PHYS 615 – Activity 4.3: Conservative Forces, Potential Energy, continued

1. *Potential Energy of Gravity on Earth*

   On Earth's surface, we now that the force of gravity is $F_G = mg(-\hat{y})$ if our coordinate system chosen such that the y directions points vertically up.

   We have learned (though not proved) that the condition for the work integral to be path independent is that the force field is irrational, ie., $\nabla \times \vec{F} = 0$, where the "del" (a.k.a. "nabla") operator is

   $$\nabla \equiv \hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z}$$

   (a) Let's check that the curl of $\vec{F}_G$ is indeed zero. We'll make our life a little easier by assuming we live in a 2-d (x-y) world, so we only need to check the z component of the curl.

   Write our the z component of the curl of a force $\vec{F}$ in Cartesian coordinates, where $\vec{F}$ is known in components, that is $\vec{F} = F_x\hat{x} + F_y\hat{y}$.

   Solution:

   $$(\nabla \times \vec{F})_z = \partial_x F_y - \partial_y F_x$$

   (b) Now plug in the force of gravity $\vec{F}_G$ specifically and show that the z component of the curl is indeed zero.

   Solution:

   $$\begin{aligned}(\nabla \times \vec{F}_G)_z &= \partial_x F_{G,y} - \partial_y F_{G,x} \\ &= \partial_x(-mg) - \partial_y(0) \\ &= 0\end{aligned}$$

   (c) Show that the potential energy belonging the the force of gravity above (which we now know exists, since the force of gravity depends only on $\vec{r}$ (in fact, not even that), and because its curl vanishes) is

   $$U_G = mgy + C$$

   where we typically pick the constant of integration C such that the potential energy is zero at a height of $y = 0$.

   Solution: Since F_G is constant:

   $$\begin{aligned}U &= -\int_{\vec{r}_0}^{\vec{r}} \vec{F}_G \cdot d\vec{r}' \\ &= -\vec{F}_G \cdot (\vec{r} - \vec{r}_0) \\ &= -(-mg\hat{y}) \cdot (x\hat{x} + y\hat{y} - (x_0\hat{x} - y_0\hat{y})) \\ &= mgy - mgy_0\end{aligned}$$

   where again one usually picks a reference height of $y_0 = 0$.

(d) Given $U_G = mgy$, derive the corresponding force from the relation $F = -\nabla U$.

Solution:

$$\begin{aligned}
-\nabla U_G &= -\frac{\partial U_G}{\partial x}\hat{x} - \frac{\partial U_G}{\partial y}\hat{y} \\
&= -\frac{\partial mgy}{\partial x}\hat{x} - \frac{\partial mgy}{\partial y}\hat{y} \\
&= 0 - mg\hat{y} \\
&= mg(-\hat{y})
\end{aligned}$$

We did indeed get our force of gravity back.

2. *Potential Energy for Newton's Law of Gravity*

Away from Earth's surface, we now that the force that mass M (you can consider it to be located at the origin) exerts on mass m at position $\vec{r}$ is given by Newton's Law of Gravity

$$\vec{F}_G = -G\frac{mM}{r^2}\hat{r}$$

(a) Using our usual Cartesian coordinate system, show that the x component of F_G is given by

$$F_{G,x} = -G\frac{mM}{r^2}\frac{x}{r} = -GmM\frac{x}{r^3}$$

(In order to do so, you may need to remember that $\hat{r}$ is the same direction is $\vec{r}$, but it has been shortened by some factor so that it's of unit length.)

Solution: $\hat{r} = \frac{\vec{r}}{r}$, where r is the magnitude of $\vec{r}$ – you can easily check that the magnitude of $\hat{r}$ is indeed 1. If we know the position in Cartesian coordinates x, y, z, then $\vec{r} = x\hat{x} + y\hat{y} + z\hat{z}$.

$$\begin{aligned}
\vec{F}_G &= -G\frac{mM}{r^2}\hat{r} \\
&= -G\frac{mM}{r^2}\frac{\vec{r}}{r} \\
&= -G\frac{mM}{r^2}\frac{x\hat{x} + y\hat{y} + z\hat{z}}{r}
\end{aligned}$$

So we can see the x component is indeed $F_{G,x} = -G\frac{mM}{r^2}\frac{x}{r}$.

(b) Write the magnitude r of $\vec{r} = x\hat{x} + y\hat{y} + z\hat{z}$ in terms of its components x, y, z.

