TUTORIAL 5 · [COMPUTATION AND LOGIC]

OBJECTIVES

In this tutorial, you will:

- learn more about *sequents* and *combining predicates*;
- derive new rules in the sequent calculus for *exclusive or*, *implication*, and *equivalence*;
- do proofs in *sequent calculus*.

TASKS

Exercises 1–5 are mandatory. Exercise 6 is optional.

SUBMIT a file called `cl-tutorial-5` with your answers (image or pdf) and the file `cl-tutorial-5-QuickCheck.hs` with your code.

DEADLINE Saturday, 24th of October, 4 PM UK time

Good Scholarly Practice

Please remember the good scholarly practice requirements of the University regarding work for credit.

You can find guidance at the School page

http://web.inf.ed.ac.uk/infweb/admin/policies/academic-misconduct.

This also has links to the relevant University pages. Please do not publish solutions to these exercises on the internet or elsewhere, to avoid others copying your solutions.

For any two predicates a and b, we define **exclusive or** as:

$$a \oplus b = (a \land \neg b) \lor (\neg a \land b)$$

We derive a rule for $\oplus$ by reducing the sequent $\vDash (a \land \neg b) \lor (\neg a \land b)$:

$$\frac{\vDash a, b \qquad a, b \vDash}{\vDash a \oplus b}$$

We add this derived rule to sequent calculus as:

$$\frac{\Gamma \vDash a, b, \Delta \qquad \Gamma, a, b \vDash \Delta}{\Gamma \vDash a \oplus b, \Delta} \oplus R$$

Reduce the sequent $(a \land \neg b) \lor (\neg a \land b) \vDash$ and use the result to derive the rule $(\oplus L)$.

SOLUTION TO EXERCISE 1

We reduce $(a \wedge \neg b) \vee (\neg a \wedge b) \vDash$

$$\dfrac{\dfrac{\dfrac{a \vDash b}{a, \neg b \vDash}\,\neg L}{a \wedge \neg b \vDash}\,\wedge L \qquad \dfrac{\dfrac{b \vDash a}{\neg a, b \vDash}\,\neg L}{\neg a \wedge b \vDash}\,\wedge L}{(a \wedge \neg b) \vee (\neg a \wedge b) \vDash}\,\vee L$$

and we obtain the rule:

$$\dfrac{\Gamma, a \vDash b, \Delta \qquad \Gamma, b \vDash a, \Delta}{\Gamma, a \oplus b \vDash \Delta}\,\oplus L$$

EXERCISE 2

For any two predicates a and b, we define **implication** (also known as *conditional*) as:

$$a \to b = \neg a \vee b$$

Reduce the sequents

$$\models \neg a \vee b$$

and

$$\neg a \vee b \models$$

and use the results to write rules $(\to R)$ and $(\to L)$, respectively.

SOLUTION TO EXERCISE 2

We reduce $\models \neg a \lor b$

$$\dfrac{\dfrac{a \models b}{\models \neg a, b}\ \neg R}{\models \neg a \lor b}\ \lor R$$

and we obtain the rule

$$\dfrac{\Gamma, a \models b, \Delta}{\Gamma \models a \to b, \Delta} \to R$$

We reduce $\neg a \lor b \models$

$$\dfrac{\dfrac{\models a}{\neg a \models}\ \neg L \qquad b \models}{\neg a \lor b \models}\ \lor L$$

and we obtain the rule

$$\dfrac{\Gamma \models a, \Delta \qquad \Gamma, b \models \Delta}{\Gamma, a \to b \models \Delta} \to L$$

EXERCISE 3

MANDATORY | BEFORE TUTORIAL SESSION

For any two predicates a and b, we define **equivalence** (also known as *biconditional*) as:

$$a \leftrightarrow b = (a \wedge b) \vee (\neg a \wedge \neg b)$$

Reduce the sequents

$$\models (a \wedge b) \vee (\neg a \wedge \neg b)$$

and

$$(a \wedge b) \vee (\neg a \wedge \neg b) \models$$

and use the results to write rules $(\leftrightarrow R)$ and $(\leftrightarrow L)$, respectively.

