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Communication

# How good are branching rules in DPLL?

Ming Ouyang

*Department of Computer Science, Rutgers University, New Brunswick, NJ 08903, USA*

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**Abstract**

The Davis–Putnam–Logemann–Loveland algorithm is one of the most popular algorithms for solving the satisfiability problem. Its efficiency depends on its choice of a branching rule. We construct a sequence of instances of the satisfiability problem that fools a variety of "sensible" branching rules in the following sense: when the instance has $n$ variables, each of the "sensible" branching rules brings about $\Omega(2^{n/5})$ recursive calls of the Davis–Putnam–Logemann–Loveland algorithm, even though only O(1) such calls are necessary. 

*Keywords*: Branching rules; DPLL

## 1. The SAT problem

A *truth assignment* is a mapping $f$ that assigns 0 or 1 to each variable in its domain; we shall enumerate all the variables in this domain as $x_1, \ldots, x_n$. The *complement* $\bar{x}_i$ of each such variable $x_i$ is defined by $f(\bar{x}_i) = 1 - f(x_i)$ for all truth assignments $f$; both $x_i$ and $\bar{x}_i$ are called *literals*; if $u = \bar{x}_i$ then $\bar{u} = x_i$. A *clause* is a set of (distinct) literals and a *formula* is a family of (not necessarily distinct) clauses. A truth assignment *satisfies* a clause if it maps at least one of its literals to 1; the assignment *satisfies* a formula if and only if it satisfies each of its clauses. A formula is called *satisfiable* if it is satisfied by at least one truth assignment; otherwise it is called *unsatisfiable*. The problem of recognizing satisfiable formulas is known as the *satisfiability problem*, or SAT for short.

## 2. The Davis–Putnam–Logemann–Loveland algorithm

Given a formula $\mathscr{F}$ and a literal $u$ in $\mathscr{F}$, we let $\mathscr{F}|u$ denote the "residual formula" arising from $\mathscr{F}$ when $f(u)$ is set at 1: explicitly, this formula is obtained from $\mathscr{F}$ by


