PHYS 615 – HW 9

Types of homework questions

- RQ (Reading questions): prompt you to go back to the text and read and think about the text more carefully and explain in your own words. While not directly tested in quizzes, can help you think more deeply about quiz questions.
- BF (Building foundations): gives you an opportunity to build and practice foundational skills that you have, presumably, seen before.
- TQQ (typical quiz questions): Similar questions (though perhaps longer or shorter) will be asked on quizzes. But the difficulty level and skills tested will be similar.
- Design (D): These are questions in which you are given a desired outcome and asked to figure out how to make it happen. These will often also be TQQ's, but always starting with desired motion/behavior as the given.
- COMP (Computing): computing questions often related to TQQ but will never be asked on a quiz (since debugging can take so long). You will need to do at least four computing questions over the semester
- FC (free choice): allows you to decide where to put your time. Any of the following are possible: work through a section of the text or a lecture in detail; redo a problem from before; do an unassigned problem in the text; extend a computing project; try a problem using a different analytical approach (e.g. forces instead of conservation of energy).
- ACT (in-class activity): These questions are repeats of questions (or similar to) that occurred in a previous in-class activity.
- **Standard Reading Questions**: How does the reading connect with what you already know? What was something new? Ask an "I wonder" question OR give an example applying the idea in the reading.

Please remember to say something about the "Check/Learn" part at the end of solving a problem!

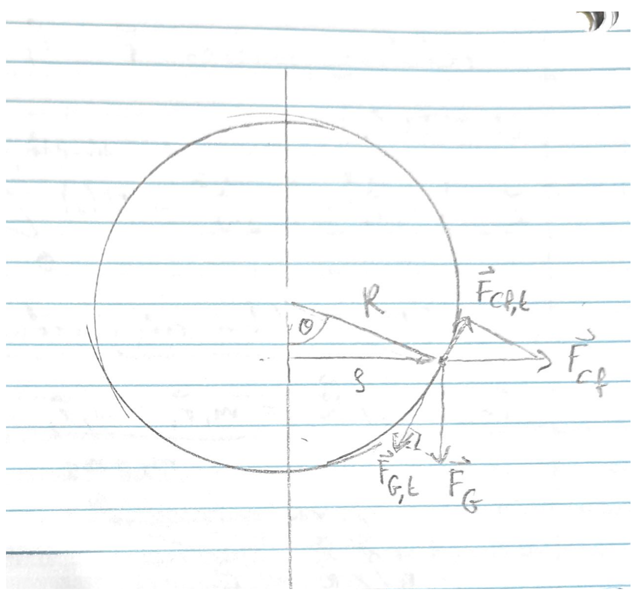
Full credit will be given at 75% of the total points possible, so you can choose a subset of problems (you can do more / all, but the score is capped at 75%)

This homework contains some previous group activities. I'm including them here in order to try to help gradescope, but you can of course hand in the original paper version I handed out in class.

1. RQ/TQQ (10 points) *Bead on Spinning Hoop*

   As mentioned in the text's example 7.6, one can understand the equilibrium positions part-way up the hoop by thinking in terms of centrifugal force, which is a perfectly legitimate way to make Newton's Laws work in non-inertial frames of reference. In particular in a frame corotating with the bead, the bead feels a centrifugal force $F_C = m\omega^2\rho$ (which points outward, in contrast to centripetal acceleration which points inward). The bead is also subject to gravity and the normal force of the wire. Verify that at the equilibrium points given by Taylor (7.71), the tangential components of the forces balance each other. (A free-body diagram will help)

   For extra credit, show that Newton's 2nd Law is also satisfied for an outside observer which looks at the spinning hoop – that outside observer is in an inertial frame of reference, so Newton's Laws hold without any extra fictitious force like centrifugal force.



   Solution:

   The sketch shows gravity and centrifugal force acting on the bead, and in particular their tangential components $F_{G,t}$ and $F_{cf,t}$.

   Those components have magnitude $F_{G,t} = mg\sin\theta$ and $F_{cf,t} = m\omega^2\rho\cos\theta$. The sketch also shows that the distance to the axis is $\rho = R\sin\theta$.

   Setting these two equal (they are supposed to cancel):

$$mg\sin\theta = m\omega^2 R\sin\theta\cos\theta \Longrightarrow g = \omega^2 R\cos\theta$$

   which rearranges to Eq. (7.76)

2. TQQ (15 points) *Bead on rotating rod*

   The center of a long frictionless rod is pivoted at the origin, and the rod is forced to rotate in the horizontal plane with constant angular velocity ω. Write down the Lagrangian for a bead with mass m threaded on the rod, using r as your generalized coordinate, where (r, ϕ) are the polar coordinates of the bead. (Notice that ϕ is not an independent variable since it is fixed by the rotation of the rod to be $\phi = \omega t$.) Solve Lagrange's equation for $r(t)$. What happens if the bead is initially at rest at the origin? If it is released from any point $r_0 > 0$, show that $r(t)$ eventually grows exponentially. Explain your results in terms of the centrifugal force $m\omega^2 r$.

