PHYS 615 – Activity 6.1: Calculus of Variations

1. *Circumference of a quarter circle*

   As we already discussed in class, unsurprisingly the circumference of a quarter circle is a quarter of the circumference of a full circle, so $L = \frac{1}{4}2\pi R = \frac{\pi}{2}R$.

   But we can calculate to get some practice with line integrals. As shown in the text, in general one can find the length of the path given by function $y(x)$ between points 1 and 2 as

$$L = \int_1^2 ds = \int_{x_1}^{x_2} \sqrt{1 + y'^2}dx \tag{1}$$

   (a) Show that the function describing a quarter circle (with its center at the origin) is (it's okay to use $R = 1$ here and in the following to make life a little bit easier, if you prefer.)

$$y(x) = \sqrt{R^2 - x^2}$$

   Solution: Any point (x, y) on the (quarter) circle satisfies the Pythagorean Theorem $x^2 + y^2 = R^2$, so we can get the equation above by solving for y.

   (b) Calculate the derivative $y' \equiv \frac{dy}{dx}$ and show that it is equal to

$$y'(x) = \frac{-x}{\sqrt{R^2 - x^2}}$$

   Solution: This should just require knowing the derivative of $x^{1/2}$ and the chain rule.

   (c) Plug y' into the integral to calculate L above and simplify. Show that your integral can be written as

$$L = \int_0^R \frac{1}{\sqrt{1 - (x/R)^2}}\,dx$$

   Solution: Just some algebra...

   (d) To actually solve this integral, use the substitution $x/R = \sin u$.

   Solution: Since $x = R\sin u$, $dx = R\cos u\,du$.

$$\begin{aligned}
L &= \int_0^R \frac{1}{\sqrt{1 - (x/R)^2}}\,dx \\
&= \int_{u(0)}^{u(R)} \frac{R\cos u}{\sqrt{1 - \sin^2 u}}du \\
&= \int_{u(0)}^{u(R)} \frac{R\cos u}{\cos u}du \\
&= R(u(R) - u(0)) = R(\arcsin 1 - \arcsin 0) = \frac{\pi}{2}R
\end{aligned}$$

   So yeah, we got the right answer.

The Euler-Lagrange Equation
Here is the big take-away from this chapter (ie., Calculus of Variations):
An integral of the form

$$S = \int_{x_1}^{x_2} f(y, y', x) dx$$

taken along a path $y = y(x)$ (and $y' = y'(x)$) is stationary with respect to variations of that path if and only if $y(x)$ satisfies the Euler-Lagrange equation

$$\frac{\partial f}{\partial y} - \frac{d}{dx}\frac{\partial f}{\partial y'} = 0$$

2. *The shortest path*

   We have seen that the length of a path given by $y = y(x)$ is

   $$L = \int_1^2 ds = \int_{x_1}^{x_2} \sqrt{1 + y'^2} dx$$

   (a) Write down the equation for $f(y, y', x)$ so that this integral for L takes the standard form S where the Euler Lagrange equations apply (see above).
   Solution:

   $$f(y, y', x) = \sqrt{1 + y'^2}$$

   (b) Find the partial derivatives $\frac{\partial f}{\partial y}$ and $\frac{\partial f}{\partial y'}$.
   Solution:

   $$\frac{\partial f}{\partial y} = 0$$

   $$\frac{\partial f}{\partial y'} = \frac{y'}{\sqrt{1 + y'^2}}$$

   (c) Plug these partial derivatives into the Euler-Lagrange equation (since we're looking for the shortest path, ie., one with minimum length, the function we're looking for is definitely stationary).
   Solution:

   $$0 - \frac{d}{dx}\frac{y'}{\sqrt{1 + y'^2}} = 0$$

   (d) You should be able to see that the Euler-Lagrange equation says that the $x-$derivative of some fraction involving y' is zero, which means that it doesn't depend on x – and the only thing it might depend on in x in the first place. So it must be constant. Set the constant term equal to C_1 and solve for $y'(x)$. You should be able to show that

   $$y'(x) = C_2$$

Solution:

$$\begin{aligned}
\frac{y'}{\sqrt{1+y'^2}} &= C_1 \\
y'^2 &= C_1^2(1+y'^2) \\
y'^2(1-C_1^2) &= C_1^2 \\
y' &= \pm\frac{C_1}{\sqrt{1-C_1^2}} \equiv C_2
\end{aligned}$$

(e) So apparently, $y'(x) = C_2 = const$. Use calculus to find $y(x)$ and show that you get something like $y(x) = mx + b$. If that is what you did get, did you just show that the shortest path between two points is a straight line?

Solution:

$$y'(x) = C_2 \Longrightarrow y(x) = \int C_2 dx = C_2 x + C_3$$

So that is exactly the function for a straight line (with $m = C_2, b = C_3$), and yes, we did just show that the shortest path between two points is a straight line. (If given two specific points, these two points will determine the specific values for C_2 and C_3, since of course are solution $y(x)$ must start/end at those two points.)

3. Find the equation of the path joining the origin O to the point $P(1,1)$ in the x-y plane that makes the integral

$$\int_O^P (y'^2 + yy' + y^2)dx$$

stationary.

Solution: We have the standard form of the variational problem with $f(y, y', x) = y'^2 + yy' + y^2$. So taking the partial derivatives

$$\frac{\partial f}{\partial y} = y' + 2y$$

and

$$\frac{\partial f}{\partial y'} = 2y' + y$$

. Putting this into the Euler-Lagrange equation

$$\begin{aligned}
y' + 2y - \frac{d}{dx}(2y' + y) &= 0 \\
y' + 2y &= 2y'' + y' \\
y &= y''
\end{aligned}$$

The general solution is $y(x) = A\sinh x + B\cosh x$, and since $y(0) = 0$ and $y(1) = 1$, $B = 0$ and $A = 1/(\sinh 1)$, so the path is given by

$$y(x) = \frac{\sinh x}{\sinh 1} = \frac{e^x - e^{-x}}{e - e^{-1}}$$