PHYS 615 – HW 2

Types of homework questions

- RQ (Reading questions): prompt you to go back to the text and read and think about the text more carefully and explain in your own words. While not directly tested in quizzes, can help you think more deeply about quiz questions.
- BF (Building foundations): gives you an opportunity to build and practice foundational skills that you have, presumably, seen before.
- TQQ (typical quiz questions): Similar questions (though perhaps longer or shorter) will be asked on quizzes. But the difficulty level and skills tested will be similar.
- Design (D): These are questions in which you are given a desired outcome and asked to figure out how to make it happen. These will often also be TQQ's, but always starting with desired motion/behavior as the given.
- COMP (Computing): computing questions often related to TQQ but will never be asked on a quiz (since debugging can take so long). You will need to do at least four computing questions over the semester
- FC (free choice): allows you to decide where to put your time. Any of the following are possible: work through a section of the text or a lecture in detail; redo a problem from before; do an unassigned problem in the text; extend a computing project; try a problem using a different analytical approach (e.g. forces instead of conservation of energy).
- **Standard Reading Questions**: How does the reading connect with what you already know? What was something new? Ask an "I wonder" question OR give an example applying the idea in the reading.

Please remember to say something about the "Check/Learn" part at the end of solving a problem!

Full credit will be given at 75% of the total points possible, so you can choose a subset of problems (you can do more / all, but the score is capped at 75%)

1. RQ/COMP (5 points) **Euler's Method:** If you'd like to learn more about Euler's method, select this question. Either read https://tutorial.math.lamar.edu/classes/de/eulersmethod.aspx. Or watch Khan Academy on Euler's Method. Answer the standard three reading questions above. This is a good opportunity to review Activity 1.3.

2. COMP (15 points – **required**) Write code to solve a differential equation We will solve example 1.2 for a skateboard in a half pipe. We want to solve the differential equation given in Eq. 1.51:

$$\ddot{\phi} = -\frac{g}{R}\sin\phi \tag{1}$$

But for the time being, we'll actually just do the small-angle approximation, so that we know the analytic solution:

$$\ddot{\phi} = -\frac{g}{R}\phi \tag{2}$$

(a) The first step is to change the second order diff eq into 2 first order equations by defining $\omega = \dot{\phi}$. This gives two coupled equations for $\dot{\omega}$ and $\dot{\phi}$. Write those equations out. (Notation: The text defines a different ω – a constant, which we called ω_0 instead in order to not confuse it, since it's just a constant, not a function of time.)

$$\omega_0 = \sqrt{\frac{g}{R}} \tag{3}$$

(b) Set expectations: Sketch your expectation for ϕ and ω as a function of time. That is, do they oscillate? Go to zero? Go to some other asymptotic value? Go to infinity?

(c) Write code: The simplest method to solve a differential equation is Euler's method. Write code using Euler's method that integrates the differential equation for 20 s, given the initial $\phi_0 = .1$ (in radians), initial $\omega = 0$, g (9.8 m/s) and R (5 m) using Euler's method. Run the code using two different values for Δt = 0.1 and 0.01 (We will be doing more accurate integration methods with Runge Kutta later.) [Keep this code. We will be adding on to it throughout the semester!] Plot the results. Are the results in line with your expectations from above?

Solution: See https://github.com/germasch/hw/blob/main/notebooks/euler-skateboard.ipynb (Note: Sometimes github has issues displaying Jupyter notebooks. In that case, reload the page and it usually works the 2nd time around.)

3. RQ/BF (5 points) Go back to your differential equation text book or to this page https://tutorial.math.lamar.edu/classes/de/separable.aspx to (re)learn about separable differential equations. Answer the standard reading questions above. Note that Paul gives a more rigorous way to move the dt to the other side of the equation.

4. TQQ (5 points) **A simple differential equation:** Solve the differential equation

$$\ddot{x} = \frac{g}{\mu_s} \tag{4}$$

(Not that it matters for the following, but this comes from the $a = \frac{g}{\mu_s}$ that we found for problem about the block pushing on another block such that it doesn't fall down.)

While this might sound scary, this is just another way of saying "calculate the (time) integral of the rhs to get $v(t) = \dot{x}$, and then integrate again to get $x(t)$. Remember constants of integration.

(a) Find the general solution to the ODE.

(b) How would you verify that what you get is indeed a correct solution to the differential equation?

(c) Why was this differential equation relatively easy to solve as opposed to the one we got from the skateboard in a half pipe?

(d) How many constants of integration do you have in your solution? Does that match the order of the differential equation (the order of this ODE is 2, since the highest derivative in it is the 2nd derivative.) I'll often use "ODE" (ordinary differential equation) as an acronym for "differential equation".

Solution:

4. $\ddot{x} = \frac{g}{\mu_s}$

(a) This equation can be integrated directly, or one could see this as already separated variables:

$$\int \ddot{x}\, dt = \int \frac{g}{\mu_s}\, dt$$

$$\dot{x} = \frac{g}{\mu_s} t + C_0$$

again: $\int \dot{x}\, dt = \int \left(\frac{g}{\mu_s} t + C_0\right) dt$

$$\underline{\underline{x(t) = \frac{1}{2}\frac{g}{\mu_s} t^2 + C_0 t + C_1}}$$

(b) Calculate derivatives, plug into ODE

(c) Because the r.h.s doesn't depend on x (nor on t, though after the first integration it does, but that's still not a problem)

(d) 2 constants of integration, consistent with 2^{nd} order ODE.

