PHYS 615 – Activity 7.2: Lagrangian for two Particles

Let's consider two particles, at (x_1, y_1, z_1) and (x_2, y_2, z_2), respectively. The kinetic energy is then the sum of the kinetic energies for each particle, and we've seen before that we can get the forces from a single potential $U(x_1, y_1, z_1, x_2, y_2, z_2)$.
The Lagrangian is defined just as before, but now depends on 6 coordinates (and, possibly, time):

$$\mathcal{L} = T - U = \frac{1}{2}m_1(\dot{x}_1^2 + \dot{y}_1^2 + \dot{z}_1^2) + \frac{1}{2}m_2(\dot{x}_2^2 + \dot{y}_2^2 + \dot{z}_2^2) - U(x_1, y_1, z_1, x_2, y_2, z_2) \tag{1}$$

As before, a change to generalized coordinates $q_1, q_2, \ldots, q_n$, may be advantageous, but the Euler-Lagrange equation will hold either with the original Cartesian coordinates or the new generalized coordinates:

$$\frac{\partial \mathcal{L}}{\partial q_i} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_i} \tag{2}$$

1. *Two masses and a spring*

   (a) Write down the Lagrangian $\mathcal{L}(x_1, x_2, \dot{x}_1, \dot{x}_2)$ for particles of equal mass m that move along the x-axis (only). The two particles are connected by a spring with potential energy $\frac{1}{2}kx^2$, where x is the extension of the spring $x = x_1 - x_2 - l$, l being the equilibrium length of the spring.

   Solution: No surprises in the Lagrangian:

   $$\mathcal{L} = \frac{1}{2}m(\dot{x}_1^2 + (\dot{x}_2^2) - \frac{1}{2}k(x_1 - x_2 - l)^2$$

   (b) Rewrite $\mathcal{L}$ in terms of the new variables $X = \frac{1}{2}(x_1 + x_2)$ and x.

   Solution: We first have to express x_1 and x_2 in terms of X and x: Doubling the X equation and adding it to the x equation makes x_2 cancel out:

   $$2X + x = (x_1 + x_2) + x_1 - x_2 - l \Longrightarrow \frac{2X + x + l}{2} = x_1$$

   and similarly we get $x_2 = \frac{2X - x - l}{2}$.
   Now we can find $\dot{x}_1$ and $\dot{x}_2$:

   $$\dot{x}_1 = \dot{X} + \frac{1}{2}\dot{x} \qquad \dot{x}_2 = \dot{X} - \frac{1}{2}\dot{x}$$

   Plugging this into our $\mathcal{L}$ from above and simplifying a bit:

   $$\mathcal{L} = \frac{1}{2}m\left(2\dot{X}^2 + 2\frac{1}{4}\dot{x}^2\right) - \frac{1}{2}kx^2 = m\dot{X}^2 + \frac{1}{4}m\dot{x}^2 - \frac{1}{2}kx^2$$

(c) Write down the two Euler-Lagrange equations.

Solution:

$$\frac{\partial \mathcal{L}}{\partial X} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{X}} \Longrightarrow 0 = 2M\ddot{X}$$

$$\frac{\partial \mathcal{L}}{\partial x} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{x}} \Longrightarrow -kx = \frac{1}{2}m\ddot{x}$$

(d) Solve for $X(t)$ and $x(t)$ and describe the motion. Is there a connection to what we saw about the center of mass earlier in this class?

Solution: The equation for X is the equation for the motion of the center of mass – and since there are no external forces, total momentum is conserved, ie., the center of mass moves at constant velocity, as the solution for this equation says: $X(t) = V_0 t + X_0$.

The x equation looks very similar to our usual harmonic oscillator equation, though it has only $(m/2)$ instead of the full mass. So the solution is, e.g., $x_0 \cos()\omega t + \phi_0)$ with $\omega = \sqrt{2k m}$. The factor of 2 can be explained, e.g., like this: Let's assume the two masses are equally stretching the spring beyond it's equilibrium position. Once one lets go, they're both moving toward each other, compressing the spring, which makes them stop and go back outward. What happens is that the center of the spring will stay at rest, and on each side, half the spring is fixed on one side and attached to a mass m on the other side – that's the usual kind of oscillations we've been considering. But since that's only a half spring, it's spring constant is double $(2k)$ of the full spring. (You may want to think about why that is).