PHYS 615 – Activity 2.5: Gyromotion of a charged particle

Given the Lorentz Force on a particle of charge q and mass m in a uniform magnetic field $\vec{B}$, $\vec{F} = q\vec{v} \times \vec{B}$, we can use Newton's 2nd Law to get the equation of motion

$$m\dot{\vec{v}} = q\vec{v} \times \vec{B} \tag{1}$$

We can always choose our coordinates such that the z direction is aligned with the magnetic field, so without loss of generality, $\vec{B} = B\hat{z}$.

1. Derive the equations of motion for the components of our velocity vector $\vec{v}$:

$$\dot{v}_x = \omega_0 v_y \tag{2}$$
$$\dot{v}_y = -\omega_0 v_x \tag{3}$$
$$\dot{v}_z = 0 \tag{4}$$

   What is ω_0? (The motion we're looking at here is called cyclotron motion or gyro motion. Not too surprisingly, then, $omega_0$ is called the "cyclotron frequency" or "gyro frequency".)

2. The equation of motion for v_z is not coupled to the rest of the motion, and hence can be solved individually. Use your method of choice to find $v_z(t)$. What is the meaning of the constant of integration?

In the following, we'll forget about the fact that v_z even exists and focus entirely on the "transverse" velocity $v_x\hat{x} + v_y\hat{y}$. We can put the v_z part back in the very end.
As stated in class, there are many ways to solve the coupled linear homogeneous system of ODEs for v_x, v_y. We'll use complex numbers, so that we can combine the v_x and v_y into a single complex number $\eta = v_x + iv_y$. This is particularly convenient because first of all, any complex number can be written as $A + iB$, where the real and imaginary parts A and B are essentially the same as x and y coordinates of that number in the complex plane. But a complex number can also be written as $Ce^{i\theta} = C\cos\theta + iC\sin\theta$, which is basically the same thing as polar coordinates with a radius of C and an angle of θ.

3. v_x and v_y are functions of time, which means $\eta = v_x + iv_y$ is a function of time as well. Calculate its time derivative $\dot{\eta}$ in terms of $\dot{v}_x$ and $\dot{v}_y$. Then use Eqns. (2), (3) to replace those time derivatives with their respective r.h.s. After a bit of algebra involving i, you should get

$$\dot{\eta} = -i\omega_0\eta \tag{5}$$

4. Use separation of variables to solve this ODE (the fact that η is complex-valued really doesn't make a difference for that.) Use the initial condition $\eta(t = 0) = \eta_0$.

Without more context, this is still kinda difficult to make intuitive sense of, so let's look at the particle's position. Again, life is easier and more elegant if we use a complex number for the particle's position (x, y), ie., $\xi = x + iy$.

5. Show that the equation $\dot{\xi} = \eta$ is equivalent to the system $\dot{x} = v_x$ and $\dot{y} = v_y$.

6. Now solve $\dot{\xi} = \eta$ by a method of your choice. Calling the constant of integration ξ_{gc}, you should get something like

$$\xi(t) = i\frac{\eta_0}{\omega_0}e^{-i\omega_0 t} + \xi_{gc} \tag{6}$$

   I used the subscript "gc" because its meaning is that is is the center of the gyro motion, which is called "gyro center".

7. At this point, let's put in some (made-up) numbers and make a sketch. (You could do this on a computer, too).

   Let's say $\xi_{gc} = 4 + 3i, \omega_0 = 2\pi, \eta_0 = 4\pi$. Set $t = 0, 0.1, 0.2, 0.3, \ldots$ and calculate $\xi(t)$ as well as $\eta(t)$. Mark the points $\xi(t)$ on the complex plane using the real and imaginary parts. For each of those points, indicate the corresponding velocities based on $\eta(t)$ by drawing little arrows. Hopefully things start making sense at that point. [If your calculator doesn't support complex numbers, you may have to use Euler's formula for $e^{i\theta}$ to do the i parts yourself and have the calculator just do the calculations with real numbers, sine, cosine, etc.

8. Let's call $v_0 \equiv |\eta_0|$. (Why does that make sense?) Find an expression for the gyro radius, that is, the distance between the particle's position (at any time t) and its gyro center ξ_{gc}. To do so, calculate $|\xi(t) - \xi_{gc}|$.

9. What happens if you set $\eta_0 = 4\pi i$ in the example above, ie., you shift the phase by $i = e^{i\pi/2}$?

10. Let's put the z direction back in. Solve the ODE(s) in the z direction to find $z(t)$ given the initial conditions $v_z(0) = v_{z0}$ and $z(0) = 0$. Describe the particle's 3-d trajectory.