None

Mathematics for Machine Learning

Session 20: Trigonometric functions

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Most material taken from Chapters 2 and 7 of Keisler, H. Jerome. *Elementary Calculus: An Infinitesimal Approach*. 2012.

Applets programmed with the help of ChatGPT

In [1]:

```
import numpy as np
import matplotlib.pyplot as plt
from ipywidgets import interact
```

Trigonometric function

Consider a unit circle, with a point (x, y) on the circle.

In [2]:

```
import numpy as np
import matplotlib.pyplot as plt
from ipywidgets import interact
from matplotlib.patches import Arc

def unit_circle(theta_deg=0):
    theta = np.radians(theta_deg)
    x = np.cos(theta)
    y = np.sin(theta)

    # Plot setup
    fig, ax = plt.subplots(figsize=(10, 10))  # Larger plot
    ax.set_xlim(-1.5, 1.5)
    ax.set_ylim(-1.5, 1.5)
    ax.set_aspect('equal', 'box')

    # Draw the unit circle
    circle = plt.Circle((0, 0), 1, color='black', fill=False)
    ax.add_artist(circle)

    # Draw the angle theta
    ax.plot([0, x], [0, y], color='black')  # Hypotenuse
    ax.plot([0, x], [0, 0], color='blue', linestyle='--')  # Adjacent (cosine)
    ax.plot([x, x], [0, y], color='green', linestyle='--')  # Opposite (sine)

    # Draw colored arc on the circumference
    arc = Arc((0, 0), 2, 2, theta1=0, theta2=theta_deg, color='purple', linewidth=2)
    ax.add_artist(arc)

    # Calculate position for the theta label just outside the arc
    arc_x = 1.1 * np.cos(theta / 2)  # Slightly outside the unit circle
    arc_y = 1.1 * np.sin(theta / 2)
    ax.text(arc_x, arc_y, r'$\theta$', color='purple', fontsize=16, ha='center', va='center')

    # Add annotations for sine and cosine
    ax.text(x / 2, 0, r'$\cos(\theta)$', color='blue', fontsize=16, ha='center', va='center')
    ax.text(x + 0.05, y / 2, r'$\sin(\theta)$', color='green', fontsize=16, ha='left', va='center')

    # Add labels
    ax.text(1.1, 0, "1", fontsize=16, ha='left', va='center')
    ax.text(0, 0, "(0, 0)", fontsize=12, ha='center', va='center')

    # Axes
    ax.axhline(0, color='black', linewidth=0.5)
    ax.axvline(0, color='black', linewidth=0.5)

    plt.title("sine and cosine", fontsize=18)
    plt.grid(False)
    plt.show()

# Interactive slider
interact(unit_circle, theta_deg=(0, 360, 1))
```

```
interactive(children=(IntSlider(value=0, description='theta_deg', max=360), Output()), _dom_classes=('widget-i…
```

Out[2]:

```
<function __main__.unit_circle(theta_deg=0)>
```

We measure the angle in *radians*, i.e., the length of the arc at the circumference of the unit circle inside the circle.

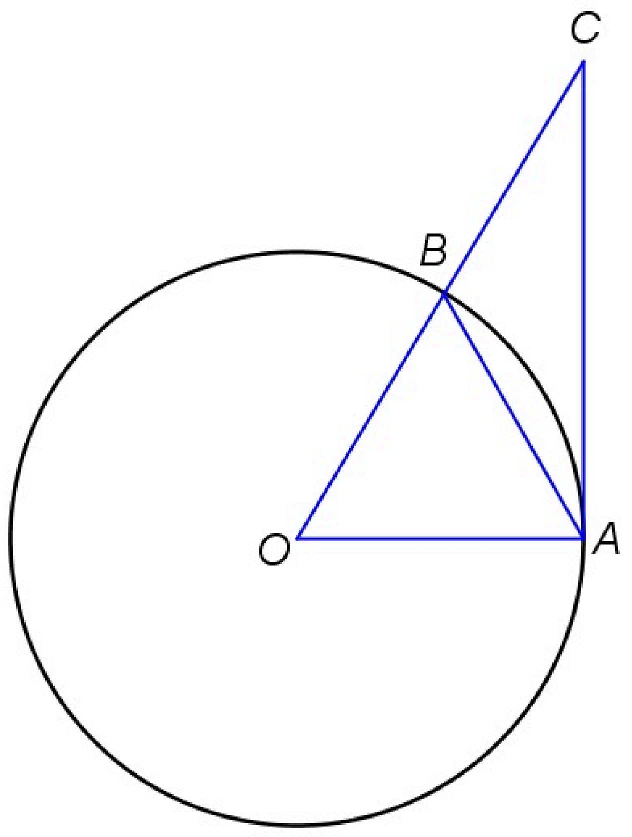
$$\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}} \tag{1}$$

