PHYS 615 – Activity 15.3: Relativistic Velocity Addition

1. *A rocket shooting bullets*

   A rocket traveling at speed $\frac{1}{2}c$ relative to frame S shoots forward bullets at speed $\frac{3}{4}c$ relative to the rocket. What is the speed of the bullets relative to frame S?

   Solution:

   $$v_x = \frac{v'_x + V}{1 + \frac{V}{c^2}v'_x} = \frac{\frac{5}{4}c}{1 + \frac{3}{8}} = \frac{\frac{5}{4}}{\frac{11}{8}}c = \frac{10}{11}c$$

2. *A rocket shooting laser pulses*

   A rocket traveling at speed $\frac{1}{2}c$ relative to frame S shoots forward laser pulses at speed of light. What is the speed of the laser pulses relative to frame S?

   Solution: Same thing with c instead of $\frac{3}{4}c$:

   $$v_x = \frac{v'_x + V}{1 + \frac{V}{c^2}v'_x} = \frac{c + \frac{1}{2}c}{1 + \frac{1}{2}} = c$$

3. *A rocket shooting perpendicular laser pulses*

   A rocket traveling at speed V in the x direction relative to frame S shoots laser pulses along the y' direction relative to its rest frame. What is the speed of the laser pulses relative to frame S?

   Solution: This time, we have to find both velocity components:

$$v_x = \frac{v'_x + V}{1 + \frac{V}{c^2}v'_x} = \frac{V}{1} = V$$
$$v_y = \frac{v'_y}{\gamma(1 + \frac{V}{c^2}v'_x)} = \frac{c}{\gamma}$$

   So the speed is given by

$$v^2 = v_x^2 + v_y^2 = V^2 + \frac{c^2}{\gamma^2} = V^2 + c^2\left(1 - \frac{V^2}{c^2}\right) = c^2$$

   The speed is the speed of light, as it should be, in any frame, since it's a light pulse.

4. *(bonus) A rocket shooting laser pulses at an angle*

   A rocket traveling at speed V in the x direction relative to frame S shoots laser pulses in the x'-y' plane in its rest frame, at an angle θ above the x' direction. What is the speed of the laser pulses relative to frame S?

   Solution: Let's presume the pulse is at an angle θ from the $x - axis$, that is, $v'_x = c\cos\theta$ and $v'_y = c\sin\theta$.

$$v_x = \frac{v'_x + V}{1 + \frac{V}{c^2}v'_x} = \frac{c\cos\theta + V}{1 + \frac{V}{c^2}c\cos\theta} = \frac{\cos\theta + \frac{V}{c}}{1 + \frac{V}{c}\cos\theta}c$$
$$v_y = \frac{v'_y}{\gamma(1 + \frac{V}{c^2}v'_x)} = \frac{c\sin\theta}{\gamma(1 + \frac{V}{c^2}c\cos\theta)} = \frac{\sin\theta}{\gamma(1 + \frac{V}{c}\cos\theta)}c$$

   So again let's find the speed

$$\begin{aligned}
v^2 = v_x^2 + v_y^2 &= \frac{c^2}{(1 + \frac{V}{c}\cos\theta)^2}\left(\left(\cos\theta + \frac{V}{c}\right)^2 + \left(\frac{\sin\theta}{\gamma}\right)^2\right) \\
&= \frac{c^2}{(1+\beta\cos\theta)^2}\left(\cos^2\theta + 2\beta\cos\theta + \beta^2 + (1-\beta^2)\sin^2\theta\right) \\
&= \frac{c^2}{(1+\beta\cos\theta)^2}\left(1 + 2\beta\cos\theta + \beta^2 - \beta^2\sin^2\theta\right) \\
&= \frac{c^2}{(1+\beta\cos\theta)^2}\left(1 + 2\beta\cos\theta + \beta^2(1 - \sin^2\theta)\right) \\
&= \frac{c^2}{(1+\beta\cos\theta)^2}\left(1 + 2\beta\cos\theta + \beta^2\cos^2\theta\right) \\
&= \frac{c^2}{(1+\beta\cos\theta)^2}\left(1 + \beta\cos\theta\right)^2 \\
&= c^2
\end{aligned}$$

Once again, we end up with speed of light, as we kinda knew we should...