



SPATIO-TEMPORAL EVENT MODELING

Lecture Series

COLLABORATIVE INTELLIGENCE

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github.com/gerritgr/spatio-temporal-lecture

Space and Time

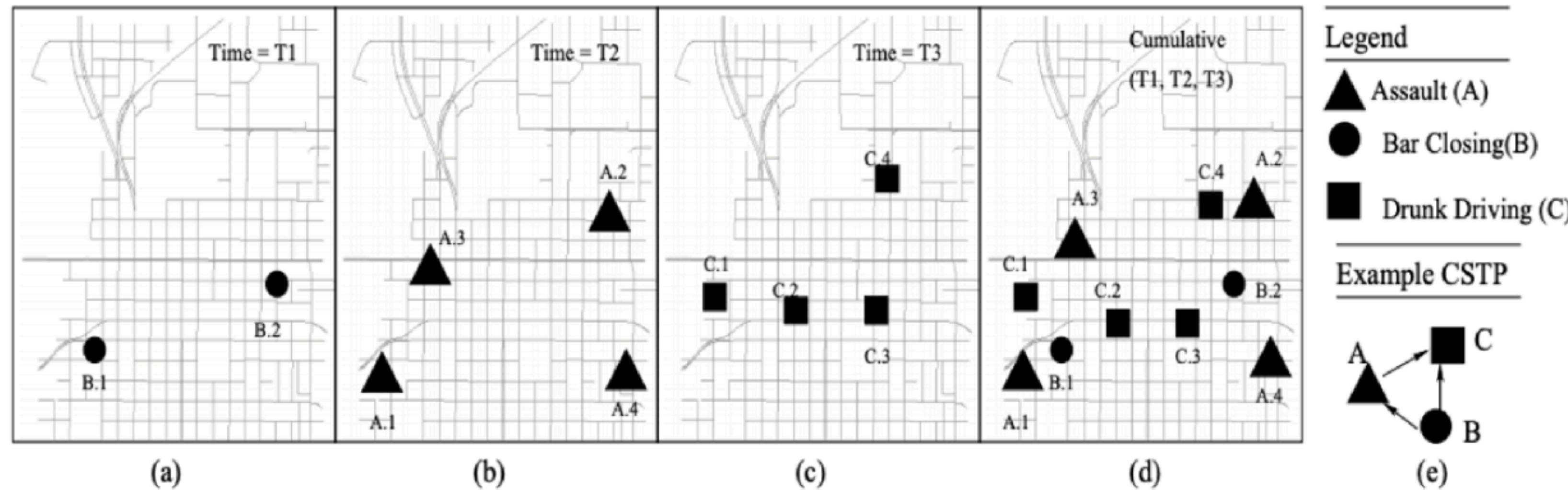
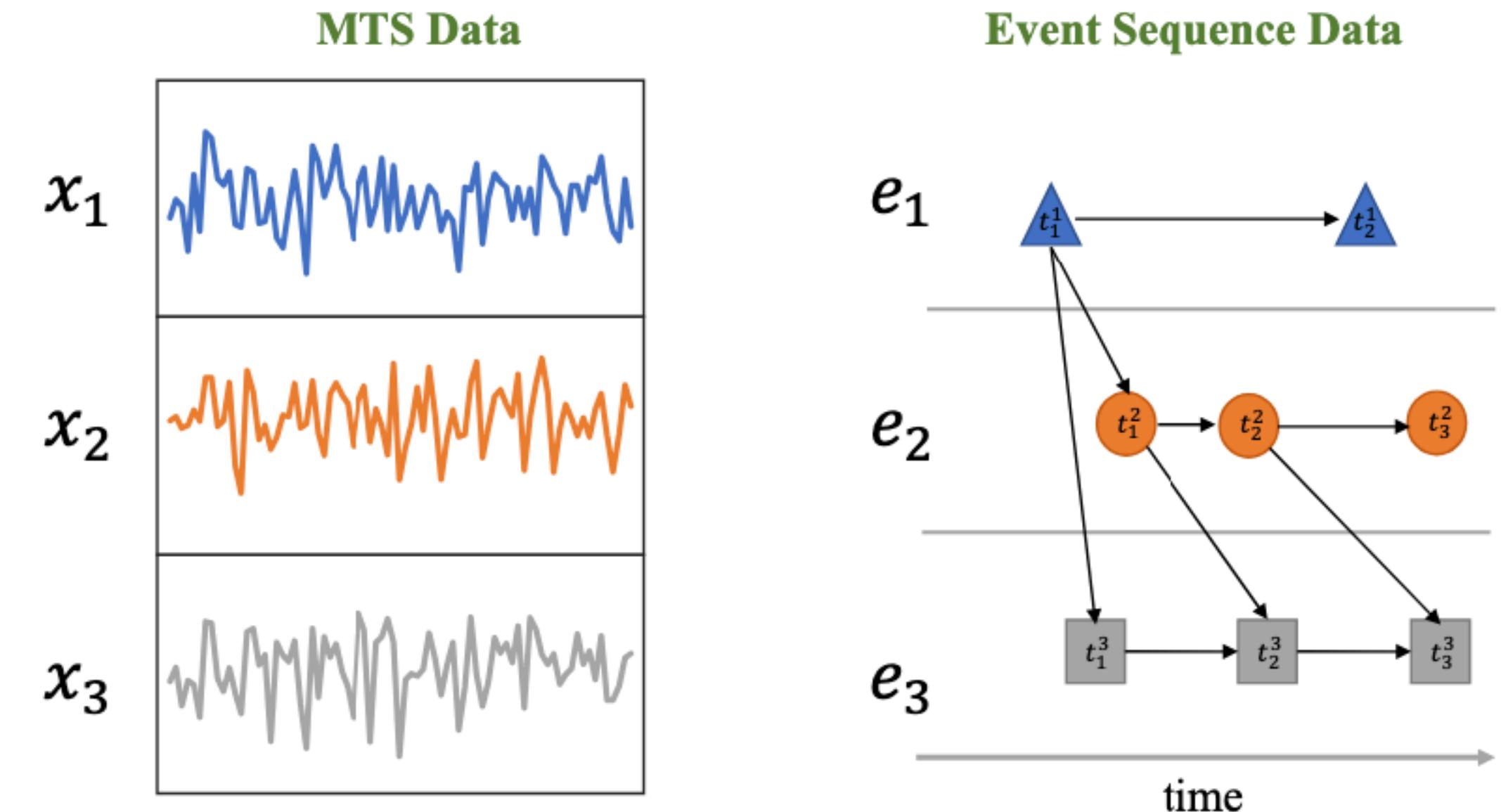
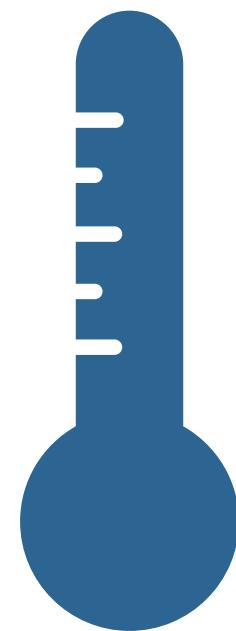


Fig. 2. Illustrative ST Crime Dataset and Cascading ST Pattern. (Best in color)

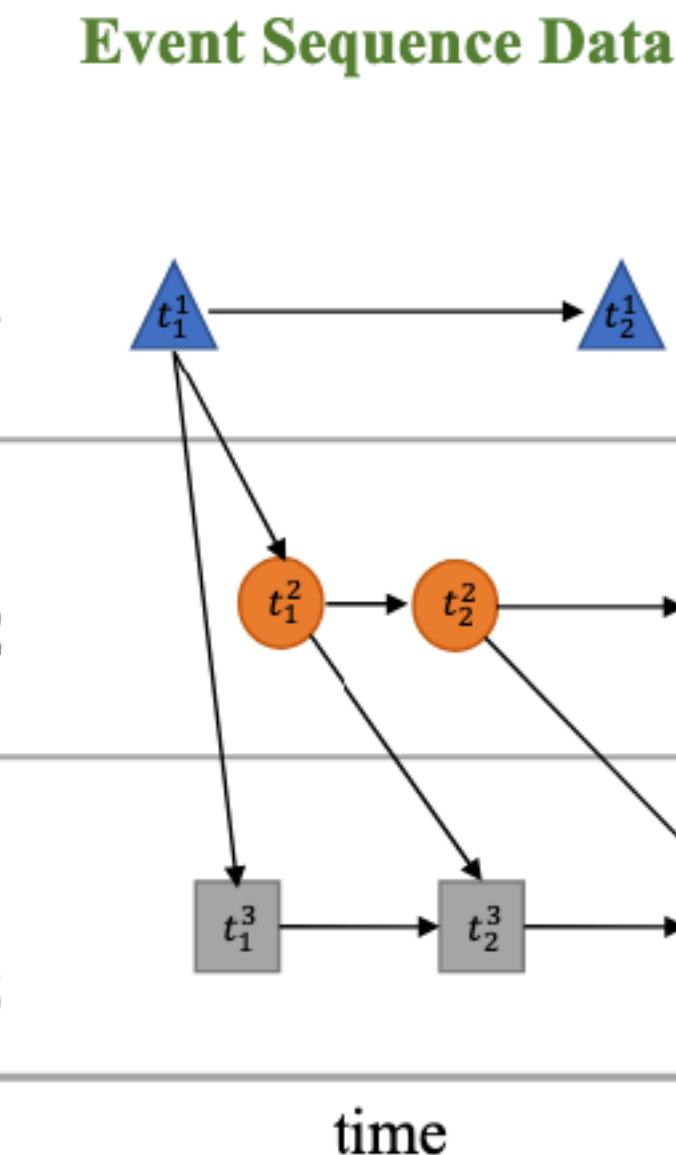
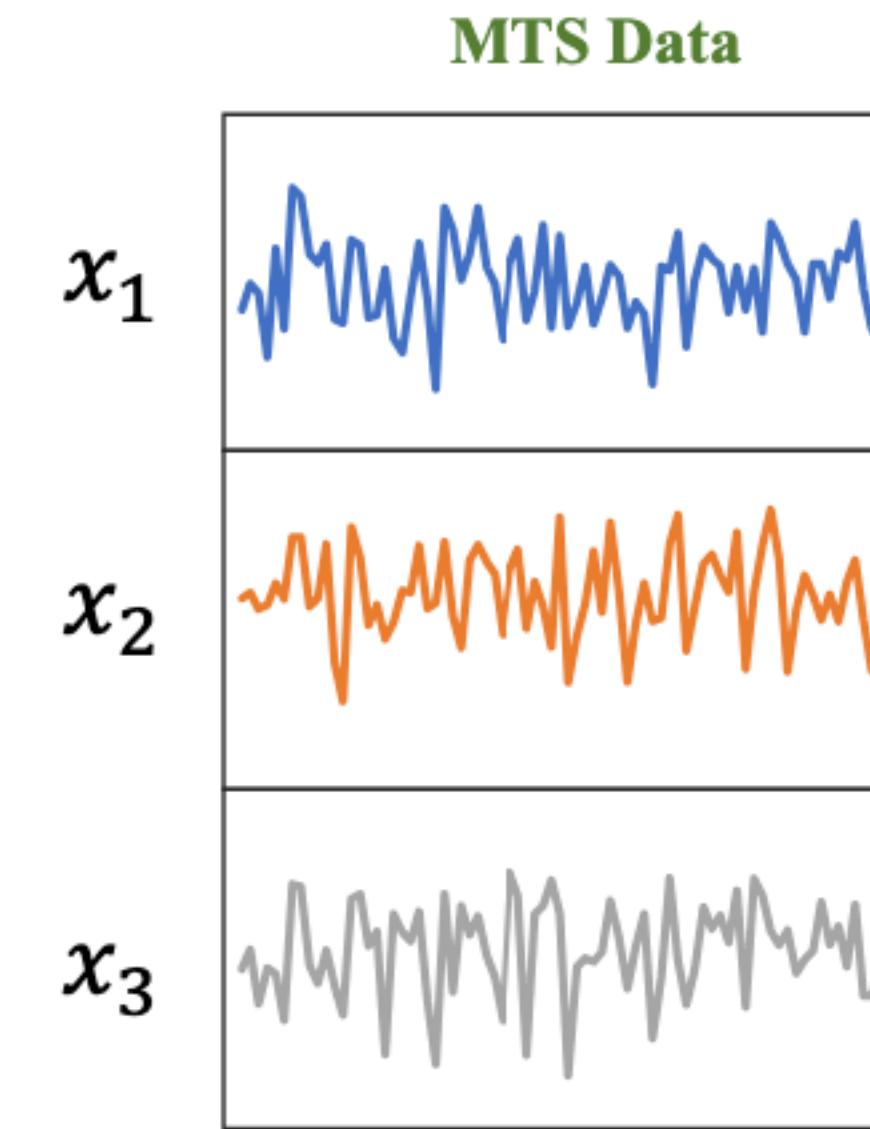
Time-Series vs Event Data



Time-Series vs Event Data

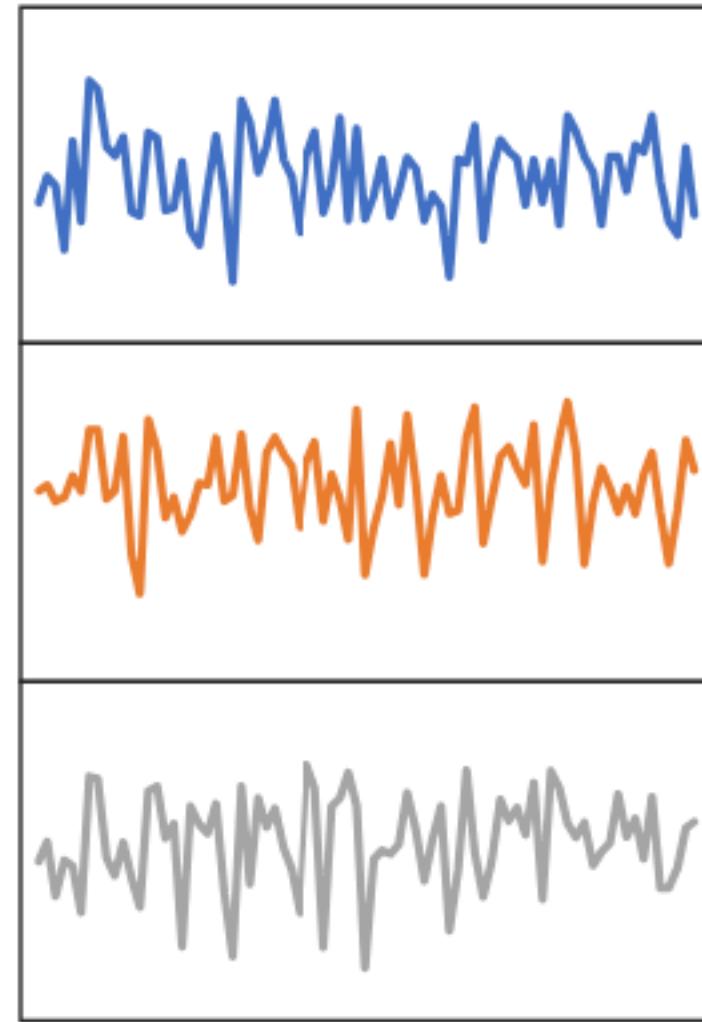


Temperature can be **continuously** monitored at various weather stations.



Lightning strike are **events** with specific times and locations.

Time-Series vs Event Data



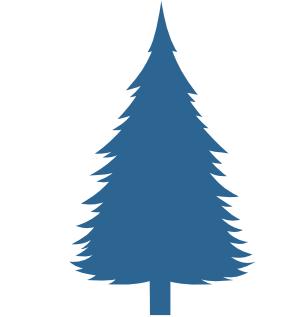
vehicle counts
traffic density



energy consumption
noise



air quality
soil moisture



stock prices
trading volume

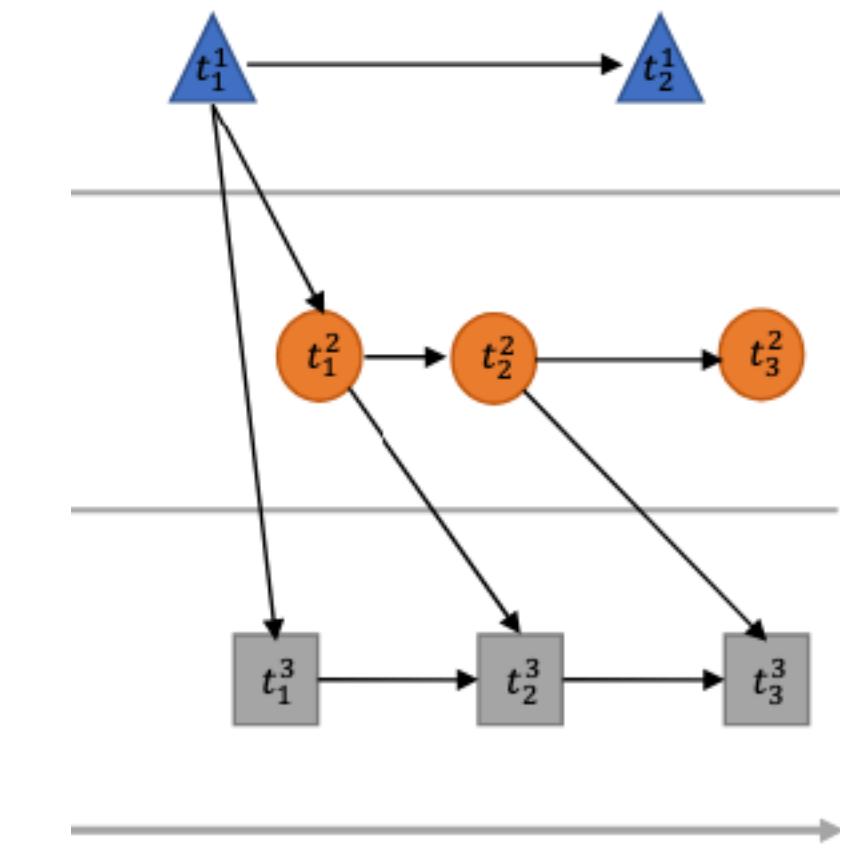


accidents
traffic jams

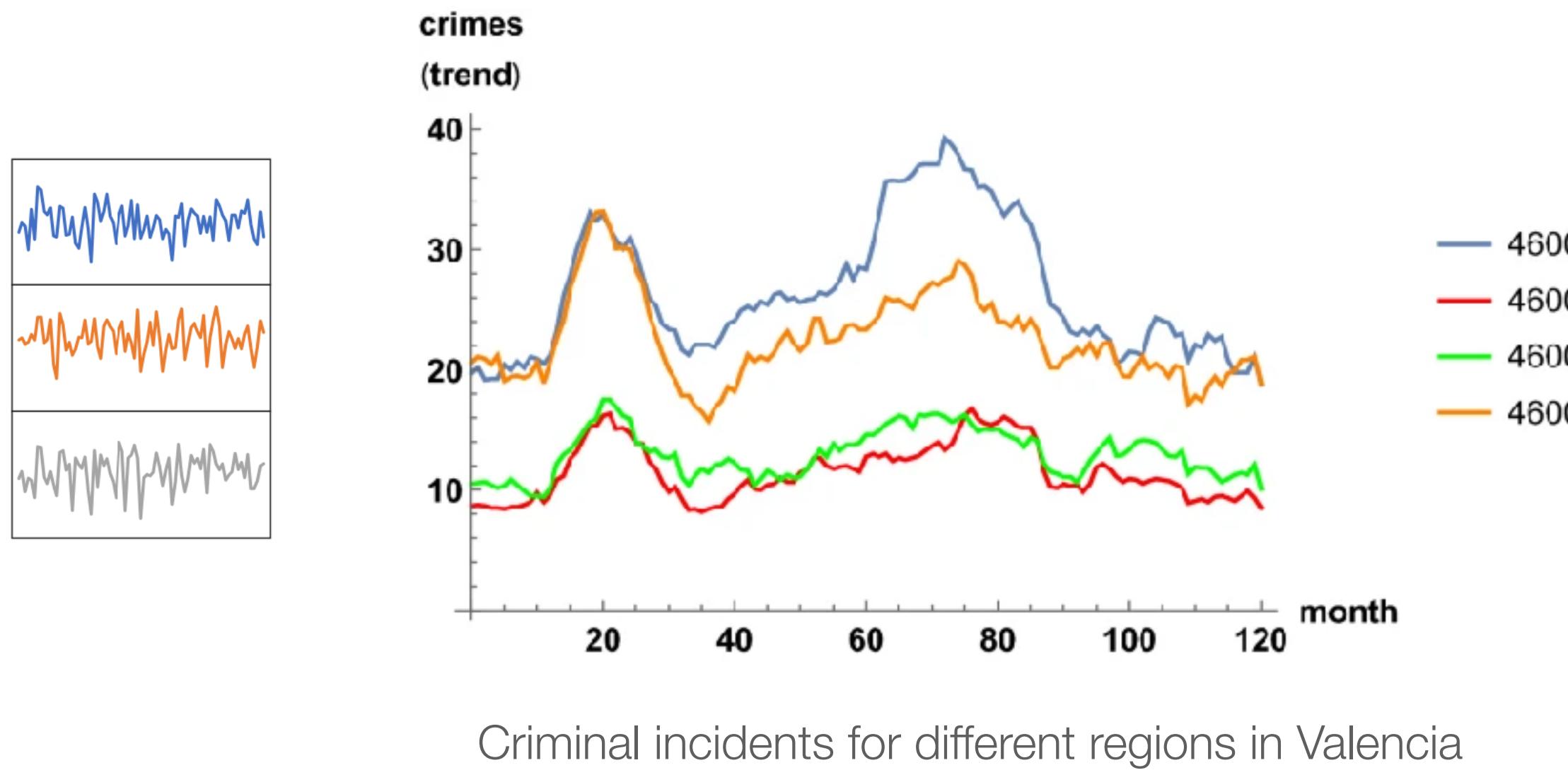
public gathering
infrastructure failure

forest fire
oil spills

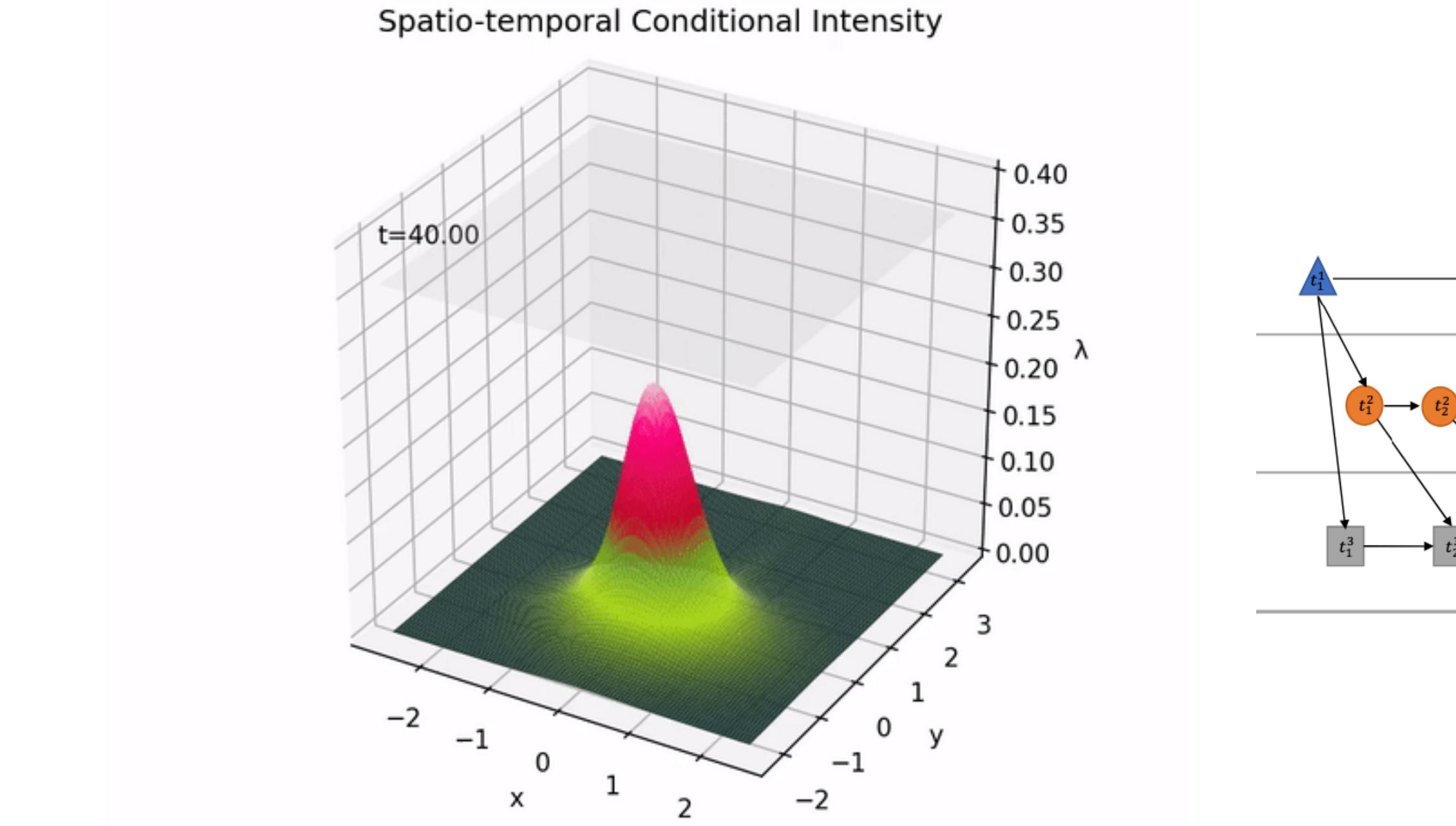
large trades
market crashes



Models

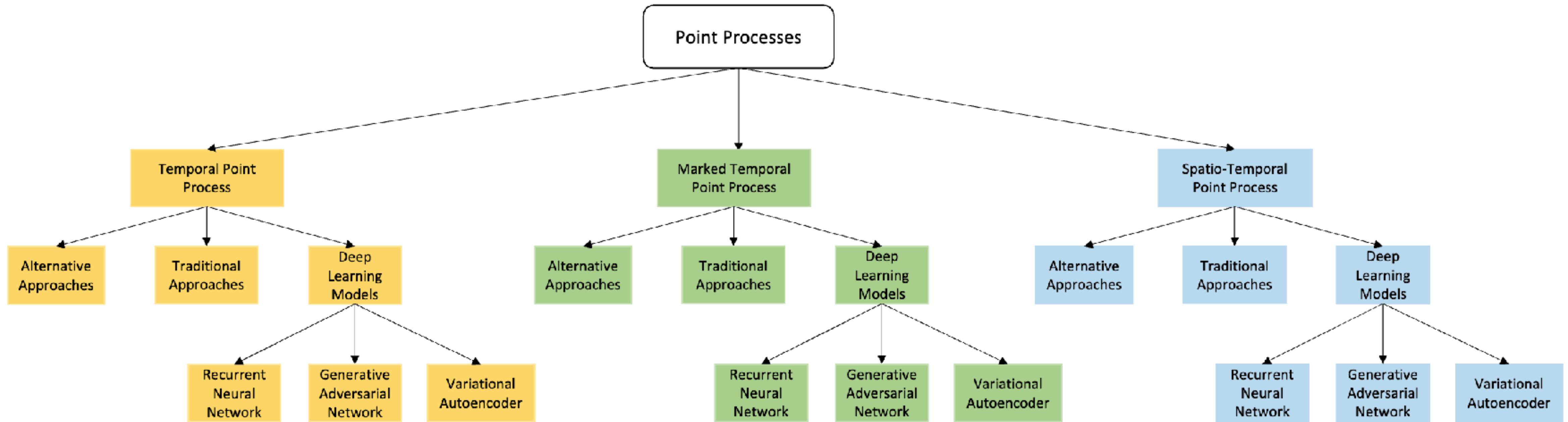


(Stochastic/Partial) **Differential Equations**
Gaussian Processes

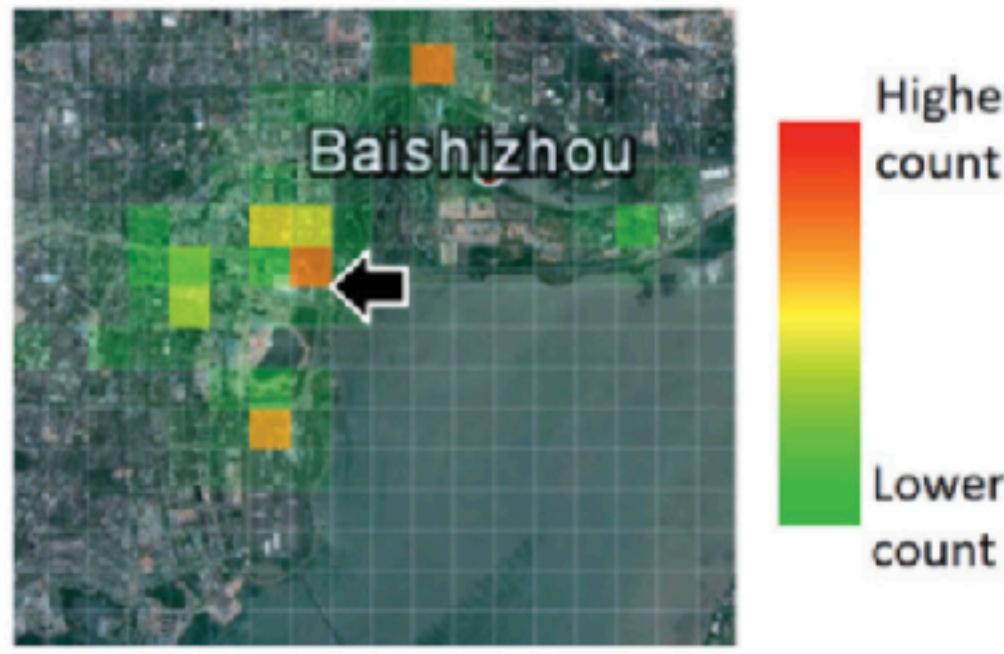


(Marked) (Spatio-) **Temporal Point Processes**

Predictions on Graphs



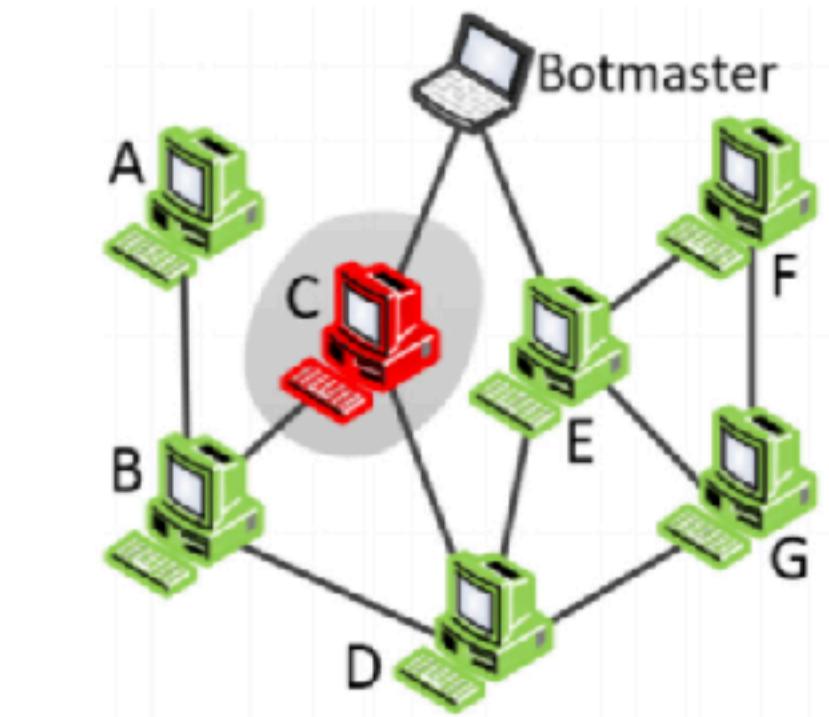
Notions of Space and Time



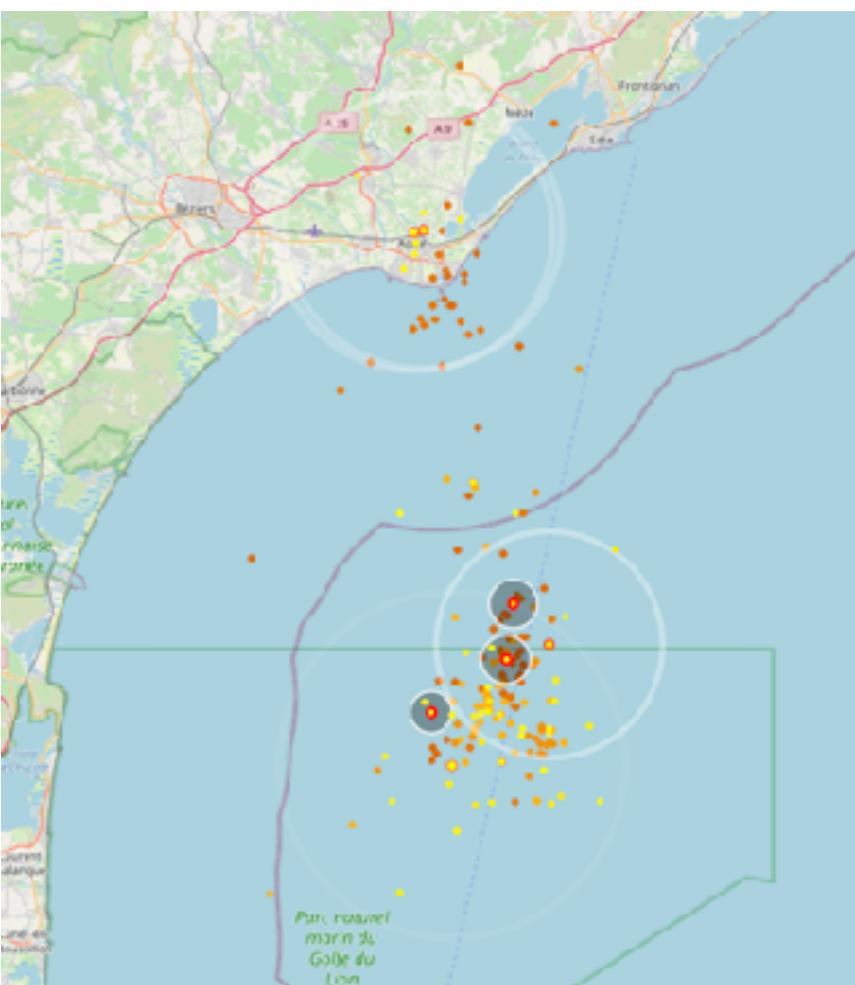
(a) Raster-based event prediction



(b) Point-based event prediction in Euclidean and non-Euclidean spaces

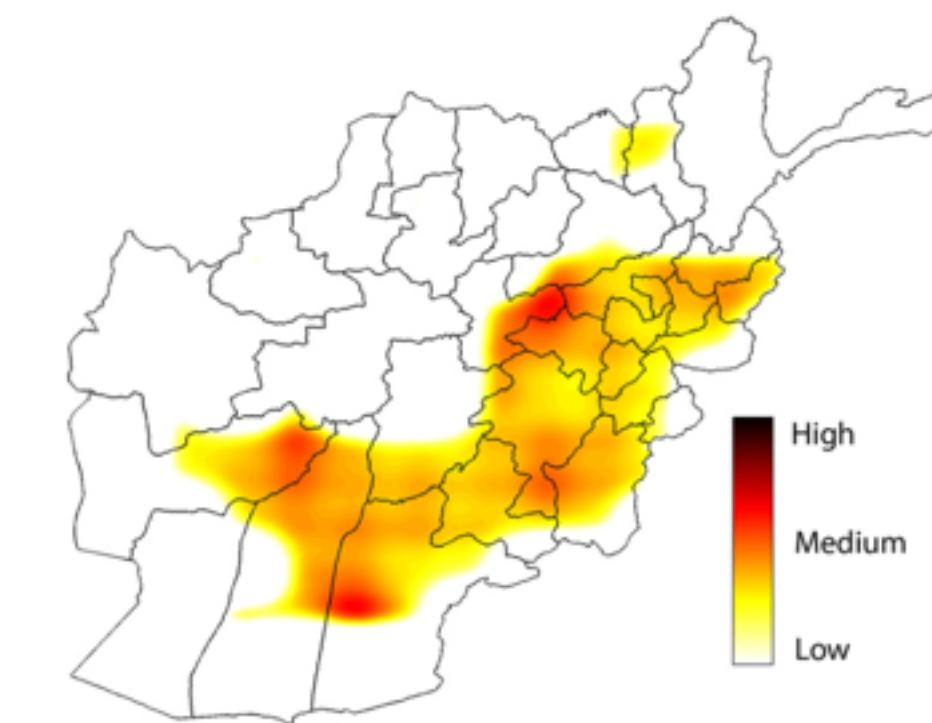


What makes spatio-temporal data special?



lightningmaps.org

Not i.i.d.



Point Process Modeling of the Afghan War Diary (Andrew Zammit-Mangion et al.)

Arrow of time (temporal priority principle)

"A cause must precede its effect"

What makes spatio-temporal data special?

Arrow of time (temporal priority principle)

Caution: Depends on abstraction



Event A: **rooster crows**

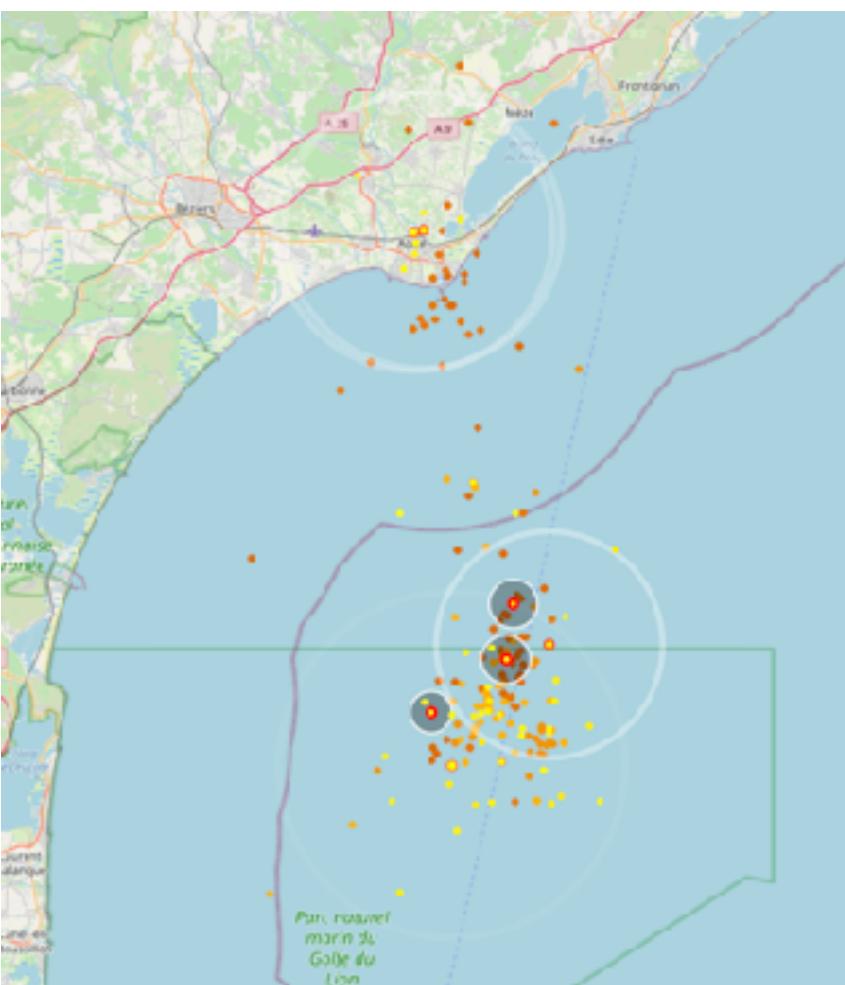


precedes (causes?)

