PHYS 615 – Activity 15.2: Time Dilation, Length Contraction, Lorentz transform

1. *Ball in Train Car*

   Analogous to how we derived time dilation, let's do the same gedanken experiment for throwing up a ball from the train car's floor, which then hits the ceiling where it gets reflected back and hits the floor again a short time later, so we can just use our regular (non-relativistic) physics.

   For simplicity, let's assume there is no gravity. Given the ball's vertical speed is $v_y = const$, and the height of the train car is h, what time $\delta t'$ does an observer in the train car measure?

   Now, let's do that calculation again from the ground, where the observer(s) stand still while the train goes by at a speed of V?

   Solution: Inside the train, the speed of the ball is always v_y – both as the ball moves up and as it moves down, and the distance it travels is $2h$. Since $v = \Delta s/\Delta t$:

$$\Delta t = \frac{2h}{v_y}$$

   From the outside, the ball does not only have an initialy y velocity, but also moves together with the train at V in the x direction, so $v = \sqrt{V^2 + v_y^2}$. It also travels a longer distance – as it moves up by h, the top of the train that it hits has moved $V\Delta t/2$, so the distance it moved is $\sqrt{(V\Delta t/2)^2 + h^2}$ – and then after the reflection it moves that same distance back down and to the right. So this time:

$$\Delta t = \frac{2\sqrt{(V\Delta t)^2 + h^2)}}{\sqrt{V^2 + v_y^2}}$$

   This needs a bit of work to solve for Δt:

$$\Delta t(\sqrt{V^2 + v_y^2}) = 2\sqrt{(V\Delta t/2)^2 + h^2} \tag{1}$$
$$\Delta t^2(V^2 + v_y^2) = 4((V\Delta t/2)^2 + h^2) \tag{2}$$
$$\Delta t^2 V^2 + \Delta t^2 v_y^2 = V^2\Delta t^2 + 4h^2) \tag{3}$$
$$\Delta t^2 v_y^2 = 4h^2) \tag{4}$$
$$\Delta t = \frac{2h}{v_y} \tag{5}$$
$$\tag{6}$$

   This was meant as a demonstration how in non-relativistic physics, velocities just get added, and that leads to the time being measured in one frame is the same as the one measured in another frame, which is how we usually think about stuff in everyday life – because it works that way. But it only works if velocities can just be added, which causes trouble when it comes to the speed of light, because then one would observed light traveling at speeds greater than the speed of light... Which is why instead, one needs to give up the concept of one single time.

2. *Time Dilation*

   What is the factor γ for a speed of $0.99c$? As observed from the ground, by how much would a clock traveling at this speed differ from the ground-based clock after one hour (one hour measured by the latter, that is)?

   Solution:
$$\gamma = \frac{1}{\sqrt{1-\beta^2}} \approx 7.1$$

   The proper time for the traveling clock (its proper time) would be given by $t_0 = t/\gamma \approx 8.5$ minutes. That is, the difference between the two clocks would be 51.5 minutes.

3. *Rent-a-rocket*

   When he returns his Hertz rent-a-rocket after one week's cruising in the galaxy, Spock is shocked to be billed for a 3 week rental. Assuming he traveled straight out, and then straight back, always at the same speed, how fast was he traveling?

   Solution: Apparently, Spock must have been traveling at $\gamma = 3$. So $\beta = \sqrt{1-(1/gamma)^2} \approx 0.94$. That is, he has been traveling at $0.94c$.

4. *Length Contraction*

   As a meter stick rushes past me (with velocity $\vec{v}$ parallel to the stick), I measure it's length to be 80 cm. What is v?

   Solution: Being a meter stick, its proper length is $l_0 = 1$ m. Its apparent length is 0.8 m. So

$$l = \frac{l_0}{\gamma} \Longrightarrow \gamma = 1.25$$

   Solving for β from γ, we get $\beta = 0.6$ or $V = 0.6c$.

5. *Lorentz Transformation*

   Solve the Lorentz transformation equations

$$\begin{aligned} x' &= \gamma(x - Vt) \\ y' &= y \\ z' &= z \\ t' &= \gamma\left(t - \frac{V}{c^2}x\right) \end{aligned}$$

   to give x, y, z, t in terms of x', y', z', t'. Compare to what you would have obtained if you had swapped primed and unprimed quantities.

   Solution: The equations for y' and z' can be trivially reinterpreted as equations for y and z. But x and t are coupled:

$$\begin{aligned} x' &= \gamma(x - Vt) \Longrightarrow x = \frac{x' + \gamma Vt}{\gamma} \\ t' &= \gamma\left(t - \frac{V}{c^2}\left(\frac{x'}{\gamma} + Vt\right)\right) \\ t' &= \gamma\left(t - \frac{Vx'}{\gamma c^2} + \frac{V^2}{c^2}t\right) \\ t' &= \gamma t - \frac{Vx'}{c^2} + \gamma\frac{V^2}{c^2}t \\ t' &= \gamma\left(1 - \frac{V^2}{c^2}\right)t - \frac{Vx'}{c^2} \\ t' &= \frac{t}{\gamma} - \frac{Vx'}{c^2} \\ t &= \gamma\left(t' + \frac{Vx'}{c^2}\right) \end{aligned}$$

Now we can find x, too:

$$
\begin{aligned}
x &= \frac{x' + \gamma V t}{\gamma} \\
x &= \frac{x' + \gamma V \gamma \left(t' + \frac{V x'}{c^2}\right)}{\gamma} \\
\gamma x &= x' + \gamma V \gamma \left(t' + \frac{V x'}{c^2}\right) \\
\gamma x &= x' + \gamma^2 V t' + \gamma^2 \frac{V^2}{c^2} x' \\
\gamma x &= (1 + \gamma^2 \beta^2) x' + \gamma^2 V t' \\
\gamma x &= (1 + \frac{\beta^2}{1 - \beta^2}) x' + \gamma^2 V t' \\
\gamma x &= \frac{1 - \beta^2 + \beta^2}{1 - \beta^2} x' + \gamma^2 V t' \\
\gamma x &= \frac{1}{1 - \beta^2} x' + \gamma^2 V t' \\
\gamma x &= \gamma^2 x' + \gamma^2 V t' \\
x &= \gamma (x' + V t')
\end{aligned}
$$

So yeah, fortunately we can just swap primed/unprimed and swap the sign of V, that's less work ;)