PHYS 615 – Activity 3.2: Momentum, Center of Mass, Angular Momentum

1. A shell traveling with speed v_0 exactly horizontally and due north explodes into two equal mass fragments. It is observed that just after the explosion, one fragment is traveling vertically up with speed v_0. What is the velocity of the other fragment?

   Solution: I'm using $\hat{y}$: due North, $\hat{z}$: up

   Before the collision:

   $$\vec{P}_i = mv_0\hat{y}$$

   After the collision:

   $$\vec{P}_f = \frac{m}{2}v_0\hat{z} + \frac{m}{2}\vec{v}_{2f}$$

   Setting them equal (because momentum is approximately conserved for the short time of the explosion):

   $$\begin{aligned} mv_0\hat{y} &= \frac{m}{2}v_0\hat{z} + \frac{m}{2}\vec{v}_{2f} \\ \vec{v}_{2f} &= \frac{mv_0\hat{y} - \frac{m}{2}v_0\hat{z}}{m/2} \\ &= 2mv_0\hat{y} - v_0\hat{z} \end{aligned}$$

2. Consider a gun of mass M (when unloaded) that fires a shell of mass m with muzzle speed v (That is, the shell's speed relative to the gun is v.) Assuming that the gun is completely free to recoil (no external forces on gun or shell), use conservation of momentum to show that the shell's speed relative to the ground is $v/(1 + m/M)$.

   Solution: Let's call V the magnitude of gun's velocity after firing (moving to the left), and v_G the speed of the bullet relative to the ground (to the right).

   Conservation of momentum:

   $$0 = -MV + mv_G$$

   Muzzle velocity is $v = v_G + V$, solve for $V = v - v_G$ and plug into momentum equation:

   $$\begin{aligned} 0 &= -M(v - v_G) + mv_G \\ Mv &= (m + M)v_G \\ v_G &= \frac{Mv}{m + M} = \frac{v}{1 + m/M} \end{aligned}$$

3. The masses of the Earth and the Sun are $M_E \approx 6.0 \times 10^{24}$ kg and $M_S \approx 2.0 \times 10^{30}$ kg and their center-to-center distance is 1.5×10^8 km. Find the position of their center of mass and comment. (The radius of the Sun is $R_S \approx 7.0 \times 10^5$ km.)

Solution: Putting the Sun at $x_S = 0$, and Earth at $x_E = 1.5 \times 10^8$ km. (Ie, but the x axis of the coordinate system on the Sun-Earth line.)

$$x_{CM} = \frac{m_S x_S + m_E x_E}{m_S + m_E} = \frac{m_E}{m_S + m_E} x_E = 450\,\text{km}$$

So the center of the mass of the system is still within the Sun's radius.

4. The center of mass of a triangular plate.

   An isosceles triangular plate has a base of length b, a height of h, and a thickness of d, as well as a mass of M. It is made from a uniform material (e.g., wood).

   (a) Find the plate's volume in terms of the quantities given above. Before actually doing so exactly, think about what result you would expect from dimensional analysis.

   Solution: A volume should be m^3, and it should depend on all of the dimensions, so that means an answer taking the product of all three lengths seems like a good expectation. An easy way to do this is to use the area of the triangle $A = \frac{1}{2}bh$ and multiply by thickness to get $V = \frac{bhd}{2}$. Or one can use a bunch of calculus to get the same.

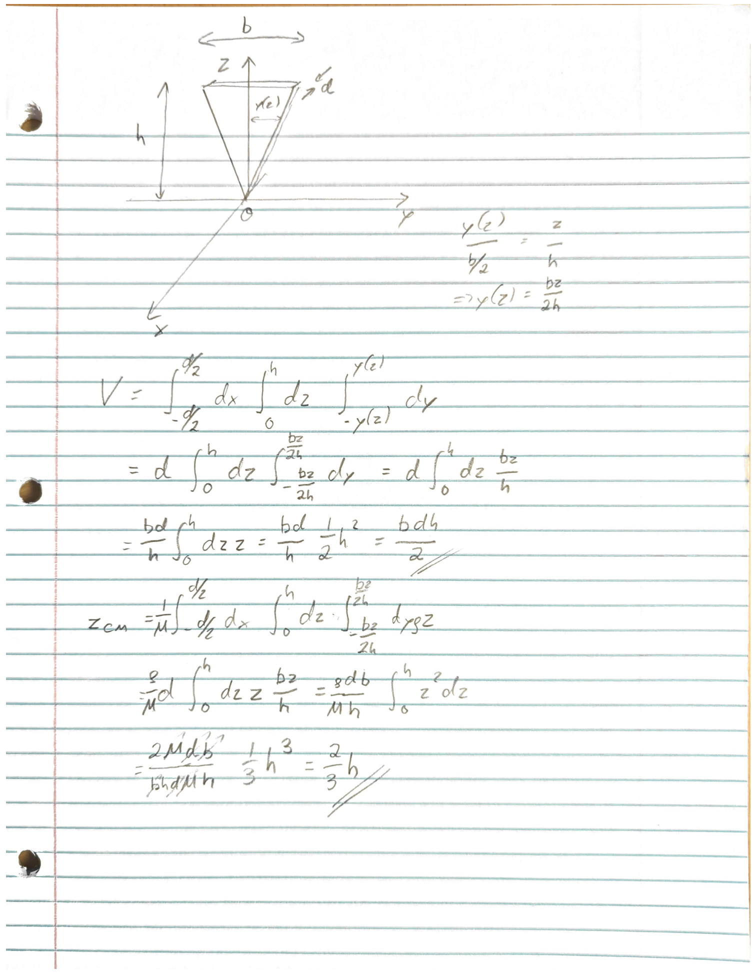
   (b) Find an expression for the plate's density in terms of the quantities given above.

