 \end{bmatrix}$$

$$\mathbf{x} = \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}$$

$$A\mathbf{x} = 2 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + 1 \begin{bmatrix} -1 \\ 2 \end{bmatrix} - 1 \begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$



Conversely, A may project a low-dimensional vector into a higher-dimensional space.

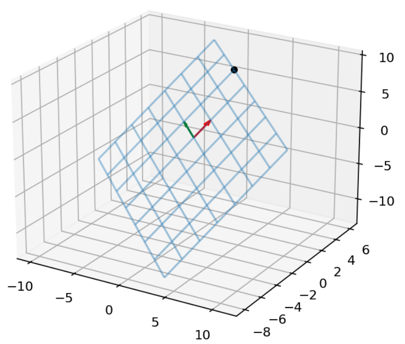
$$A = \begin{bmatrix} 1 & -2 \\ 1 & 1 \\ 2 & 1 \end{bmatrix}$$

$$\mathbf{x} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

$$A\mathbf{x} = 3 \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} + 1 \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \\ 7 \end{bmatrix}$$

In [8]:

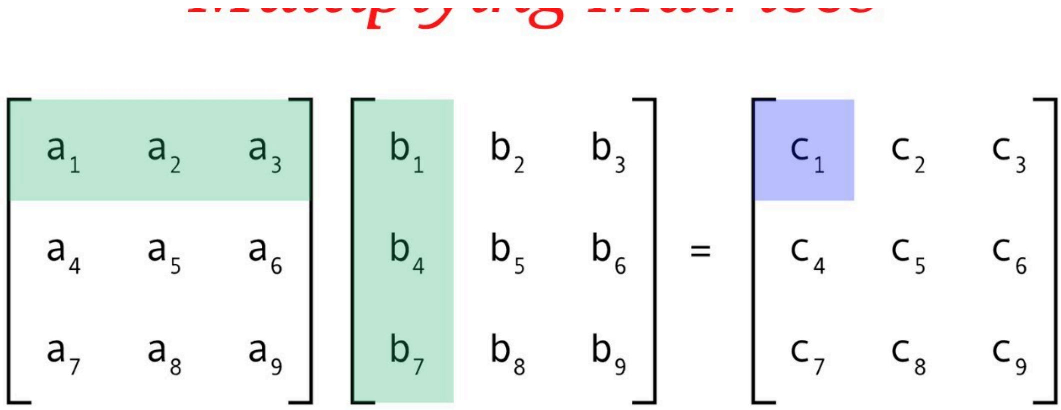
```
fig = plt.figure(figsize=(12,6))
ax = fig.add_subplot(121, projection='3d')




x = np.arange(-4, 4, 1)
y = np.arange(-4, 4, 1)
X,Y = np.meshgrid(x,y)
Xt = X - 2* Y
Yt = X + Y
Z = 2*X + Y

ax.plot_wireframe(Xt, Yt, Z, alpha=0.4)
ax.quiver((0,),(0,),(0,),(1,),(1,),(2,), color='red', length=1)
ax.quiver((0,),(0,),(0,),(-2,),(1,),(1,), color='green', length=1)
ax.scatter3D((1,), (4,), (7,), color="black")
plt.show()
```



Matrix multiplication

If A is an $m \times n$ matrix and B is a $n \times o$ matrix, than the **matrix product** AB is an $m \times o$ matrix.

$$\begin{aligned}(AB)_{i,j} &= \sum_k a_{i,k} b_{kj} \\ &= A_{i,\_} \cdot B_{\_,j}\end{aligned}$$



properties of matrix multiplication

- matrix multiplication is **associative**

$$(AB)C = A(BC)$$

- matrix multiplication is **not commutative**. It is possible that

$$AB \neq BA$$

Example

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} = \begin{bmatrix} 4 & 5 & 6 \\ 7 & 8 & 9 \\ 1 & 2 & 3 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 3 & 1 & 2 \\ 6 & 4 & 5 \\ 9 & 7 & 8 \end{bmatrix}$$

(There are cases where commutativity holds, but you cannot rely on it.)

special matrices

- a **diagonal matrix** is a matrix where all entries except the *main diagonal* (all $a_{i,i}$) are zero

In [9]:

```
Matrix([
    [1,0,0],
    [0,2,0],
    [0,0,3]
])
```

Out[9]:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

In [10]:

```
Matrix([
    [3,0,0],
```

```
    [0,2,0],
    [0,0,1],
    [0,0,0]
])
```

