

# Live Exercise 1: Hotelling with quadratic transportation costs

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## Solutions

### Group exercise ( 20 minutes)

- Work in groups of 2-3.
- Show your steps (the algebra is the point).
- Assume full market coverage throughout (everyone buys from one of the two firms).

### Problem: Quadratic transport costs and cost asymmetry

Consider a Hotelling line of length 1 with consumers uniformly distributed on  $[0, 1]$ . Each consumer buys one unit from the firm that gives higher utility.

There are two firms with *fixed locations*:

- Firm 1 at  $x_1 = a$
- Firm 2 at  $x_2 = 1 - a$

where  $0 \leq a < \frac{1}{2}$ . Define the distance between firms  $d \equiv x_2 - x_1 = 1 - 2a$ .

Consumers have quasi-linear utility

$$U_i(x) = \bar{u} - p_i - t(x - x_i)^2,$$

where  $t > 0$  and  $\bar{u}$  is large enough that the outside option never binds. Firm  $i$  has constant marginal cost  $c_i$  and chooses price  $p_i$  simultaneously.

### (a) Marginal consumer and demands

Let  $x^*$  be the consumer indifferent between the two firms (assume  $x^* \in (a, 1 - a)$ ).

1. Write the indifference condition defining  $x^*$ .
2. Solve for  $x^*$  as a function of  $(p_1, p_2, t, a)$ .
3. Use your result to write demands  $q_1(p_1, p_2)$  and  $q_2(p_1, p_2)$ .

*Hint:* For  $x \in (a, 1 - a)$ , squared distances are always positive; no absolute values are needed.

### (b) Best responses in prices

Firm  $i$ 's profit is

$$\pi_i(p_i, p_j) = (p_i - c_i) q_i(p_i, p_j).$$

1. Take the first-order condition for Firm 1 and simplify it into a linear best response  $BR_1(p_2)$ .
2. Do the same for Firm 2.

**(c) Price equilibrium (Nash in prices)**

Solve the system of best responses to get equilibrium prices  $(p_1^*, p_2^*)$ . Then compute equilibrium market shares  $(q_1^*, q_2^*)$ .

1. Show that the *price gap* satisfies  $p_2^* - p_1^* = \frac{c_2 - c_1}{3}$ .
2. How does *distance*  $d$  affect the average markup? (Be explicit.)

**(d) When does one firm capture the whole market? (optional challenge)**

Your answer in (a)–(c) assumed an interior split ( $x^* \in (a, 1 - a)$ ).

1. Derive the condition on the cost difference  $\Delta c \equiv c_2 - c_1$  (relative to  $t$  and  $d$ ) under which the interior split is valid.
2. Give a short economic interpretation: what happens if  $|\Delta c|$  is “too large”?

**Suggested solution (sketch)**

**(a)**

Indifference:

$$\bar{u} - p_1 - t(x - a)^2 = \bar{u} - p_2 - t(1 - a - x)^2.$$

Cancel  $\bar{u}$  and rearrange:

$$p_2 - p_1 = t[(x - a)^2 - (1 - a - x)^2].$$

Expanding (or using  $(x - a)^2 - (1 - a - x)^2 = (1 - 2a)(2x - 1) = d(2x - 1)$ ) gives

$$x^* = \frac{1}{2} + \frac{p_2 - p_1}{2td}.$$

For an interior split, demands are

$$q_1 = x^*, \quad q_2 = 1 - x^*.$$

**(b)**

Using  $q_1 = \frac{1}{2} + \frac{p_2 - p_1}{2td}$ , we have  $\frac{\partial q_1}{\partial p_1} = -\frac{1}{2td}$ . FOC for Firm 1:

$$0 = \frac{\partial \pi_1}{\partial p_1} = q_1 + (p_1 - c_1) \frac{\partial q_1}{\partial p_1} = q_1 - \frac{p_1 - c_1}{2td}.$$

So  $p_1 - c_1 = 2td q_1$ , i.e.

$$p_1 - c_1 = 2td \left( \frac{1}{2} + \frac{p_2 - p_1}{2td} \right) = td + p_2 - p_1 \quad \Rightarrow \quad BR_1(p_2) = \frac{p_2 + c_1 + td}{2}.$$

Similarly,

$$BR_2(p_1) = \frac{p_1 + c_2 + td}{2}.$$

(c)

Solve  $2p_1 = p_2 + c_1 + td$  and  $2p_2 = p_1 + c_2 + td$  to get

$$p_1^* = td + \frac{2c_1 + c_2}{3}, \quad p_2^* = td + \frac{c_1 + 2c_2}{3}.$$

Price gap:  $p_2^* - p_1^* = \frac{c_2 - c_1}{3}$ .

Market shares:

$$q_1^* = \frac{1}{2} + \frac{c_2 - c_1}{6td}, \quad q_2^* = \frac{1}{2} - \frac{c_2 - c_1}{6td}.$$

Distance  $d$  raises the common “differentiation” term  $td$  in prices, hence increases average markups and profits.

(d)

Interior split requires  $q_1^*, q_2^* \in (0, 1)$ , i.e.

$$\left| \frac{c_2 - c_1}{6td} \right| < \frac{1}{2} \quad \Leftrightarrow \quad |c_2 - c_1| < 3td.$$

If  $|c_2 - c_1|$  is too large relative to  $td$ , the low-cost firm can profitably set a price that leaves the rival with (essentially) zero demand.