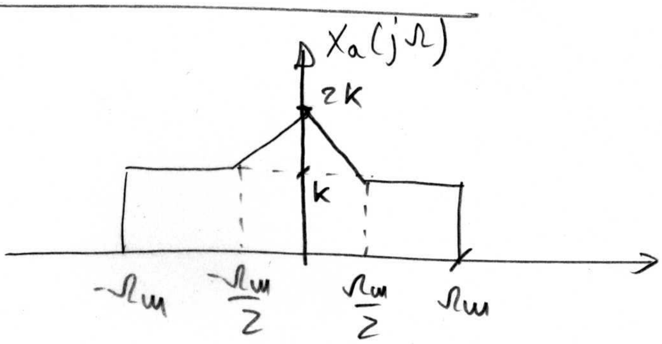
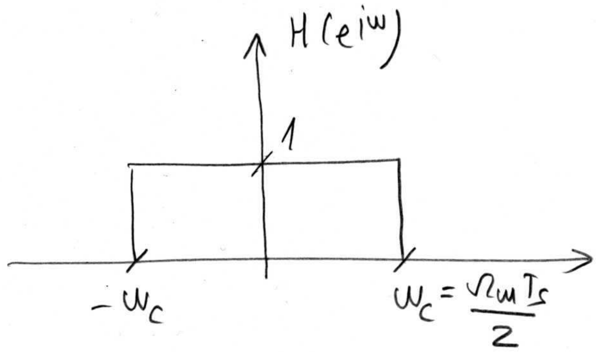
PROBLEM 1





Nyquist rate:

$$\boxed{\begin{aligned}\Omega_s &= 2\Omega_m \\ T_s &= \frac{2\pi}{\Omega_s} = \frac{\pi}{\Omega_m}\end{aligned}}$$

$\Rightarrow$ The cut-off of $H(e^{j\omega})$ will be:

$$\omega_c = \frac{\Omega_m T_s}{2} = \frac{\Omega_m}{2} \cdot \frac{\pi}{\Omega_m} = \frac{\pi}{2}$$

reconstruction filter
$\parallel$

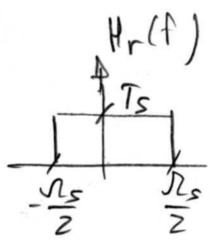
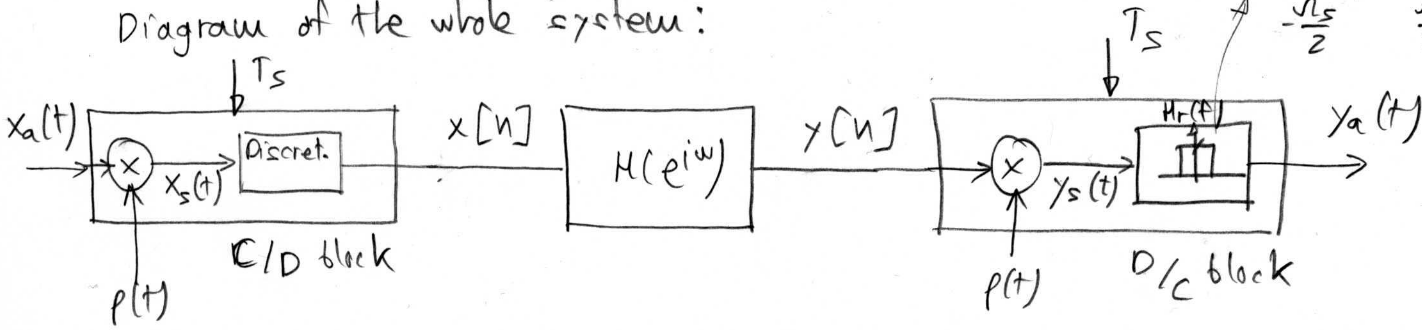
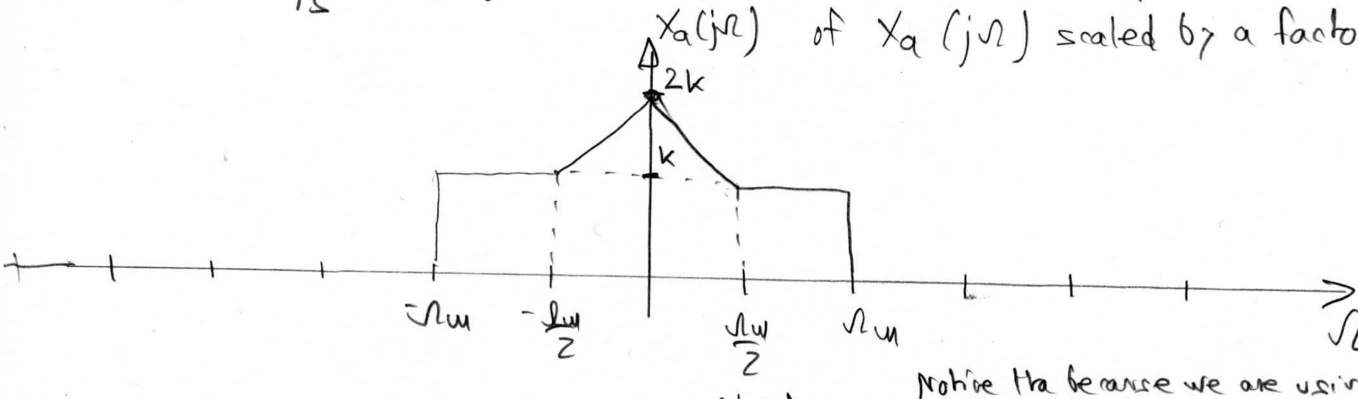


Diagram of the whole system:

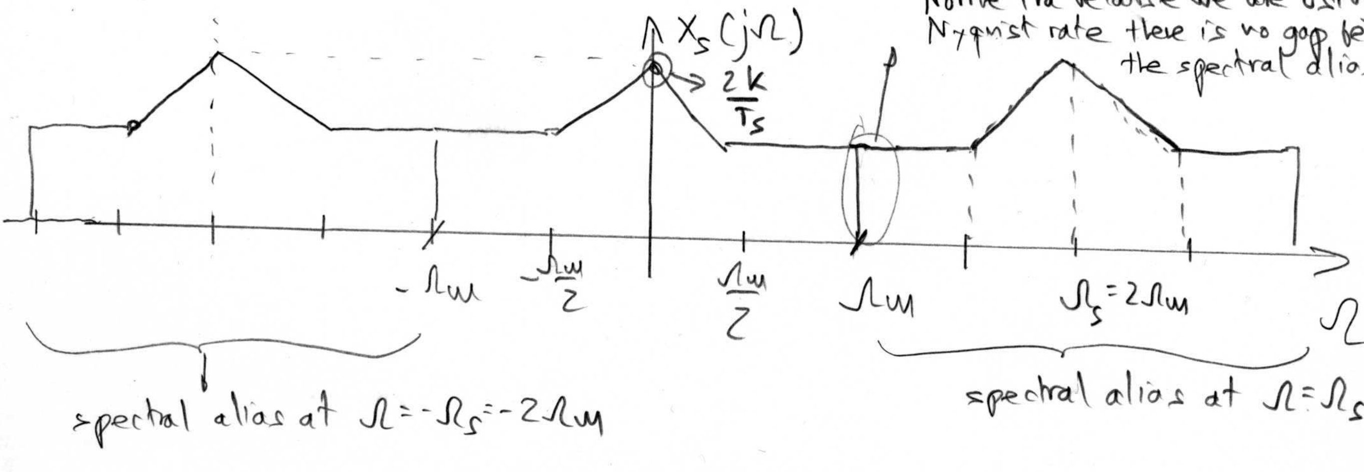


They are asking only for $X_a(j\Omega)$ but we SHOULD GO step by step:

