PHYS 615 – HW 7

Types of homework questions

- RQ (Reading questions): prompt you to go back to the text and read and think about the text more carefully and explain in your own words. While not directly tested in quizzes, can help you think more deeply about quiz questions.
- BF (Building foundations): gives you an opportunity to build and practice foundational skills that you have, presumably, seen before.
- TQQ (typical quiz questions): Similar questions (though perhaps longer or shorter) will be asked on quizzes. But the difficulty level and skills tested will be similar.
- Design (D): These are questions in which you are given a desired outcome and asked to figure out how to make it happen. These will often also be TQQ's, but always starting with desired motion/behavior as the given.
- COMP (Computing): computing questions often related to TQQ but will never be asked on a quiz (since debugging can take so long). You will need to do at least four computing questions over the semester
- FC (free choice): allows you to decide where to put your time. Any of the following are possible: work through a section of the text or a lecture in detail; redo a problem from before; do an unassigned problem in the text; extend a computing project; try a problem using a different analytical approach (e.g. forces instead of conservation of energy).
- ACT (in-class activity): These questions are repeats of questions (or similar to) that occurred in a previous in-class activity.
- **Standard Reading Questions**: How does the reading connect with what you already know? What was something new? Ask an "I wonder" question OR give an example applying the idea in the reading.

Please remember to say something about the "Check/Learn" part at the end of solving a problem!

Full credit will be given at 75% of the total points possible, so you can choose a subset of problems (you can do more / all, but the score is capped at 75%)

This homework contains some previous group activities. I'm including them here in order to try to help gradescope, but you can of course hand in the original paper version I handed out in class.

1. TQQ (20 points) *Geodesic*

   The shortest path between two points on a *curved surface*, such as the surface of a sphere or a cylinder, is called a **geodesic**.

   (a) Use cylindrical coordinates (r, ϕ, z) to show that the length of a path joining two points on a cylinder of radius R is

   $$L = \int_{z_1}^{z_2} \sqrt{R^2\phi'(z)^2 + 1}\, dz \tag{1}$$

   if (z_1, ϕ_1) and (z_2, ϕ_2) specify the two points and we assume the path is expressed as $\phi = \phi(z)$.

   (Cylindrical coordinates are using the usual polar coordinates (r, ϕ) in the x-y plane and in addition keep the usual Cartesian z coordinate to cover all of 3-d space. People often use the letter "rho" ρ instead of r for polar coordinates – your choice.)

   Solution: On the surface of the cylinder, when z changes by dz, the position moves up by a distance dz. When ϕ changes, the position moves across by a distance of $Rd\phi$. ρ (a.k.a. r) does not change. These two displacements are orthogonal, so we can use the Pythagorean Theorem to find the total length of the displacement:

   $$ds = \sqrt{(dz)^2 + (Rd\phi)^2} = \sqrt{(dz)^2 + (R\phi' dz)^2} = \sqrt{1 + (R\phi')^2}dz$$

   Adding up all these little replacement leads to the intgral expression for L.

(b) Find the shortest path between two given points on a cylinder.

Note: The calculus of variations part of this problem is relatively straightforward, but interpreting the result is a bit more challenging. How many solutions are there? (Hint: many) Are they all actually shortest? (Hint: Probably not)

Think about these examples, with $R = 1$. Find and describe the specific shortest path to connect these points, and also describe other possible stationary paths:

i. $(\phi = 0, z = 0) \Longrightarrow (\phi = \pi/2, z = 1)$
ii. $(\phi = 0, z = 1) \Longrightarrow (\phi = \pi/2, z = 2)$
iii. $(\phi = 0, z = 0) \Longrightarrow (\phi = \pi, z = 1)$
iv. $(\phi = 0, z = 0) \Longrightarrow (\phi = 3\pi, z = 1)$

Solution: Given

$$L = \int_{z_1}^{z_2} \sqrt{R^2\phi'(z)^2 + 1}\, dz$$

, for a stationary path we must have

$$0 = \frac{\partial f}{\partial z} = \frac{d}{dz}\frac{\partial f}{\partial \phi'}$$

where $f(\phi, \phi', z) = \sqrt{R^2\phi'(z)^2 + 1}$.
Since the z derivative is zero, that means that $\frac{\partial f}{\partial \phi'}$ is constant:

$$C = \frac{\partial f}{\partial \phi'} = \frac{R^2\phi'}{\sqrt{R^2\phi'^2 + 1}}$$

One can see (or solve for ϕ' first) that $\phi'(z)$ is constant, which I'll call m, hence $\phi(z)$ is found by a simple integration

$$\phi(z) = mz + b$$

where b is another constant of integration.