5. TQQ, D (5 points) **Back to the skateboard problem**

   In class, we used the small angle approximation to derive the ODE

   $$\ddot{\phi} = -\omega_0^2 \phi \tag{5}$$

   (where ω_0 is some presumably known parameter, ie., the angular frequency). We used some educated guessing to find the general solution

   $$\phi(t) = A \cos \omega_0 t + B \sin \omega_0 t \tag{6}$$

   (a) How would you go about verifying that this truly is a solution to the ODE above? (You don't have to do it, though it might not hurt to do so.)

   (b) Find the angular velocity $\omega(t) = \dot{\phi}$, that is, how fast the skateboard is actually moving at a given point in time.

   Note that there is some possible confusion here, with all the ω's. ω_0 is the (constant) angular frequency of the oscillation itself, so that's basically telling you how many times per second the skateboard is going back and forth. (A better way might be to think in terms of the regular frequency f, related by $\omega_0 = 2\pi f$.) On the other hand, the angular velocity tells you how fast the angle ϕ is changing at a given point in time. E.g., if you put the skateboard at $30°$ and let go at time 0, at that very time, the angle is not changing yet, so angular velocity is still 0, but it's just about to increase (well, technically, it decreases since it goes to a negative value, as the angle starts to decrease towards 0, ie., the bottom of the halfpipe.)

   (c) Let's say the skateboard is being held at an angle of ϕ_0 (at rest), and then let go at time $t = 0$. Determine the constants A, B in the general solution above.

   Solution:

5. (a) Calculate derivatives, plug into ODE

(b) $\omega(t) = \frac{d}{dt}\left(A\cos\omega_0 t + B\sin\omega_0 t\right)$

$= -A\omega_0 \sin\omega_0 t + B\omega_0 \cos\omega_0 t$

(c) $\phi(0) = \phi_0$

$A\underbrace{\cos\omega_0 0}_{1} + \cancel{B\sin\omega_0 0} = \phi_0$

$\Rightarrow A = \phi_0$

$\omega(0) = 0$

$\cancel{-A\omega_0\sin\omega_0 0} + B\omega_0\cos\omega_0 0 = 0$

$\Rightarrow B = 0$

So $\omega(t) = \phi_0\cos(\omega_0 t)$

6. TQQ (4 points) Look at the following differential equations. If they are separable, put them in the form that allows us to integrate them (but you do not need to do the integration). If they are not separable, state that. Take t and v to be our independent and dependent variables, a, b are constants.

   (a)
   $$\frac{dv}{dt} = av^2 + t^3$$
   (b)
   $$\frac{dv}{dt} = b + t^3$$
   (c)
   $$\frac{dv}{dt} = \frac{1}{b + v^3}$$
   (d)
   $$\frac{dv}{dt} = c\sqrt{vt}$$

   Solution:

   (a) not possible
   (b)
   $$dv = (b + t^3)dt$$
   (c)
   $$\frac{dv}{b + v^3} = dt$$
   (d)
   $$\frac{dv}{\sqrt{v}} = c\sqrt{t}dt$$

7. BF (5 points) From the definition of Taylor series, derive the MacLaurin series (expanding around $t = 0$) for e^t and e^{it} (where $i^2 = -1$) out to the fourth term.

   Solution:

$$\begin{aligned}
e^t &= e^0 + t\left.\frac{d(e^t)}{dt}\right|_0 + \frac{1}{2}t^2\left.\frac{d^2(e^t)}{dt^2}\right|_0 + \frac{1}{3!}t^3\left.\frac{d^3(e^t)}{dt^3}\right|_0 + \frac{1}{4!}t^4\left.\frac{d^4(e^t)}{dt^4}\right|_0 \\
&= 1 + t + \frac{1}{2}t^2 + \frac{1}{6}t^3 + \frac{1}{24}t^4 \\
e^{it} &= e^0 + t\left.\frac{d(e^{it})}{dt}\right|_0 + \frac{1}{2}t^2\left.\frac{d^2(e^{it})}{dt^2}\right|_0 + \frac{1}{3!}t^3\left.\frac{d^3(e^{it})}{dt^3}\right|_0 + \frac{1}{4!}t^4\left.\frac{d^4(e^{it})}{dt^4}\right|_0 \\
&= 1 + it - \frac{1}{2}t^2 - \frac{1}{6}it^3 + \frac{1}{24}t^4
\end{aligned}$$

8. TQQ (5 points) The book derives $v(t)$ for an object falling and subject to linear drag. The solution is (Eq 2.33)

$$v(t) = v_t \left(1 - e^{-t/\tau}\right)$$

where $\tau = m/b$ and $v_t = mg/b$.

(a) Argue that in the case of linear drag, we would expect $v(t) \approx gt$ for short times.