Solution:

$$r = \sqrt{x^2 + y^2 + z^2}$$

(c) Show that

$$\frac{\partial r}{\partial x} = \frac{x}{r}$$

Solution:

$$\begin{aligned}\frac{\partial r}{\partial x} &= \frac{\partial}{\partial x}(x^2+y^2+z^2)^{1/2} \\ &= \frac{1}{2}(x^2+y^2+z^2)^{-1/2}(2x) \\ &= \frac{x}{r}\end{aligned}$$

(d) Show that the z component of $\nabla \times \vec{F}_G$ equals zero.
Solution:

$$\begin{aligned}(\nabla \times \vec{F}_G)_z &= \partial_x F_{G,y} - \partial_y F_{G,x} \\ &= \partial_x(-GmM\frac{y}{r^3}) - \partial_y(-GmM\frac{x}{r^3}) \\ &= -GmM\left(y\partial_x r^{-3} - x\partial_y r^{-3}\right) \\ &= -GmM\left(y(-3)r^{-4}\frac{x}{r} - x(-3)r^{-4}\frac{y}{r}\right) \\ &= -GmM\left((-3)r^{-5}yx - (-3)r^{-5}xy\right) \\ &= 0\end{aligned}$$

(e) At this point, one could redo this for the x and y components of the curl, but it's essentially the same calculation. Given that we have now shown that $\nabla \times \vec{F}_G = 0$ and $\vec{F}_G = \vec{F}_G(\vec{r})$, Newton's force of gravity is conservative. So we can calculate the corresponding potential energy

$$U_G = -\int_{\vec{r}_0}^{\vec{r}} \vec{F}_G(\vec{r}') \cdot d\vec{r}'$$

Do so and show that the result is

$$U_G = -GmM\left(\frac{1}{r} - \frac{1}{r_0}\right)$$

In order to make things a bit easier, let's first assume $\vec{r}$ and $\vec{r}_0$ are along the same direction, so $\vec{r} = r\hat{r}$ and $\vec{r}_0 = r_0\hat{r}$. Since we know that F_G is conservative you can choose any path you want, ideally a simple one ;)
Solution: I'm picking a straight line, so my integral becomes 1-d

$$\begin{aligned}U_G &= -\int_{\vec{r}_0}^{\vec{r}} \vec{F}_G(\vec{r}') \cdot d\vec{r}' \\ &= -(-GmM)\int_{r_0}^{r} \frac{1}{r'^2}\hat{r} \cdot dr'\hat{r} \\ &= GmM\int_{r_0}^{r} \frac{dr'}{r'^2} \\ &= GmM\left(\frac{-1}{r} - \frac{-1}{r_0}\right)\end{aligned}$$

(f) Generally speaking, $\vec{r}$ and $\vec{r}_0$ may not be along the same direction. Show that your result above still holds in this case.

Hint: You can pick any path you want, so you might want to pick one that has two parts: The part you've already done, and a missing piece, where the integral is easy to calculate.

Solution: Starting at $\vec{r}_0$, I'd move along a circle, staying at a distance of r_0 from the origin until I get to the ray in the $\vec{r}$ direction, ie., I'll go to $r_0\hat{r}$. Then I follow the straight line to $\vec{r}$, so the integral for the 2nd part is the same as above. For the 1st segment, $\vec{F}_G$ is towards the center of my circle, but my motion is tangential, ie., perpendicular to $\vec{F}_G$, so the integral is zero and this segment does not contribute anything, leaving me with just the 2nd segment where I already know the answer.

(g) For Newton's Law of Gravity (and Coulomb's Law), the reference location is usually put at $r_0 = \infty$. Show that the potential energy is then

$$U_G = -G\frac{mM}{r}$$

Solution: Just needs plugging in r_0.

(h) Finally, let's think about $\vec{F} = -\nabla U_G$. What do you expect do get? Do the calculation just for one component, say the x component F_x.

Solution:

$$\begin{aligned}
F_x &= (-\nabla U_G)_x \\
&= -\partial_x U_G \\
&= -\partial_x \left(-G\frac{mM}{r}\right) \\
&= GmM\frac{-1}{r^2}\partial_x r \\
&= GmM\frac{-1}{r^2}\frac{x}{r} \\
&= -GmM\frac{x}{r^3}
\end{aligned}$$

This is the x component of the force of gravity that we found earlier – as it should be.