PHYS 615 – Activity 2.3: Quadratic Drag

Let's look at horizontal motion in the presence of drag. This is quite a common situation, e.g., a car coasting and being slowed down by air resistance (in addition, rolling friction can also be significant, though at highway speeds drag dominates).

1. Given the formula for quadratic drag $\vec{F}_{D,quad} = -cv^2\hat{v}$, assume the only motion is in the horizontal direction, ie., gravity is canceled out by the normal force in the y direction. Use Newton's 2nd Law to derive the equation of motion

$$\dot{v}_x = -\frac{c}{m}v_x^2 \tag{1}$$

Solution:

$$\begin{aligned} m\dot{v}_x &= -cv_x^2 \\ \dot{v}_x &= -\frac{c}{m}v_x^2 \end{aligned}$$

($v^2 = v_x^2$ since $v_y = 0$)

Since the only direction we'll be dealing with in the following is the x direction, we'll save ourselves some typing / writing effort and skip the $_x$ subscripts:

$$\dot{v} = -\frac{c}{m}v^2 \tag{2}$$

2. Using the formula for quadratic drag given above, determine the units for the constant c.

   Solution: $F_{D,quad}$ has units of $\mathrm{N} = \mathrm{kg\,m/s^2}$, so

$$\frac{\mathrm{kg\,m}}{\mathrm{s}^2} = [c]\frac{\mathrm{m}^2}{\mathrm{s}^2} \Longrightarrow [c] = \frac{\mathrm{kg}}{\mathrm{m}}$$

3. The formula for drag that people usually use when applying it to a real problem is $F_{D,quad} = \frac{1}{2}\rho v^2\, C_D\, A$, where ρ is the density of the fluid, A is the cross sectional area of the object experiencing the drag, v is the speed of the object, and C_D is a dimensionless number that describes how aerodynamic (or not) the object's shape is. So Taylor's c constant is a shortcut for $c = \frac{1}{2}\rho C_D A$. Make sure that the units here are also consistent with what you found before.

   Solution:

$$[c] = 1 \cdot \frac{\mathrm{kg}}{\mathrm{m}^3} \cdot 1 \cdot \mathrm{m}^2 = \frac{\mathrm{kg}}{\mathrm{m}}$$

   which is, fortunately, the same.

4. Our ODE above has two parameters, m and c. Previously, with linear drag, we were able to find a "typical time" by dimensional analysis, ie. we could find $\tau = m/b$ just by trying to combine m and b in a way that we end up with a quantity with units of "seconds".

   Try to find a typical time based on m and c's units. You may not succeed. You may be able to find a typical length scale, though.

   Solution: Given the units of kg for m and $\mathrm{kg/m}$ for c, there's no way to get seconds. But we can do $L = m/c$ to get a length scale. It's not all that clear what significance it may have, though.

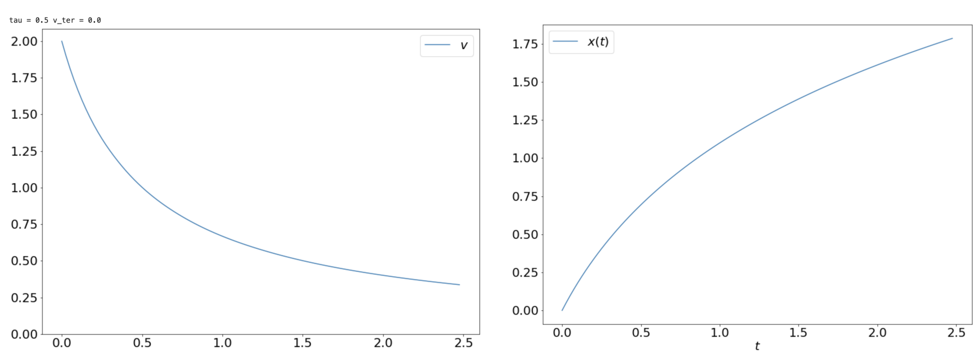
5. While the equation by itself is not enough to find a typical time, knowing only m and c is also not enough to determine the motion of (e.g.) a given car, either, since it is missing an initial condition. If we suppose we are given $v_0 \equiv v(0)$, we now have one more parameter to work with. Find a typical time τ based on m, c, and v_0.

   Solution: Given that we now have a length, and a length/time (speed), we can get seconds by doing $\tau = \frac{L}{v_0} = \frac{m}{cv_0}$.

At this point, we have some idea as to what possible length and time scales might be involved in this motion, but there's a lot more detail yet to figure out.

6. Sketch two graphs for what you expect the motion to look like, that is, $v(t)$ and $x(t)$. How does this compare to the situation with linear drag? (There's no need to try to be precise here, it's just about general characteristics of the motion.)

   Solution:



   The above are actual solutions, but the main characteristics we should expect are that velocity starts at some initial values and the decays at a slowing rate, since as velocity goes down, drag goes down. For position, it should increase more quickly in the beginning, as velocity is large, and then increases more and more slowly over time.

7. Time to do some math. Solve the ODE given above given the initial condition $v(0) = v_0$. This can be done analytically without too much trouble.

Solution:

$$\begin{aligned}
\frac{dv}{dt} &= -\frac{c}{m}v^2 \\
\frac{dv}{v^2} &= -\frac{c}{m}\,dt \\
\int \frac{dv}{v^2} &= \int -\frac{c}{m}\,dt \\
\frac{-1}{v} &= -\frac{c}{m}t + C \\
v &= \frac{1}{\frac{c}{m}t - C}
\end{aligned}$$

This would have been less painful if I had used definite integrals, but well, I wanted to be different... I still have to determine C from $v(0) = v_0$:

$$\begin{aligned}
v_0 &= \frac{1}{0 - C} \\
C &= -\frac{1}{v_0}
\end{aligned}$$

Let's plug C back in:

$$\begin{aligned}
v &= \frac{1}{\frac{c}{m}t + \frac{1}{v_0}} \\
&= \frac{v_0}{v_0\frac{c}{m}t + v_0\frac{1}{v_0}} \\
&= \frac{v_0}{\frac{cv_0}{m}t + 1}
\end{aligned}$$

(See the text after eq. (2.47) for how to do this with definite integrals to get the initial condition directly incorporated into the solution.)

8. Write your solution in the form

$$v(t) = \frac{v_0}{1 + \frac{t}{\tau}} \tag{3}$$

How do you have to define τ in order to make this work out? How does this compare to the τ that you found above?

Solution: My final equation pretty much is already in this form, and I can see that τ must be $\frac{m}{cv_0}$. This is in fact the typical time I found above from dimensional analysis.

9. Integrate one more time to find $x(t)$. This integral requires a little bit of work to solve by hand, but no major tricks.

Solution: This pretty much only requires knowing that the antiderivative of $1/u$ is $\ln u$, so one can either use substitution $u = 1 + t/\tau$, or just try $\ln(1 + t/\tau)$, which almost works and one just needs to put a constant factor of τ to cancel out the inner derivative:

$$x(t) = x_0 + v_0 \tau \ln\left(1 + \frac{t}{\tau}\right)$$

Can you find the length scale we found above (by dimensional analysis) in your solution?

Solution: It is there: $v_0 \tau = v_0 \frac{m}{c v_0} = \frac{m}{c}$.