

Lumping the Approximate Master Equation for Multistate Processes on Complex Networks

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Complex Networks

Networks are everywhere

- Friendship networks
- Online social networks
- Telecommunication networks
- Infrastructure networks
- ...

Motivation

Understand spreading phenomenon of

- Infectious diseases
- Computer viruses
- Rumours/opinions/emotions
- Blackouts
- ...

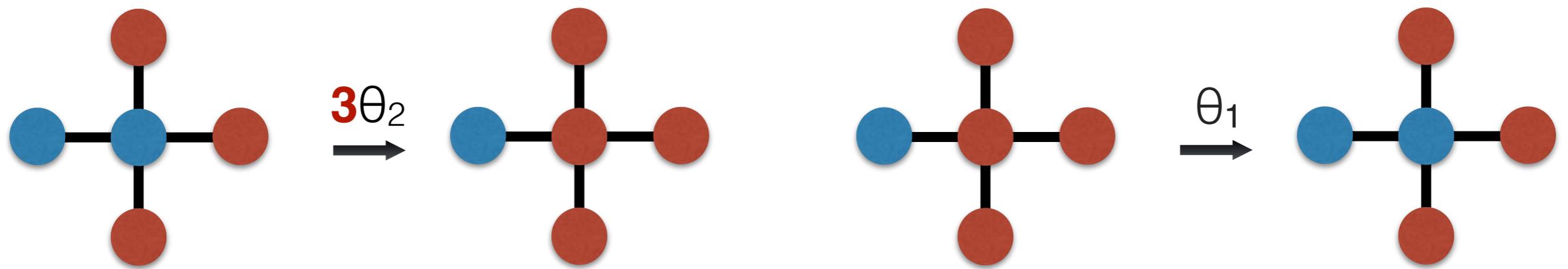
Spreading Process

- Fixed graph topology
- Continuous time dynamics
- Nodes occupy node-states
- Node-states change randomly w.r.t. rules

Classical example: SIS model

- 2 node-states (infected, susceptible)
- 2 rules (infection, recovery)

SIS Model



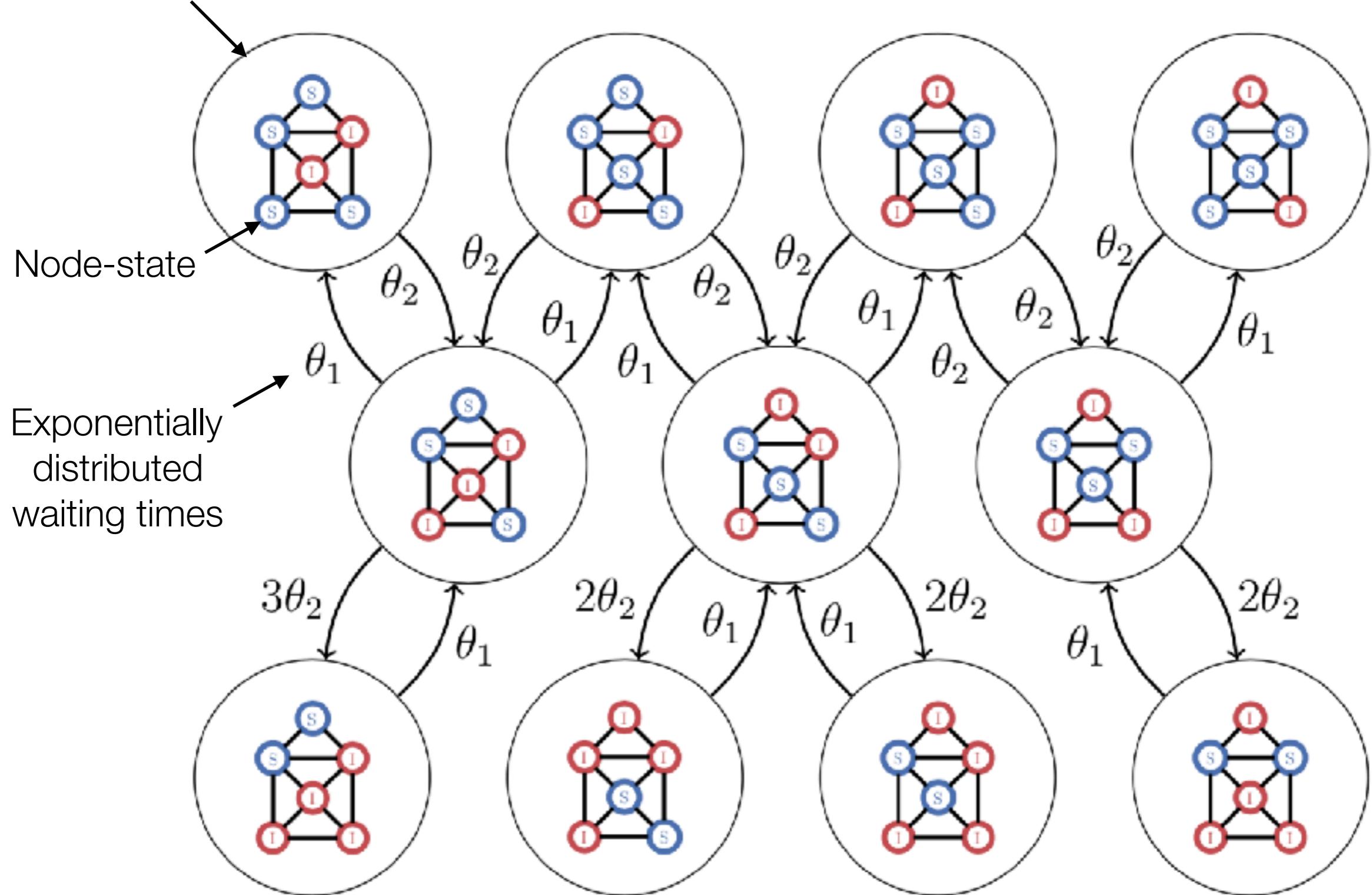
Infection (edge-based)

Recovery (node-based)

- 2 node-states (**i**nfected, **s**usceptible)
- 2 rules (infection, recovery)

SIS Semantics

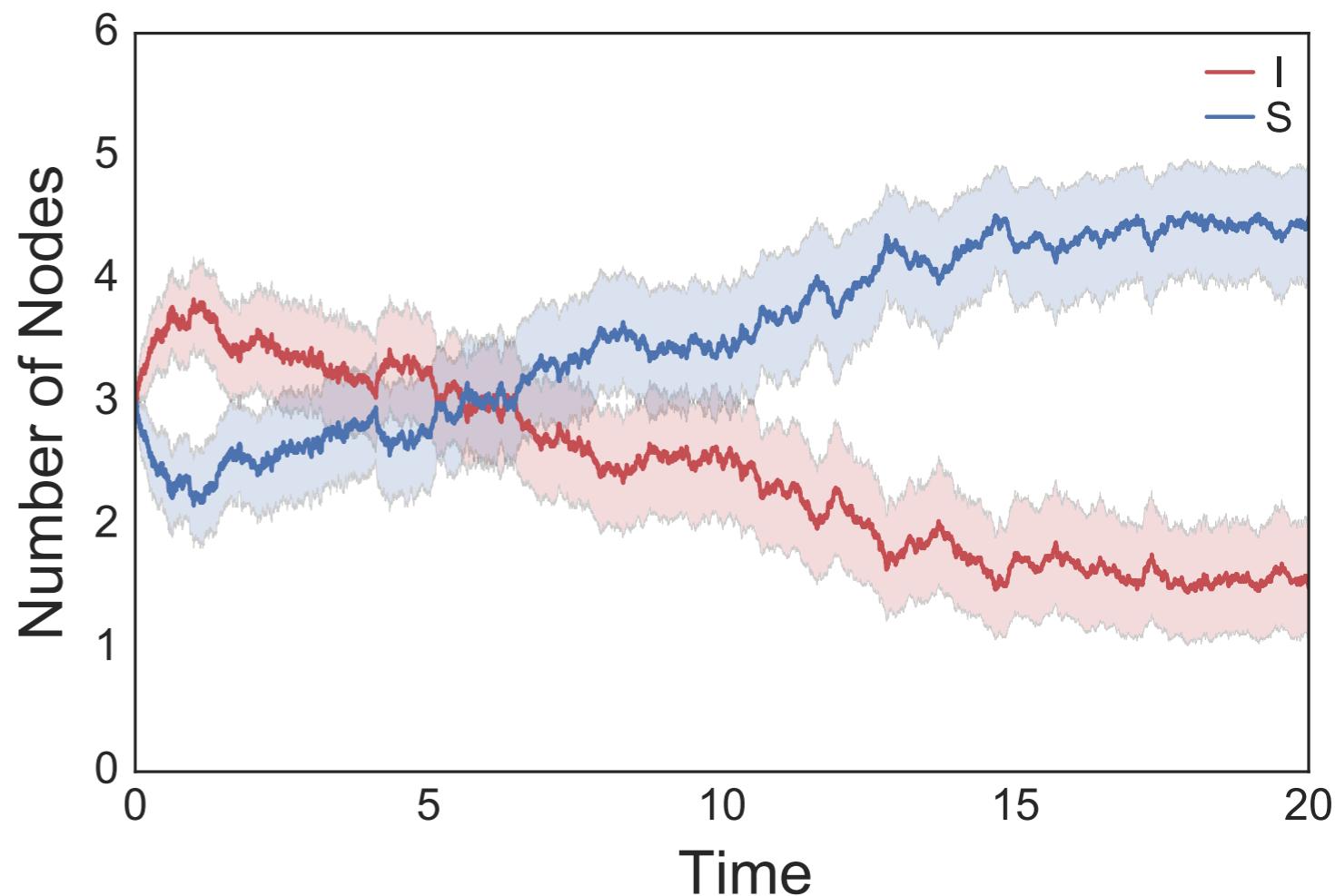
Network-state



Fraction of Infected Nodes

Measure of interest:

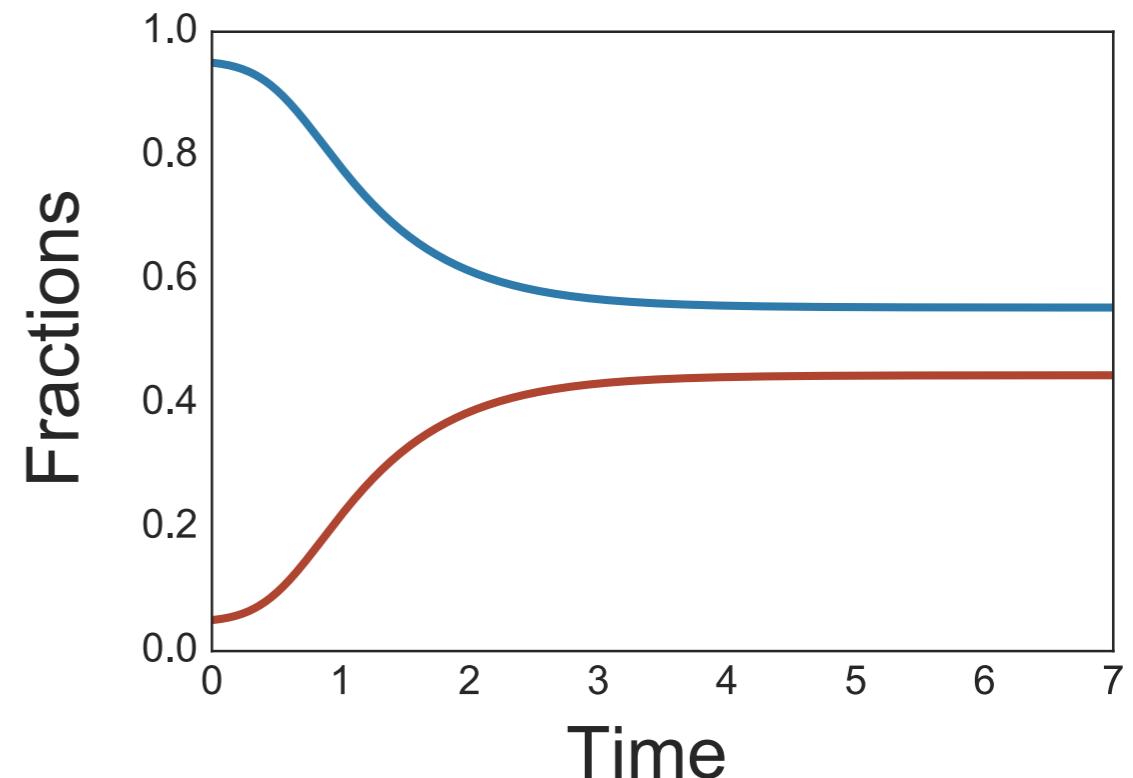
- Fractions of nodes in each node-state
- Can be estimated with Monte-Carlo simulation (expensive)



