Live Exercise 1: Hotelling with quadratic transportation costs

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Solutions

Group exercise (20 minutes)

- Work in groups of 2–3.
- Show your steps (the algebra is the point).
- Assume full market coverage throughout (everyone buys from one of the two firms).

Problem: Quadratic transport costs and cost asymmetry

Consider a Hotelling line of length 1 with consumers uniformly distributed on $[0, 1]$. Each consumer buys one unit from the firm that gives higher utility.

There are two firms with *fixed locations*:

- Firm 1 at $x_1 = a$
- Firm 2 at $x_2 = 1 - a$

where $0 \leq a < \frac{1}{2}$. Define the distance between firms $d \equiv x_2 - x_1 = 1 - 2a$.

Consumers have quasi-linear utility

$$U_i(x) = \bar{u} - p_i - t(x - x_i)^2,$$

where $t > 0$ and $\bar{u}$ is large enough that the outside option never binds. Firm i has constant marginal cost c_i and chooses price p_i simultaneously.

(a) Marginal consumer and demands

Let x^* be the consumer indifferent between the two firms (assume $x^* \in (a, 1 - a)$).

1. Write the indifference condition defining x^*.
2. Solve for x^* as a function of (p_1, p_2, t, a).
3. Use your result to write demands $q_1(p_1, p_2)$ and $q_2(p_1, p_2)$.

Hint: For $x \in (a, 1 - a)$, squared distances are always positive; no absolute values are needed.

(b) Best responses in prices

Firm i's profit is

$$\pi_i(p_i, p_j) = (p_i - c_i)\, q_i(p_i, p_j).$$

1. Take the first-order condition for Firm 1 and simplify it into a linear best response $BR_1(p_2)$.
2. Do the same for Firm 2.

(c) Price equilibrium (Nash in prices)

Solve the system of best responses to get equilibrium prices (p_1^*, p_2^*). Then compute equilibrium market shares (q_1^*, q_2^*).

1. Show that the *price gap* satisfies $p_2^* - p_1^* = \frac{c_2 - c_1}{3}$.
2. How does *distance* d affect the average markup? (Be explicit.)

(d) When does one firm capture the whole market? (optional challenge)

Your answer in (a)–(c) assumed an interior split ($x^* \in (a, 1-a)$).

1. Derive the condition on the cost difference $\Delta c \equiv c_2 - c_1$ (relative to t and d) under which the interior split is valid.
2. Give a short economic interpretation: what happens if $|\Delta c|$ is "too large"?

Suggested solution (sketch)

(a)

Indifference:

$$\bar{u} - p_1 - t(x-a)^2 = \bar{u} - p_2 - t(1-a-x)^2.$$

Cancel $\bar{u}$ and rearrange:

$$p_2 - p_1 = t\big[(x-a)^2 - (1-a-x)^2\big].$$

Expanding (or using $(x-a)^2 - (1-a-x)^2 = (1-2a)(2x-1) = d(2x-1)$) gives

$$x^* = \frac{1}{2} + \frac{p_2 - p_1}{2td}.$$

For an interior split, demands are

$$q_1 = x^*, \qquad q_2 = 1 - x^*.$$

(b)

Using $q_1 = \frac{1}{2} + \frac{p_2 - p_1}{2td}$, we have $\frac{\partial q_1}{\partial p_1} = -\frac{1}{2td}$. FOC for Firm 1:

$$0 = \frac{\partial \pi_1}{\partial p_1} = q_1 + (p_1 - c_1)\frac{\partial q_1}{\partial p_1} = q_1 - \frac{p_1 - c_1}{2td}.$$

So $p_1 - c_1 = 2td\, q_1$, i.e.

$$p_1 - c_1 = 2td\left(\frac{1}{2} + \frac{p_2 - p_1}{2td}\right) = td + p_2 - p_1 \quad \Rightarrow \quad BR_1(p_2) = \frac{p_2 + c_1 + td}{2}.$$

Similarly,

$$BR_2(p_1) = \frac{p_1 + c_2 + td}{2}.$$

(c)

Solve $2p_1 = p_2 + c_1 + td$ and $2p_2 = p_1 + c_2 + td$ to get

$$p_1^* = td + \frac{2c_1 + c_2}{3}, \qquad p_2^* = td + \frac{c_1 + 2c_2}{3}.$$

Price gap: $p_2^* - p_1^* = \frac{c_2 - c_1}{3}$.

Market shares:

$$q_1^* = \frac{1}{2} + \frac{c_2 - c_1}{6td}, \qquad q_2^* = \frac{1}{2} - \frac{c_2 - c_1}{6td}.$$

Distance d raises the common "differentiation" term td in prices, hence increases average markups and profits.

(d)

Interior split requires $q_1^*, q_2^* \in (0,1)$, i.e.

$$\left|\frac{c_2 - c_1}{6td}\right| < \frac{1}{2} \quad \Leftrightarrow \quad |c_2 - c_1| < 3td.$$

If $|c_2 - c_1|$ is too large relative to td, the low-cost firm can profitably set a price that leaves the rival with (essentially) zero demand.