$X_s(j\Omega) = \frac{1}{T_s} \sum X_a(j(\Omega - k\Omega_s)) \equiv$ $X_s(j\Omega)$ is just a periodic extension of $X_a(j\Omega)$ scaled by a factor $\frac{1}{T_s}$



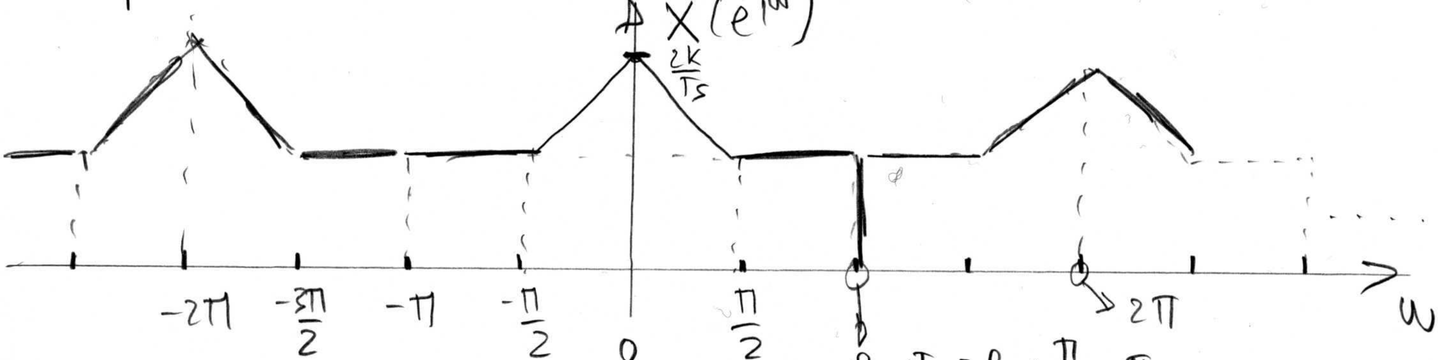
Notice tha because we are using the Nyquist rate there is no gap between the spectral aliases



The next step is to create the discrete sequence $x[n]$ from the sampled continuous-time signal $x_s(t)$:

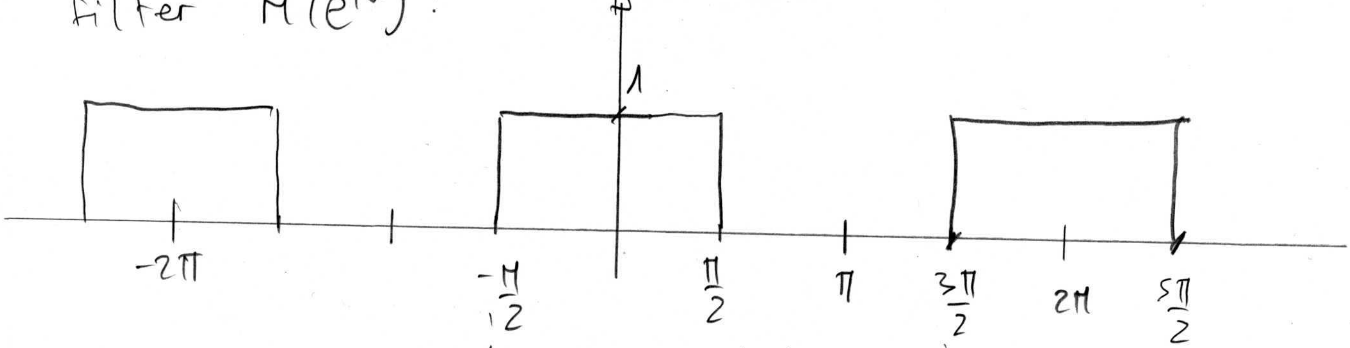
$$x[n] = x_s(n \cdot T_s) \Rightarrow X(e^{j\omega}) = X_s(j\Omega)\Big|_{\Omega = \frac{\omega}{T_s}}$$

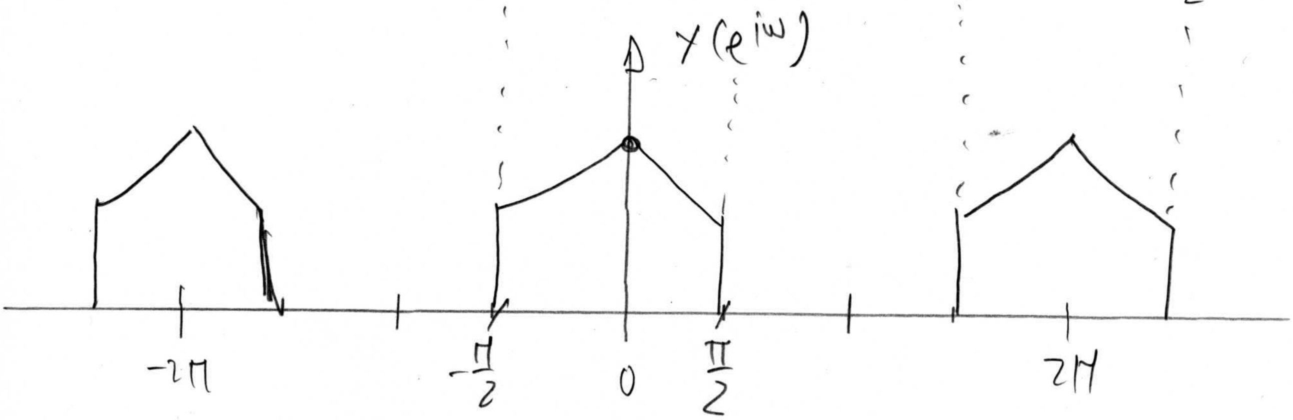
so this means that the DTFT $X(e^{j\omega})$ has exactly the same shape as $X_s(j\Omega)$ but we simply make the change $\omega = T_s \cdot \Omega$