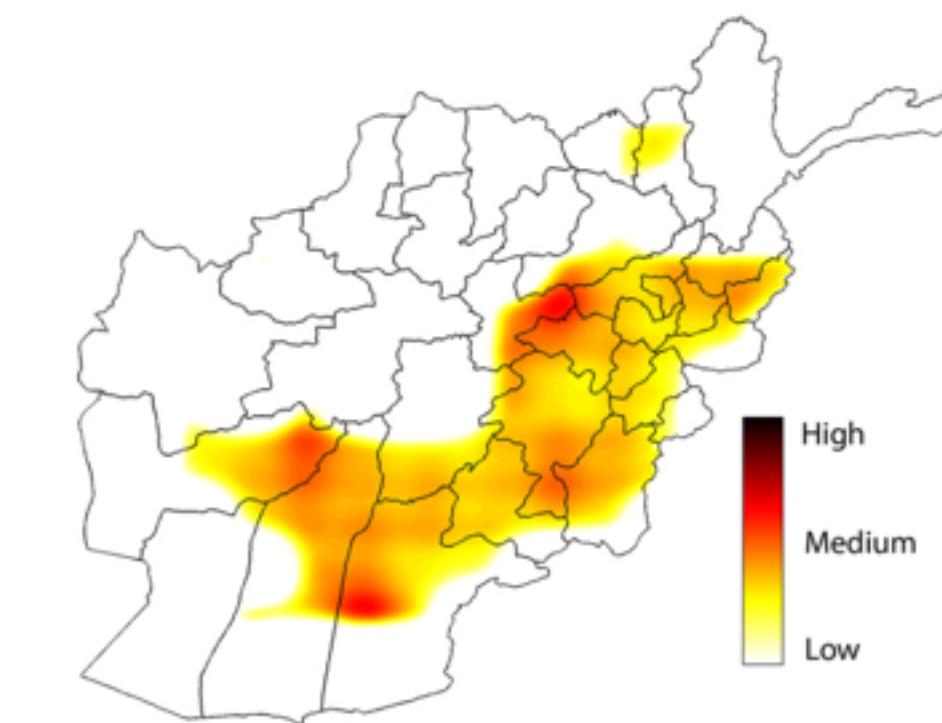
Event B: **sun rises**

Fix: Consider morning light (caused by the sun rising) as an additional event.

What makes spatio-temporal data special?



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Point Process Modeling of the Afghan War Diary (Andrew Zammit-Mangion et al.)

Clustering: Positive correlation in space

(positive feedback mechanisms,
latent cause)

Self-Excitement: Positive correlation in time

(resource is depleted,
environment adapts)

Self-Correction: negative correlation in time

What makes spatio-temporal data special?

Periodicity and Seasonality:

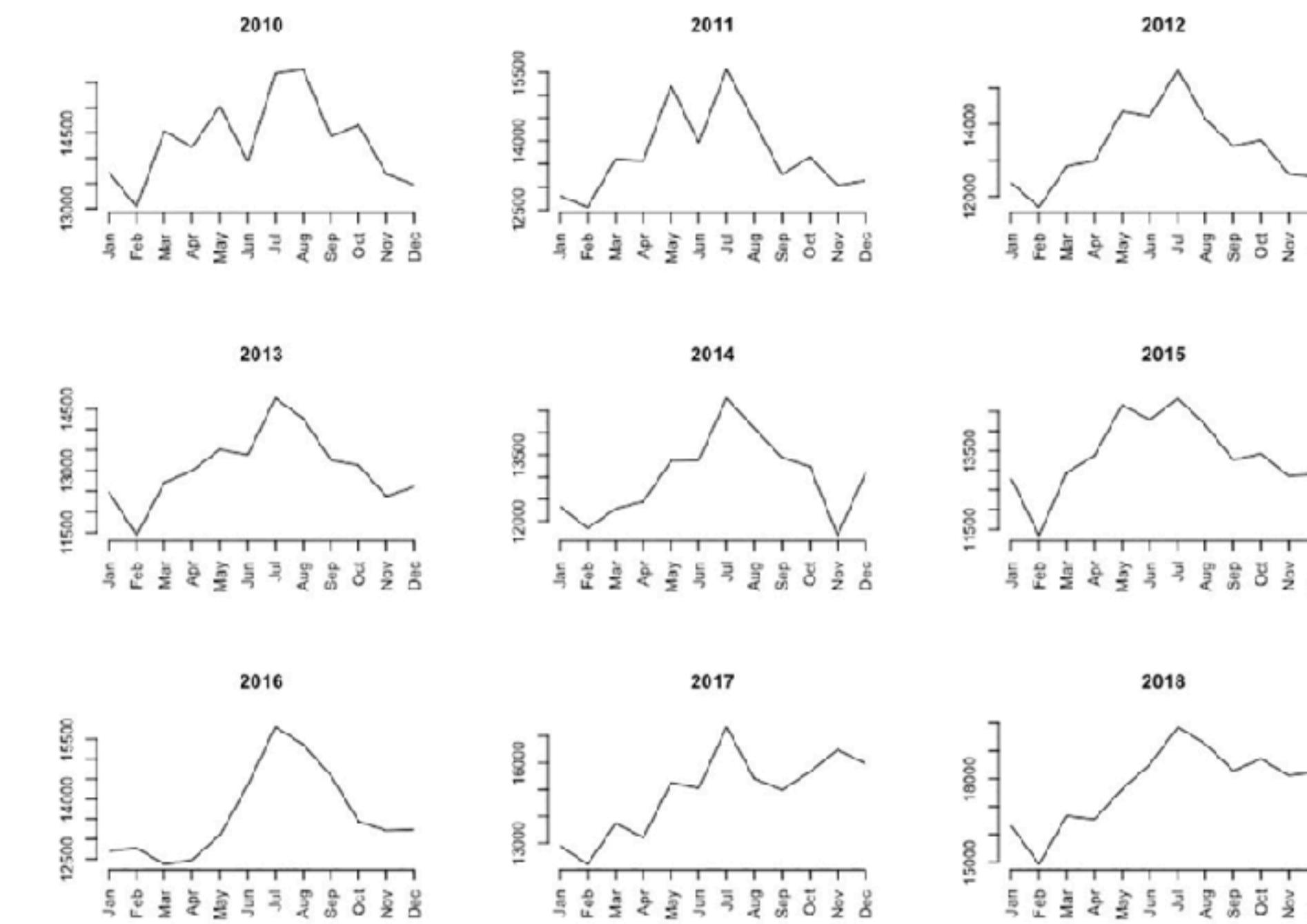
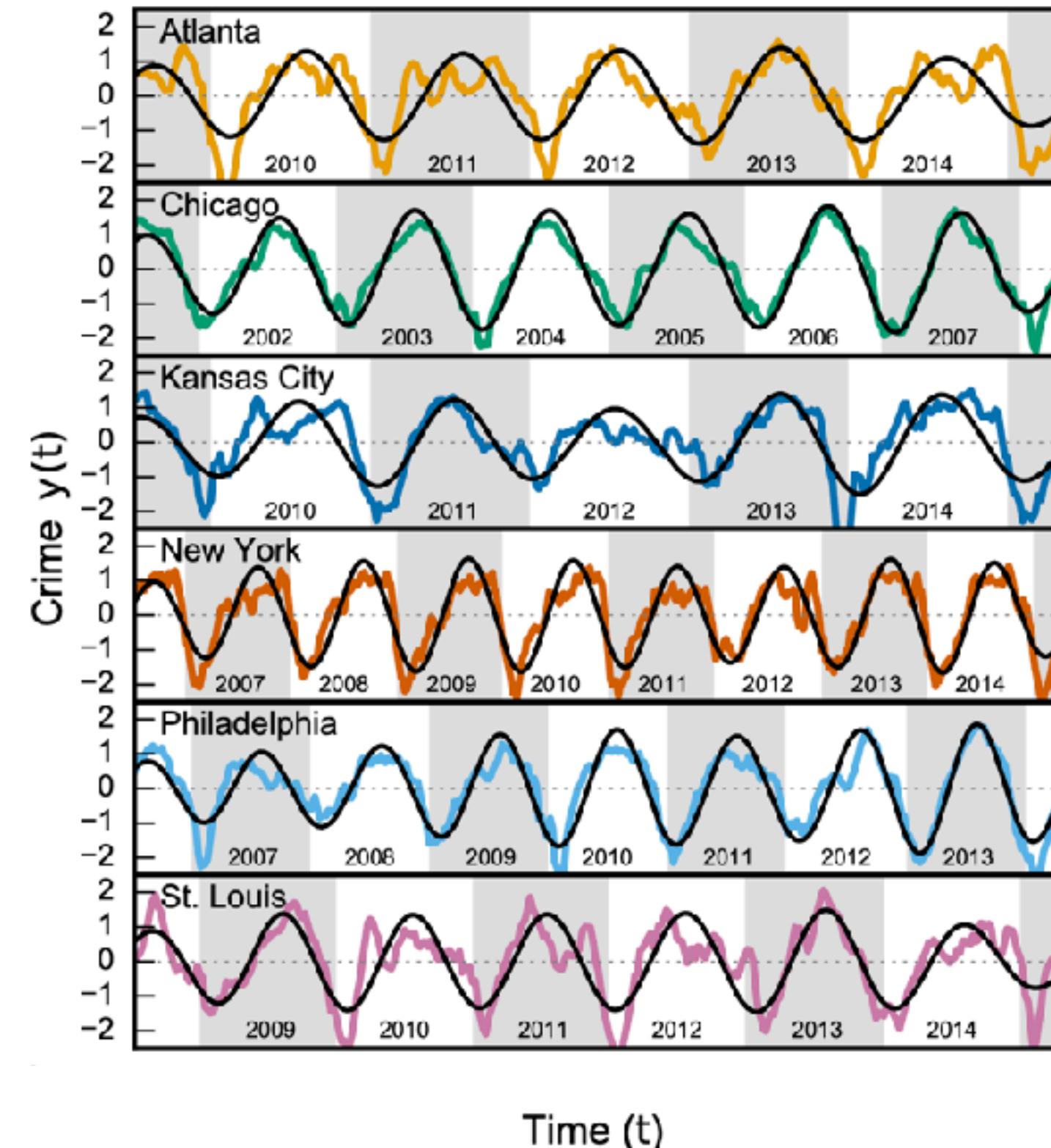


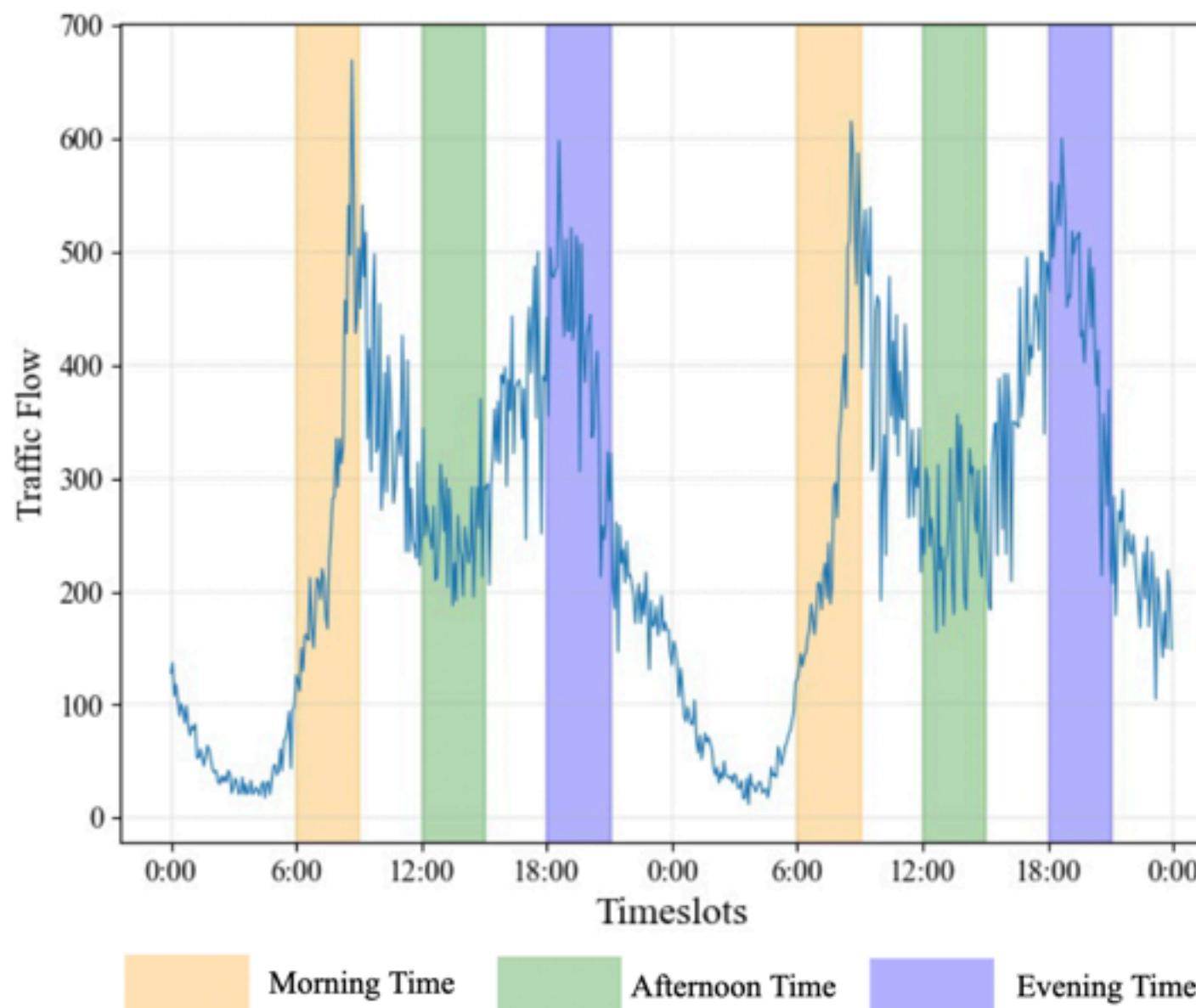
Fig 6. Seasonal trend from January to December of the monthly total of crimes in Barcelona, by year.

Spatio-temporal variations in the urban rhythm: the travelling waves of crime (Oliveira et al.)

The effect of seasonality in predicting the level of crime. A spatial perspective (Delgado et al.)

What makes spatio-temporal data special?

Periodicity and Seasonality:



STG4Traffic: A Survey and Benchmark of Spatial-Temporal Graph Neural Networks for Traffic Prediction (Luo et al.)

Self-Exciting

Does a robbery make it more or less likely that another robbery will occur the next day?

- Signal to criminals
- Acts of retaliation

Self-Correcting

- Vigilance
- Police presence
- Community engagement



<https://comic-denkblase.de/70-jahre-panzerknacker>

Self-Exciting

Self-Correcting

The flashing of a firefly

Suicide

Crime

Earthquake

A goal in a soccer match

Terrorist attack

Large jump in the stock price of a company

Neuron firing

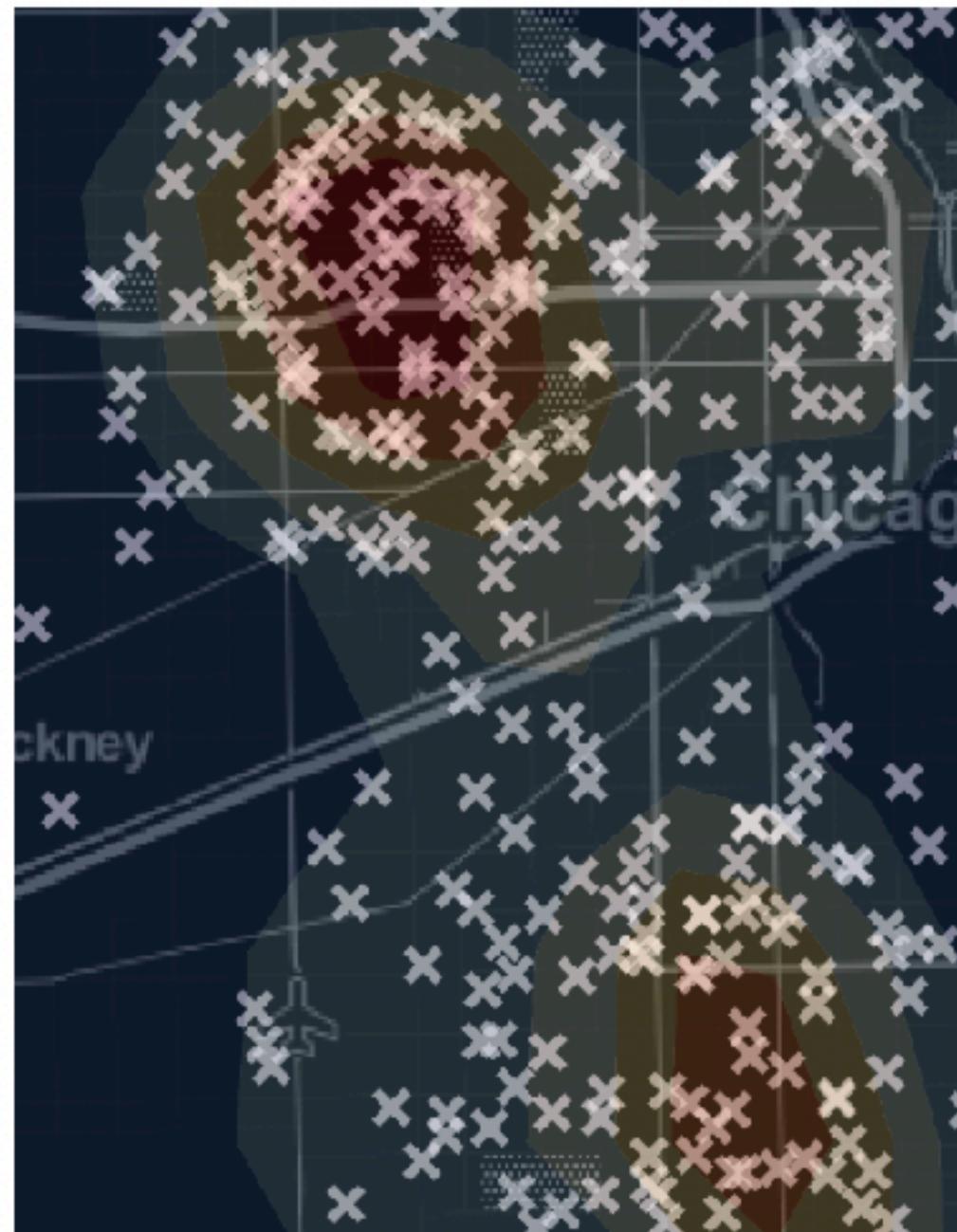
A person coughs

Rainfall

Traffic congestion

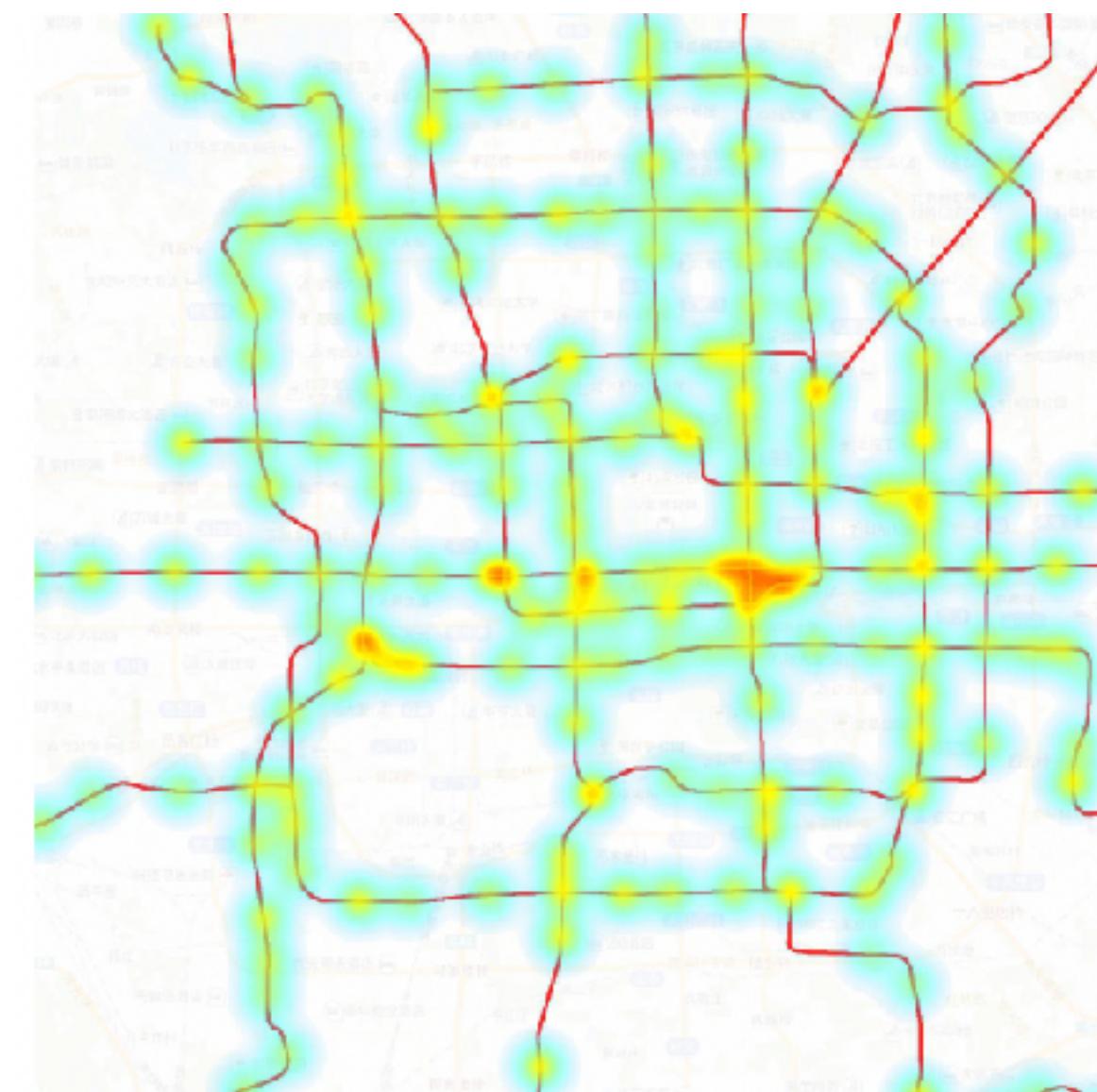
Applications

Prediction:



Crime

Deep Mixture Point Processes: Spatio-temporal Event Prediction with Rich Contextual Information (Okawa et al.)



Traffic

GSTNet: Global Spatial-Temporal Network for Traffic Flow Prediction (Fang et al.)

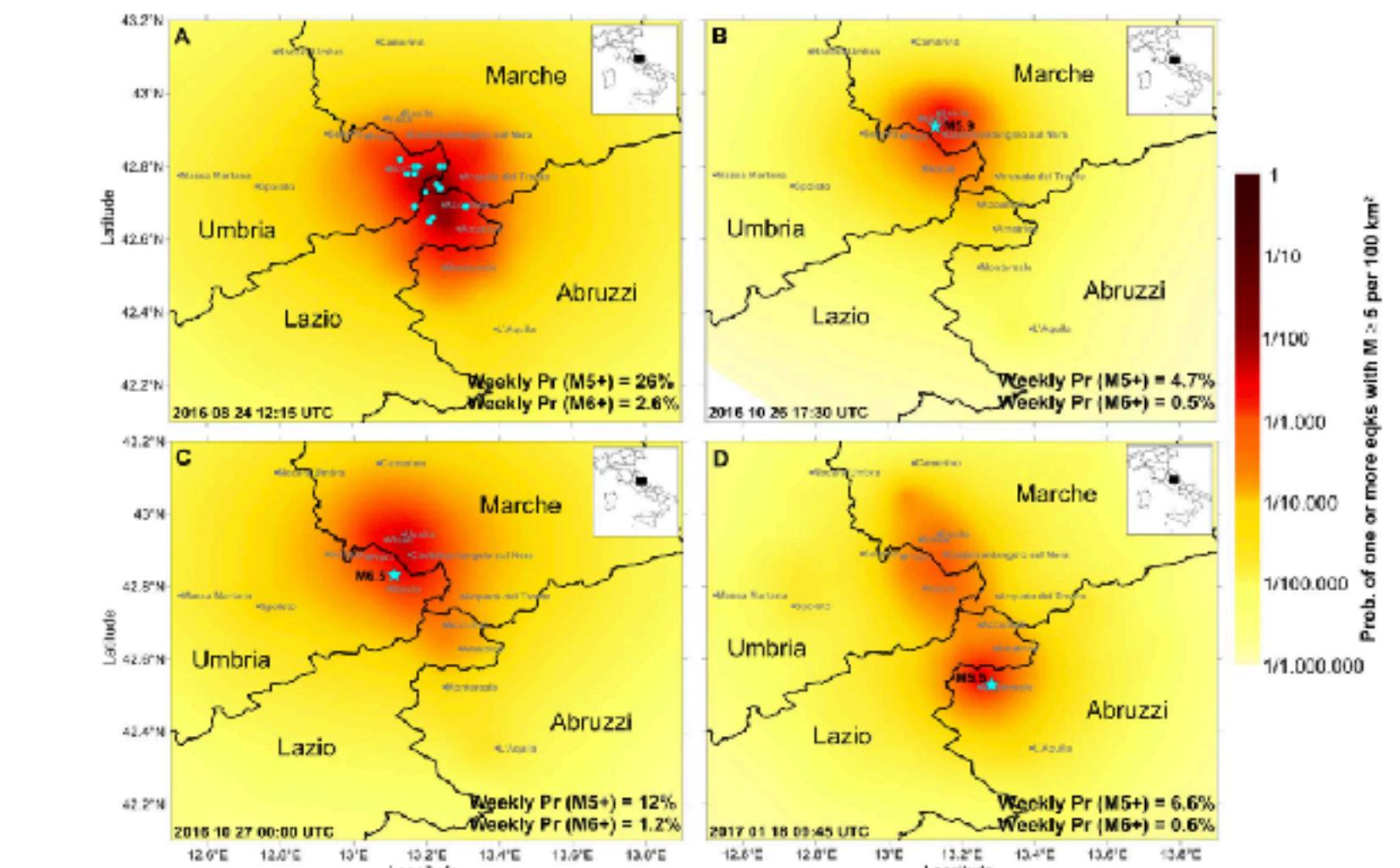


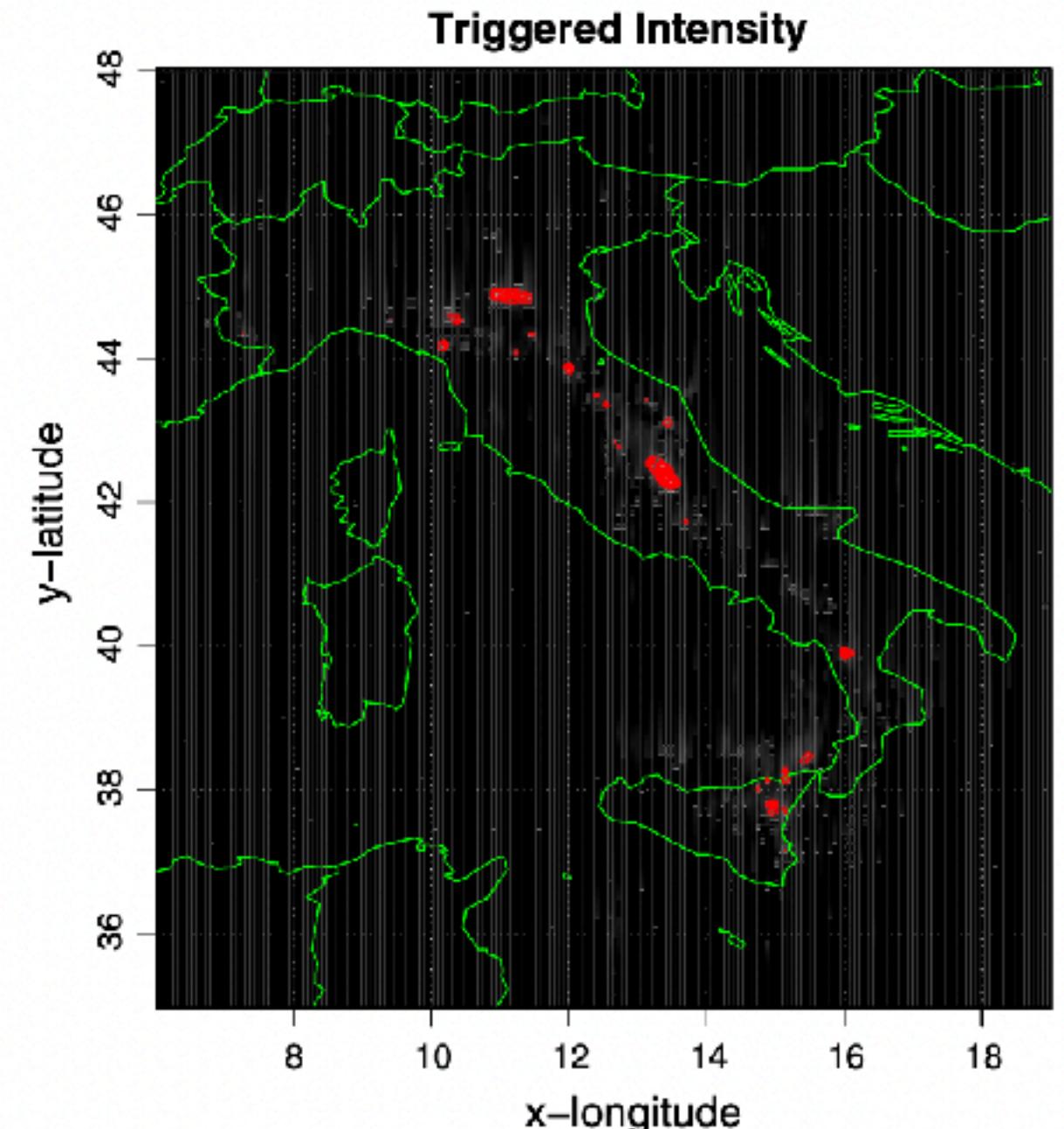
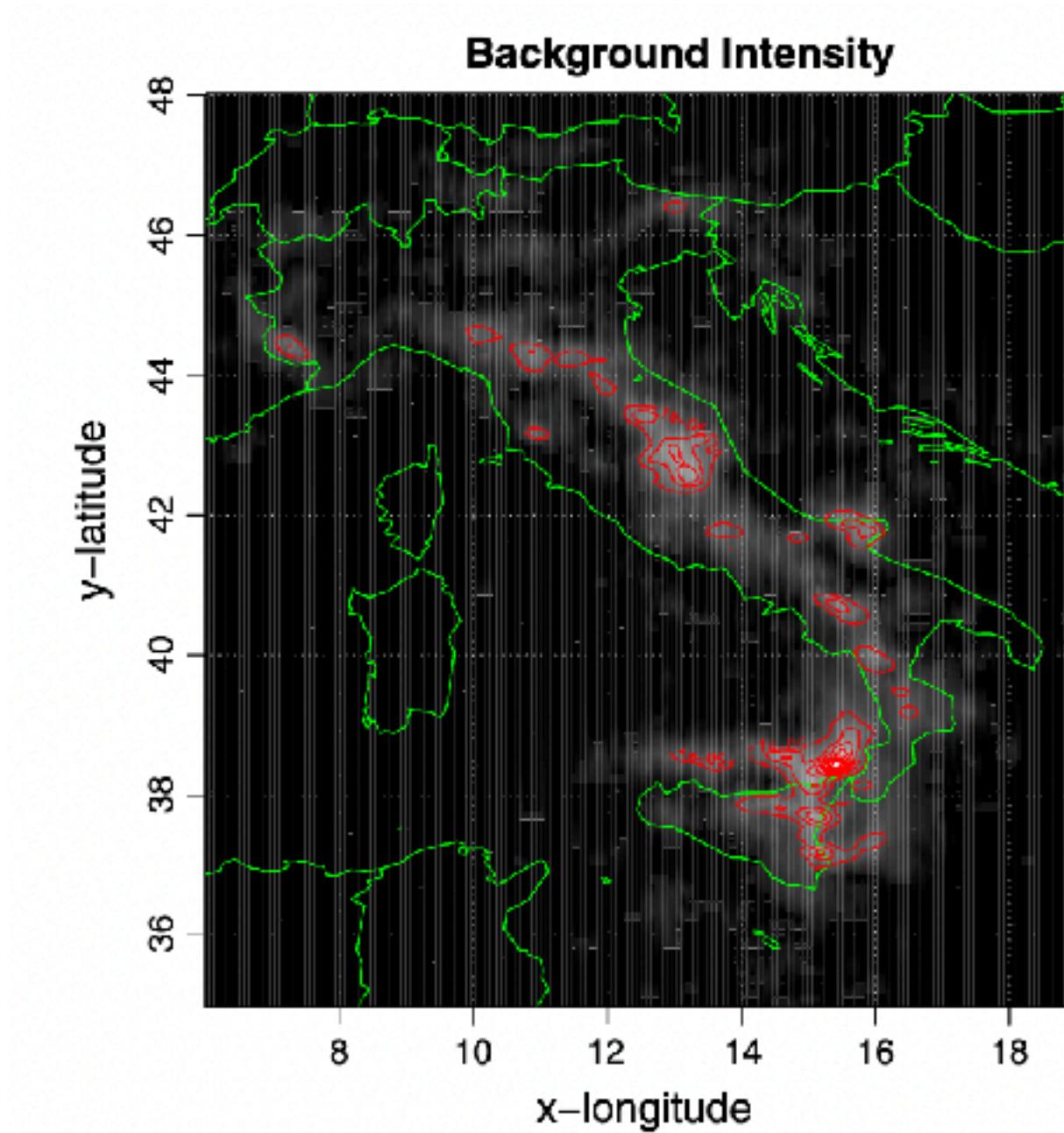
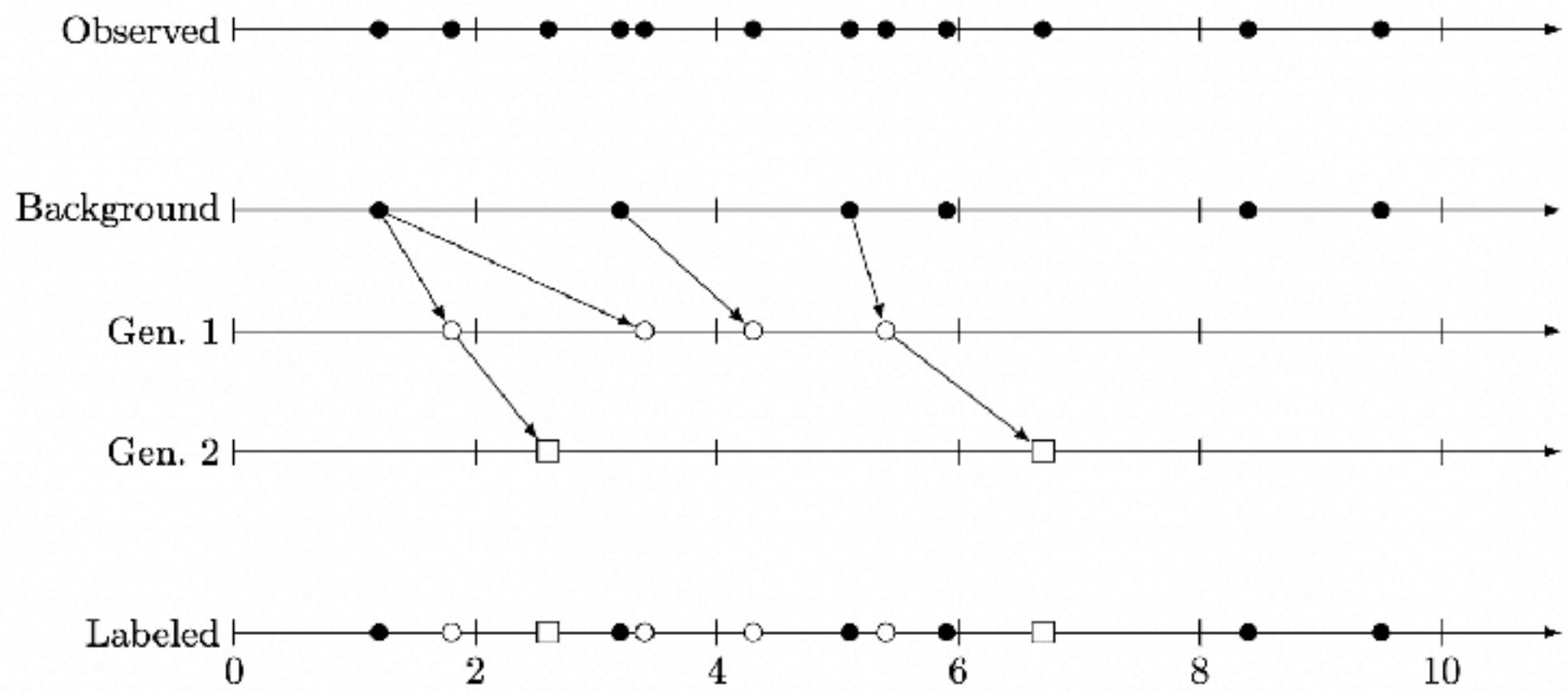
Fig. 2. Some examples of weekly forecasts (the number of the forecasts are reported on Table 1). (A) Forecast number 3, a few hours after the Amatrice earthquake and the M3.51 earthquake (blue-green circles) that occurred in the forecasting time window. (B) Forecast number 15, before the M5.9 earthquake (blue-green star) that occurred on October 26. (C) Forecast number 18, before the Norcia M6.5 earthquake (blue-green star) that occurred on October 30. (D) Forecast number 35, before the Campotosto M5.5 earthquake (blue-green star) that occurred on January 18.

Earthquakes

Earthquake forecasting during the complex Amatrice-Norcia seismic sequence (Marzocchi et al.)

