PHYS 615 – Activity 7.1: Unconstrained Lagrangian

The **Lagrangian** $\mathcal{L}$ is defined as

$$\mathcal{L} = T - U \tag{1}$$

were T is kinetic energy and U is potential energy.
Hamilton's Principle: The actual path which a particle follows between two points 1 and 2 in a given time interval t_1 to t_2 is such that the action integral

$$S = \int_{t_1}^{t^2} \mathcal{L} dt \tag{2}$$

is stationary when taken along the actual path.
This in turns means (by calculus of variations) that the **Euler-Lagrange equations** hold, where $q_1, q_2, \ldots$ are *generalized coordinates* and $i = 1, 2, \ldots$:

$$\frac{\partial \mathcal{L}}{\partial q_i} = \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{q}_i} \tag{3}$$

1. *Projectile motion*

   Write down the Lagrangian for a projectile (neglect air resistance) in terms of Cartesian coordinates with z being the upward vertical direction. Find the three Euler-Lagrange equations and show that they are exactly what you'd expect for the equations of motion.

2. *Harmonic oscillator*

   Write down the Lagrangian for a one-dimensional particle moving along the x direction subject to a force $F = -kx$ (where k is a positive constant.) Find the Euler-Lagrange equation and solve it.

3. *Inclined plane*

   Consider a mass m moving on a frictionless plane that's inclined at an angle α over the horizontal. Write down the Lagrangian in terms of coordinates x measured horizontally across the slope and y measured down the slope. (Treat the system as two-dimensional, but include gravitational potential energy.) Find the two Euler-Lagrange equations and show that they are what you should have expected.

4. *Polar coordinates*

Polar coordinates (r, ϕ), are defined by the transformation

$$x = r\cos\phi$$
$$y = r\sin\phi$$

(a) Show that the kinetic energy in polar coordinates is

$$T = \frac{1}{2}m(\dot{r}^2 + (r\dot{\phi})^2)$$

(b) Show that in polar coordinates

$$d\vec{r} = dr\,\hat{r} + r d\phi\,\hat{\phi}$$

This can be done using calculus, or by drawing some pictures.

(c) Show that in polar coordinates

$$\nabla f = \frac{df}{dr}\hat{r} + \frac{1}{r}\frac{df}{d\phi}\hat{\phi}$$

This can be done by computing df in two ways, and setting them equal:
(1) $df = \nabla f \cdot d\vec{r}$
(2) Writing down df using the generic multi-dimensional chain rule.

5. Show that angular momentum is conserved for a single particle (e.g., a planet) that is subject to (only) a central force. Do so in polar coordinates, where $U = U(r, \phi) = U(r)$ – the last equality holds because of our assumption that the force is central (why?).

   Write down the Lagrangian, and, using the Euler-Lagrange equations, find the equations of motion – one of which should show directly that angular momentum is conserved (it might be a good idea to write down angular momentum $l_z = (\vec{r} \times \vec{p})_z$ in polar coordinates first.)