PHYS 615 – Activity 7.4: More Constrained Lagrangians

Here's what the text says about solving constrained motion problems the Lagrangian way:

1. Write down the kinetic and potential energies, and hence the Lagrangian $\mathcal{L} = T - U$, using any convenient inertial reference frame.
2. Choose a convenient set of n generalized coordinates $q_1, q_2, \ldots q_n$, and find expressions for the original coordinates of step 1 in terms of your chosen generalized coordinates. (Steps 1 and 2 can be done in either order.)
3. Rewrite $\mathcal{L}$ in terms of $q_1, q_2, \ldots q_n$ and $\dot{q}_1, \dot{q}_2, \ldots \dot{q}_n$.
4. Write down the n Lagrange equations.

1. *The Atwood Machine (with a real pulley)*

   Note: Example 7.3 in the text is the Atwood machine with a massless pulley, so this problem is an extension of it.

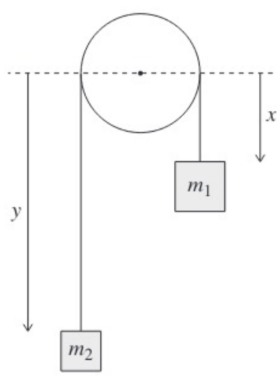
   

   Figure 7.6 An Atwood machine consisting of two masses, m_1 and m_2, suspended by a massless inextensible string that passes over a massless, frictionless pulley of radius R. Because the string's length is fixed, the position of the whole system can be specified by a single variable, which we can take to be the distance x.

   As shown in this figure, the Atwood machine has two masses connected by a string that runs over a pulley. The question for today is, what is the acceleration of the two masses?

   This problem can be solved (and you've probably done at least one way before) using Newton's Laws, or conservation of energy. But, unsurprisingly, the goal is to solve it the Lagrangian way.

(a) Follow the standard process to find the Lagrangian and find the equation of motion, which should give you the acceleration we're looking to find.

Note: The kinetic energy of a pulley with inertia I (assumed to be known) is $\frac{1}{2}I\omega^2$, where ω is the pulley's angular velocity.

Solution: As the caption suggests, I'll express everything in terms of x. Since the lenght of the string is constant, $y = C - x$ – this only goes into the into the potential energy and the actual value of C doesn't matter. Also, as x increases by dx, the angle of the pulley ϕ must increase in a way that $ds = Rd\phi = dx$, which means that $\omega = \dot{\phi} = \dot{x}/R$, ie., the tangential speed of the pulley's edge must match the speed of the string and the two masses.

$$T = \frac{1}{2}m_1\dot{x}^2 + \frac{1}{2}m_2\dot{y}^2 + \frac{1}{2}I\omega^2 = \frac{1}{2}(m_1 + m_2 + \frac{I}{R^2})\dot{x}^2$$

For the potential energy adding a constant doesn't matter, so I'll ignore C and I'll also ignore the potential energy for the pulley since that is constant as well.

$$U = m_1 g(-x) + m_2 g(-y) + const = -g(m_1 - m_2)x + const$$

So here's the Lagrangian in terms of my one generalized coordinate x:

$$\mathcal{L} = \frac{1}{2}(m_1 + m_2 + \frac{I}{R^2})\dot{x}^2 + (m_1 - m_2)gx$$

Lagrange equation:

$$\frac{\partial \mathcal{L}}{\partial x} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{x}} \quad (1)$$

$$(m_1 - m_2)g = \frac{d}{dt}(m_1 + m_2 + \frac{I}{R^2})\dot{x} \quad (2)$$

$$(m_1 - m_2)g = (m_1 + m_2 + \frac{I}{R^2})\ddot{x} \quad (3)$$

$$\frac{m_1 - m_2}{m_1 + m_2 + \frac{I}{R^2}}g = \ddot{x} \quad (4)$$

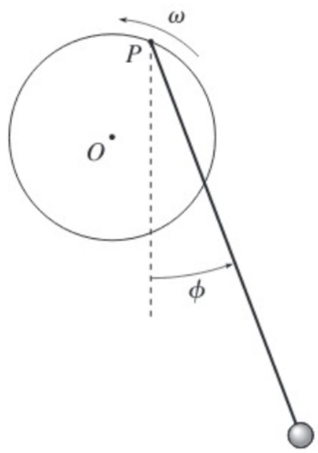
As expected, acceleration is constant, has the right units, and shows the right behavior in various limiting cases (e.g., equal masses: no acceleration, massless pulley: $\ddot{x} = \frac{m_1 - m_2}{m_1 + m_2}g$, etc.)