Lumping the **Approximate Master Equation** for Multistate Processes on Complex Networks

Approximate Master Equation (AME)

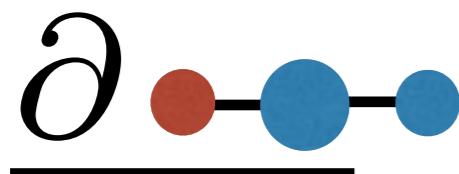
- ODEs to approximate expected fractions over time
- One ODE for each node-state and possible neighbourhood



AME

one ODE for each node-state and neighbourhood

Fraction of **susceptible** nodes with 1 **susceptible** and 1 **infected** neighbour



$$\frac{\partial}{\partial t}$$

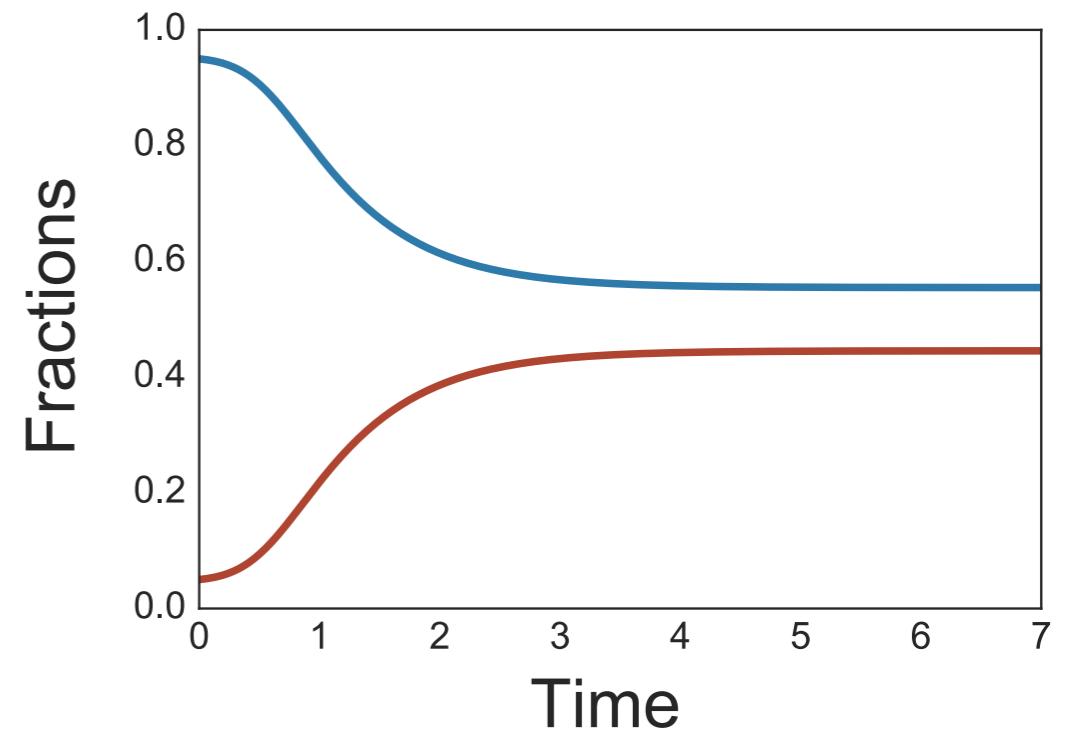
A horizontal line with two circular nodes. The left node is red (infected) and the right node is blue (susceptible), with a single line segment connecting them.

$$\frac{\partial}{\partial t}$$

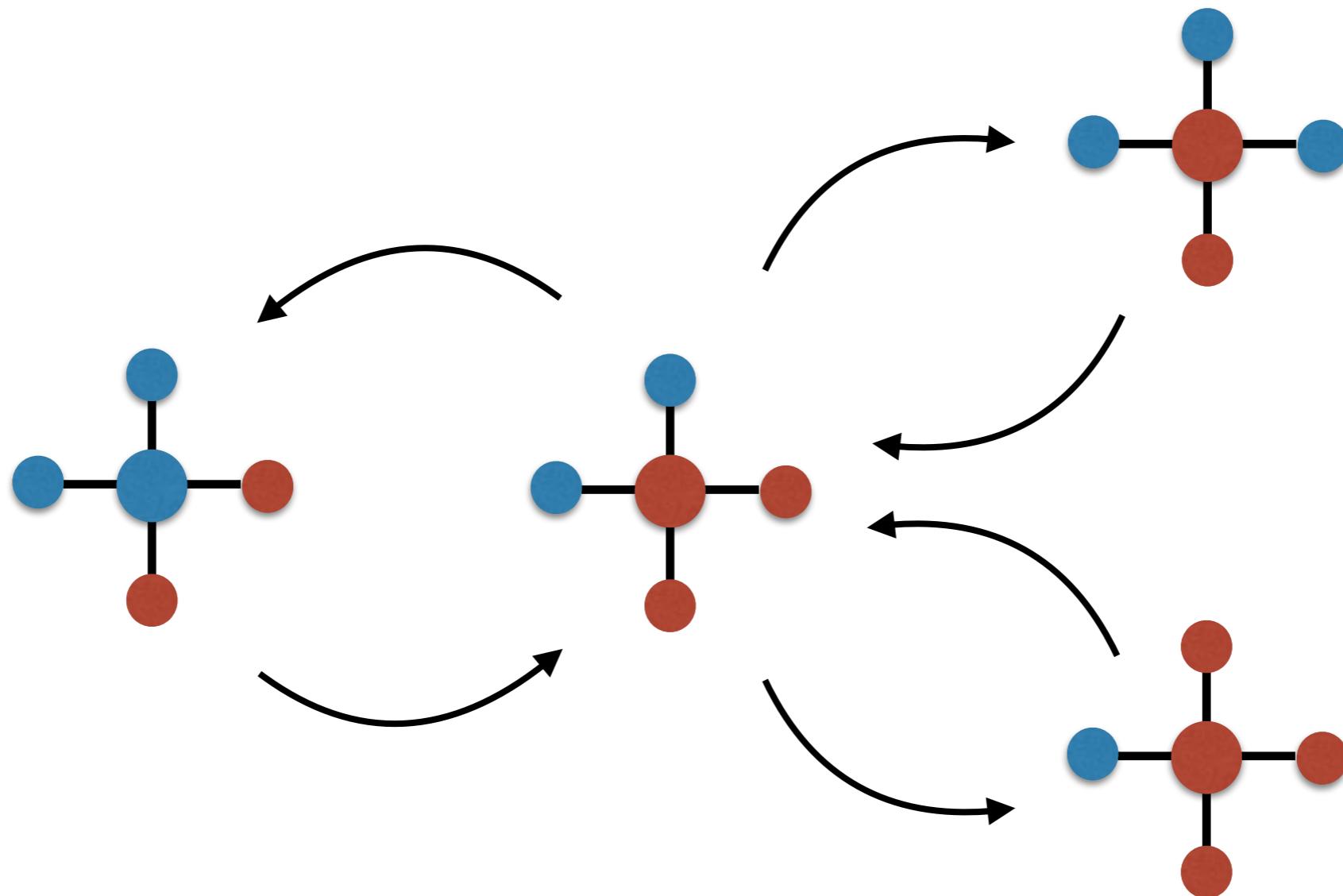
A triangular arrangement of three circular nodes, all colored red (infected).