Applications

De-clustering:

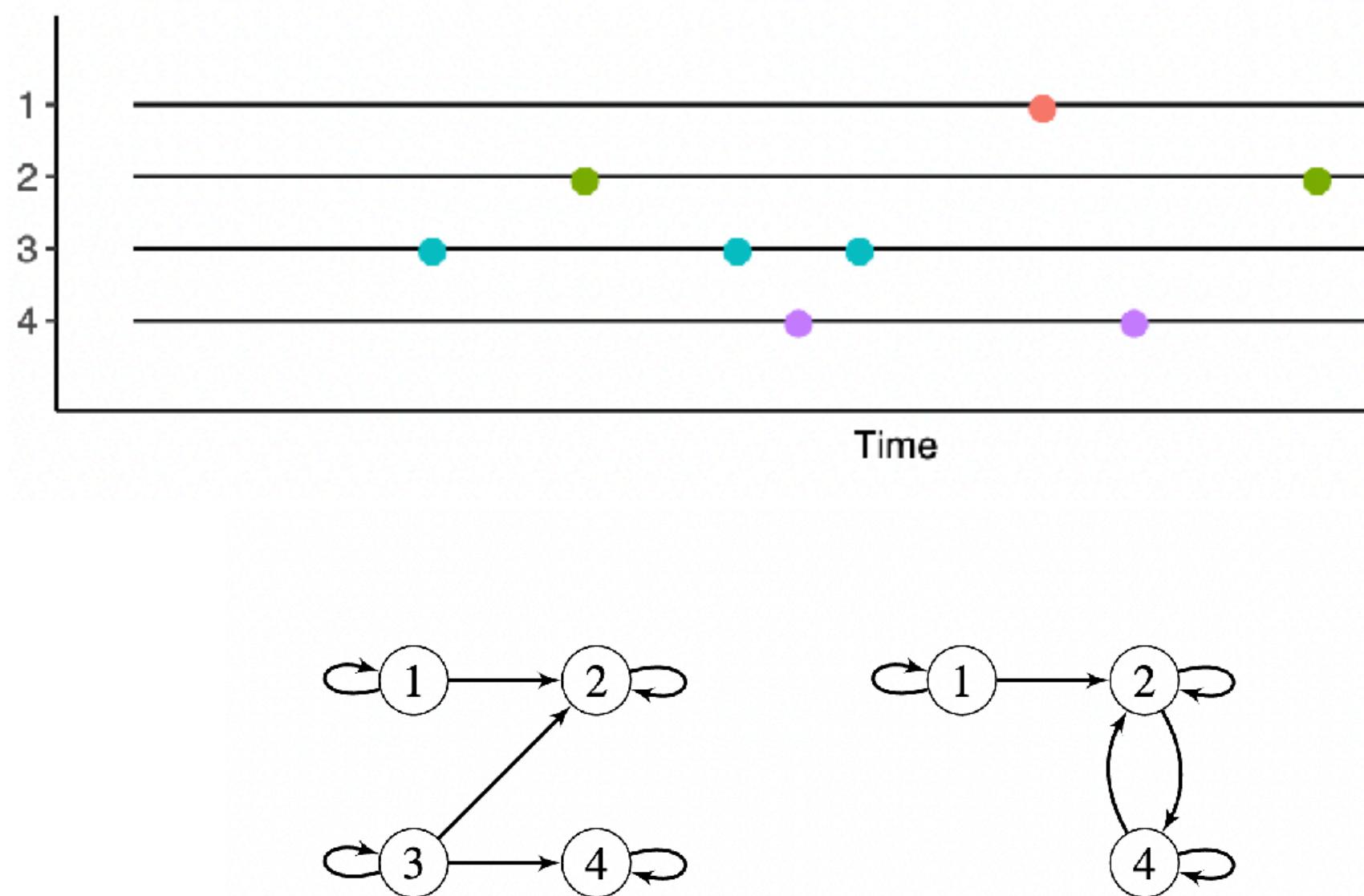


A Review of Self-Exciting Spatio-Temporal Point Processes and Their Applications (Reinhart)

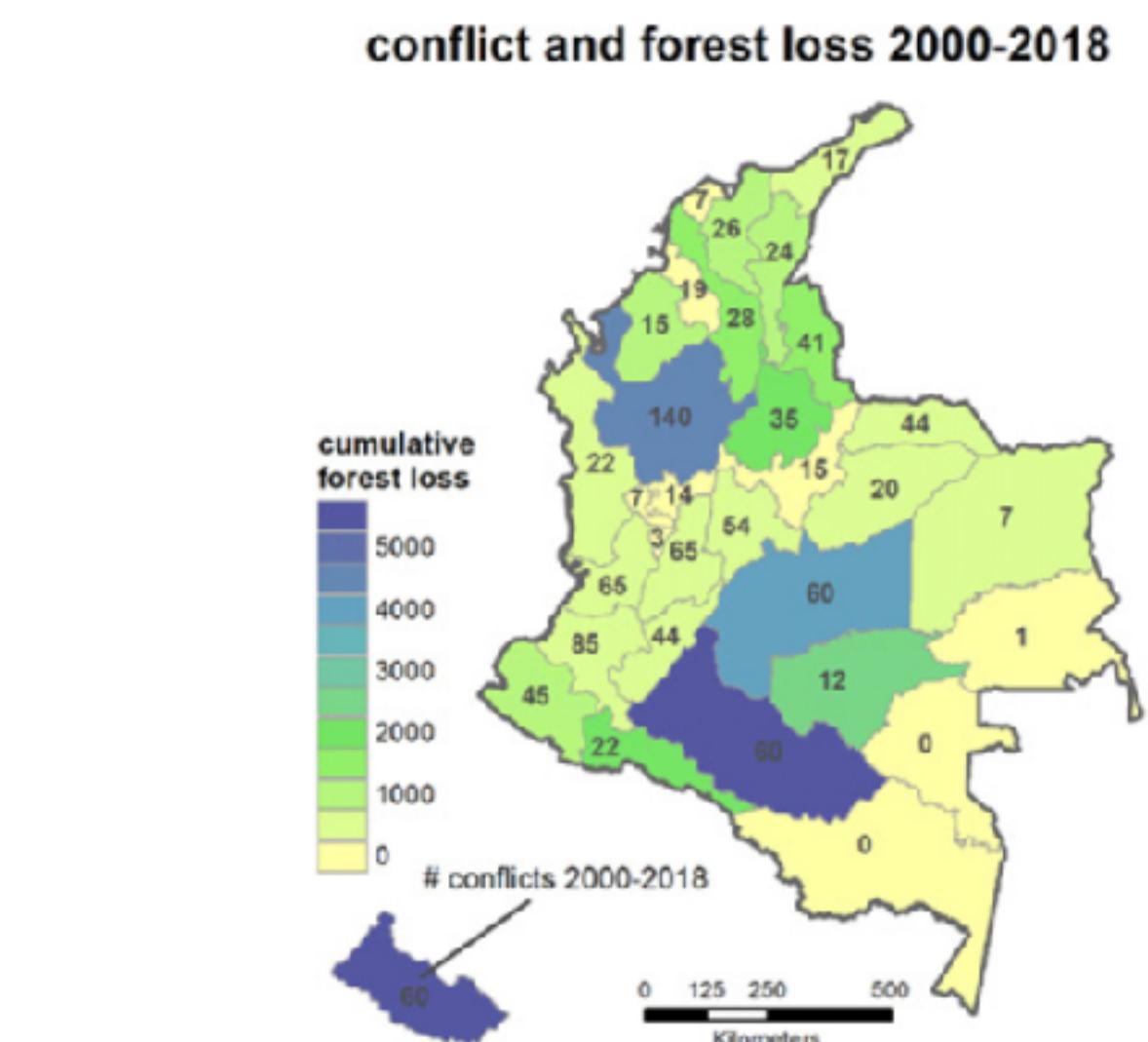
Alternated estimation in semi-parametric space-time branching-type point processes with application to seismic catalogs (Adelfio et al.)

Applications

Causal Discovery:



Causal screening in dynamical systems (Mogensen)



Toward Causal Inference for Spatio-Temporal Data:
Conflict and Forest Loss in Colombia (Christiansen et al.)

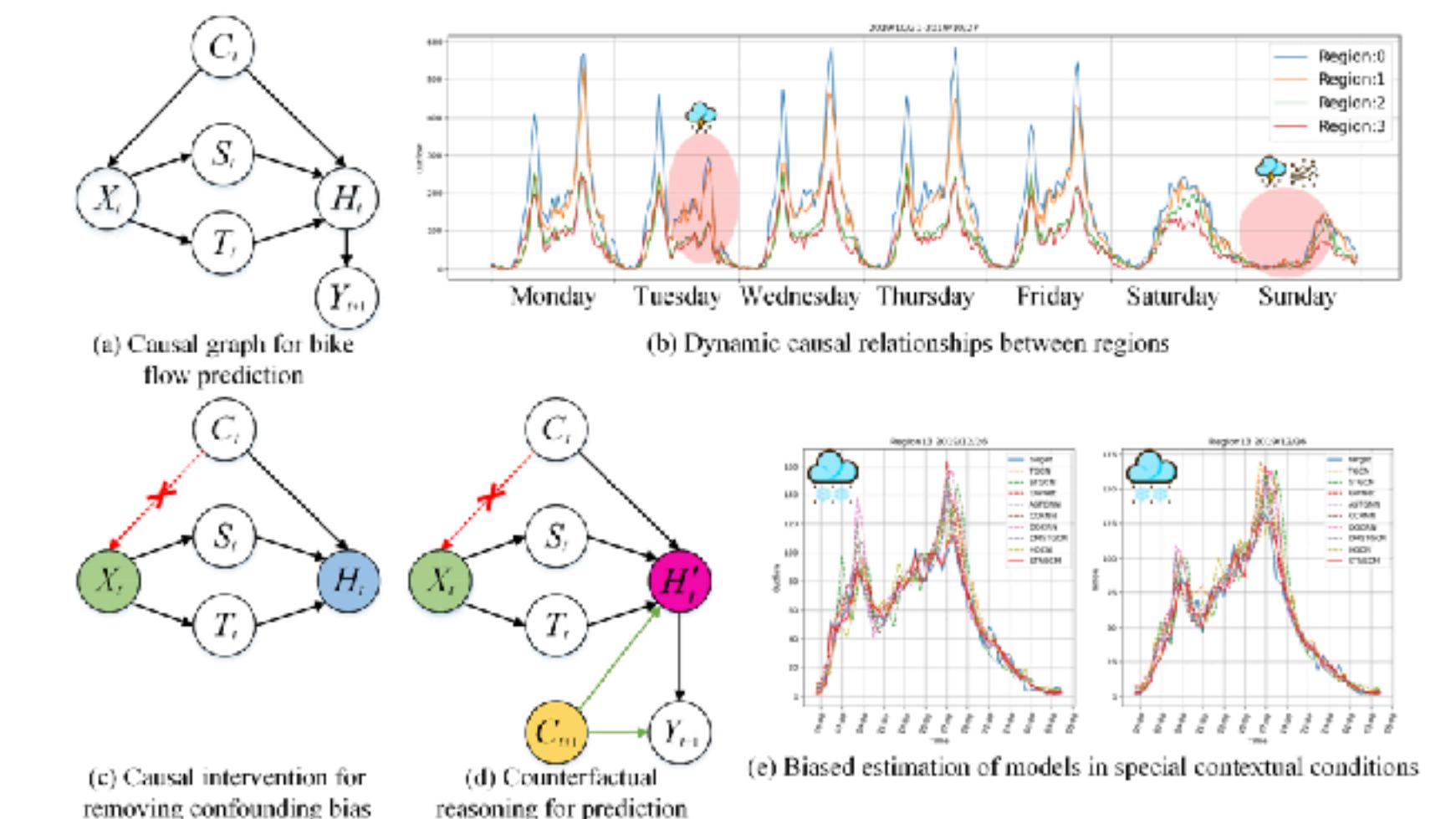
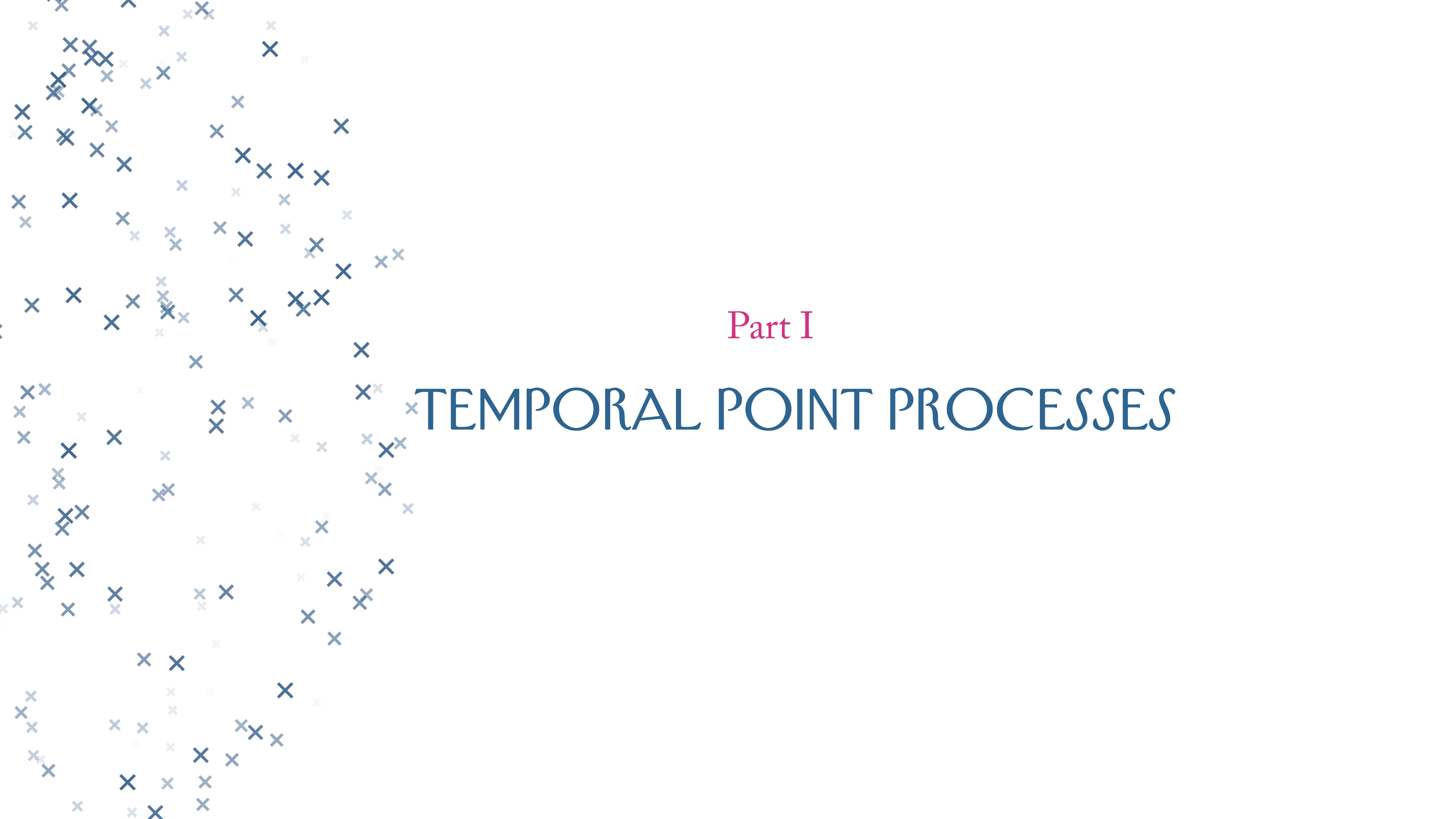


Figure 1: The change process of bike flow from the perspective of causality.

Spatio-temporal Neural Structural Causal Models
for Bike Flow Prediction (Deng et al.)

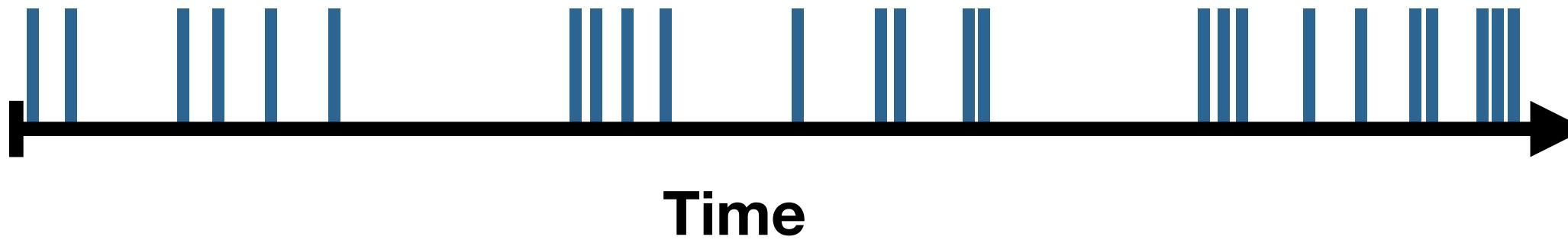


TEMPORAL POINT PROCESSES

Part I

Models of TPPs

Spike train of a single neuron



<https://mark-kramer.github.io/Case-Studies-Python/08.html>

Event sequence: $H = [t_0, t_1, \dots, t_n]$ ($t_i < t_{i+1}$ and $t_i \in \mathbb{R}_{\geq 0}$).

E.g., $H = [0.1, 0.5, 3.4, 4.2]$

t_i is called the event time, timestamp, or spike time

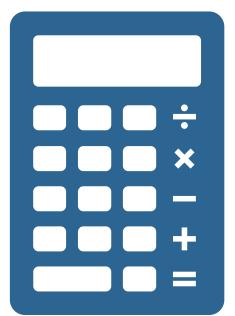
Inter-event time or waiting time: $t_{i+1} - t_i$

H_t is the sequence up to a time horizon t .

E.g., $H_4 = [0.1, 0.5, 3.4]$

Models of TPPs

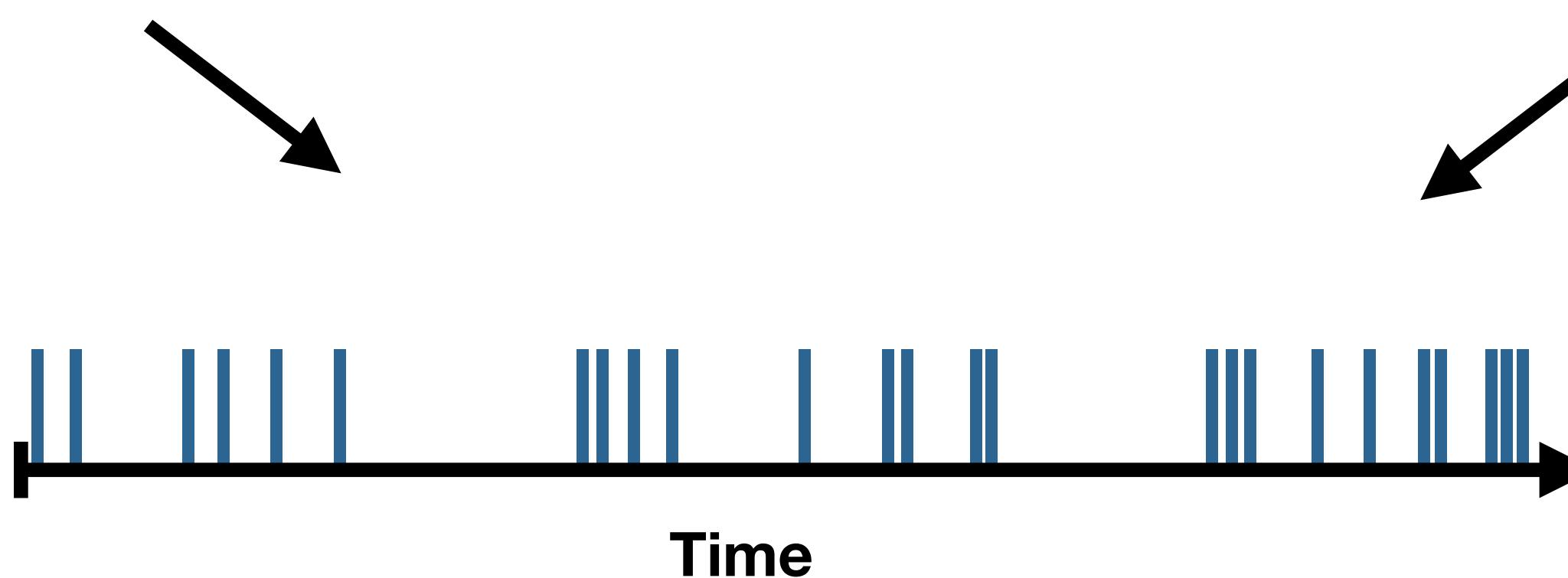
How do we specify TPPs?



Specify mathematical model (probability measure)

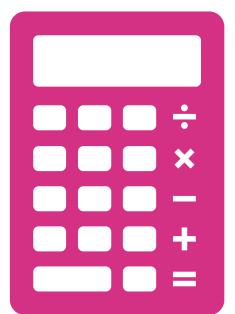


Build simulation algorithm

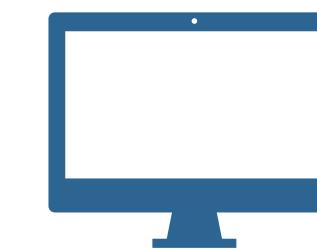


Models of TPPs

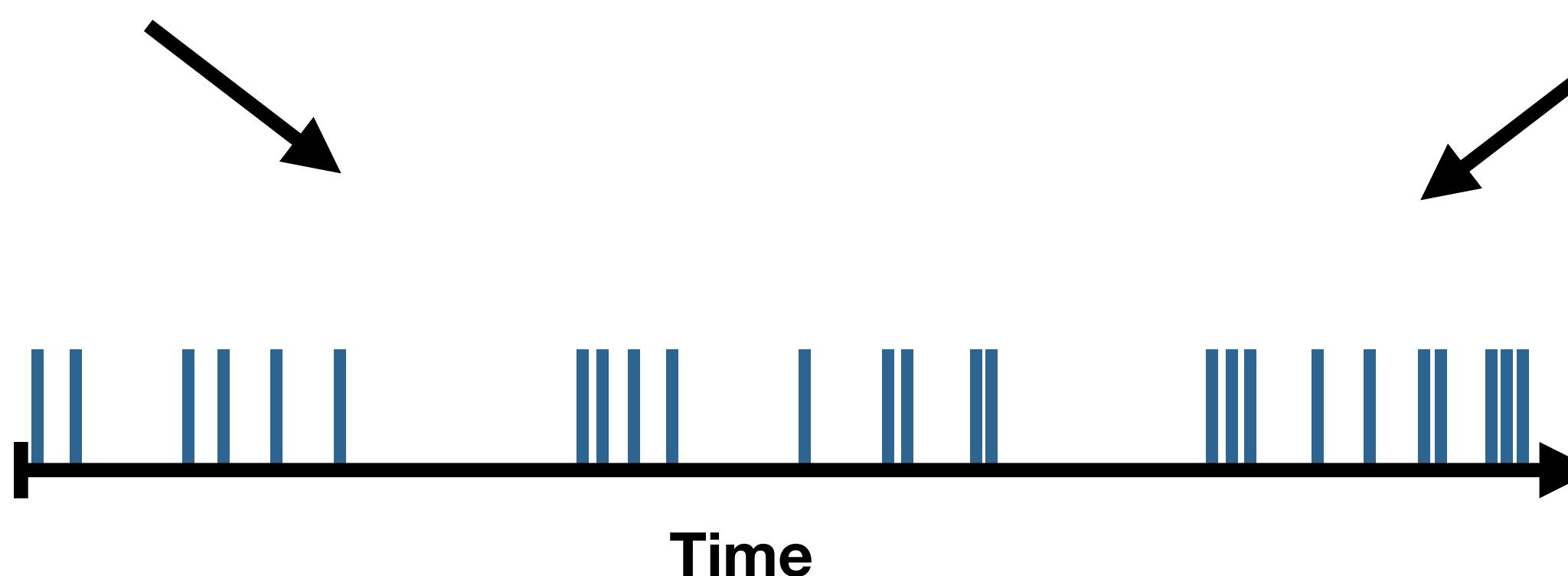
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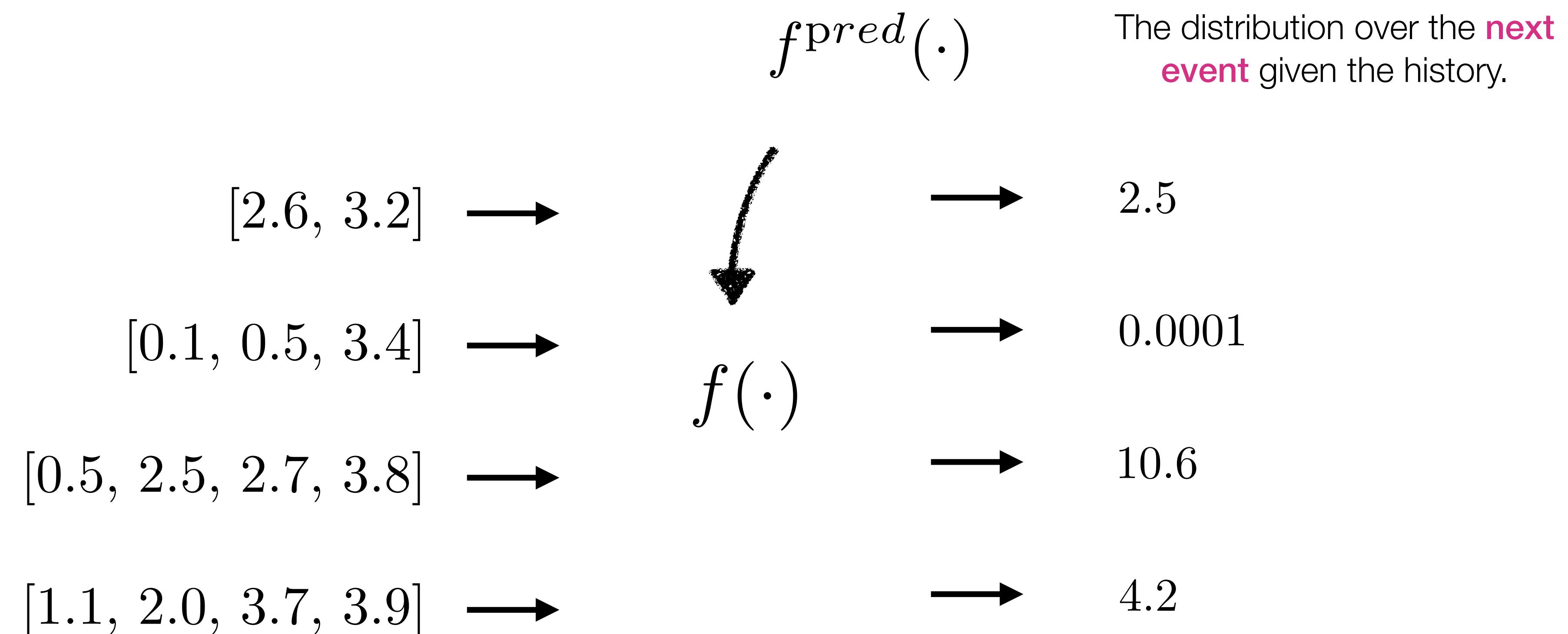
Models of TPPs

How do we specify TPPs?

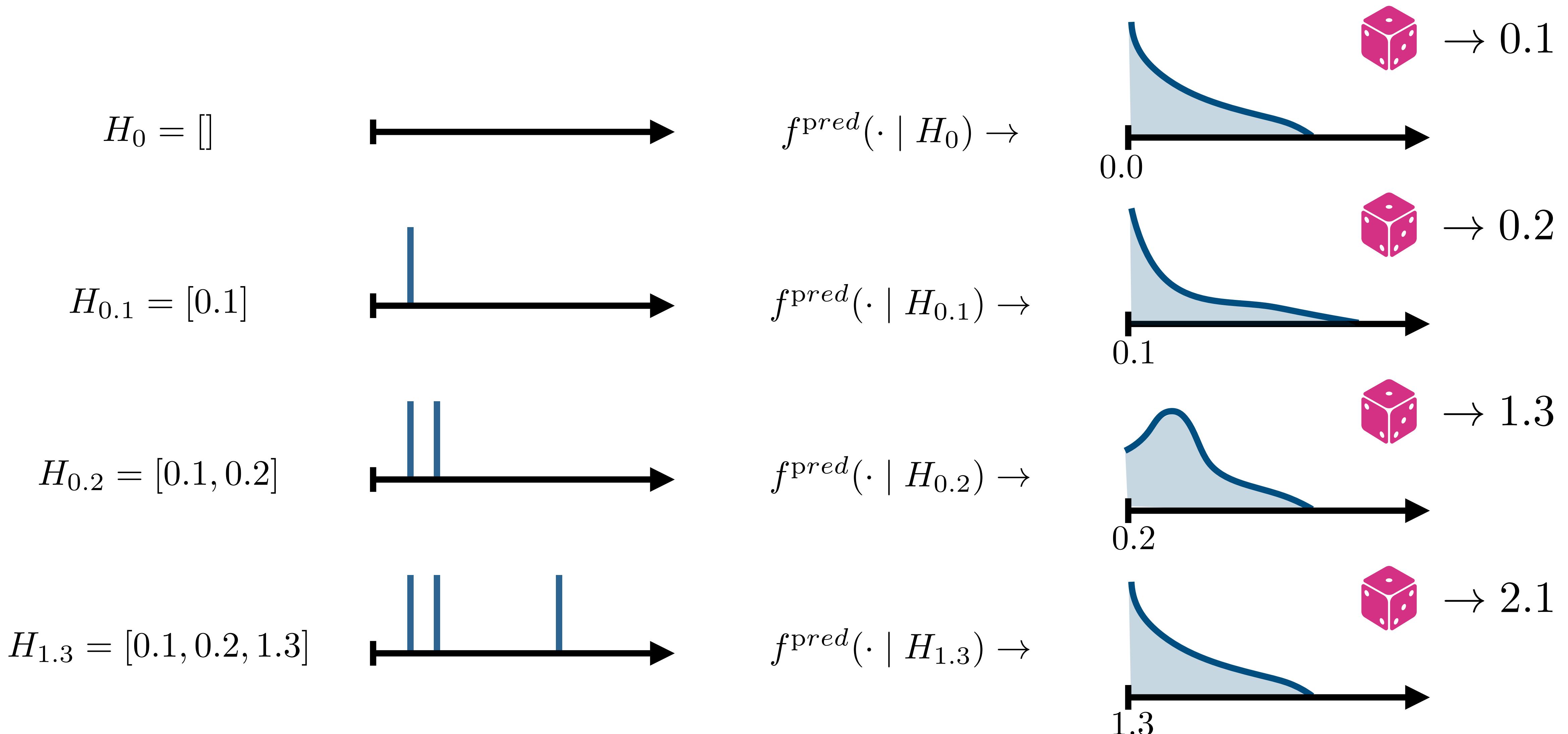
Use **statistical model** or **neural network** to specify.

Build likelihood function for all (finite) sequences of fixed horizon:

$f(\cdot)$ integrates to 1 over the space of all sequences.



Models of TPPs



Models of TPPs

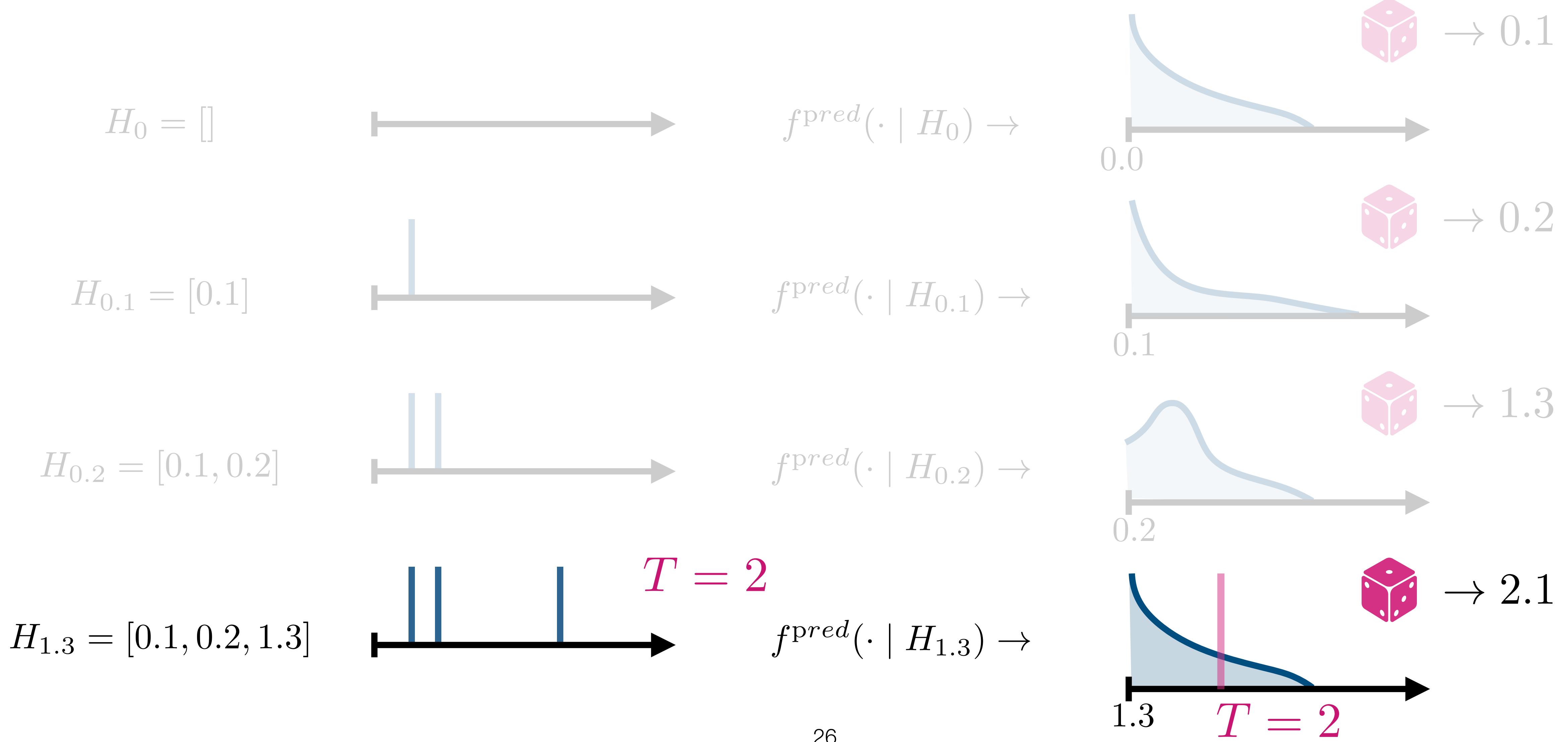
Likelihood of event sequence H .

represents the probability that no events occur from the last observed event time t_n up to the time horizon T .

$$f(H) = \left(\prod_{i=1}^n f^{\text{pred}}(t_{i+1} \mid H_{t_i}) \right) (1 - F^{\text{pred}}(T \mid H_{t_n})) .$$

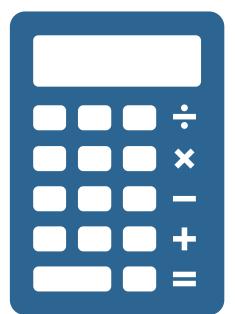
represents the probability density of the next event occurring at time t_{i+1} given the history of all previous events up to time t_i .

Models of TPPs



Models of TPPs

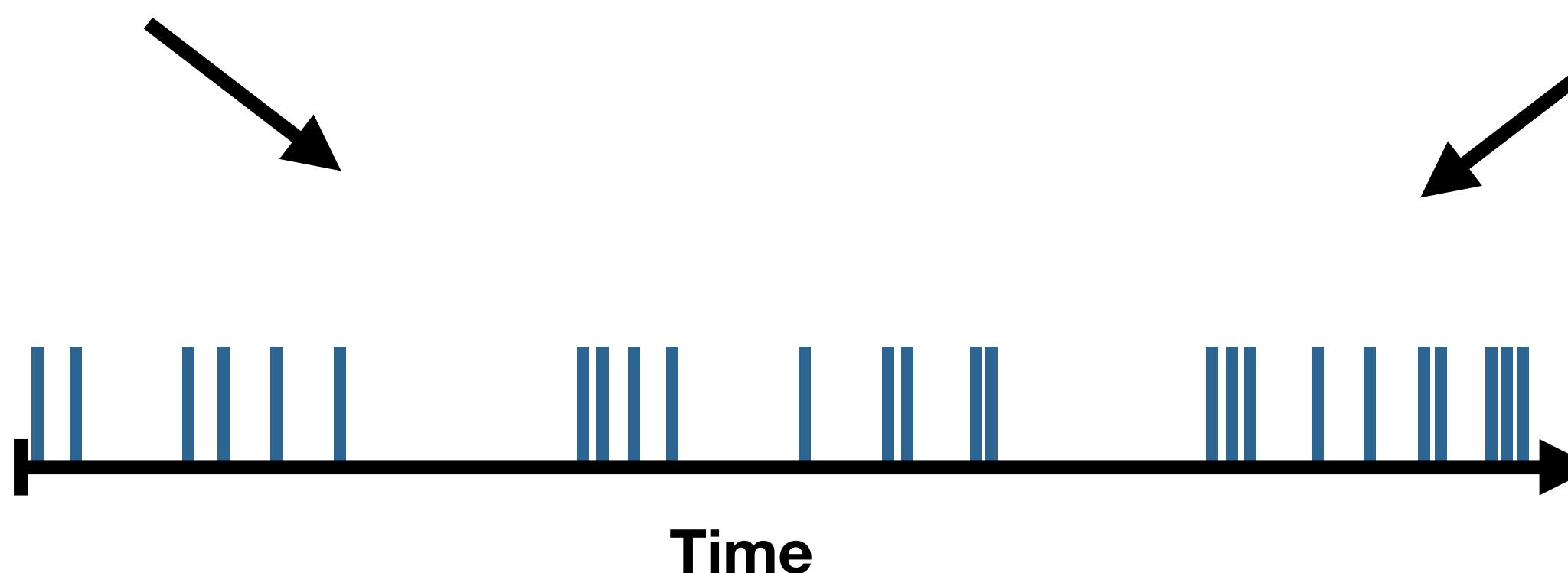
How do we specify TPPs?