$$\cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}} \tag{2}$$

$$\tan\theta = \frac{\text{opposite}}{\text{adjacent}} \tag{3}$$

$$\cot\theta = \frac{\text{adjacent}}{\text{opposite}} \tag{4}$$

The derivative of $\sin$



(source: Wikipedia)

Let θ be the angle between OA and OB, and let the radius of the circle be 1. Then the height of the triangle OAB is $\sin\theta$, and its area is $\frac{\sin\theta}{2}$. The sector of the circle between OA and OB has the area $\frac{\theta}{2}$. The length $|AC|$ is $\tan\theta$, so its area is $\frac{\tan\theta}{2}$. It is obvious from the sketch that

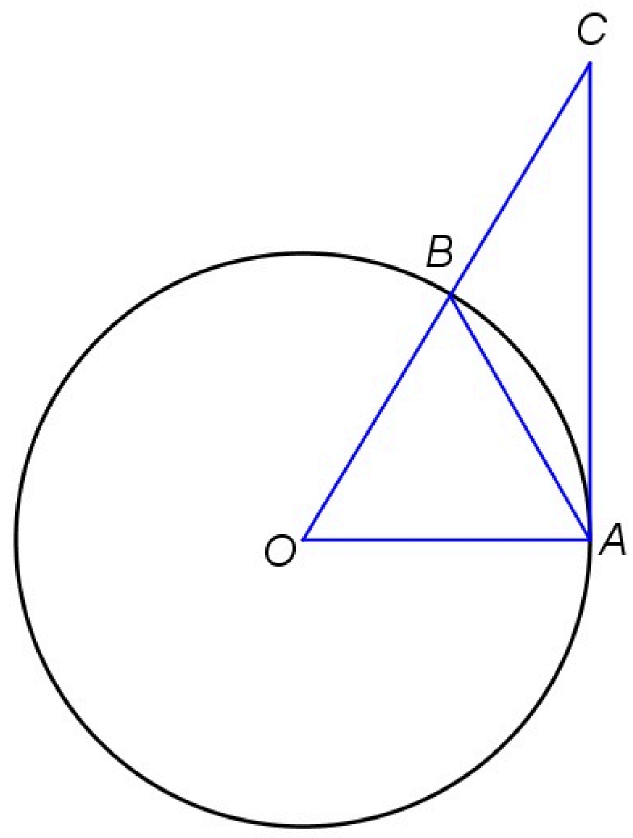
$$\frac{\sin\theta}{2} < \frac{\theta}{2} < \frac{\tan\theta}{2},$$

and therefore

$$\sin\theta < \theta < \tan\theta = \frac{\sin\theta}{\cos\theta}.$$

Dividing everything by $\sin\theta$ (assuming $\sin\theta \neq 0$ gives us

$$1 < \frac{\theta}{\sin\theta} < \frac{1}{\cos\theta}$$



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In [3]:

```
import matplotlib.pyplot as plt
import numpy as np

# Redraw with a larger plot size and increased font size for the labels
fig, ax = plt.subplots(figsize=(10, 10))  # Increased figure size
ax.set_aspect('equal')
```