   Solution:

   $$\rho = \frac{M}{V} = \frac{2M}{bhd}$$

   (c) Find the position of the center of mass for the plate. It probably makes sense to orient the triangle similarly to the cone in Taylor's Example 3.2, ie., have the tip at the origin.

   Solution:



$$\frac{y(z)}{b/2} = \frac{z}{h} \Rightarrow y(z) = \frac{bz}{2h}$$

$$V = \int_{-d/2}^{d/2} dx \int_0^h dz \int_{-y(z)}^{y(z)} dy$$
$$= d \int_0^h dz \int_{-\frac{bz}{2h}}^{\frac{bz}{2h}} dy = d \int_0^h dz \frac{bz}{h}$$
$$= \frac{bd}{h} \int_0^h dz\, z = \frac{bd}{h} \frac{1}{2} h^2 = \frac{bdh}{2}$$

$$z_{CM} = \frac{1}{M} \int_{-d/2}^{d/2} dx \int_0^h dz \int_{-\frac{bz}{2h}}^{\frac{bz}{2h}} dy\, \rho z$$
$$= \frac{\rho}{M} d \int_0^h dz\, z \frac{bz}{h} = \frac{\rho d b}{M h} \int_0^h z^2 dz$$
$$= \frac{2 \cancel{M} \cancel{d} \cancel{b}}{\cancel{b} h \cancel{d} \cancel{M} h} \frac{1}{3} h^3 = \frac{2}{3} h$$

x_{CM} and y_{CM} could be computed following the same process, but it is easy to see by symmetry that those will be zero.

The angular momentum of a single particle at position $\vec{r}$ is defined as

$$\vec{l} = \vec{r} \times \vec{p}$$

5. Check that the following equality is true by evaluating explicitly the z component of both the left hand side and the right hand side.

$$\dot{\vec{l}} = \frac{d}{dt}(\vec{r} \times \vec{p}) = \dot{\vec{r}} \times \vec{p} + \vec{r} \times \dot{\vec{p}}$$

Solution: Let's do the z component of the l.h.s.:

$$l_z = y p_z - z p_y$$
$$\dot{l}_z = \dot{y} p_z + y \dot{p}_z - \dot{z} p_y - z \dot{p}_y$$

Then the r.h.s.:

$$\left[\dot{\vec{r}} \times \vec{p} + \vec{r} \times \dot{\vec{p}}\right]_z = \dot{y} p_z - \dot{z} p_y + y \dot{p}_z - z \dot{p}_y$$

They do indeed match!

6. Show that $\dot{\vec{r}} \times \vec{p}$ is zero (think back to the definition of momentum).

   Solution: $\vec{p} = m\vec{v} = m\dot{\vec{r}}$, so $\dot{\vec{r}} \times \vec{p} = \dot{\vec{r}} \times m\dot{\vec{r}} = \vec{0}$.

7. Derive the "2nd Law of angular motion"

$$\dot{\vec{l}} = \vec{\Gamma} \equiv \vec{r} \times \vec{F}_{net}$$

   where the term on the r.h.s. is used to define "torque" $\vec{\Gamma}$. (Taylor uses capital gamma Γ for torque.) Compare to the regular 2nd Law, written in terms of momentum as $\dot{\vec{p}} = \vec{F}_{net}$.

   Solution: We're pretty much there:

$$\dot{\vec{l}} = \frac{d}{dt}(\vec{r} \times \vec{p}) = \dot{\vec{r}} \times \vec{p} + \vec{r} \times \dot{\vec{p}} = \vec{r} \times \dot{\vec{p}} = \vec{r} \times \vec{F}_{net}$$

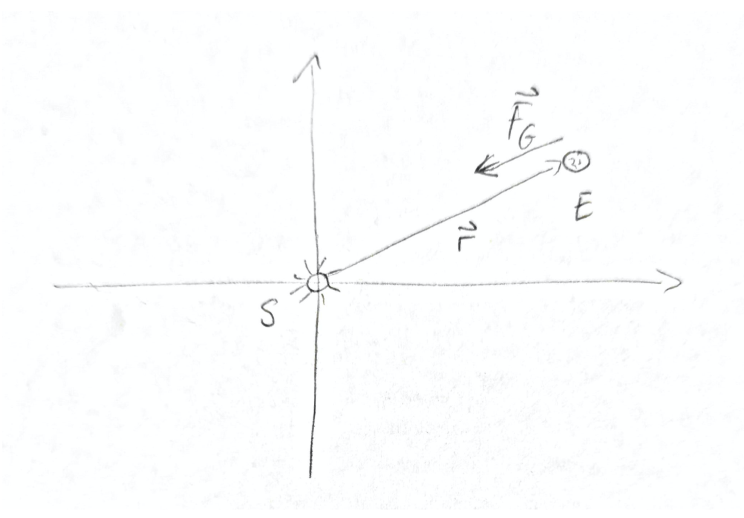
8. Newton's Law of gravity is

$$\vec{F}_G = G\frac{mM}{r^2}(-\hat{r})$$

   Sketch the situation of the Earth rotating around the Sun. Put the Sun at the origin and draw vectors $\vec{r}$ indicating the position of Earth, and $\vec{F}_G$ showing the gravitionational force exerted by the Sun onto Earth.

   What is the torque that the Sun exerts on Earth?

   Solution:



$$\Gamma = \vec{r} \times G\frac{mM}{r^2}(-\hat{r}) = \vec{0}$$

   since $\vec{r}$ and $\hat{r}$ are parallel.