PHYS 615 – Activity 2.4: Vertical Quadratic Drag

This topic has been dragging on for quite a bit (pun intended), but we're about to call it done. The final thing missing, which we can actually do analytically, is motion in the vertical direction in the presence of quadratic drag and gravity.
In the following, we'll look at the case where an object is dropped vertically in the presence of quadratic drag. There's still the case of throwing / shooting something up into the sky, which will follow the same steps, but it'll happen in your homework.

1. Given the formula for quadratic drag $\vec{F}_{D,quad} = -cv^2\hat{v}$, assume the only motion is in the vertical direction. Gravity is also present. Use Newton's 2nd Law to derive the equation of motion

$$\dot{v} = g - \frac{c}{m}v^2 \tag{1}$$

where the down direction is taken to be positive. Again, we're abbreviating v_y to just v here.

Solution: This looks a lot like what we did in the previous activity:

$$m\dot{v}_x = mg - cv_x^2$$
$$\dot{v}_x = g - \frac{c}{m}v_x^2$$

and then change v_x to v.

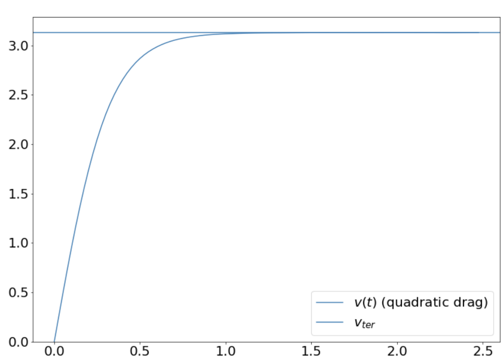
2. Find the terminal velocity v_{ter} using the equation above in terms of m, g and c. Make sure the units work out. (In fact, we could have found v_{ter} by just dimensional analysis from the three quantities above.)

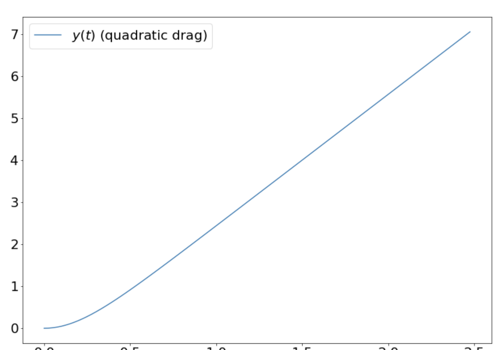
Solution: At terminal velocity, the velocity doesn't change anymore, so $\dot{v} = 0$:

$$0 = \dot{v} = g - \frac{c}{m}v_{ter}^2 \qquad \Longrightarrow \qquad v_{ter} = \sqrt{\frac{mg}{c}}$$

3. Sketch two graphs for what you expect the motion to look like, that is, $v(t)$ and $y(t)$. How does this compare to the situation with linear drag? (There's no need to try to be precise here, it's just about general characteristics of the motion.)

Solution:





The above are actual solutions, but the main characteristics we should expect are that velocity starts at 0 (that's our initial condition and then increases quickly (at a rate of g), and then that increase starts to slow down and eventually approximates the terminal velocity. Position changes more quickly as the object speeds up, but then approaches a straight line as speed approaches a constant terminal velocity.

Time to do some math. Solve the ODE given above given the initial condition $v(0) = 0$. This can be done similarly to other ODEs we've solved, that is, with separation of variables, but one of the integrals is quite non-trivial, so instead you'll just need to verify the solution I'll give you.

4. One thing which is quite similar to what happened with linear drag is that the differential equation we have now resembles the one from last class for the horizontal case: We again have an additional term (g), which makes it inhomogeneous. We do have the solutions to the homogeneous equation from the last activity. We also have a particular solution (terminal velocity). Can we just add those solutions together to get the general solution to our ODE? Why or why not? If you're not sure, you could try plugging things in.

   Solution: You could try try plugging in $v(t) = v_{ter} + v_{homogeneous}(t)$ – That'll work nicely for time derivative term, but when calculating v^2, one doesn't just get $v_{ter}^2 + v_{homogeneous}^2$, but also a mixed term and that messes everything up. For a linear ODE, once can always construct the solutions of the inhomogeneous equation and a particular solution, but his doesn't work for nonlinear ODEs, as is the case here.

5. Let's simplify the equation a bit. Rewrite it so that it looks like this:

$$\dot{v} = g\left(1 - \left(\frac{v}{v_{ter}}\right)^2\right) \tag{2}$$

   Solution: This only requires factoring out g and recognizing the definition of v_{ter}.

6. Show that the following is in fact a solution to the ODE above, and that it satisfies the initial condition $v(0) = 0$.

$$v(t) = v_{ter} \tanh \frac{t}{\tau} \tag{3}$$

   Determine the constant τ to make this solution work.

   This may come in handy: $\frac{d}{dx} \tanh x = \frac{1}{\cosh^2 x}$ and $\cosh^2 x - \sinh^2 x = 1$.

   Solution:

$$\frac{dv}{dt} = \frac{v_{ter}}{\tau} \frac{1}{\cosh^2 t/\tau}$$

Plugging this into the ODE:

$$\begin{aligned}
\frac{v_{ter}}{\tau}\frac{1}{\cosh^2 t/\tau} &= g\left(1-\tanh^2 t/\tau\right) \\
v_{ter} &= g\tau\left(1-\tanh^2 t/\tau\right)\cosh^2 t/\tau \\
v_{ter} &= g\tau\left(\cosh^2 t/\tau - \sinh^2 t/\tau\right) \\
v_{ter} &= g\tau \\
\tau &= \frac{v_{ter}}{g} = \sqrt{\frac{m}{cg}}
\end{aligned}$$

7. Now find $y(t)$ by integrating one more time. This can be done by the substitution $u = \cosh t/\tau$, or by trying a solution of the form $\ln\cosh t/\tau$.

$$\frac{du}{dt} = \frac{1}{\tau}\sinh t/\tau \qquad \Longrightarrow \qquad du = \frac{1}{\tau}\sinh t/\tau \qquad dt$$

Using substitution:

$$\begin{aligned}
y(t) &= \int v(t)dt \\
&= \int v_{ter}\tanh t/\tau \, dt \\
&= v_{ter}\int \frac{\sinh t/\tau}{\cosh t/\tau}\, dt \\
&= \tau v_{ter}\int \frac{du}{u} \\
&= \tau v_{ter}\ln\cosh t/\tau + C
\end{aligned}$$

Given the initial condition $y(0) = 0$, we find $C = 0$.