Out[10]:

$$\begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

In [11]:

```
Matrix([
    [1,0,0,0],
    [0,2,0,0],
    [0,0,3,0]
])
```

Out[11]:

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \end{bmatrix}$$

- square diagonal matrices have interesting properties
  - multiplying a matrix A from the *left* with a diagonal matrix multiplies each *row* of A with the corresponding diagonal entry
  - multiplying a matrix A from the *right* with a diagonal matrix multiplies each *column* of A with the corresponding diagonal entry

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 8 & 10 & 12 \\ 21 & 24 & 27 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 4 & 9 \\ 4 & 10 & 18 \\ 7 & 16 & 27 \end{bmatrix}$$

identity matrix

- a special case is $\mathbf{I}$, the diagonal matrix with only 1s at the diagonal. It is called the **identity matrix**

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

(NB: This is an example where commutativity happens to hold.)

- Strictly speaking, there is an $n \times n$ identity matrix for each number n of dimensions. In mathematical contexts, we usually rely that the value of n determined by the context. When programming, you have to be pedantic about theses things, of course.

inverse matrix

- Given a square matrix A, the **inverse Matrix** A^{-1} – if it exists – reduces A to $\mathbf{I}$.

$$AA^{-1} = A^{-1}A = \mathbf{I}$$

- Note that A^{-1} is both the left and the right multiplicative inverse.
- example:

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{bmatrix}$$

- example for a matrix without inverse:

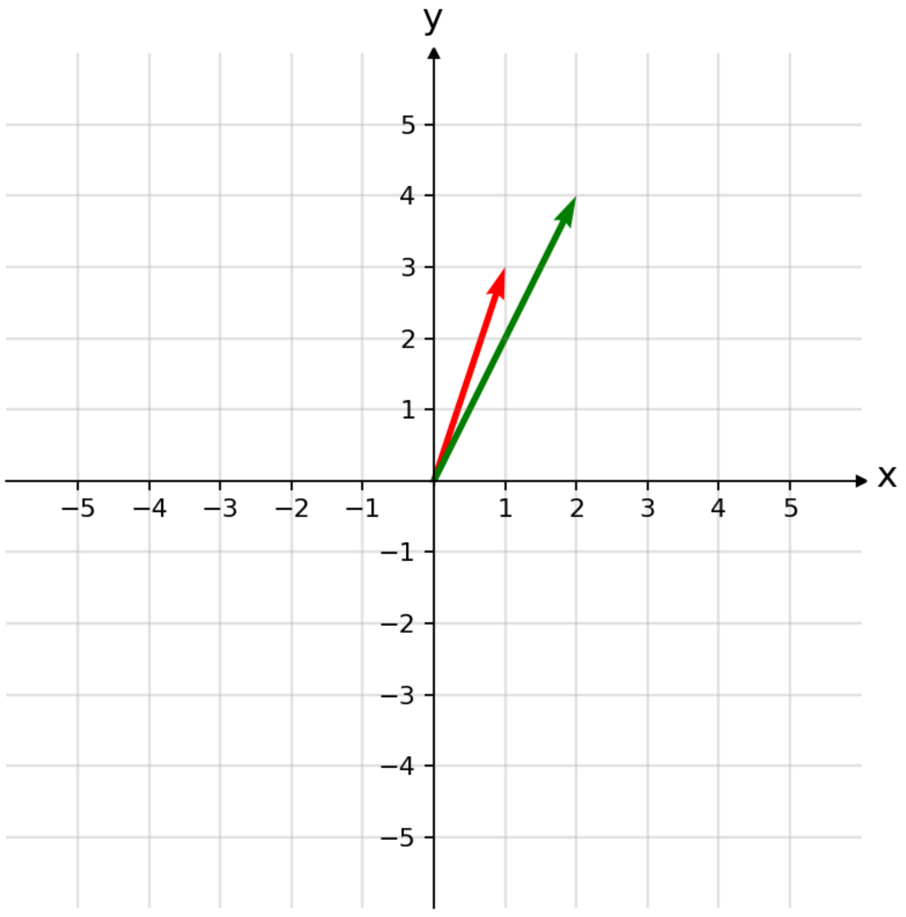
$$B = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$$
$$B^{-1} \text{ is undefined}$$

Why is this so?

