How the Contributions of Archimedes Led to the Development of Calculus

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Mathematics is one of the most useful and influential areas of study. Many people do not realize the extent to which mathematics is used in our world today, whether it be to make safer cars, create new computer technologies, or build skyscrapers. However, in order to get to the point that mathematics is at today, countless people had to make many contributions to the different fields of mathematics. For example, Carl Friedrich Gauss made advancements in many areas, especially analysis and number theory. Euler contributed to almost all areas of mathematics, especially trigonometry, graph theory, and number theory, and he introduced many of the mathematical notations that we use today, such as the function notation . Joseph-Louis Lagrange made huge advancements in the fields of analysis and differential equations. However, it is not only the more recent mathematicians that have made great contributions to mathematics; people even living in ancient times made enormous advancements, such as Pythagoras and Euclid, who both lived in the ancient Greek world. Another ancient Greek mathematician is Archimedes, who is often hailed as one of the greatest mathematicians of all time. Archimedes, a brilliant mathematician and inventor of mechanical devices, mainly through his work with potential infinity in calculating the areas and volumes of objects, influenced Johannes Kepler, leading to the development of calculus.

Archimedes was born around 287 BC in Syracuse, Sicily. He was the son of Pheidias, an astronomer, and he was a friend (and possible relative) of King Hiero of Syracuse. Archimedes most likely attended Euclid’s school in Alexandria, but he lived the rest of his life in Syracuse. Because of the brilliance Archimedes demonstrated throughout his life, and because of how famous he became, there are many myths about events that may or may not have occurred during his life. One such famous myth is the story of the gold crown: the king had given gold to a goldsmith to make a crown. However, when the king received the crown, he thought the goldsmith may have taken some of the gold for himself and had used some silver in the making of the crown. So the king asked Archimedes to figure out if indeed the crown was not made of pure gold. While taking a bath, Archimedes noticed that the water became displaced as he immersed his body. He realized he could use this fact (that the amount of water displaced is equal to the volume of the object) to solve the problem of the crown, and immediately got out of the bath and ran out naked, crying out “*Eureka!*” [“I have found it out”]. While this makes for a good story, most people doubt that it actually happened – it seems like too trivial a discovery, and it was not even mentioned in Archimedes’ work *On Floating Bodies*.1 But because Archimedes was (and is) so famous, these sorts of legends have accrued around him.

Other sorts of legends revolve around Archimedes’ involvement in the defense of Syracuse against the Roman siege during the Second Punic War. Archimedes is said to have created four major weapons (one is less probable than the other three). The first is called the “death ray,” but people have doubted this weapon’s existence/effectiveness. Legend has it “that he set the Roman ships on fire by using mirrors arranged in a parabola to reflect the sunlight on a single burning point.”2 The reason many people do not believe this weapon actually existed (and/or worked) is because the ships could not have moved in the water, and the sun’s ray must have been focused on the exact same spot on the ship the entire time. Another reason is because it is not mentioned by the three most credible historians: Polybius, Livy, and Plutarch.3 Because this legend is so interesting, an MIT class tested it out on wood and got the wood to ignite.4 The TV show “Mythbusters” then conducted the experiment with a real ship in water – they “were able to cause charring and smoldering in a 1-2 foot wide swath along much of the boat’s length. After three passes over the boat, the hull was penetrated and a small open flame was achieved.”5 So the death ray’s existence has been deemed definitely possible, although not very probable. The second weapon is Archimedes’ claw (or the “ship-shaker”). This invention was essentially a grappling hook capable of lifting a Roman ship out of the water. The third weapon was a series of poles that “reached over the city walls and dropped heavy stones onto the ships.”6 A final method was the use of catapults to launch stones at the Roman ships. Reviel Netz and William Noel, in *The Archimedes Codex*, quote from the historian Polybius:

But Archimedes, who had prepared machines constructed to carry to any distance, so damaged the assailant at long range, as they sailed up [the first attempt was by sea] with his more powerful catapults as to throw them into much difficulty and distress; and as soon as these machines shot too high he continued using smaller and smaller ones as the range became shorter and, finally, so thoroughly shook their courage that he put a complete stop to their advance… [The Romans gave up the assault, and so Polybius sums up:] Such a great and marvelous thing does the genius of one man show itself to be… The Romans, strong as they were both by sea and land, had every hope of capturing the town at once if one old man of Syracuse were removed; but as long as he was present, they did not venture even to attempt to attack.7

These catapults implemented the principle of the lever, “where distance is traded for force.”8 Throughout his life, Archimedes became extremely knowledgeable with the principles behind how levers work. Indeed, he is even recorded as saying, “Give me a place to stand and I will move the earth.” However, even with Archimedes’ unprecedented defense of the Syracuse, the city eventually fell to the Roman general Marcus Claudius Marcellus when the Romans attacked from land (rather than by sea), and Archimedes died during that time (around 212 BC). There are many theories surrounding exactly how Archimedes died. One theory is that he was drawing diagrams in the sand (which he often did) when a Roman soldier approached him and commanded him to meet Marcellus. However, Archimedes refused because he said he needed to finish his diagram. The soldier was so enraged that he killed Archimedes on the spot.

Through these mechanical feats, we can see examples of Archimedes’ ingenuity. Even so, “Archimedes has many discoveries and inventions to his credit, but he considered his theoretical work as his main triumph.”9 Archimedes demonstrates his brilliance through his work in various different aspects of geometry, in which he proved many new mathematical theorems. His areas of inquiry in geometry include spheres and cylinders, circles, conoids and spheroids, spirals, planes, and parabolas. One example of an Archimedean theorem is that “the area of any circle is equal to a right-angled triangle in which one of the sides about the right angle is equal to the radius, and the base [i.e. the other of the sides about the right angle] to the circumference [of the circle]”10 (found in Archimedes’ *Measurement of the Circle*). What this theorem is saying is that if you have a circle with radius and circumference , then the area of that circle (AC) is equivalent to the area of a triangle with sides of length and : AC  (the now well-known equation for the area of a circle). Archimedes’ method for proving this relationship between the circle and triangle is straightforward, but ingenious. He denotes the area of the circle to be *K* and the area of the triangle in question to be *Δ*. He first supposes that *K* > *Δ* and shows that it leads to a contradiction. He then supposes that *K* < *Δ* and shows that it also leads to a contradiction, which leads to the only possible conclusion that

*K* = *Δ*.11 (This method used is called a proof by contradiction, or a reduction to the impossible. Archimedes often used this technique in his proofs.) This is only one simple example of an Archimedean theorem. He proved countless other theorems, many of which are extremely complex. Additionally, he explored hydrostatics and optics, and he even created a new number system in order to denote large numbers (found in his work *The Sandreckoner*). Through all of this, we begin to gain an understanding of Archimedes’ great intelligence and depth of knowledge.

One of the most important contributions Archimedes made to mathematics was his use of potential infinity in his proofs. Netz and Noel in *The Archimedes Codex* elegantly describe an example of this process as an imaginary dialogue between Archimedes and a critic:

Archimedes packs a curved object so that a certain area has been left out, an area greater than the size of a grain of sand. A critic comes along and says: “There is still a difference greater than the size of a grain of sand.” “Is that right?” exclaims Archimedes. “All right then, I shall apply my mechanism successively several more times,” and then the area left out is smaller than the grain of sand. “Wait a minute,” says the critic, “the area left out is still greater than a hair’s width.” Archimedes goes on, and so on, and on and on it goes. The difference always becomes smaller than any given magnitude mentioned by the critic. This dialogue goes on *indefinitely*. This is *potential infinity*.12

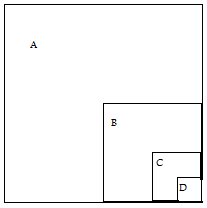
So although Archimedes (and Greek mathematicians in general) did not deal with actual infinity, he used potential infinity extensively.