To find a particular stationary path, say i., one has to determine the two constants of integration such that the path (which is a straight line) goes through both points:

$$\phi(0) = 0 \qquad \text{and} \qquad \phi(1) = \pi/2$$

So

$$m(0) + b = 0 \qquad \text{and} \qquad m(1) + b = \pi/2$$

which has the solution $m = \pi/2$ and $b = 0$, ie., $\phi(z) = (\pi/2)z$.
Similarly, we find that for ii. $\phi(z) = (\pi/2)(z - 1)$, iii. $\phi(z) = \pi z$ and iv. $\phi(z) = 3\pi z$.

All we know at this time is that these paths are stationary – they may not necessarily even be a local minimum, and even more so they may not be shortest path overall.

The last path (even though I actually meant to write $3\pi/2$, rather than 3π) meant to nudge you there. (And it might help to actually draw these paths on a piece of paper and roll it into a cylinder.) Actually, the point $(\phi, z) = (3\pi, 1)$ is the very same point as $(\pi, 1)$ – both of them are $180°$ from the origin. But the "stationary" path I found above is actually much longer for iv. than for iii. In the first case, I spiral up from $z = 0$ to $z = 1$ in a half turn. In iv., I go a turn and a half around the cylinder to go from $z = 0$ to $z = 1$.

In general, there are (infinitely) many different straight line paths that go from one point to another on a cylindrical surface, but of course only one is the shortest one. Well, sometimes, there are two shortest paths, in fact for the example I picked here, that' the case, since for the shortest path I need to talk half a turn, I could go that in either direction. That is, $\phi(z) = -\pi z$ is also a shortest path.

Going from $(0, 0)$ to $(3\pi/2, 1)$ (ie., three quarters of a turn, the solution is $\phi(z) = (3\pi/2)z$ – but it's actually shorter to go the other way, where it's only a quarter turn – the path there would be $\phi(z) = (-\pi/2)z$, ending up at $(-\pi/2, 1)$, but that's actually the very same point as $(3\pi/2, 1)$.

2. RQ / TQQ (20 points) *The Brachistochrone*

   Summarize the derivation of the brachistochrone solution, as shown in the text Example 6.2. Describe the process followed, the choice of variables, the equations used, etc.

   In addition fill in these gaps:

   (a) Derive the formula for the speed of the roller coaster $v = \sqrt{2gy}$.

   (b) Evaluate the integral (6.23) using the substitution (6.24).

   (c) (see Fig. 6.5) Derive the equation for a cycloid – that is, the curve traced out by a point on the rim of a wheel of radius a, and show that it is the same as the brachistochrone (6.26).

   Solution: Well, it is already written down in the text, so I'll not repeat all of it here.

   To get the formula for $v(y)$, one can just use conservation of energy, where at $y = 0$, there is no kinetic energy and also no gravitational potential energy mgh, because I'm putting my $h = 0$ at $y = 0$. So $E_i = 0$. Later, there is kinetic energy $\frac{1}{2}mv^2$ and also potential energy $mgh = mg(-y)$, because y is measured downward. So by conservation of energy

$$0 = E_i = E_f = \frac{1}{2}mv^2 - mgy$$

   which can be easily solved to find $v = \sqrt{2gy}$.

