PHYS 615 – Activity 4.4: Energy in 1-d

1. *Free Fall via Conservation of Energy*

   A ball is dropped from an initial height of $y(0) = h$. It is initally at res. I'm using the y coordinate as the one coordinate that describes the motion. As usual, I'm choosing the y direction to be vertically up (though this isn't necessarily the most convenient choice for what's to come).

   (a) Find $y(t)$ that describes the ball motion using whatever approach you like. Make sure the initial conditions $y(0) = h$, $v_y(0) = 0$ are satisfied.

   Solution: I know from PHYS 407 that this is just motion at constant acceleration, $y(t) = -\frac{1}{2}gt^2 + v_{0y}t + y_0 = -\frac{1}{2}gt^2 + h$.

   Now we'll do it using conservation of energy.

   (b) Write out $T + U(y) = E$, where the total energy E is a constant that's yet to be determined. That is, write T in terms of $\dot{y}$, and $U(y)$ as the gravitational potential energy for a particle at height y.

   Solution:

   $$\frac{1}{2}m\dot{y}^2 + mgy = E$$

   Note: I chose $U(y = 0) = 0$ (that's not the only choice possible, but it won't matter).

   (c) Plug in what you know at time $t = 0$ to determine E.

   Solution:

   $$0 + mgh = E$$

   So we found E and can rewrite the equation as

   $$\frac{1}{2}m\dot{y}^2 + mgy = mgh$$

   (d) Solve the equation for $\dot{y}$. That is, do some algebra to keep only $\dot{y}$ on the l.h.s. of your equation and move the rest to the r.h.s.

   Solution:

   $$\begin{aligned} \frac{1}{2}m\dot{y}^2 &= mgh - mgy \\ \dot{y}^2 &= 2g(h - y) \\ \dot{y} &= -\sqrt{2g(h - y)} \end{aligned}$$

   (e) Double check that the sign of $\dot{y}$ come out correctly. What sign should $\dot{y}$ have for the motion at hand. Does it? If not, remember that when you take the root to get rid of a square, there are two possible solutions (that differ in sign).

   Solution: I already put a $-$ sign in front of the square root, so I'm good, since velocity should definitely be negative (down).

(f) Use separation of variables to solve this ODE. Make sure to either use definite integrals or determine your constant of integration.

Solution:

$$\begin{aligned}
\dot{y} &= -\sqrt{2g(h-y)} \\
\frac{dy}{-\sqrt{2g(h-y)}} &= dt \\
\int_h^y \frac{dy'}{-\sqrt{2g(h-y')}} &= \int_0^t dt' \\
(-2)(-\sqrt{2g(h-y')})\frac{-1}{2g}\Big|_h^y &= t \\
\frac{-1}{g}\sqrt{2g(h-y)} - \frac{-1}{g}\sqrt{2g(h-h)} &= t \\
\frac{-1}{g}\sqrt{2g(h-y)} &= t \\
\frac{1}{g^2}2g(h-y) &= t^2 \\
h-y &= \frac{1}{2}gt^2 \\
y &= -\frac{1}{2}gt^2 + h
\end{aligned}$$

I'd say it's pretty clear here that the PHYS 407 way is much simpler, but it's always nice if things work out...

2. *Oscillation via Conservation of Energy* [This is an optional part of this activity, which you probably want to leave for the end, and if you do it, it can be used as free-choice problem down the road.]

A block of mass m is attach to a horizontal spring with spring constant k. Friction can be neglected. We place the origin $x = 0$ at the equilbrium position of the spring, so $U(x) = \frac{1}{2}kx^2$.

At time $t = 0$, the mass is sitting at the origin and we give it a sudden push so that it starts moving to the right up to some maximum displacement x_{max} and then starts moving back and continues to oscillate about the origin.

Work on separate paper!

(a) Write down the equation for conservation of energy and solve it to give the velocity $\dot{x}$ in terms of the position x and total energy E.

(b) Show that $E = \frac{1}{2}kx_{max}^2$ and substitute this into your $\dot{x}$ equation for E.

(c) Solve the ODE by separation of variables. The integral you'll get is non-trivial, but it becomes much simpler with the substitution $x = x_{max}\sin\theta$.