SOLUTION TO EXERCISE 3

We reduce $\models (a \wedge b) \vee (\neg a \wedge \neg b)$

$$
\dfrac{
  \dfrac{
    \dfrac{\dfrac{}{a \models a}\, I}{\models a, \neg a}\, \neg R
    \qquad
    \dfrac{b \models a}{\models a, \neg b}\, \neg R
  }{\models a, \neg a \wedge \neg b}\, \wedge R
  \qquad
  \dfrac{
    \dfrac{a \models b}{\models b, \neg a}\, \neg R
    \qquad
    \dfrac{\dfrac{}{b \models b}\, I}{\models b, \neg b}\, \neg R
  }{\models b, \neg a \wedge \neg b}\, \wedge R
}{
  \dfrac{\models a \wedge b, \neg a \wedge \neg b}{\models (a \wedge b) \vee (\neg a \wedge \neg b)}\, \vee R
}\, \wedge R
$$

and we obtain the rule:

$$
\dfrac{\Gamma, b \models a, \Delta \qquad \Gamma, a \models b, \Delta}{\Gamma \models a \leftrightarrow b, \Delta} \leftrightarrow R
$$

SOLUTION TO EXERCISE 3 (CONT.)

We reduce $(a \wedge b) \vee (\neg a \wedge \neg b) \vDash$

$$\dfrac{\dfrac{a, b \vDash}{a \wedge b \vDash}\wedge L \qquad \dfrac{\dfrac{\dfrac{\vDash a, b}{\neg b \vDash a}\neg L}{\neg a, \neg b \vDash}\neg L}{\neg a \wedge \neg b \vDash}\wedge L}{(a \wedge b) \vee (\neg a \wedge \neg b) \vDash}\vee L$$

and we obtain the rule:

$$\dfrac{\Gamma, a, b \vDash \Delta \qquad \Gamma \vDash a, b, \Delta}{\Gamma, a \leftrightarrow b \vDash \Delta} \leftrightarrow L$$

EXERCISE 4

MANDATORY | BEFORE TUTORIAL SESSION

Use reductions to decide whether the following equations are universally valid:

1. $a \leftrightarrow b = (a \to b) \wedge (b \to a)$
2. $(a \to b) \to c = a \to (b \to c)$
3. $(a \leftrightarrow b) \leftrightarrow c = a \leftrightarrow (b \leftrightarrow c)$
4. $(a \oplus b) \oplus c = a \oplus (b \oplus c)$

Recall that in order to decide whether an equation $x = y$ is universally valid, we need to write proofs which reduce the sequents $x \vDash y$ and $y \vDash x$.

SOLUTION TO EXERCISE 4

1. We reduce $a \leftrightarrow b \vDash (a \to b) \wedge (b \to a)$

$$
\cfrac{
  \cfrac{
    \cfrac{
      \cfrac{}{a, b \vDash b}\,I \qquad \cfrac{}{a \vDash a, b}\,I
    }{a \leftrightarrow b, a \vDash b}\,{\leftrightarrow}L
  }{a \leftrightarrow b \vDash a \to b}\,{\to}R
  \qquad
  \cfrac{
    \cfrac{
      \cfrac{}{a, b \vDash a}\,I \qquad \cfrac{}{b \vDash a, b}\,I
    }{a \leftrightarrow b, b \vDash a}\,{\leftrightarrow}L
  }{a \leftrightarrow b \vDash b \to a}\,{\to}R
}{a \leftrightarrow b \vDash (a \to b) \wedge (b \to a)}\,{\wedge}R
$$

SOLUTION TO EXERCISE 4 (CONT.)

and $(a \to b) \land (b \to a) \vDash a \leftrightarrow b$

$$\dfrac{\dfrac{\dfrac{}{b \to a, a \vDash a, b}\,I \quad \dfrac{}{b \to a, a, b \vDash b}\,I}{a \to b, b \to a, a \vDash b}\,{\to}L \qquad \dfrac{\dfrac{}{a \to b, b \vDash b, a}\,I \quad \dfrac{}{a \to b, b, a \vDash a}\,I}{a \to b, b \to a, b \vDash a}\,{\to}L}{\dfrac{a \to b, b \to a \vDash a \leftrightarrow b}{(a \to b) \land (b \to a) \vDash a \leftrightarrow b}\,{\land}L}\,{\leftrightarrow}R$$

Since both sequents are universally valid, the equation holds.