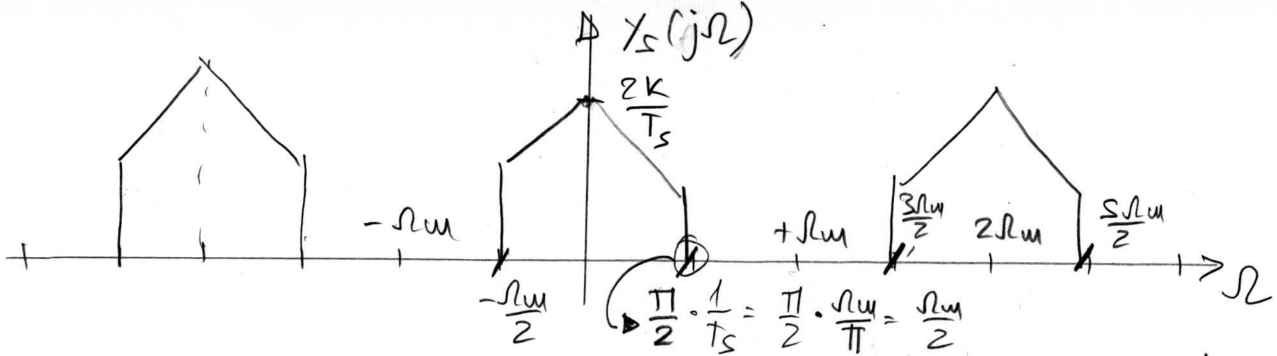


$\Omega_m \cdot T_s = \Omega_m \cdot \frac{\pi}{\Omega_m} = \pi$

To obtain $Y(e^{j\omega})$ we multiply $X(e^{j\omega})$ by the freq. response of the filter $H(e^{j\omega})$:



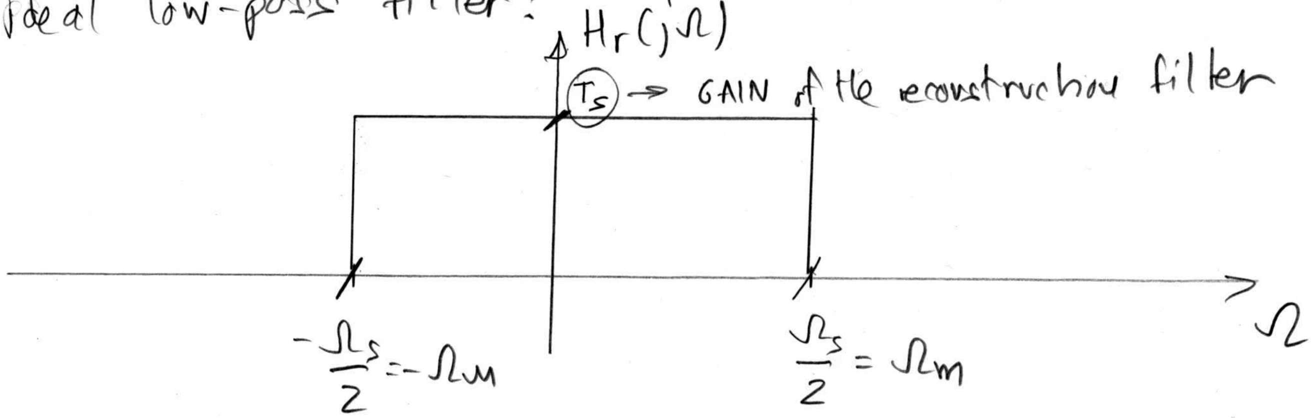




Now: $y_s(t) = \sum_{k=-\infty}^{\infty} y[k]\, \delta(t-kT_s) \Rightarrow Y_s(j\Omega) = Y(e^{j\omega})\Big|_{\omega=\Omega \cdot T_s}$

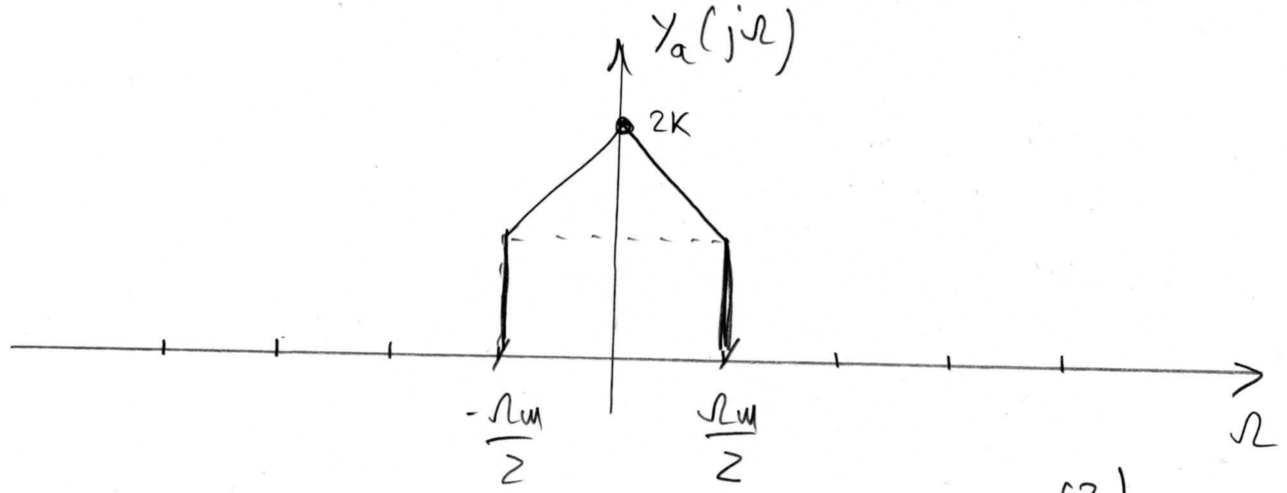
which means that $Y(e^{j\omega})$ has exactly the same shape as $Y_s(j\Omega)$ but we make the change of variable $\Omega = \frac{\omega}{T_s}$.

And finally we obtain the output $Y_a(j\Omega)$ by multiplying $Y_s(j\Omega)$ with the reconstruction filter which is just an ideal low-pass filter:



(*) The cutoff of the reconstruction filter can be either $\Omega_c = \Omega_c' = \frac{\Omega_s}{2}$ or $\Omega_c = \Omega_c'' = \Omega_M$. Usually we will have that both possibilities are just the same (like in this case) but this is not always true. Unless it is said otherwise in the problem you can choose to use Ω_c' or Ω_c''. Usually the best is to choose the value of Ω_c that lets pass a SINGLE copy of the original signals spectrum.