$$\frac{\partial}{\partial t}$$

Numerical integration



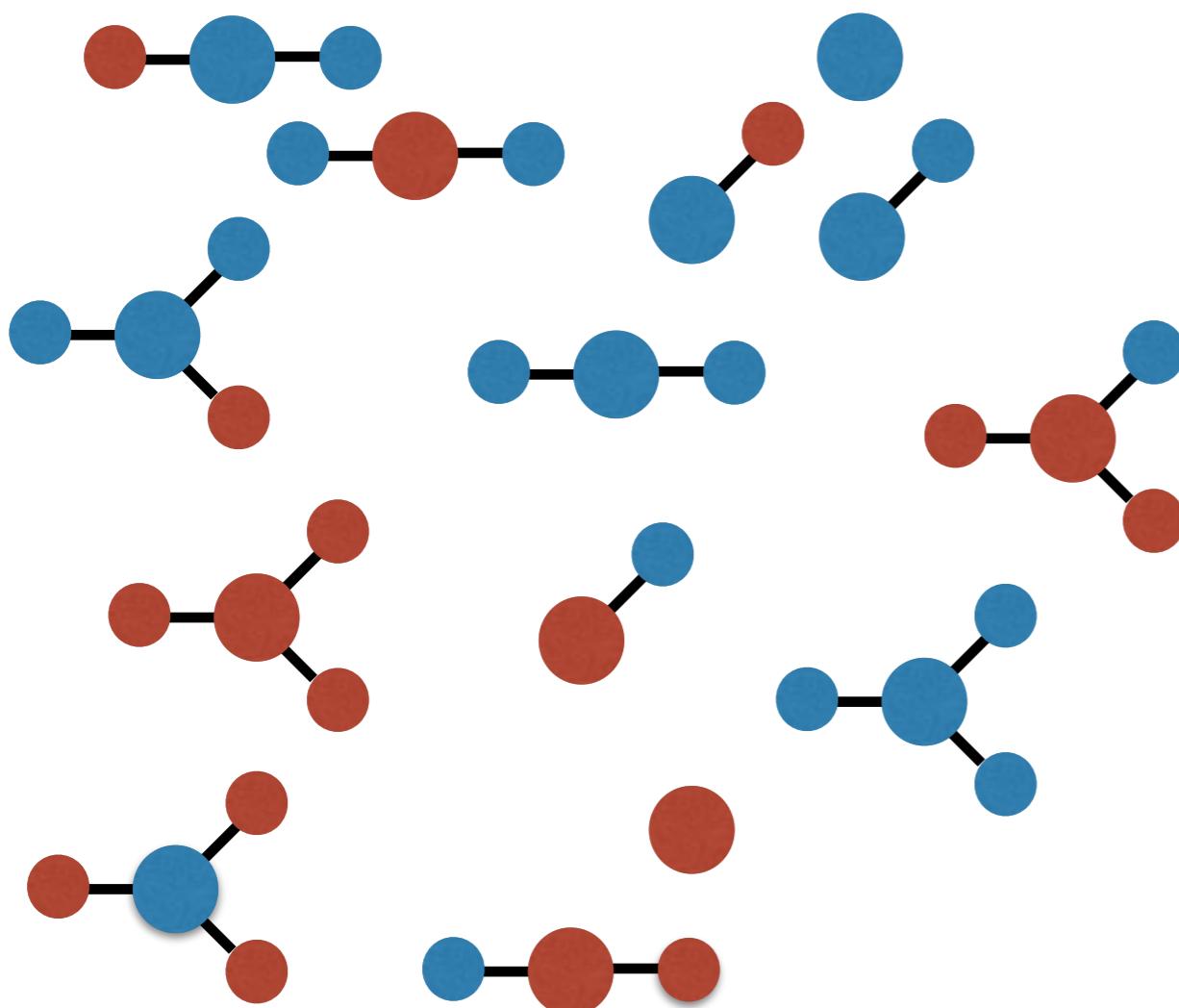
AME



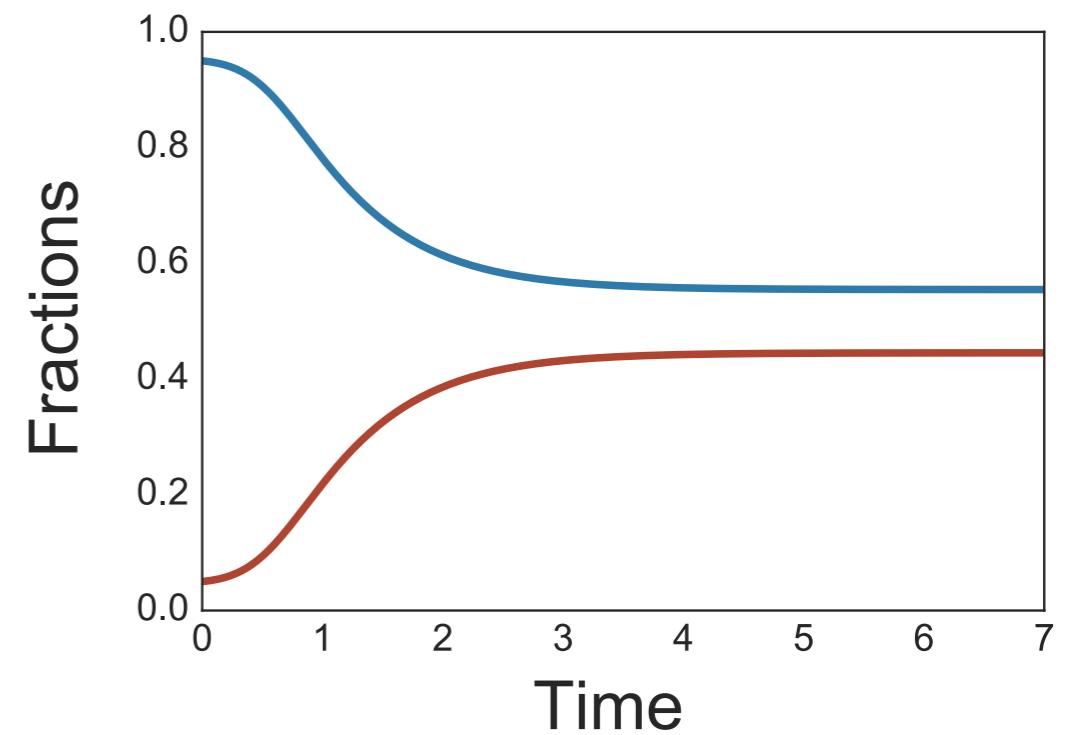
AME

one ODE for each node-state and neighbourhood

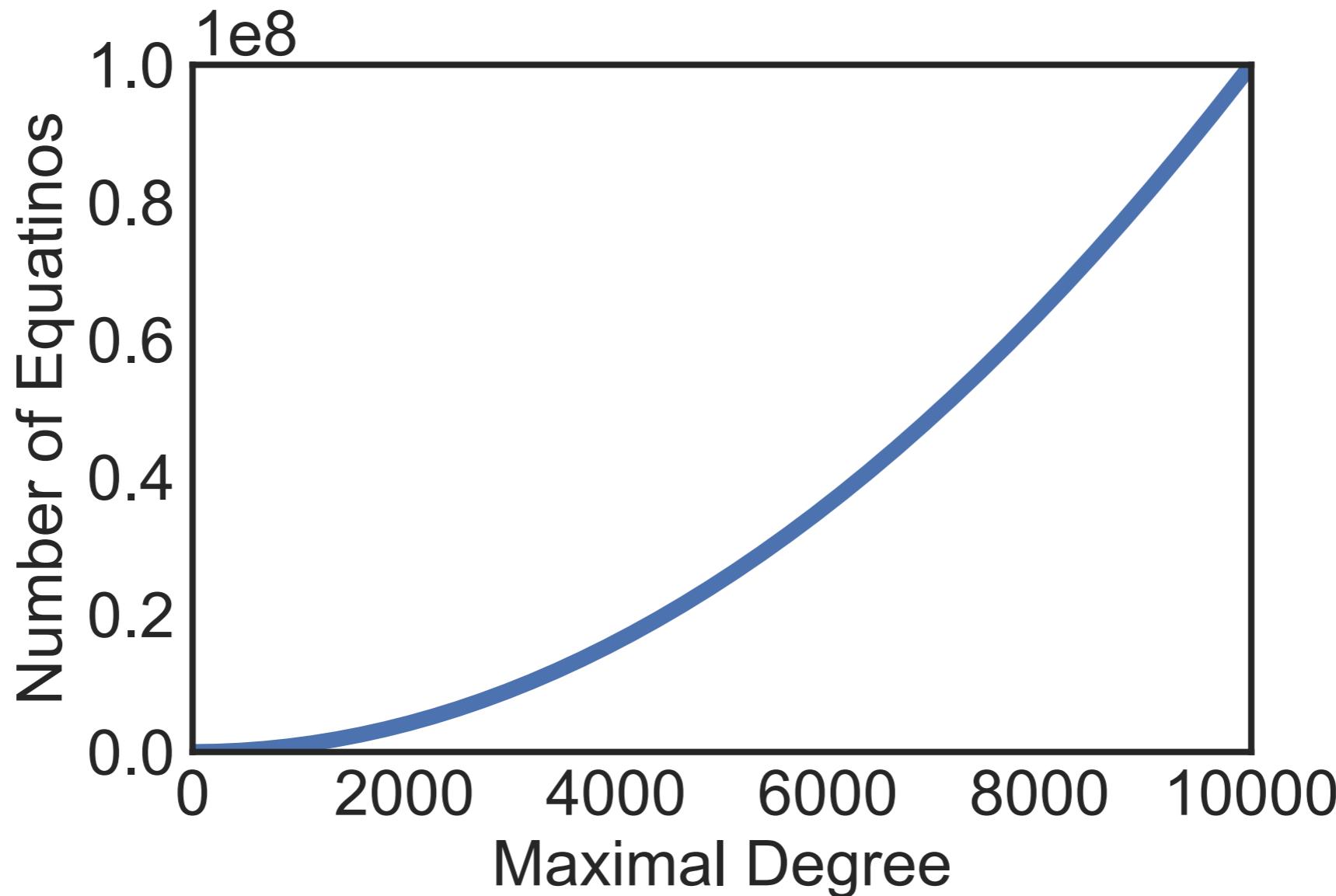
Fraction of **susceptible** nodes with 1 **susceptible** and 1 **infected** neighbour



Numerical integration

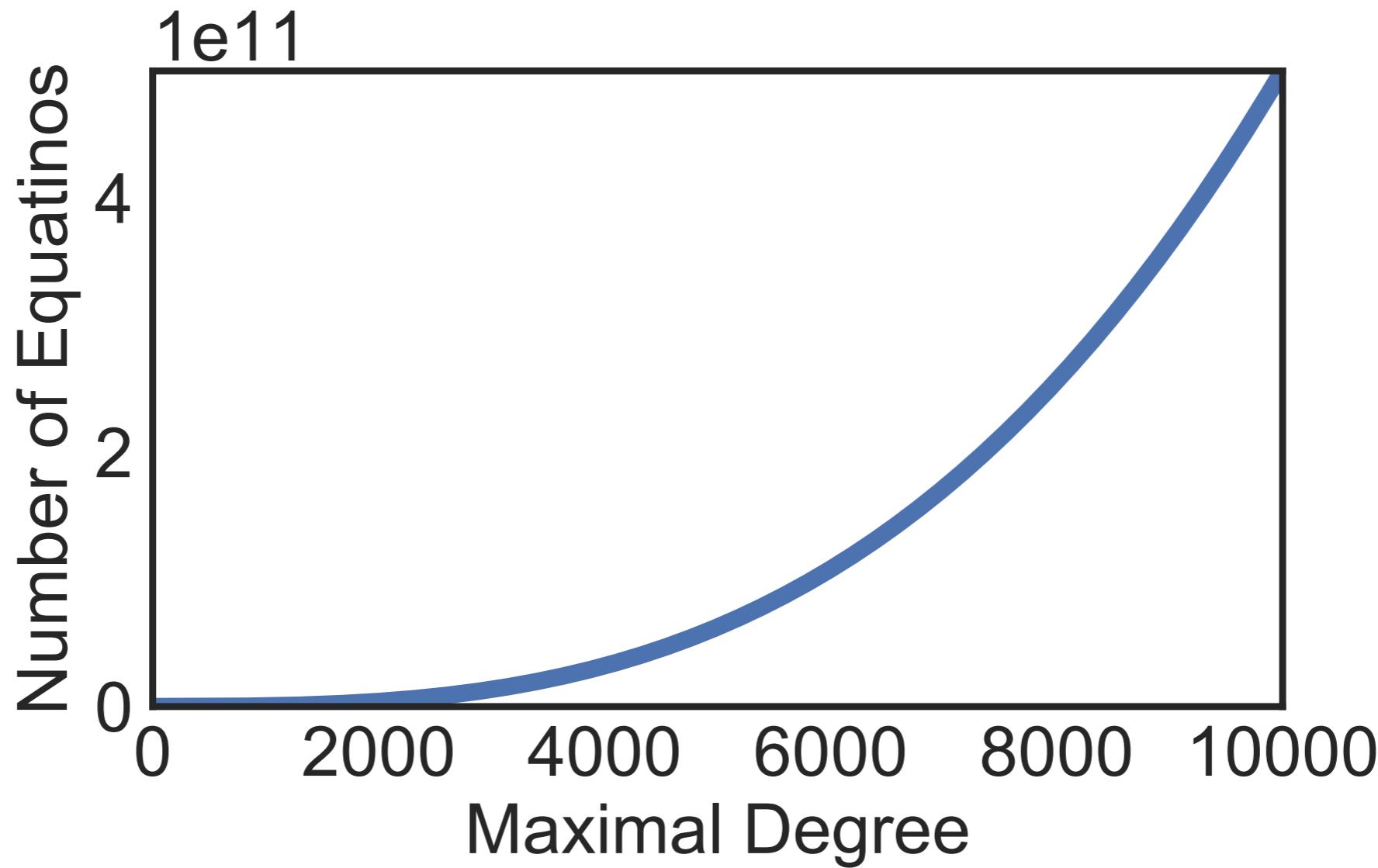


Number of Equations



2 node-states (SIS)

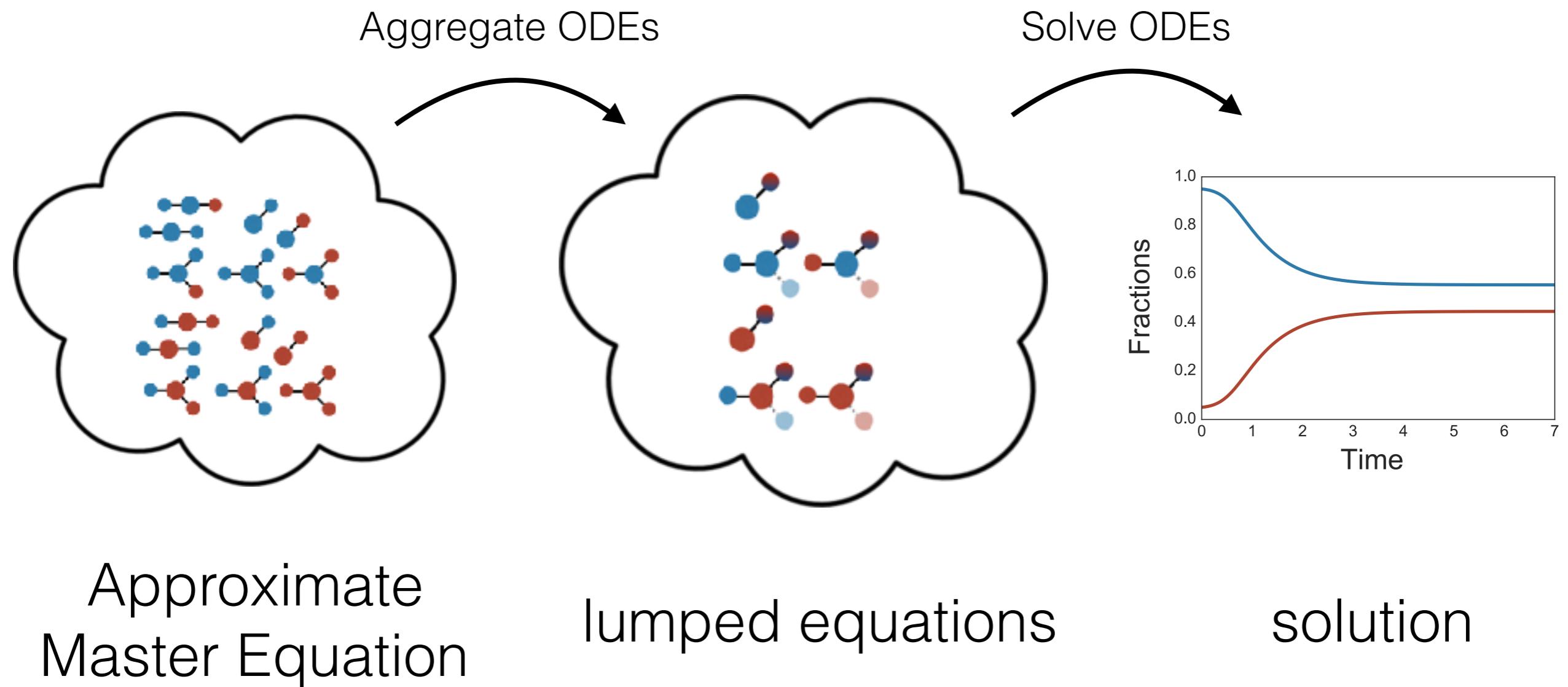
Number of Equations



3 node-states (SIR)

