PHYS 615 – Activity 7.3: Constrained Lagrangian

Here's what the text says about solving constrained motion problems the Lagrangian way:

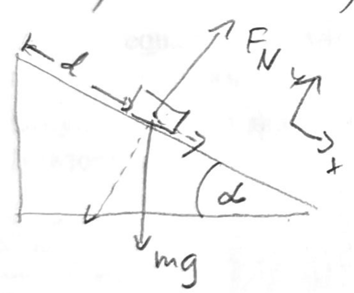
1. Write down the kinetic and potential energies, and hence the Lagrangian $\mathcal{L} = T - U$, using any convenient inertial reference frame.
2. Choose a convenient set of n generalized coordinates $q_1, q_2, \ldots q_n$, and find expressions for the original coordinates of step 1 in terms of your chosen generalized coordinates. (Steps 1 and 2 can be done in either order.)
3. Rewrite $\mathcal{L}$ in terms of $q_1, q_2, \ldots q_n$ and $\dot{q}_1, \dot{q}_2, \ldots \dot{q}_n$.
4. Write down the n Lagrange equations.

1. *The good old inclined plane*

   (a) A block that is initially at rest is sliding down a distance d on frictionless plane. The plane is inclined over the horizontal at an angle of α. Use Newton's 2nd Law and Kinematics to find a formula for how long that takes.

   First, lay out the steps you will take to get an answer, and then actually follow your plan. By the way, it may be worth noticing that one of the intermediate steps here is also to find the "equation of motion", though you may not quite realize that this is what you've been doing all along.

FBD, 2nd Law, find a, do Kinematics.



"y": $0 = ma_y = F_N - mg\cos\alpha$

"x": $0 = ma = mg\sin\alpha$

$\Rightarrow a = g\sin\alpha$

equation of motion

$d = \frac{1}{2}at^2 + \cancel{v_0 t}$

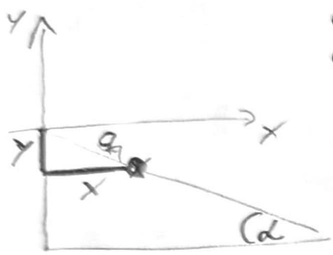
$t = \sqrt{\frac{2d}{a}}$

(b) Let's do it the Lagrangian way. Write down the Lagrangian $\mathcal{L}(x, y, \dot{x}, \dot{y}, t)$ for the block, where x and y are the relevant horizontal and vertical direction, respectively (The problem is 2-d to start with, so we'll ignore z).

$$\mathcal{L} = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) - mgy$$

(c) Since the motion of the block is constrained to be on the inclined plane, let's change to a new variable q_1 which measures the distance down the plane that the particle has moved.

Express x and y in terms of q_1. Given those equations, also find $\dot{x}$ and $\dot{y}$. Use these equations to rewrite your Lagrangian $\mathcal{L}(q_1, \dot{q}_1, t)$ to be in terms of the generalized coordinate q_1, which means the constraint is automatically satisfied.



$$y = -q_1 \sin\alpha \qquad x = q_1 \cos\alpha$$

$$\dot{y} = -\dot{q}_1 \sin\alpha \qquad \dot{x} = \dot{q}_1 \cos\alpha$$

$$\mathcal{L} = \frac{1}{2}m(\dot{q}_1^2 \cos^2\alpha + \dot{q}_1^2 \sin^2\alpha) + mgq_1 \sin\alpha$$

$$= \frac{1}{2}m\dot{q}_1^2 + mgq_1 \sin\alpha$$

(d) Find the equation of motion from the Lagrangian, and solve it so that you can answer the actual question this problem is looking for (how long...?).

Do your equation and motion and final answer match what you got doing things the Newtonian way?

$$\frac{\partial \mathcal{L}}{\partial q_1} = mg\sin\alpha = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_1} = m\ddot{q}_1$$

Exactly the same equation of motion, so I don't have to redo $t = \sqrt{\frac{2d}{g}}$

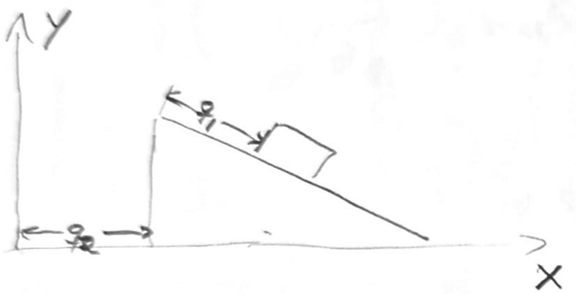
2. *A block sliding on a sliding wedge*

This problem is quite similar to what we just did – a block is sliding down a frictionless wedge. However, that wedge is not fixed, but can itself slide (frictionless) on a horizontal table. The question is going to be the same – How long does it take for the block to slide down a distance d (starting from rest).