So finally we obtain:

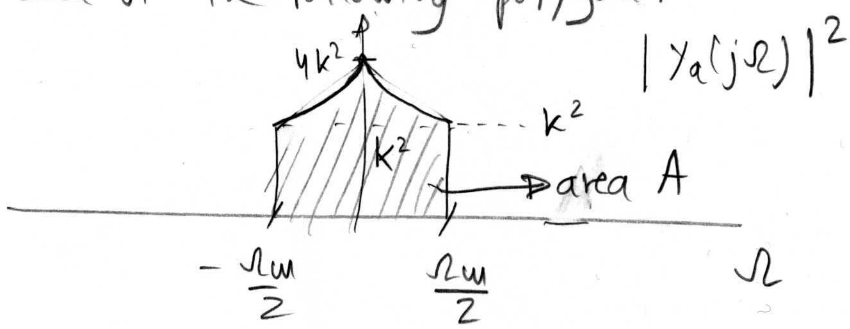


Energy of $y_a(t)$?

$$E_y = \int_{-\infty}^{\infty} |y_a(t)|^2 \, dt \overset{\text{Parseval}}{=} \frac{1}{2\pi} \int_{-\infty}^{\infty} |Y_a(j\Omega)|^2 \, d\Omega = \boxed{\frac{1}{2\pi} \cdot A} = E_y \quad \text{Eq. (2)}$$

where

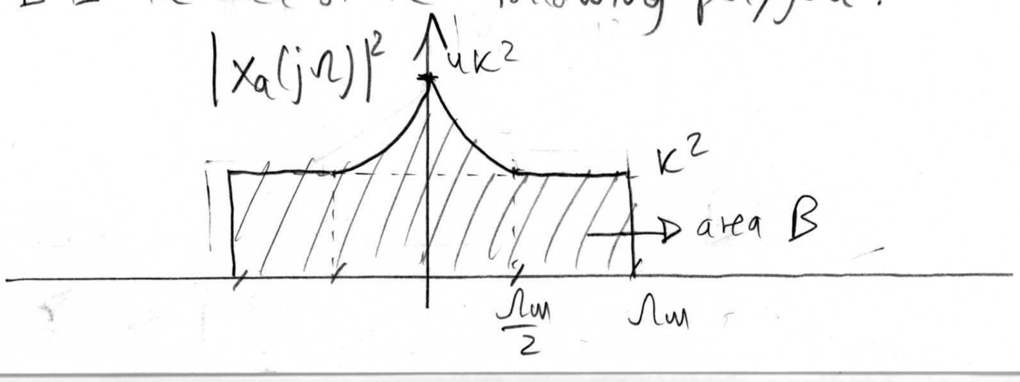
A is the area of the following polygon:



in order to calculate A we need to know the value of K^2. For that we use the information that $x_a(t)$ has unit-energy:

$$E_x = \frac{1}{2\pi} \int_{-\infty}^{\infty} |X_a(j\Omega)|^2 \, d\Omega = \frac{1}{2\pi} \cdot B = 1 \Rightarrow \boxed{B = 2\pi} \quad \text{Eq. (1)}$$

where B is the area of the following polygon:



The analytical expression for $|X_q(j\Omega)|^2$ is:

$$|X_q(j\Omega)|^2 = \begin{cases} K^2 & \frac{\Omega_m}{2} < |\Omega| < \Omega_m \\ \left(\frac{2K}{\Omega_m}\Omega + 2K\right)^2 & -\frac{\Omega_m}{2} < \Omega_m < 0 \\ \left(\frac{-2K}{\Omega_m}\Omega + 2K\right)^2 & 0 < \Omega < \frac{\Omega_m}{2} \\ 0 & \text{otherwise} \end{cases}$$

so we can calculate analytically:

$$B = \int_{-\Omega_m}^{\Omega_m} |X_q(j\Omega)|^2 = \Omega_m K^2 + 2 \cdot \int_0^{\frac{\Omega_m}{2}} \left(\frac{2k}{\Omega_m}\Omega + 2k\right)^2 d\Omega = \ldots$$

$$\ldots = \Omega_m K^2 + 2 \cdot \frac{7\Omega_m}{6} k^2 = \frac{10\Omega_m}{3} K^2$$

which means that:

$$B = \frac{10\Omega_m}{3} K^2 = 2\pi \quad \rightarrow \text{see Eq. (1)}$$

$$\Downarrow$$

$$\boxed{K^2 = \frac{3\pi}{5\Omega_m}}$$

and then the area A will be:

$$A = B - \Omega_m \cdot K^2 = \frac{10\Omega_m}{3} K^2 - \Omega_m K^2 = \frac{7\Omega_m}{3} K^2 =$$

$$= \frac{7\cancel{\Omega_m}}{\cancel{3}} \cdot \frac{\cancel{3}\pi}{5\cancel{\Omega_m}} = \boxed{\frac{7\pi}{5} = A}$$

$$K^2 = \frac{3\pi}{5\Omega_m}$$

And finally the energy of $y_a(t)$ will be:

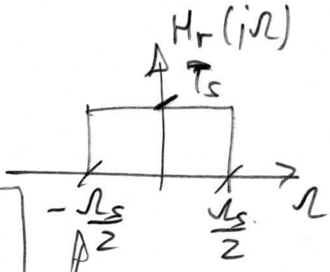
see Eq. (2) $\rightarrow$ $$\boxed{E_y = \frac{1}{2\pi} \cdot A = \frac{7}{10}}$$

PROBLEM 2

$$x_a(t) = s_a(t) + \varepsilon_a(t)$$

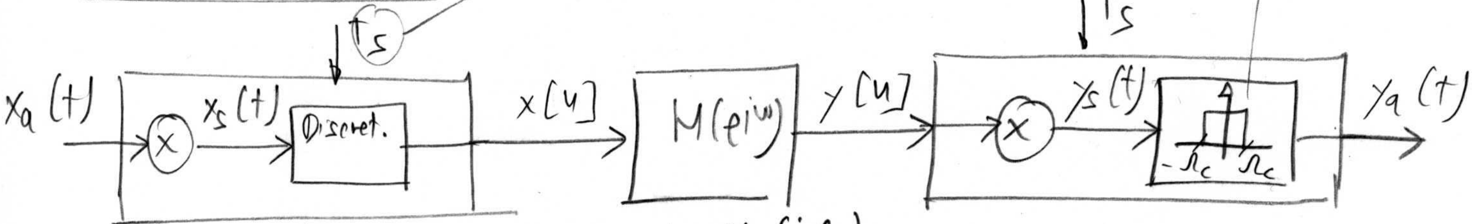
$f_\varepsilon = 60\,Hz \Rightarrow \Omega_\varepsilon = 2\pi f_\varepsilon = 120\pi$ rad/sg