Lumping the Approximate Master Equation for Multistate Processes on Complex Networks

Lumping



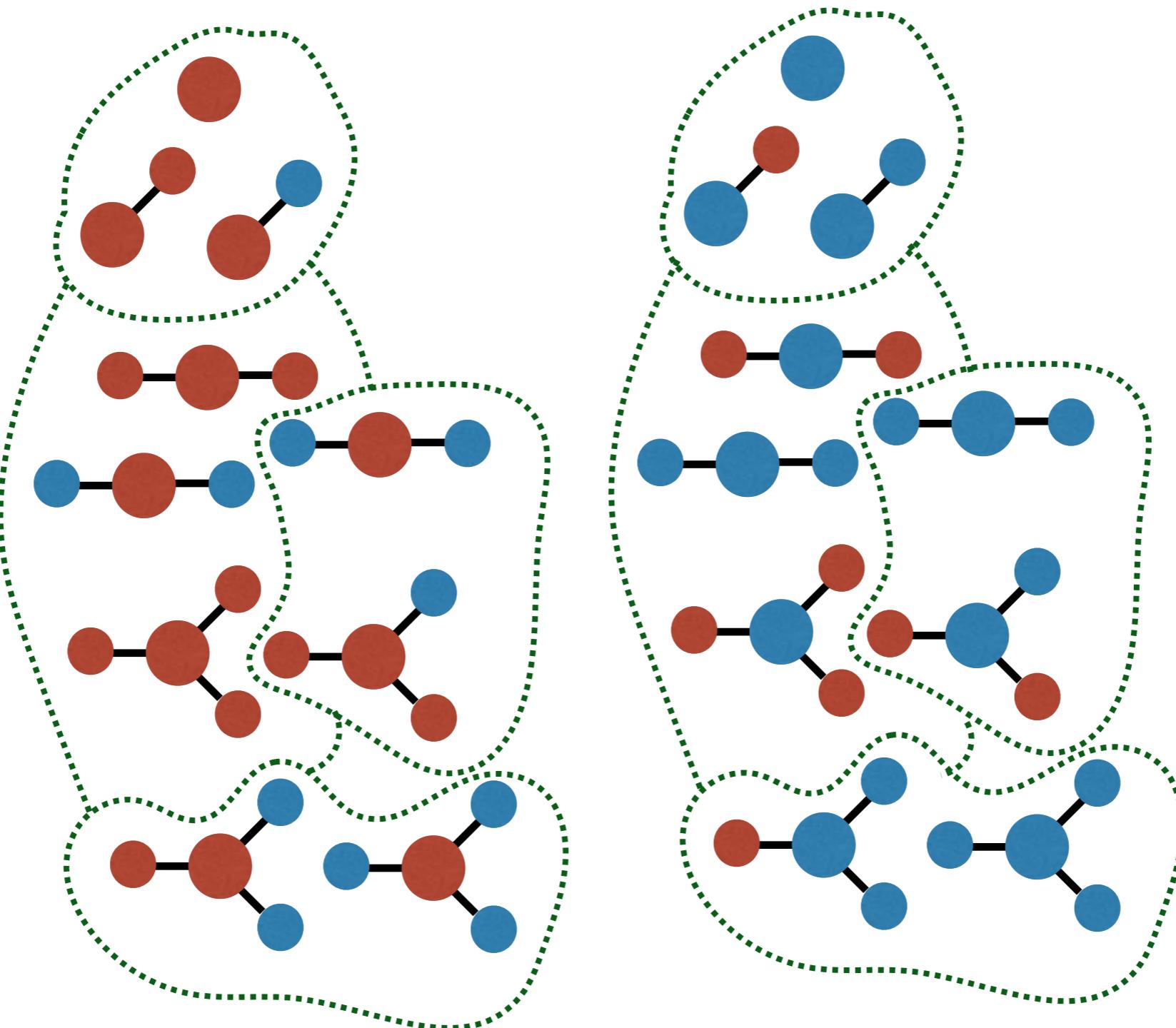
Approximate
Master Equation

lumped equations

solution

Partitioning

Solve one
equation for
each
partition



Lumping

$$\frac{\partial x_1}{\partial t} = x_1 + x_2$$

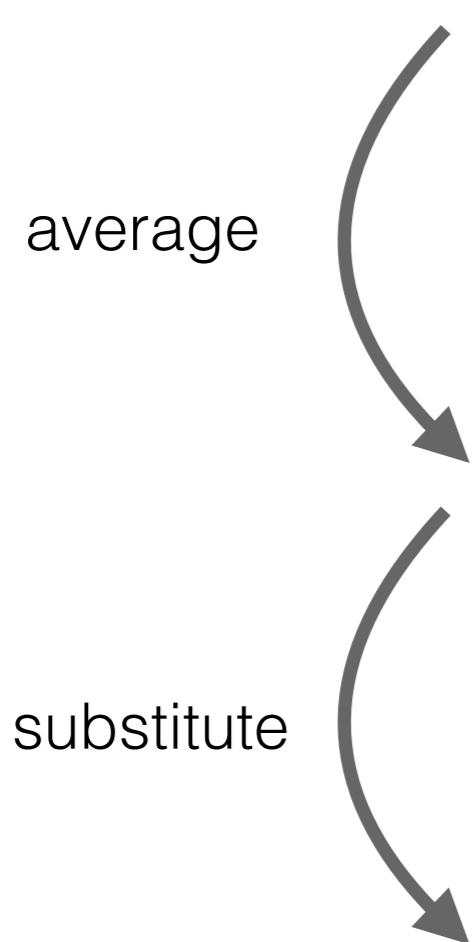
$$\frac{\partial x_2}{\partial t} = x_1 \cdot x_2$$

average



$$\frac{\partial x_{12}}{\partial t} = \frac{1}{2}(x_1 + x_2) + \frac{1}{2}(x_1 \cdot x_2)$$

Lumping



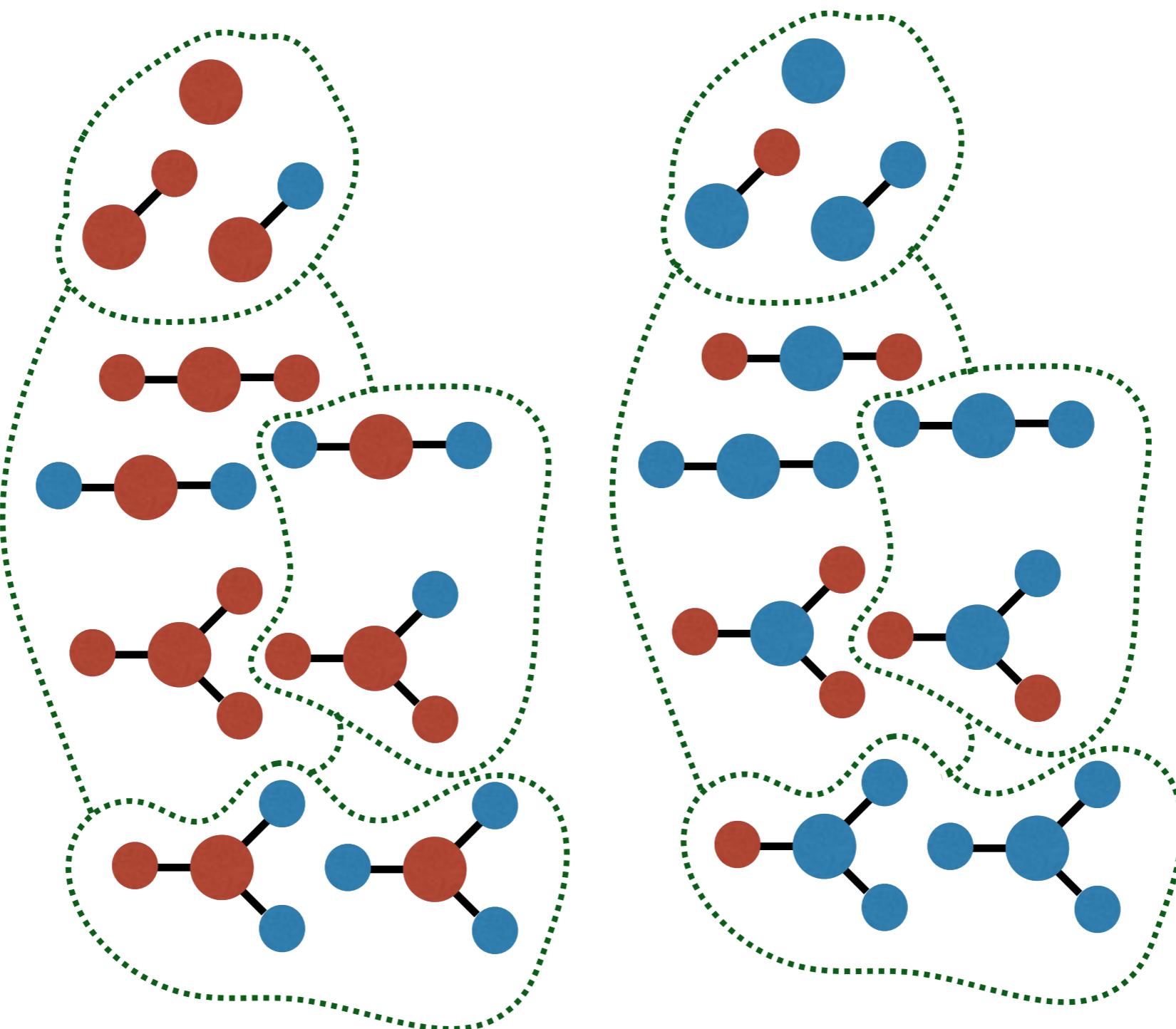
$$\frac{\partial x_1}{\partial t} = x_1 + x_2$$

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$$\frac{\partial x_{12}}{\partial t} = \frac{1}{2}(x_1 + x_2) + \frac{1}{2}(x_1 \cdot x_2)$$

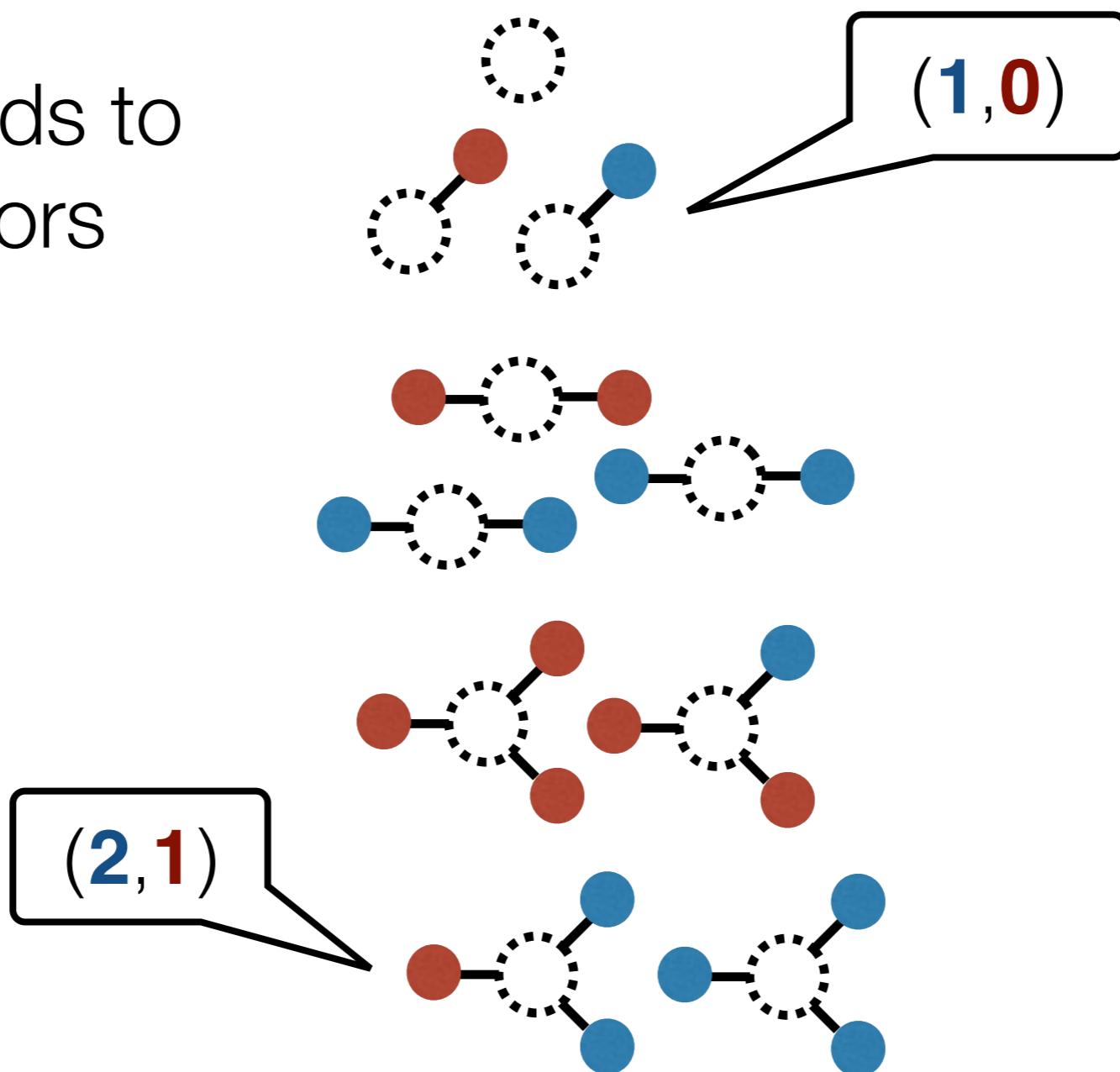
$$\frac{\partial x_{12}}{\partial t} = \frac{1}{2}(x_{12} + x_{12}) + \frac{1}{2}(x_{12} \cdot x_{12})$$

