PHYS 615 – Activity 8.1: Two-Body Problems

1. *Changing to center of mass and relative position*

   In class, we have found the Lagrangian for two bodies of mass m_1 and m_2 that are subject to (only) an internal, central and conservative force:

$$\mathcal{L} = T - U = \frac{1}{2}m_1\dot{\vec{r}}_1^2 + \frac{1}{2}m_2\dot{\vec{r}}_2^2 - U(|\vec{r}_1 - \vec{r}_2|) \tag{1}$$

   We would like to change from expressing $\mathcal{L}$ in terms of the coordinates $\vec{r}_1$ and $\vec{r}_2$ to new coordinates: the position of the center of mass $\vec{R}$ and the relative position $\vec{r} = \vec{r}_1 - \vec{r}_2$.

   We know the position of the center of mass to be

$$\vec{R} = \frac{m_1\vec{r}_1 + m_2\vec{r}_2}{m_1 + m_2} \tag{2}$$

   (a) Show that we can express our original coordinates in terms of the new ones as

$$\vec{r}_1 = \vec{R} + \frac{m_2}{m_1 + m_2}\vec{r} \qquad \vec{r}_2 = \vec{R} - \frac{m_1}{m_1 + m_2}\vec{r} \tag{3}$$

   Solution:

$$\vec{r} = \vec{r}_1 - \vec{r}_2 \implies m_2\vec{r} = m_2\vec{r}_1 - m_2\vec{r}_2$$
$$(2) \implies (m_1 + m_2)\vec{R} = m_1\vec{r}_1 + m_2\vec{r}_2$$

   Adding the two equations on the right (chosen such that $\vec{r}_2$ cancels out):

$$m_2\vec{r} + (m_1 + m_2)\vec{R} = (m_1 + m_2)\vec{r}_1$$
$$\vec{R} + \frac{m_2}{m_1 + m_2}\vec{r} = \vec{r}_1$$

   That's $\vec{r}_1$, so now we can find $\vec{r}_2$:

$$\vec{r} = \vec{r}_1 - \vec{r}_2 \implies \vec{r}_2 = \vec{r}_1 - \vec{r} = \vec{R} + \frac{m_2}{m_1 + m_2}\vec{r} - \vec{r} = \vec{R} - \frac{m_1}{m_1 + m_2}\vec{r}$$

   (b) Given the transformation you just confirmed, express $\mathcal{L}$ in terms of the new coordinates $\vec{R}$, $\vec{r}$, and its time derivatives.

   Show that you get something of the form

$$\mathcal{L} = \frac{1}{2}M\dot{\vec{R}}^2 + \frac{1}{2}\mu\dot{\vec{r}}^2 - U(r) \tag{4}$$

   Determine expressions for M and μ.

   Solution: First, we need the dotted quantities:

$$\dot{\vec{r}}_1 = \dot{\vec{R}} + \frac{m_2}{m_1 + m_2}\dot{\vec{r}} \qquad \dot{\vec{r}}_2 = \dot{\vec{R}} - \frac{m_1}{m_1 + m_2}\dot{\vec{r}}$$

Plugging these in and doing a bunch of algebra:

$$\begin{aligned}
\mathcal{L} = T - U &= \frac{1}{2}m_1\dot{\vec{r}}_1^2 + \frac{1}{2}m_2\dot{\vec{r}}_2^2 - U(|\vec{r}_1 - \vec{r}_2|) \\
&= \frac{1}{2}m_1(\dot{\vec{R}} + \frac{m_2}{m_1+m_2}\dot{\vec{r}})^2 + \frac{1}{2}m_2(\dot{\vec{R}} - \frac{m_1}{m_1+m_2}\dot{\vec{r}})^2 - U(r) \\
&= \frac{1}{2}(m_1+m_2)\dot{\vec{R}}^2 + \frac{1}{2}m_1\frac{m_2}{m_1+m_2}\dot{\vec{r}}^2 + \frac{1}{2}m_2\frac{m_1}{m_1+m_2}\dot{\vec{r}}^2 - U(r) \\
&= \frac{1}{2}(m_1+m_2)\dot{\vec{R}}^2 + \frac{m_1 m_2}{m_1+m_2}\dot{\vec{r}}^2 - U(r)
\end{aligned}$$

We can see that $M = m_1 + m_2$ and $\mu = \frac{m_1 m_2}{m_1+m_2}$.