(b) Recall that we have called the Lagrangian equation of motion a generalization of Newton's 2nd Law, where $\vec{F} = \frac{d}{dt}\vec{p}$ turns into a "generalized force equals the time derivative of a generalized momentum". What are generalized force and generalized momentum in this case? Can they be interpreted physically? (You might also consider the simpler case of the massless pulley where $I = 0$.)

Solution: We can see that the generalized force is $(m_1 - m_2)g$, and that has the units of a usual force, and it is essentially the net force on the two masses where gravity forces partially balance each other and only the difference is what makes things move. The generalized momentum is $m_1 + m_2 + \frac{I}{R^2}\dot{x}$, which in the case of $I = 0$ may be more easily interpreted as the total momentum of the two masses both moving together with the same speed – and for $I \neq 0$, this takes into account that the pulley itself moves, too (rotationally).

2. *Pendulum attached to a wheel*

The figure below shows a simple pendulum (mass m, length l) whose point of support P is attached to the edge of a wheel (center O, radius R) that is forced to rotate at a fixed angular velocity ω. At $t = 0$, the point P is level with O on the right.



Write down the Lagrangian and find the equation of motion for the angle ϕ.

Hint: Be careful when writing down the kinetic energy T. A safe way to get to the velocity is to write down the position of the bob at time t and then differentiate.

Check that your answer makes sense in the special case that $\omega = 0$.

Solution: The position of the support P can be written down in Cartesian coordinates easily (origin is at O):

$$x_P = R\cos\omega t, \qquad y_P = R\sin\omega_t$$

. The position of the pendulum relative to P is $\Delta x = l\sin\phi$ and $\Delta y = -l\cos\phi$. So together that gives the position of the bob as

$$x = R\cos\omega t + l\sin\phi, \qquad y = R\sin\omega t - l\cos\phi$$

We'll need the velocity, too:

$$\dot{x} = -R\omega\sin\omega t + l\dot{\phi}\cos\phi, \qquad \dot{y} = R\omega\cos\omega_t + l\dot{\phi}\sin\phi$$

Since potential energy is $U = mgy$, we can now write down the Lagrangian:

$$\begin{aligned}
\mathcal{L} &= \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) - mgy \\
&= \frac{1}{2}m(R^2\omega^2\sin^2\omega t - 2R\omega l\dot{\phi}\sin\omega t\cos\phi + l^2\dot{\phi}^2\cos^2\phi \\
&\qquad R^2\omega^2\cos^2\omega t + 2R\omega l\dot{\phi}\cos\omega t\sin\phi + l^2\dot{\phi}^2\sin^2\phi) - mg(R\sin\omega t - l\cos\phi) \\
&= \frac{1}{2}m(R^2\omega^2 + l^2\dot{\phi}^2 + 2R\omega l\dot{\phi}(-\sin\omega t\cos\phi + \cos\omega t\sin\phi)) - mg(R\sin\omega t - l\cos\phi) \\
&= \frac{1}{2}m(R^2\omega^2 + l^2\dot{\phi}^2 + 2R\omega l\dot{\phi}\sin(\phi - \omega t)) - mg(R\sin\omega t - l\cos\phi)
\end{aligned}$$

Where I used the addition theorem $\sin(\alpha+\beta) = \sin\alpha\cos\beta + \sin\beta\cos\alpha$.

Now we're ready for the Lagrange equation:

$$\frac{\partial \mathcal{L}}{\partial \phi} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\phi}} \tag{5}$$

$$mR\omega l\dot{\phi}\cos(\phi-\omega t) - mgl\sin\phi = \frac{d}{dt}(ml^2\dot{\phi} + mR\omega l\sin(\phi-\omega t)) \tag{6}$$

$$R\omega l\dot{\phi}\cos(\phi-\omega t) - gl\sin\phi = \frac{d}{dt}(l^2\dot{\phi} + R\omega l\sin(\phi-\omega t)) \tag{7}$$

$$R\omega l\dot{\phi}\cos(\phi-\omega t) - gl\sin\phi = l^2\ddot{\phi} + R\omega l\cos(\phi-\omega t)(\dot{\phi}-\omega) \tag{8}$$

$$-gl\sin\phi = l^2\ddot{\phi} + R\omega l\cos(\phi-\omega t)(-\omega) \tag{9}$$

$$-g\sin\phi + R\omega^2\cos(\phi-\omega t) = l\ddot{\phi} \tag{10}$$

Fortunately, we do not have to solve this equation. But we can check the case of $\omega = 0$, which should be the regular pendulum – and it is:

$$-\frac{g}{l}\sin\phi = \ddot{\phi}$$