Partitioning



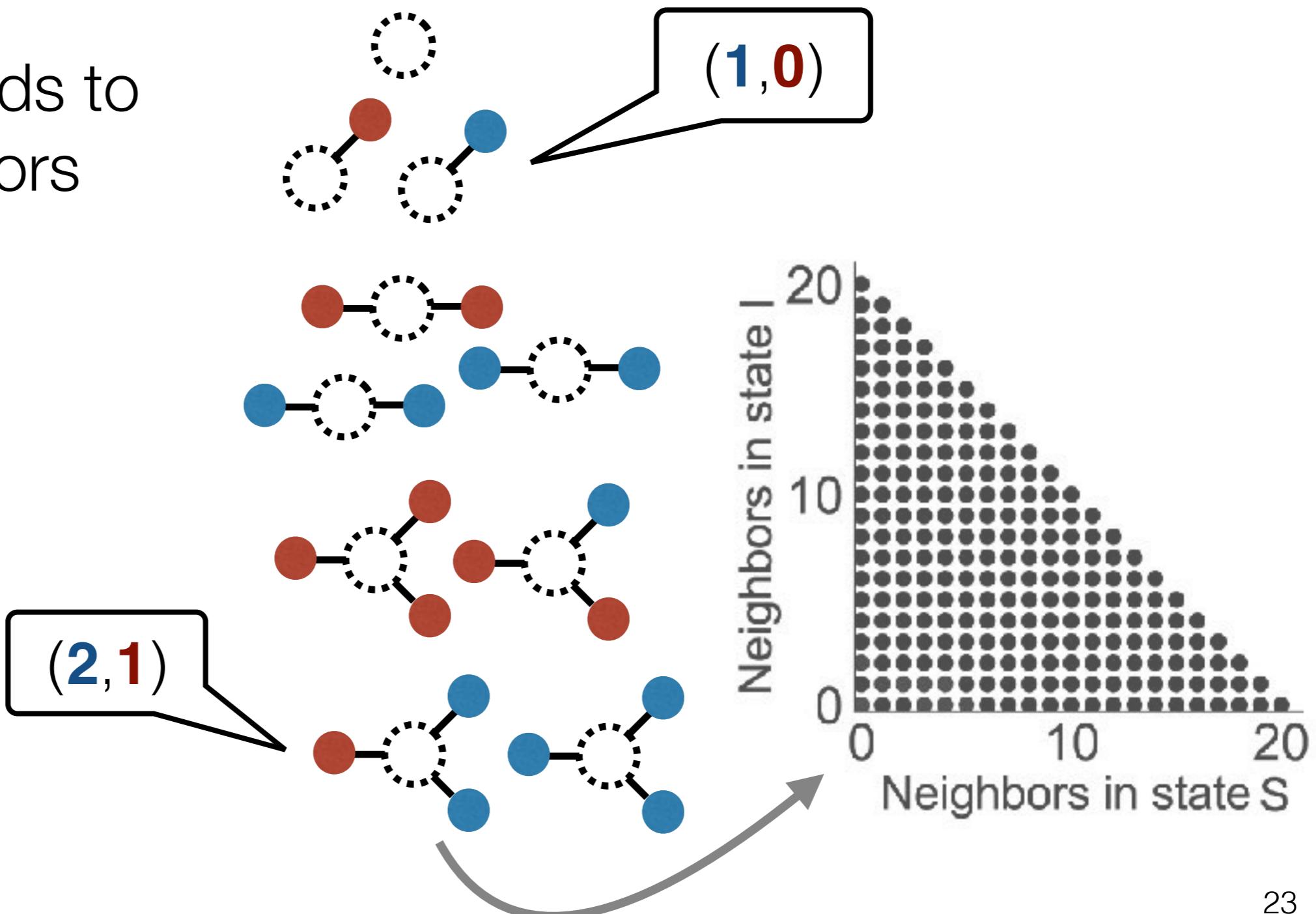
Partitioning

Map
neighbourhoods to
integer vectors

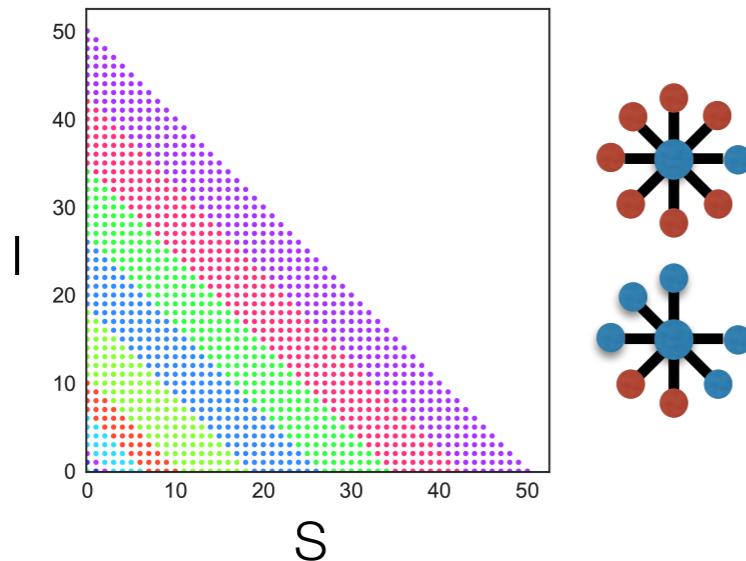


Partitioning

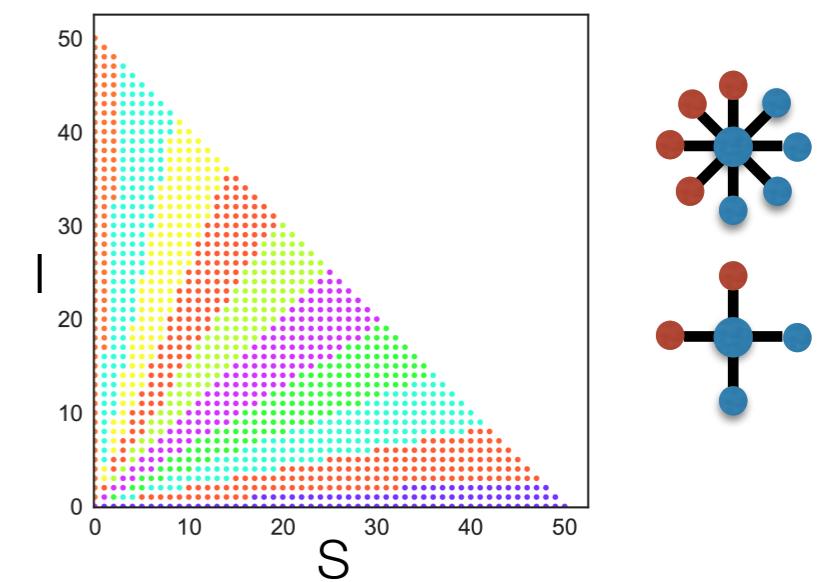
Map
neighbourhoods to
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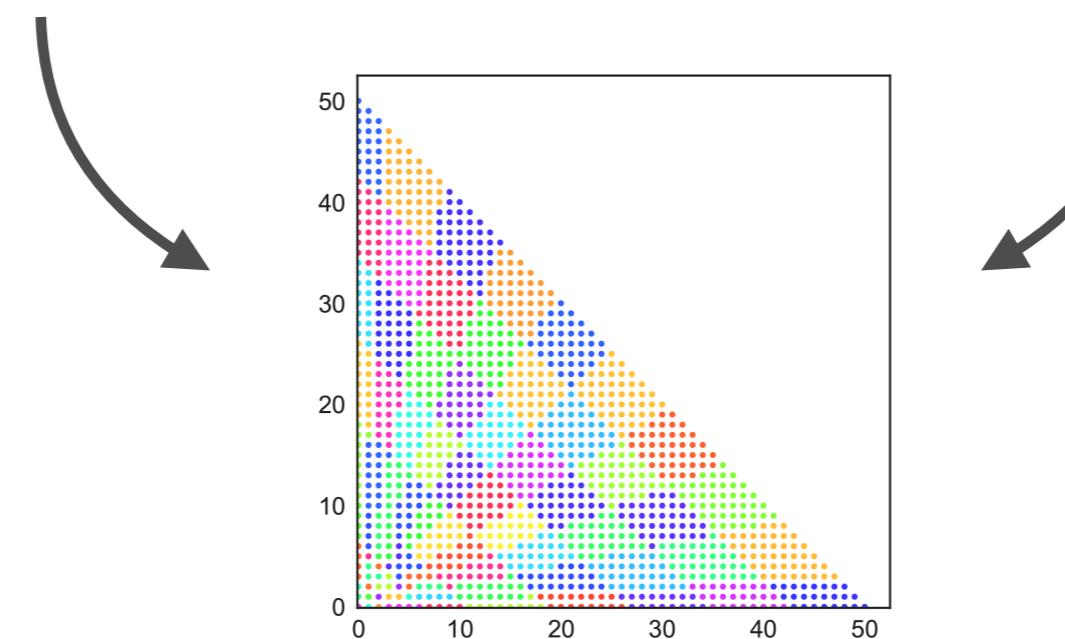
Partitioning



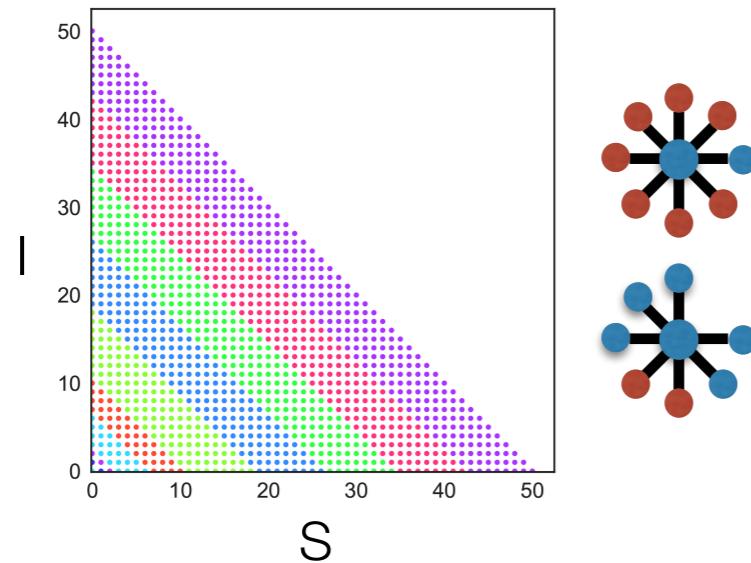
Cluster degrees k based on
degree-distribution $P(k)$