Example of the “good” (invertable) matrix:

In [12]:
```
xs = [0, 2, -3, -1.5]
ys = [0, 3, 1, -2.5]
colors = ['m', 'g', 'r', 'b']

# Select length of axes and the space between tick labels
xmin, xmax, ymin, ymax = -5, 5, -5, 5
ticks_frequency = 1

fig, ax = plt.subplots(figsize=(6,6))
# Set identical scales for both axes
ax.set(xlim=(xmin-1, xmax+1), ylim=(ymin-1, ymax+1), aspect='equal')

# Set bottom and left spines as x and y axes of coordinate system
ax.spines['bottom'].set_position('zero')
ax.spines['left'].set_position('zero')

# Remove top and right spines
ax.spines['top'].set_visible(False)
ax.spines['right'].set_visible(False)

# Create 'x' and 'y' labels placed at the end of the axes
ax.set_xlabel('x', size=14, labelpad=-24, x=1.03)
ax.set_ylabel('y', size=14, labelpad=-21, y=1.02, rotation=0)
```

```
# Create custom major ticks to determine position of tick labels
x_ticks = np.arange(xmin, xmax+1, ticks_frequency)
y_ticks = np.arange(ymin, ymax+1, ticks_frequency)
ax.set_xticks(x_ticks[x_ticks != 0])
ax.set_yticks(y_ticks[y_ticks != 0])

# Create minor ticks placed at each integer to enable drawing of minor
# lines: note that this has no effect in this example with ticks_freque
ax.set_xticks(np.arange(xmin, xmax+1), minor=True)
ax.set_yticks(np.arange(ymin, ymax+1), minor=True)


# Draw major and minor grid lines
ax.grid(which='both', color='grey', linewidth=1, linestyle='-', alpha=(

# Draw arrows
arrow_fmt = dict(markersize=4, color='black', clip_on=False)
ax.plot((1), (0), marker='>', transform=ax.get_yaxis_transform(), **ar
ax.plot((0), (1), marker='^', transform=ax.get_xaxis_transform(), **ar

ax.quiver((0,),(0,), (1,), (3,), units="xy", scale=1, color='red')
ax.quiver((0,),(0,), (2,), (4,), units="xy", scale=1, color='green')
```

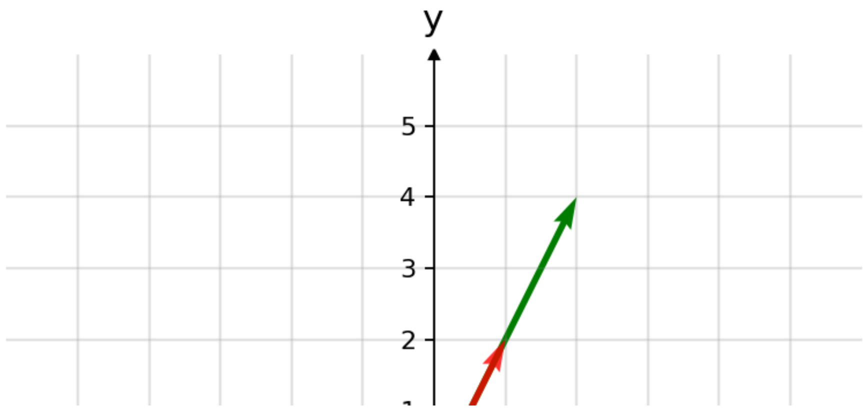


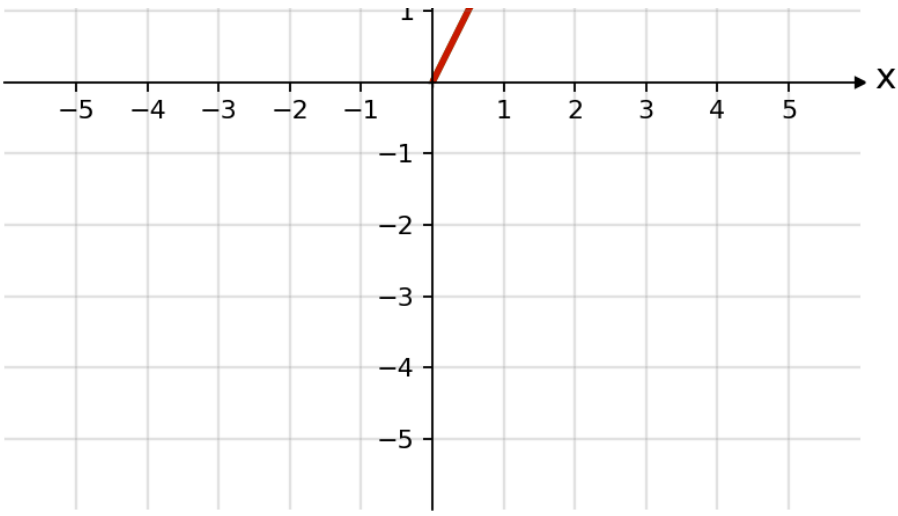
Out[12]: `<matplotlib.quiver.Quiver at 0x731d7c36edf0>`