(a) If you're up for it, you could try solving this problem the Newtonian way. But really, what we want to do use the Lagrangian approach, and we'll be able to reuse quite a bit of what we just did.

There are now two things moving – the wedge and the block. Let's call the mass of the wedge M (the block's mass is m). Write down the Lagrangian in terms of regular Cartesian coordinates (x_2, y_2) for the wedge and (x_1, y_1) for the block (and their time derivatives).

$$\mathcal{L} = \frac{1}{2} m (\dot{x}_1^2 + \dot{y}_1^2) + \frac{1}{2} M (\dot{x}_2^2 + \dot{y}_2^2) - m g y_1$$



(b) There are two degrees of freedom in the system – the wedge can move horizontally left or right, and the block can slide up or down the wedge. We can use the same q_1 to indicate how far down the block has moved. And then we'll use q_2 to be the horizontal distance that the wedge itself has moved.

As before, the next step is to write down the coordinates of the block and the wedge in terms of q_1, and q_2. Then take the time derivative to find $\dot{x}_1, \dot{y}_1, \dot{x}_2, \dot{y}_2$, too. Use those to find the Lagrangian in terms of your generalized coordinates.

$$x_2 = q_2 \quad y_2 = 0 \qquad x_1 = q_2 + q_1 \cos\alpha \quad y_1 = -q_1 \sin\alpha$$

$$\dot{x}_2 = \dot{q}_2 \quad \dot{y}_2 = 0 \qquad \dot{x}_1 = \dot{q}_2 + \dot{q}_1 \cos\alpha \quad \dot{y}_1 = -\dot{q}_1 \sin\alpha$$

$$\mathcal{L} = \frac{1}{2} m (\dot{q}_2^2 + 2\dot{q}_2\dot{q}_1 \cos\alpha + \dot{q}_1^2 \cos^2\alpha + \dot{q}_1^2 \sin^2\alpha)$$
$$+ \frac{1}{2} M \dot{q}_2^2 + m g q_1 \sin\alpha$$

(c) Find the equations of motion for $q_1(t)$ and $q_2(t)$. This will give two coupled ODEs – but one can use one to eliminate, say, q_2 from the other, so that one ends up with a single differential equation for q_1 – and it is one that's easy to solve.

$$\frac{\partial \mathcal{L}}{\partial q_2} = 0 = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_2} = \frac{d}{dt}\underbrace{\left(m\dot{q}_1 \cos\alpha + m\dot{q}_2 + M\dot{q}_2\right)}_{\text{total momentum (conserved)}}$$

$$0 = (m+M)\ddot{q}_2 + m\ddot{q}_1\cos\alpha \quad (1)$$

$$\frac{\partial \mathcal{L}}{\partial q_1} = mg\sin\alpha = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_1} = \frac{d}{dt}\left(m\dot{q}_2\cos\alpha + m\dot{q}_1\right)$$

$$g\sin\alpha = \ddot{q}_2\cos\alpha + \ddot{q}_1 \quad (2)$$

$$(1) \Rightarrow \ddot{q}_2 = -\frac{m}{M+m}\cos\alpha\,\ddot{q}_1$$

plug into (2):

$$g\sin\alpha = \left(-\frac{m}{M+m}\cos^2\alpha + 1\right)\ddot{q}_1$$

$$\Rightarrow \ddot{q}_1 = \frac{g\sin\alpha}{1 - m\dfrac{\cos^2\alpha}{M+m}}$$

(d) In the process, you presumably have found an equation for the acceleration (down the plane) of the block:

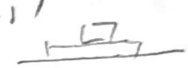
$$\ddot{q}_1 = \frac{g \sin \alpha}{1 - \frac{m \cos^2 \alpha}{m+M}}$$

Let's do some checks and limiting cases.

- Are the units correct?
- Limiting case where one should have a pretty good idea of what the expected behavior is: $\alpha \to 0$, $\alpha \to 90°$, $M \to \infty$, $M \to 0$. For each case, think about what behavior you'd expect, and then work out the answer you get from the equation above in the respective limit. Do things make sense?

Yes, units of $\ddot{q}_1$ are units of g, which is m/s^2

- $\alpha \to 0$ $\quad \ddot{q}_1 = 0$ (horizontal wedge, nothing happens!)



- $\alpha \to 90°$ $\quad \sin\alpha \to 1$, $\cos\alpha \to 0$, so $\ddot{q}_1 = g$ ✓

(free fall)

- $M \to \infty$: wedge doesn't really move:

$$\ddot{q}_1 = \frac{g \sin\alpha}{1 - \underbrace{\frac{m \cos^2\alpha}{m+M}}_{\to 0}} = g\sin\alpha$$ ✓ (usual inclined plane)

- $M \to 0$: wedge will just easily get out of the way:

$$\ddot{q}_1 = \frac{g \sin\alpha}{1 - \frac{m\cos^2\alpha}{m}} = g$$ just free fall

✓