One concrete example of this is found in *Quadrature of the Parabola*. Archimedes’ Proposition 23 says “Given a series of areas *A*, *B*, *C*, *D*, …, *Z*, of which A is the greatest, and each is equal to four times the next in order, then *A* + *B* + *C* + … + *Z* + *Z* = *A*.”13 (As a side-note: just because the areas are labeled *A*,...,*Z* does not mean that there are only 26 areas.) The hypothesis of this proposition can be visualized as a series of concentric squares, in which the next square in the sequence is situated in one of the quadrants of the previous square (see Figure 1). In order to prove this proposition, Archimedes begins by defining new variables: take areas *b*, *c*, *d*, … *z* such that *b* = *B*, *c* = *C*, *d* = *D*, etc. So since *b* = *B* and *B* = *A* (by definition), by adding these two equations we get that *B* + *b* = *B* + *A*. By substitution, we then get that *B* + *b* = ( *A*) + *A* = *A*. Similarly, we get *C* + *c* = *B*, *D* + *d* = *C*, and so on. Using these equations, we can get these two new equations:

① (*B* + *C* + *D* + … + *Z*) + (*b* + *c* + *d* + … + *z*) = (*A* + *B* + *C* + … + *Y*)

Fig. 1

Heath, *The Works of Archimedes*, 250.

② *b* + *c* + *d* + … + *y* = (*B* + *C* + *D* + … + *Y*).

If we subtract ① - ②, we get *B* + *C* + *D* + … + *Z* + *z* = *A*. Since, by definition, *z* = *Z*, then *B* + *C* + *D* + … + *Z* + *Z* = *A*. Finally, by adding *A* to both sides of this equation, we get

*A* + *B* + *C* + *D* + … + *Z* + *Z* = *A*, and thus we have proven the proposition.14 Looking at this proof, it is clear to see that this equation will hold true for any arbitrarily large amount of numbers in the series, because the cancellation when the subtraction is done will yield the same result every time. Since someone can choose as many numbers as he or she wishes, this is an example of potential infinity; however, this does not deal with actual infinity, because the summation of numbers terminates at a certain point (at *Z*, in our example). Archimedes used this proposition in his proof that “the area of a parabolic segment is four-thirds of the triangle with the same base and vertex.”15

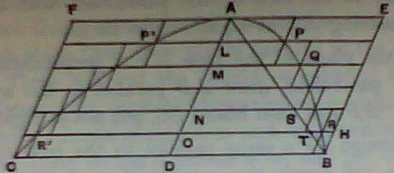
Archimedes also worked with finding the volumes of objects such as paraboloids of revolution, hyperboloids of revolution, and spheroids, and he wrote about his findings in *On Conoids and Spheroids*. I will refer exclusively to his work with paraboloids. (A paraboloid of revolution is a three-dimensional object created by revolving a parabola about its axis.) One of his proofs is that “any segment of a paraboloid of revolution is half as large again as the cone or segment of a cone which has the same base and the same axis.”16 Without going in to the gory details of the proof, here are the important steps along the way: Archimedes previously proved (in *On Spirals*) that if there are *n* terms of an arithmetical sequence *h*, 2*h*, 3*h*, … , *nh* then

③ *h* + 2*h* + 3*h* + … + *nh* > *n*2*h*

and ④ *h* + 2*h* + 3*h* + … + (*n* - 1) *h* < *n*2*h*.

Fig. 2

Heath, *The Works of Archimedes*, 132.

Archimedes then “inscribes and circumscribes to the segment of the paraboloid figures made up of small cylinders [see Figure 2] whose axes lie along the axis of the segment and divide it into any number of equal parts.”17 Call *c* the length of the axis *AD* of the segment. Supposing there are *n* cylinders in the circumscribed figure, and supposing these cylinders all have length *h*, then

*c* = *nh*. Archimedes proceeds to prove that

⑤ =

> 2 (by equation ④)

and ⑥ =

< 2 (by equation ③).

