PHYS 615 – Activity 8.2: Kepler Orbits

1. *Effective Potential*

   (a) By examining the effective potential energy

   $$U_{eff} = -\frac{\gamma}{r} + \frac{l^2}{2\mu r^2}$$

   find the radius r_0 at which a planet with angular momentum l can orbit the Sun in a circular orbit with fixed radius. (Look at dU_{eff}/dr)

   Solution: $r = r_0 = const$ implies that $\dot{r} = 0$, and hence $\ddot{r} = 0$ as well. Since $\mu\ddot{r} = -dU_{eff}/dr$, that means we need $dU_{eff}/dr = 0$.

   $$\frac{dU_{eff}}{dr} = \frac{\gamma}{r^2} - \frac{l^2}{\mu r^3}$$

   So to find that fixed radius r_0:

   $$0 = \left.\frac{dU_{eff}}{dr}\right|_{r_0} = \frac{\gamma}{r_0^2} - \frac{l^2}{\mu r_0^3} \Longrightarrow r_0 = \frac{l^2}{\gamma\mu}$$

   (b) Show that this circular orbit is stable, in the sense that a small nudge will only cause small radial oscillations. (Look at d^2U_{eff}/dr^2). Show that the period of these oscillations is equal to the planet's orbital period.

   Solution: The orbit at $r = r_0$ will be stable if U_{eff} has a minimum there. We already know the first derivative of U_{eff} is zero, now we still need to make sure the 2nd derivative is positive. Let's find the 2nd derivative first:

   $$\frac{d^2U_{eff}}{dr^2} = \frac{-2\gamma}{r^3} + \frac{3l^2}{\mu r^4}$$

   Plug in r_0:

   $$\left.\frac{d^2U_{eff}}{dr^2}\right|_{r_0} = \frac{-2\gamma}{r_0^3} + \frac{3l^2}{\mu r_0^4} = \frac{\gamma}{r_0^3}$$

   So that is in fact positive.

   What this means is that if we perturb the orbit a little bit from r_0, acceleration will move us back towards r_0, but overshoot, and we're doing a small oscillation around the true r_0. In order to look at that more closely, we can approximate $Ueff$ by its Taylor expansion around r_0:

   $$\begin{aligned} U_{eff}(r) &= U_{eff}(r_0) + (r - r_0)\frac{dU_{eff}}{dr} + \frac{1}{2}(r - r_0)^2\frac{d^2U_{eff}}{dr^2} \\ &= U_{eff}(r_0) + \frac{\gamma}{2r_0^3}(r - r_0)^2 \end{aligned}$$

Putting this into the equation of motion:

$$\mu\ddot{r} = -\frac{dU_{eff}}{dr} = -\frac{\gamma}{r_0^3}(r - r_0)$$

which is the differential equation for oscillations that we've seen many times before, and we can read that $\omega_0 = \sqrt{\gamma/\mu r_0^3}$. This is actually the same value as the angular velocity of the planet circling the Sun.

2. *Ellipses*

We have proved that any Kepler orbit can be written in the form

$$r(\phi) = \frac{c}{1 + \epsilon \cos\phi}$$

where $c > 0$ and $\epsilon \geq 0$. For the case that $0 \leq \epsilon < 1$, rewrite this equation in Cartesian coordinates (x, y) and prove that the equation can be cast in the form

$$\frac{(x+d)^2}{a^2} + \frac{y^2}{b^2} = 1$$

which the equation of an ellipse. Verify these expressions:

$$a = \frac{c}{1-\epsilon^2}, \qquad b = \frac{c}{\sqrt{1-\epsilon^2}}, \qquad d = a\epsilon$$

Draw a picture of the ellipse and indicate all the lengths (a, b, c, d).

Solution: We'll end up using that $x = r\cos\phi$ and $r^2 = x^2 + y^2$.

First, multiply the orbit equation by its denominator

$$\begin{aligned}
r + \epsilon r \cos\phi &= c \\
r + \epsilon x &= c \\
r &= c - \epsilon x \\
r^2 &= (c - \epsilon x)^2 \\
x^2 + y^2 &= c^2 - 2\epsilon c x + \epsilon^2 x^2
\end{aligned}$$

In some sense we're done, since the equation is now expressed in terms of x and y – but it's not the standard form yet, so that'll require some more fun and completing the square to get it there. I'll introduce $d = \frac{\epsilon c}{1-\epsilon^2}$.

$$\begin{aligned}
x^2 + y^2 &= c^2 - 2\epsilon c x + \epsilon^2 x^2 \\
(1-\epsilon^2)x^2 + 2\epsilon c x + y^2 - c^2 &= 0 \\
x^2 + 2\frac{\epsilon c}{(1-\epsilon^2)}x + \frac{y^2}{(1-\epsilon^2)} - \frac{c^2}{(1-\epsilon^2)} &= 0 \\
x^2 + 2\frac{\epsilon c}{(1-\epsilon^2)}x + \frac{(\epsilon c)^2}{(1-\epsilon^2)^2} - \frac{(\epsilon c)^2}{(1-\epsilon^2)^2} + \frac{y^2}{(1-\epsilon^2)} - \frac{c^2}{(1-\epsilon^2)} &= 0 \\
\left(x + \frac{\epsilon c}{(1-\epsilon^2)}\right)^2 - \frac{(\epsilon c)^2}{(1-\epsilon^2)^2} + \frac{y^2}{(1-\epsilon^2)} - \frac{c^2}{(1-\epsilon^2)} &= 0 \\
\left(x + \frac{\epsilon c}{(1-\epsilon^2)}\right)^2 + \frac{y^2}{(1-\epsilon^2)} &= \frac{(\epsilon c)^2}{(1-\epsilon^2)^2} + \frac{c^2}{(1-\epsilon^2)} \\
\left(x + \frac{\epsilon c}{(1-\epsilon^2)}\right)^2 + \frac{y^2}{(1-\epsilon^2)} &= \frac{\epsilon^2 c^2 + c^2 - c^2\epsilon^2}{(1-\epsilon^2)^2} \\
\left(x + \frac{\epsilon c}{(1-\epsilon^2)}\right)^2 + \frac{y^2}{(1-\epsilon^2)} &= \frac{c^2}{(1-\epsilon^2)^2}
\end{aligned}$$

Time to use some constants to get this under control: I'll introduce $d = \frac{\epsilon c}{1-\epsilon^2}$, and $a = \frac{c}{1-\epsilon^2}$

$$(x+d)^2 + \frac{y^2}{(1-\epsilon^2)} = a^2$$
$$\frac{(x+d)^2}{a^2} + \frac{y^2}{(1-\epsilon^2)a^2} = 1$$

Almost there – let's use $b = \sqrt{1-\epsilon^2}a$, too:

$$\frac{(x+d)^2}{a^2} + \frac{y^2}{b^2} = 1$$

It turns out that the constants we introduced really match those given in the problem ;)