Specify mathematical model (probability measure)



Build simulation algorithm

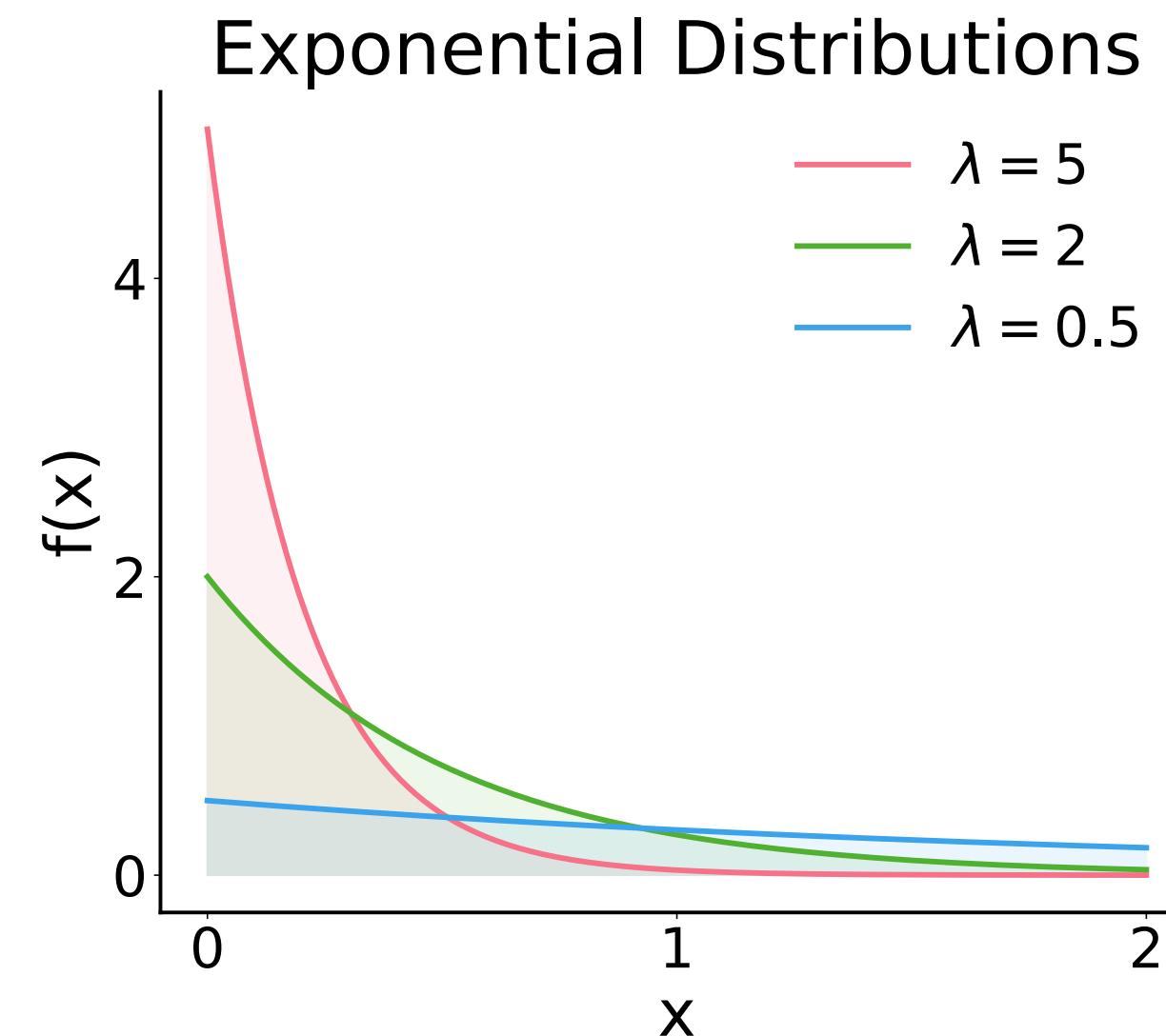


Simulating TPPs

Homogenous TPP: Inter-event times follow **exponential** distribution

Heterogeneous TPP: Inter-event times follow **arbitrary** distribution

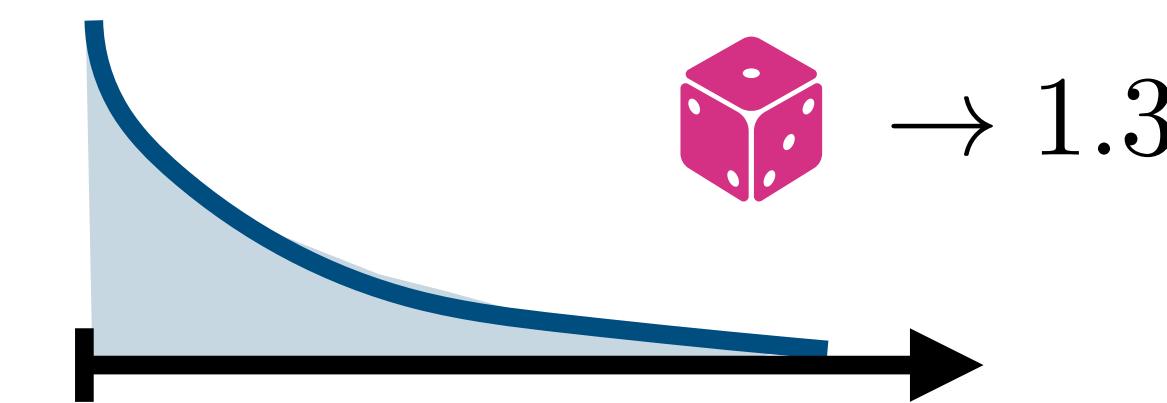
$$f(x; \lambda) = \begin{cases} \lambda e^{-\lambda x} & x \geq 0, \\ 0 & x < 0 \end{cases}$$



$$H_{0.2} = [0.1, 0.2]$$

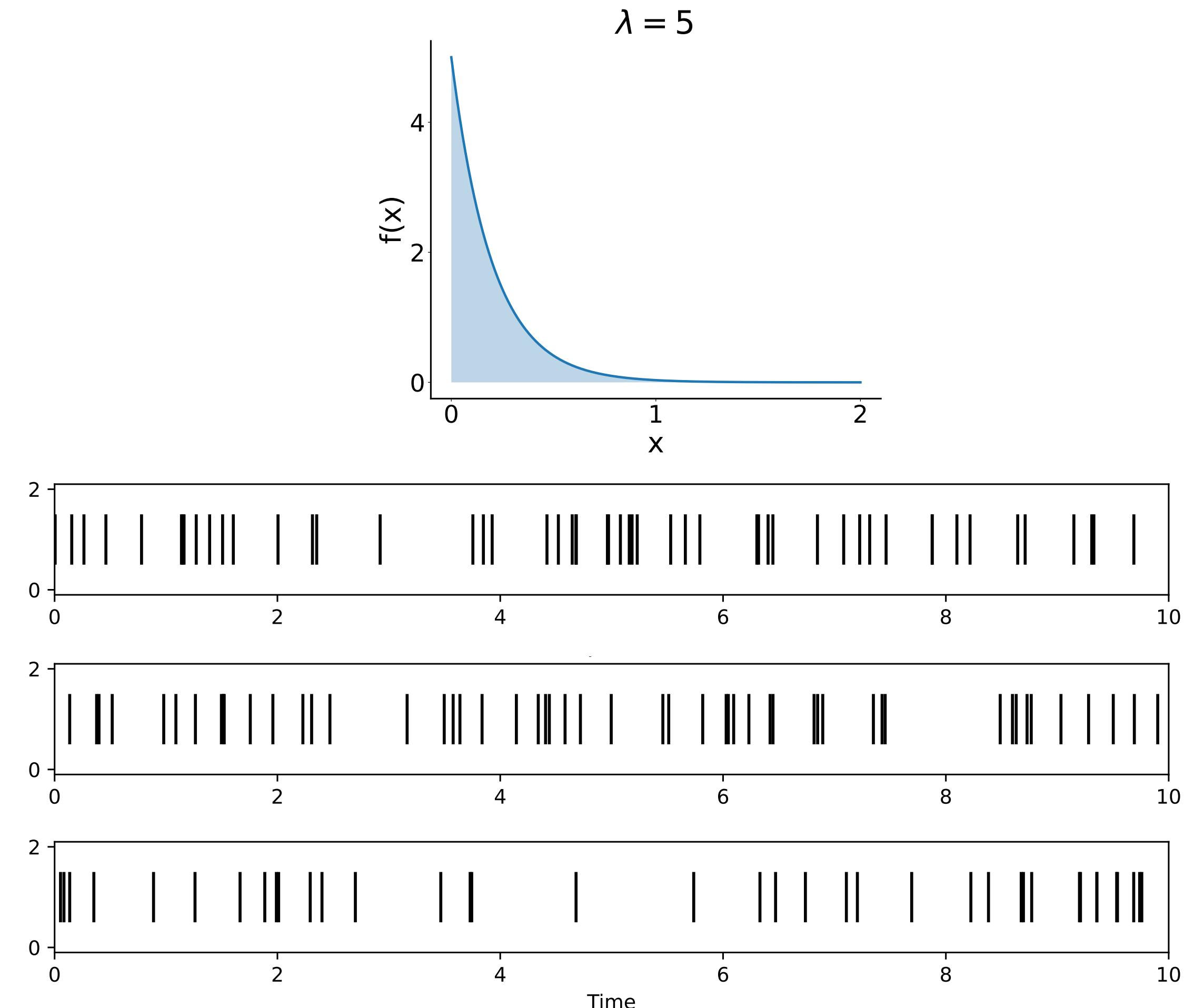
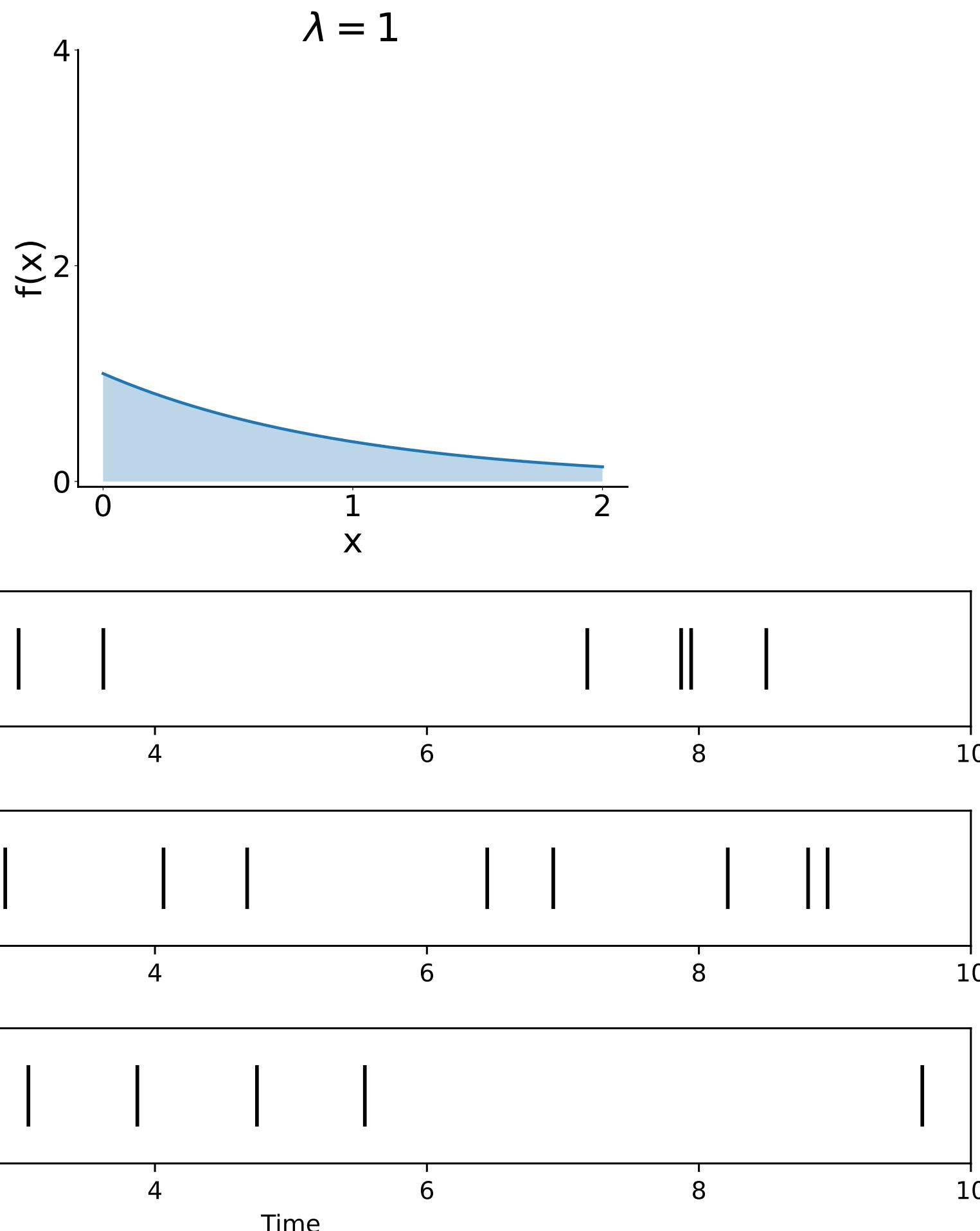


$$f^{\text{pred}}(\cdot \mid H_{0.2}) \rightarrow$$



Simulating TPPs

Homogenous TPP: Inter-event times follow **exponential** distribution



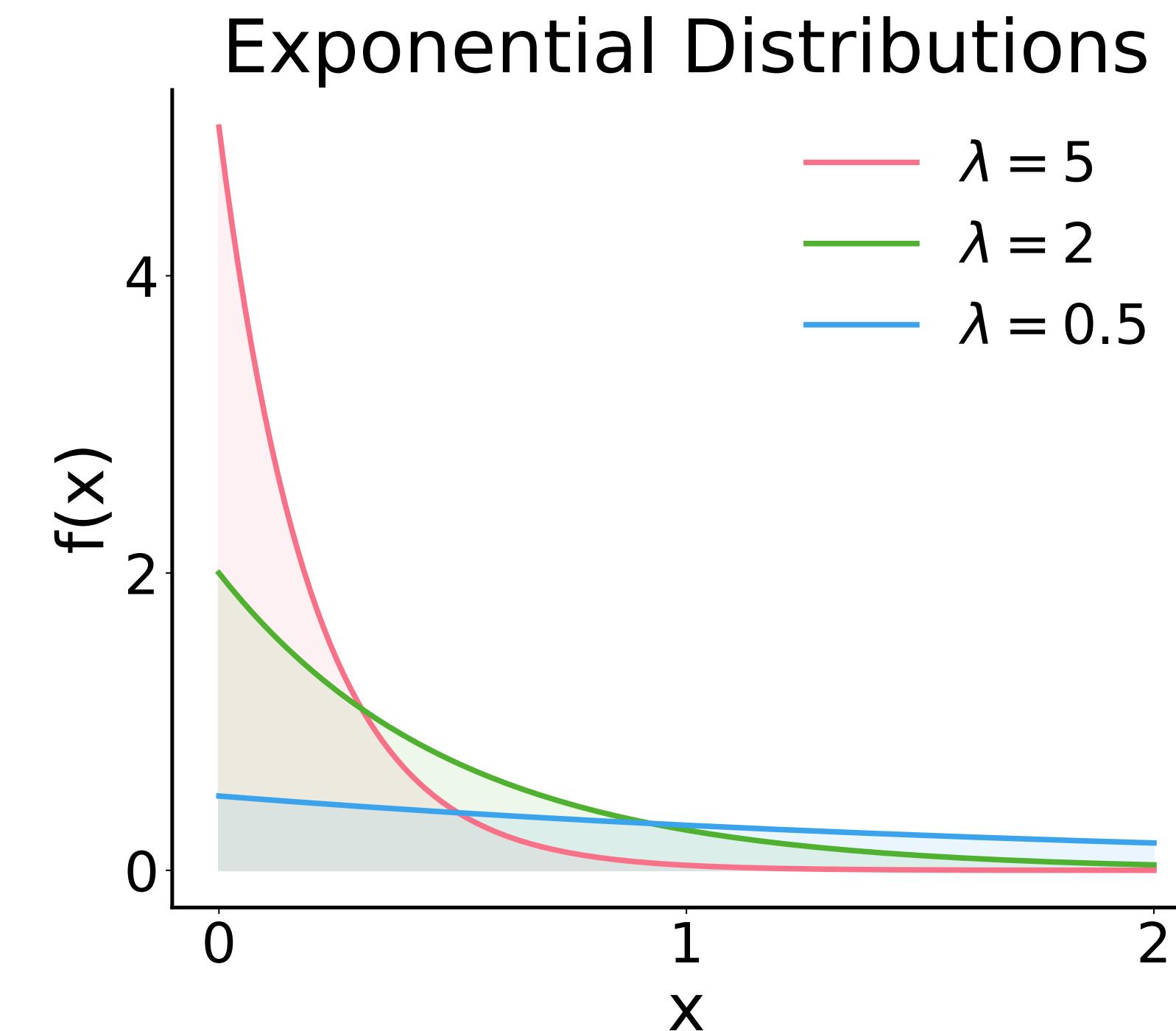
The Exponential Distribution

How do we sample from the exponential distribution?

- Use our favorite library
 - e.g., `numpy.random.exponential`
- Inverse transform
 - Sample U uniformly at random in $[0, 1]$

$$X = -\frac{1}{\lambda} \ln(U)$$

- Step-wise
 - Based on time discretization



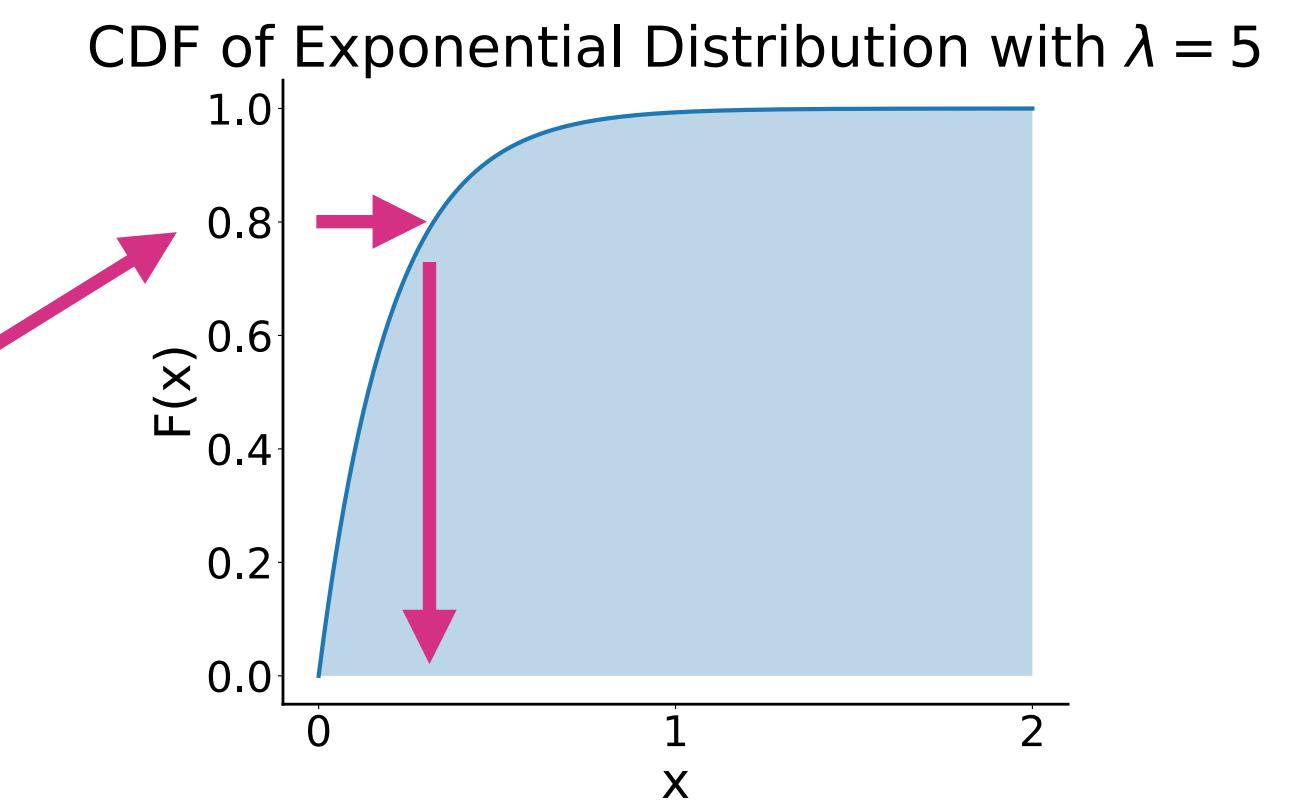
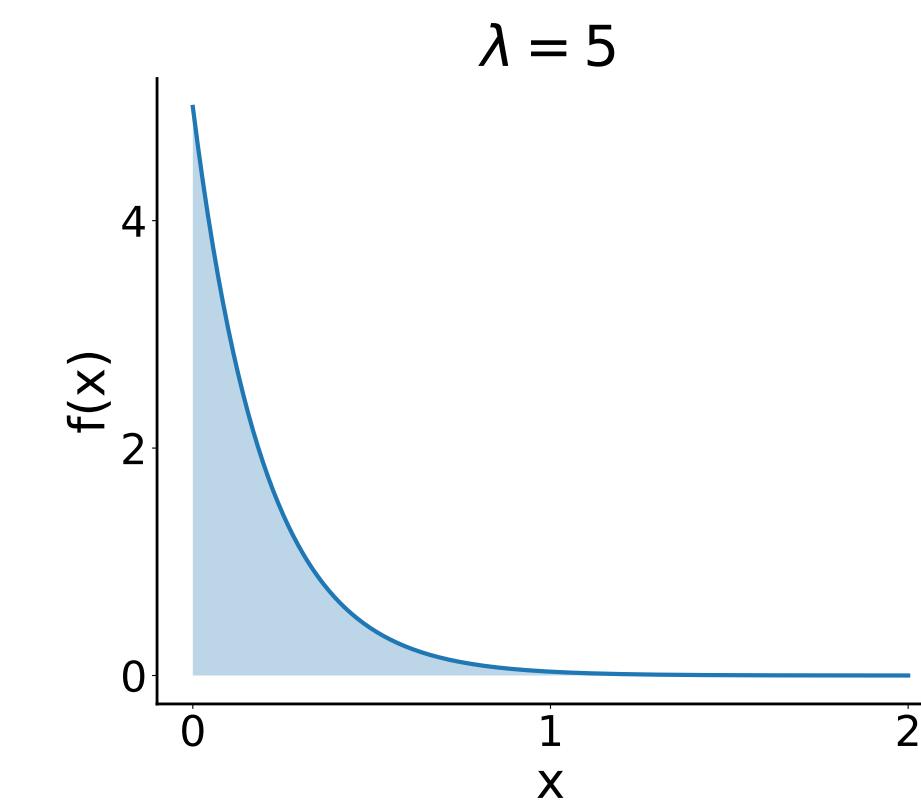
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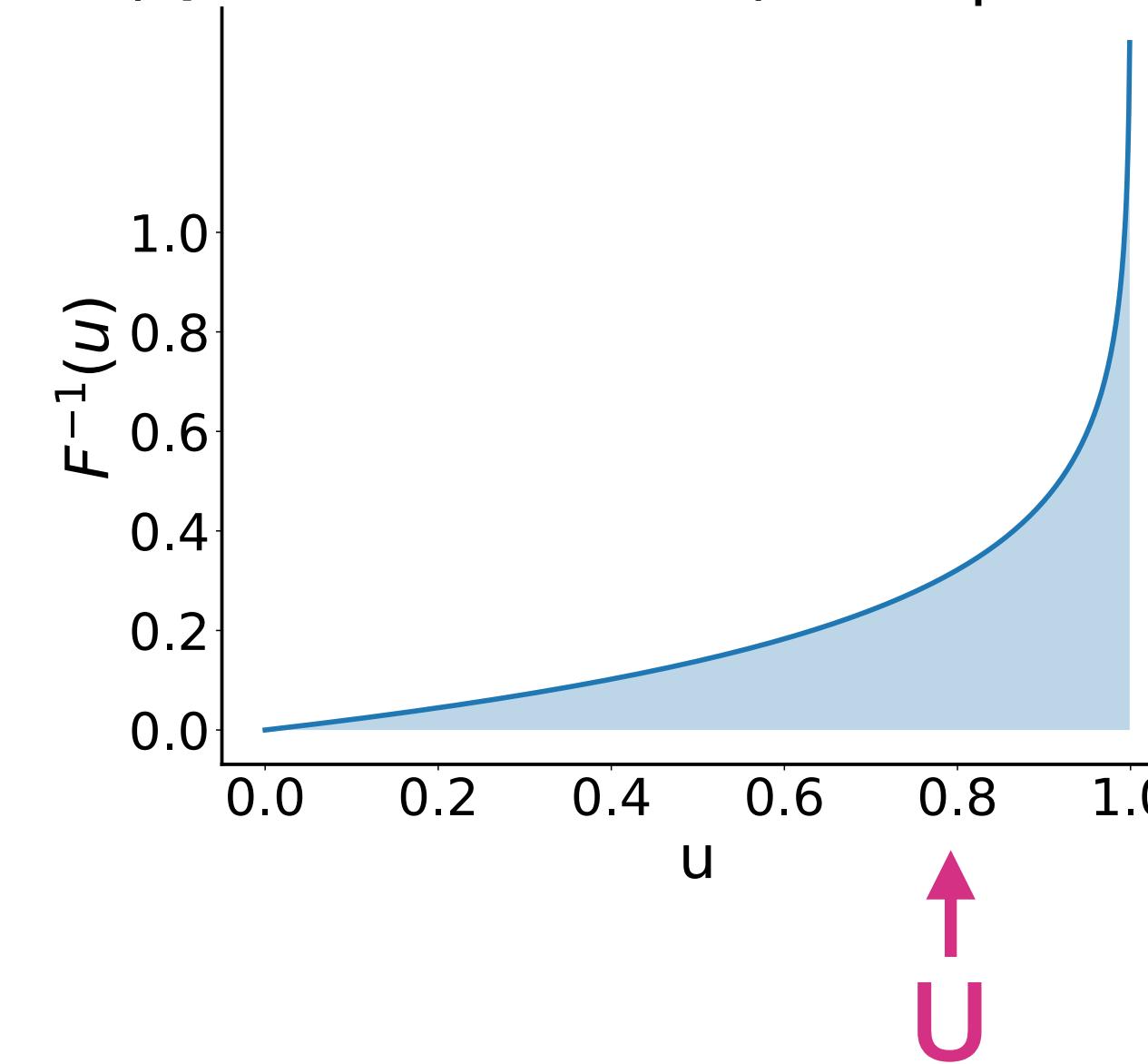
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Inverse CDF (Quantile Function) of Exponential Distribution



The Exponential Distribution

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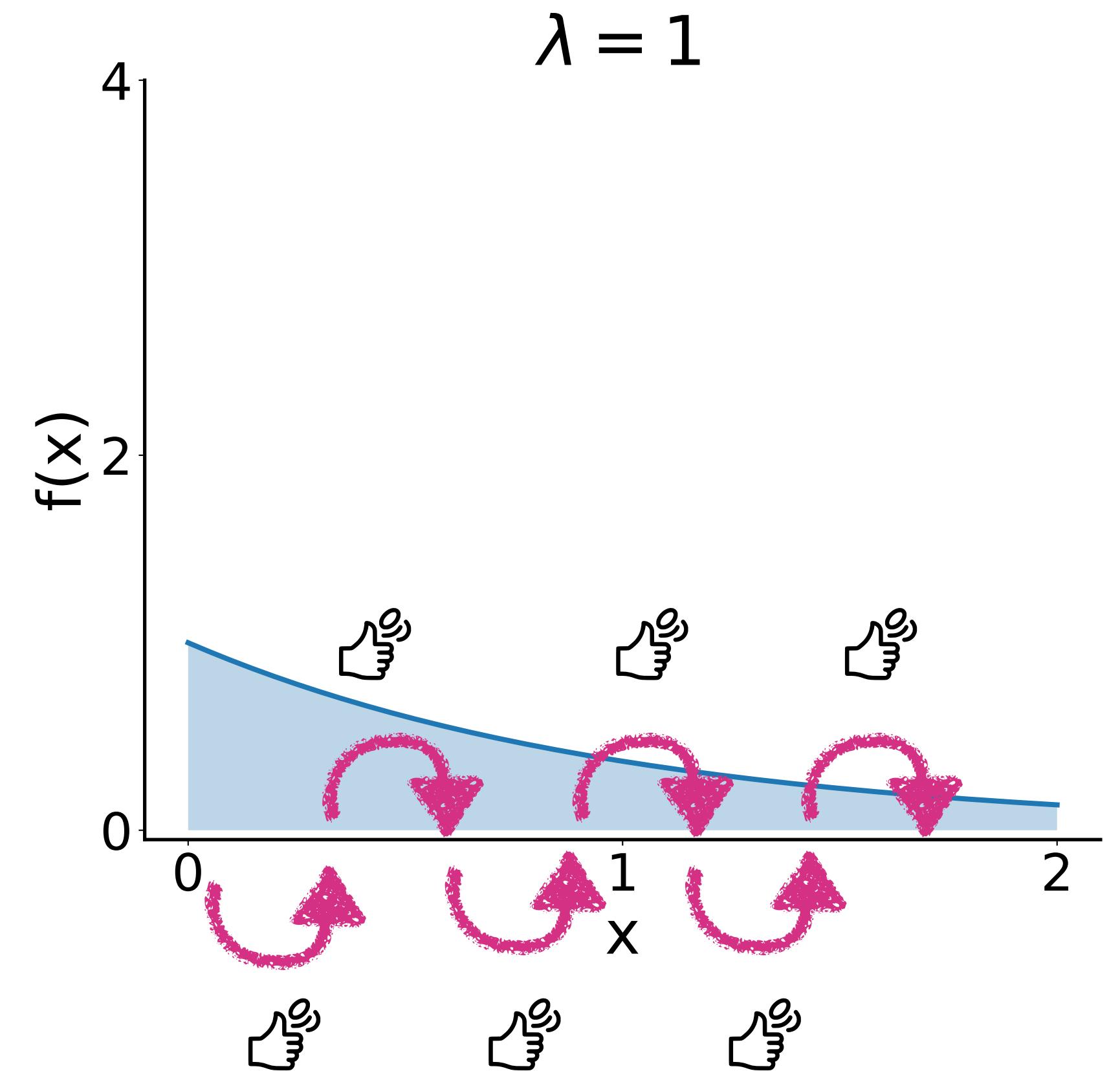
- Step-wise
 - Based on time discretization

The Exponential Distribution

1. **Discretize Time:** Choose a small time step Δt .
2. **Biased Coin:** At each time step, use a biased coin to decide whether an event occurs. The probability p of the event occurring in each time step is $p = \lambda\Delta t$.
3. **Simulate Until Event:** Continue this process until the event occurs.

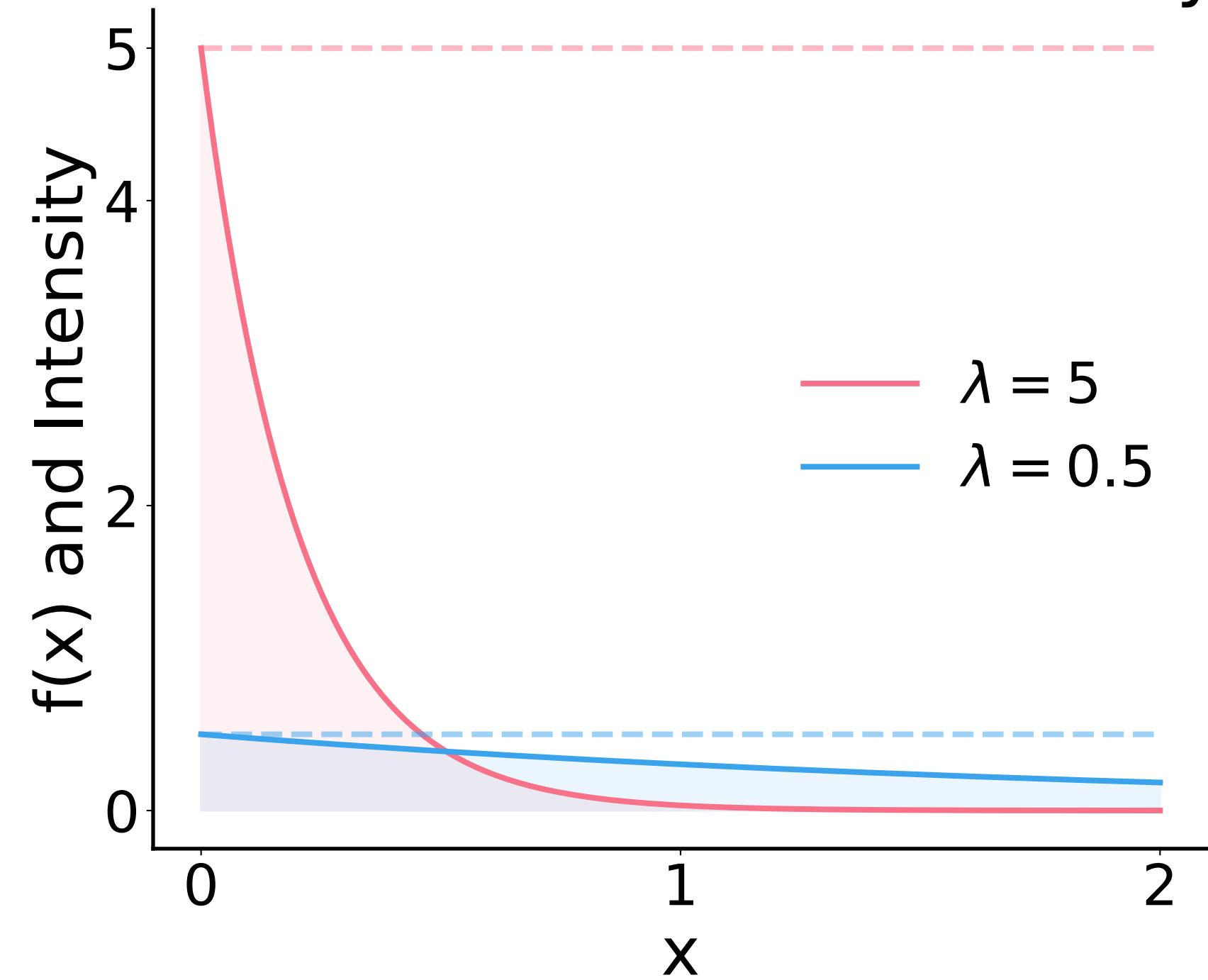
The **intensity** (in a homogeneous TPP) is constant

$$\lambda = \lambda(t) = \lim_{\Delta t \rightarrow 0} \frac{P(\text{Event in } [t, t + \Delta t))}{\Delta t}$$

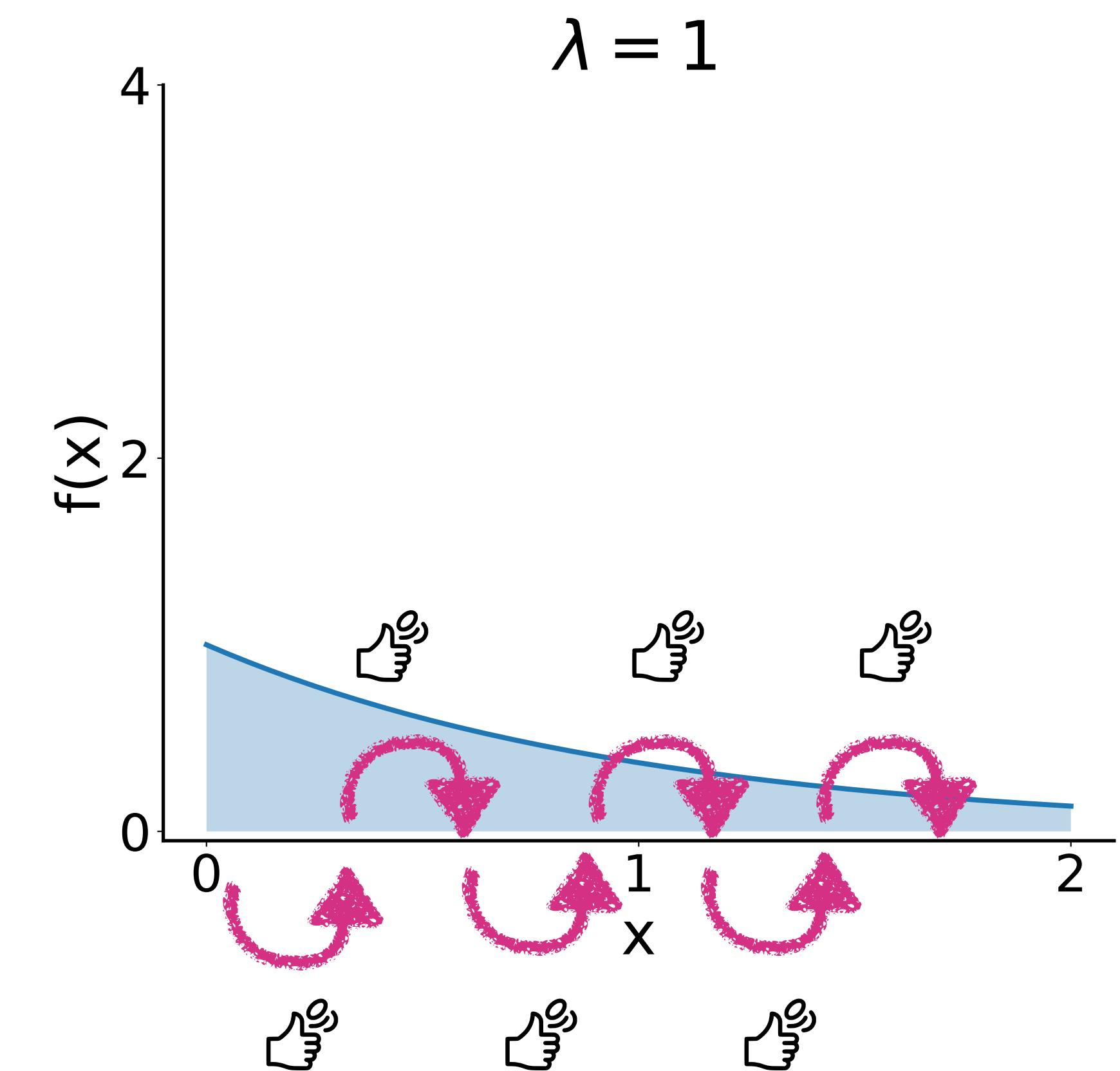


The Exponential Distribution

Exponential Distributions and Intensity Functions



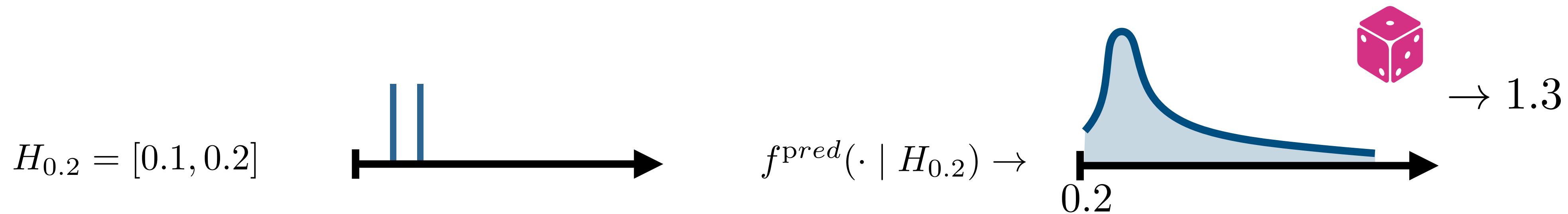
$$\lambda = \lambda(t) = \lim_{\Delta t \rightarrow 0} \frac{P(\text{Event in } [t, t + \Delta t))}{\Delta t}$$



Simulating TPPs

Homogenous TPP: Inter-event times follow **exponential** distribution

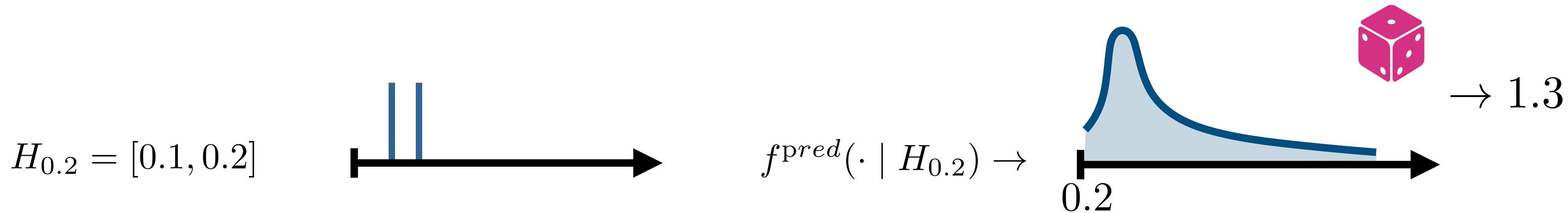
Heterogeneous TPP: Inter-event times follow **arbitrary** distribution



Goal:

- Allow arbitrary inter-event time distributions.
- Enable the event time distribution to depend on the history, allowing us to model effects such as periodicity, self-excitation, and self-correction.