SOLUTION TO EXERCISE 4 (CONT.)

2. We reduce $a \to (b \to c) \vDash (a \to b) \to c$

$$
\dfrac{
  \dfrac{
    \dfrac{}{a \to b, c \vDash c}\,I
    \qquad
    \dfrac{\dfrac{}{b \vDash b, c}\,I \qquad \vDash a, b, c}{a \to b \vDash b, c}\,{\to}L
  }{a \to b, b \to c \vDash c}\,{\to}L
  \qquad
  \dfrac{b \vDash a, c \qquad \vDash a, c}{a \to b \vDash a, c}\,{\to}L
}{
  \dfrac{a \to (b \to c), a \to b \vDash c}{a \to (b \to c) \vDash (a \to b) \to c}\,{\to}R
}\,{\to}L
$$

The proof shows that the conclusion is true whenever all three highlighted premises $\vDash a, b, c$, $b \vDash a, c$, and $\vDash a, c$ are true. One counterexample is a universe containing a thing x for which $b\,x$ is true and $a\,x$, $c\,x$ are false.

SOLUTION TO EXERCISE 4 (CONT.)

Before solving tasks 3 and 4, we make the following observations:

- The rules for equivalence and exclusive or are very similar. In fact, the premises of $\leftrightarrow R$ are the same as the ones of $\oplus L$, and the premises of $\leftrightarrow L$ are the same as those of $\oplus R$.

  This is not surprising, as these operations are each other's negation: they correspond to `==` and `=/=` on Booleans.

  It thus suffices to treat just one of equations 3 and 4; the proofs are essentially the same.

SOLUTION TO EXERCISE 4 (CONT.)

- There are other solutions to this type of exercise than the one we have seen so far, and they are sometimes shorter. For example, to show that an equation of the form $\varphi = \psi$ is universally valid, we can show that $x \vDash \varphi$ and $x \vDash \psi$ both reduce to the same collection of simple sequents (and hence to each other), and then substitute φ for x and ψ for x to derive the two implications. You can watch here a video showing how to prove the associativity of equivalence (task 3).
- Because the associativity equations involve 3 predicates, the proof trees are larger than those we have seen so far. Their structure is still straightforward, but they do require more time to write down. If you consider it too tedious, you can automate the process by writing a Haskell program. The file `xorAssoc.hs` contains an implementation of the reduction of equation 4. While we didn't expect you to write such an implementation, thinking outside the box is always good; this sort of solutions are welcome.

SOLUTION TO EXERCISE 4 (CONT.)

4. Let's prove now the associativity of exclusive or using the same method as for the previous exercises. A shorter proof can be obtained using the method presented in the video mentioned on the previous slide.

We first reduce $a \oplus (b \oplus c) \vDash (a \oplus b) \oplus c$ to 4 premises:

$$\dfrac{\dfrac{a \vDash b, c, (a \oplus b) \oplus c \qquad a, b, c \vDash (a \oplus b) \oplus c}{a \vDash b \oplus c, (a \oplus b) \oplus c}\,{\scriptstyle \oplus R} \qquad \dfrac{b \vDash c, a, (a \oplus b) \oplus c \qquad c \vDash b, a, (a \oplus b) \oplus c}{b \oplus c \vDash a, (a \oplus b) \oplus c}\,{\scriptstyle \oplus L}}{a \oplus (b \oplus c) \vDash (a \oplus b) \oplus c}\,{\scriptstyle \oplus L}$$

