PHYS 615 – Activity 4.1: Kinetic Energy

1. The dot product

   (a) There are a number of ways of computing the dot product of two vectors. The first gives the dot product in terms of the vectors' magntitudes and the angle in between:

   $$\vec{a} \cdot \vec{b} = ab\cos\theta$$

   where a is the magnitude of $\vec{a}$, ie, $a = |\vec{a}|$ and same for b (which is the notation we've been using quite a bit already), and θ is the angle between the two vectors.

   A second way is convenient of the vectors' components are known, ie., $\vec{a} = a_x\hat{x} + a_y\hat{y} + a_z\hat{z}$ and similarly for $\vec{b}$. In that case,

   $$\vec{a} \cdot \vec{b} = a_x b_x + a_y b_y + a_z b_z$$

   Show that the second way gives the same result as the first. In order to do so, given two vectors $\vec{a}$ and $\vec{b}$ with an angle of θ in between, choose your coordinate system such that the x axis points in the direction of $\vec{a}$, which makes it easy to write $\vec{a}$ in components. Then choose the y axis such that $\vec{b}$ lies in the $x - y$ plane. What's the angle between $\vec{b}$ and the x axis? Use that to write $\vec{b}$ in components. Now that you have the components, use the 2nd way to calculate the dot product, and compare to the 1st formula.

   Solution: Given the choice of x axis, $\vec{a}$ has only an x component: $\vec{a} = a\hat{x}$. Given that the angle between $\vec{b}$ and the x axis is θ, SOCAHTOA gives $\vec{b} = b\cos\theta\hat{x} + b\sin\theta\hat{y}$. So

   $$\vec{a} \cdot \vec{b} = a_x b_x + a_y b_y + a_z b_z = ab\cos\theta + 0\, b\sin\theta + 0 = ab\cos\theta$$

   which is indeed the same as the first way.

   (b) Show that $a^2 = \vec{a} \cdot \vec{a}$. (Again, a is the magnitude of $\vec{a}$.) This can be done using either of the two ways of calculating the dot product shown above.

   Solution: First way: $\vec{a} \cdot \vec{a} = aa\cos 0 = a^2$, since the angle between a vector and itself is 0 degrees (they point in the same direction.)

   Second way: $\vec{a} \cdot \vec{a} = a_x a_x + a_y a_y + a_z a_z = a_x^2 + a_y^2 + a_z^2 = a^2$ by way of the Pythagorean Theorem.

(c) Show that a product rule holds for the derivative $\frac{d}{dt}(\vec{a} \cdot \vec{b})$ of the dot product (you can use the dot product from calculus, where we know it's true for scalar functions). $\vec{a}$ and $\vec{b}$ are considered to be functions of time.

Solution:

$$\frac{d}{dt}(\vec{a} \cdot \vec{b}) = \frac{d}{dt}\sum_i a_i b_i = \sum_i \frac{d\, a_i b_i}{dt} = \sum_i (\dot{a}_i b_i + a_i \dot{b}_i) = \sum_i \dot{a}_i b_i + \sum_i a_i \dot{b}_i = \dot{\vec{a}} \cdot \vec{b} + \vec{a} \cdot \dot{\vec{b}}$$

where I used the sum notation to save myself some typing (not sure that really worked out...)

(d) Show that $\frac{d}{dt}v^2 = 2\vec{v} \cdot \dot{\vec{v}}$, where again $\vec{v}$ is a function of time $\vec{v} = \vec{v}(t)$. How does that compare to what the chain rule would give if v was simply a scalar function?

Solution:

$$\frac{d}{dt}v^2 = \frac{d}{dt}(\vec{v} \cdot \vec{v}) = \dot{\vec{v}} \cdot \vec{v} + \vec{v} \cdot \dot{\vec{v}} = 2\vec{v} \cdot \dot{\vec{v}}$$

This looks very similar to what we'd usually get with the chain rule for scalar functions – derivative of v^2 is $2v$, multiplied by the inner derivative $\dot{v}$.

2. Taylor Sec. 4.1 derived the (infinitesimal) change in kinetic energy (which we now call T) to be

$$dT = \vec{F}_{net} \cdot d\vec{r} \tag{1}$$

The term on the right is what we call *"work"* done by the force $\vec{F}_{net}$ over a displacement of $d\vec{r}$.