(b) Use the Taylor expansion of the exponential function to verify that this solution meets that expectation.

(c) What do we mean by "short" times in this problem? "Short" compared to what? And how does the math set this time scale?

Solution:

(a) As the object starts from rest, its initial speed will be zero, and for some (short) time, it'll still be small (but growing). As long as its small, the drag force is small, and so we can neglect it and just have (almost) constant acceleration due to gravity, ie., $v(t) \approx gt$.

(b) The Taylor expansion gives $e^{-t/\tau} \approx 1 - \frac{t}{\tau}$. Plugging that in:

$$v(t) = v_t \left(1 - e^{-t/\tau}\right) \approx v_t(1 - (1 - t/\tau)) = \frac{v_t}{\tau} t = gt$$

(c) τ sets the the time scale. We can see this by plugging in the expansion of $v(t) \approx v_t \frac{t}{\tau}$:

$$\dot{v} = \frac{1}{\tau}(-v + v_t) \approx \frac{1}{\tau}\left(-v_t \frac{t}{\tau} + v_t\right) = \frac{v_t}{\tau}\left(1 - \frac{t}{\tau}\right)$$

Here we can see that acceleration is approximately constant if $t \ll \tau$ ie., $t/\tau \ll 1$.

9. BF, TQQ (10 points) Introduction/Review of Hyperbolic functions. These functions will arise several times this year as solutions to problems, so we will begin by becoming a bit more familiar with them. Beginning with the definitions of the hyperbolic functions:

$$\cosh x = \frac{e^x + e^{-x}}{2}$$
$$\sinh x = \frac{e^x - e^{-x}}{2}$$
$$\tanh x = \frac{\sinh x}{\cosh x}$$

(a) Evaluate e^x, e^{-x}, $\sinh x$, $\cosh x$, $\tanh x$ at $x = -\infty, 0, \infty$.

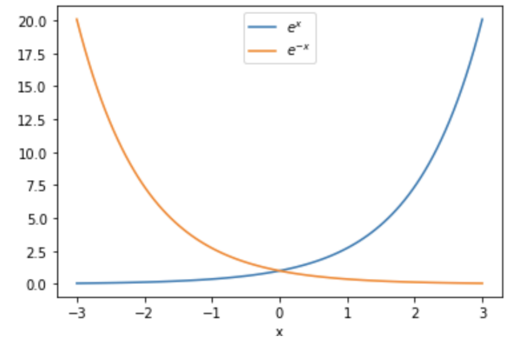
Solution:

$f(x)$	$f(\infty)$	$f(0)$	$f(\infty)$
e^x	0	1	∞
e^{-x}	∞	1	0
$\sinh x$	$-\infty$	0	∞
$\cosh x$	∞	1	∞
$\tanh x$	-1	0	1

(b) Sketch e^x and e^{-x} for both positive and negative values of x.

Solution:

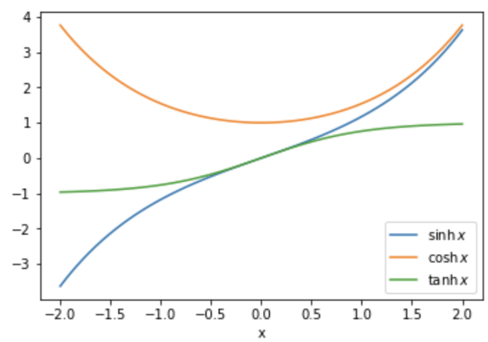
[124]:
```python
x = np.linspace(-2, 2, 100)
plt.plot(x, np.exp(x), label='$e^x$')
plt.plot(x, np.exp(-x), label='$e^{-x}$')
plt.xlabel("x")
plt.legend();
```



(c) Sketch $\cosh x$, $\sinh x$, $\tanh x$.

Solution:

[129]:
```python
x = np.linspace(-2, 2, 100)
plt.plot(x, np.sinh(x), label='$\sinh x$')
plt.plot(x, np.cosh(x), label='$\cosh x$')
plt.plot(x, np.tanh(x), label=r'$\tanh x$')
plt.xlabel("x")
plt.legend();
```



(d) Show $\cosh^2 x - \sinh^2 x = 1$.

Solution:

$$\begin{aligned}
\cosh^2 x - \sinh^2 x &= (\cosh x - \sinh x)(\cosh x + \sinh x) \\
&= e^{-x} e^{x} \\
&= 1
\end{aligned}$$

(e) Show that the derivative of the cosh is the sinh, and vice versa.

Solution:

$$\begin{aligned}
\frac{d \cosh x}{dx} &= \frac{1}{2}\left(\frac{de^x}{dx} + \frac{de^{-x}}{dx}\right) \\
&= \frac{1}{2}\left(e^x - e^{-x}\right) = \sinh x
\end{aligned}$$

and similarly for the derivative of sinh.

10. FC (10 points) (free choice): allows you to decide where to put your time. Any of the following are possible: work through a section of the text or a lecture in detail; polish up a group work assignment from class; redo a problem from before; do an unassigned problem in the text; extend a computing project; try a problem using a different analytical approach (e.g. forces instead of conservation of energy).