PHYS 615 – Activity 7.2: Lagrangian for two Particles

Let's consider two particles, at (x_1, y_1, z_1) and (x_2, y_2, z_2), respectively. The kinetic energy is then the sum of the kinetic energies for each particle, and we've seen before that we can get the forces from a single potential $U(x_1, y_1, z_1, x_2, y_2, z_2)$.
The Lagrangian is defined just as before, but now depends on 6 coordinates (and, possibly, time):

$$\mathcal{L} = T - U = \frac{1}{2}m_1(\dot{x}_1^2 + \dot{y}_1^2 + \dot{z}_1^2) + \frac{1}{2}m_2(\dot{x}_2^2 + \dot{y}_2^2 + \dot{z}_2^2) - U(x_1, y_1, z_1, x_2, y_2, z_2) \tag{1}$$

As before, a change to generalized coordinates $q_1, q_2, \ldots, q_n$, may be advantageous, but the Euler-Lagrange equation will hold either with the original Cartesian coordinates or the new generalized coordinates:

$$\frac{\partial \mathcal{L}}{\partial q_i} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_i} \tag{2}$$

1. *Two masses and a spring*

   (a) Write down the Lagrangian $\mathcal{L}(x_1, x_2, \dot{x}_1, \dot{x}_2)$ for particles of equal mass m that move along the x-axis (only). The two particles are connected by a spring with potential energy $\frac{1}{2}kx^2$, where x is the extension of the spring $x = x_1 - x_2 - l$, l being the equilibrium length of the spring.

   (b) Rewrite $\mathcal{L}$ in terms of the new variables $X = \frac{1}{2}(x_1 + x_2)$ and x.

(c) Write down the two Euler-Lagrange equations.

(d) Solve for $X(t)$ and $x(t)$ and describe the motion. Is there a connection to what we saw about the center of mass earlier in this class?