Simulating TPPs



Goal:

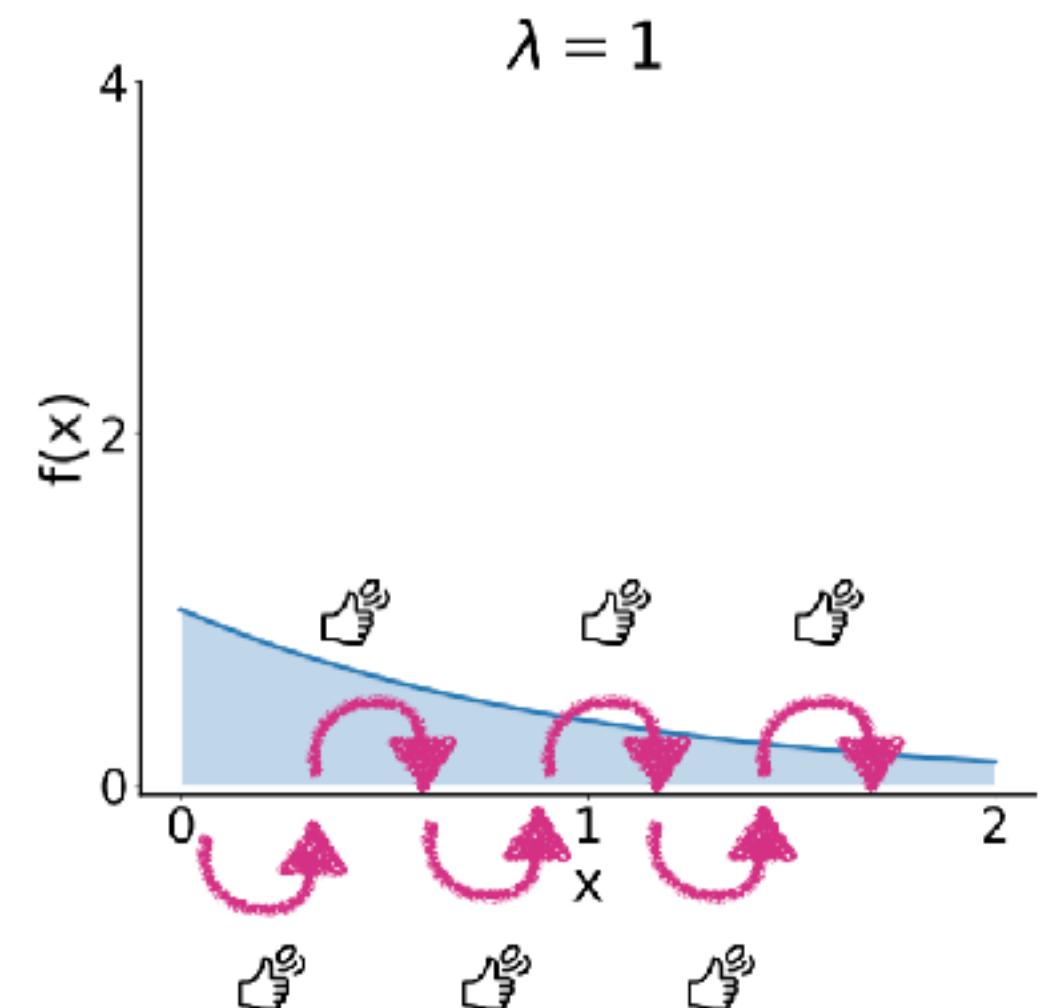
- Allow arbitrary inter-event time distributions.
- Enable the event time distribution to depend on the history, allowing us to model effects such as periodicity, self-excitation, and self-correction.

Construct a time and history dependent intensity function:

$$\lambda : \mathbb{R}_{\geq 0} \times \mathcal{H} \rightarrow \mathbb{R}_{\geq 0}$$

Method:

$$\lambda(t, H_t) = \lim_{\Delta t \rightarrow 0} \frac{P(\text{Event in } [t, t + \Delta t) \mid H_t)}{\Delta t}$$

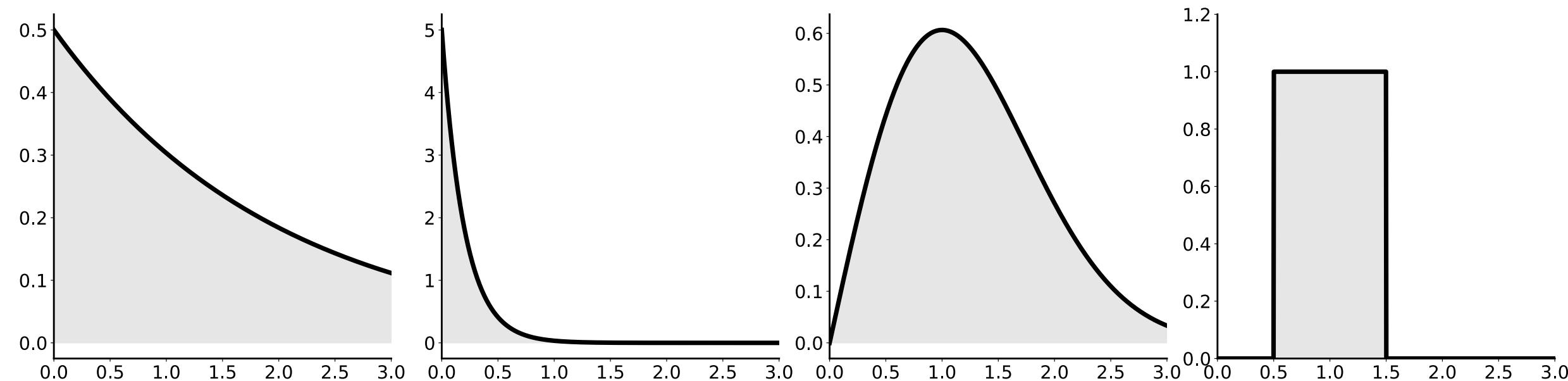


The probability of an event happening changes over time

Intensity Function

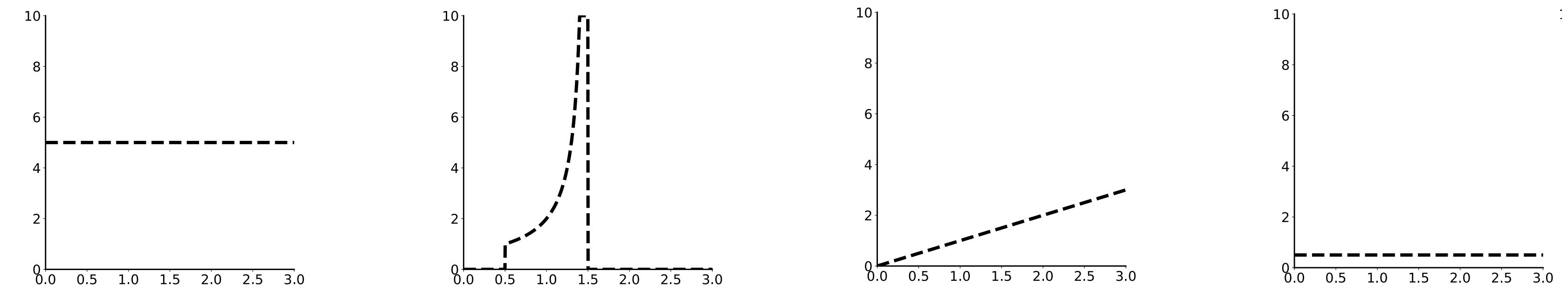
Predictive distribution:

PDF over inter-event times.



Intensity Function:

Instantaneous rate of event happening at t .



Questions:

Which PDF corresponds to each intensity function

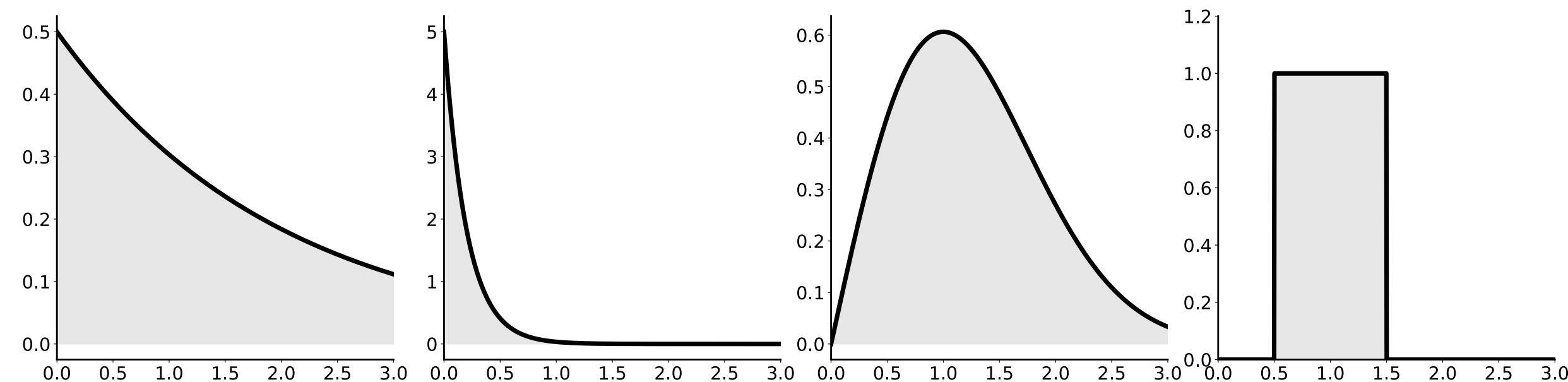
What are the properties of intensity functions?

How can we convert them into each other?

Intensity Function

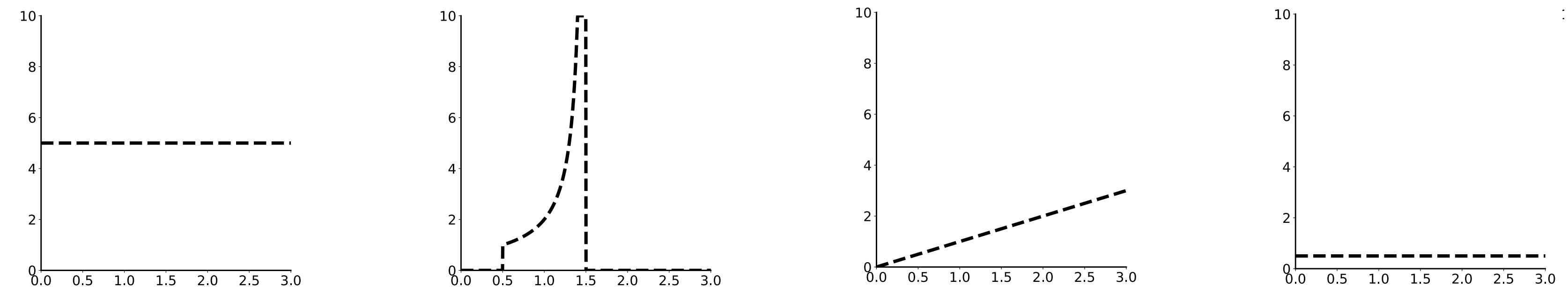
Predictive distribution:

PDF over inter-event times.



Intensity Function:

Instantaneous rate of event happening at t .



Questions:

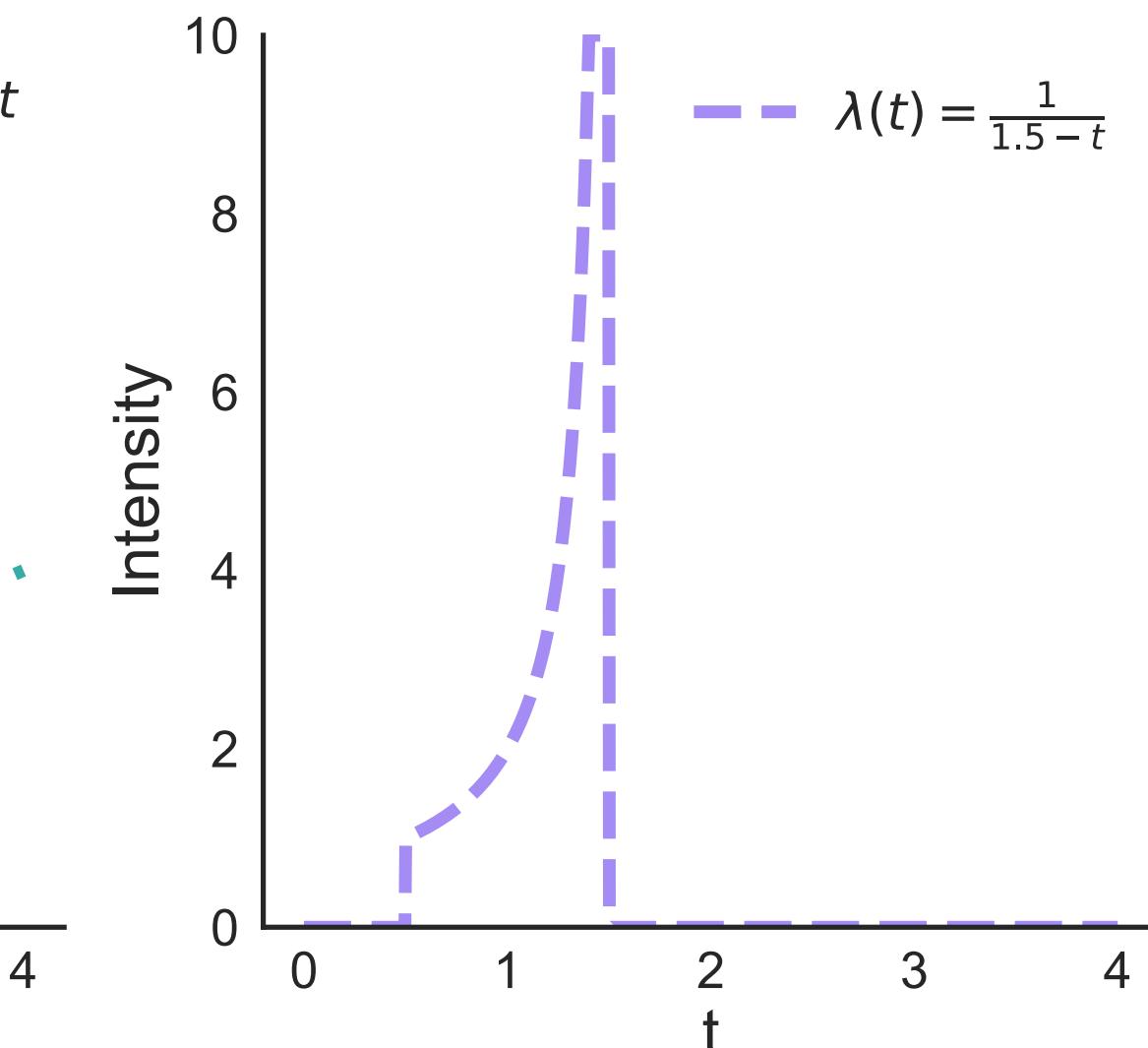
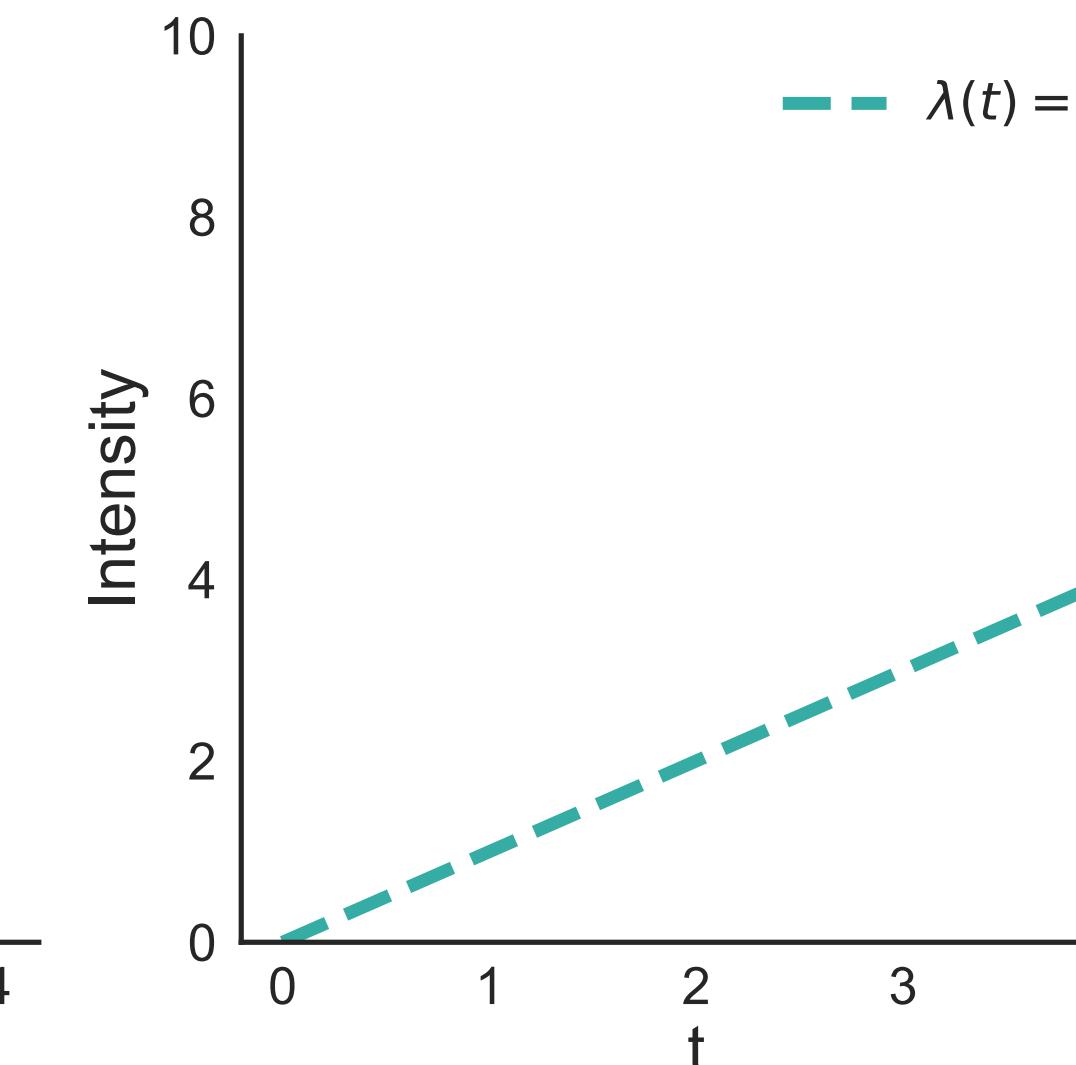
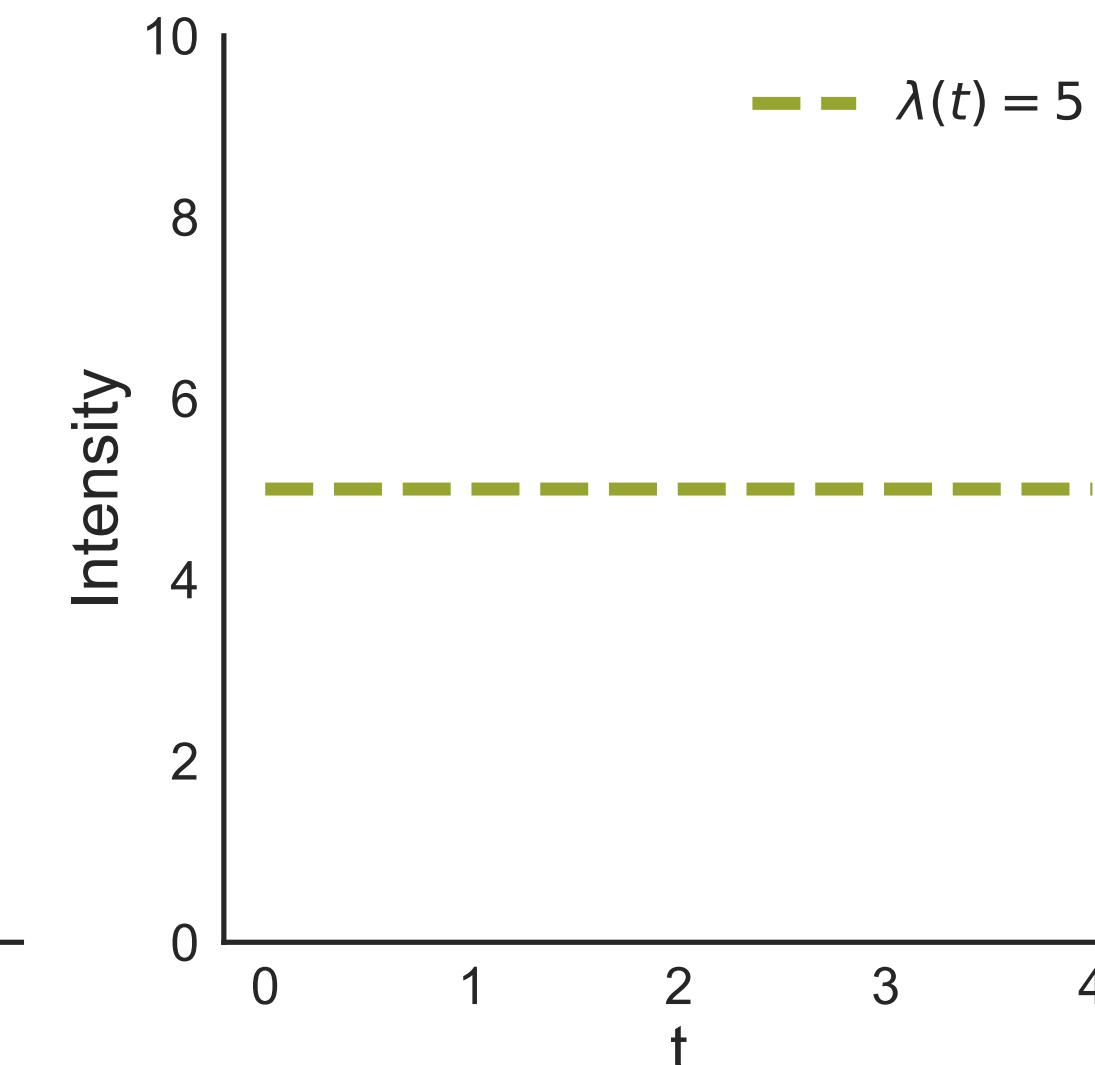
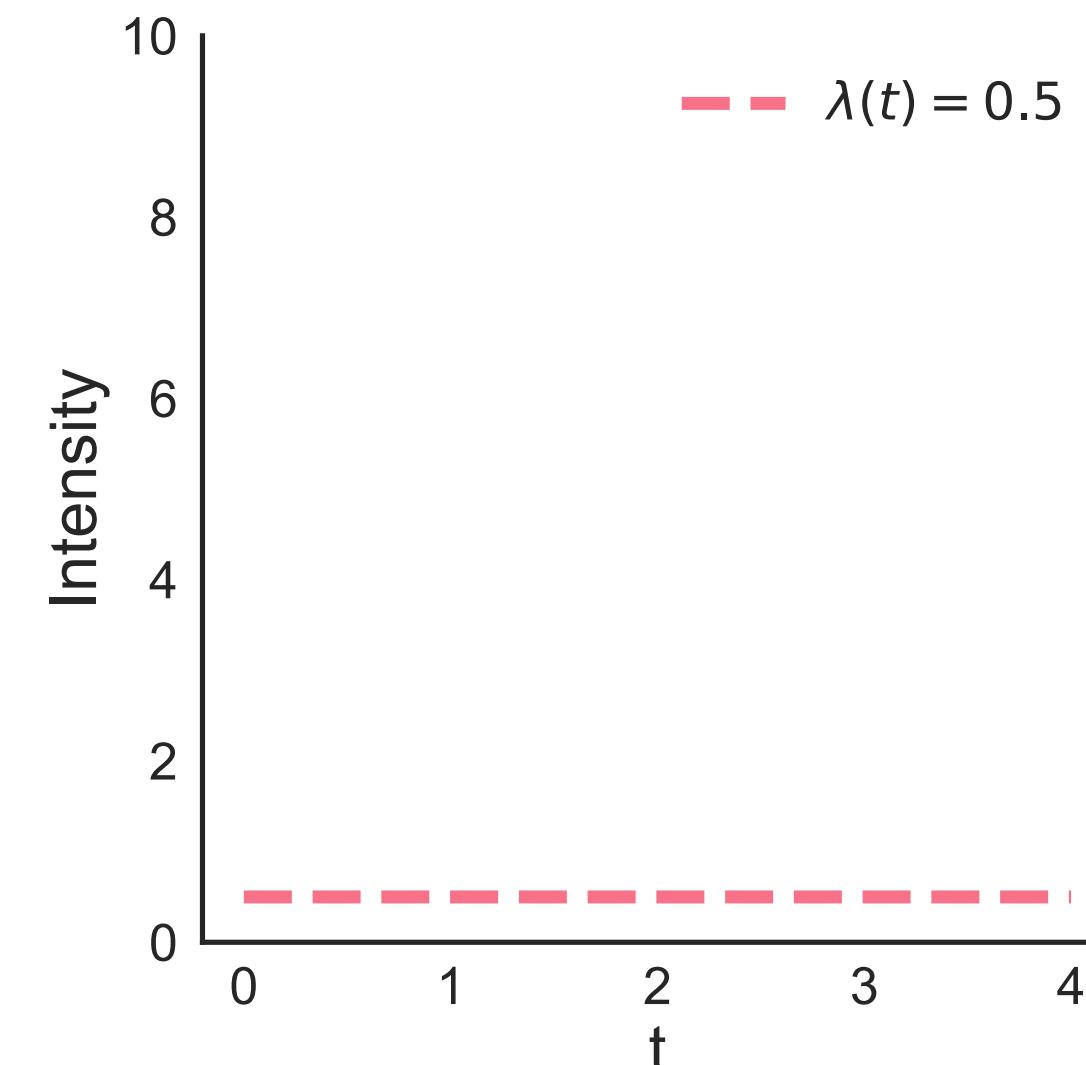
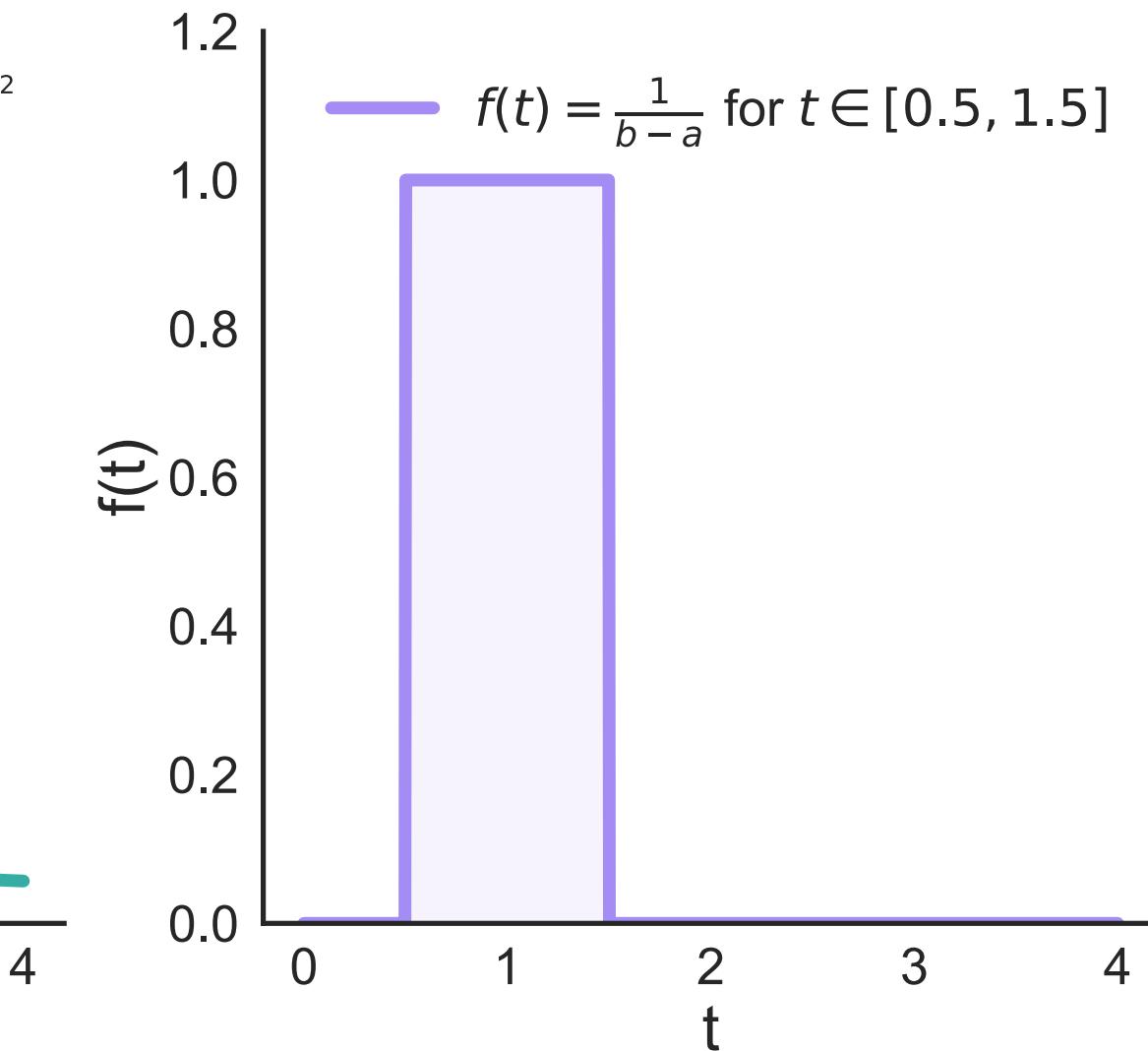
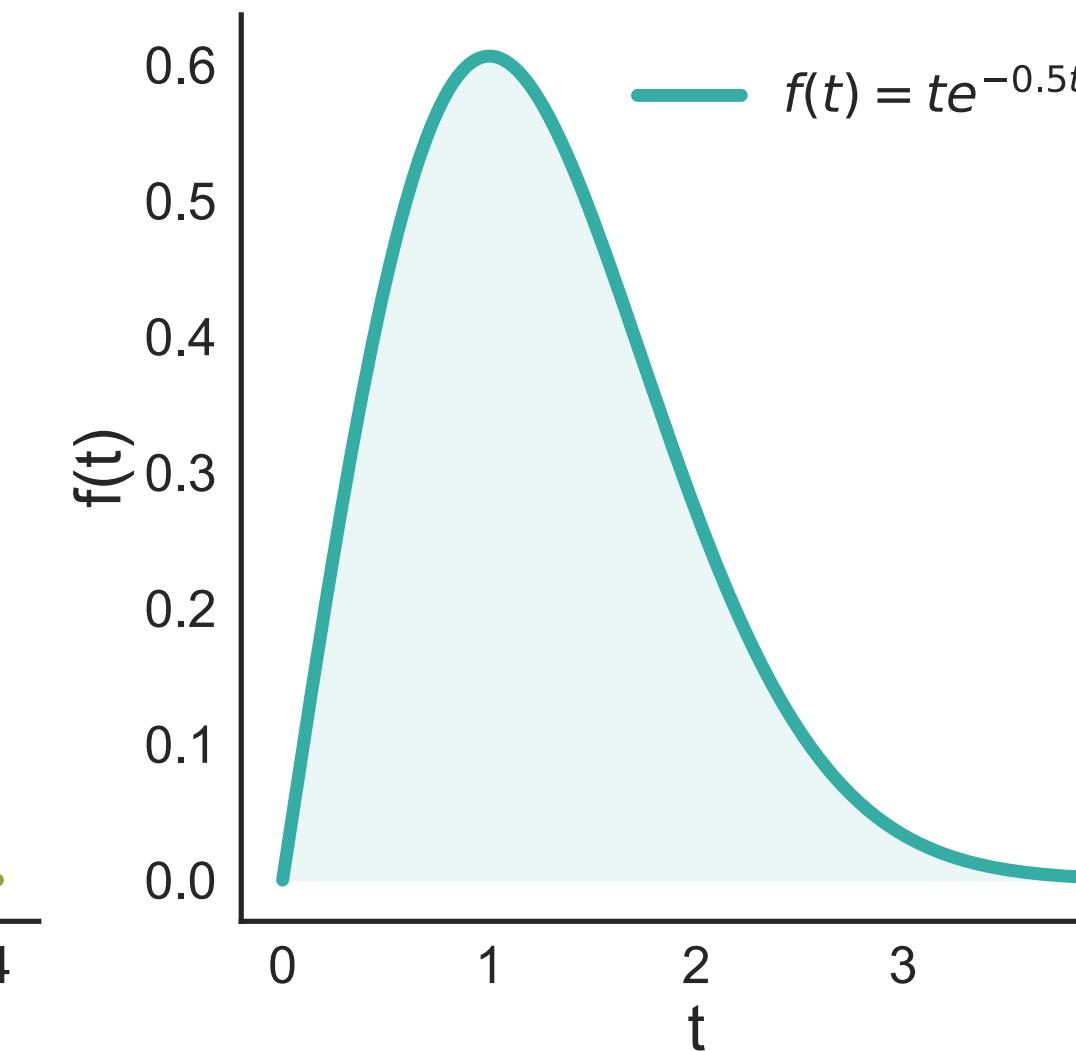
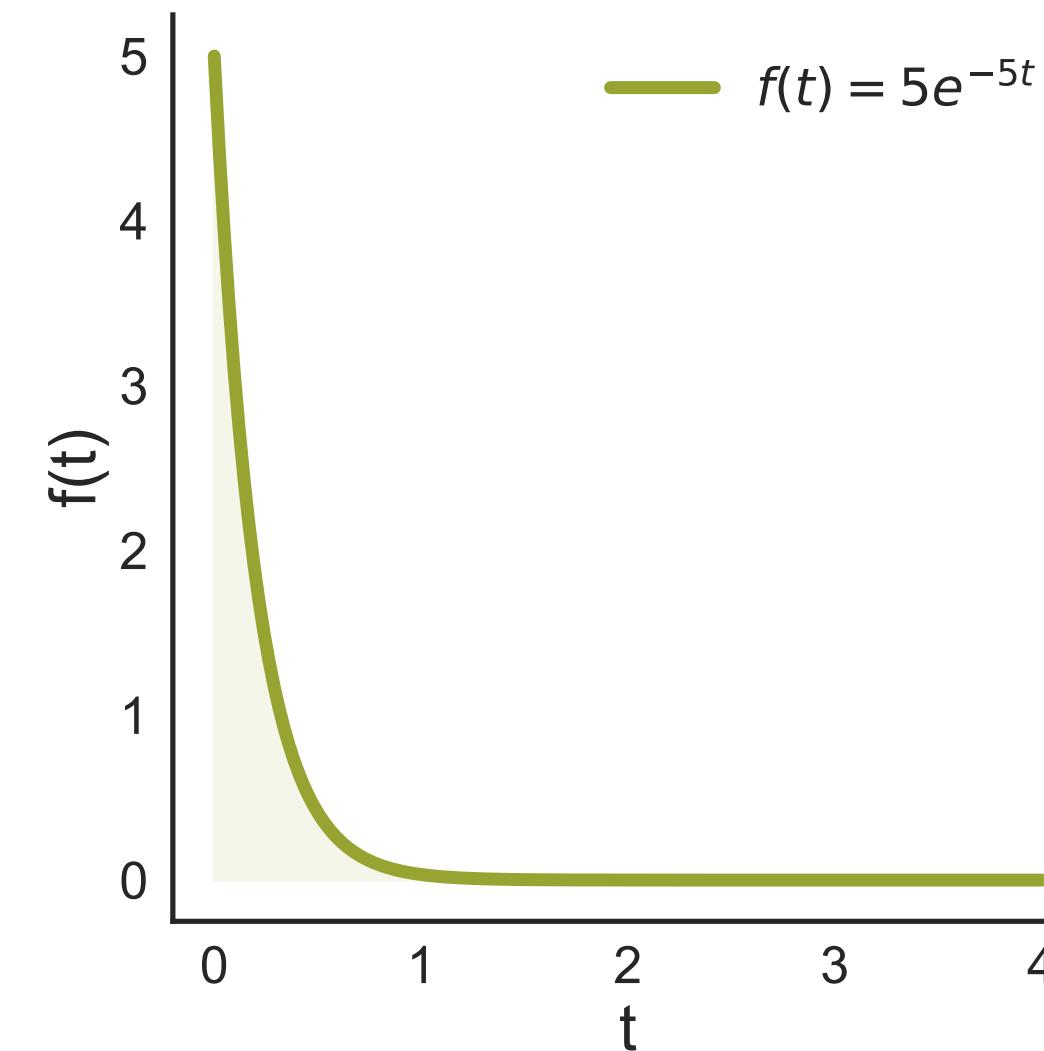
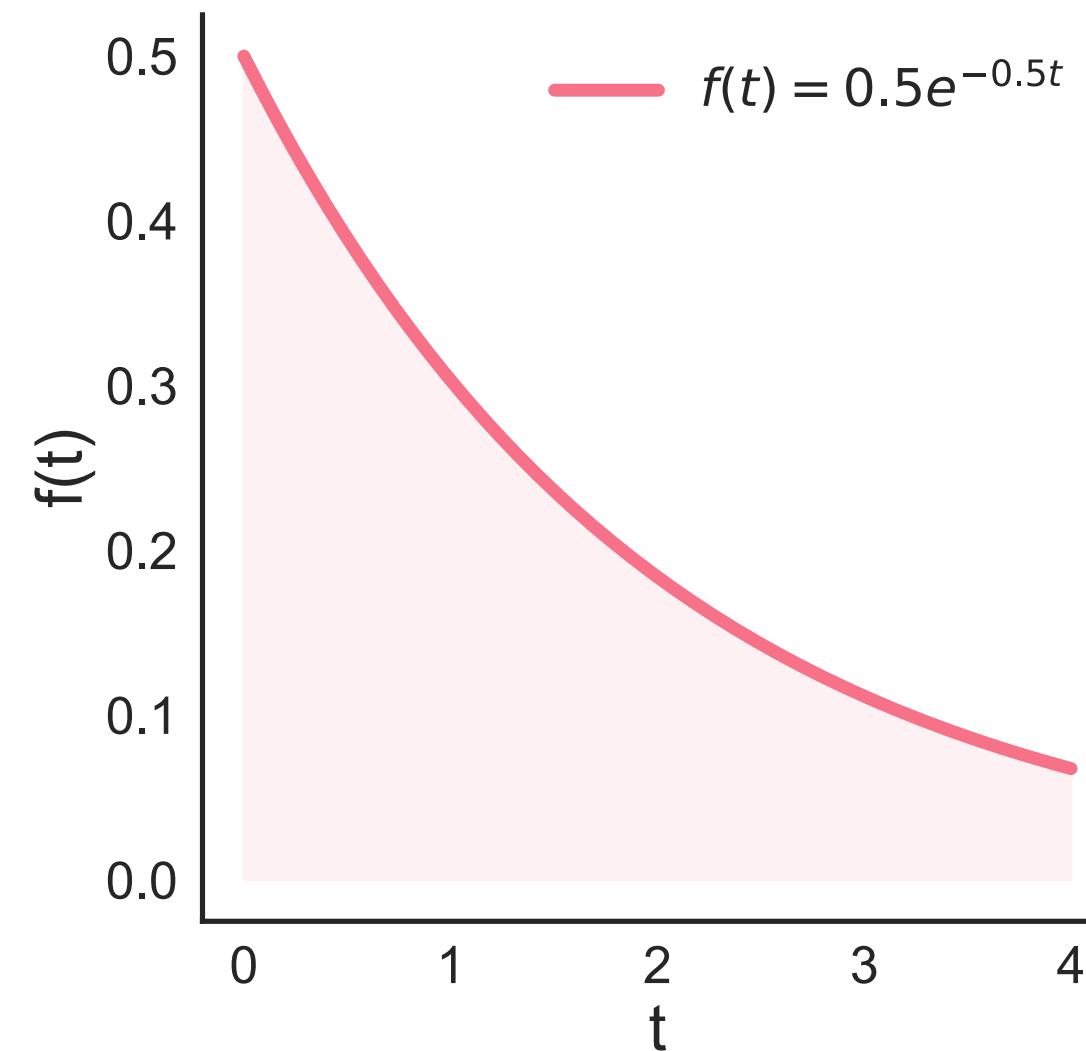
Which PDF corresponds to each intensity function

What are the properties of intensity functions?