(d) Show that you get simple harmonic motion with a period of $\tau = 2\pi\sqrt{m/k}$.

3. *Horizontal vs Vertical Oscillation*

We again have a mass m and a spring with spring constant k, but this time, the spring is oriented vertically and the mass is attached to the bottom of the spring.

(a) Write down the (total) potential energy for the mass as a function of y (positive pointing up as usual). Put the origin at the equilbrium position of the spring, so $U_{spr} = \frac{1}{2}ky^2$, and choose your gravitational potential energy such that it is zero at $y = 0$ as well.

Solution:

$$U(y) = mgy + \frac{1}{2}ky^2$$

(b) Find the equilibrium position y_0 of the mass given that both gravity and spring force act, ie., using the $U(y)$ that you just found.

Solution: The condition for equilbrium is

$$0 = \frac{dU}{dy} = mg + ky_0 \qquad \Longrightarrow \qquad y_0 = -\frac{mg}{k}$$

(c) Now rewrite your original formula for $U(y)$ in terms of $\delta y = y - y_0$, ie., in terms of the perturbation from the equilbrium position you just found.

Solution:

$$\begin{aligned}
U(\delta y) &= mg(y_0 + \delta y) + \frac{1}{2}k(y_0 + \delta y)^2 \\
&= mg(y_0 + \delta y) + \frac{1}{2}k(y_0^2 + 2y_0\delta y + \delta y^2) \\
&= mg(-\frac{mg}{k} + \delta y) + \frac{1}{2}k\left(\frac{(mg)^2}{k^2} - 2\frac{mg}{k}\delta y + \delta y^2\right) \\
&= mg(-\frac{mg}{k} + \delta y) + \frac{1}{2}\frac{(mg)^2}{k} - mg\delta y + \frac{1}{2}k\delta y^2 \\
&= -\frac{(mg)^2}{2k} + \frac{1}{2}k\delta y^2
\end{aligned}$$

(d) How different is what you found from the potential energy of a spring alone? (E.g., the spring from the previous problem, which is moving things horizontally, so gravity is not a factor)?

Solution: Other than a constant, $U(\delta y)$ looks exactly like the potential energy for a single spring. And adding a constant to potential energy doesn't change the physics. So what we just did is verify that mass hanging on a spring behaves just like the horizontal situation, only that the equilibrium position gets shifted.

4. *Block on a cylinder*

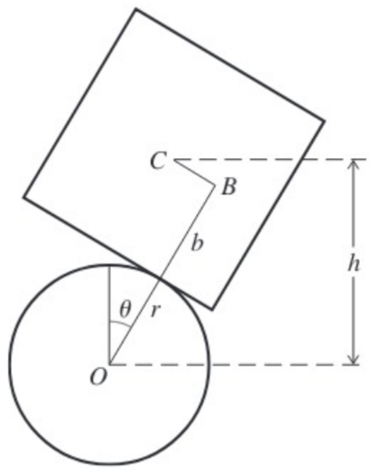


Figure 4.14 A cube, of side $2b$ and center C, is placed on a fixed horizontal cylinder of radius r and center O. It is originally put so that C is centered above O, but it can roll from side to side without slipping.

(a) Given the figure above (from the textbook), find the x and y coordinates of point B, and then point C, in terms of r, b and θ.

Solution:

$$
\begin{aligned}
x_B &= (r+b)\sin\theta \\
y_B &= (r+b)\cos\theta \\
x_C &= x_B - r\theta\cos\theta = (r+b)\sin\theta - r\theta\cos\theta \\
y_C &= y_B + r\theta\sin\theta = (r+b)\cos\theta + r\theta\sin\theta
\end{aligned}
$$

Where I used that the point where the block touches the cylinder has shifted by the arc $r\theta$ as it ”rolled down” by θ.

(b) Show that $U(\theta) = mg[(r+b)\cos\theta + r\theta\sin\theta]$. (Note: the y coordinate of C should come in handy here.)

Solution:

$$U = mgh = mgy_C = mg[(r+b)\cos\theta + r\theta\sin\theta]$$

(c) Find the derivative $dU/d\theta$.