   Solution: Since the bead is threaded onto the rod, the angle ϕ changes together with the rod at angular velocity ω, ie., $\phi = \omega t$ and hence $\dot{\phi} = \omega$. Since the motion is in a horizontal plane, gravitational potential energy is constant and can hence be set to zero.

$$\mathcal{L} = \frac{1}{2}m(\dot{r}^2 + (r\dot{\phi})^2) = \frac{1}{2}m(\dot{r}^2 + \omega^2 r^2)$$

   Lagrange's equation of motion:

$$m\omega^2 r = \frac{d}{dt}(m\dot{r}) = m\ddot{r}$$

   m cancels out and this equation, looks quite similar to the small-angle pendulum and other oscillations that we've been dealing with – however, it is missing a minus sign. So we're looking for a function whose second derivative is pretty much the function itself, without a minus sign, and so we'll try $e^{\lambda t}$. That works, since

$$\omega^2 e^{\lambda t} = \lambda^2 e^{\lambda t} \implies \lambda = \pm\omega.$$

   So our general solution looks like $r(t) = Ae^{\omega t} + Be^{-\omega t}$.

   In order to find the constants, we know that $r_0 = r(0) = A + B$, which isn't quite enough. But we also know that the bead's initial velocity is zero. $v(t) = \dot{r}(t) = A\omega e^{\omega t} - Be^{-\omega t}$. $0 = v(0) = A\omega - B\omega$, so $A = B$ and hence $A = B = r_0/2$. That gives the solution:

$$r(t) = \frac{r_0}{2}(e^{\omega t} + e^{-\omega t})$$

   This could be written in terms of cosh, but since we're supposed to show that this shows exponential growth for large times, that's more obvious in this form, since for large t, $e^{-\omega t} \implies 0$, so only the exponential growth term remains. Except when $r_0 = 0$, in which case the solution is $r(t) = 0$ (but it's unstable).

   The bead is accelerated outward by centrifugal force $F_C = m\omega^2 r$. That is, the farther out the bead already is, that is for growing $r(t)$, the stronger the force becomes that makes it accelerate outward farther. That is the typical underlying mechanics for exponential growth (the more covid cases already exist, the more people are infectious, so more additional people get infected, which go on to infect even more people...)

3. TQQ (15 points) *Pendulum in an elevator*

Using the usual angle ϕ as a generalized coordinate, write down the Lagrangian for a simple pendulum of length l, bob of mass m, suspended from the ceiling of an elevator that is accelerating upward at constant a. (Be careful when writing down T – it is probably safest to write the bob's velocity in component form.) Find the Lagrange equation of motion and show that it is the same as that for a normal, non-accelerating pendulum, except that g has been replaced by $g + a$. In particular, the angular frequency of small angle oscillations is $\sqrt{(g+a)/l}$.

Solution: The Lagrangian can be written down directly in Cartesian coordinates:

$$\mathcal{L} = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) - mgy$$

The bob's position relative to the pivot at the elevator ceiling is what we've seen before:

$$x = l\sin\phi \qquad y = -l\cos\phi$$

However, from the inertial reference frame outside the elevator, the y position of the pivot is accelerating upward at a together with the elevator, like $\frac{1}{2}at^2$. So in the proper reference frame

$$x = l\sin\phi \qquad y = -l\cos\phi + \frac{1}{2}at^2$$

Let's calculate time derivatives:

$$\dot{x} = l\dot{\phi}\cos\phi \qquad \dot{y} = l\dot{\phi}\sin\phi + at$$

So now we can express the Lagrangian including the constraint by using the single variable ϕ:

$$\begin{aligned}\mathcal{L} &= \frac{1}{2}m(l^2\dot{\phi}^2\cos^2\phi + l^2\dot{\phi}^2\sin^2\phi + a^2t^2 + 2lat\dot{\phi}\sin\phi - mg\left(-l\cos\phi + \frac{1}{2}at^2\right) \\ &= \frac{1}{2}m(l^2\dot{\phi}^2 + a^2t^2 + 2lat\dot{\phi}\sin\phi + mg\left(l\cos\phi - \frac{1}{2}at^2\right)\end{aligned}$$

Time for the Lagrange equation:

$$\frac{\partial\mathcal{L}}{\partial\phi} = mlat\dot{\phi}\cos\phi - mgl\sin\phi$$

$$\frac{\partial\mathcal{L}}{\partial\dot{\phi}} = ml^2\dot{\phi} + mlat\sin\phi$$

So the equation of motion is

$$mlat\dot{\phi}\cos\phi - mgl\sin\phi = \frac{d}{dt}(ml^2\dot{\phi} + mlat\sin\phi) = ml^2\ddot{\phi} + mla\sin\phi + mlat\dot{\phi}\cos\phi$$

Which nicely simplifies to

$$-m(g+a)l\sin\phi = ml^2\ddot{\phi}$$

or in the small angle approximation

$$-\omega_0^2\phi = \ddot{\phi}$$

with $\omega_0 = \sqrt{(g+a)/l}$, as we wanted to show.

4. FC (10 points) (free choice): allows you to decide where to put your time. Any of the following are possible: work through a section of the text or a lecture in detail; polish up a group work assignment from class; redo a problem from before; do an unassigned problem in the text; extend a computing project; try a problem using a different analytical approach (e.g. forces instead of conservation of energy).

   Solution:

5. TQQ / ACT (20 points) Hand in Activity 7.3

   Solution: See Activity 7.3 solution.

6. TQQ / ACT (30 points) Hand in Activity 7.4

   Solution: See Activity 7.4 solution.