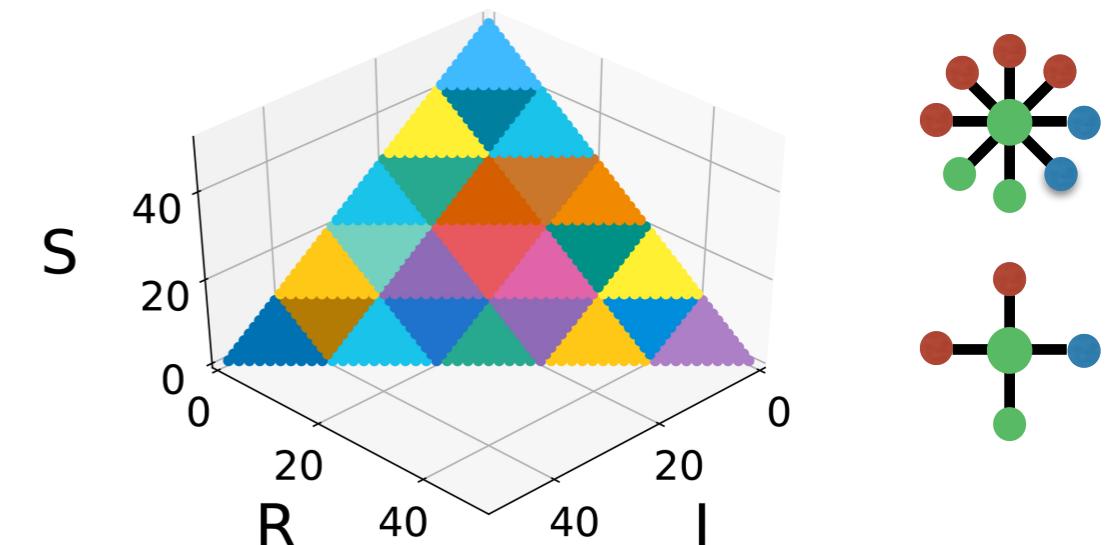
Cluster neighbourhood vectors
based on proportionality



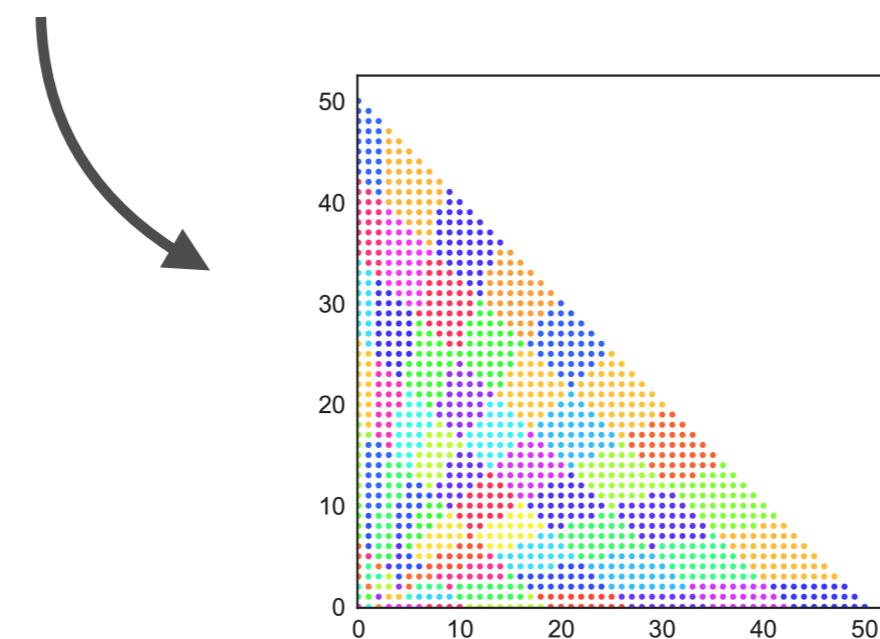
Partitioning 3D



Cluster degrees k based on
degree-distribution $P(k)$



Cluster neighbourhood vectors
based on proportionality



How Many Partitions?

Simple Stopping Heuristic

- Start with small number of partitions
- Increase partition number and solve model as long as solution changes

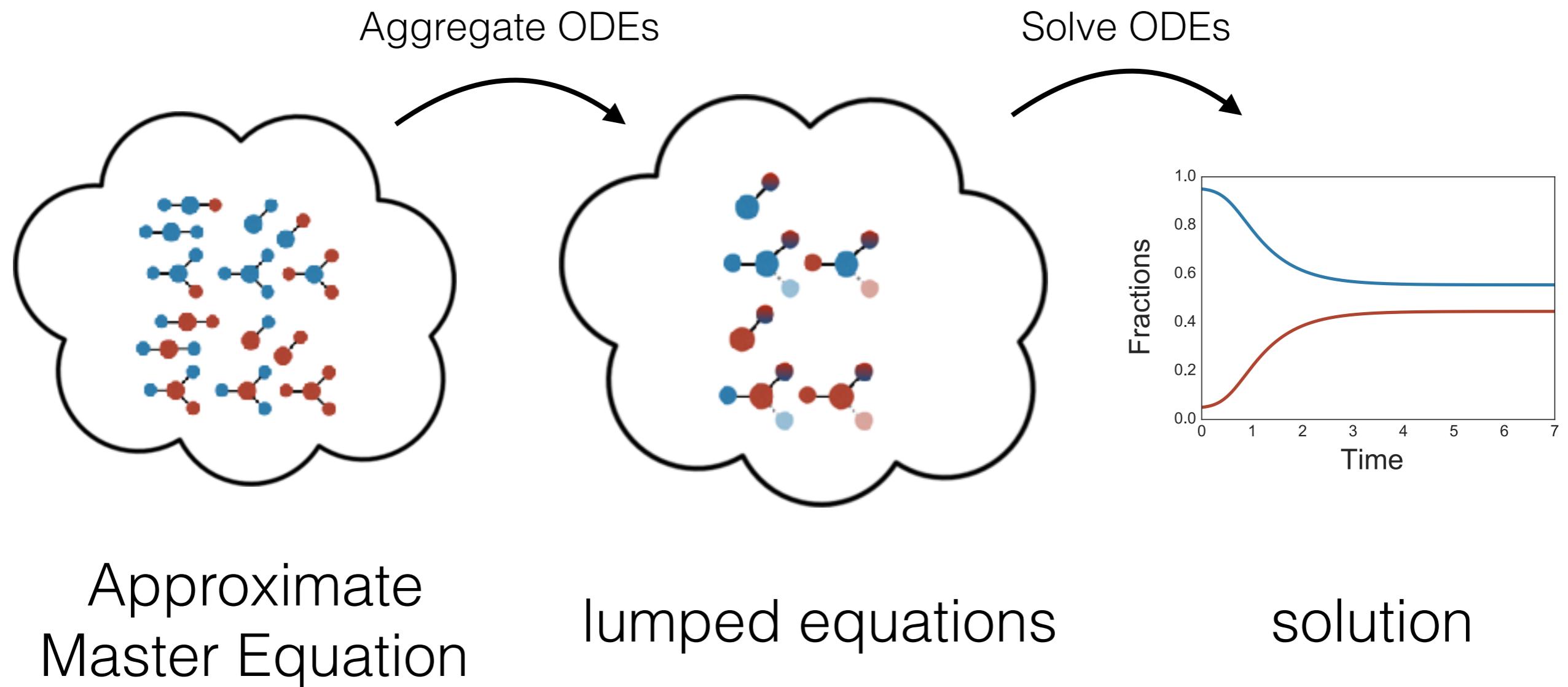
How Many Partitions?

Simple Stopping Heuristic

- Start with small number of partitions
- Increase partition number and solve model as long as solution changes

Difference between consecutive solutions can be used to predict overall lumping error

Lumping

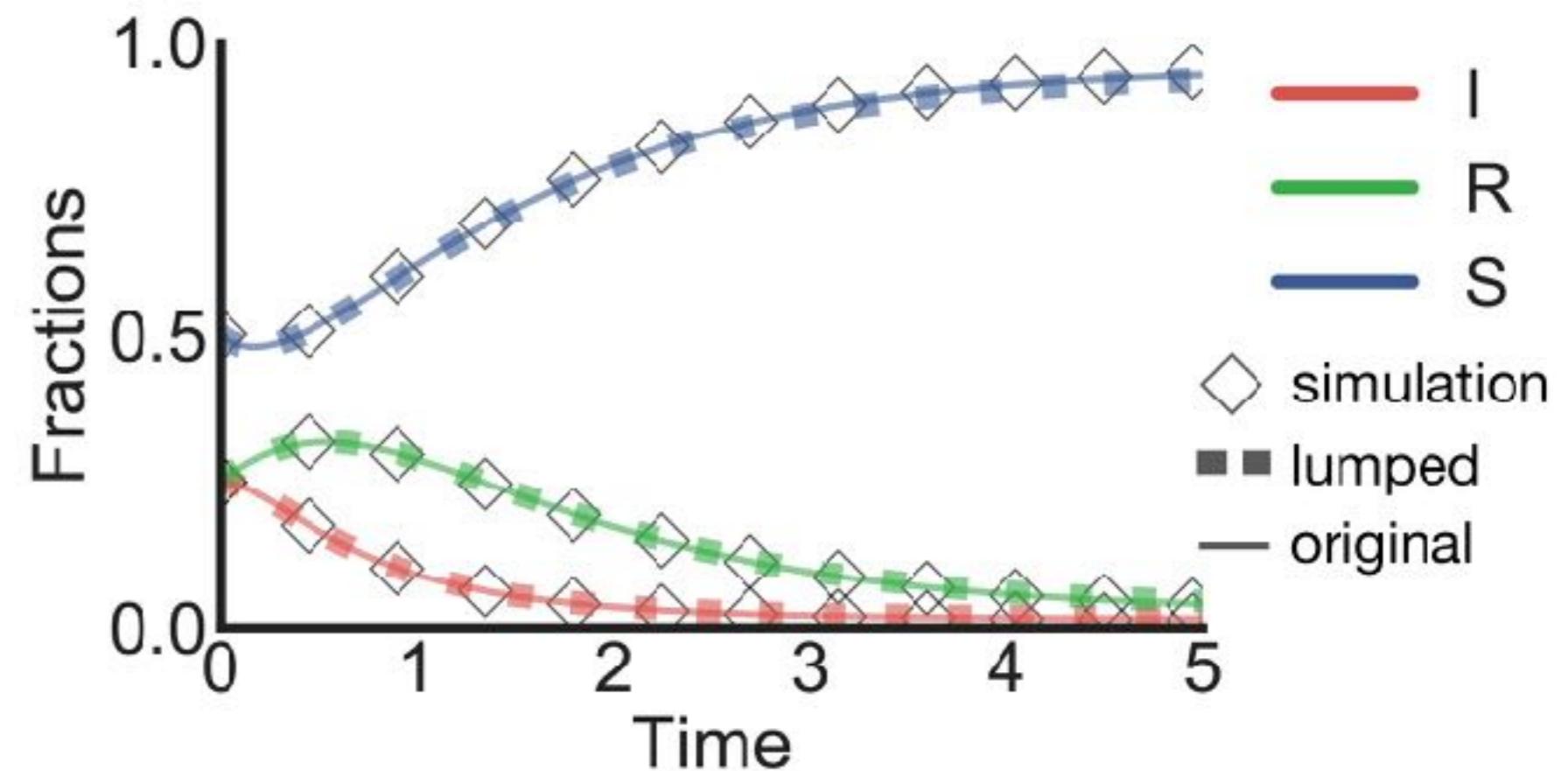
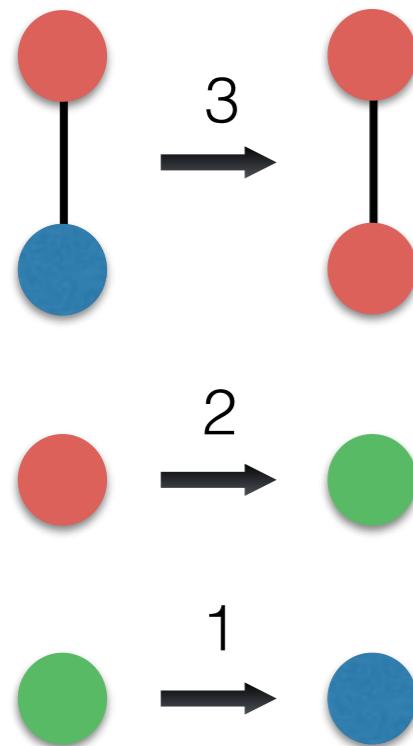