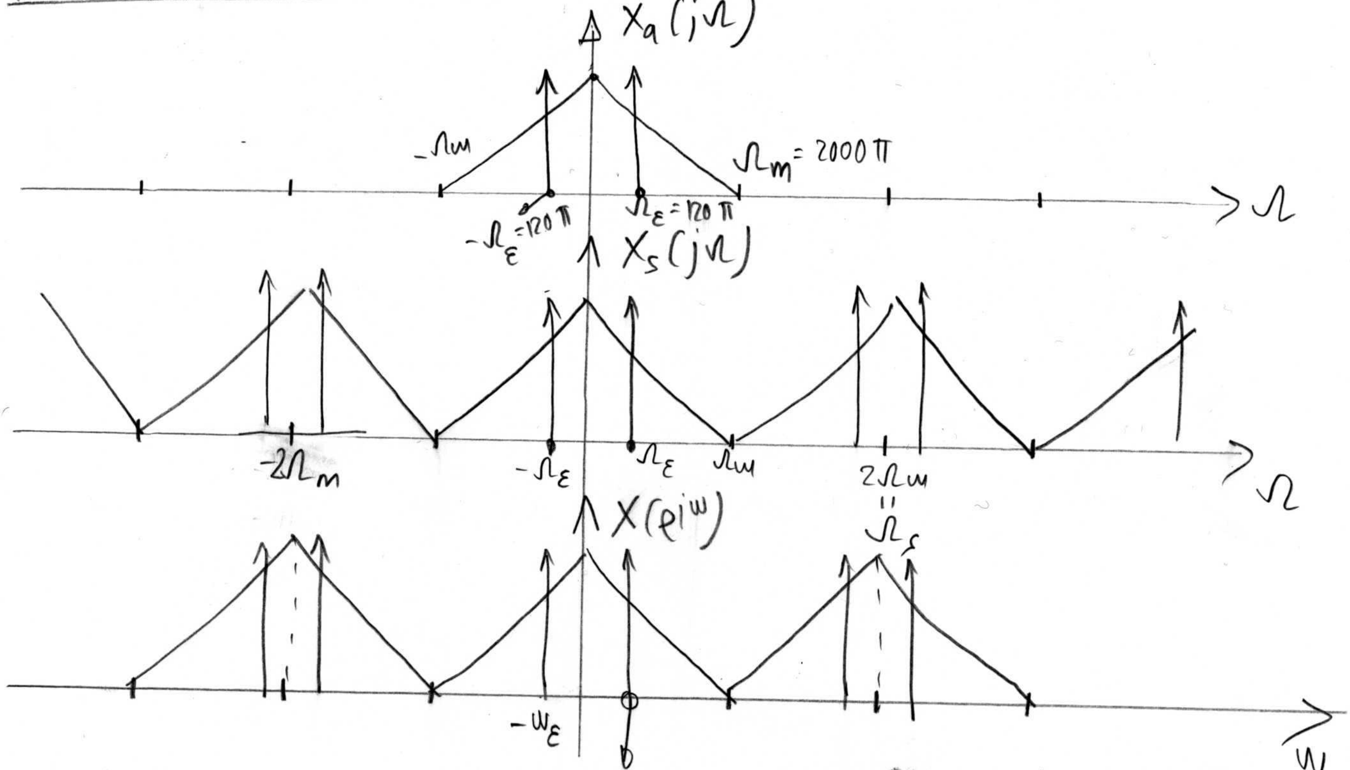
$f_{max} = 1000\,Hz \Rightarrow \Omega_m = 2\pi \cdot f_{max} = 2000\pi$ rad/sg



$$\boxed{\Omega_s \overset{\text{Nyquist}}{=} 2\Omega_m = 4000\pi \Rightarrow T_s = \frac{2\pi}{\Omega_s} = 5\cdot 10^{-4}\ \text{sec.}}$$

System diagram:





$$\omega_\varepsilon = \Omega_\varepsilon \cdot T_s = 120\pi \cdot 5\cdot 10^{-4} = 0'06\pi \ \ \text{rad/sg}$$

In order to remove the interference we need to enforce that

$$H(e^{j\omega_\varepsilon}) = H(e^{-j\omega_\varepsilon}) = 0$$

Moreover they are telling us that our filter must be in time domain:

$$y[n] = x[n] + a\,x[n-1] + b\,x[n-2]$$

which means that the freq. response of the filter must be :

$$H(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})} = 1 + a e^{-j\omega} + b e^{-j\omega \cdot 2} \quad \rightarrow \underline{\text{Eq. (1)}}$$

clearly $H(e^{j\omega})$ is a second order polynomial of $e^{-j\omega}$ which has two zeros. These two zeros must correspond to the frequency of the noise $\pm\omega_\varepsilon$. Then $H(e^{j\omega})$ must be :

$$H(e^{j\omega}) = \underbrace{(1 - e^{j\omega_\varepsilon} e^{-j\omega})(1 - e^{-j\omega_\varepsilon} e^{-j\omega})} = \ldots\ldots$$

NOTE : The sign in $e^{-j\omega}$ is negative because we want a polynomial of $e^{-j\omega}$ (see Eq. (1)). If you would have used instead $e^{+j\omega}$ then we would have that: $H'(e^{j\omega}) = b + ae^{j\omega} + e^{+j\omega 2}$ and then

in time domain $y'[n] = bx[n] + ax[n+1] + 1 \cdot x[n+2]$.

In practice this filter is just an anticausal version of the filter that they are giving us and we have that:

$$y[n] = y'[n-2] = x[n] + ax[n-1] + bx[n-2]$$

This illustrates that for every anti-causal LTI system we can build a causal equivalent version by delaying the output of the anticausal filter by the proper amount.

$$\ldots = 1 - e^{-j\omega_\varepsilon} e^{-j\omega} - e^{j\omega_\varepsilon} e^{-j\omega} + e^{-j2\omega} = 1 - (e^{-j\omega_\varepsilon} + e^{j\omega_\varepsilon}) e^{-j\omega} + e^{-j2\omega} =$$

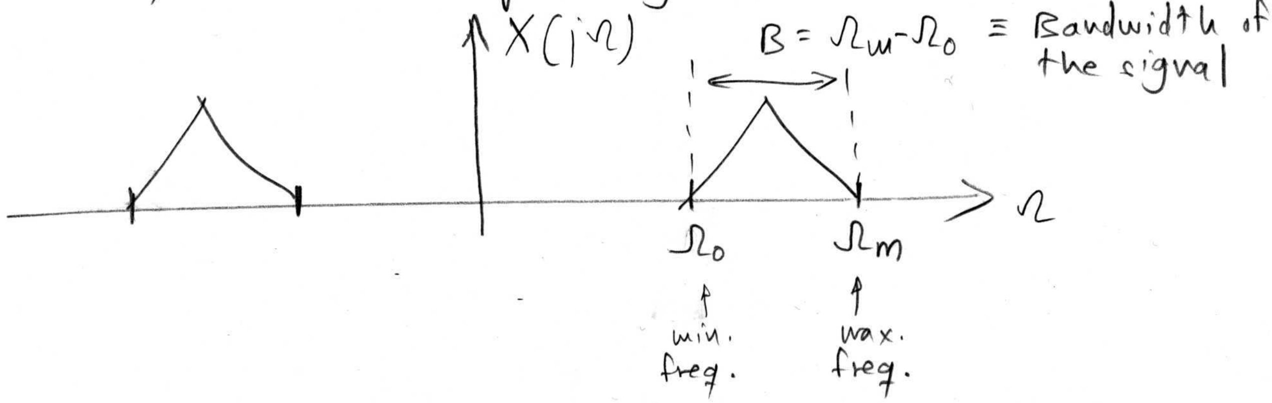
$$= \boxed{1 - \underbrace{(2\cos\omega_\varepsilon)}_{a = -2\cos\omega_\varepsilon} \cdot e^{-j\omega} + \underbrace{e^{-j2\omega}}_{b=1} = H(e^{j\omega})}$$

$$\Rightarrow \boxed{\begin{array}{l} a = -2\cos\omega_\varepsilon = -2\cos 0'06\pi \\ b = 1 \end{array}}$$

PROBLEM 3

IMPORTANT: In general the smallest possible sampling rate (without aliasing) for bandpass signals is smaller that $2\Omega_m$ where Ω_m is the maximum frequency in the signal. The process to determine the minimum Ω_s for bandpass signals is the following:

Assume that you have a bandpass signal with CTFT:



The first thing to do is to calculate:

$$r = \left\lfloor \frac{\Omega_m}{B} \right\rfloor$$

where $\lfloor \cdot \rfloor$ is the "floor" operator.