Example of the “bad” (non-invertable) matrix:

In [13]:

```
xs = [0, 2, -3, -1.5]
ys = [0, 3, 1, -2.5]
colors = ['m', 'g', 'r', 'b']

# Select length of axes and the space between tick labels
xmin, xmax, ymin, ymax = -5, 5, -5, 5
ticks_frequency = 1

fig, ax = plt.subplots(figsize=(6,6))
# Set identical scales for both axes
ax.set(xlim=(xmin-1, xmax+1), ylim=(ymin-1, ymax+1), aspect='equal')

# Set bottom and left spines as x and y axes of coordinate system
ax.spines['bottom'].set_position('zero')
ax.spines['left'].set_position('zero')

# Remove top and right spines
ax.spines['top'].set_visible(False)
ax.spines['right'].set_visible(False)

# Create 'x' and 'y' labels placed at the end of the axes
ax.set_xlabel('x', size=14, labelpad=-24, x=1.03)
ax.set_ylabel('y', size=14, labelpad=-21, y=1.02, rotation=0)

# Create custom major ticks to determine position of tick labels
x_ticks = np.arange(xmin, xmax+1, ticks_frequency)
y_ticks = np.arange(ymin, ymax+1, ticks_frequency)
ax.set_xticks(x_ticks[x_ticks != 0])
ax.set_yticks(y_ticks[y_ticks != 0])

# Create minor ticks placed at each integer to enable drawing of minor
# lines: note that this has no effect in this example with ticks_freque
ax.set_xticks(np.arange(xmin, xmax+1), minor=True)
ax.set_yticks(np.arange(ymin, ymax+1), minor=True)


# Draw major and minor grid lines
ax.grid(which='both', color='grey', linewidth=1, linestyle='-', alpha=(

# Draw arrows
arrow_fmt = dict(markersize=4, color='black', clip_on=False)
ax.plot((1), (0), marker='>', transform=ax.get_yaxis_transform(), **ar
ax.plot((0), (1), marker='^', transform=ax.get_xaxis_transform(), **ar

ax.quiver((0,),(0,), (2,), (4,), units="xy", scale=1, color='green')
ax.quiver((0,),(0,), (1,), (2,), units="xy", scale=1, color='red', alph
```





Out[13]: <matplotlib.quiver.Quiver at 0x731d7c289850>

examples of an invertable and a non-invertable matrix in 3d

$$A = \begin{bmatrix} 1 & -4 & 2 \\ -2 & 1 & 3 \\ 2 & 6 & 8 \end{bmatrix}$$

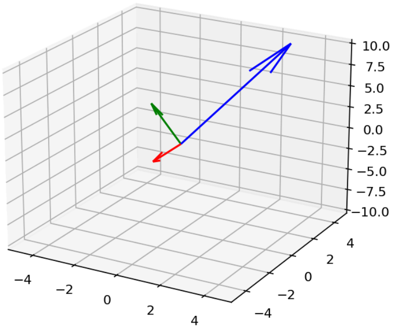
$$A^{-1} = \begin{bmatrix} \frac{5}{63} & -\frac{22}{63} & \frac{1}{9} \\ -\frac{11}{63} & -\frac{2}{63} & \frac{1}{18} \\ \frac{1}{9} & \frac{1}{9} & \frac{1}{18} \end{bmatrix}$$