How can we convert them into each other?

The probability of an event happening changes over time

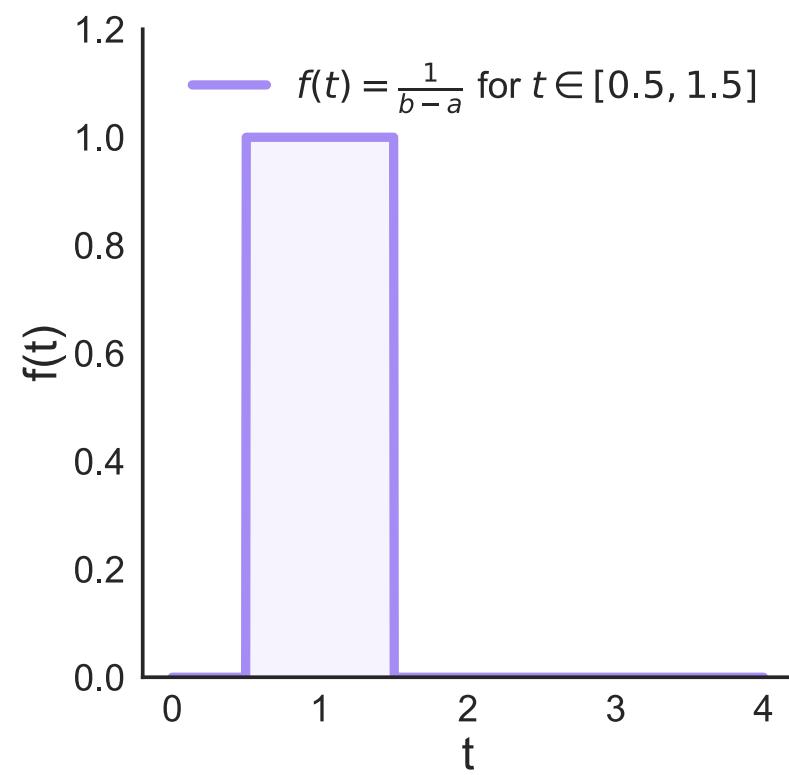
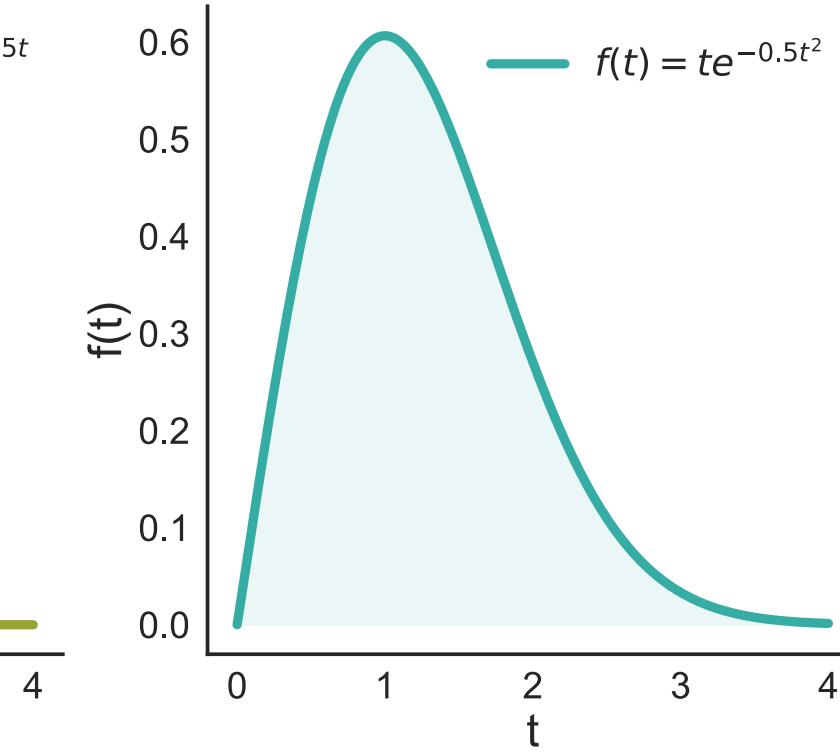
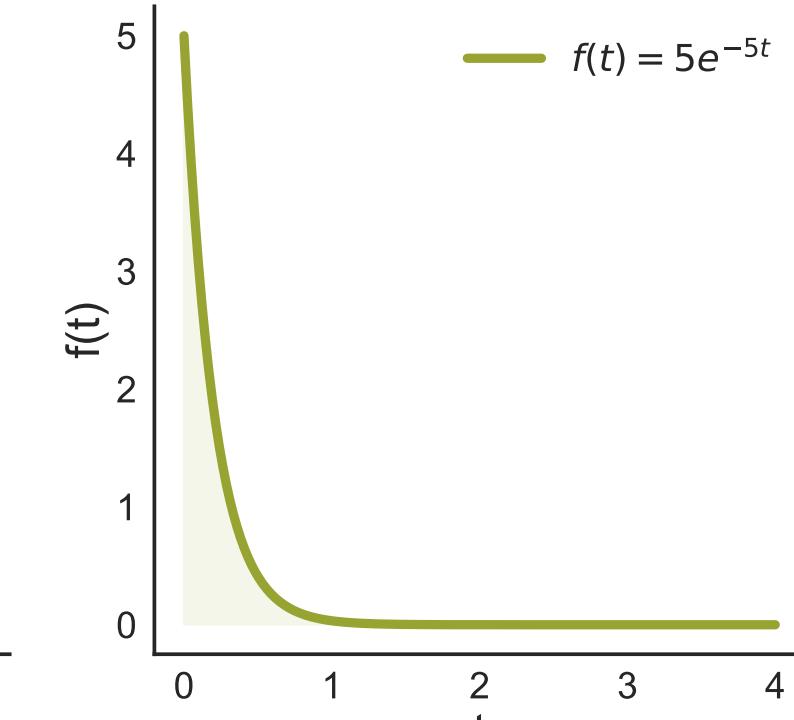
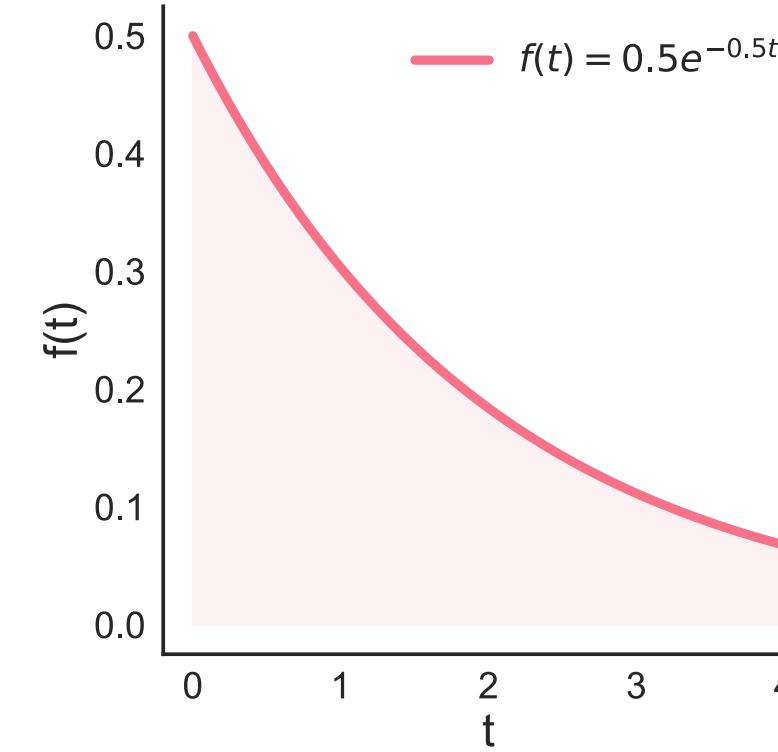
Intensity Function



Intensity Function

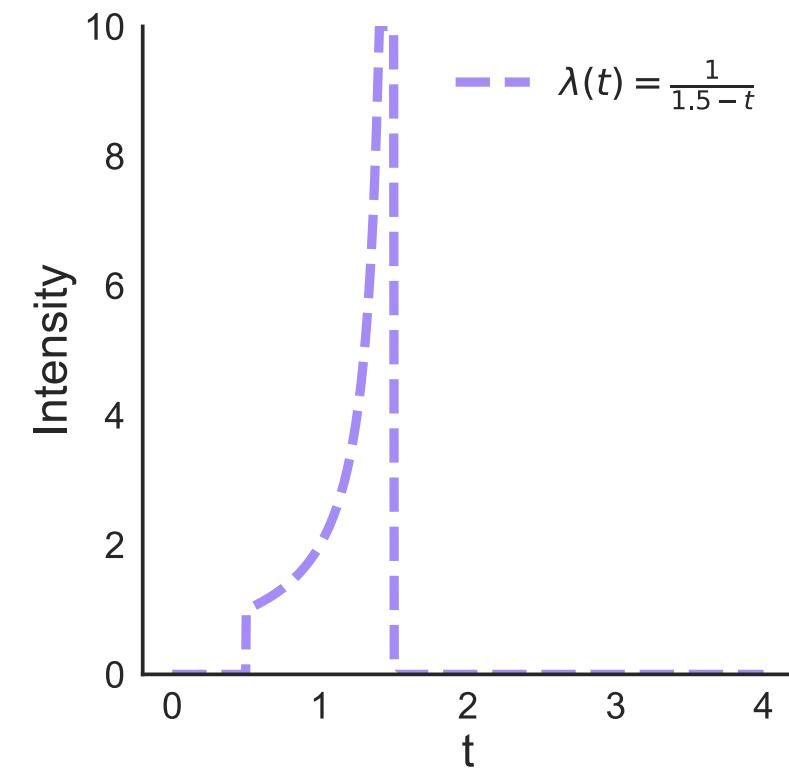
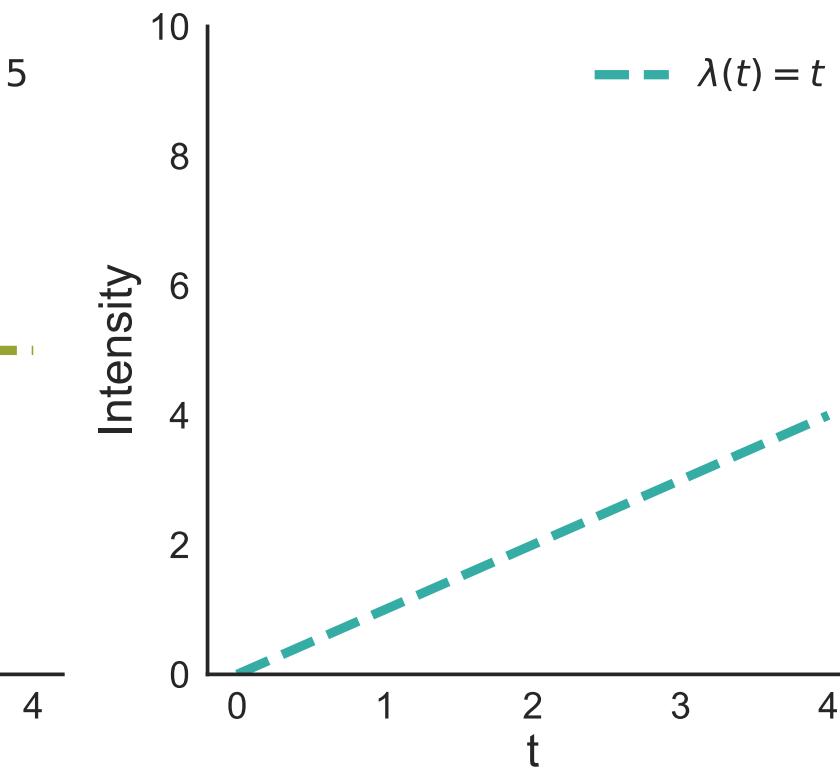
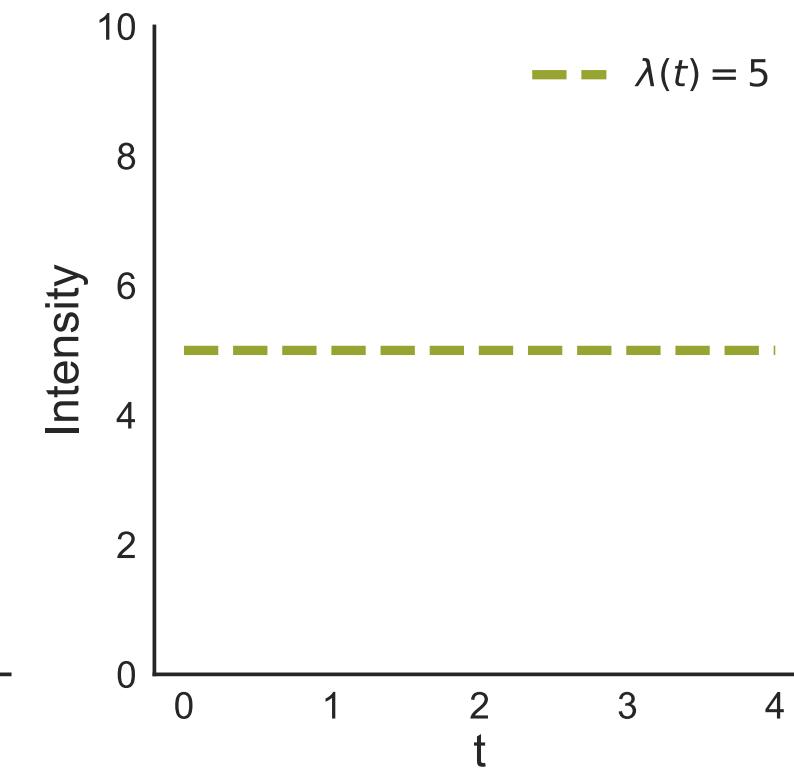
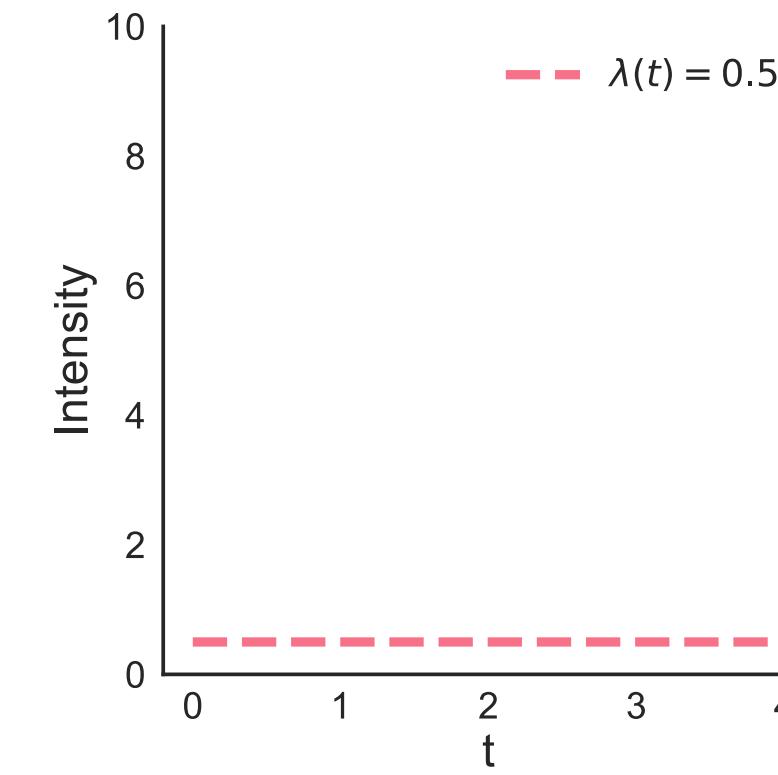
Predictive distribution:

PDF over inter-event times.



Intensity Function:

Instantaneous rate of event happening at t .



Questions:

Which PDF corresponds to each intensity function

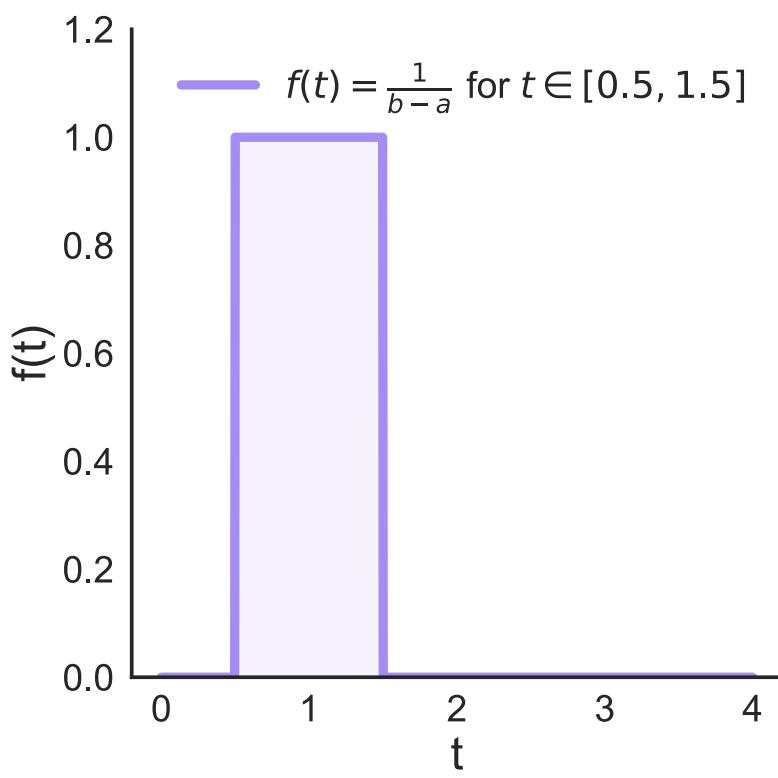
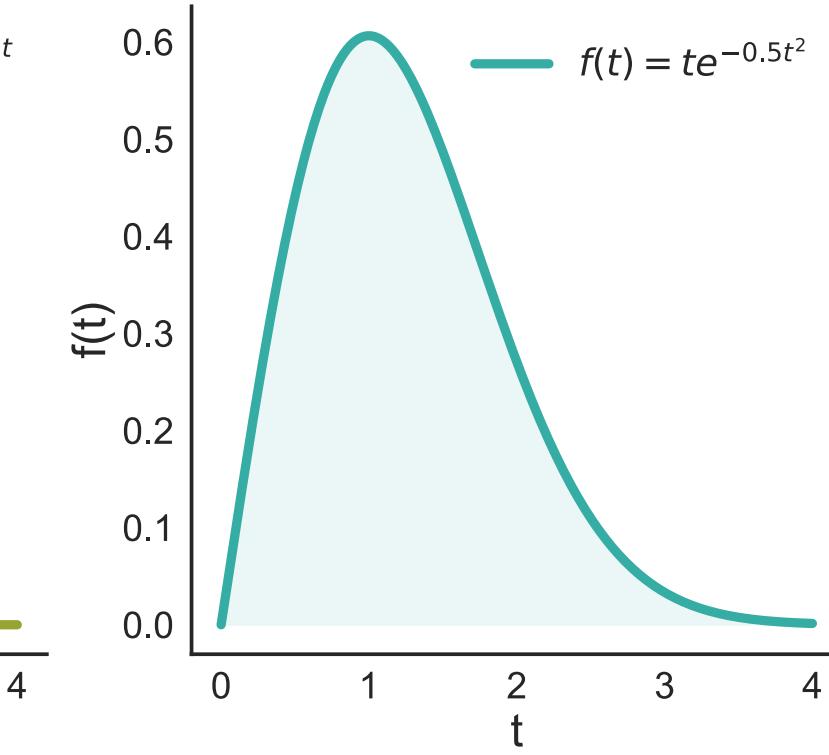
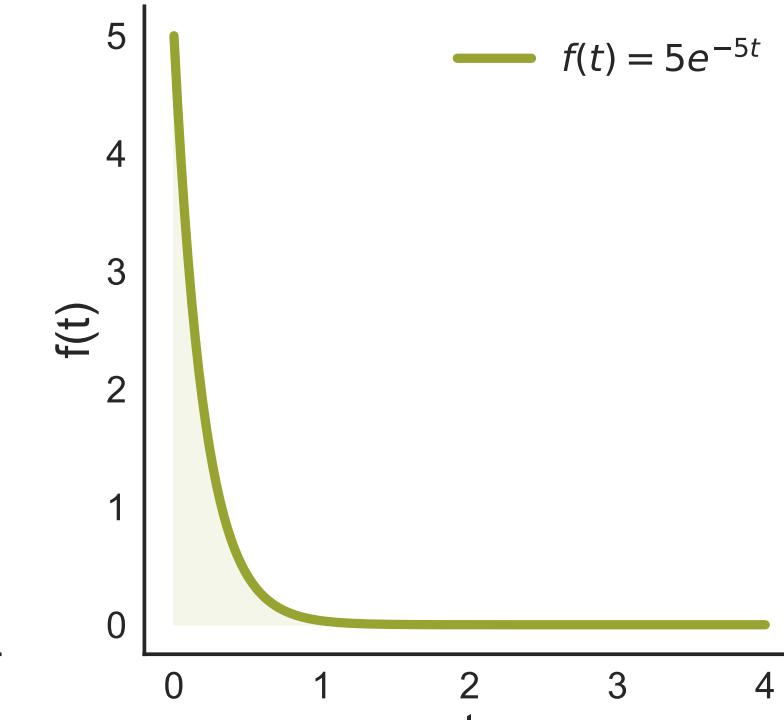
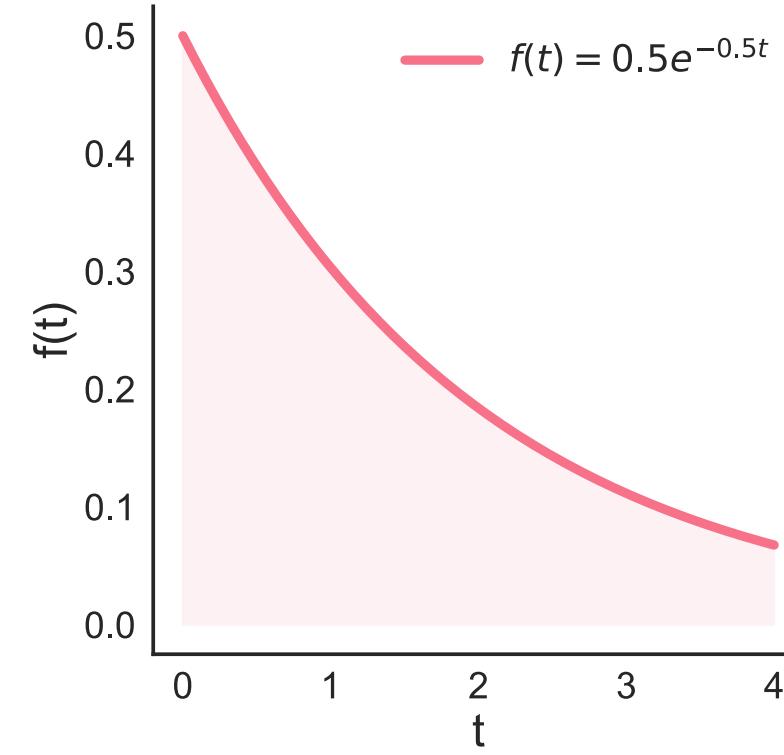
What are the properties of intensity functions?

How can we convert them into each other?

Intensity Function

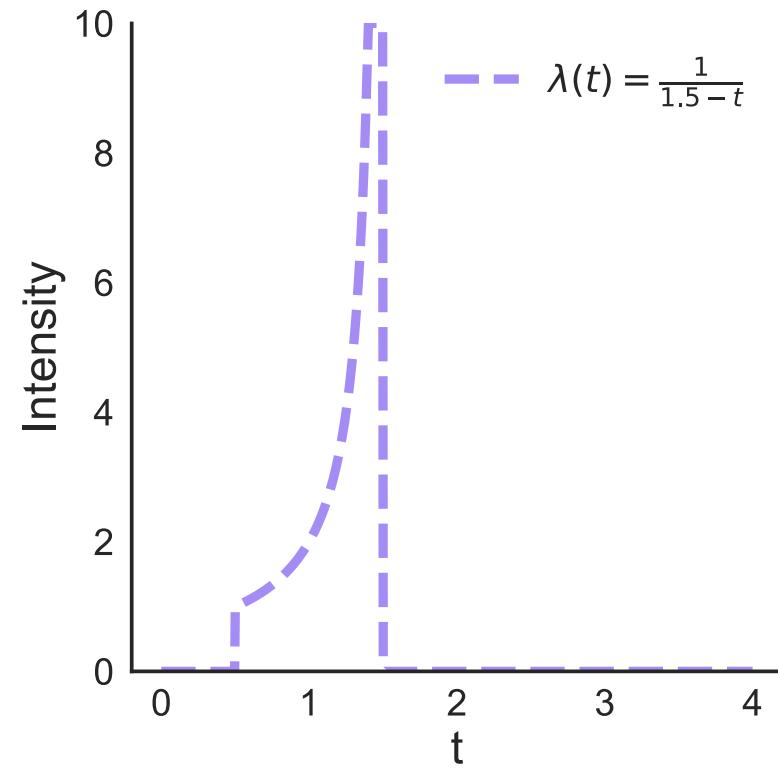
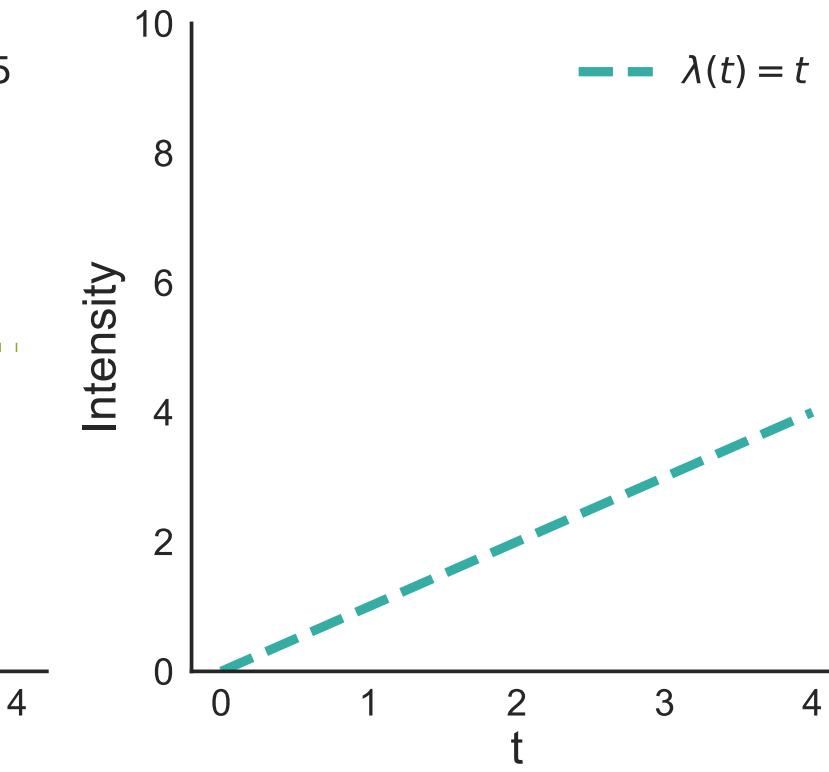
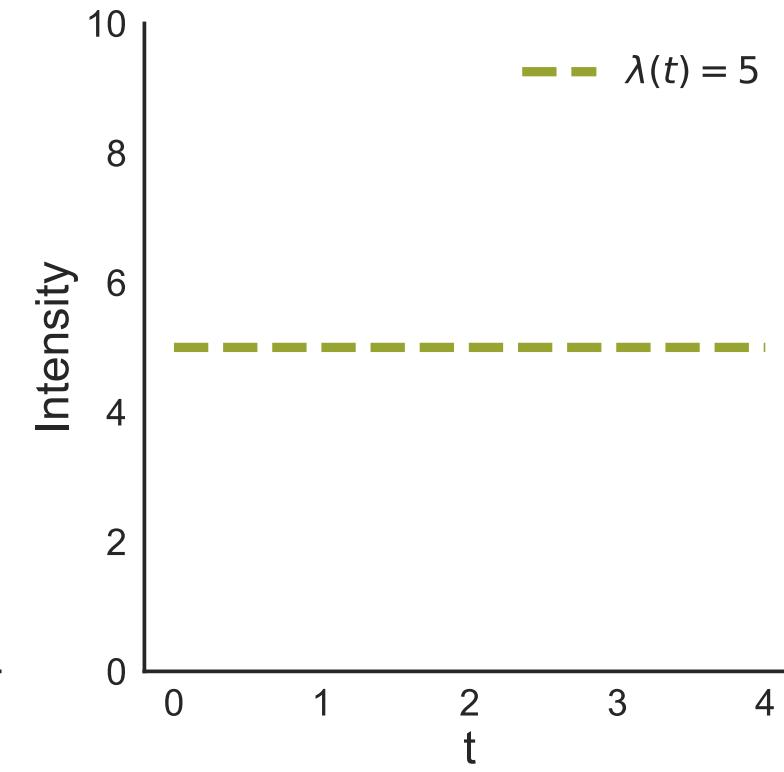
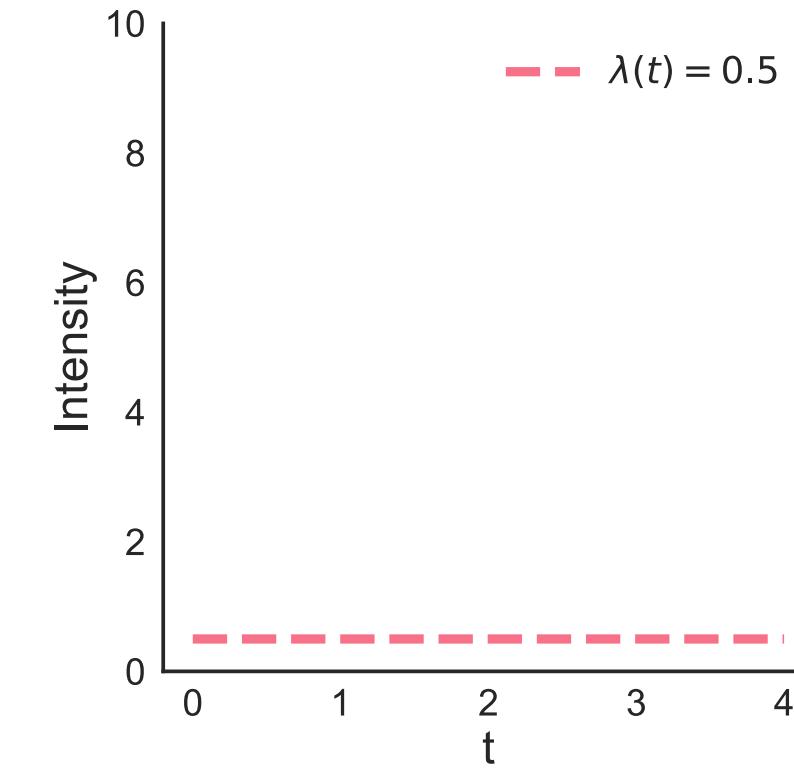
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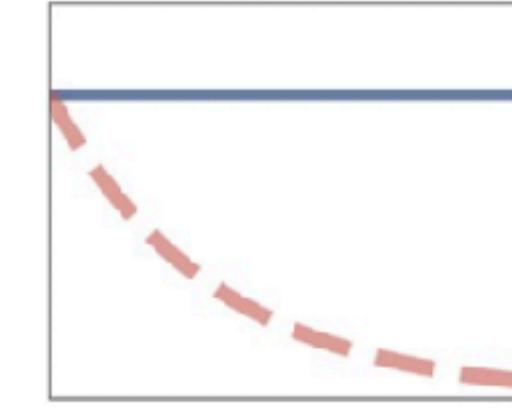
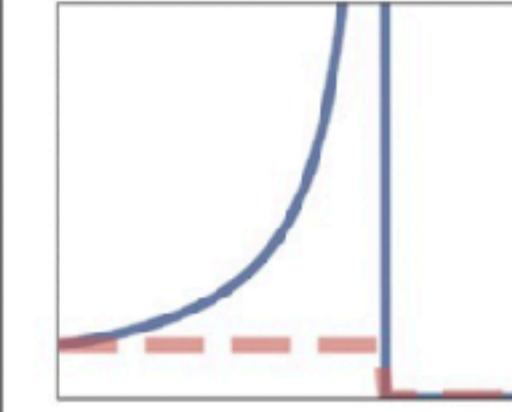
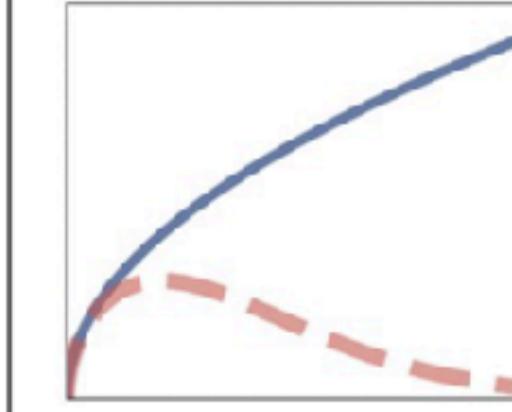
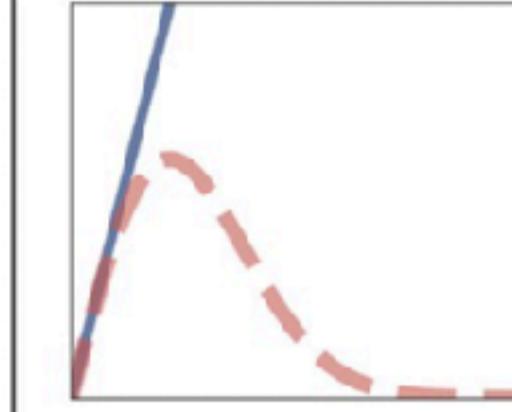
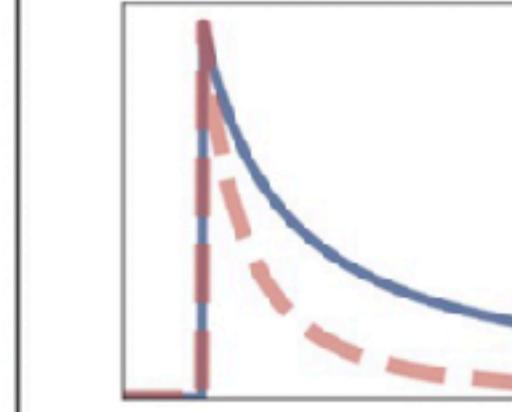
Intensity Function

Predictive distribution:

PDF over inter-event times.

Intensity Function:

Instantaneous rate of event happening at t .

	Exponential	Uniform	Weibull	Rayleigh	Power law
y-axis: γ (dashed), λ (solid) x-axis: time					
Parameters	$\lambda \in \mathbb{R}_{>0}$	$a, b \in \mathbb{R}_{>0}$ $a < b$	$c, u \in \mathbb{R}_{>0}$	$\sigma \in \mathbb{R}_{>0}$	$\alpha, t_{\min} \in \mathbb{R}_{>0}$ $\alpha > 1$
Intensity $\lambda(t)$	λ	$\frac{\mathbb{1}_{t \in [a,b]}}{1 - \frac{t}{b-a}}$	$cu(tu)^{c-1}$	$\frac{t}{\sigma^2}$	$\mathbb{1}_{t \geq t_{\min}} \frac{\alpha-1}{t} \frac{1}{t^{1/\alpha}}$
PDF $\gamma(t)$	$\lambda e^{-\lambda t}$	$\mathbb{1}_{t \in [a,b]}$	$cu(tu)^{c-1} e^{-(tu)^c}$	$\frac{t}{\sigma^2} e^{-\frac{t^2}{2\sigma^2}}$	$\mathbb{1}_{t \geq t_{\min}} \frac{\alpha-1}{t_{\min}} \left(\frac{t}{t_{\min}}\right)^{-\alpha}$

Efficient simulation of non-Markovian dynamics on complex networks (Großmann et al.)

Questions:

Which PDF corresponds to each intensity function

What are the properties of intensity functions?

How can we convert them into each other?