Integrating both sides, we get the *work-energy theorem*:

$$\Delta T = \int_1^2 \vec{F}_{net} \cdot d\vec{r} \equiv W_{net} \tag{2}$$

Since the net force is the sum of all forces acting on an object, we can correspondingly split up the net work into the sum of the work done by each respective force $W = \int \vec{F} \cdot d\vec{r}$.

(a) Does gravity always do work whenever an object moves from some position 1 to position 2, or can it be zero?

Solution: It can be zero. E.g., a cart rolling horizontally: The displacement is perpendicular to the force of gravity, hence the dot product is zero.

(b) Describe a situation where gravity does positive work. How does the speed of the object change in this case?

Solution: Dropping a ball from rest. Gravity is down, displacement is down. The ball will speed up (it better do so, kinetic energy must be increasing.)

(c) Describe a situation where gravity does negative work. How does the speed of the object change in this case?

Solution: Throwing a ball up into the sky. Gravity is down, displacement is up (or maybe up at an angle). The ball will slow down (it better, kinetic energy must be decreasing.)

(d) There are two situations where work is easier to calculate than having to do an actual line integral:

$$W = \vec{F} \cdot \Delta r \qquad \text{if the force is constant} \tag{3}$$

$$W = \int_{x_1}^{x_2} F_x dx \qquad \text{if the problem is 1-d} \tag{4}$$

Justify these two simpler formulas.

Solution: If the force is constant, it can be pulled out of the integral:

$$\int_1^2 \vec{F} \cdot d\vec{r} = \vec{F} \cdot \int_1^2 d\vec{r} = \vec{F} \cdot (\vec{r}_2 - \vec{r}_1) = \vec{F} \cdot \Delta\vec{r}$$

If the problem is 1-d, $\vec{F} = F_x \hat{x}$, and the integral must go from x_1 to x_2:

$$\int_1^2 \vec{F} \cdot dr = \int_{x_1}^{x_2} F_x \hat{x} \cdot dx \hat{x} = \int_{x_1}^{x_2} F_x (\hat{x} \cdot \hat{x}) dx = \int_{x_1}^{x_2} F_x dx$$

3. Line integrals

   Evaluate the work done

$$W = \int_O^P \vec{F} \cdot d\vec{r} = \int_O^P (F_x dx + F_y dy) \tag{5}$$

   by the two-dimensional force $\vec{F} = x^2\hat{x} + 2xy\hat{y}$ along three paths starting at the origin O and ending at point $P = (1,1)$, similarly to Taylor's Fig. 4.24(a).

   (a) The path goes along the x axis to $Q = (1,0)$ and then straight up to P.

   Solution:

$$\begin{aligned} W &= \int_O^Q \vec{F} \cdot d\vec{r} + \int_Q^P \vec{F} \cdot d\vec{r} \\ &= \int_0^1 x^2 dx + \int_0^1 2 \cdot 1 \cdot y\, dy \\ &= \frac{1}{3} + \frac{1}{2}2 = \frac{4}{3} \end{aligned}$$

   (b) The path goes along the y axis to $R = (0,1)$ and then straight over to P.

   Solution:

$$\begin{aligned} W &= \int_O^R \vec{F} \cdot d\vec{r} + \int_R^P \vec{F} \cdot d\vec{r} \\ &= \int_0^1 2 \cdot 0 \cdot y\, dy + \int_0^1 x^2 dx \\ &= 0 + \frac{1}{3} = \frac{1}{3} \end{aligned}$$

   (c) The path is described by $y = x^2$, and because of that, you can replace $dy = 2xdx$ and convert the whole integral to an integral over x.

   Solution:

$$\begin{aligned} W &= \int_O^P \vec{F} \cdot d\vec{r} \\ &= \int_0^1 (x^2 dx + 2xx^2\, 2xdx) \\ &= \int_0^1 (x^2 + 4x^4) dx \\ &= \frac{1}{3} + \frac{4}{5} = \frac{17}{15} \end{aligned}$$

(d) The path is described by parameterically as $x = t^3$, $y = t^2$. In this case, rewrite x, y, dx, dy in terms of t and dt, and convert the integral to an integral over t.

Solution:

$$
\begin{aligned}
W &= \int_O^P \vec{F} \cdot d\vec{r} \\
&= \int_0^1 ((t^3)^2(3t^2)dt + 2t^3t^2(2tdt)) \\
&= \int_0^1 (3t^8 + 4t^6)dt \\
&= \frac{3}{9} + \frac{4}{7} = \frac{19}{21}
\end{aligned}
$$