PHYS 615 – Activity 3.1: Conservation of Momentum

After having dealt with the motion of a single particle for a while, let's look at the situation where we have multiple particles, that is, N of them. For each of them, Newton's 2nd Law holds, that is,

$$m_i \dot{\vec{v}}_i = \vec{F}_{net,on\,i} = \sum_{j \neq i} \vec{F}_{j\,on\,i} + \vec{F}_{ext,on\,i} \tag{1}$$

where I break up the forces that act on particle i into forces exerted by some other particle in our system j and external forces exerted from outside the system. Note: $\vec{F}_{j\,on\imath}$ is the sum of all forces that j exerts on i – there could be more than one, or there could be none, too. Similarly, there may be more than one external force acting on i, so again $\vec{F}_{ext,on\,i}$ is the sum of all external forces acting on i.

1. Let's go back to the first problem we looked at in this class, the hand H pushing a block A on top of table T, while block A is pushing a block B which is elevated over the table.

   This problem has a multitude of forces acting, normal forces (the pushing force from the hand is a normal force, too), friction, and gravity. To make things a bit simpler, let's neglect friction between block A and the table – but there is friction between the two blocks.

2. Draw two FBDs, one for each of the two blocks. Label your forces like $F_{N,T\,on\,A}$. *Solution:* TBD

3. Write out Newton's 2nd Law for each of the two blocks, keep it in vector form. Basically, go look at your FBD, find all the forces acting on a given block and write $\vec{F}_{net,on\,X}$ explicitly as the sum of those forces.

   Solution:

$$\begin{aligned}
\vec{F}_{net,\,on\,A} &= \vec{F}_{N,B\,on\,A} + \vec{F}_{f,B\,on\,A} + \vec{F}_{N,T\,on\,A} + \vec{F}_{N,H\,on\,A} + \vec{F}_{G,E\,on\,A} \\
\vec{F}_{net,\,on\,B} &= \vec{F}_{N,A\,on\,B} + \vec{F}_{f,A\,on\,B} + \vec{F}_{G,E\,on\,B}
\end{aligned}$$

4. Our system consists of the two blocks. For each block, write out Newton's 2nd Law in the form given above, that is

$$m_A \dot{\vec{v}}_A = \vec{F}_{B\,on\,A} + \vec{F}_{ext,on\,A} \tag{2}$$

   and similarly for block B. Write out the categorized forces on the r.h.s. in terms of the actual forces from the FBD.

   Solution:

$$\begin{aligned}
m_A \dot{\vec{v}}_A &= \vec{F}_{net,\,on\,A} = \vec{F}_{B\,on\,A} + \vec{F}_{ext,on\,A} \\
m_B \dot{\vec{v}}_B &= \vec{F}_{net,\,on\,B} = \vec{F}_{A\,on\,B} + \vec{F}_{ext,on\,B}
\end{aligned}$$

Comparison with the previous part shows that

$$\begin{aligned}
\vec{F}_{B\,on\,A} &= \vec{F}_{N,B\,on\,A} + \vec{F}_{f,B\,on\,A}\\
\vec{F}_{ext\,on\,A} &= \vec{F}_{N,T\,on\,A} + \vec{F}_{N,H\,on\,A} + \vec{F}_{G,E\,on\,A}\\
\vec{F}_{A\,on\,B} &= \vec{F}_{N,A\,on\,B} + \vec{F}_{f,A\,on\,B}\\
\vec{F}_{ext\,on\,B} &= \vec{F}_{G,E\,on\,B}
\end{aligned}$$

5. Is it true that $\vec{F}_{B\,on\,A} = -\vec{F}_{A\,on\,B}$? (justify your answer). Is it generally true that $\vec{F}_{i\,on\,j} = -\vec{F}_{j\,on\,i}$ for any two particles?

   Solution: It is true, because the two forces making up either of these two terms are themselves third-law pairs, so there sums are also equal and opposite.

6. Now add up the two equations from 4. Simplify.

   Solution:

$$\begin{aligned}
m_A\dot{\vec{v}}_A + m_B\dot{\vec{v}}_B &= \vec{F}_{B\,on\,A} + \vec{F}_{ext,on\,A} + \vec{F}_{A\,on\,B} + \vec{F}_{ext,on\,B}\\
&= \vec{F}_{ext,on\,A} + \vec{F}_{ext,on\,B}
\end{aligned}$$

   The point here is that indeed the internal forces cancel out and only the external ones remain.

What you've hopefully seen happen here is that the internal forces all canceled out, and only the external forces remained. And that is, basically, what we call of conservation of momentum, except we have to cast things in terms of momentum first:

7. As you hopefully remember, the momentum for some particle i is defined as $\vec{p}_i \equiv m_i\vec{v}_i$. Find $\dot{\vec{p}}_i$, and hopefully you'll recognize that in the 2nd Law equations you found above.

   Solution:

$$\dot{\vec{p}}_i = m_i\dot{\vec{v}}_i$$

   which is exactly the l.h.s. that occurred above (for $i = A, B$).

8. Let's define total momentum for a system of N particles

$$\vec{p} \equiv \sum_{i=1}^{N} \vec{p}_i \tag{3}$$

   Express $\dot{\vec{p}}$ in terms of the time derivatives of the individual momenta $\dot{\vec{p}}_i$.

   Solution:

$$\dot{\vec{p}} = \sum_{i=1}^{N} \dot{\vec{p}}_i$$

9. In the case of two particles, like the two blocks above, show that

$$\dot{\vec{p}} = \sum_{i=1}^{2} \vec{F}_{ext,on\,i} \tag{4}$$

Solution: We've mostly just done that, for A, B

$$\begin{aligned}
\dot{\vec{p}} &= \sum_{i=1}^{N} \dot{\vec{p}}_i \\
&= \vec{F}_{2\,on\,1} + \vec{F}_{ext,on\,1} + \vec{F}_{1\,on\,2} + \vec{F}_{ext,on\,2} \\
&= \vec{F}_{ext,on\,1} + \vec{F}_{ext,on\,2}
\end{aligned}$$

10. Now let's do it again for three particles. Write out the three individual equations for $\dot{\vec{p}}_i$ of the form of Eq.(1). Add up all three equations and show that you can derive

$$\dot{\vec{p}} = \vec{F}_{ext} \equiv \sum_{i=1}^{3} \vec{F}_{ext,on\,i} \tag{5}$$

where I introduced the sum of all external forces acting on the system $\vec{F}_{ext}$.

Solution:

$$\begin{aligned}
\dot{\vec{p}} &= \sum_{i=1}^{3} \dot{\vec{p}}_i \\
&= \vec{F}_{2\,on\,1} + \vec{F}_{3\,on\,1} + \vec{F}_{ext,on\,1} + \\
&\quad \vec{F}_{1\,on\,2} + \vec{F}_{3\,on\,2} + \vec{F}_{ext,on\,2} + \\
&\quad \vec{F}_{1\,on\,3} + \vec{F}_{2\,on\,3} + \vec{F}_{ext,on\,3} \\
&= \vec{F}_{ext,on\,1} + \vec{F}_{ext,on\,2} + \vec{F}_{ext,on\,3} \\
&= \vec{F}_{ext}
\end{aligned}$$

11. (Challenge) It seems quite believable that by the same token conservation of momentum in the form above holds for $N = 4, 5, \ldots$ particles, but you could try to actually prove it to be true for any N, e.g., by using mathematical induction. (You can use that $\vec{F}_{i\,on\,j} = -\vec{F}_{j\,on\,i}$ as a consequence of Newton's 3rd Law.)