QUESTION 2 :

Our sequence sampled at $f_s = 10$ Hz

$$x[n] = \cos\left(n \cdot \frac{\pi}{8}\right) = \cos\Big(n \underbrace{\left(\frac{\pi}{8} + 2\pi \cdot k\right)}_{\omega_k}\Big) \qquad \underbrace{k \in \mathbb{Z}}_{k \text{ is an integer}}$$

Therefore all the frequencies $\frac{\pi}{8} + 2\pi k$ for k any integer are equivalent.

remember that the relationship between discrete time frequencies (ω) and continuous-time frequencies (Ω) is :

$$\omega = \Omega \cdot T_s \Rightarrow \Omega = \frac{\omega}{T_s} = \omega \cdot f_s$$

So we have that the following continuous-time frequencies :

$$\Omega_k = \omega_k \cdot f_s = \left(\frac{\pi}{8} + 2\pi k\right) \cdot 10 = \frac{5\pi}{4} + 20\pi k$$

will map into the same discrete frequency when sampled at $f_s = 10$ Hz.

For instance :

$$k = 0 \Rightarrow \Omega_0 = \frac{5\pi}{4} \Rightarrow \omega_0 = \frac{5\pi}{4 \cdot f_s} = \frac{\pi}{8}$$ → the one given in the question

$$k = 1 \Rightarrow \Omega_1 = \frac{85\pi}{4} \Rightarrow \omega_1 = \frac{85\pi}{4 \cdot f_s} = \frac{85\pi}{4 \cdot 10} = \frac{17\pi}{8} = \underbrace{\frac{\pi}{8} + 2\pi}_{\equiv \text{ equivalent to } \frac{\pi}{8}}$$

(*) Note that $Z\{\cdot\}$ means z-transform of $\cdot$

QUESTION 3

a) $x_a[n] = \alpha^n \mu[-n-1]$

ROC

$$Z\{x_a[n]\} = -Z\{-(\alpha^n)\mu[-n-1]\} = \frac{-1}{1-\alpha z^{-1}} \qquad |z| < \alpha$$

using the table of z-transform pairs

b) $x_b[n] = (n+1)\alpha^n \mu[-n-1] = n\alpha^n\mu[-n-1] + \alpha^n\mu[-n-1]$

$$Z\{x_b[n]\} = \underbrace{-Z\{-n\alpha^n\mu[-n-1]\}}_{\frac{-\alpha z^{-1}}{(1-\alpha z^{-1})^2}} \underbrace{- Z\{-\alpha^n\mu[-n-1]\}}_{\frac{-1}{(1-\alpha z^{-1})}} = \dots$$

ROC $|z| < \alpha$ ROC $|z| < \alpha$

$$\dots = \frac{-\alpha z^{-1} - (1-\alpha z^{-1})}{(1-\alpha z^{-1})^2} = \frac{-1}{(1-\alpha z^{-1})^2}$$

$\boxed{\text{ROC } |z| < \alpha}$

QUESTION 4 :

$$H(z) = \frac{3-7z^{-1}+5z^{-2}}{3-\frac{5}{2}z^{-1}+z^{-2}} \qquad \begin{matrix}\text{ROC} \\ |z| > P_{max}\end{matrix}$$

greatest pole

$$x[n] = \mu[n] \Rightarrow X(z) = \frac{1}{1-z^{-1}} \qquad \begin{matrix}\text{ROC} \\ |z| > 1\end{matrix}$$

$$Y(z) = X(z)\cdot H(z) = \frac{3-7z^{-1}+5z^{-2}}{(1-z^{-1})(3-\frac{5}{2}z^{-1}+z^{-2})} \qquad \begin{matrix}\text{ROC} \\ |z| > 1\end{matrix}$$

In order to obtain $y[n]$ we need to invert $Y(z)$. For doing that we first have to find all the poles of $Y(z)$. We already know that $p_1 = 1$ and p_2 and p_3 will just be the roots of the second order polynomial $3-\frac{5}{2}z^{-1}+z^{-2}$:

$$3-\frac{5}{2}z^{-1}+z^{-2} = 0$$

$\Downarrow$ we change variable $x = z^{-1}$

Remember the variable change!

$$3-\frac{5}{2}x+x^2 = 0$$

$$\Downarrow$$

$$x = \frac{\frac{5}{2} \pm \sqrt{\frac{25}{4}-12}}{2} = \begin{cases} x_1 = \frac{5}{4} + j\cdot\sqrt{\frac{23}{16}} = p_2^{-1} \Rightarrow \dots \\ x_2 = \frac{5}{4} - j\cdot\sqrt{\frac{23}{16}} = p_3^{-1} \Rightarrow \dots \end{cases}$$

$$p_2 = \frac{1}{\frac{5}{4}+j\sqrt{\frac{23}{16}}} = \frac{\frac{5}{4}-j\sqrt{\frac{23}{16}}}{\left(\frac{5}{4}\right)^2+\frac{23}{16}} = \frac{5}{12} - j\sqrt{\frac{23}{144}}$$

multiplying by $\frac{5}{4}-j\sqrt{\frac{23}{16}}$ in numerator and denominator

NOTICE that p_2 and p_3 are inside the unit circle because $H(z)$ was stable

$$p_3 = \frac{5}{12} + j\sqrt{\frac{23}{144}}$$

$\hookrightarrow$ p_2 must be the conjugate of p_1 because x_1 & x_2 are conjugates