Approximate
Master Equation

lumped equations

solution

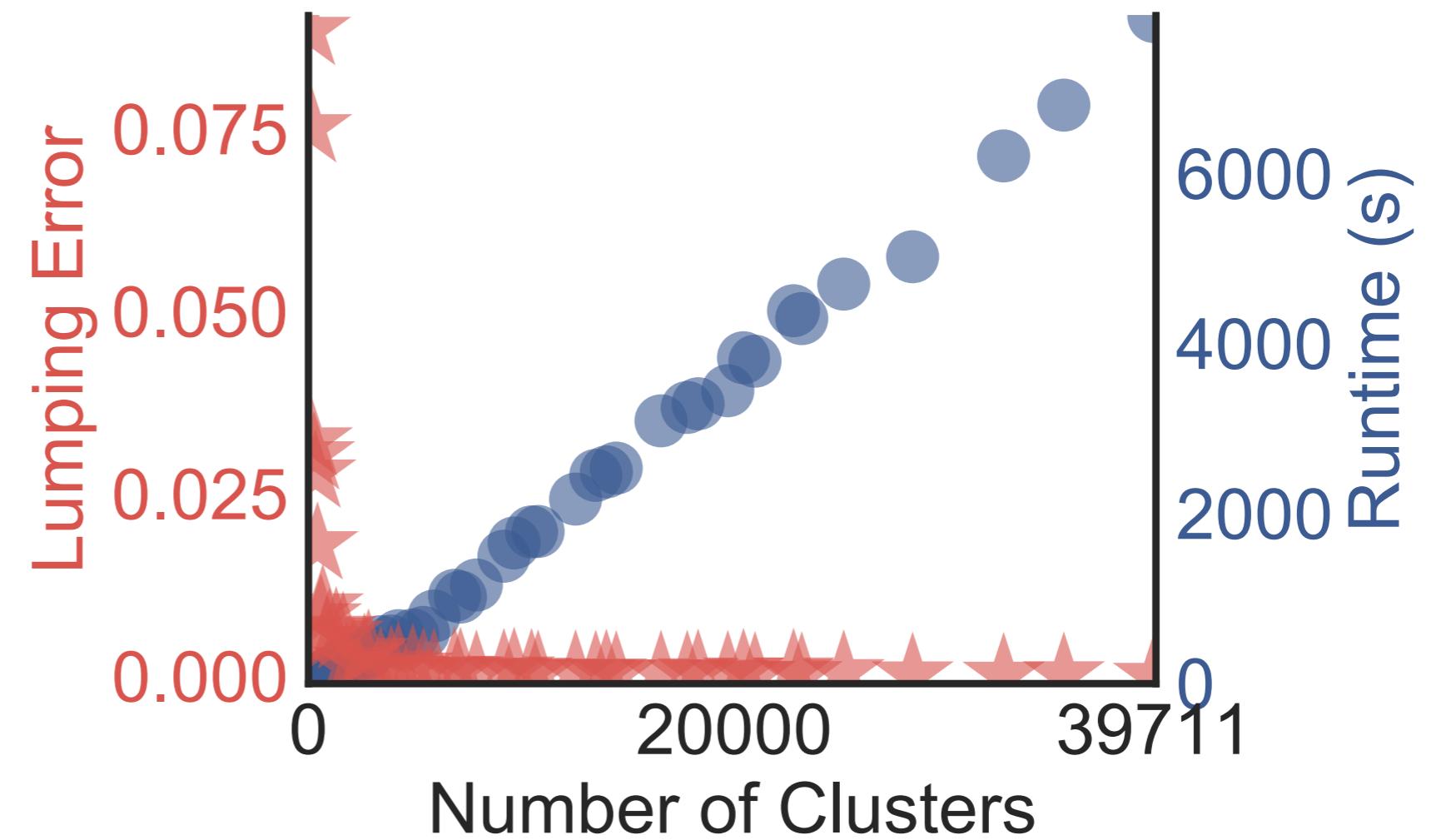
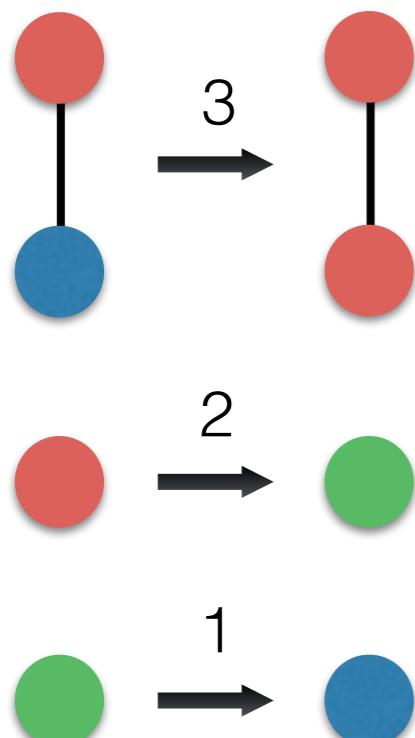
Results SIR



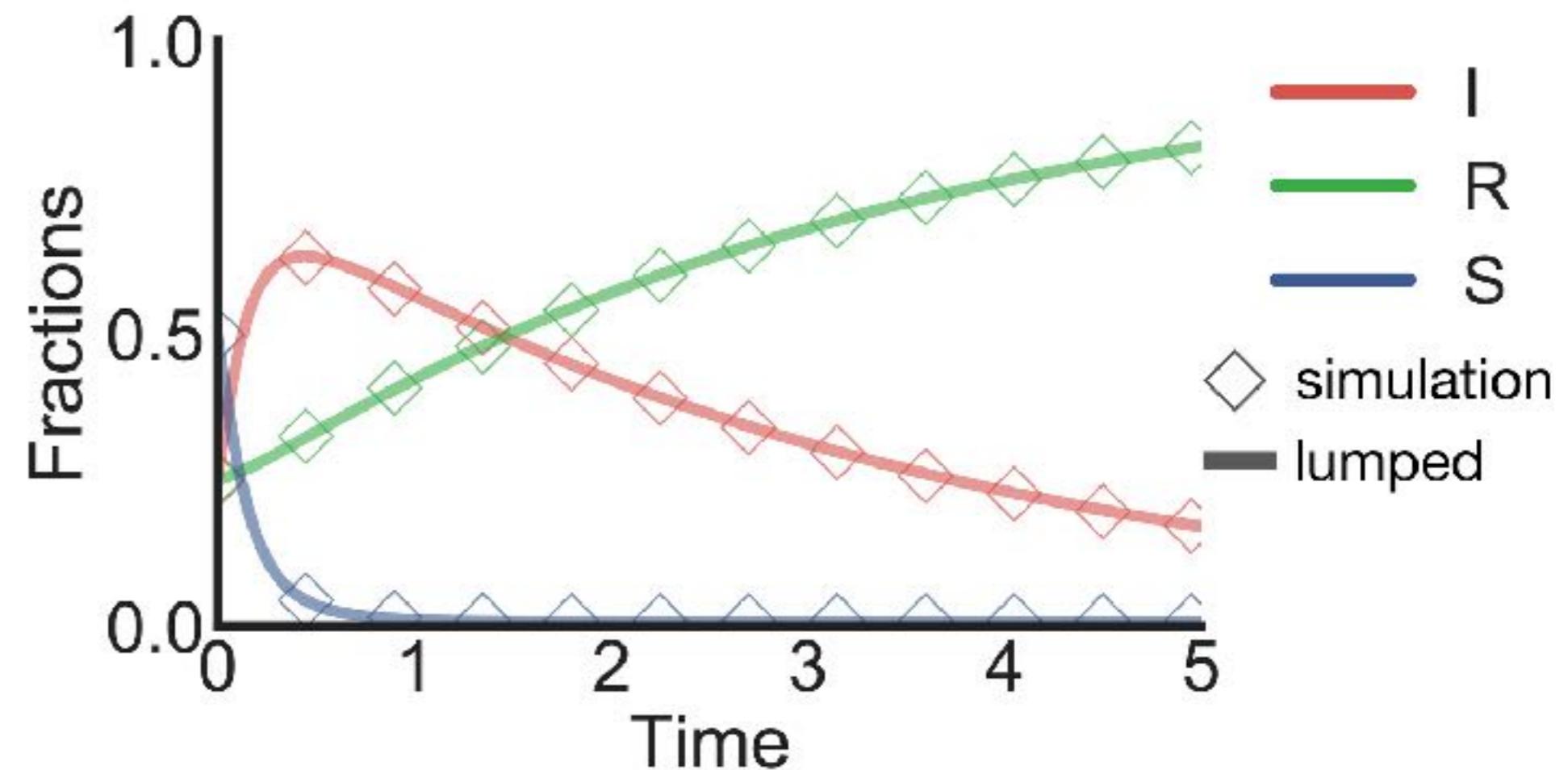
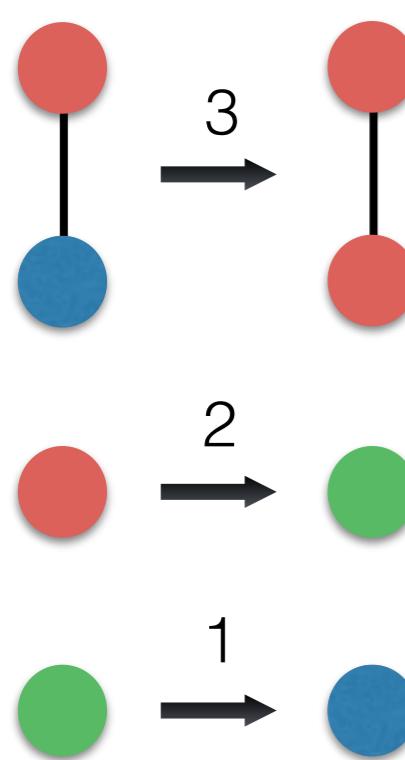
5 373 ODEs vs 119 133 ODEs

