PHYS 615 – Activity 6.1: Calculus of Variations

1. *Circumference of a quarter circle*

   As we already discussed in class, unsurprisingly the circumference of a quarter circle is a quarter of the circumference of a full circle, so $L = \frac{1}{4}2\pi R = \frac{\pi}{2}R$.

   But we can calculate to get some practice with line integrals. As shown in the text, in general one can find the length of the path given by function $y(x)$ between points 1 and 2 as

$$L = \int_1^2 ds = \int_{x_1}^{x_2} \sqrt{1 + y'^2} dx \tag{1}$$

   (a) Show that the function describing a quarter circle (with its center at the origin) is (it's okay to use $R = 1$ here and in the following to make life a little bit easier, if you prefer.)

$$y(x) = \sqrt{R^2 - x^2}$$

   (b) Calculate the derivative $y' \equiv \frac{dy}{dx}$ and show that it is equal to

$$y'(x) = \frac{-x}{\sqrt{R^2 - x^2}}$$

   (c) Plug y' into the integral to calculate L above and simplify. Show that your integral can be written as

$$L = \int_0^R \frac{1}{\sqrt{1 - (x/R)^2}} \, dx$$

(d) To actually solve this integral, use the substitution $x/R = \sin u$.

The Euler-Lagrange Equation
Here is the big take-away from this chapter (ie., Calculus of Variations):
An integral of the form

$$S = \int_{x_1}^{x_2} f(y, y', x)dx$$

taken along a path $y = y(x)$ (and $y' = y'(x)$) is stationary with respect to variations of that path if and only if $y(x)$ satisfies the Euler-Lagrange equation

$$\frac{\partial f}{\partial y} - \frac{d}{dx}\frac{\partial f}{\partial y'} = 0$$

2. *The shortest path*

   We have seen that the length of a path given by $y = y(x)$ is

$$L = \int_1^2 ds = \int_{x_1}^{x_2} \sqrt{1 + y'^2}dx$$

   (a) Write down the equation for $f(y, y', x)$ so that this integral for L takes the standard form S where the Euler Lagrange equations apply (see above).

   (b) Find the partial derivatives $\frac{\partial f}{\partial y}$ and $\frac{\partial f}{\partial y'}$.

   (c) Plug these partial derivatives into the Euler-Lagrange equation (since we're looking for the shortest path, ie., one with minimum length, the function we're looking for is definitely stationary).

(d) You should be able to see that the Euler-Lagrange equation says that the $x-$derivative of some fraction involving y' is zero, which means that it doesn't depend on x – and the only thing it might depend on in x in the first place. So it must be constant. Set the constant term equal to C_1 and solve for $y'(x)$. You should be able to show that

$$y'(x) = C_2$$

(e) So apparently, $y'(x) = C_2 = const$. Use calculus to find $y(x)$ and show that you get something like $y(x) = mx + b$. If that is what you did get, did you just show that the shortest path between two points is a straight line?

3. Find the equation of the path joining the origin O to the point $P(1, 1)$ in the x-y plane that makes the integral

$$\int_O^P (y'^2 + yy' + y^2)dx$$

stationary.