$$B = \begin{bmatrix} 1 & -4 & 2 \\ -2 & 1 & 3 \\ 2 & 6 & -10 \end{bmatrix}$$

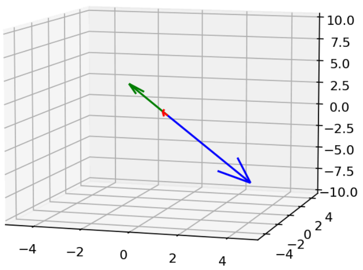
$$B^{-1} \text{ is undefined}$$

In [14]:

```
fig = plt.figure(figsize=(12,6))
ax = fig.add_subplot(121, projection='3d')



ax.set_xlim(-5,5)
ax.set_ylim(-5,5)
ax.set_zlim(-10,10)
ax.quiver((0,),(0,),(0,),(1,),(-4,),(2,), color='red', length=1)
ax.quiver((0,),(0,),(0,),(-2,),(1,),(3,), color='green', length=1)
ax.quiver((0,),(0,),(0,),(2,),(6,),(8,), color='blue', length=1)

plt.show()
```



In [15]:
```
fig = plt.figure(figsize=(12,6))
ax = fig.add_subplot(121, projection='3d')



ax.set_xlim(-5,5)
ax.set_ylim(-5,5)
ax.set_zlim(-10,10)
ax.quiver((0,),(0,),(0,),(1,),(-4,),(2,), color='red', length=1)
ax.quiver((0,),(0,),(0,),(-2,),(1,),(3,), color='green', length=1)
ax.quiver((0,),(0,),(0,),(2,),(6,),(-10,), color='blue', length=1)

plt.show()
```



- a matrix is invertible if the **column space** has dimensionality n
- it is not invertible if the column space has dimensionality $< n$

Matrices with python and numpy

- vector

In [16]:
```
import numpy as np
x = np.array([1,2,3])
x
```

Out[16]:
```
array([1, 2, 3])
```

- matrix

In [17]:
```
A = np.array([
    [1,2, 3],
    [4,5,6],
    [7,8,9]
])
A
```

Out[17]:
```
array([[1, 2, 3],
       [4, 5, 6],
       [7, 8, 9]])
```

vector algebra

In [18]:
```
y = np.array([3,10,4])

x + y
```

Out[18]:
```
array([ 4, 12,  7])
```

In [19]:
```
3*x
```

Out[19]:
```
array([3, 6, 9])
```

In [20]:
```
2*x - 3*y
```

Out[20]:
```
array([ -7, -26,  -6])
```

applying a matrix to a vector

In [21]:
```
A, x
```

Out[21]:
```
(array([[1, 2, 3],
        [4, 5, 6],
        [7, 8, 9]]),
 array([1, 2, 3]))
```

In [22]:
```
A @ x
```

Out[22]:
```
array([14, 32, 50])
```

matrix transposition

In [23]:
```
A
```

Out[23]:
```
array([[1, 2, 3],
       [4, 5, 6],
       [7, 8, 9]])
```

In [24]:
```
A.T
```

Out[24]:
```
array([[1, 4, 7],
       [2, 5, 8],
       [3, 6, 9]])
```

matrix multiplication

In [25]:
```
B = np.array([
    [4,1,0],
    [1,0,2],
    [4,5,6]
])
```

In [26]:
```
A, B
```

Out[26]:
```
(array([[1, 2, 3],
        [4, 5, 6],
        [7, 8, 9]]),
 array([[4, 1, 0],
        [1, 0, 2],
        [4, 5, 6]]))
```

In [27]:
```
A @ B
```

Out[27]:
```
array([[18, 16, 22],
       [45, 34, 46],
       [72, 52, 70]])
```

identity matrix

In [28]:
```
np.eye(3)
```

Out[28]:
```
array([[1., 0., 0.],
       [0., 1., 0.],
       [0., 0., 1.]])
```

diagonal matrix

In [29]:
```
np.diag([2,3, 4])
```

Out[29]:
```
array([[2, 0, 0],
       [0, 3, 0],
       [0, 0, 4]])
```

inverse matrix

In [30]:
```
A = np.array(
[
    [1, -4, 2],
    [-2, 1, 3],
    [2,6,8]
])
A
```

Out[30]:
```
array([[ 1, -4,  2],
       [-2,  1,  3],
       [ 2,  6,  8]])
```

In [31]:
```
np.linalg.inv(A)
```

Out[31]:
```
array([[ 0.07936508, -0.34920635,  0.11111111],
       [-0.17460317, -0.03174603,  0.05555556],
       [ 0.11111111,  0.11111111,  0.05555556]])
```