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DPLL(𝓕)
{
  while (𝓕 includes a clause of length at most one)
  { if (𝓕 includes an empty clause) return UNSATISFIABLE;
    {v} = any clause of length one;
    𝓕 = 𝓕|v;
  }
  while (there is a monotone literal)
  { v = any monotone literal;
    𝓕 = 𝓕|v;
  }
  if (𝓕 is empty) return SATISFIABLE;
  choose a literal u in 𝓕;
  if (DPLL(𝓕|u) = SATISFIABLE) return SATISFIABLE;
  if (DPLL(𝓕|ū) = SATISFIABLE) return SATISFIABLE;
  return UNSATISFIABLE;
}
```

Fig. 1. Definition of the Davis–Putnam–Logemann–Loveland algorithm.

- removing all the clauses that contain $u$,
- deleting $\bar{u}$ from all the clauses that contain $\bar{u}$.
- removing both $u$ and $\bar{u}$ from the list of literals.

Trivially, $\mathscr{F}$ is satisfiable if and only if at least one of $\mathscr{F}|u$ and $\mathscr{F}|\bar{u}$ is satisfiable.

It is customary to refer to the number of literals in a clause as the *length* (rather than size) of this clause. Clauses of length one are called *unit clauses*. Trivially, if a formula $\mathscr{F}$ includes a unit clause $\{u\}$, then every truth assignment $f$ that satisfies $\mathscr{F}$ must have $f(u)=1$. Hence, $\mathscr{F}$ is satisfiable if and only if $\mathscr{F}|u$ is satisfiable.

A literal $u$ in a formula $\mathscr{F}$ is called *monotone* if $\bar{u}$ appears in no clause of $\mathscr{F}$. Trivially, if $u$ is a monotone literal and if $\mathscr{F}$ is satisfiable, then $\mathscr{F}$ is satisfied by a truth assignment $f$ such that $f(u)=1$. Hence, $\mathscr{F}$ is satisfiable if and only if $\mathscr{F}|u$ is satisfiable.

These observations are the backbone of an algorithm for solving SAT, designed by Davis, Logemann, and Loveland [1] and evolved from an earlier proposal by Davis and Putnam [2]: see Fig. 1.

## 3. Branching rules

Each recursive call of DPLL may involve a choice of a literal $u$; algorithms for making these choices are referred to as *branching rules*. It is customary to represent each call of DPLL($\mathscr{F}$) by a node of a full binary tree: if a call represented by a node $\gamma$ brings about calls of DPLL($\mathscr{F}|u$) and DPLL($\mathscr{F}|\bar{u}$), then the call of DPLL($\mathscr{F}|u$) is represented by the left child of $\gamma$ and the call of DPLL($\mathscr{F}|\bar{u}$) is represented by the right child of $\gamma$, else $\gamma$ is a leaf. The shape and size of this tree depends not only on the input formula $\mathscr{F}$, but also on the branching rule $B$ that is used; we shall let $T(\mathscr{F},B)$ denote the tree.

To see how dramatically a choice of $B$ can influence the size of $T(\mathscr{F},B)$, take the formula with variables $x_1,x_2,\dots,x_n$ and clauses

$$\{x_i,x_{n-1},x_n\},\{x_i,\bar{x}_{n-1},x_n\},\{x_i,x_{n-1},\bar{x}_n\},\{x_i,\bar{x}_{n-1},\bar{x}_n\},\quad (i=1,2,\dots,n-2),$$

$$\{\bar{x}_j,\bar{x}_{j+1},\bar{x}_{j+2},\dots,\bar{x}_{n-3},\bar{x}_{n-2}\}\quad (j=1,2,\dots,n-3).$$

It is easy to see that this formula, $\mathscr{G}$, is unsatisfiable and that, with branching rules

**MIN**: "let $u$ be the variable with the smallest subscript", and

**MAX**: "let $u$ be the variable with the largest subscript",

$|T(\mathscr{G},\mathbf{MIN})| = 2^{n-1}-1$ but $|T(\mathscr{G},\mathbf{MAX})| = 7$. However, one might object that, unlike the branching rules commonly used in DPLL, both **MIN** and **MAX** are artificial: they disregard the structure of the (residual) formula from which $u$ is to be chosen.