Archimedes has also previously proved that “by increasing *n* sufficiently, the inscribed and circumscribed figure can be made to differ by less than any assignable volume.”18 It is then concluded that (*cylinder* *CE*) = 2 (*segment*), which then implies that (*segment ABC*) = (*cone ABC*), and thus ends the proof. This result has a direct relationship with calculus: “The proof is therefore equivalent to the assertion that if *h* is indefinitely diminished and *n* indefinitely increased, while *nh* remains equal to *c*,”19 then = *c*2, which is the same as saying = *c*2, so Archimedes’ result can also be achieved using calculus.20

Indeed, these are not the only examples of the relationship between Archimedes’ work and modern calculus. Netz and Noel conclude that five of Archimedes works (*Quadrature of the Parabola*, *Sphere and Cylinder* (2 books), *Spiral Lines*, and *Conoids and Spheroids­*) “form a certain unity, as together they constitute the cornerstone of the calculus.”21 Some people even suspect that “Archimedes would have invented calculus if the Greeks had only possessed a more tractable mathematical notation.”22 But even if Archimedes did not invent calculus, his works certainly led to its development. Archimedes influenced many mathematicians to come; one such mathematician was Johannes Kepler:

The Latin translation of many of Archimedes’ works by Federico Commandino in 1588 contributed greatly to the spread of knowledge of them, which was reflected in the work of the foremost mathematicians and physicists of the time, including Johannes Kepler (1571-1630) and Galileo Galilei (1564-1642)… Without the background of the rediscovered ancient mathematicians, among whom Archimedes was paramount, the development of mathematics in Europe in the century between 1550 and 1650 is inconceivable.23

Because Kepler was instrumental in the development of calculus, and Archimedes had such an influence on Kepler, this implies that Archimedes’ works led to the invention of calculus.

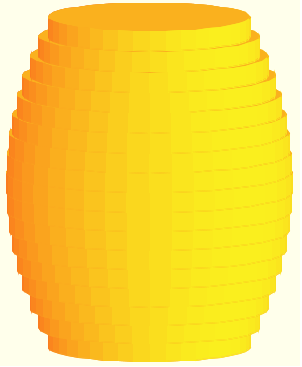
Kepler was born in Germany on December 27, 1572 and died in 1630. He attended the University of Tubingen to study to become a Lutheran minster. He is best known for his three laws of planetary motion, his founding of modern optics, and his derivation of logarithms.24 However, one of his greatest contributions to mathematics stemmed from an event that occurred in his personal life – his second wedding ceremony (his first wife had died, and he then remarried in 1613). For this wedding, Kepler had purchased a barrel of wine, but the wine merchant’s method for measuring the volume of the barrel upset Kepler.25 (The volume was “estimated by means of a rod slipped diagonally through the bung-hole.”26) This led Kepler to investigate how to calculate areas and volumes of these kinds of objects: “The result was a study of the volumes of solids of revolutions in which Kepler, basing himself on the work of Archimedes, used a resolution into ‘indivisibles’. This method… is part of the ancestry of the infinitesimal calculus.”27 Specifically, in order “to determine the volume of a wine barrel accurately, Kepler thought of the wine in a full barrel, or of any solid body, as made up of numerous thin sheets arranged in layers, and treated the volume as the sum of the volumes of these leaves. In the case of a wine barrel, each of these leaves was a cylinder.”28 (See Figure 3.) By decreasing the heights of each of the cylinders, thereby increasing the number of cylinders, he was able to get a better and better approximation for the volume; by taking infinitely many cylinders, with heights approaching zero, he could find the exact volume. Kepler wrote about these studies on volume in his book *New Solid Geometry of Wine Barrels* (*Nova Stereometria Doliorum Vinariorum*): “This book is a systematic work on the calculation of areas and volumes by infinitesimal techniques. Building on the results of Archimedes, it focuses on solids of revolution and includes calculations of exact or approximate volumes of over ninety such solids.”29 (Today, to find the volume of such figures, we would use integration.) Additionally, Kepler “took up a problem of differential calculus, the problem of maximums: What is the best design for a barrel in order to maximize its volume?”30 (Today we know that to find the maximum (or minimum) of a function, we find the derivative and set it equal to zero, because at a maximum (or minimum), the tangent line to the curve has a slope of zero.) Kepler took a practical problem of finding the volume of a wine barrel and unintentionally contributed to the invention of calculus.