2. *The two-body equations of motion*

   Now that we have expressed our Lagrangian as

$$\mathcal{L} = \frac{1}{2}M\dot{\vec{R}}^2 + \frac{1}{2}\mu\dot{\vec{r}}^2 - U(r) \tag{5}$$

   with M being the total mass and μ being called the reduced mass, find the Lagrange equations of motion, and solve them in as far as possible.

   Describe what these equations mean physically.

   Solution: I'll do this in terms of components (X, Y, Z) of $\vec{R}$ and (x, y, z) for $\vec{r}$.

$$\frac{\partial\mathcal{L}}{\partial X} = \frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{X}} \Longrightarrow 0 = M\ddot{X} \tag{6}$$

$$\frac{\partial\mathcal{L}}{\partial Y} = \frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{Y}} \Longrightarrow 0 = M\ddot{Y} \tag{7}$$

$$\frac{\partial\mathcal{L}}{\partial Z} = \frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{Z}} \Longrightarrow 0 = M\ddot{Z} \tag{8}$$

$$\frac{\partial\mathcal{L}}{\partial x} = \frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{x}} \Longrightarrow -\frac{\partial U}{\partial x} = \mu\ddot{x} \tag{9}$$

$$\frac{\partial\mathcal{L}}{\partial y} = \frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{y}} \Longrightarrow -\frac{\partial U}{\partial y} = \mu\ddot{x} \tag{10}$$

$$\frac{\partial\mathcal{L}}{\partial z} = \frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{z}} \Longrightarrow -\frac{\partial U}{\partial z} = \mu\ddot{x} \tag{11}$$

$$\tag{12}$$

   Putting this back into vector form, we just have Newton's 2nd Law, separately for the center of mass and for the relative motion:

$$0 = M\ddot{\vec{R}} \qquad \vec{F} = -\nabla U = \mu\ddot{\vec{r}}$$

   (The equation for the center of mass is just an indication of conservation of total momentum, as one would think given there are no external forces.)

3. *Angular Momentum*

   Since we've just seen that the center of mass is moving at constant velocity, we can move our frame of reference to be at the center of mass and it's still an inertial reference frame, and in that frame the center of mass is naturally sitting at rest, so $\vec{R} = 0$.

   Find an expression for the total angular momentum $\vec{L}$ about the center of mass, ie., the origin in our new frame of reference. That is, write down the sum of the angular momenta of particle 1 and particle 2, using the original coordinates.

   Using the coordinate transform from part 1., express $\vec{r}_1$ and $\vec{r}_2$ in terms of $\vec{r}$.

   Combine the two things you have just found to show that total angular momentum is

$$\vec{L} = \vec{r} \times \mu\dot{\vec{r}}$$

Solution:

$$\vec{L} = \vec{r}_1 \times m_1\dot{\vec{r}}_1 + \vec{r}_1 \times m_1\dot{\vec{r}}_1$$

In the center of mass frame, $\vec{R} = 0$. So

$$\vec{r}_1 = \frac{m_2}{m_1 + m_2}\vec{r} \qquad \vec{r}_2 = -\frac{m_1}{m_1 + m_2}$$

Plugging this in:

$$\begin{aligned}
\vec{L} &= \vec{r}_1 \times m_1\dot{\vec{r}}_1 + \vec{r}_1 \times m_1\dot{\vec{r}}_1 &(13)\\
&= \frac{m_2}{m_1 + m_2}\vec{r} \times m_1\frac{m_2}{m_1 + m_2}\dot{\vec{r}} + \frac{-m_1}{m_1 + m_2}\vec{r} \times m_2\frac{-m_1}{m_1 + m_2}\dot{\vec{r}} &(14)\\
&= \frac{m_1 m_2^2}{(m_1 + m_2)^2}\vec{r} \times \dot{\vec{r}} + \frac{m_1^2 m_2}{(m_1 + m_2)^2}\vec{r} \times \dot{\vec{r}} &(15)\\
&= \frac{m_1 m_2}{(m_1 + m_2)^2}(m_2 + m_1)\vec{r} \times \dot{\vec{r}} &(16)\\
&= \frac{m_1 m_2}{m_1 + m_2}\vec{r} \times \dot{\vec{r}} &(17)\\
&= \vec{r} \times \mu\dot{\vec{r}} &(18)
\end{aligned}$$

4. *External Forces*

   Although the main topic of the current chapter is the the motion of two particles subject to no external forces, many of the ideas easily extend to more general situations.

   To illustrate this, consider the following: Two masses m_1 and m_2 move in a uniform gravitational field $\vec{g} = -g\hat{z}$ and interact via a potential energy $U(r)$.

