PHYS 615 – Quiz 6: Two-body Problems, Kepler Orbits

Name: ____________________________________

Instructions: You have 40 minutes to work on this quiz.
If you get stuck on one part, still try the other parts. Some parts are independent; for dependent parts, I'll give you full credit if your process is correct, even in your input from another part is incorrect.
Possibly useful physics equations:

$$\vec{F}_{net} = m\vec{a} \qquad F_{fk} = \mu_k F_N \qquad F_{fs} \leq \mu_s F_N \qquad F_G = mg \qquad \vec{F}_{kind,A\,on\,B} = -\vec{F}_{kind,B\,on\,A}$$

$$\vec{F}_{D,quad} = -cv^2\hat{v} \qquad \vec{F}_{D,lin} = -bv\hat{v}$$

$$\vec{p} = m\vec{v} \qquad \dot{\vec{p}} = \vec{F} \qquad \vec{l} = I\vec{\omega} = \vec{r}\times\vec{p} \qquad \dot{\vec{l}} = \vec{\Gamma} = \vec{r}\times\vec{F}$$

$$\vec{R}_{CM} = \frac{1}{M}\int_M \vec{r}dm = \frac{m_1\vec{r}_1 + m_2\vec{r}_2}{m_1+m_2} \text{ (in the case of two bodies)}$$

$$\mathcal{L} = T - U \qquad T = \frac{1}{2}mv^2 \qquad T_{rot} = \frac{1}{2}I\omega^2$$

$$\text{reduced mass } \mu = \frac{m_1 m_2}{m_1+m_2}$$

$$\text{effective potential energy } U_{eff}(r) = U(r) + U_{cf}(r) = U(r) + \frac{l^2}{2\mu r^2}$$

For a planet or comet, the force is $F = Gm_1m_2/r^2 = \gamma/r^2$. The orbit is described by

$$r(\phi) = \frac{c}{1+\epsilon\cos\phi} \quad \text{where } c = l^2/\gamma\mu \text{ and } \epsilon \geq 0 \text{ is called eccentricity.}$$

The eccentricity ϵ is related to energy by

$$E = \frac{\gamma^2\mu}{2l^2}(\epsilon^2 - 1)$$

An elliptical orbit can be described in Cartesian coordinates as

$$\frac{(x+d)^2}{a^2} + \frac{y^2}{b^2} = 1$$

with these constants:

$$a = \frac{c}{1-\epsilon^2}, \qquad b = \frac{c}{\sqrt{1-\epsilon^2}}, \qquad d = a\epsilon$$

Possibly useful math equations:

$$\frac{d}{dx}x^n = nx^{n-1} \qquad \int x^n dx = \frac{1}{n+1}x^{n+1} \qquad \text{(for any } n \neq 0\text{, including fractions.)}$$

$S = \int_{t_1}^{t_2} \mathcal{L}(q,\dot{q},t)dt$ is stationary with respect to variations of the path $q(t)$ iff $\frac{\partial\mathcal{L}}{\partial q} = \frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{q}}$
Polar coordinates: $x = r\cos\phi$, $y = r\sin\phi$.

1. *Relative and CM coordinates*

   (a) *(10 points)* Given the relations $\vec{r}_1 = \vec{R} + \frac{m_2}{M}\vec{r}$ and $\vec{r}_2 = \vec{R} - \frac{m_1}{M}\vec{r}$, show that the center of mass for the two objects at $\vec{r}_1$ and $\vec{r}_2$ is indeed $\vec{R}$, and that their relative position is indeed $\vec{r}$.

   (b) *(20 points)*

   Starting from the kinetic energy of the system of the two masses at $\vec{r}_1$ and $\vec{r}_2$, show that their kinetic energy can be expressed as

$$T = \frac{1}{2}M\dot{\vec{R}}^2 + \frac{1}{2}\mu\dot{\vec{r}}^2$$

(c) *(10 points)*

When is it useful to switch from the coordinates of the two bodies $\vec{r}_1$ and $\vec{r}_2$ to CM position $\vec{R}$ and relative position $\vec{r}$? Explain.

2. *(30 points) Kepler orbits*

What kind of orbits can a planet or comet moving in our solar system follow? Make a table that describes the orbit by its geometry and relate that to eccentricity ϵ and energy of the body. Also include a sketch for each case.

3. *(30 points) Satellite orbit*

   The height of a satellite at perigee is 300 km above the Earth's surface ($R_E = 6400$ km), and it is 3000 km at apogee. Find the orbit's eccentricity. If we take the orbit to define the x-y plane and the major axis in the x direction, with the Earth at the origin, what is the satellite's height when it crosses the y axis?