PHYS 615 – Activity 6.2: Calculus of Variations II

The Euler-Lagrange Equation

Here is, again, the big take-away from this chapter (ie., Calculus of Variations):
An integral of the form

$$S = \int_{x_1}^{x_2} f(y, y', x) dx$$

taken along a path $y = y(x)$ (and $y' = y'(x)$) is stationary with respect to variations of that path if and only if $y(x)$ satisfies the Euler-Lagrange equation

$$\frac{\partial f}{\partial y} - \frac{d}{dx}\frac{\partial f}{\partial y'} = 0$$

[The following is copied, mostly literally, from the text.]
The procedure for using the Euler-Lagrange equation is this:

- Set up the problem such that the quantity whose stationary path you seek is expressed as an integral in the standard form

  $$S = \int_{x_1}^{x_2} f(y, y', x) dx$$

  where $f(y, y', x)$ is the function appropriate to your problem.

- Write down the Euler-Lagrange equation (see above) in terms of the specific function $f(y, y', x)$ for your problem.

  $$\frac{\partial f}{\partial y} - \frac{d}{dx}\frac{\partial f}{\partial y'} = 0$$

- Finally, solve (if possible) the differential equation for the function $y(x)$ that you just wrote down – the solution(s) $y(x)$ will be the required stationary path.

1. *Geodesic*

The shortest path between two points on a *curved surface*, such as the surface of a sphere, is called a **geodesic**. To find a geodesic, one first has to set up an integral that gives the length of a path on the surface in question. This will always be similar to the integral S above, but may be more complicated (depending on the nature of the surface) and may involve different coordinates than x and y. To illustrate this, use spherical coordinates (r, θ, ϕ) to show that the length of a path joining two points on a sphere of radius R is

$$L = R \int_{\theta_1}^{\theta_2} \sqrt{1 + \sin^2\theta\, \phi'(\theta)^2} d\theta \tag{1}$$

if (θ_1, ϕ_1) and (θ_2, ϕ_2) specify the two points and we assume the path is expressed as $\phi = \phi(\theta)$.

(a) Before deriving the formula above, let's first do an example, with the sphere being Earth. On Earth the coordinates typically used are: ϕ is the "longitude", that is the east-west part of the location, and it is measured with 0 at the "prime meridian", the north-south line that passes through Greenwich, London. The range of ϕ in geography is $180°W \ldots 0 \ldots 180°E$. In the usual spherical coordinates we have phi go from 0 to $360\circ = 2\pi$, so we'll use that, but this isn't really fundamentally different.

In geography, the other quantity used is latitude, that is how far north or south from the equator the location is. The range goes from $90°N$ (north pole) through 0 (equator) to $90°S$ (south pole). FWIW, UNH is at $43.1340°N, 70.9264°W$. Spherical coordinates, which we'll use, are a bit different in that θ, sometimes called co-latitude goes from 0 (north pole) through $90°$ (equator) to $180°$ (south pole).

Calculate the length of the of the prime meridian, that is, the straight line on Earth that connects the north pole and the south pole and passes through Greenwich.

Earth's radius is $R_E = 3959$ mi ≈ 4000 mi, but you don't actually need to plug in any numbers, unless you care to get an answer in miles. Express the prime meridian as a path as used in the interal for L above, that is, find the function $\phi(\theta)$ and also the limits θ_1 and θ_2.

Evaluate the integral above for this specific path, and check it against what you'd find using basic geometry.

Solution: The prime meridian starts at the north pole $\theta_1 = 0$ and ends at the south pole $\theta_2 = \pi$. Longitude is zero all along, so $\phi(\theta) = 0$.

That makes the integral quite trivial since given $\phi(\theta)$ is constant, $\phi' = 0$, but hey, why complain...?

$$L = R_E \int_0^{\pi} \sqrt{1 + \sin^2\theta\, \phi'(\theta)^2} d\theta = R_E \int_0^{\pi} d\theta = \pi R_E$$

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And since going from north pole to south pole, we're halfway around Earth, and the full circle would be $2\pi R_E$, that is what we'd expect.

(b) In order to derive the general expression for the length of the path, it makes sense to start with $L = \int_1^2 ds$. Now we'll need to find a how long that small segment of path ds is where we start at some (R, θ, ϕ) and end at $(R, \theta + d\theta, \phi + d\phi)$. It should help to sketch the situation, which should look like a small triangle (with a right angle), so that you can use the Pythagorean Theorem. It may be a bit tricky to find the length of the two sides of the triangle first. Once you got that, though, it's down to replacing $d\phi$ by using $d\phi/d\theta = \phi'$.

Solution: The small triangle will have one side taking us south by the distance $Rd\theta$. The other side takes us east by $R \sin\theta\, d\phi$. So

$$ds = \sqrt{(Rd\theta)^2 + (R\sin\theta\, d\phi)^2} = \sqrt{1 + \sin^2\theta\, \phi'(\theta)^2} d\theta$$

Therefore we do in fact the total path length as

$$L = R\int_{\theta_1}^{\theta_2} \sqrt{1 + \sin^2\theta\, \phi'(\theta)^2} d\theta$$

(c) Now that we have confirmed the expression for the length of a path on the surface of a sphere, show that the shortest path between two points is a great circle.

Hints: The integrand $f(\phi, \phi', \theta)$ does not actually depend on ϕ. So as with the shortest path in a plane, one partial derivative in the Euler-Lagrange equation is zero, which means that the other partial derivative is constant. This gives you ϕ' as a function of θ. You can avoid doing the integral by the following trick: There is no loss of generality in choosing your z axis to pass through point 1. Show that with this choice the constant is necessarily zero, and describe the corresponding geodesics.

Solution: We find that our independent variable is θ, our dependent variable is $\phi = \phi(\theta)$ and $f(\phi, \phi', \theta) = \sqrt{1 + \sin^2\theta\, \phi'(\theta)^2}$. Hence

$$\frac{\partial f}{\partial \phi} = 0$$

and therefore

$$\frac{d}{d\theta}\frac{\partial f}{\partial \phi'} = 0$$

which means the partial derivative term itself is constant:

$$C = \frac{\partial f}{\partial \phi'} = \frac{\sin^2\theta\phi'}{\sqrt{1 + \sin^2\theta\, \phi'^2}}$$

Without loss of generality, we can choose our coordinate system such that $\theta_1 = 0$ for the starting point. Since $\sin\theta_1 = 0$, that means C has to be zero (it is constant, ie., the same for any θ and corresponding ϕ'). That in turn means that $\phi' = 0$ everywhere (except maybe at $\theta = 0$ and π), so $\phi = const$. The lines described by $\phi = const$ are meridians, going purely north-south, and as such are clearly great circles, which is what we meant to show.

If our starting point is not at $\theta = 0$, ie., the north pole, we just start over but this time pick a coordinate system where the starting point is at $\theta = 0$ – and then what just saw is that the shortest path has to be on a great circle starting from that point. It's likely not a meridian in the Earth North-South sence, since our coordinate system has been moved, but it is a great circle, and when rotating back into the original coordinate system, a great circle is still a great circle, so the shortest path does have to be along a great circle, as we wanted to show.