Next we deal with each of the premises:

$$\dfrac{\dfrac{\dfrac{}{a, b \vDash b, c}\,I \qquad \dfrac{}{a \vDash a, b, c}\,I}{a \vDash a \oplus b, b, c}\,\oplus R \qquad \dfrac{}{a \oplus b, c, a \vDash b, c}\,I}{a \vDash b, c, (a \oplus b) \oplus c}\,\oplus R$$

SOLUTION TO EXERCISE 4 (CONT.)

$$\dfrac{\dfrac{\dfrac{}{a, b, c \vDash b}\,I \qquad \dfrac{}{a, b, c \vDash a}\,I}{a, b, c, a \oplus b \vDash}\,\oplus L \qquad \dfrac{}{a, b, c \vDash c, a \oplus b}\,I}{a, b, c \vDash (a \oplus b) \oplus c}\,\oplus R$$

$$\dfrac{\dfrac{\dfrac{}{b \vDash c, a, b}\,I \qquad \dfrac{}{b, a \vDash c, a}\,I}{b \vDash c, a, a \oplus b}\,\oplus R \qquad \dfrac{}{b, c, a \oplus b \vDash c, a}\,I}{b \vDash c, a, (a \oplus b) \oplus c}\,\oplus R$$

SOLUTION TO EXERCISE 4 (CONT.)

$$
\dfrac{\dfrac{\dfrac{}{c, a \vDash b, a}\ I \qquad \dfrac{}{c, b \vDash b, a}\ I}{c, a \oplus b \vDash b, a}\ \oplus L \qquad \dfrac{}{c \vDash a \oplus b, c, b, a}\ I}{c \vDash b, a, (a \oplus b) \oplus c}\ \oplus R
$$

Combining these 5 proof trees, it follows that $a \oplus (b \oplus c) \vDash (a \oplus b) \oplus c$ is universally valid.

You can check that $(a \oplus b) \oplus c \vDash a \oplus (b \oplus c)$ is also universally valid in a similar way, or by using the Haskell program in `xorAssoc.hs`.

EXERCISE 5

We can write proofs to reduce sequents of the form $\vDash \varphi$ in order to derive simpler equivalent expressions to φ. To achieve this, we need to move every predicate in the premises of the derived rule to the right of the turnstile, using negation.

For example, for the rule for exclusive or

$$\frac{\vDash a, b \qquad a, b \vDash}{\vDash (a \land \neg b) \lor (\neg a \land b)} \quad \text{we obtain} \quad \frac{\vDash a, b \qquad \vDash \neg a, \neg b}{\vDash (a \land \neg b) \lor (\neg a \land b)}$$

which is equivalent to

$$\frac{\vDash a \lor b \qquad \vDash \neg a \lor \neg b}{\vDash (a \land \neg b) \lor (\neg a \land b)}$$

It follows that $(a \land \neg b) \lor (\neg a \land b)$ is equivalent to $(a \lor b) \land (\neg a \lor \neg b)$.

EXERCISE 5 (CONT.)

In this way we produce a conjunction of disjunctions of literals, where a **literal** is a predicate or the negation of a predicate.

We say that an expression is in **conjunctive normal form (CNF)** if it consists of a conjunction of disjunctions of literals.

Using the technique outlined on the previous slide, find equivalent CNFs for the following expressions:

1. $r \leftrightarrow (a \wedge b)$
2. $r \leftrightarrow (a \vee b)$
3. $r \leftrightarrow (a \rightarrow b)$
4. $r \leftrightarrow (\neg a)$

SOLUTION TO EXERCISE 5

1.

$$
\dfrac{\dfrac{\dfrac{\vDash a, \neg r}{r \vDash a}\,\neg L \quad \dfrac{\vDash b, \neg r}{r \vDash b}\,\neg L}{r \vDash a \wedge b}\,\wedge R \qquad \dfrac{\dfrac{\vDash r, \neg a, \neg b}{a, b \vDash r}\,\neg L}{a \wedge b \vDash r}\,\wedge L}{\vDash r \leftrightarrow (a \wedge b)}\,\leftrightarrow R
$$

$r \leftrightarrow (a \wedge b)$ is equivalent to $(a \vee \neg r) \wedge (b \vee \neg r) \wedge (r \vee \neg a \vee \neg b)$

SOLUTION TO EXERCISE 5 (CONT.)

