

## PHYS 615 – Activity 7.4: More Constrained Lagrangians

Here's what the text says about solving constrained motion problems the Lagrangian way:

1. Write down the kinetic and potential energies, and hence the Lagrangian  $\mathcal{L} = T - U$ , using any convenient inertial reference frame.
2. Choose a convenient set of  $n$  generalized coordinates  $q_1, q_2, \dots, q_n$ , and find expressions for the original coordinates of step 1 in terms of your chosen generalized coordinates. (Steps 1 and 2 can be done in either order.)
3. Rewrite  $\mathcal{L}$  in terms of  $q_1, q_2, \dots, q_n$  and  $\dot{q}_1, \dot{q}_2, \dots, \dot{q}_n$ .
4. Write down the  $n$  Lagrange equations.

### 1. *The Atwood Machine (with a real pulley)*

Note: Example 7.3 in the text is the Atwood machine with a massless pulley, so this problem is an extension of it.

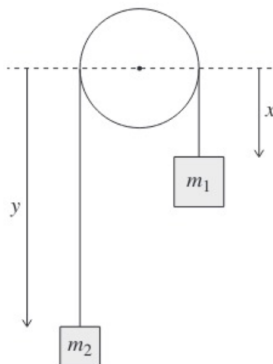


Figure 7.6 An Atwood machine consisting of two masses,  $m_1$  and  $m_2$ , suspended by a massless inextensible string that passes over a massless, frictionless pulley of radius  $R$ . Because the string's length is fixed, the position of the whole system can be specified by a single variable, which we can take to be the distance  $x$ .

As shown in this figure, the Atwood machine has two masses connected by a string that runs over a pulley. The question for today is, what is the acceleration of the two masses?

This problem can be solved (and you've probably done at least one way before) using Newton's Laws, or conservation of energy. But, unsurprisingly, the goal is to solve it the Lagrangian way.

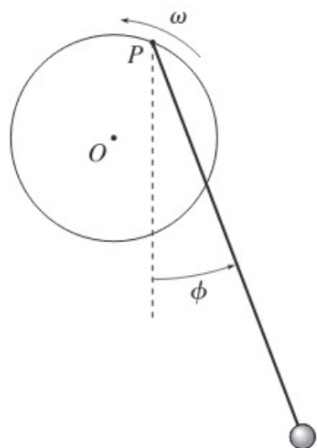
- (a) Follow the standard process to find the Lagrangian and find the equation of motion, which should give you the acceleration we're looking to find.

Note: The kinetic energy of a pulley with inertia  $I$  (assumed to be known) is  $\frac{1}{2}I\omega^2$ , where  $\omega$  is the pulley's angular velocity.

- (b) Recall that we have called the Lagrangian equation of motion a generalization of Newton's 2nd Law, where  $\vec{F} = \frac{d}{dt}\vec{p}$  turns into a "generalized force equals the time derivative of a generalized momentum". What are generalized force and generalized momentum in this case? Can they be interpreted physically? (You might also consider the simpler case of the massless pulley where  $I = 0$ .)

## 2. Pendulum attached to a wheel

The figure below shows a simple pendulum (mass  $m$ , length  $l$ ) whose point of support  $P$  is attached to the edge of a wheel (center  $O$ , radius  $R$ ) that is forced to rotate at a fixed angular velocity  $\omega$ . At  $t = 0$ , the point  $P$  is level with  $O$  on the right.



Write down the Lagrangian and find the equation of motion for the angle  $\phi$ .

Hint: Be careful when writing down the kinetic energy  $T$ . A safe way to get to the velocity is to write down the position of the bob at time  $t$  and then differentiate.

Check that your answer makes sense in the special case that  $\omega = 0$ .