   To solve the integral, we'll use the given substitution

$$y = a(1 - \cos\theta) \Longrightarrow dy = a \sin\theta \, d\theta$$

$$
\begin{aligned}
x &= \int \sqrt{\frac{y}{2a-y}}dy & (2)\\
&= \int \sqrt{\frac{a(1-\cos\theta)}{2a-(a-a\cos\theta)}}a\sin\theta\, d\theta & (3)\\
&= \int \sqrt{\frac{a(1-\cos\theta)}{a+a\cos\theta}}a\sin\theta\, d\theta & (4)\\
&= \int \sqrt{\frac{1-\cos\theta}{1+\cos\theta}}a\sin\theta\, d\theta & (5)\\
&= \int \sqrt{\frac{(1-\cos\theta)(1-\cos\theta)}{(1+\cos\theta)(1-\cos\theta)}}a\sin\theta\, d\theta & (6)\\
&= \int \sqrt{\frac{(1-\cos\theta)^2}{1-\cos^2\theta}}a\sin\theta\, d\theta & (7)\\
&= \int \sqrt{\frac{(1-\cos\theta)^2}{\sin^2\theta}}a\sin\theta\, d\theta & (8)\\
&= \int a(1-\cos\theta)\, d\theta = a(\theta-\sin\theta) + const & (9)\\
& & (10)
\end{aligned}
$$

The equation for a point on the rim of a rotating wheel is $(x, y) = (-a\sin\theta, -a\cos\theta)$ (phase shifted so that the point at $\theta = 0$ is a dead top center (the negative sign for y is because y points downward). I'm then shifting it down by a, ie., adding a to y such that at the initial $\theta = 0$, the point is at the origin. Finally, as the wheel is rotating, it is also moving (rolling) to the right. As θ goes from 0 to 2π, the wheel travels its circumference $2\pi a$ to the right, so the x coordinate shifts by θa. Putting this together,

$$(x, y) = (-a\sin\theta + a\theta, -a\cos\theta + a)$$

which is indeed $x(\theta)$ and $(y(\theta)$ found for the brachistochrone.

3. FC (10 points) (free choice): allows you to decide where to put your time. Any of the following are possible: work through a section of the text or a lecture in detail; polish up a group work assignment from class; redo a problem from before; do an unassigned problem in the text; extend a computing project; try a problem using a different analytical approach (e.g. forces instead of conservation of energy).

4. TQQ / ACT (30 points) Hand in Activity 6.1

   Solution: See Activity 6.1 solution.

5. TQQ / ACT (20 points) Hand in Activity 6.2

   Solution: See Activity 6.2 solution.

Solution:

PHYS 615 – Activity 6.1: Calculus of Variations

1. *Circumference of a quarter circle*

   As we already discussed in class, unsurprisingly the circumference of a quarter circle is a quarter of the circumference of a full circle, so $L = \frac{1}{4} 2\pi R = \frac{\pi}{2} R$.

   But we can calculate to get some practice with line integrals. As shown in the text, in general one can find the length of the path given by function $y(x)$ between points 1 and 2 as

$$L = \int_1^2 ds = \int_{x_1}^{x_2} \sqrt{1 + y'^2} dx \tag{1}$$

   (a) Show that the function describing a quarter circle (with its center at the origin) is (it's okay to use $R = 1$ here and in the following to make life a little bit easier, if you prefer.)

$$y(x) = \sqrt{R^2 - x^2}$$

   (b) Calculate the derivative $y' \equiv \frac{dy}{dx}$ and show that it is equal to

$$y'(x) = \frac{-x}{\sqrt{R^2 - x^2}}$$

   (c) Plug y' into the integral to calculate L above and simplify. Show that your integral can be written as

$$L = \int_0^R \frac{1}{\sqrt{1 - (x/R)^2}} dx$$

(d) To actually solve this integral, use the substitution $x/R = \sin u$.

The Euler-Lagrange Equation
Here is the big take-away from this chapter (ie., Calculus of Variations):
An integral of the form

$$S = \int_{x_1}^{x_2} f(y, y', x) dx$$

taken along a path $y = y(x)$ (and $y' = y'(x)$) is stationary with respect to variations of that path if and only if $y(x)$ satisfies the Euler-Lagrange equation

$$\frac{\partial f}{\partial y} - \frac{d}{dx}\frac{\partial f}{\partial y'} = 0$$

2. *The shortest path*

   We have seen that the length of a path given by $y = y(x)$ is

$$L = \int_1^2 ds = \int_{x_1}^{x_2} \sqrt{1 + y'^2} dx$$

   (a) Write down the equation for $f(y, y', x)$ so that this integral for L takes the standard form S where the Euler Lagrange equations apply (see above).