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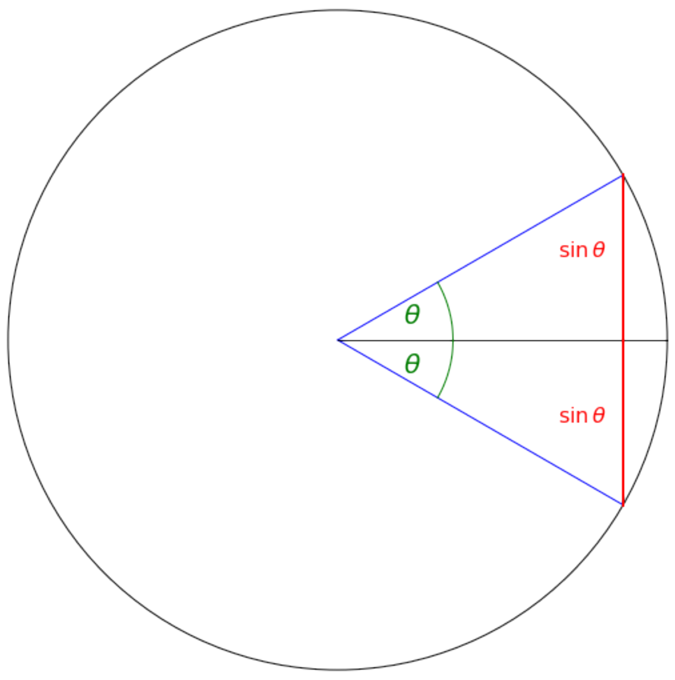
$\frac{\sin}{\theta}$

```
ax.axis('off')

# Draw the main circle
circle = plt.Circle((0, 0), 1, color='black', fill=False)
ax.add_artist(circle)

# Define the angle θ (in radians) - make it smaller
theta = np.pi / 6  # 30 degrees

# Plot the radius lines
# Upper triangle
ax.plot([0, np.cos(theta)], [0, np.sin(theta)], color='blue', linestyle='-', linewidth=1)  # First radius
ax.plot([0, 1], [0, 0], color='black', linestyle='-', linewidth=1)  # Second radius
# Lower triangle (mirrored)
ax.plot([0, np.cos(theta)], [0, -np.sin(theta)], color='blue', linestyle='-', linewidth=1)  # First radius
ax.plot([0, 1], [0, 0], color='black', linestyle='-', linewidth=1)  # Second radius

# Highlight the sine components
ax.plot([np.cos(theta), np.cos(theta)], [0, np.sin(theta)], color='red', linestyle='-', linewidth=2)  # Upper sine
ax.plot([np.cos(theta), np.cos(theta)], [0, -np.sin(theta)], color='red', linestyle='-', linewidth=2)  # Lower sine

# Draw arcs for θ
arc_radius = 0.35
arc = np.linspace(0, theta, 100)
# Upper arc
ax.plot(arc_radius * np.cos(arc), arc_radius * np.sin(arc), color='green', linewidth=1)
# Lower arc (mirrored)
ax.plot(arc_radius * np.cos(arc), -arc_radius * np.sin(arc), color='green', linewidth=1)

# Label θ near the arcs
ax.text(0.2, 0.05, r'$\theta$', color='green', fontsize=20)  # Upper label
ax.text(0.2, -0.1, r'$\theta$', color='green', fontsize=20)  # Lower label

# Label sin θ (moved to the left of the red line)
ax.text(np.cos(theta) - 0.2, np.sin(theta) / 2, r'$\sin \theta$', color='red', fontsize=16)  # Upper sine label
ax.text(np.cos(theta) - 0.2, -np.sin(theta) / 2, r'$\sin \theta$', color='red', fontsize=16)  # Lower sine label

# Set the limits of the plot
ax.set_xlim(-1.1, 1.1)
ax.set_ylim(-1.1, 1.1)

# Show the plot
plt.show()
```



betw the two blue lines is 2θ, and the lengt of the red line is $2\sin\theta$. If $\sin\theta$ is infinit

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In other words the propo of the secar of an angle to the lengt of the arc is infinit close to 1 if the secar is infinit