PHYS 615 – Activity 7.1: Unconstrained Lagrangian

The **Lagrangian** $\mathcal{L}$ is defined as

$$\mathcal{L} = T - U \tag{1}$$

were T is kinetic energy and U is potential energy.
Hamilton's Principle: The actual path which a particle follows between two points 1 and 2 in a given time interval t_1 to t_2 is such that the action integral

$$S = \int_{t_1}^{t^2} \mathcal{L} dt \tag{2}$$

is stationary when taken along the actual path.
This in turns means (by calculus of variations) that the **Euler-Lagrange equations** hold, where $q_1, q_2, \ldots$ are *generalized coordinates* and $i = 1, 2, \ldots$:

$$\frac{\partial \mathcal{L}}{\partial q_i} = \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{q}_i} \tag{3}$$

1. *Projectile motion*

   Write down the Lagrangian for a projectile (neglect air resistance) in terms of Cartesian coordinates with z being the upward vertical direction. Find the three Euler-Lagrange equations and show that they are exactly what you'd expect for the equations of motion.

   Solution:

   $$\mathcal{L} = T - U = \frac{1}{2} m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - mgz$$

   The (Euler-) Lagrange equations are

   $$\begin{aligned} 0 &= \frac{\partial \mathcal{L}}{dx} = \frac{d}{dt} \frac{\partial \mathcal{L}}{d\dot{x}} = \frac{d}{dt}(m\dot{x}) = m\ddot{x} \\ 0 &= \frac{\partial \mathcal{L}}{dy} = \frac{d}{dt} \frac{\partial \mathcal{L}}{d\dot{y}} = \frac{d}{dt}(m\dot{y}) = m\ddot{y} \\ -mg &= \frac{\partial \mathcal{L}}{dz} = \frac{d}{dt} \frac{\partial \mathcal{L}}{d\dot{z}} = \frac{d}{dt}(m\dot{z}) = m\ddot{z} \end{aligned}$$

   These are exaclty the equations of motion I know from Newton's 2nd Law $mg(-\hat{z}) = \vec{F}_G = \vec{F}_{net} = m\vec{a} = m\ddot{\vec{r}}$.

2. *Harmonic oscillator*

   Write down the Lagrangian for a one-dimensional particle moving along the x direction subject to a force $F = -kx$ (where k is a positive constant.) Find the Euler-Lagrange equation and solve it.

   Solution: I remember the potential energy for a spring to be $U_{spring} = \frac{1}{2}kx^2$, though if I did not, I'd just integrate $-F$ over space and get just that.

   $$\mathcal{L} = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}kx^2$$

   Equation of motion:

   $$-kx = \frac{\partial \mathcal{L}}{dx} = \frac{d}{dt}\frac{\partial \mathcal{L}}{d\dot{x}} = \frac{d}{dt}(m\dot{x}) = m\ddot{x}$$

   So I end up with the differential equation $\ddot{x} = -\omega_0^2 x$ with $\omega_0 = \sqrt{k/m}$, and we've solved it enough times that I can just write down the solution: $x(t) = A\cos\omega t + B\sin\omega t$.

3. *Inclined plane*

   Consider a mass m moving on a frictionless plane that's inclined at an angle α over the horizontal. Write down the Lagrangian in terms of coordinates x measured horizontally across the slope and y measured down the slope. (Treat the system as two-dimensional, but include gravitational potential energy.) Find the two Euler-Lagrange equations and show that they are what you should have expected.

   Solution: Kinetic energy is straight forward, but for the gravitational potential energy, we need to know the height of the particle, which is $z = -y\sin\alpha$ (I'm using positive z being the vertically up direction.) So $U = mg(-y\sin\alpha)$.

   $$\mathcal{L} = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) + mgy\sin\alpha$$

   Equations of motion:

   $$\begin{aligned} 0 &= \frac{\partial \mathcal{L}}{dx} = \frac{d}{dt}\frac{\partial \mathcal{L}}{d\dot{x}} = \frac{d}{dt}(m\dot{x}) = m\ddot{x} \\ mg\sin\alpha &= \frac{\partial \mathcal{L}}{dy} = \frac{d}{dt}\frac{\partial \mathcal{L}}{d\dot{y}} = \frac{d}{dt}(m\dot{y}) = m\ddot{y} \end{aligned}$$

   There is no force in the horizontal (x) direction, as I expected, and down the slope we get the usual $g\sin\alpha$ for the acceleration, which also makes sense, including in the limits of $\alpha = 0$ and $90°$.

