PHYS 615 – Activity 4.2: Conservative Forces, Potential Energy

1. *Conditions for a force to be conservative*

   What are the two conditions for a force to be called *conservative*?

   Solution: A force $\vec{F}$ acting on a single particle is *conservative* if and only if it satisfies two conditions:

   (i) $\vec{F} = \vec{F}(\vec{r})$ depends only on the particle's position $\vec{r}$ (and not on velocity $\vec{v}$, or the time t, or any other variable).

   (ii) For any two points 1 and 2, the work $W(1 \to 2)$ done by $\vec{F}$ is the same for all paths between 1 and 2.

2. *Potential Energy for a spring*

   The force exerted by a spring $\vec{F}_{spr}$ is given by Hooke's Law $\vec{F}_{spr} = -k\vec{r}$, where $\vec{r}$ indicates how much the spring is stretched (or compressed) from its equilibrium length.

   (a) For each of the two conditions that $\vec{F}_{spr}$ be conservative, is it satisified / not satisfied / not known at this point?

   Solution: It is clear that the spring force depends only on position, not on $\vec{v}$ or t, (the spring constant k is constant, as its name says.) In terms of path independence, we don't quite know yet.

   (b) Since a spring is usually stretched or compressed along its axis, you can assume that the only motion that happens is in the x direction. Calculate the work done by the spring on a particle attached to its end as it moves from x_1 to x_2.

   Solution:

   $$W_{spr} = \int_1^2 \vec{F}_{spr} \cdot d\vec{r} = \int_{x_1}^{x_2} (-kx)dx = -\left.\frac{1}{2}kx^2\right|_{x_1}^{x_2} = -\frac{1}{2}kx_2^2 + \frac{1}{2}kx_1^2$$

   (c) Is the spring force conservative?

   Solution: Yes, since we can see that work only depends on initial and final position, not on the path.

   (d) Find the potential energy for a spring U_{spr}.

   Solution: U_{spr} is just the negative of the anti-derivative we found above:

   $$U_{spr} = \frac{1}{2}kx^2$$

   (It may worth confirming that the work W_{spr} is in fact equal to $-\Delta U = -(U(x_2) - U(x_1))$.)

3. *Friction force*

   A block is sliding across a plane (inclined or not). Since the normal force is constant, kinetic friction $F_{fk} = \mu_k F_N$ is also constant.

It would appear that (a) the friction force is constant and (b) the motion is 1-d, and in both of those cases we can more easily calculate the work done by the friction force than having to do full-blown line integrals. So that makes one think that the force of kinetic friction should be conservative. Is that true? What, if anything, have we missed?

Solution: Well, if the force of kinetic friction really were constant, then yeah, it should be conservative. But actually, only its magnitude is constant here – the direction will change depending on which way the block moves – it will always be opposite to the motion, slowing the block down. One way one could express this is to write $\vec{F}_{fk} = \mu_k F_N(-\hat{v})$, where $\hat{v}$ is the unit vector in the direction of the blocks velocity $\vec{v}$. And that causes two issues: (1) The friction now does depend on $\vec{v}$, not just $\vec{r}$. (2) It makes the integral path dependent, even in 1-d: E.g., if initial and final position are the same, and the block doesn't move at all, the work done is of course zero. But if the blocks slides forward a meter, and then backward a meter, it again has the same initial and final position. But during both parts of the slide, work is done, and its negative, and the total work is hence even more negative and clearly different from the zero we got when the path was just "sitting still".

4. A block is sliding down a distance of d on an inclined plane – it starts from rest. Find an expression of the block's final speed in terms of d, the angle of inclination θ, and the coefficient of kinetic friction μ_k.

   We'll do this multiple ways.

   (a) Draw a free-body diagram. Identify all forces acting on the block.

   Solution: See Fig. 4.6 in Taylor

   (b) You probably want to choose your coordinate system to be parallel / perpendicular to the inclined plane, though you could choose horizontal / vertical if you're up for some more math.

   Write down Newton's 2nd Law for both the x component and the y component of the net force. Express the net force in terms of the forces on your FBD, and put in what you know about the acceleration components.

   Solve to find the acceleration of the block a.

   Solution: I'm choosing x down the plane and y perpendicular to the plane.

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\begin{aligned}
0 &= F_{net,y} = F_N - mg\cos\theta && \Longrightarrow F_N = mg\cos\theta \\
ma &= F_{net,x} = mg\sin\theta - mu_k F_N = mg(\sin\theta - \mu\cos\theta) && \Longrightarrow a = g(\sin\theta - \mu\cos\theta)
\end{aligned}
$$

   (c) Give your acceleration a and distance d, find the final speed v_f of the block.

   Solution: I could find the time first and then plug that into $x(t)$, but it's easier to use the 3rd kinematics law ($v_i = 0$):

$$v_f^2 - v_i^2 = 2ad \qquad \Longrightarrow v_f = \sqrt{2gd(\sin\theta - \mu\cos\theta)}$$

   (d) So in the above, we didn't use energy at all (though the 3rd kinematics law is closely related to the work-energy theorem). So let's do it using energy.

For all three forces, figure out whether they are conservative, and if so, use their potential energy. If a force is not conservative, use the work done by that force. Write down

$$\Delta E = \Delta(T + U) = W_{nc}$$

and plug in kinetic and potential energy at the initial and final times, as well as the work done by non-conservative force(s). (You will still need to find the friction force in order to be able to calculate the work done by it. Fortunately, you've already done that above, so you can reuse that result.)

Solve for v_f. Do you expect to get the same answer? Did you?

Solution: The force of gravity is conservative and we know its potential energy mgh. We can also see from the sketch that $h = d \sin\theta$.

The normal force is generally not conservative, but in the current situation, the motion is perpendicular to the normal force, so the normal force does no work, so we don't need to include it in the energy balance.

Friction is constant along the path the block is taking, at a magnitude of $F_{fk} = \mu mg \cos\theta$. It is opposite to the displacement of the block, so $W_{nc} = W_{fk} = -\mu mgd \cos\theta$.

Putting it all together:

$$\begin{aligned}
(T_f + U_f) - (T_i + U_i) &= W_{nc} \\
\frac{1}{2}mv_f^2 - mgd\sin\theta &= -\mu mgd\cos\theta \\
\frac{1}{2}mv_f^2 &= mgd(\sin\theta - \mu\cos\theta) \\
v_f &= \sqrt{2gd(\sin\theta - \mu\cos\theta)}
\end{aligned}$$

That is the same answer, and that's what I was hoping for.