Fig. 3

Cardil, “Kepler: The Volume of a Wine Barrel.”

In conclusion, Archimedes has proven himself to be one of the top mathematicians of all time – he was even able to apply his mathematics practically with the defense of Syracuse against the Romans, and he was able to rigorously prove many new, innovative theorems. One of Archimedes’ greatest contributions to mathematics was his use of potential infinity in his proofs, because Kepler studied Archimedes’ works, and these works aided Kepler in finding areas and volumes, since Kepler also used infinitesimal techniques. Kepler’s method of breaking an object up into infinitely many pieces with negligibly small heights is the backbone of calculus as we know it today. Netz and Noel conclude: “And so, since Archimedes led more than anyone else to the formation of the calculus and since he was the pioneer of the application of mathematics to the physical world, it turns out that Western science is but a series of footnotes to Archimedes. Thus, it turns out that Archimedes is the most important scientist who ever lived.”31

*(3,272 words)*

Notes

1. Reviel Netz and William Noel, *The Archimedes Codex* (Philadelphia: Da Capo Press, 2007), 34.

2. Sherman Stein, *Archimedes: What Did He Do Besides Cry Eureka?* (Washington D.C.: The Mathematical Association of America, 1999), 5.

3. E.J. Dijksterhuis, *Archimedes*, trans. C. Dikshoorn (Princeton, NJ: Princeton University Press, 1987), 26-28.

4. University Press, “Archimedes,” *Ancient Greece*, http://www.ancientgreece.com/s/People/Archimedes/ (accessed October 1, 2012).

5. Massachusetts Institute of Technology, “Archimedes Death Ray: Testing with Mythbusters,” http://web.mit.edu/2.009/www/experiments/deathray/10\_Mythbusters.html (accessed November 4, 2012).

6. Bellevue College, “Archimedes,” http://scidiv.bellevuecollege.edu/math/archimedes.html (accessed October 1, 2012).

7. Netz and Noel, *The Archimedes Codex*, 60.

8. Stein, *Archimedes: What Did He Do Besides Cry Eureka?*, 5.

9. University Press, “Archimedes,” http://www.ancientgreece.com/s/People/Archimedes/.

10. Dijksterhuis, *Archimedes*, 222.

11. Ibid., 222-223.

12. Netz and Noel, *The Archimedes Codex*, 185.

13. Thomas Little Heath, *The Works of Archimedes* (New York: Dover Publications, 1912), 249.

14. Ibid., 250.

15. Ibid., cxliii.

16. Ibid., 131.

17. Ibid., cxlvii.

18. Ibid., cxlviii.

19. Ibid., cxlviii.

20. Ibid., cxlviii.

21. Netz and Noel, *The Archimedes Codex*, 41

22. Eric Weisstein, “Archimedes of Syracuse,” *Science World*, http://scienceworld.wolfram.com/biography/Archimedes.html (accessed October 1, 2012).

23. Gerald Toomer, “Archimedes,” *Encyclopaedia Britannica Online*, http://www.britannica.com/EBchecked/topic/32808/Archimedes (accessed October 1, 2012).

24. “Johannes Kepler,” *NASA*, http://kepler.nasa.gov/Mission/JohannesKepler/#anchor778225 (accessed November 8, 2012).

25. Roberto Cardil, “Kepler: The Volume of a Wine Barrel,” *The Mathematical Association of America*, http://mathdl.maa.org/mathDL/46/?pa=content&sa=viewDocument&nodeId=3499&pf=1 (accessed October 1, 2012).

26. J.V. Field, “Johannes Kepler,” *University of St. Andrew’s*, http://www-history.mcs.st-and.ac.uk/Biographies/Kepler.html (accessed October 1, 2012).

27. Ibid.

28. Cardil, “Kepler: The Volume of a Wine Barrel,” http://mathdl.maa.org/mathDL/46/?pa=content&sa=viewDocument&nodeId=3499&pf=1.

29. Ibid.

30. Ibid.

31. Netz and Noel, *The Archimedes Codex*, 29.

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