   (a) Show that the Lagrangian can be separated into the sum of two parts – one which depends only on $\vec{R}$ and its time derivative, and another, which only depends on $\vec{r}$ and its time derivative.

   Solution: Everything is the same as in the case of no external forces, except that we now have a gravitational potential energy for both of the particles in addition to their interaction, that is

   $$U = U(r) + m_1 g z_1 + m_2 g z_2$$

   When rewriting things in terms of CM and relative motion, again everything is the same, but we get an additional potential energy term MgZ, which due to the definition of the center of mass coordinate is exactly as the sum of the separate two particle gravitational potential energy.

   $$\mathcal{L} = \frac{1}{2}M\dot{\vec{R}}^2 - MgZ + \frac{1}{2}\mu\dot{\vec{r}}^2 - U(r)$$

   (b) Write down Lagrange's equations for the three center-of-mass coordinates X, Y, Z and describe the motion of the center of mass.

Solution:

$$\frac{\partial \mathcal{L}}{\partial X} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{X}} \Longrightarrow 0 = M\ddot{X} \tag{19}$$
$$\frac{\partial \mathcal{L}}{\partial Y} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{Y}} \Longrightarrow 0 = M\ddot{Y} \tag{20}$$
$$\frac{\partial \mathcal{L}}{\partial Z} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{Z}} \Longrightarrow -Mg = M\ddot{Z} \tag{21}$$
$$\tag{22}$$

This means the center of mass undergoes the usual projectile motion trajectory, as the equation of motion is

$$-Mg\hat{z} = M\ddot{\vec{R}}$$

(c) Write down Lagrange's motion for the three relative coordinates x, y, z and show that the motion of $\vec{r}$ is the same as that of a single particle with mass μ, position $\vec{r}$ and potential energy $U(r)$.

Solution: Well, we've just done this:

$$\frac{\partial \mathcal{L}}{\partial x} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{x}} \Longrightarrow -\frac{\partial U}{\partial x} = \mu\ddot{x} \tag{23}$$
$$\frac{\partial \mathcal{L}}{\partial y} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{y}} \Longrightarrow -\frac{\partial U}{\partial y} = \mu\ddot{x} \tag{24}$$
$$\frac{\partial \mathcal{L}}{\partial z} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{z}} \Longrightarrow -\frac{\partial U}{\partial z} = \mu\ddot{x} \tag{25}$$
$$\tag{26}$$

A.k.a. Newton's 2nd Law:

$$\vec{F} = -\nabla U = \mu\ddot{\vec{r}}$$

5. *Two particles and a spring*

This problem builds upon the previous one.

Two particles of mass m_1 and m_2 are joined by a massless spring of natural length l and spring constant k. Initially, m_2 is resting on a table, and I'm holding m_1 vertically above m_2 at height l. At time $t = 0$, I project m_1 upward with initial velocity v_0. Find the positions of the two masses at any subsequent time t (before either mass returns to the table) and describe the motion. [Assume that v_0 is small enough that the two masses never collide.]

Solution: The motion is confined to the vertical directions, so let's just keep z and Z. From the previous problem, and the potential nergy of the spring $U = \frac{1}{2}k(z - L)^2$:

$$\mathcal{L} = \frac{1}{2}M\dot{Z}^2 - MgZ + \frac{1}{2}\mu\dot{z}^2 - \frac{1}{2}k(z - l)^2$$

So the Lagrange equations of motion (again, like above) are:

$$0 = \ddot{Z} \qquad \text{and} \qquad -k(z - l) = \mu\ddot{z}$$

The solutions are:

$$Z(t) = -\frac{1}{2}gt^2 + V_0 t + Z_0$$

and

$$z(t) = l + A\sin\omega t + B\cos\omega t$$

where $\omega = \sqrt{k/\mu}$. We still have to find the constants of integration. Initially, $v_1 = v_0$ and $v_2 = 0$, as well as $z_1 = l, z_2 = 0$. That means, $Z_0 = \frac{m_1 l}{m_1+m_2}$, $z(0) = l$, and $V_0 = \frac{m_1 v_0}{M}$, $v(0) = v_0$.

That means $B = 0$ and since $v(t) = \omega A\cos\omega t$, $A = v_0/\omega$. So the solutions is:

$$Z(t) = -\frac{1}{2}gt^2 + \frac{m_1}{M}v_0 t + Z_0$$

and

$$z(t) = l + \frac{v_0}{\omega}\sin\omega t$$

The CM moves up and the back down like any object in free fall, while the two masses oscillate about the CM.