One branching rule that does *not* disregard the structure of the formula from which $u$ is to be chosen has been proposed by Jeroslow and Wang [7]; it goes as follows. Let $d_k(\mathscr{F},u)$ be the number of clauses of length $k$ in $\mathscr{F}$ which contain $u$. The Jeroslow–Wang branching rule (**JW**) associates a weight with each literal $u$,

$$w(\mathscr{F},u) = \sum_k 2^{-k} d_k(\mathscr{F},u),$$

and chooses the literal with the largest weight. (If there is a tie, then, among the literals with the largest weight, the literal with the smallest subscript is chosen.) In our example,

$$w(\mathscr{G},x_i) = \begin{cases} 1/2 & \text{if } i=1,\dots,n-2,\\ (n-2)/4 & \text{if } i=n-1 \text{ or } n,\end{cases}$$

$$w(\mathscr{G},\bar{x}_i) = \begin{cases} (2^i-1)/2^{n-2} & \text{if } i=1,\dots,n-3,\\ (2^{n-3}-1)/2^{n-2} & \text{if } i=n-2,\\ (n-2)/4 & \text{if } i=n-1 \text{ or } n.\end{cases}$$

Therefore, **JW** chooses $x_{n-1}$. The residual formulas $\mathscr{G}|x_{n-1}$ and $\mathscr{G}|\bar{x}_{n-1}$ are identical; their clauses are

$$\{x_i,x_n\},\{x_i,\bar{x}_n\}\quad (i=1,2,\dots,n-2),$$
$$\{\bar{x}_j,\bar{x}_{j+1},\bar{x}_{j+2},\dots,\bar{x}_{n-3},\bar{x}_{n-2}\}\quad (j=1,2,\dots,n-3).$$

Therefore, **JW** chooses $x_n$ for both $\mathscr{G}|x_{n-1}$ and $\mathscr{G}|\bar{x}_{n-1}$. Since all of the resulting four residual formulas contain the empty clause, $|T(\mathscr{G},\mathbf{JW})| = 7$.

This branching rule is based on the intuition that shorter clauses are more important than longer ones and, particularly, that clauses of length $k$ are twice as important as clauses of length $k+1$. (The idea of progressively halving the weighting factors was used by Johnson [8] some fifteen years earlier in an approximation algorithm for MAX-SAT.)

Hooker and Vinay [6] proposed a variation (**HV**) on **JW**, which they called "two-sided Jeroslow–Wang rule": among all the literals $u$ such that $w(\mathscr{F},u) \geqslant w(\mathscr{F},\bar{u})$, choose one that maximizes $w(\mathscr{F},u) + w(\mathscr{F},\bar{u})$.

Van Gelder and Tsuji [10] proposed another variation (**vGT**): among all the literals $u$ such that $w(\mathscr{F},u) \geqslant w(\mathscr{F},\bar{u})$, choose one that maximizes $w(\mathscr{F},u) * w(\mathscr{F},\bar{u})$.

Dubois et al. [5] developed a program called C-SAT, which was shown to be one of the most efficient programs in solving the formulas from the DIMACS benchmarks [3]. Their branching rule (**C-SAT**) is as follows. Define

$$w(\mathscr{F},u) = \sum_k \ln\left(1 + \frac{1}{4^k - 2^{k+1}}\right) d_k(\mathscr{F},u)$$

and

$$W(\mathscr{F},u) = w(\mathscr{F},u) + \sum_{\{u,v\} \in \mathscr{F}} w(\mathscr{F},\bar{v}),$$

among all the literals $u$ such that $W(\mathscr{F},u) \geqslant W(\mathscr{F},\bar{u})$, choose one that maximizes

$$W(\mathscr{F},u) + W(\mathscr{F},\bar{u}) + 1.5 \min(W(\mathscr{F},u), W(\mathscr{F},\bar{u})).$$

These four branching rules, **JW**, **HV**, **vGT**, and **C-SAT**, share the following property.

**Sensible property.** If all clauses have length three and if $v, w$ are literals such that $d_3(v) < d_3(w)$, $d_3(\bar{v}) < d_3(\bar{w})$, then do *not* choose $v$.

## 4. Fooling a family of branching rules

**Theorem.** *For every nonnegative integer $t$, there is an unsatisfiable formula $\mathscr{H}$ with $5t+21$ variables such that $|T(\mathscr{H},\mathbf{MAX})| = 111$, but $|T(\mathscr{H},\mathbf{B})| > 2^t$ for every branching rule* **B** *with the Sensible Property.*

**Proof.** Write $r = 5t$ and consider the following formula, $\mathscr{H}_t$, with variables $x_1, \ldots, x_r, \ldots, x_{r+21}$. For each $s = 0, 1, \ldots, t-1$, there are eight clauses involving $x_{5s+1}, x_{5s+2}, x_{5s+3}, x_{5s+4}, x_{5s+5}$ and their complements,

$$\{x_{5s+1}, x_{5s+2}, x_{5s+3}\}, \{x_{5s+1}, x_{5s+3}, x_{5s+4}\}, \{x_{5s+1}, x_{5s+4}, x_{5s+5}\}, \{x_{5s+1}, x_{5s+2}, x_{5s+5}\}$$