Solution:

$$\frac{dU}{d\theta} = mg[-(r+b)\sin\theta + r\sin\theta + r\theta\cos\theta] = mg[-b\sin\theta + r\theta\cos\theta]$$

(d) Show that $\theta = 0$ is in fact an equilbrium position.

Solution: Just needs plugging in $\theta = 0$ to show $\frac{dU}{d\theta}|_{\theta=0} = 0$.

(e) Find an expression for

$$\left.\frac{d^2U}{d\theta^2}\right|_{\theta=0}$$

When is the equilbrium stable vs unstable?

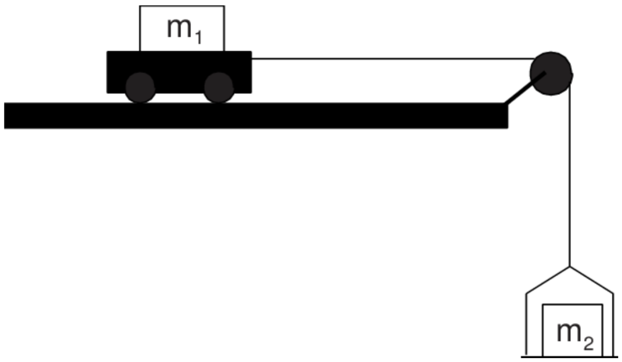
Solution:

$$\frac{d^2U}{d\theta^2} = mg[-b\cos\theta + r\cos\theta - r\theta\sin\theta] = mg[(r-b)\cos\theta - r\theta\sin\theta]$$

Plugging in zero, we get $mg(r-b)$, which is great than zero (ie., U is at a minimum, equilbrium is stable) if $r > b$, and unstable if $r < b$.

5. *Modified Atwood Machine*

You've probably seen it before:



Mass m_2 is initially at rest and at a height of h above ground. How fast is it hitting the ground? We'll do this with conservation of energy.

(a) What kind of forces are involved here? List the forces acting on the cart (m_1) and m_2. Neglect friction.

Solution: On the cart, we have tension, gravity, and normal force. On m_2, we have tension and gravity.

(b) Which of these forces are conservative? For those we'll use potential energy.

Solution: Gravity is conservative. We don't really need gravitational potential energy for the cart, though, since its height does not change, so it's just a constant.

(c) Which of these forces are non-conservative, and are actually doing work as the motion happens?

Solution: The normal force doesn't do any work, since it acts perpendicular to the motion, so it doesn't contribute to energy/work. Tension does do work, though.

(d) Write down conservation of energy for the cart and m_2 separately, e.g., $\Delta(T_1 + U_1) = W_{nc,1}$ and in particular write out the work done by non-conservative forces.
Solution:

$$\Delta(T_1 + U_1) = F_T h$$

The tension force on the cart points to the right, and it moves to the right by a distance of $\Delta x = h$ as the weight m_2 moves down a distance of h.
Similarly,

$$\Delta(T_1 + U_1) = -F_T h$$

where the minus sign happens because tension on the weight m_2 points up, but the motion is down.

(e) Argue that $W_{nc,1} = -W_{nc,2}$.
Solution: The way I've written it out, it's pretty obvious ;)

(f) Add the two energy equations together and show that you get something like $\Delta(T + U) = 0$, ie., $T + U = E = const$. What is T, U?

Solution: This works out nicely if we say $T = T_1 + T_2$ and $U = U_1 + U_2$. And I'm just going to set $U_1 = 0$, since it's constant and I can choose my constant to make life easier.

(g) Find expressions for E at the initial and final (just about hitting the ground) time, and since energy is conserved, set them equal.

Solution: Kinetic Energy is zero initially, so $0 + m_2gh = E_i$. At the final time:

$$E_f = T + 0 = \frac{1}{2}m_1v_1^2 + \frac{1}{2}m_2v_2^2$$

Setting them equal, and realizing that $v_1 = v_2 \equiv v$:

$$\begin{aligned} \frac{1}{2}(m_1 + m_2)v^2 &= m_2gh \\ v &= \sqrt{\frac{m_2}{m_1 + m_2}2gh} \end{aligned}$$

Looks reasonable (slower than free fall, and in the limit of $m_1 \to 0$, we do get the free-fall result.)