# PHYS 615 – Quiz 4: Calculus of Variations

Name: ___________________________________

**Instructions:** You have 40 minutes to work on this quiz.
If you get stuck on one part, still try the other parts. Some parts are independent; for dependent parts, I'll give you full credit if your process is correct, even in your input from another part is incorrect.
Possibly useful physics equations:

$$\vec{F}_{net} = m\vec{a} \qquad F_{fk} = \mu_k F_N \qquad F_{fs} \leq \mu_s F_N \qquad F_G = mg \qquad \vec{F}_{kind,A\,on\,B} = -\vec{F}_{kind,B\,on\,A}$$

$$\vec{F}_{D,quad} = -cv^2\hat{v} \qquad \vec{F}_{D,lin} = -bv\hat{v}$$

$$\vec{p} = m\vec{v} \qquad \dot{\vec{p}} = \vec{F} \qquad \vec{l} = I\vec{\omega} = \vec{r} \times \vec{p} \qquad \dot{\vec{l}} = \vec{\Gamma} = \vec{r} \times \vec{F}$$

$$\vec{R}_{CM} = \frac{1}{M}\int_M \vec{r}dm$$

Possibly useful math equations:

$$\frac{d}{dx}x^n = nx^{n-1} \qquad \int x^n dx = \frac{1}{n+1}x^{n+1} \qquad \text{(for any } n \neq 0\text{, including fractions.)}$$

$S = \int_{x_1}^{x_2} f(y, y', x)dx$ is stationary with respect to variations of the path $y(x)$ iff $\frac{\partial f}{\partial y} = \frac{d}{dx}\frac{\partial f}{\partial y'}$

1. *Shortest Path*

   (a) (20 points) Show that in polar coordinates, the length of a path described by $\phi(r)$ is given by

   $$L = \int_{r_1}^{r_2} \sqrt{1 + r^2\phi'^2} dr$$

   . Make sure to show and explain your derivation.

   (b) (20 points) Derive the differential equation for $\phi(r)$ that describes the shortest path between two points in polar coordinates, $(r_1, \phi_1)$ and $(r_2, \phi_2)$. Prepare to solve this equation – one "integration" can be done trivially by realizing that if the derivative of some term is zero, then that term has to equal a constant. Find an expression for $\phi'$, ie., an equation that has $\phi'$ on the l.h.s. and an expression in terms of $r$ and constants on the r.h.s.

(c) (20 points) Describe the process for solving the differential equation you just found (but you don't need to do it). If you did not get an answer from the previous part, use this differential equation:

$$\phi' = \frac{K}{r\sqrt{r^2 - K^2}}$$

(d) (20 points) Verify that the following function is a solution to the differential equation given above.

$$r = \frac{K}{\cos(\phi - \phi_0)}$$

where $K$ and $\phi_0$ are constants. Note that I wrote it as $r(\phi)$ rather than $\phi(r)$. You could solve the equation above for $\phi$ and go from there, but as physicists we know we can just do $d\phi/dr = \frac{1}{dr/d\phi}$ and that may help.

(e) (10 points **bonus – leave for after you finished the rest**) Pick some values for $K$ and $\phi_0$ – e.g., $K = 2$ and $\phi_0 = 0$ and sketch the path described by $r(\theta)$ above. To make it a bit more general pick another $\phi_0$, say $30°$ and again sketch that path. What kind of path would you expect in the first place?

2. (20 points) Derive the differential equation that describes solutions $y(x)$ for which

$$S = \int_{x_1}^{x_2} x\sqrt{1 - y'^2}dx$$

is stationary. Since the integrand does not depend on $y$ itself, you can express it in terms of (some expression) = const. Solve that for $y'$, but you don't need to solve the actual differential equation to find $y(x)$.