PHYS 615 – HW 6

Types of homework questions

- RQ (Reading questions): prompt you to go back to the text and read and think about the text more carefully and explain in your own words. While not directly tested in quizzes, can help you think more deeply about quiz questions.
- BF (Building foundations): gives you an opportunity to build and practice foundational skills that you have, presumably, seen before.
- TQQ (typical quiz questions): Similar questions (though perhaps longer or shorter) will be asked on quizzes. But the difficulty level and skills tested will be similar.
- Design (D): These are questions in which you are given a desired outcome and asked to figure out how to make it happen. These will often also be TQQ's, but always starting with desired motion/behavior as the given.
- COMP (Computing): computing questions often related to TQQ but will never be asked on a quiz (since debugging can take so long). You will need to do at least four computing questions over the semester
- FC (free choice): allows you to decide where to put your time. Any of the following are possible: work through a section of the text or a lecture in detail; redo a problem from before; do an unassigned problem in the text; extend a computing project; try a problem using a different analytical approach (e.g. forces instead of conservation of energy).
- ACT (in-class activity): These questions are repeats of questions (or similar to) that occurred in a previous in-class activity.
- **Standard Reading Questions**: How does the reading connect with what you already know? What was something new? Ask an "I wonder" question OR give an example applying the idea in the reading.

Please remember to say something about the "Check/Learn" part at the end of solving a problem!

Full credit will be given at 75% of the total points possible, so you can choose a subset of problems (you can do more / all, but the score is capped at 75%)

This homework contains some previous group activities. I'm including them here in order to try to help gradescope, but you can of course hand in the original paper version I handed out in class.

1. COMP (15 points) Runge-Kutta integration

   Back quite a while ago, we solved the skateboard in a half pipe problem numerically, and we noticed that even in the linear (small-angle) approximation, the amplitude seemed to grow over time quite substantially unless we used a tiny timestep: https://github.com/germasch/hw/blob/main/notebooks/euler-skateboard.ipynb

   Our next goal is going to be solving projectile motion with quadratic drag, but before we try to do so (down the road), let's try to get our code to give us a more accurate numerical approximate solution.

   Follow the tutorial at https://lpsa.swarthmore.edu/NumInt/NumIntSecond.html through "Example 1". As a first step, implement the 2nd order Runge-Kutta method to solve the 1st order ODE $\dot{y} = -2y$. (As usual, you can do so in Matlab or Python).

   Once it is working, you can now apply it to the skateboard problem, which we have also written as 1st order ODE previously. Compare the solution we got previously from the Euler method to this hopefully improved method for a timestep of 0.1 and 0.01.

   Please hand in your code on Canvas, and include a brief write-up on the results you got (here or on canvas).

2. TQQ (10 points) Jumping on a merry-go-round [Sorry, this is problem kinda late, I had hoped to have you do this in class at the end of Chapter 3, but well, better late than never...]

   Sarah, with mass m and speed v, runs toward a merry-go-round and jumps on at its edge. Sarah and the merry-go-round (mass M, radius R, and moment of inertial $I = \frac{1}{2}MR^2$) then spin together with angular velocity ω_f. If Sarah's initial velocity is tangent to the circular merry-go-round, what is ω_f?

3. TQQ (10 points) A particle of mass m is moving on a frictionless horizontal table and is attached to a string, whose other end passes through a hole in the table, where I am holding it. Initially, the particle is moving in a circle of radius r_0 with angular velocity ω_0, but I now pull the string down through the hole until a length r remains between the hole and the particle.

   (a) What's the particle's angular velocity now?

   (b) Assuming that I pull the string so slowly that we can approximate the particle's path by a circle of slowly changing radius, calculate the work I did pulling the string.

   (c) Compare your answer to part (b) with the particle's gain in kinetic energy.

4. TQQ / ACT (25 points) Hand in Activity 4.1

5. TQQ / ACT (25 points) Hand in Activity 4.2

6. TQQ / ACT (25 points) Hand in Activity 4.3

PHYS 615 – Activity 4.1: Kinetic Energy

1. The dot product

   (a) There are a number of ways of computing the dot product of two vectors. The first gives the dot product in terms of the vectors' magntitudes and the angle in between:

   $$\vec{a} \cdot \vec{b} = ab\cos\theta$$

   where a is the magnitude of $\vec{a}$, ie, $a = |\vec{a}|$ and same for b (which is the notation we've been using quite a bit already), and θ is the angle between the two vectors.

   A second way is convenient of the vectors' components are known, ie., $\vec{a} = a_x\hat{x} + a_y\hat{y} + a_z\hat{z}$ and similarly for $\vec{b}$. In that case,

   $$\vec{a} \cdot \vec{b} = a_x b_x + a_y b_y + a_z b_z$$

   Show that the second way gives the same result as the first. In order to do so, given two vectors $\vec{a}$ and $\vec{b}$ with an angle of θ in between, choose your coordinate system such that the x axis points in the direction of $\vec{a}$, which makes it easy to write $\vec{a}$ in components. Then choose the y axis such that $\vec{b}$ lies in the $x - y$ plane. What's the angle between $\vec{b}$ and the x axis? Use that to write $\vec{b}$ in components. Now that you have the components, use the 2nd way to calculate the dot product, and compare to the 1st formula.