and

$$\{\bar{x}_{5s+1}, \bar{x}_{5s+2}, \bar{x}_{5s+3}\}, \{\bar{x}_{5s+1}, \bar{x}_{5s+3}, \bar{x}_{5s+4}\}, \{\bar{x}_{5s+1}, \bar{x}_{5s+4}, \bar{x}_{5s+5}\}, \{\bar{x}_{5s+1}, \bar{x}_{5s+2}, \bar{x}_{5s+5}\};$$

in addition, there are 22 clauses involving $x_{r+1},\dots,x_{r+21}$ and their complements,

$$\{x_{r+1},x_{r+7},x_{r+13}\},\{x_{r+2},x_{r+8},x_{r+14}\},\{x_{r+3},x_{r+9},x_{r+15}\},\{x_{r+4},x_{r+10},x_{r+16}\},$$
$$\{x_{r+5},x_{r+11},x_{r+17}\},\{x_{r+6},x_{r+12},x_{r+18}\},\{\bar{x}_{r+1},x_{r+7},x_{r+13}\},\{\bar{x}_{r+2},x_{r+8},x_{r+14}\},$$
$$\{\bar{x}_{r+3},x_{r+9},x_{r+15}\},\{\bar{x}_{r+4},x_{r+10},x_{r+16}\},\{\bar{x}_{r+5},x_{r+11},x_{r+17}\},\{\bar{x}_{r+6},x_{r+12},x_{r+18}\},$$
$$\{\bar{x}_{r+7},x_{r+13},x_{r+19}\},\{\bar{x}_{r+8},x_{r+14},x_{r+19}\},\{\bar{x}_{r+9},x_{r+15},x_{r+20}\},\{\bar{x}_{r+10},x_{r+16},x_{r+20}\},$$
$$\{\bar{x}_{r+11},x_{r+17},x_{r+21}\},\{\bar{x}_{r+12},x_{r+18},x_{r+21}\},\{\bar{x}_{r+13},\bar{x}_{r+14},x_{r+19}\},\{\bar{x}_{r+15},\bar{x}_{r+16},x_{r+20}\},$$
$$\{\bar{x}_{r+17},\bar{x}_{r+18},x_{r+21}\},\{\bar{x}_{r+19},\bar{x}_{r+20},\bar{x}_{r+21}\}.$$

These last 22 clauses alone constitute an unsatisfiable formula ([4], bottom of p. 56). Altogether, $\mathscr{H}_t$ has $8t+22$ clauses and each of these clauses has length three. It is a routine matter to verify that $|T(\mathscr{H}_t,\mathbf{MAX})|=111$; we will use induction on $t$ to prove that $|T(\mathscr{H}_t,\mathbf{B})|>2^t$ for every branching rule $\mathbf{B}$ with the Sensible Property.

Trivially, $|T(\mathscr{H}_0,\mathbf{B})|>1$. Since

$$d_3(\mathscr{H}_t,x_i)=\begin{cases}4 & \text{if } i=5s+1,\ s=0,\dots,t-1,\\ 2 & \text{if } i=5s+j,\ s=0,\dots,t-1,\ j=2,\dots,5,\\ 1 & \text{if } i=r+1,\dots,r+6,\\ 2 & \text{if } i=r+7,\dots,r+12,\\ 3 & \text{if } i=r+13,\dots,r+21,\end{cases}$$

$$d_3(\mathscr{H}_t,\bar{x}_i)=\begin{cases}4 & \text{if } i=5s+1,\ s=0,\dots,t-1,\\ 2 & \text{if } i=5s+j,\ s=0,\dots,t-1,\ j=2,\dots,5,\\ 1 & \text{if } i=r+1,\dots,r+21,\end{cases}$$

any branching rule $\mathbf{B}$ with the Sensible property, given $\mathscr{H}_t$ with $t>0$, chooses some $x_{5s+1}$ with $0\leqslant s<t$ to be $u$. In $\mathscr{H}_t|x_{5s+1}$, the four literals $\bar{x}_{5s+2},\bar{x}_{5s+3},\bar{x}_{5s+4},\bar{x}_{5s+5}$ are monotone; in $\mathscr{H}_t|\bar{x}_{5s+1}$, the four literals $x_{5s+2},x_{5s+3},x_{5s+4},x_{5s+5}$ are monotone. These monotone literals (together with the clauses containing them) are removed from the formulas by DPLL before the branching rule is consulted again. The resulting formulas.

$$((((\mathscr{H}_t|x_{5s+1})|\bar{x}_{5s+2})|\bar{x}_{5s+3})|\bar{x}_{5s+4})|\bar{x}_{5s+5}$$
$$\text{and}\quad ((((\mathscr{H}_t|\bar{x}_{5s+1})|x_{5s+2})|x_{5s+3})|x_{5s+4})|x_{5s+5}$$

are identical and isomorphic to $\mathscr{H}_{t-1}$ via $\sigma$ defined by

$$\sigma(x_i)=\begin{cases}x_i & \text{if } i\leqslant 5s,\\ x_{i-5} & \text{if } i>5s+5.\end{cases}$$

Hence $|T(\mathscr{H}_t,\mathbf{B})|=1+|T(\mathscr{H}_t|x_{5s+1},\mathbf{B})|+|T(\mathscr{H}_t|\bar{x}_{5s+1},\mathbf{B})|=1+2|T(\mathscr{H}_{t-1},\mathbf{B})|$; now, by the induction hypothesis, $|T(\mathscr{H}_t,\mathbf{B})|>1+2(2^{t-1})>2^t$. □

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