In [32]:
```
B = np.array(
[
    [1, -4, 2],
    [-2, 1, 3],
    [2,6,-10]
])
```

In [33]:
```
B
```

Out[33]:
```
array([[  1,  -4,   2],
       [ -2,   1,   3],
       [  2,   6, -10]])
```

In [34]:
```
from numpy.linalg import LinAlgError

try:
    np.linalg.inv(B)
except LinAlgError:
    print("matrix is not invertible")
```

```
matrix is not invertible
```

Python and SymPy

In [35]:
```
import sympy
from sympy import Matrix
```

- creating a vector

In [36]:
```
x = Matrix([1,2,3])
x
```

Out[36]:

$$\begin{bmatrix}1\\2\\3\end{bmatrix}$$

- creating a matrix

In [37]:
```
A = Matrix(
[
    [1,2, 3],
    [4,5,6],
```

```
    [4,5,6],
    [7,8,9]
])
A
```

Out[37]:

$$\begin{bmatrix}1 & 2 & 3\\4 & 5 & 6\\7 & 8 & 9\end{bmatrix}$$

vector algebra

In [38]:

```
y = Matrix([3,10,4])
y
```

Out[38]:

$$\begin{bmatrix}3\\10\\4\end{bmatrix}$$

In [39]:

```
x+y
```

Out[39]:

$$\begin{bmatrix}4\\12\\7\end{bmatrix}$$

In [40]:

```
3*x
```

Out[40]:

$$\begin{bmatrix}3\\6\\9\end{bmatrix}$$

In [41]:

```
2 * x - 3 * y
```

Out[41]:

$$\begin{bmatrix}-7\\-26\\-6\end{bmatrix}$$

applying a vector to a matrix

In [42]:

```
A * x
```

Out[42]:

$$\begin{bmatrix}14\\32\\50\end{bmatrix}$$

matrix transposition

In [43]:

```
A.T
```

Out[43]:

$$\begin{bmatrix}1 & 4 & 7\\2 & 5 & 8\\3 & 6 & 9\end{bmatrix}$$

matrix multiplication

In [44]:
```
B = Matrix(
[
    [4,1,0],
    [1,0,2],
    [4,5,6]
])
B
```

Out[44]:

$$\begin{bmatrix}4 & 1 & 0\\1 & 0 & 2\\4 & 5 & 6\end{bmatrix}$$

In [45]:
```
A * B
```

Out[45]:

$$\begin{bmatrix}18 & 16 & 22\\45 & 34 & 46\\72 & 52 & 70\end{bmatrix}$$

identity matrix

In [46]:
```
sympy.eye(3)
```

Out[46]:

$$\begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix}$$

diagonal matrix

In [47]:
```
sympy.diag(1,2,3)
```

Out[47]:

$$\begin{bmatrix}1 & 0 & 0\\0 & 2 & 0\\0 & 0 & 3\end{bmatrix}$$

matrix inverse

In [48]:
```
A = Matrix(
[
    [1, -4, 2],
    [-2, 1, 3],
    [2,6,8]
])
```

In [49]:
```
A.inv()
```

Out[49]:

$$\begin{bmatrix}\frac{5}{63} & -\frac{22}{63} & \frac{1}{9}\\-\frac{11}{63} & -\frac{2}{63} & \frac{1}{18}\\\frac{1}{9} & \frac{1}{9} & \frac{1}{18}\end{bmatrix}$$

In [50]:
```
B = Matrix(
[
    [1, -4, 2],
    [-2, 1, 3],
    [2,6,-10]
])
```

In [51]:
```
try:
    B.inv()
except ValueError:
    print("B is not invertible")
```

```
B is not invertible
```

Symbolic computation with SymPy

The real strength of Sympy is that it can calculate with variables as well as with numbers.

In [52]:
```
from sympy import symbols
a,b,c,d = symbols('a b c d')
```

In [53]:
```
A = Matrix([
    [a, b],
    [c,d]
])
A
```

Out[53]:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

In [54]:
```
A.inv()
```

Out[54]:

$$\begin{bmatrix} \frac{d}{ad-bc} & -\frac{b}{ad-bc} \\ -\frac{c}{ad-bc} & \frac{a}{ad-bc} \end{bmatrix}$$