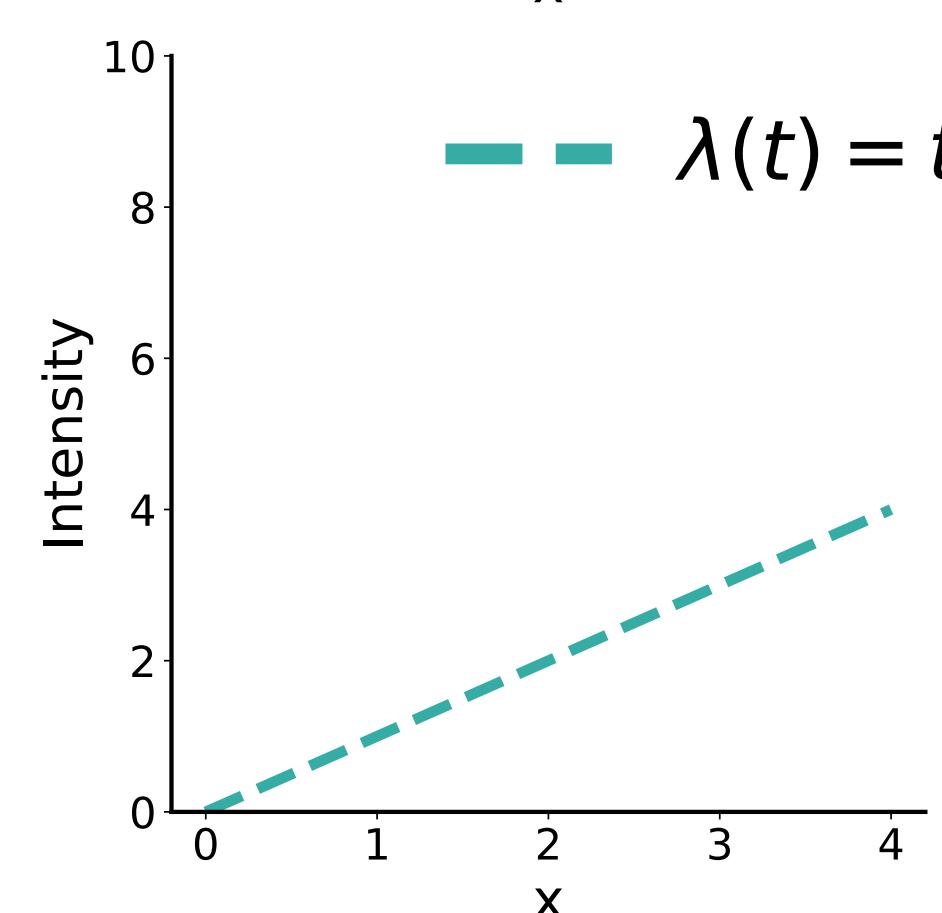
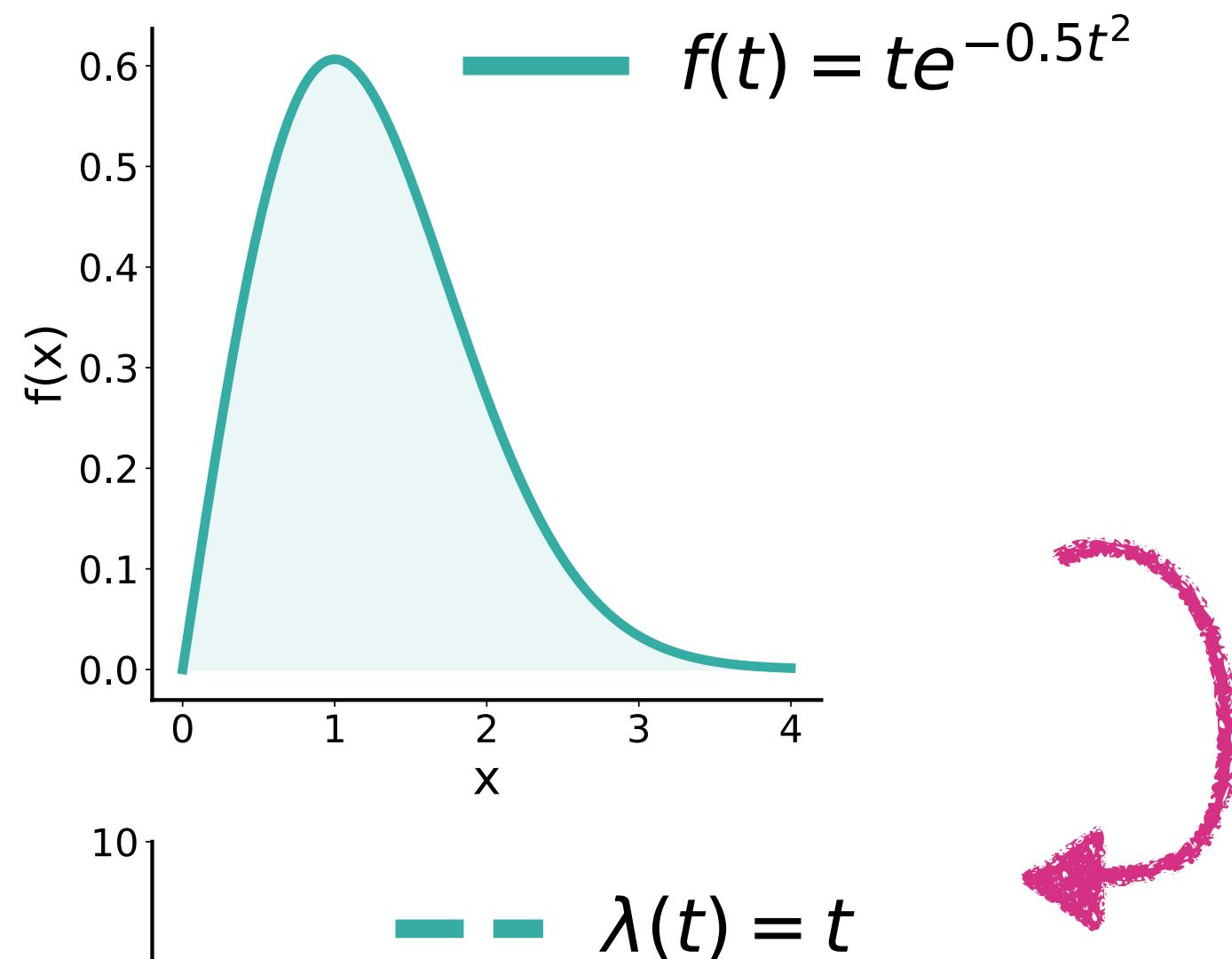
Intensity Function

Predictive distribution:

PDF over inter-event times.

Intensity Function:

Instantaneous rate of event happening at t .



$$\lambda(t) = \frac{f(t)}{\text{P(No event until } t\text{)}} = \frac{f(t)}{1 - F(t)}$$

$$F(t) = \int_0^t f(s) ds$$

Questions:

Which PDF corresponds to each intensity function

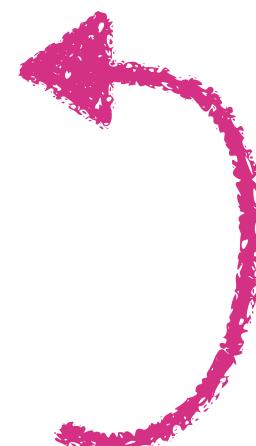
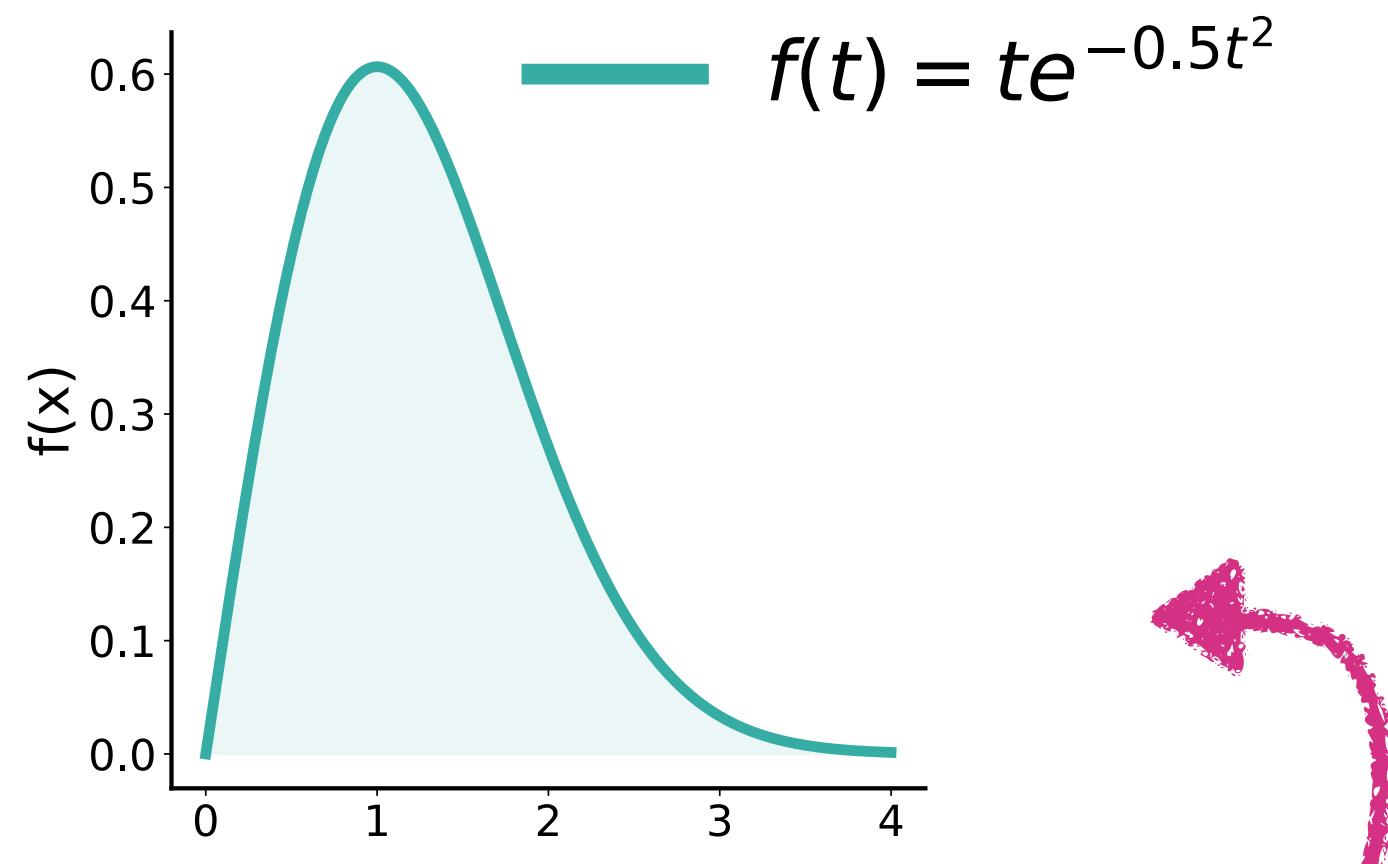
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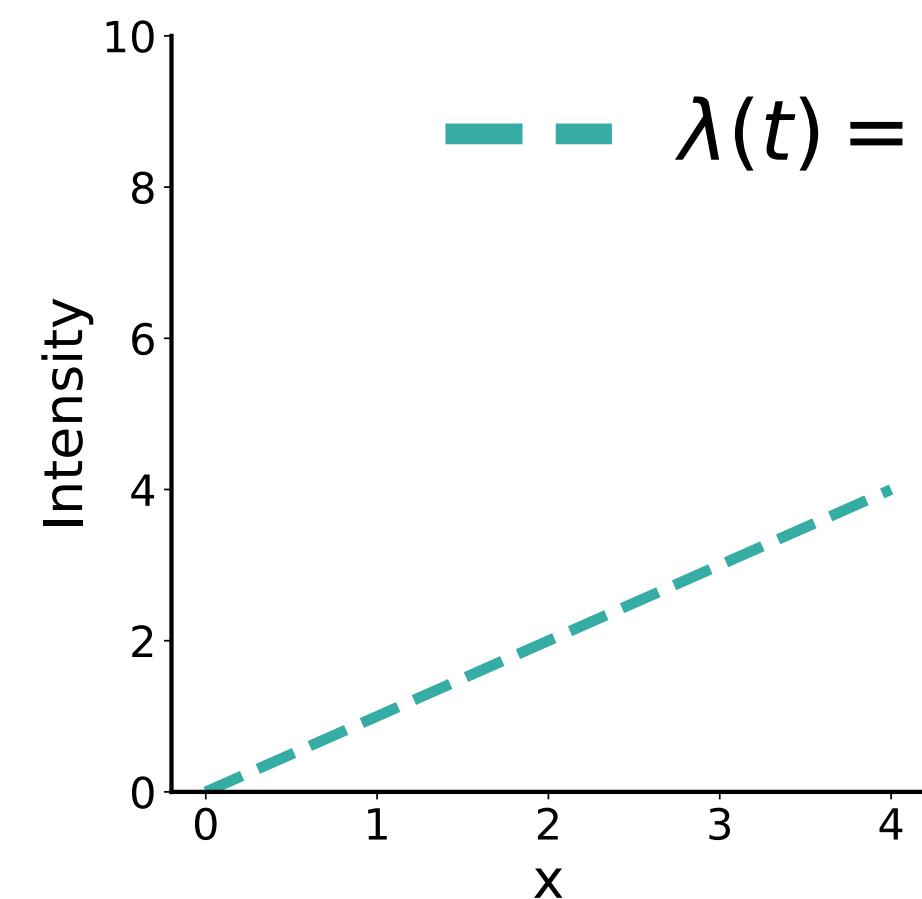
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Intensity Function:

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$$f(t) = \lambda(t)e^{-\int_0^t \lambda(s) ds}$$

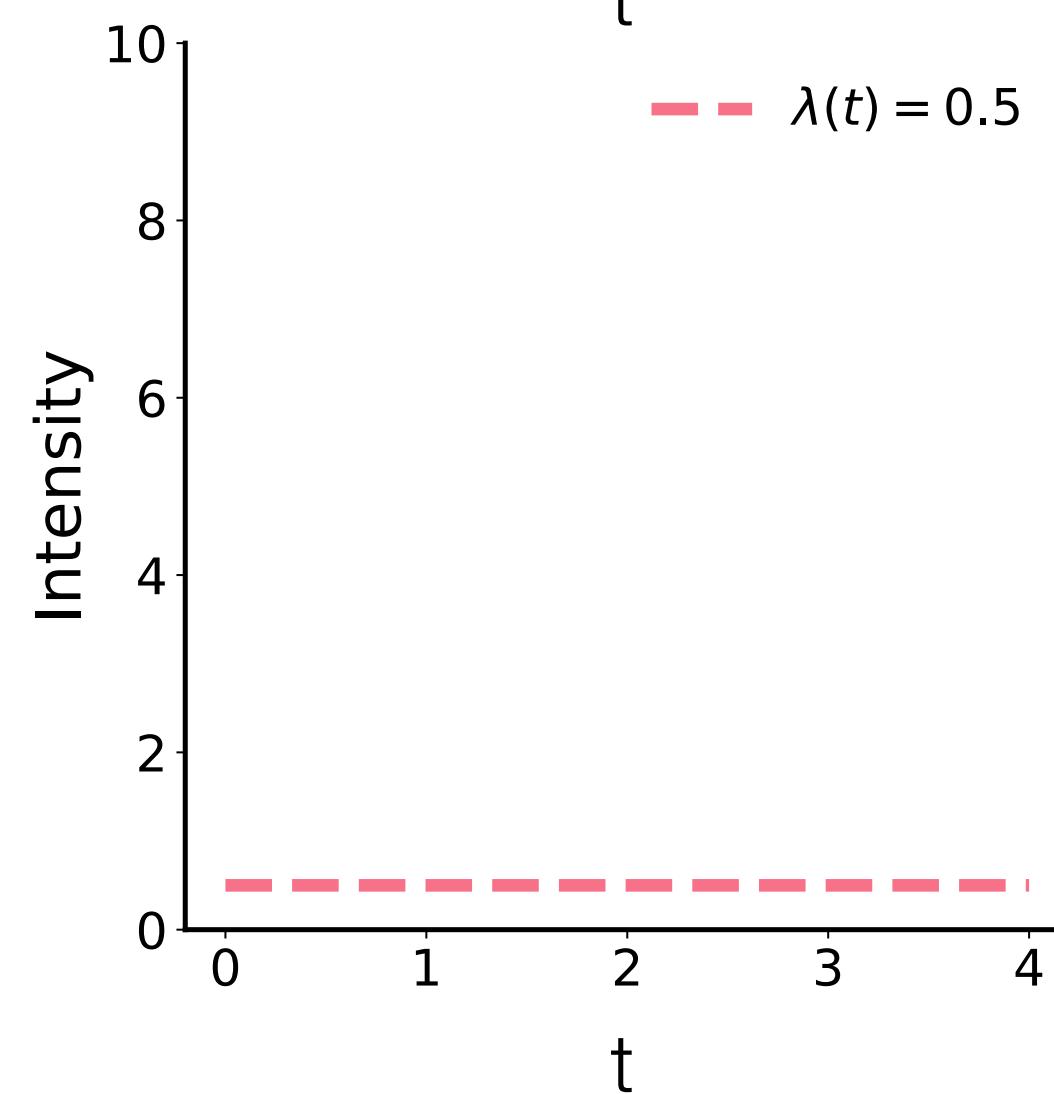
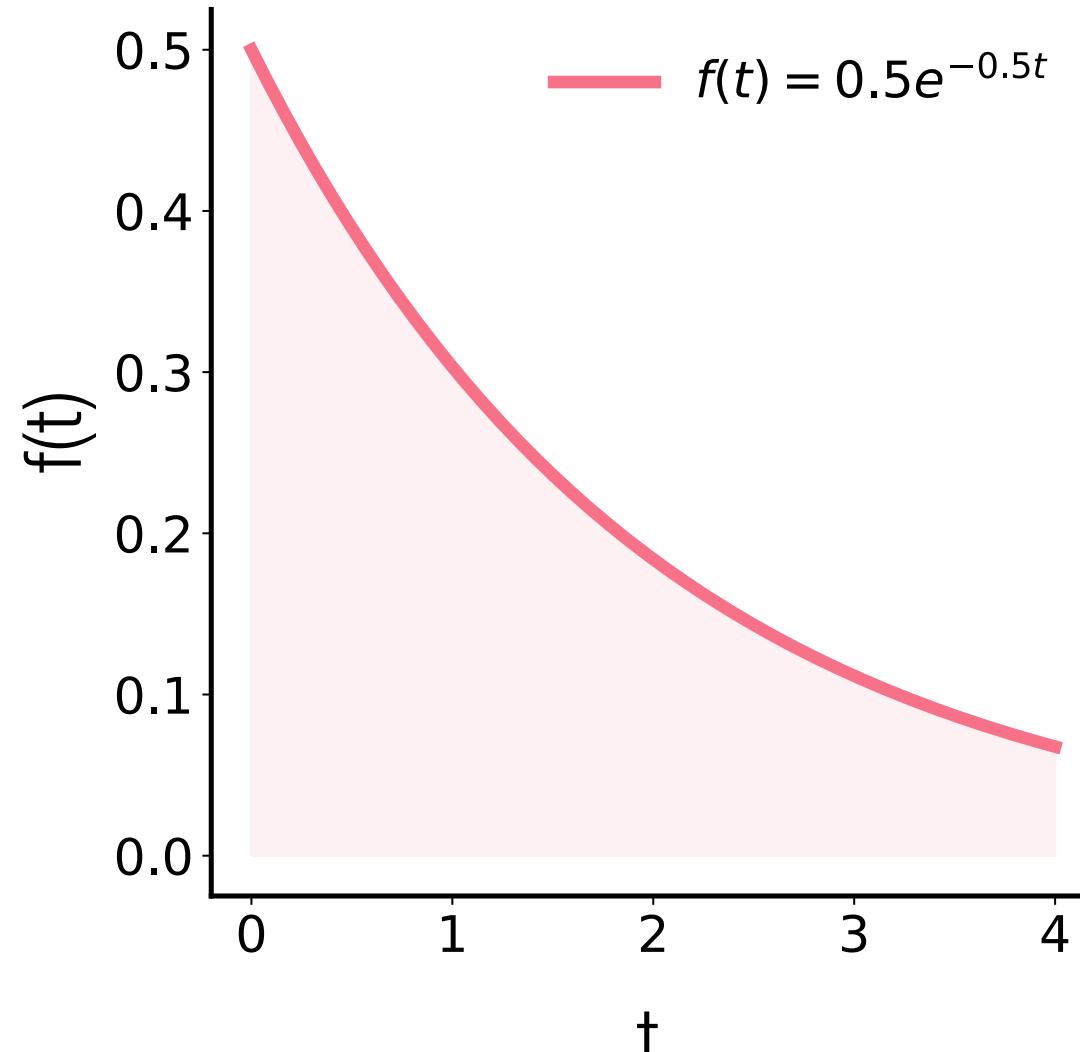
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Intensity Function



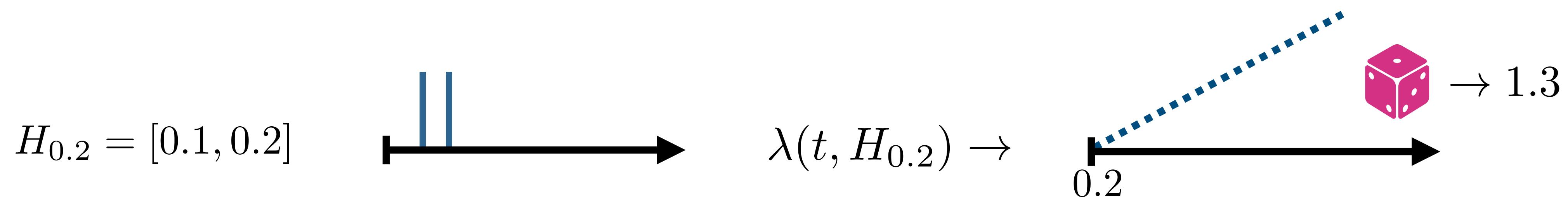
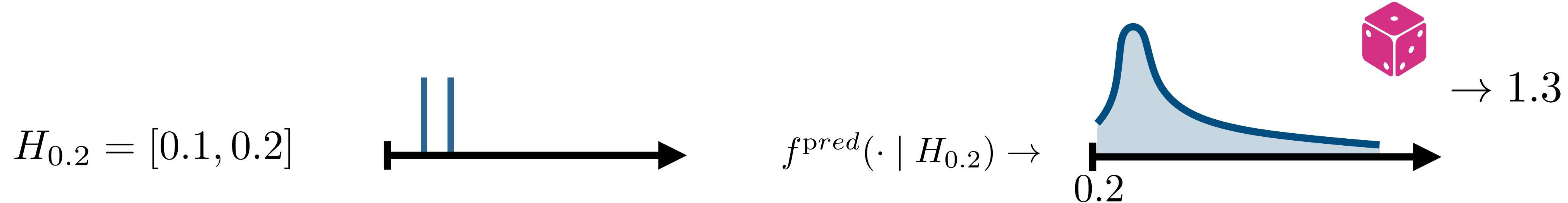
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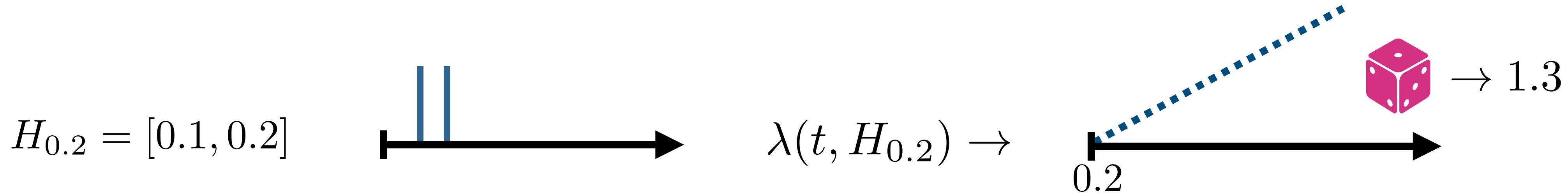
$$F(t) = \int_0^t f(s) ds$$

- $e^{-\int_0^t \lambda(s) ds}$ is called the *exponential decay*. It is the probability that no event has occurred up to time t .
- $1 - F(t)$ is called the *survival function*. This term represents the probability that the event has not occurred by time t .

Simulating TPPs



Simulating TPPs



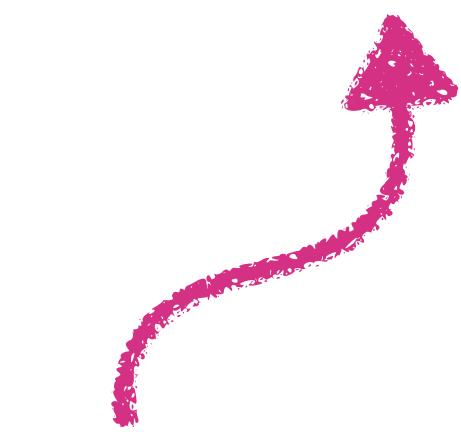
How do we sample from $\lambda(\cdot | H)$?

Simulation Algorithm:

Initialize $H = []$

Until horizon is reached:

- Sample next event time t_i using $\lambda(\cdot | H)$ ($t_i > \max(H)$)
- Add t_i to H



$$\lambda(t) = \lim_{\Delta t \rightarrow 0} \frac{P(\text{Event in } [t, t + \Delta t])}{\Delta t}$$

Simulating TPPs

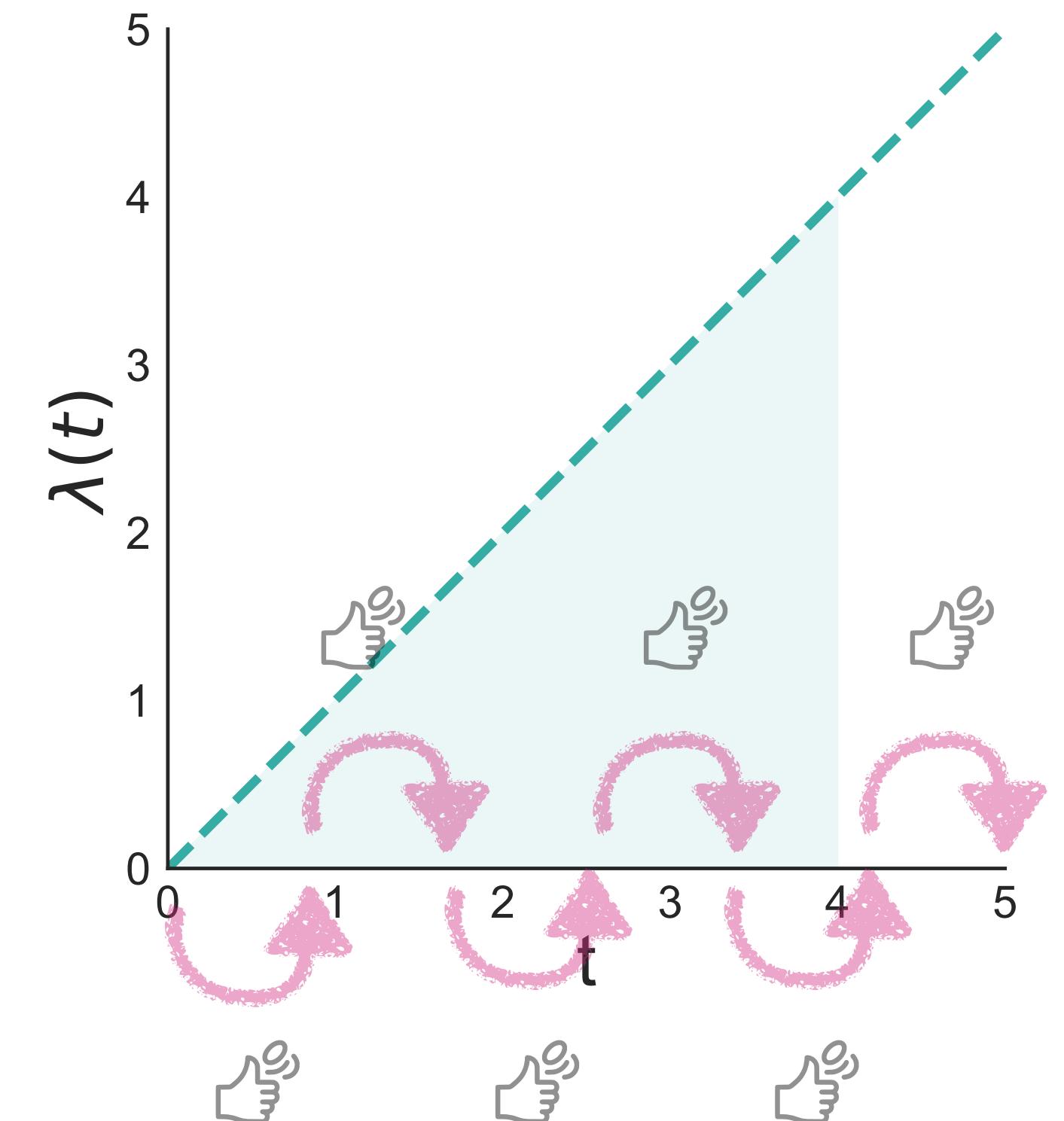
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Until horizon is reached:

- Sample next event time t_i using $\lambda(\cdot | H)$ ($t_i > \max(H)$) How do we sample from $\lambda(\cdot | H)$?
- Add t_i to H

Method 1:

1. **Discretize Time:** Choose a small time step Δt .
2. **Biased Coin:** At each time step t , use a biased coin to decide whether an event occurs. The probability p_t of the event occurring in each time step is given by $p_t = \lambda(t)\Delta t$.
3. **Simulate Until Event:** Continue this process until an event occurs.



Simulating TPPs

Initialize $H = []$

Until horizon is reached:

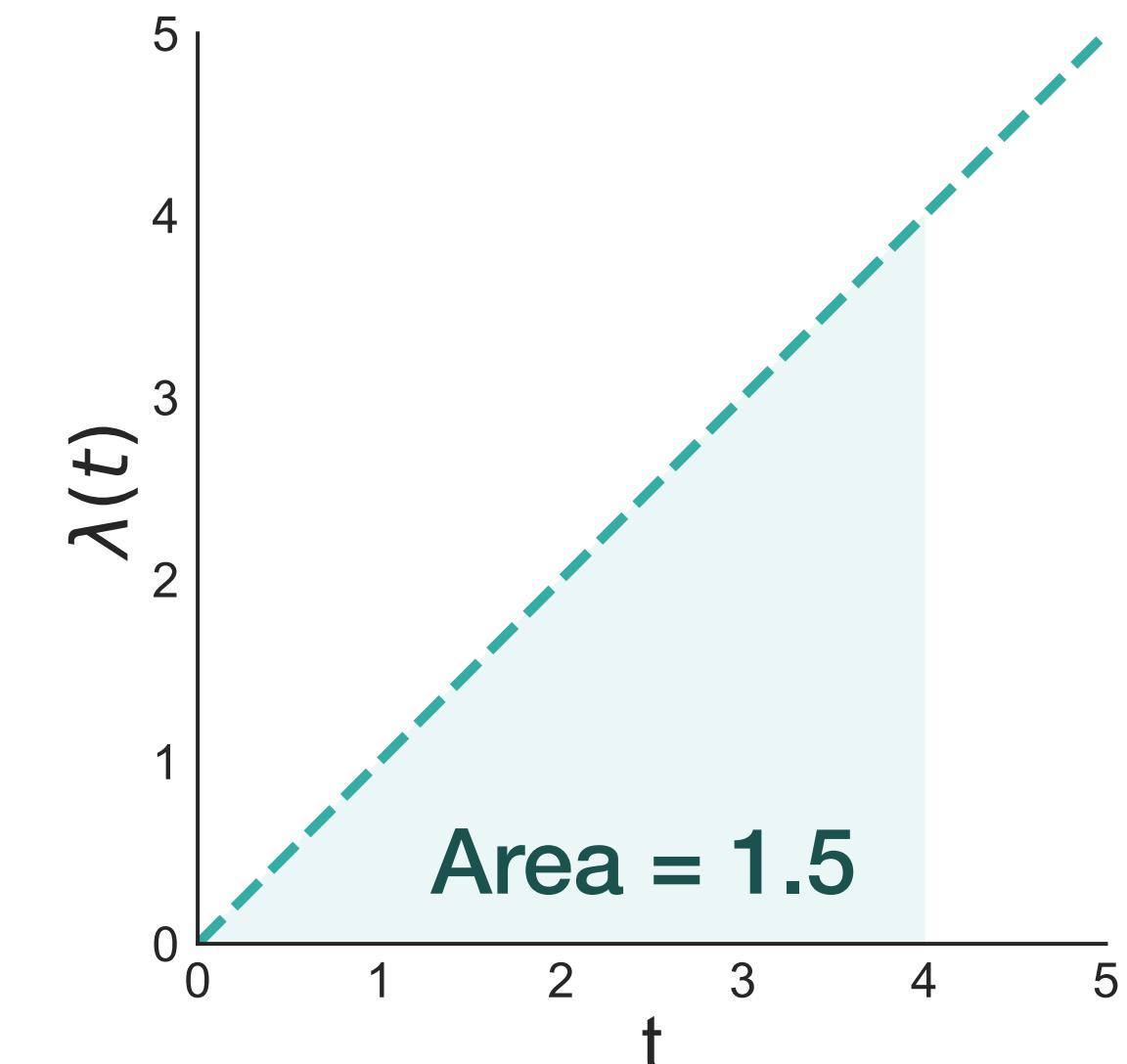
- Sample next event time t_i using $\lambda(\cdot | H)$ ($t_i > \max(H)$) How do we sample from $\lambda(\cdot | H)$?
- Add t_i to H

Method 2:

1. **Discretize Time:** Choose Δt .
2. **Draw a Random Variate:** Draw an exponentially distributed random variate E with $\lambda = 1$.
3. **Numerical Integration:** Integrate $\lambda(t)$ over time until the area under the curve of the intensity equals E . Mathematically, find the smallest T such that:

$$\int_0^T \lambda(t) dt = E.$$

4. **Event Occurrence:** The time T at which this condition is first met is the time at which the event occurs.



$$E = 1.5 \implies T = 3$$

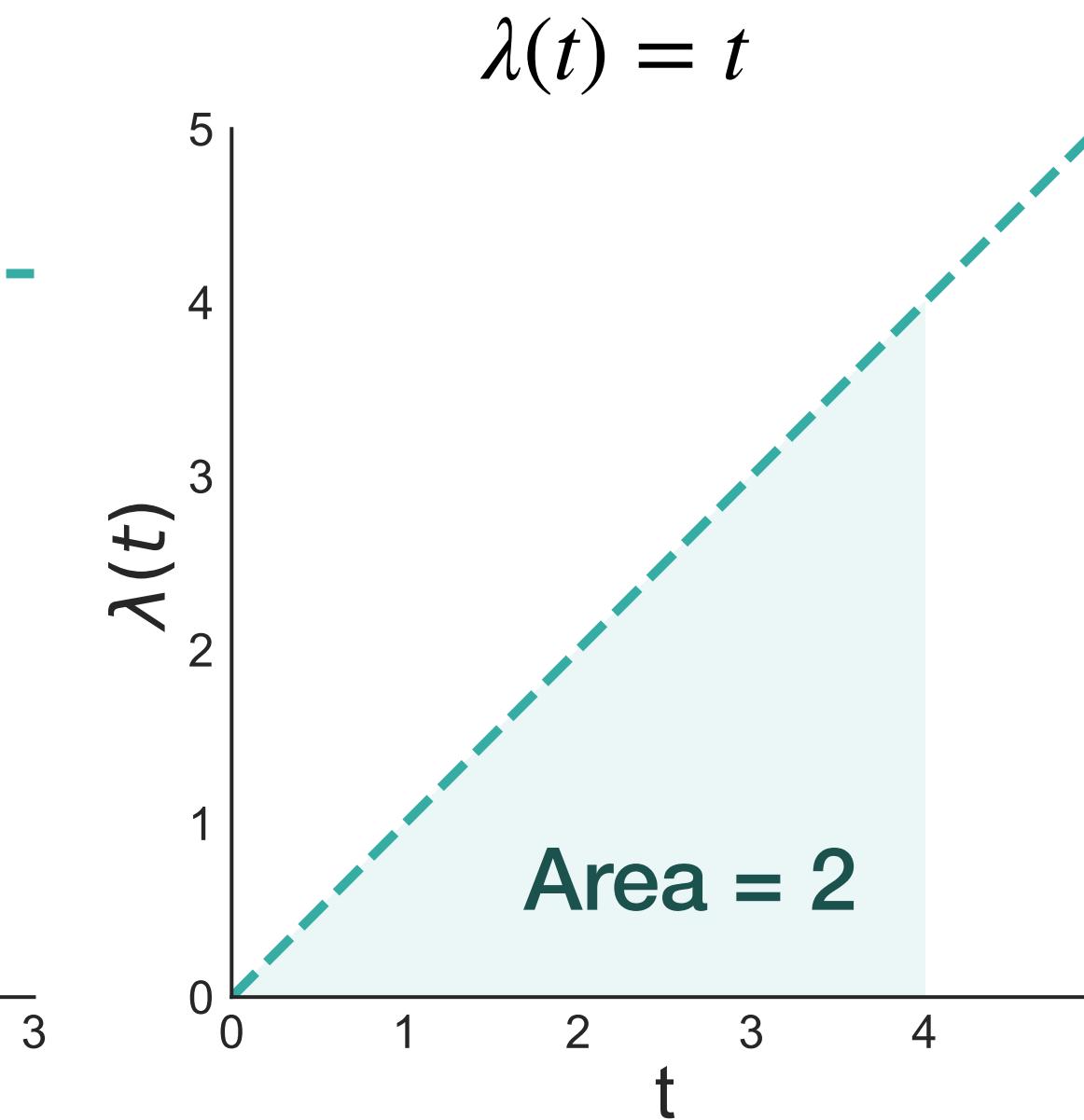
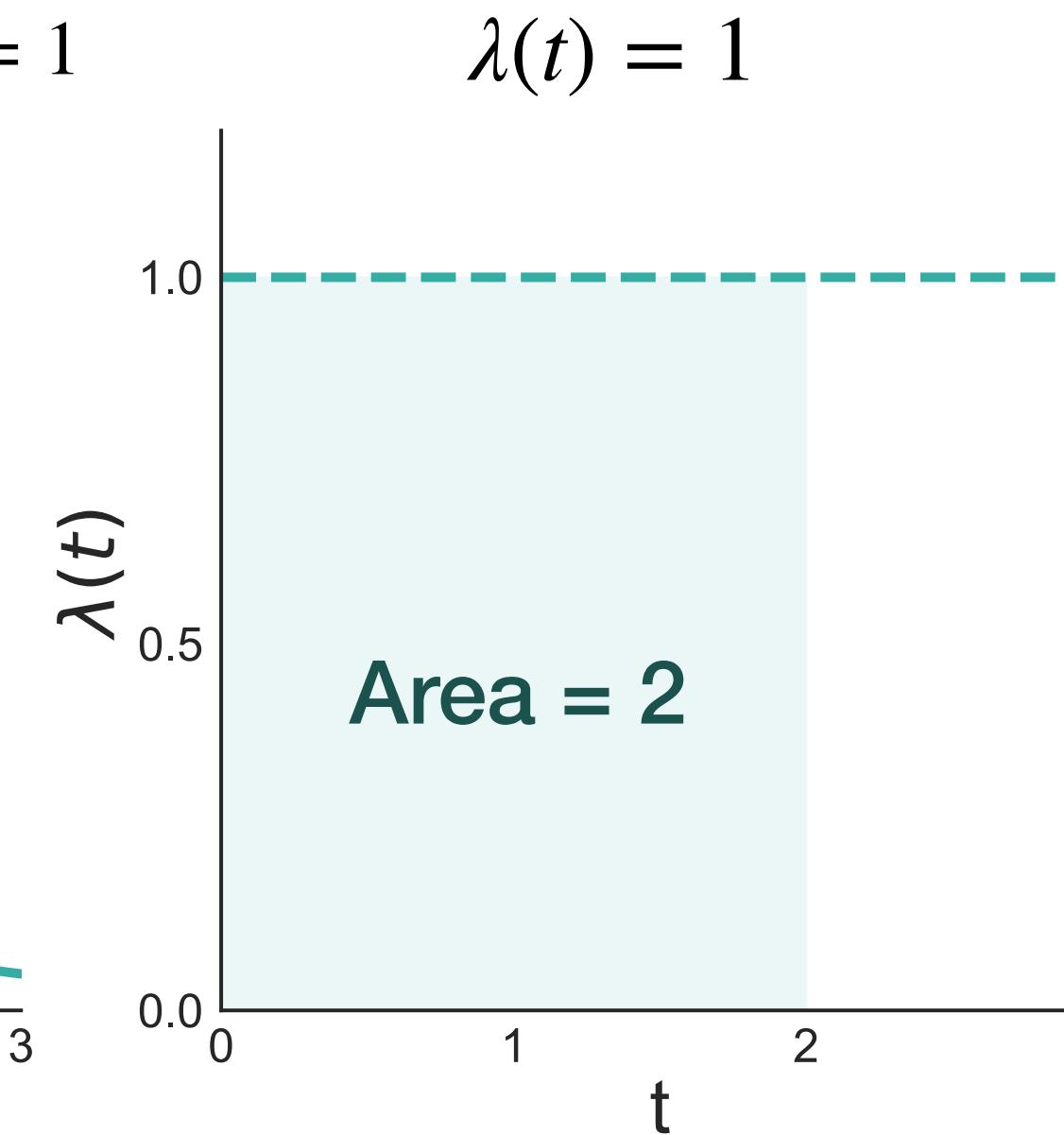
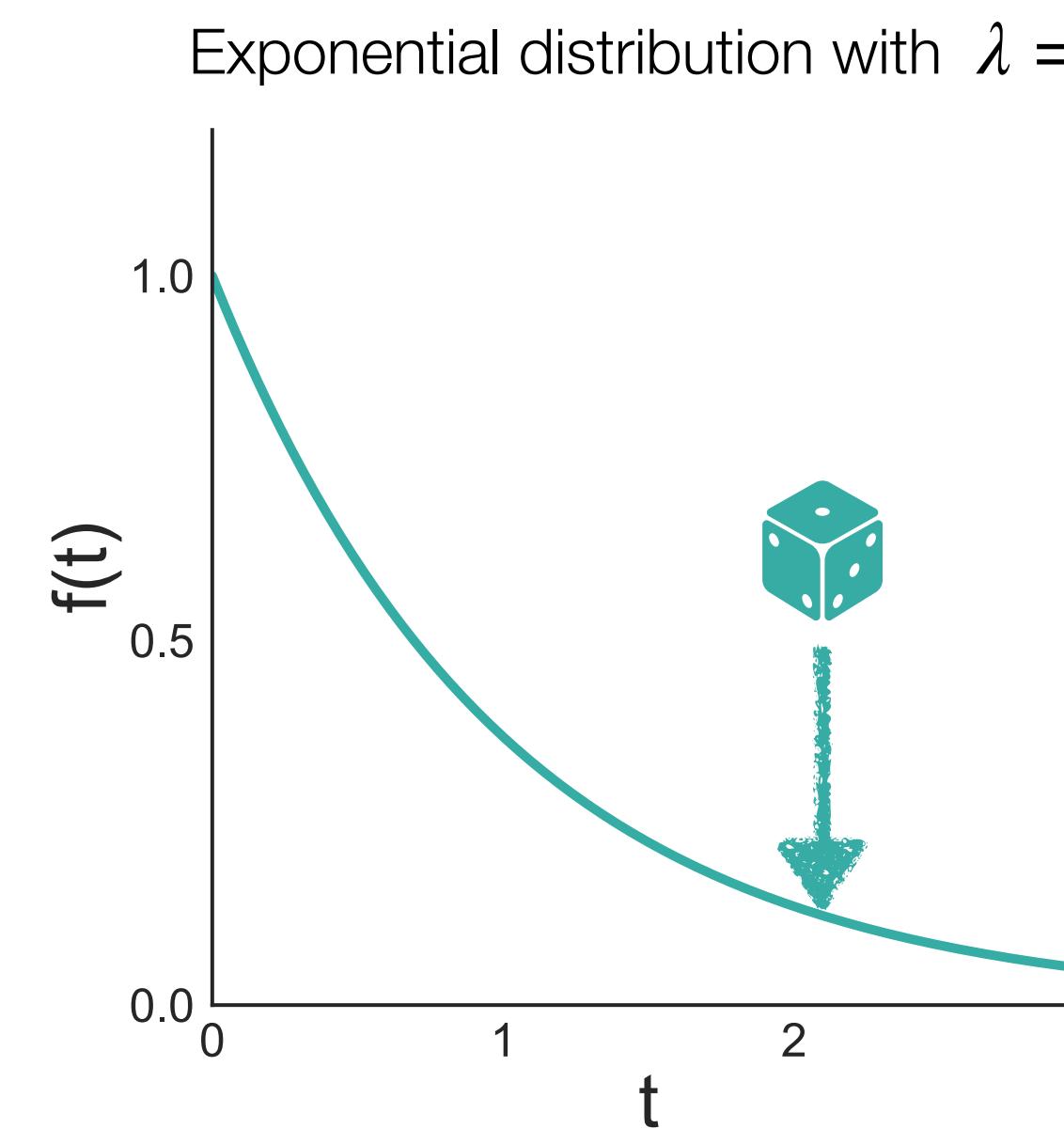
Simulating TPPs