Then we calculate the value B' such that:

$$\frac{\Omega_m}{B'} = r \Rightarrow \boxed{B' = \frac{\Omega_m}{r}}$$

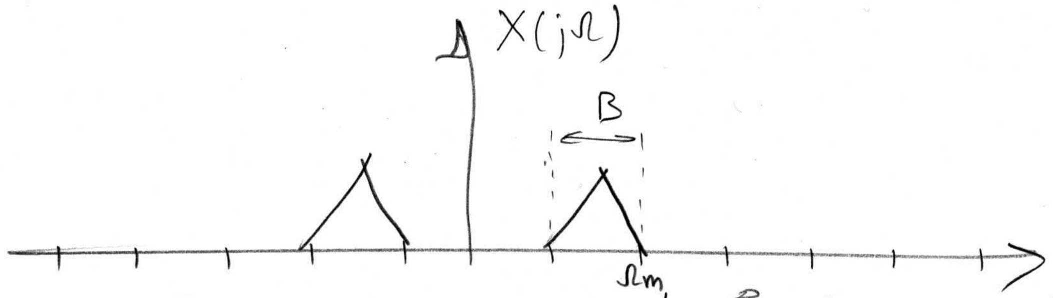
Finally, knowing B', you know that the minimum possible sampling frequency to avoid aliasing is $\boxed{\Omega_s = 2B'}$

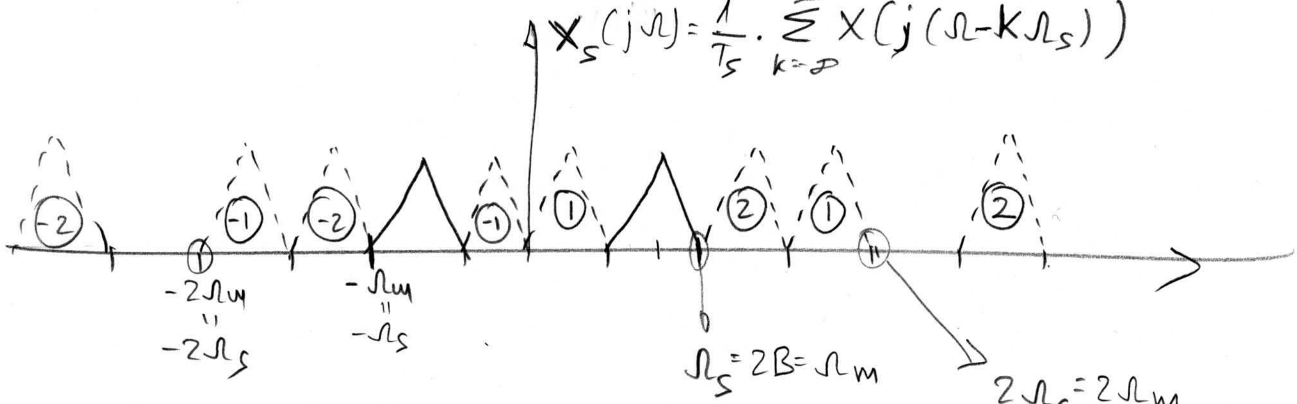
Let's check that this is true...

Let's assume that we have a signal such that $\left\lfloor \frac{\Omega_m}{B} \right\rfloor = \frac{\Omega_m}{B}$, that is, "the maximum frequency is an integer multiple of the bandwidth". In this case $B' = B \Rightarrow \Omega_s = 2B$.

Can we really use $\Omega_s = 2B$ for this signal?

We check with our example having $\Omega_M = 2B$:



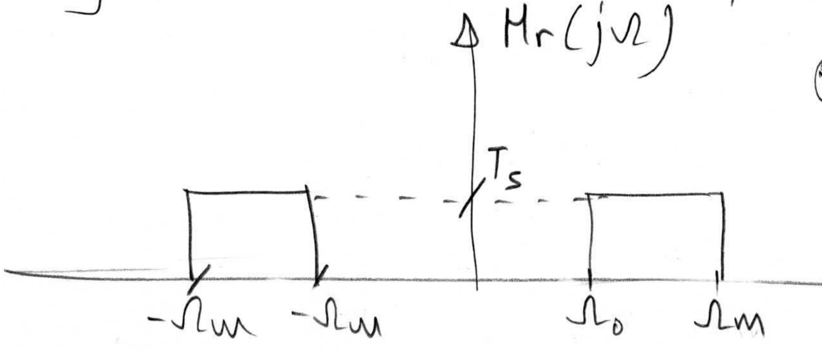


where the notation ① denotes the copies due to the alias centered in Ω_s, ② denotes those corresponding to the alias centered in $2\Omega_s$, and so on....

(-1) $\rightarrow -\Omega_s$

(-2) $\rightarrow -2\Omega_s$

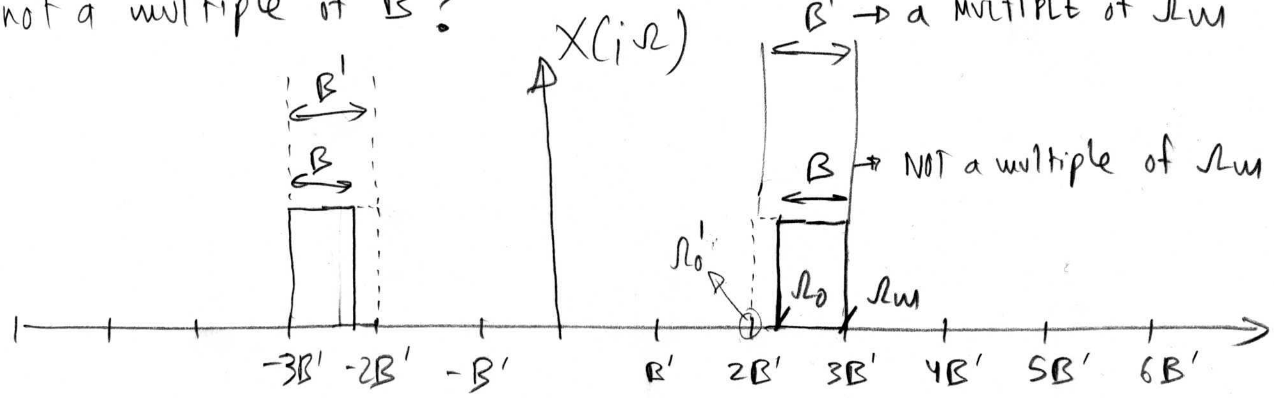
You can see that the spectral aliases DO NOT OVERLAP and that we can recover the original analog bandpass signal using an analog bandpass filter with freq. response:



(*) So the reconstruction filter for bandpass signals is a BANDPASS FILTER and not a lowpass filter as we used for baseband signals

You can check that $\Omega_s = 2B$ is enough to avoid aliasing also when Ω_m is any other multiple of B : $2B, 3B, \dots, kB \quad k \in \mathbb{Z}$

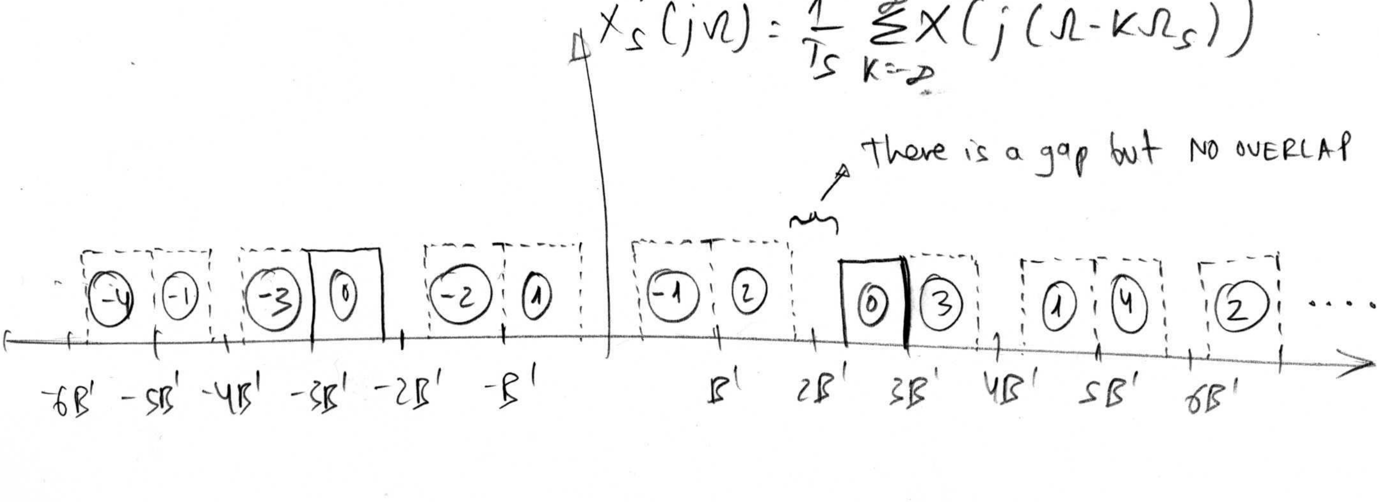
So we have concluded that when Ω_m is a multiple of B then $\Omega_s = 2B$ is the minimum sampling freq. What happens if Ω_m is not a multiple of B?



What we do is to redefine the original bandwidth to a larger value: $B' = \dfrac{\Omega_m}{\left\lfloor \frac{\Omega_m}{B} \right\rfloor} > B$ such that Ω_m is a multiple of B'. $\Rightarrow$ $\boxed{\Omega_s = 2B'}$ $\rightarrow$ this will be the sampling frequency required to avoid aliasing

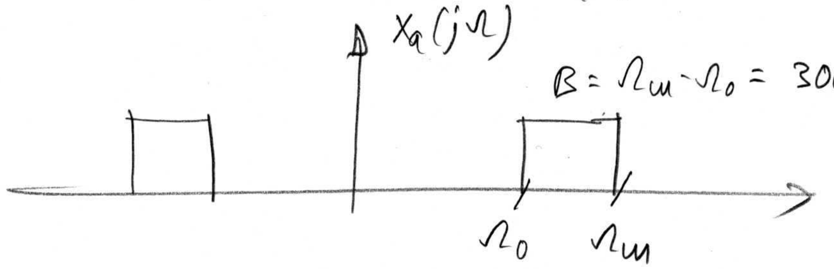
In this case, there will be a gap between spectral aliases but there will not be overlap between aliases:

$$X_s(j\Omega) = \frac{1}{T_s} \sum_{k=-\infty}^{\infty} X(j(\Omega - k\Omega_s))$$



Now we solve the cases of PROBLEM 2 :

(a) $x_a(t) \rightarrow X_a(j\Omega)$ such that $X_a(j\Omega) \neq 0 \iff 2\pi \cdot 9000 \leq |\Omega| \leq 2\pi \cdot 12000$



$B = \Omega_m - \Omega_0 = 3000 \cdot 2\pi$

We follow the steps :

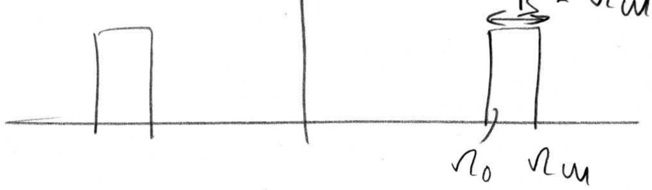
1) $r = \left\lfloor \dfrac{\Omega_m}{B} \right\rfloor = \left\lfloor \dfrac{2\pi \cdot 12000}{3000 \cdot 2\pi} \right\rfloor = 4$

2) $B' = \dfrac{\Omega_m}{r} = \dfrac{2\pi \cdot 12000}{4} = 3000 \cdot 2\pi = B$ → of course because Ω_m was already a multiple of B

3) $\boxed{\Omega_s^{min} = 2B' = 2B = 6000 \cdot 2\pi}$ $\frac{rad}{sec}$.

(b) $X_a(j\Omega) \neq 0 \iff 2\pi \cdot 18000 \leq |\Omega| \leq 2\pi \cdot 22000$

$B = \Omega_m - \Omega_0 = 4000 \cdot 2\pi$



Again the same steps :

1) $r = \left\lfloor \dfrac{\Omega_m}{B} \right\rfloor = \left\lfloor \dfrac{2\pi \cdot 22000}{2\pi \cdot 4000} \right\rfloor = 5$

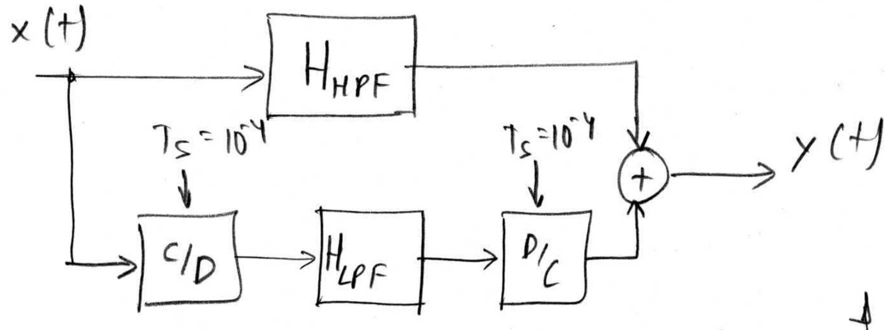
2) $B' = \dfrac{\Omega_m}{r} = \dfrac{2\pi \cdot 22000}{5} = 2\pi \cdot 4400 > B$