4. *Polar coordinates*

Polar coordinates (r, ϕ), are defined by the transformation

$$x = r\cos\phi$$
$$y = r\sin\phi$$

(a) Show that the kinetic energy in polar coordinates is

$$T = \frac{1}{2}m(\dot{r}^2 + (r\dot{\phi})^2)$$

Solution: I could say that this is immediately obvious since radial velocity is $v_r = \dot{r}$ and tangential velocity is $v_t = r\dot{p}hi$. But I can also find it from the given coordinate transform:

$$\dot{x} = \dot{r}\cos\phi - r\dot{p}hi\sin\phi$$
$$\dot{y} = \dot{r}\sin\phi + r\dot{p}hi\cos\phi$$

So

$$\begin{aligned} T &= \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) \\ &= \frac{1}{2}m(\dot{r}^2\cos^2\phi - 2\dot{r}r\dot{\phi}\cos\phi\sin\phi + r^2\dot{\phi}^2\sin^2\phi + \\ &\qquad \dot{r}^2\sin^2\phi + 2\dot{r}r\dot{\phi}\sin\phi\cos\phi + r^2\dot{\phi}^2\cos^2\phi) \\ &= \frac{1}{2}m(\dot{r}^2 + (r\dot{\phi})^2) \end{aligned}$$

(b) Show that in polar coordinates

$$d\vec{r} = dr\,\hat{r} + rd\phi\,\hat{\phi}$$

This can be done using calculus, or by drawing some pictures.

Solution: Using the (multi-variate) chain rule: (for a differential by itself – if that's not familiar you can just divide all the differentials by, say dt and it'll look more familiar.)

$$d\vec{r} = \frac{\partial\vec{r}}{\partial r}dr + \frac{\partial\vec{r}}{\partial\phi}d\phi$$

Writing our vectors in Cartesian coordinates, the first partial derivative is straightforward:

$$\frac{\partial\vec{r}}{\partial r} = \frac{\partial}{\partial r}\begin{pmatrix} r\cos\phi \\ r\sin\phi \end{pmatrix} = \begin{pmatrix} \cos\phi \\ \sin\phi \end{pmatrix} = \hat{r}$$

because that last column vector is already normalized (has magnitude 1).

$$\hat{r} = \frac{\vec{r}}{r} = \begin{pmatrix} \cos\phi \\ \sin\phi \end{pmatrix}$$

Similarly,

$$\frac{\partial \vec{r}}{\partial \phi} = \frac{\partial}{\partial \phi}\begin{pmatrix} r\cos\phi \\ r\sin\phi \end{pmatrix} = \begin{pmatrix} -r\sin\phi \\ r\cos\phi \end{pmatrix} = r\hat{\phi}$$

where this time the column vector wasn't normalized, but it's easy to see that its magnitude is r, so it can be written as r times unit vector.

Plugging these partial derivatives back into the chain rule, we do get the desired result.

(c) Show that in polar coordinates

$$\nabla f = \frac{df}{dr}\hat{r} + \frac{1}{r}\frac{df}{d\phi}\hat{\phi}$$

This can be done by computing df in two ways, and setting them equal:
(1) $df = \nabla f \cdot d\vec{r}$
(2) Writing down df using the generic multi-dimensional chain rule.

Solution: First (1):

$$df = \nabla f \cdot d\vec{r} = ((\nabla f)_r \hat{r} + (\nabla f)_\phi \hat{\phi}) \cdot (dr\,\hat{r} + rd\phi\,\hat{\phi}) = (\nabla f)_r dr + (\nabla f)_\phi r d\phi$$

Then (2):

$$df = \frac{\partial f}{\partial r}dr + \frac{\partial f}{\partial \phi}d\phi$$

Comparing the two expressions, we can see the r and ϕ components of ∇f to be exactly what we were supposed to show.

5. Show that angular momentum is conserved for a single particle (e.g., a planet) that is subject to (only) a central force. Do so in polar coordinates, where $U = U(r,\phi) = U(r)$ – the last equality holds because of our assumption that the force is central (why?).

   Write down the Lagrangian, and, using the Euler-Lagrange equations, find the equations of motion – one of which should show directly that angular momentum is conserved (it might be a good idea to write down angular momentum $l_z = (\vec{r} \times \vec{p})_z$ in polar coordinates first.)

   Solution: First of all, angular momentum as we know is $\vec{l} = \vec{r} \times \vec{p}$, so $l_z = rmv\sin_\phi = mrv_t = mr^2\omega$.

   Here's the Lagrangian:

$$\mathcal{L} = \frac{1}{2}m(\dot{r}^2 + (r\dot{\phi})^2) - U(r)$$

   To get something conserved, it seems like a good plan to use the Lagrange equation for ϕ, since the Lagrangian doesn't depend on ϕ, that is,

$$0 = \frac{\partial \mathcal{L}}{d\phi} = \frac{d}{dt}\frac{\partial \mathcal{L}}{d\dot{\phi}}$$

   Therefore,

$$\frac{\partial \mathcal{L}}{d\dot{\phi}} = mr^2\dot{\phi} = const$$

   And that is indeed exactly the angular momentum being conserved.