   (b) Show that $a^2 = \vec{a} \cdot \vec{a}$. (Again, a is the magnitude of $\vec{a}$.) This can be done using either of the two ways of calculating the dot product shown above.

(c) Show that a product rule holds for the derivative $\frac{d}{dt}(\vec{a} \cdot \vec{b})$ of the dot product (you can use the dot product from calculus, where we know it's true for scalar functions). $\vec{a}$ and $\vec{b}$ are considered to be functions of time.

(d) Show that $\frac{d}{dt}v^2 = 2\vec{v} \cdot \dot{\vec{v}}$, where again $\vec{v}$ is a function of time $\vec{v} = \vec{v}(t)$. How does that compare to what the chain rule would give if v was simply a scalar function?

2. Taylor Sec. 4.1 derived the (infinitesimal) change in kinetic energy (which we now call T) to be

$$dT = \vec{F}_{net} \cdot d\vec{r} \tag{1}$$

The term on the right is what we call *"work"* done by the force $\vec{F}_{net}$ over a displacement of $d\vec{r}$.

Integrating both sides, we get the *work-energy theorem*:

$$\Delta T = \int_1^2 \vec{F}_{net} \cdot d\vec{r} \equiv W_{net} \tag{2}$$

Since the net force is the sum of all forces acting on an object, we can correspondingly split up the net work into the sum of the work done by each respective force $W = \int \vec{F} \cdot d\vec{r}$.

(a) Does gravity always do work whenever an object moves from some position 1 to position 2, or can it be zero?

(b) Describe a situation where gravity does positive work. How does the speed of the object change in this case?

(c) Describe a situation where gravity does negative work. How does the speed of the object change in this case?

(d) There are two situations where work is easier to calculate than having to do an actual line integral:

$$W = \vec{F} \cdot \Delta r \qquad \text{if the force is constant} \tag{3}$$

$$W = \int_{x_1}^{x_2} F_x dx \qquad \text{if the problem is 1-d} \tag{4}$$

Justify these two simpler formulas.

3. Line integrals

   Evaluate the work done

$$W = \int_O^P \vec{F} \cdot d\vec{r} = \int_O^P (F_x dx + F_y dy) \tag{5}$$

   by the two-dimensional force $\vec{F} = x^2\hat{x} + 2xy\hat{y}$ along three paths starting at the origin O and ending at point $P = (1, 1)$, similarly to Taylor's Fig. 4.24(a).

   (a) The path goes along the x axis to $Q = (1, 0)$ and then straight up to P.

   (b) The path goes along the y axis to $R = (0, 1)$ and then straight over to P.

   (c) The path is described by $y = x^2$, and because of that, you can replace $dy = 2xdx$ and convert the whole integral to an integral over x.

(d) The path is described by parameterically as $x = t^3$, $y = t^2$. In this case, rewrite x, y, dx, dy in terms of t and dt, and convert the integral to an integral over t.

PHYS 615 – Activity 4.2: Conservative Forces, Potential Energy

1. *Conditions for a force to be conservative*

   What are the two conditions for a force to be called *conservative*?

2. *Potential Energy for a spring*

   The force exerted by a spring $\vec{F}_{spr}$ is given by Hooke's Law $\vec{F}_{spr} = -k\vec{r}$, where $\vec{r}$ indicates how much the spring is stretched (or compressed) from its equilibrium length.

   (a) For each of the two conditions that $\vec{F}_{spr}$ be conservative, is it satisified / not satisfied / not known at this point?

   (b) Since a spring is usually stretched or compressed along its axis, you can assume that the only motion that happens is in the x direction. Calculate the work done by the spring on a particle attached to its end as it moves from x_1 to x_2.

   (c) Is the spring force conservative?

(d) Find the potential energy for a spring U_{spr}.

3. *Friction force*

   A block is sliding across a plane (inclined or not). Since the normal force is constant, kinetic friction $F_{fk} = \mu_k F_N$ is also constant.

   It would appear that (a) the friction force is constant and (b) the motion is 1-d, and in both of those cases we can more easily calculate the work done by the friction force than having to do full-blown line integrals. So that makes one think that the force of kinetic friction should be conservative. Is that true? What, if anything, have we missed?

4. A block is sliding down a distance of d on an inclined plane – it starts from rest. Find an expression of the block's final speed in terms of d, the angle of inclination θ, and the coefficient of kinetic friction μ_k.

   We'll do this multiple ways.

   (a) Draw a free-body diagram. Identify all forces acting on the block.