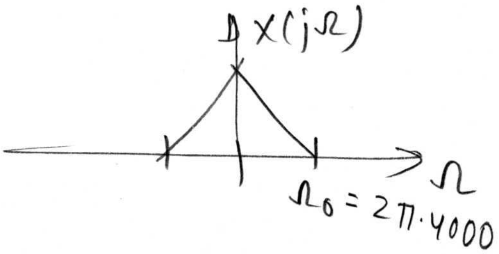
3) $\boxed{\Omega_s^{min} = 2B' = 2\pi \cdot 8800 \ \frac{rad}{sec}}$

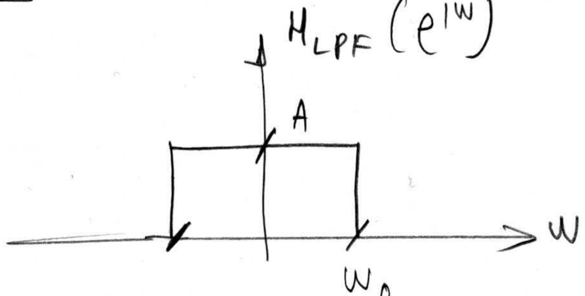
QUESTION TO THINK : If a bandpass signal has a bandwidth B we have seen that in the best case $\Omega_s^{min} = 2B$. What about the worst case ?

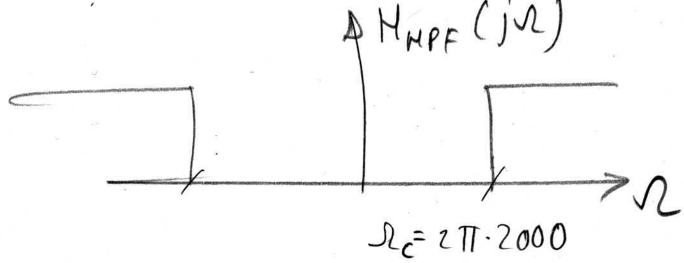
PROBLEM 4

$$H_{LPF}(e^{j\omega}) = \begin{cases} A & |\omega| \le \omega_0 \\ 0 & \text{otherwise} \end{cases}$$







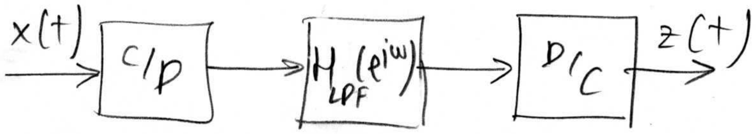


The first thing is to check if T_s is small enough to avoid aliasing:

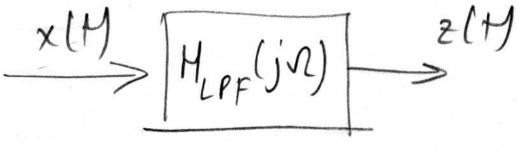
Nyquist rate: $T_s^{Nyquist} = \dfrac{2\pi}{2\cdot\Omega_0} = \dfrac{2\pi}{2\cdot 2\pi\cdot 4000} = 1'25\cdot 10^{-4} > T_s \Rightarrow$ $\boxed{\text{NO ALIASING}}$

Notice that if we had $T_s > T_s^{Nyquist}$ then perfect reconstruction would be impossible!

Because there is no aliasing in the sub-system:

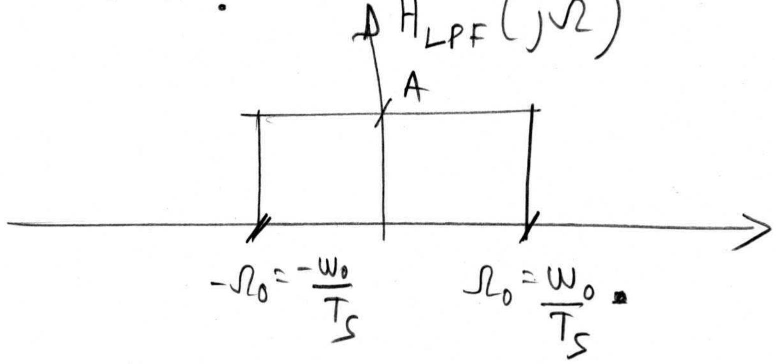


we can build an equivalent analog system:

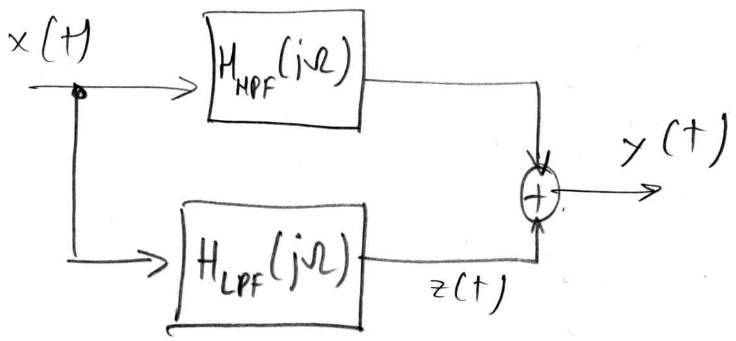


where $H_{LPF}(j\Omega) = H_{LPF}(e^{j\omega})\Big|_{\omega=\Omega\cdot T_s}$

So we have that:



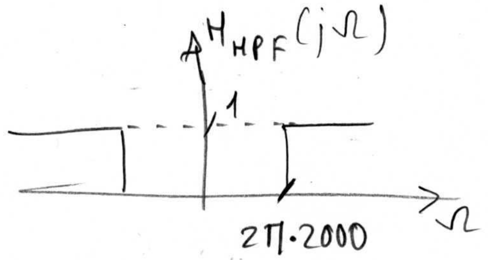
Now, since our system's diagram is:



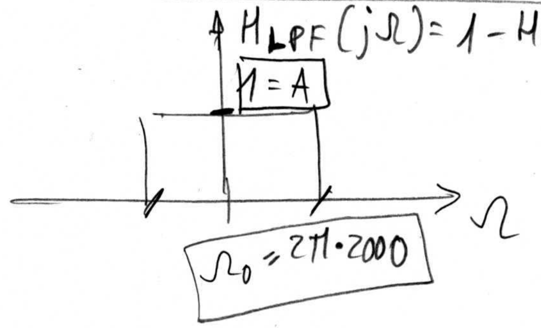
and we want that $y(t) = x(t) \quad \forall t$ then:

$$Y(j\Omega) = \underbrace{\left[H_{HPF}(j\Omega) + H_{LPF}(j\Omega)\right]}_{\text{must be equal to 1 for all frequencies}} \cdot X(j\Omega)$$

$$H_{HPF}(j\Omega) + H_{LPF}(j\Omega) = 1 \Rightarrow \boxed{H_{LPF}(j\Omega) = 1 - H_{HPF}(j\Omega)} \quad \forall f$$



$\Rightarrow$



so: $\Omega_0 = \frac{w_0}{T_s} = 2\pi \cdot 2000 \Rightarrow w_0 = 2\pi \cdot 2000 \cdot T_s$

$$\boxed{w_0 = 0'4\pi}$$