PHYS 615 – Activity 4.3: Conservative Forces, Potential Energy, continued

1. *Potential Energy of Gravity on Earth*

   On Earth's surface, we now that the force of gravity is $F_G = mg(-\hat{y})$ if our coordinate system chosen such that the y directions points vertically up.

   We have learned (though not proved) that the condition for the work integral to be path independent is that the force field is irrational, ie., $\nabla \times \vec{F} = 0$, where the "del" (a.k.a. "nabla") operator is

$$\nabla \equiv \hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z}$$

   (a) Let's check that the curl of $\vec{F}_G$ is indeed zero. We'll make our life a little easier by assuming we live in a 2-d (x-y) world, so we only need to check the z component of the curl.

   Write our the z component of the curl of a force $\vec{F}$ in Cartesian coordinates, where $\vec{F}$ is known in components, that is $\vec{F} = F_x\hat{x} + F_y\hat{y}$.

   (b) Now plug in the force of gravity $\vec{F}_G$ specifically and show that the z component of the curl is indeed zero.

   (c) Show that the potential energy belonging the the force of gravity above (which we now know exists, since the force of gravity depends only on $\vec{r}$ (in fact, not even that), and because its curl vanishes) is

$$U_G = mgy + C$$

   where we typically pick the constant of integration C such that the potential energy is zero at a height of $y = 0$.

   (d) Given $U_G = mgy$, derive the corresponding force from the relation $F = -\nabla U$.

2. *Potential Energy for Newton's Law of Gravity*

   Away from Earth's surface, we now that the force that mass M (you can consider it to be located at the origin) exerts on mass m at position $\vec{r}$ is given by Newton's Law of Gravity

$$\vec{F}_G = -G\frac{mM}{r^2}\hat{r}$$

   (a) Using our usual Cartesian coordinate system, show that the x component of F_G is given by

$$F_{G,x} = -G\frac{mM}{r^2}\frac{x}{r} = -GmM\frac{x}{r^3}$$

   (In order to do so, you may need to remember that $\hat{r}$ is the same direction is $\vec{r}$, but it has been shortened by some factor so that it's of unit length.)

   (b) Write the magnitude r of $\vec{r} = x\hat{x} + y\hat{y} + z\hat{z}$ in terms of its components x, y, z.

   (c) Show that

$$\frac{\partial r}{\partial x} = \frac{x}{r}$$

   (d) Show that the z component of $\nabla \times \vec{F}_G$ equals zero.

   (e) At this point, one could redo this for the x and y components of the curl, but it's essentially the same calculation. Given that we have now shown that $\nabla \times F_G = 0$ and $F_G = F_G(\vec{r})$, Newton's force of gravity is conservative. So we can calculate the corresponding potential energy

$$U_G = -\int_{\vec{r}_0}^{\vec{r}} \vec{F}_G(\vec{r}') \cdot d\vec{r}'$$

Do so and show that the result is

$$U_G = -GmM\left(\frac{1}{r} - \frac{1}{r_0}\right)$$

In order to make things a bit easier, let's first assume $\vec{r}$ and $\vec{r}_0$ are along the same direction, so $\vec{r} = r\hat{r}$ and $\vec{r}_0 = r_0\hat{r}$. Since we now that F_G is conservative you can choose any path you want, ideally a simple one ;)

(f) Generally speaking, $\vec{r}$ and $\vec{r}_0$ may not be along the same direction. Show that your result above still holds in this case.

Hint: You can pick any path you want, so you might want to pick one that has two parts: The part you've already done, and a missing piece, where the integral is easy to calculate.

(g) For Newton's Law of Gravity (and Coulomb's Law), the reference location is usually put at $r_0 = \infty$. Show that the potential energy is then

$$U_G = -G\frac{mM}{r}$$

(h) Finally, let's think about $\vec{F} = -\nabla U_G$. What do you expect do get? Do the calculation just for one component, say the x component F_x.