So then we have that:

$$Y(z) = \frac{3-7z^{-1}+5z^{-2}}{(1-z^{-1})(1-p_2z^{-1})(1-p_3z^{-1})} = \frac{A}{1-z^{-1}} + \frac{B}{1-p_2z^{-1}} + \frac{C}{1-p_3z^{-1}}$$

$$A = \left.\frac{3-7z^{-1}+5z^{-2}}{3-\frac{5}{2}z^{-1}+z^{-2}}\right|_{z=1} = \frac{3-7+5}{3-\frac{5}{2}+1} = \frac{2}{3}$$

remember that $(1-p_2z^{-1})(1-p_3z^{-1}) = 3-\frac{5}{2}z^{-1}+z^{-2}$

$$B = \left.\frac{3-7z^{-1}+5z^{-2}}{(1-z^{-1})(1-p_3z^{-1})}\right|_{z=p_2} =$$ this is complicated so just leave it indicated

$$C = \left.\frac{3-7z^{-1}+5z^{-2}}{(1-z^{-1})(1-p_2z^{-1})}\right|_{z=p_3} = B^*$$

And then the inverse of $Y(z)$ will be:

$$y[n] = \frac{2}{3}\mu[n] + B\cdot(p_2)^n\mu[n] + C\cdot(p_3)^n\mu[n]$$

(*) THERE IS A WAY OF EXPRESSING $y[n]$ using only real terms but this is enough for the exam.

QUESTION 5 :

$$|H(e^{i\omega})|^2 = \frac{\frac{5}{4} - \cos\omega}{\frac{10}{9} - \frac{2}{3}\cos\omega}$$

In order to find $H(z)$ we need to write $|H(e^{i\omega})|^2 = H(e^{i\omega}) H^*(e^{i\omega})$ and then obtain $H(z) = H(e^{i\omega})\Big|_{e^{i\omega}=z}$

We proceed as indicated in the hint:

$$|H(e^{i\omega})|^2 = \frac{\frac{5}{4} - \frac{1}{2}(e^{i\omega}+e^{-i\omega})}{\frac{10}{9} - \frac{1}{3}(e^{i\omega}+e^{-i\omega})} \overset{z=e^{i\omega}}{=} \frac{9}{4} \cdot \frac{5-2z-2z^{-1}}{10-3z-3z^{-1}} \overset{\substack{\text{multiplying by} \\ z \text{ the num. \& den.}}}{=} \dots$$

$$\dots = \frac{9}{4} \frac{-2z^{+2}+5z-2}{-3z^2+10z-3} = V(z)$$

We find the zeroes and poles of $V(z)$:

$$\text{NUMERATOR} = 0 \Rightarrow z = \frac{-5 \pm \sqrt{25-16}}{-4} = \frac{5}{4} \pm \frac{3}{4} = \left\{ \begin{array}{l} c_1 = 2 \\ c_2 = \frac{1}{2} \end{array} \right\} \text{zeroes}$$

$$\text{DENOMINATOR} = 0 \Rightarrow z = \frac{-10 \pm \sqrt{100-36}}{-6} = \frac{5}{3} \pm \frac{4}{3} = \left\{ \begin{array}{l} p_1 = 3 \\ p_2 = \frac{1}{3} \end{array} \right\} \text{poles}$$

So we can rewrite $V(z)$ as:

$$V(z) = \frac{9}{4} \frac{(1-c_1z^{-1})(1-c_2z^{-1})}{(1-p_1z^{-1})(1-p_2z^{-1})} \underset{\substack{p_1 = p \\ c_1 = c}}{=} \underbrace{\frac{\sqrt{3}}{\sqrt{2}} \cdot \frac{1-cz^{-1}}{1-pz^{-1}}}_{H(z) = H(e^{i\omega})} \cdot \underbrace{\frac{\sqrt{3}}{\sqrt{2}} \cdot \frac{3}{2} \cdot \frac{1-c^{-1}z^{-1}}{1-p^{-1}z^{-1}}}_{H(z^{-1}) = H^*(e^{i\omega})} \quad \underline{\text{Eq. (1)}}$$

Notice that, in this case, $H(z^{-1}) \overset{\Downarrow}{=} H(e^{-i\omega}) = H^*(e^{i\omega})$

Check it:

$$H(e^{i\omega}) = \frac{1-ce^{-i\omega}}{1-pe^{-i\omega}} \Rightarrow \left\{ \begin{array}{l} H^*(e^{i\omega}) \overset{z=e^{i\omega}}{=} \frac{1-ce^{i\omega}}{1-pe^{i\omega}} \\ H(e^{-i\omega}) = \frac{1-ce^{i\omega}}{1-pe^{i\omega}} \end{array} \right\}$$

NOTE: Explanation of the separation between $H(z)$ and $H(z^{-1})$

$$H(z) = \frac{\sqrt{3}}{2} \cdot \frac{1-cz^{-1}}{1-pz^{-1}}$$

then:

$$H(z^{-1}) = \frac{\sqrt{3}}{\sqrt{2}} \cdot \frac{1-cz}{1-pz} = \frac{\sqrt{3}}{\sqrt{2}} \frac{p\cancel{z}}{c\cancel{z}} \cdot \frac{1-c^{-1}z^{-1}}{1-p^{-1}z^{-1}} \underbrace{= \frac{\sqrt{3}}{\sqrt{2}} \cdot \frac{3}{2} \cdot \frac{1-c^{-1}z^{-1}}{1-p^{-1}z^{-1}}}_{\text{as we wrote in Eq (1)}}$$