PHYS 615 – Activity 4.5: Central Forces; Elastic Collisions

1. *Central Forces*

   In one dimension, it is clear that a force obeying Hooke's Law is conservative (since $F = -kx$ depends only on position, and that is sufficient to make it conservative in 1-d).

   Consider instead a spring that obeys Hooke's Law and has one end fixed at the origin, but whose other end is free to move in all three dimensions.

   (a) Write down the force $\vec{F}(\vec{r})$ exerted by the spring in terms of its length r and its equilibrium length r_0.

   Solution:

   $$\vec{F}(\vec{r}) = k(r - r_0)(-\hat{r})$$

   (b) Show that this force is conservative. (Hint: Is the force central? The spring does not bend, it can freely rotate about its attachment point at the origin.)

   Solution: Clearly, the force is central and spherically symmetric. That makes it automatically conservative.

2. *Elastic Collisions*

Let's consider the collision of two particles with equal masses $m_1 = m_2 = m$ (be it billiard balls or electrons). Particle 1 has an inital velocity of $\vec{v}_1$ and hits particle 2 which is initially at rest. After the collision, the two particles move at velocities $\vec{u}_1$ and $\vec{u}_2$, respectively. Show that the two balls move away from each other at an angle of $90°$.

(a) As we remember, momentum is conserved in all collisions (that's assuming they're quick enough that we can neglect external forces). Write down conservation of momentu before / after the collision, and simplify knowing that the masses are equal.

Solution:

$$m_1\vec{v}_1 = m_2\vec{u}_1 + m_2\vec{u}_2 \qquad \Longrightarrow \qquad \vec{v}_1 = \vec{u}_1 + \vec{u}_2$$

(b) Since the collision is elastic, that means that kinetic energy is conserved, too. Again write down what that means in terms of the velocities (as vectors) before/after the collision and simplify.

Solution:

$$\frac{1}{2}m_1\vec{v}_1^2 = \frac{1}{2}m_2\vec{u}_1^2 + \frac{1}{2}m_2\vec{u}_2^2 \qquad \Longrightarrow \qquad \vec{v}_1^2 = \vec{u}_1^2 + \vec{u}_2^2$$

(c) Square the equation you got from conservation of momentum and show that you get

$$\vec{v}_1^2 = \vec{u}_1^2 + 2\vec{u}_1 \cdot \vec{u}_2 + \vec{u}_2^2$$

Solution:

$$\vec{v}_1^2 = (\vec{u}_1 + \vec{u}_2) \cdot (\vec{u}_1 + \vec{u}_2) = \vec{u}_1^2 + 2\vec{u}_1 \cdot \vec{u}_2 + \vec{u}_2^2$$

(d) Comparing the squared equation you just got and conservation of kinetic energy, find a (simple) equation for $\vec{u}_1 \cdot \vec{u}_2$ that must be satisified for this elastic collision. What does this mean for the angle between the two velocities?

Solution: We can easily see that $\vec{u}_1 \cdot \vec{u}_2$ must vanish (be zero). Since, in general, both particles move, ie., have non-zero speeds, that means that the cosine of the angle between them must be zero, that is, they must be moving in perpendicular directions. (However, the equations are also valid for the case where particle 1 misses particle 2 – in that case particle 2 remains at rest and the dot product of the velocities is zero because $\vec{u}_2 = 0$.)