Truncated power-law with $\gamma = 2.5$ and $k_{\max} = 60$

Results SIR



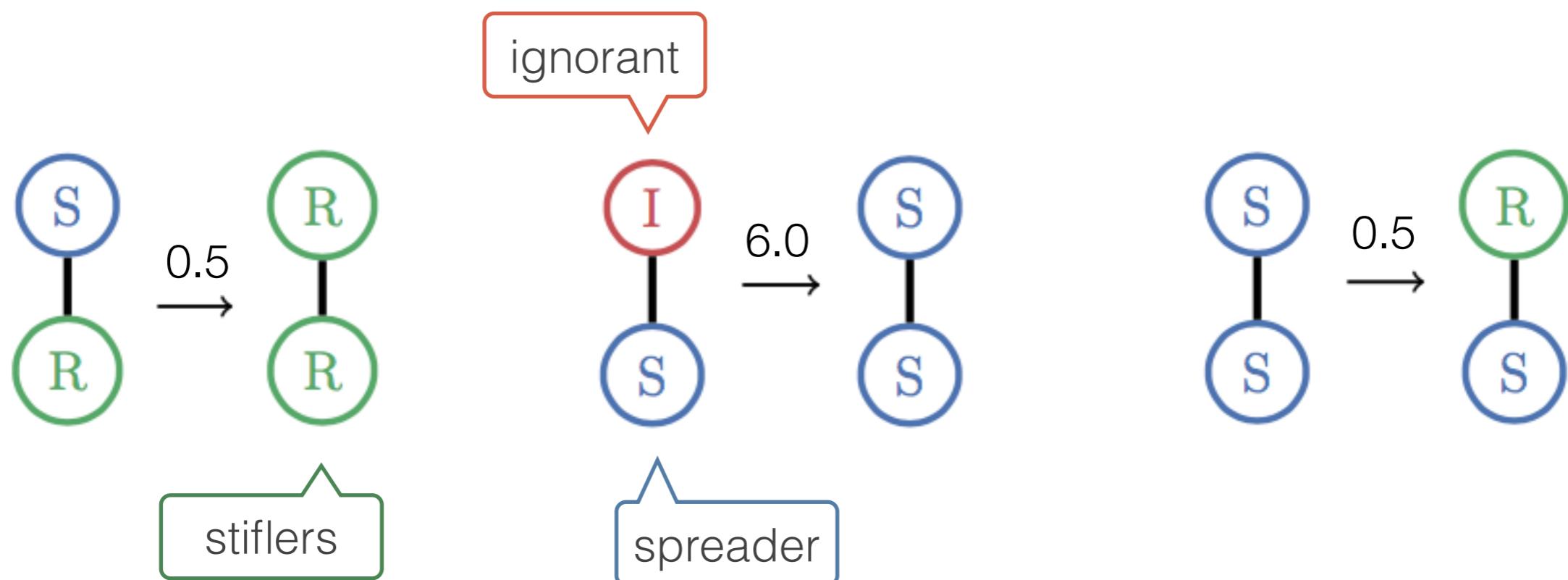
Results SIR



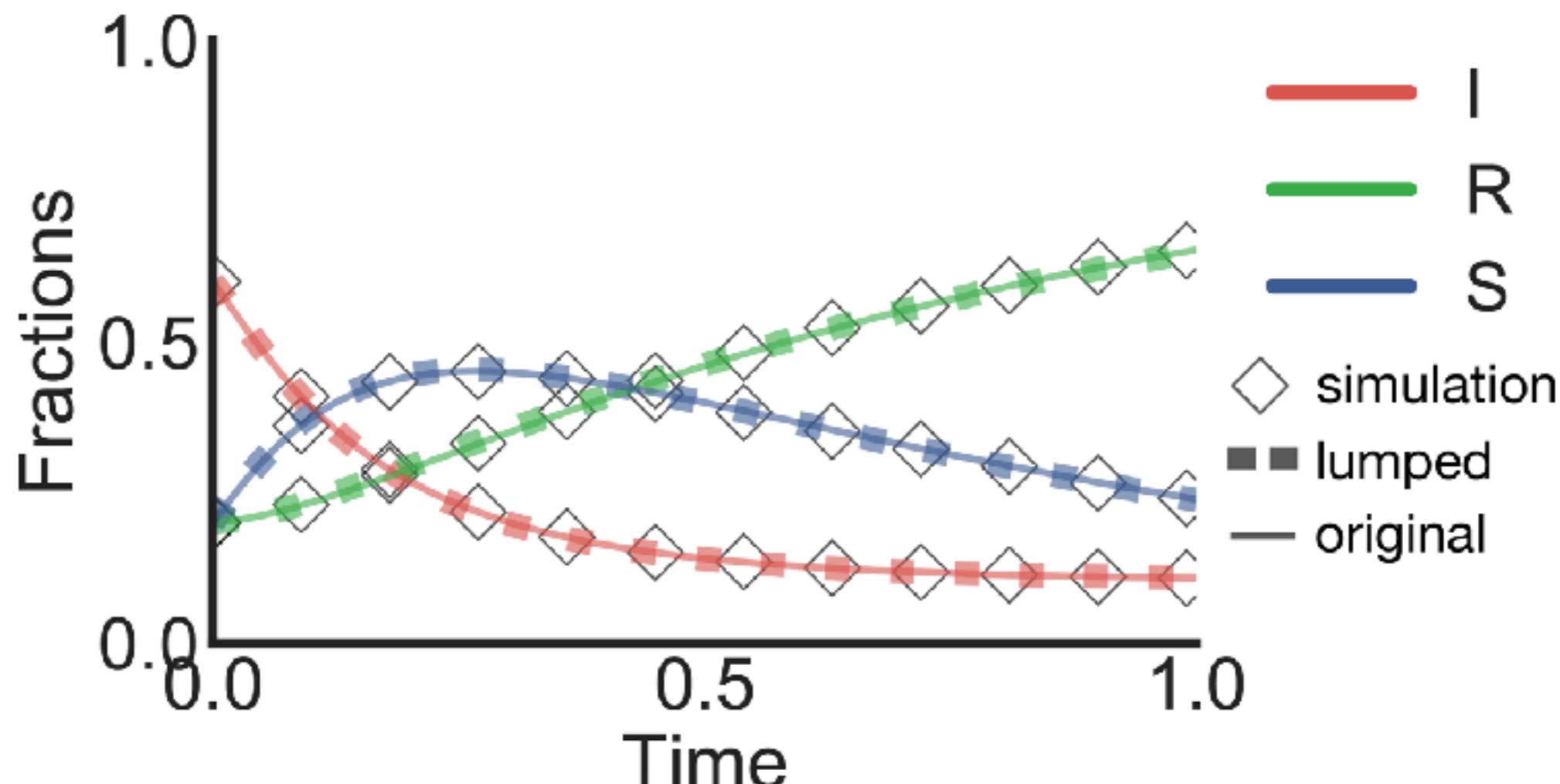
25K ODEs vs 63M ODEs

Truncated power-law with $\gamma = 2.5$ and $k_{\max} = 600$

Rumor Spreading



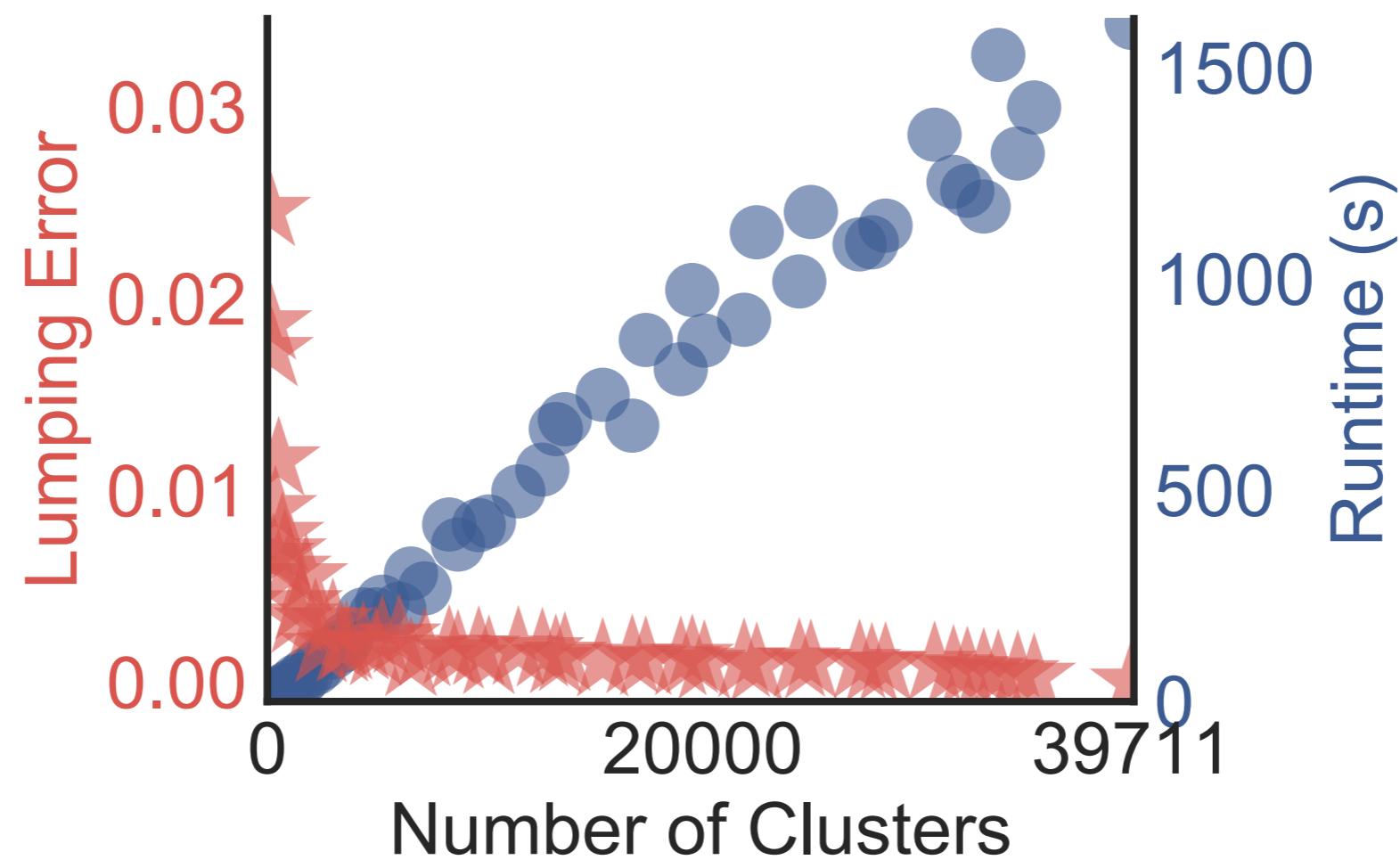
Rumor Spreading



3 096 ODEs vs 119 133 ODEs

Truncated power-law with $\gamma = 3.0$ and $k_{\max} = 60$

Rumor Spreading



Conclusion

- Lumping exploits redundancies in AME
- Massive reduction of computational time
- Lumping error of similar order as AME error

Future Work:

- Take spatial information into account

Thank you

AME

Fraction of label s with neighbourhood \mathbf{m}

ODEs

$$\frac{\partial x_{s,\mathbf{m}}}{\partial t} = \sum_{(s',f,s) \in R^{s^+}} f(\mathbf{m}) x_{s',\mathbf{m}} - \sum_{(s,f,s') \in R^{s^-}} f(\mathbf{m}) x_{s,\mathbf{m}}$$

$$+ \sum_{\substack{(s_1,s_2) \in \mathbb{S}^2 \\ s_1 \neq s_2}} \beta^{ss_1 \rightarrow ss_2} x_{s,\mathbf{m}^{\{s_1^+, s_2^-\}}} \mathbf{m}^{\{s_1^+, s_2^-\}}[s_1]$$

$$- \sum_{\substack{(s_1,s_2) \in \mathbb{S}^2 \\ s_1 \neq s_2}} \beta^{ss_1 \rightarrow ss_2} x_{s,\mathbf{m}} \mathbf{m}[s_1]$$

local

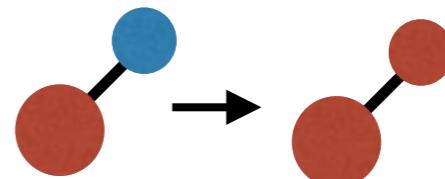
weighted average over all degrees

$$x_s(t) = \sum_{\mathbf{m} \in \mathbb{M}} x_{s,\mathbf{m}}(t) \cdot P(k_{\mathbf{m}})$$

global

$$\beta^{ss_1 \rightarrow ss_2} = \frac{\sum_{\mathbf{m} \in \mathbb{M}} P(k_{\mathbf{m}}) \sum_{(s_1,f,s_2) \in R^{s_1 \rightarrow s_2}} f(\mathbf{m}) x_{s_1,\mathbf{m}} \mathbf{m}[s]}{\sum_{\mathbf{m} \in \mathbb{M}} P(k_{\mathbf{m}}) x_{s_1,\mathbf{m}} \mathbf{m}[s]}$$

average rates of pairs



Lumping the AME

Original

$$\begin{aligned} \frac{\partial x_{s,\mathbf{m}}}{\partial t} = & \sum_{(s',f,s) \in R^{s,+}} f(\mathbf{m}) x_{s',\mathbf{m}} - \sum_{(s,f,s') \in R^{s,-}} f(\mathbf{m}) x_{s,\mathbf{m}} \\ & + \sum_{\substack{(s_1,s_2) \in \mathbb{S}^2 \\ s_1 \neq s_2}} \beta^{ss_1 \rightarrow ss_2} x_{s,\mathbf{m}^{\{s_1^+, s_2^-\}}} \mathbf{m}^{\{s_1^+, s_2^-\}}[s_1] \\ & - \sum_{\substack{(s_1,s_2) \in \mathbb{S}^2 \\ s_1 \neq s_2}} \beta^{ss_1 \rightarrow ss_2} x_{s,\mathbf{m}} \mathbf{m}[s_1] \end{aligned}$$

Lumped

$$\begin{aligned} \frac{\partial z_{s,C}}{\partial t} = & \sum_{\mathbf{m} \in C} P(k_{\mathbf{m}}|C) \cdot \left(\sum_{(s',f,s) \in R^{s,+}} f(\mathbf{m}) r_{C,k_{\mathbf{m}}} z_{s',C} \right. \\ \text{Average} \quad & - \sum_{(s,f,s') \in R^{s,-}} f(\mathbf{m}) r_{C,k_{\mathbf{m}}} z_{s,C} \\ & + \sum_{\substack{(s_1,s_2) \in \mathbb{S}^2 \\ s_1 \neq s_2}} \beta_{\mathcal{L}}^{ss_1 \rightarrow ss_2} r_{C,k_{\mathbf{m}}} z_{s,C} \mathbf{m}^{\{s_1^-, s_2^-\}}[s_1] \\ & \left. - \sum_{\substack{(s_1,s_2) \in \mathbb{S}^2 \\ s_1 \neq s_2}} \beta_{\mathcal{L}}^{ss_1 \rightarrow ss_2} r_{C,k_{\mathbf{m}}} z_{s,C} \mathbf{m}[s_1] \right) \end{aligned}$$

Substitution