   (b) Find the partial derivatives $\frac{\partial f}{\partial y}$ and $\frac{\partial f}{\partial y'}$.

   (c) Plug these partial derivatives into the Euler-Lagrange equation (since we're looking for the shortest path, ie., one with minimum length, the function we're looking for is definitely stationary).

(d) You should be able to see that the Euler-Lagrange equation says that the $x-$derivative of some fraction involving y' is zero, which means that it doesn't depend on x – and the only thing it might depend on in x in the first place. So it must be constant. Set the constant term equal to C_1 and solve for $y'(x)$. You should be able to show that

$$y'(x) = C_2$$

(e) So apparently, $y'(x) = C_2 = const$. Use calculus to find $y(x)$ and show that you get something like $y(x) = mx + b$. If that is what you did get, did you just show that the shortest path between two points is a straight line?

3. Find the equation of the path joining the origin O to the point $P(1, 1)$ in the x-y plane that makes the integral

$$\int_O^P (y'^2 + yy' + y^2)dx$$

stationary.

PHYS 615 – Activity 6.2: Calculus of Variations II

The Euler-Lagrange Equation
Here is, again, the big take-away from this chapter (ie., Calculus of Variations):
An integral of the form

$$S = \int_{x_1}^{x_2} f(y, y', x) dx$$

taken along a path $y = y(x)$ (and $y' = y'(x)$) is stationary with respect to variations of that path if and only if $y(x)$ satisfies the Euler-Lagrange equation

$$\frac{\partial f}{\partial y} - \frac{d}{dx}\frac{\partial f}{\partial y'} = 0$$

[The following is copied, mostly literally, from the text.]
The procedure for using the Euler-Lagrange equation is this:

- Set up the problem such that the quantity whose stationary path you seek is expressed as an integral in the standard form

  $$S = \int_{x_1}^{x_2} f(y, y', x) dx$$

  where $f(y, y', x)$ is the function appropriate to your problem.

- Write down the Euler-Lagrange equation (see above) in terms of the specific function $f(y, y', x)$ for your problem.

  $$\frac{\partial f}{\partial y} - \frac{d}{dx}\frac{\partial f}{\partial y'} = 0$$

- Finally, solve (if possible) the differential equation for the function $y(x)$ that you just wrote down – the solution(s) $y(x)$ will be the required stationary path.

1. *Geodesic*

The shortest path between two points on a *curved surface*, such as the surface of a sphere, is called a **geodesic**. To find a geodesic, one first has to set up an integral that gives the length of a path on the surface in question. This will always be similar to the integral S above, but may be more complicated (depending on the nature of the surface) and may involve different coordinates than x and y. To illustrate this, use spherical coordinates (r, θ, ϕ) to show that the length of a path joining two points on a sphere of radius R is

$$L = R \int_{\theta_1}^{\theta_2} \sqrt{1 + \sin^2\theta\, \phi'(\theta)^2} d\theta \tag{1}$$

if (θ_1, ϕ_1) and (θ_2, ϕ_2) specify the two points and we assume the path is expressed as $\phi = \phi(\theta)$.

(a) Before deriving the formula above, let's first do an example, with the sphere being Earth. On Earth the coordinates typically used are: ϕ is the "longitude", that is the east-west part of the location, and it is measured with 0 at the "prime meridian", the north-south line that passes through Greenwich, London. The range of ϕ in geography is $180°W \ldots 0 \ldots 180°E$. In the usual spherical coordinates we have phi go from 0 to $360\circ = 2\pi$, so we'll use that, but this isn't really fundamentally different.