   (b) You probably want to choose your coordinate system to be parallel / perpendicular to the inclined plane, though you could choose horizontal / vertical if you're up for some more math.

Write down Newton's 2nd Law for both the x component and the y component of the net force. Express the net force in terms of the forces on your FBD, and put in what you know about the acceleration components.

Solve to find the acceleration of the block a.

(c) Give your acceleration a and distance d, find the final speed v_f of the block.

(d) So in the above, we didn't use energy at all (though the 3rd kinematics law is closely related to the work-energy theorem). So let's do it using energy.

For all three forces, figure out whether they are conservative, and if so, use their potential energy. If a force is not conservative, use the work done by that force. Write down

$$\Delta E = \Delta(T + U) = W_{nc}$$

and plug in kinetic and potential energy at the initial and final times, as well as the work done by non-conservative force(s). (You will still need to find the friction force in order to be able to calculate the work done by it. Fortunately, you've already done that above, so you can reuse that result.)

Solve for v_f. Do you expect to get the same answer? Did you?

PHYS 615 – Activity 4.3: Conservative Forces, Potential Energy, continued

1. *Potential Energy of Gravity on Earth*

   On Earth's surface, we now that the force of gravity is $F_G = mg(-\hat{y})$ if our coordinate system chosen such that the y directions points vertically up.

   We have learned (though not proved) that the condition for the work integral to be path independent is that the force field is irrational, ie., $\nabla \times \vec{F} = 0$, where the "del" (a.k.a. "nabla") operator is

   $$\nabla \equiv \hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z}$$

   (a) Let's check that the curl of $\vec{F}_G$ is indeed zero. We'll make our life a little easier by assuming we live in a 2-d (x-y) world, so we only need to check the z component of the curl.

   Write our the z component of the curl of a force $\vec{F}$ in Cartesian coordinates, where $\vec{F}$ is known in components, that is $\vec{F} = F_x\hat{x} + F_y\hat{y}$.

   (b) Now plug in the force of gravity $\vec{F}_G$ specifically and show that the z component of the curl is indeed zero.

   (c) Show that the potential energy belonging the the force of gravity above (which we now know exists, since the force of gravity depends only on $\vec{r}$ (in fact, not even that), and because its curl vanishes) is

   $$U_G = mgy + C$$

   where we typically pick the constant of integration C such that the potential energy is zero at a height of $y = 0$.

   (d) Given $U_G = mgy$, derive the corresponding force from the relation $F = -\nabla U$.

2. *Potential Energy for Newton's Law of Gravity*

Away from Earth's surface, we now that the force that mass M (you can consider it to be located at the origin) exerts on mass m at position $\vec{r}$ is given by Newton's Law of Gravity

$$\vec{F}_G = -G\frac{mM}{r^2}\hat{r}$$

(a) Using our usual Cartesian coordinate system, show that the x component of F_G is given by

$$F_{G,x} = -G\frac{mM}{r^2}\frac{x}{r} = -GmM\frac{x}{r^3}$$

(In order to do so, you may need to remember that $\hat{r}$ is the same direction is $\vec{r}$, but it has been shortened by some factor so that it's of unit length.)

(b) Write the magnitude r of $\vec{r} = x\hat{x} + y\hat{y} + z\hat{z}$ in terms of its components x, y, z.

(c) Show that

$$\frac{\partial r}{\partial x} = \frac{x}{r}$$

(d) Show that the z component of $\nabla \times \vec{F}_G$ equals zero.

(e) At this point, one could redo this for the x and y components of the curl, but it's essentially the same calculation. Given that we have now shown that $\nabla \times F_G = 0$ and $F_G = F_G(\vec{r})$, Newton's force of gravity is conservative. So we can calculate the corresponding potential energy

$$U_G = -\int_{\vec{r}_0}^{\vec{r}} \vec{F}_G(\vec{r}') \cdot d\vec{r}'$$

Do so and show that the result is

$$U_G = -GmM\left(\frac{1}{r} - \frac{1}{r_0}\right)$$

In order to make things a bit easier, let's first assume $\vec{r}$ and $\vec{r}_0$ are along the same direction, so $\vec{r} = r\hat{r}$ and $\vec{r}_0 = r_0\hat{r}$. Since we now that F_G is conservative you can choose any path you want, ideally a simple one ;)

(f) Generally speaking, $\vec{r}$ and $\vec{r}_0$ may not be along the same direction. Show that your result above still holds in this case.

Hint: You can pick any path you want, so you might want to pick one that has two parts: The part you've already done, and a missing piece, where the integral is easy to calculate.

(g) For Newton's Law of Gravity (and Coulomb's Law), the reference location is usually put at $r_0 = \infty$. Show that the potential energy is then

$$U_G = -G\frac{mM}{r}$$

(h) Finally, let's think about $\vec{F} = -\nabla U_G$. What do you expect do get? Do the calculation just for one component, say the x component F_x.