2.

$$
\dfrac{
  \dfrac{\dfrac{\vDash a, b, \neg r}{r \vDash a, b}\,\neg L}{r \vDash a \vee b}\,\vee R
  \qquad
  \dfrac{\dfrac{\vDash r, \neg a}{a \vDash r}\,\neg L \quad \dfrac{\vDash r, \neg b}{b \vDash r}\,\neg L}{a \vee b \vDash r}\,\vee L
}{\vDash r \leftrightarrow (a \vee b)}\,\leftrightarrow R
$$

$r \leftrightarrow (a \vee b)$ is equivalent to $(a \vee b \vee \neg r) \wedge (r \vee \neg a) \wedge (r \vee \neg b)$

SOLUTION TO EXERCISE 5 (CONT.)

3.

$$
\dfrac{\dfrac{\dfrac{\vDash b, \neg r, \neg a}{r, a \vDash b}\ \neg L}{r \vDash a \rightarrow b}\ {\rightarrow}R \qquad \dfrac{\vDash a, r \qquad \dfrac{\vDash r, \neg b}{b \vDash r}\ \neg L}{a \rightarrow b \vDash r}\ {\rightarrow}L}{\vDash r \leftrightarrow (a \rightarrow b)}\ {\leftrightarrow}R
$$

$r \leftrightarrow (a \rightarrow b)$ is equivalent to $(b \vee \neg r \vee \neg a) \wedge (a \vee r) \wedge (r \vee \neg b)$

SOLUTION TO EXERCISE 5 (CONT.)

4.

$$\dfrac{\dfrac{\models \neg a, \neg r}{r \models \neg a}\,\neg L \qquad \dfrac{\models r, a}{\neg a \models r}\,\neg L}{\models r \leftrightarrow (\neg a)}\,\leftrightarrow R$$

$r \leftrightarrow (\neg a)$ is equivalent to $(\neg a \vee \neg r) \wedge (r \vee a)$

EXERCISE 6

OPTIONAL | BEFORE TUTORIAL SESSION

The file `cl-tutorial-5-QuickCheck.hs` contains a template for verifying the validity of sequents using `QuickCheck`. Read the file, pay attention to the comments, and don't worry if there are lines in the first part of the file that you don't understand (as in Tutorial 4).

Add to `cl-tutorial-5-QuickCheck.hs` three functions:

```
(+:+) :: Predicate u -> Predicate u -> Predicate u
(-:>) :: Predicate u -> Predicate u -> Predicate u
(<:>) :: Predicate u -> Predicate u -> Predicate u
```

that compute the exclusive or (`+:+`), implication(`-:>`), and biconditional (`<:>`) of two predicates.

SOLUTION TO EXERCISE 6

```haskell
(+:+) :: Predicate u -> Predicate u -> Predicate u
p +:+ q = (p &:& neg q) |:| (neg p &:& q)

(-:>) :: Predicate u -> Predicate u -> Predicate u
p -:> q = neg p |:| q

(<:>) :: Predicate u -> Predicate u -> Predicate u
p <:> q = (p -:> q) &:& (q -:> p)
```

EXERCISE 6 (CONT.)

OPTIONAL | BEFORE TUTORIAL SESSION

Define a function

```
(|=|) :: Predicate Thing -> Predicate Thing -> Bool
```

to check, for every predicates p and q, whether both sequents $p \models q$ and $q \models p$ are valid.

Write a property and use `QuickCheck` to verify the equivalence of $p \mathrel{\models\!\!\!|} q$ and $\models p \leftrightarrow q$, where p and q are predicates.

Can you use `QuickCheck` to verify your answers to Exercises 4 and 5?

SOLUTION TO EXERCISE 6 (CONT.)

```
(|=|) :: Predicate Thing -> Predicate Thing -> Bool
p |=| q = (p |= q) && (q |= p)

prop1 p q = (p |=| q) == ([] ||= (p <:> q))
-- quickCheck prop1
```