In geography, the other quantity used is latitude, that is how far north or south from the equator the location is. The range goes from $90°N$ (north pole) through 0 (equator) to $90°S$ (south pole). FWIW, UNH is at $43.1340°N, 70.9264°W$. Spherical coordinates, which we'll use, are a bit different in that θ, sometimes called co-latitude goes from 0 (north pole) through $90°$ (equator) to $180°$ (south pole).

Calculate the length of the of the prime meridian, that is, the straight line on Earth that connects the north pole and the south pole and passes through Greenwich.

Earth's radius is $R_E = 3959$ mi ≈ 4000 mi, but you don't actually need to plug in any numbers, unless you care to get an answer in miles. Express the prime meridian as a path as used in the interal for L above, that is, find the function $\phi(\theta)$ and also the limits θ_1 and θ_2.

Evaluate the integral above for this specific path, and check it against what you'd find using basic geometry.

Solution: The prime meridian starts at the north pole $\theta_1 = 0$ and ends at the south pole $\theta_2 = \pi$. Longitude is zero all along, so $\phi(\theta) = 0$.

That makes the integral quite trivial since given $\phi(\theta)$ is constant, $\phi' = 0$, but hey, why complain...?

$$L = R_E \int_0^{\pi} \sqrt{1 + \sin^2\theta\, \phi'(\theta)^2} d\theta = R_E \int_0^{\pi} d\theta = \pi R_E$$

.

And since going from north pole to south pole, we're halfway around Earth, and the full circle would be $2\pi R_E$, that is what we'd expect.

(b) In order to derive the general expression for the length of the path, it makes sense to start with $L = \int_1^2 ds$. Now we'll need to find a how long that small segment of path ds is where we start at some (R, θ, ϕ) and end at $(R, \theta + d\theta, \phi + d\phi)$. It should help to sketch the situation, which should look like a small triangle (with a right angle), so that you can use the Pythagorean Theorem. It may be a bit tricky to find the length of the two sides of the triangle first. Once you got that, though, it's down to replacing $d\phi$ by using $d\phi/d\theta = \phi'$.

Solution: The small triangle will have one side taking us south by the distance $Rd\theta$. The other side takes us east by $R\sin\theta\, d\phi$. So

$$ds = \sqrt{(Rd\theta)^2 + (R\sin\theta\, d\phi)^2} = \sqrt{1 + \sin^2\theta\, \phi'(\theta)^2} d\theta$$

Therefore we do in fact the total path length as

$$L = R \int_{\theta_1}^{\theta_2} \sqrt{1 + \sin^2\theta\, \phi'(\theta)^2} d\theta$$

(c) Now that we have confirmed the expression for the length of a path on the surface of a sphere, show that the shortest path between two points is a great circle.

Hints: The integrand $f(\phi, \phi', \theta)$ does not actually depend on ϕ. So as with the shortest path in a plane, one partial derivative in the Euler-Lagrange equation is zero, which means that the other partial derivative is constant. This gives you ϕ' as a function of θ. You can avoid doing the integral by the following trick: There is no loss of generality in choosing your z axis to pass through point 1. Show that with this choice the constant is necessarily zero, and describe the corresponding geodesics.

Solution: We find that our independent variable is θ, our dependent variable is $\phi = \phi(\theta)$ and $f(\phi, \phi', \theta) = \sqrt{1 + \sin^2\theta\, \phi'(\theta)^2}$. Hence

$$\frac{\partial f}{\partial \phi} = 0$$

and therefore

$$\frac{d}{d\theta}\frac{\partial f}{\partial \phi'} = 0$$

which means the partial derivative term itself is constant:

$$C = \frac{\partial f}{\partial \phi'} = \frac{\sin^2\theta\phi'}{\sqrt{1 + \sin^2\theta\, \phi'^2}}$$

Without loss of generality, we can choose our coordinate system such that $\theta_1 = 0$ for the starting point. Since $\sin\theta_1 = 0$, that means C has to be zero (it is constant, ie., the same for any θ and corresponding ϕ'). That in turn means that $\phi' = 0$ everywhere (except maybe at $\theta = 0$ and π), so $\phi = const$. The lines described by $\phi = const$ are meridians, going purely north-south, and as such are clearly great circles, which is what we meant to show.