Initialize $H = []$

Until horizon is reached:

- Sample next event time t_i using $\lambda(\cdot | H)$ ($t_i > \max(H)$) How do we sample from $\lambda(\cdot | H)$?
- Add t_i to H

Method 2:



Simulating TPPs

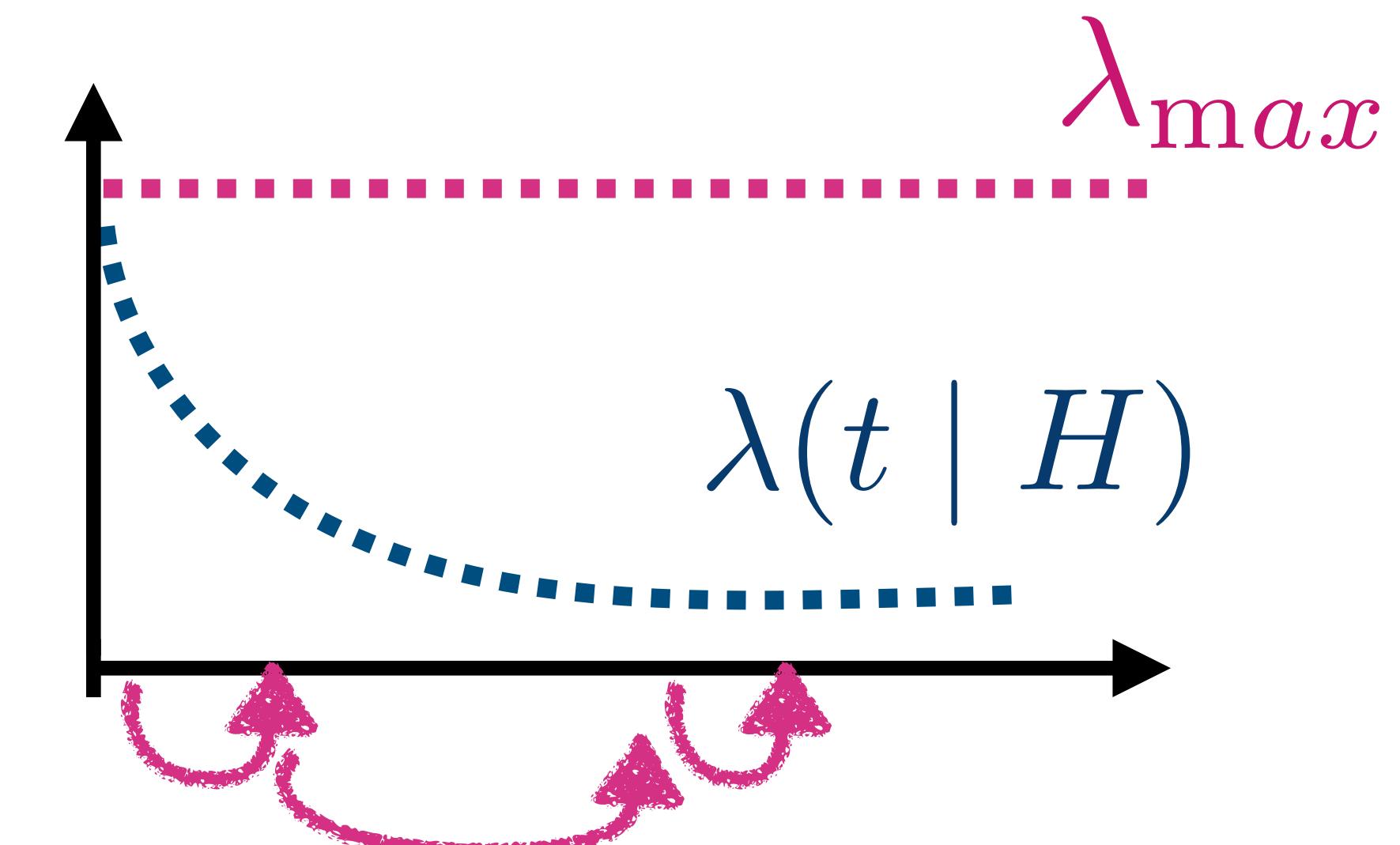
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- Add t_i to H

Method 3:

1. **Determine Maximum Intensity:** Determine an upper bound of the intensity function λ_{\max} .
2. **Initialize Time:** Set $t = 0$.
3. **Sample Waiting Time:** Draw an exponentially distributed random variate E with rate parameter λ_{\max} .
4. **Update Time:** Update the time: $t = t + E$.
5. **Acceptance Test:** Generate another uniform random number V in the interval $[0, 1]$. If $V \leq \frac{\lambda(t|H)}{\lambda_{\max}}$, accept the event at time t . Otherwise, reject the event and go back to step 3.



Simulating TPPs

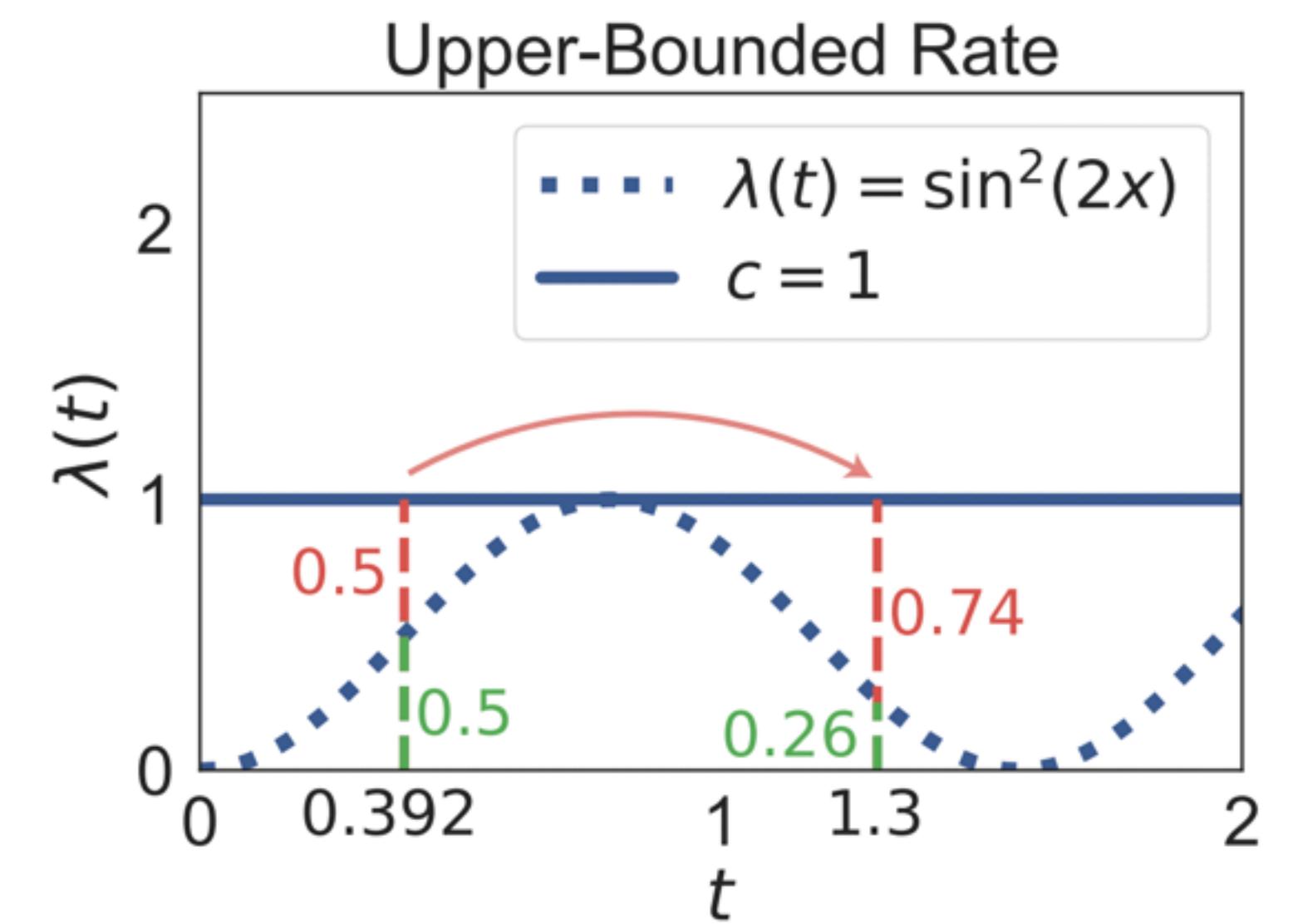
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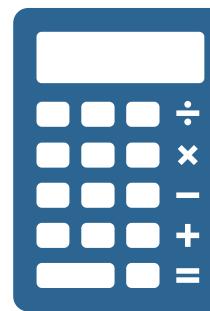
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Models of TPPs

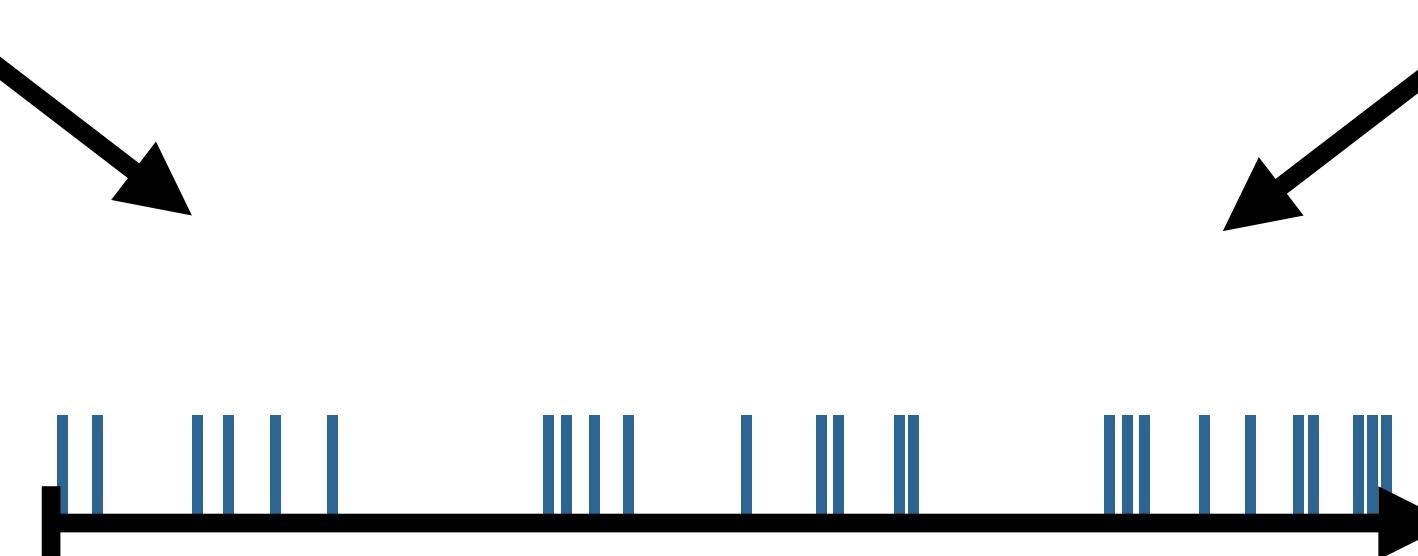
How do we specify TPPs?



Specify mathematical model (probability measure)

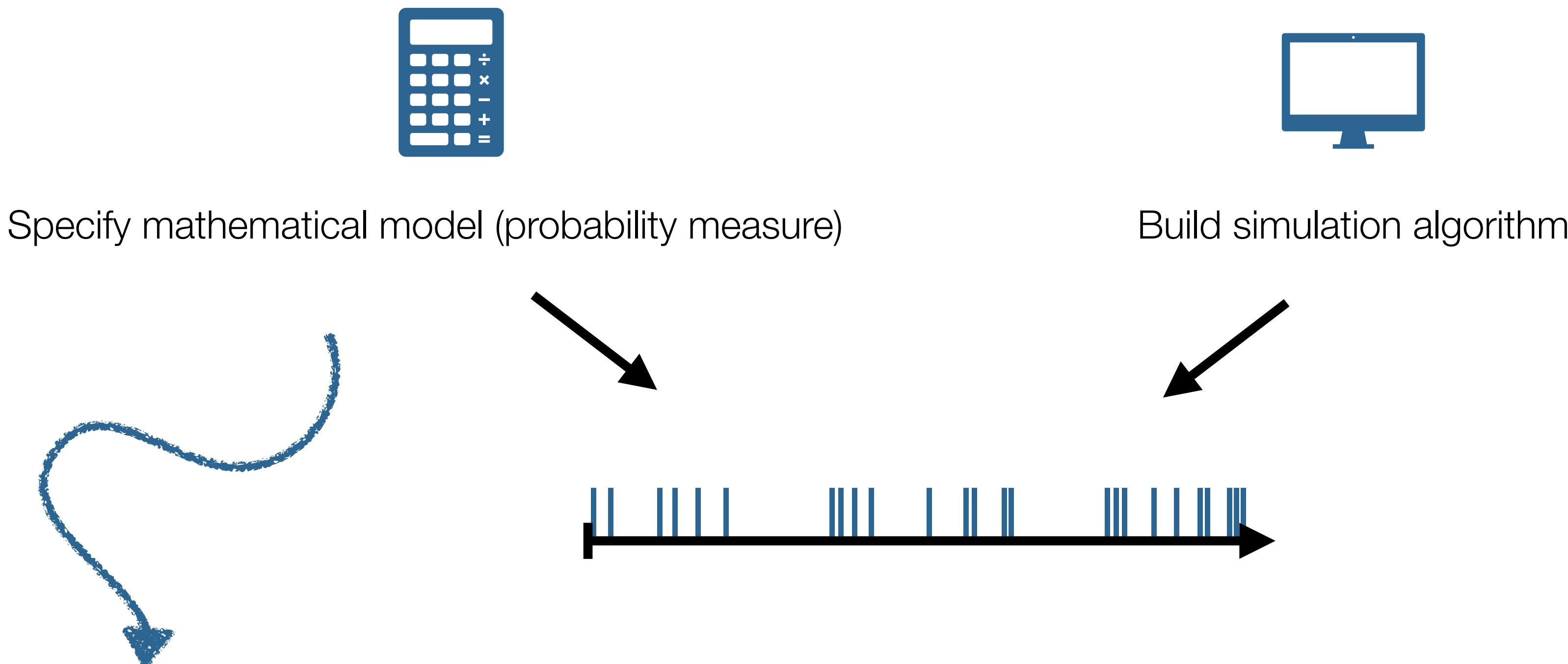


Build simulation algorithm



Models of TPPs

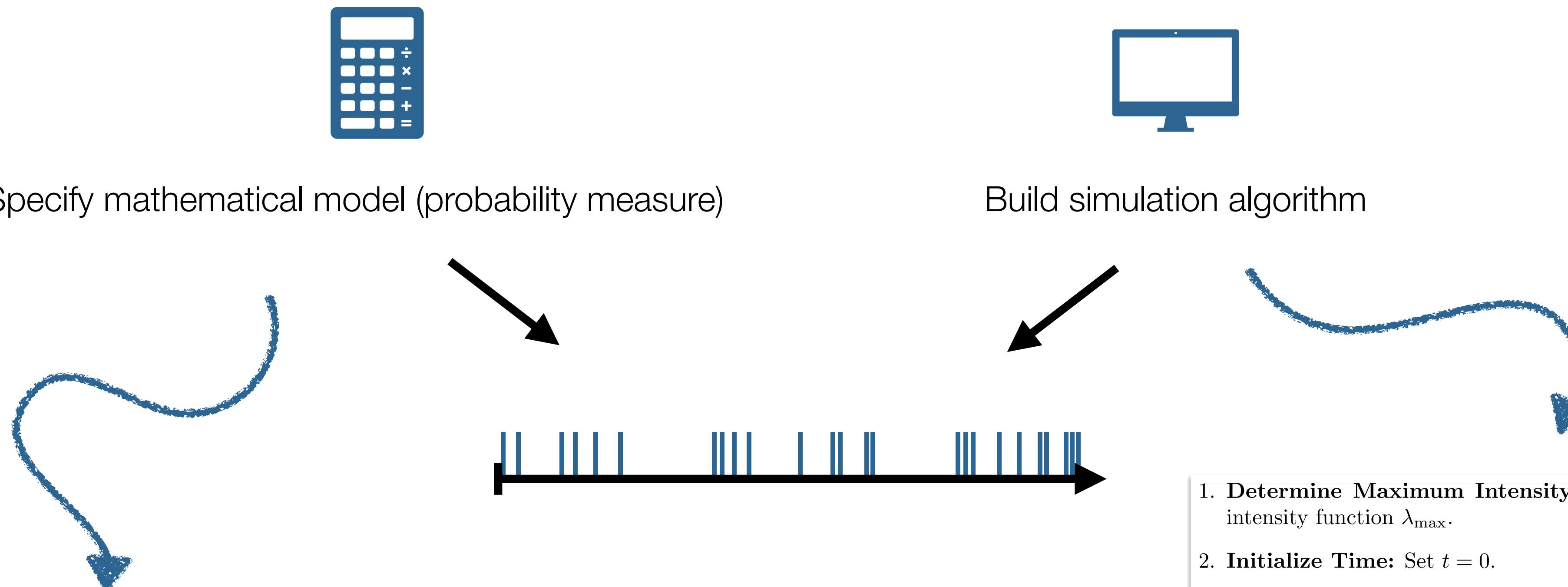
How do we specify TPPs?



$$f(H) = \left(\prod_{i=1}^n f^{\text{pred}}(t_{i+1} \mid H_{t_i}) \right) \left(1 - F^{\text{pred}}(T \mid H_{t_n}) \right) .$$

Models of TPPs

How do we specify TPPs?

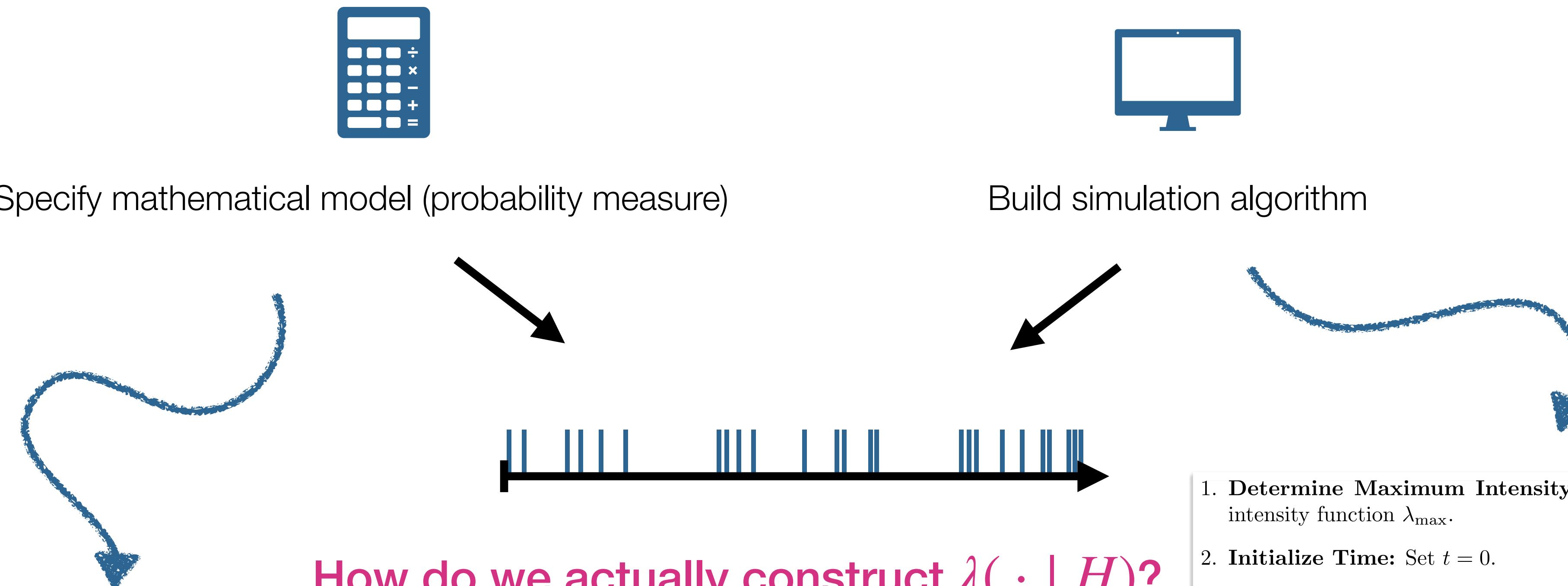


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Models of TPPs

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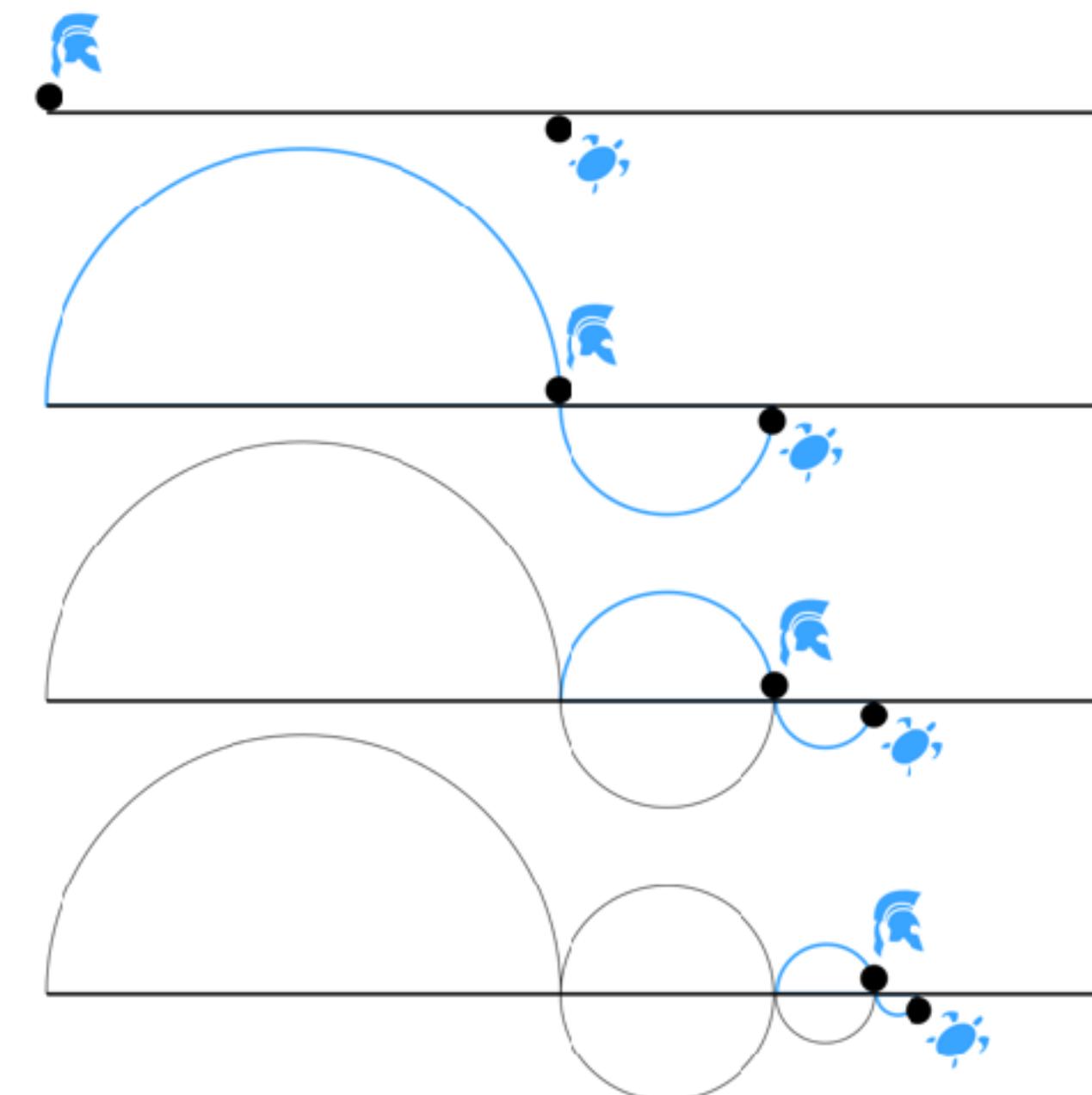
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Explosive Behavior

Can you specify a $\lambda(\cdot | H)$ such that the number of events converges to infinity in a finite amount of time?

- If H_t is empty: Converge to infinity within $[0, 0.5]$.
- If H_t is non-empty:
 - Define t_{\max} to be the maximum of H_t (the most recent event).
 - Let the intensity function convert to infinity within $(H_t - t_{\max})/2$.

Zeno behavior (Zeno of Elea) refers to the paradoxical situation where an infinite number of events occur in a finite amount of time.



Hawkes Processes

Q: How do we actually construct $\lambda(\cdot | H)$?

A: To create self-exciting dynamics: Hawkes processes

$$\lambda(t, H_t) = \mu + \sum_{i: t_i < t} \phi(t - t_i)$$

Background rate:
responsible for spontaneous events

Kernel:
The impact of each event decreases the further it is in the past.

Sum over all past events H_t

$\phi(t - t_i)$

$(t - t_i)$

Hawkes Processes

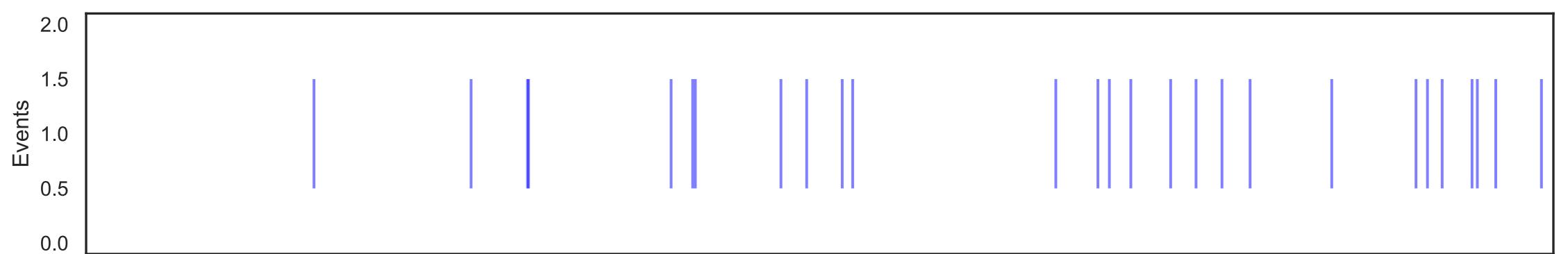
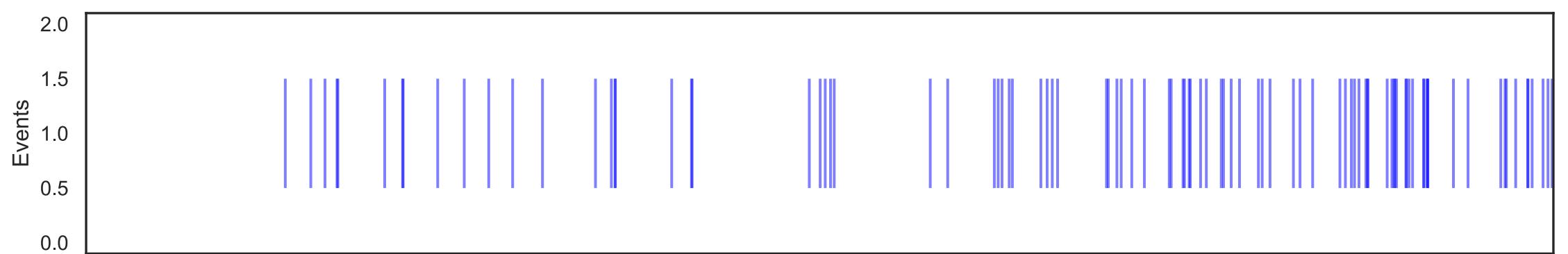
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$$\phi(t) = \alpha e^{-\beta t}$$

```
def hawkes_intensity(t, events, mu=0.25, alpha=0.7, beta=0.5):
    intensity = mu
    for t_i in events:
        if t_i < t:
            intensity += alpha * np.exp(-beta * (t - t_i))
    return intensity
```



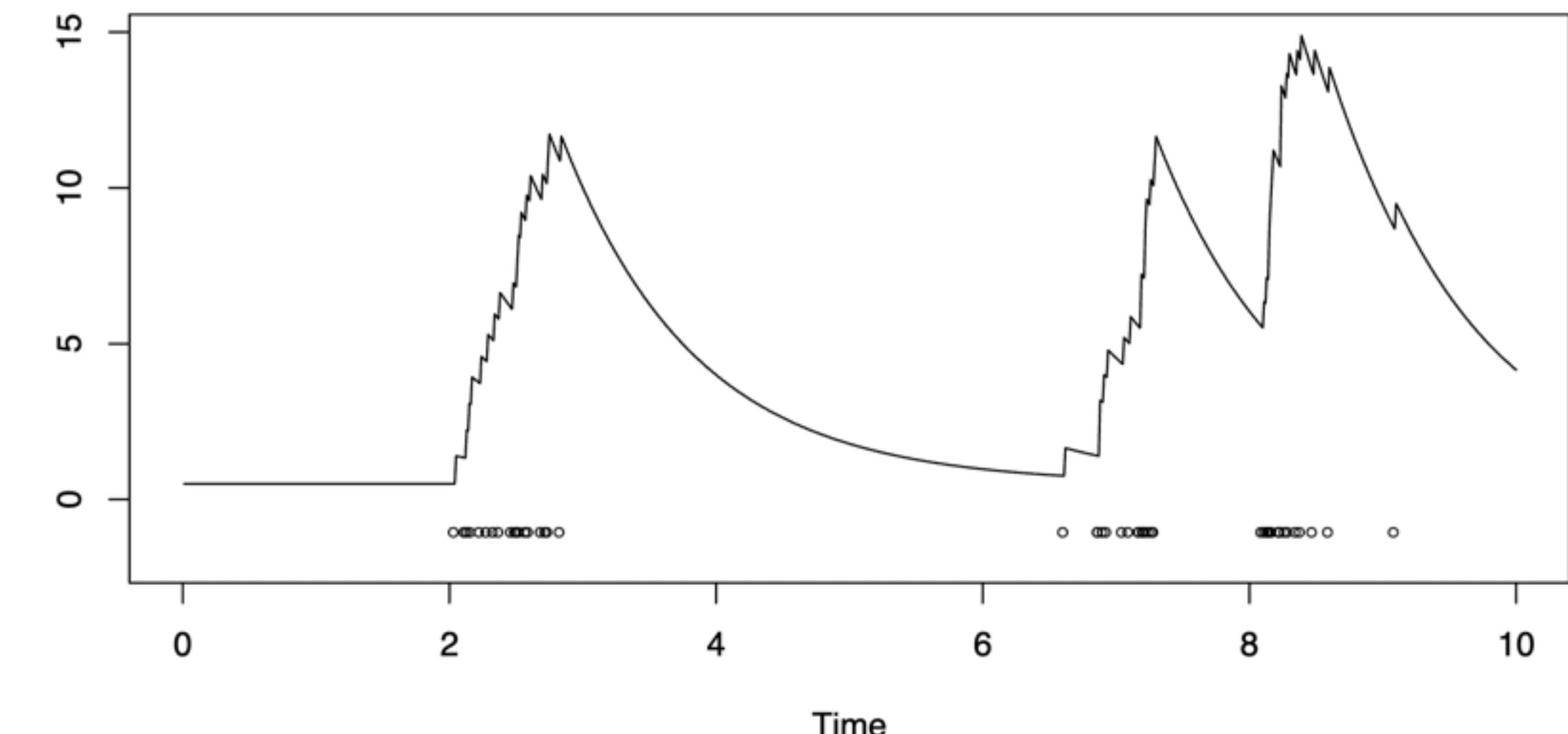
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```

Lecture Notes: Temporal Point Processes and the Conditional Intensity Function (Rasmussen et al.)

Self-Correcting Processes

Self-Exciting

$$\lambda(t, H_t) = \mu + \sum_{i:t_i < t} \phi(t - t_i)$$

Background rate:

Increases constantly.

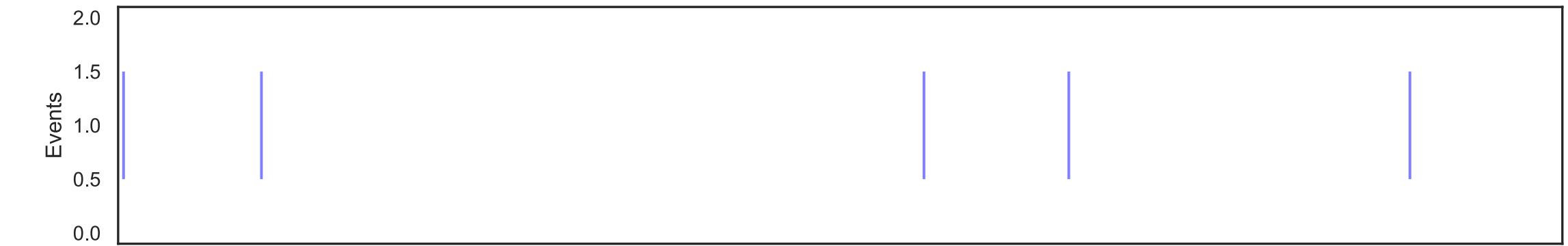
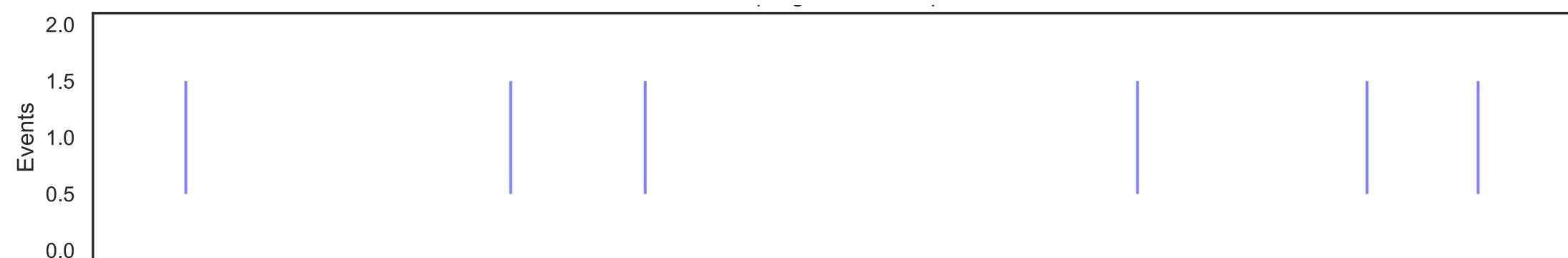
Self-Correcting

$$\lambda(t, H_t) = \exp \left(\mu t - \sum_{i:t_i < t} \alpha \right)$$

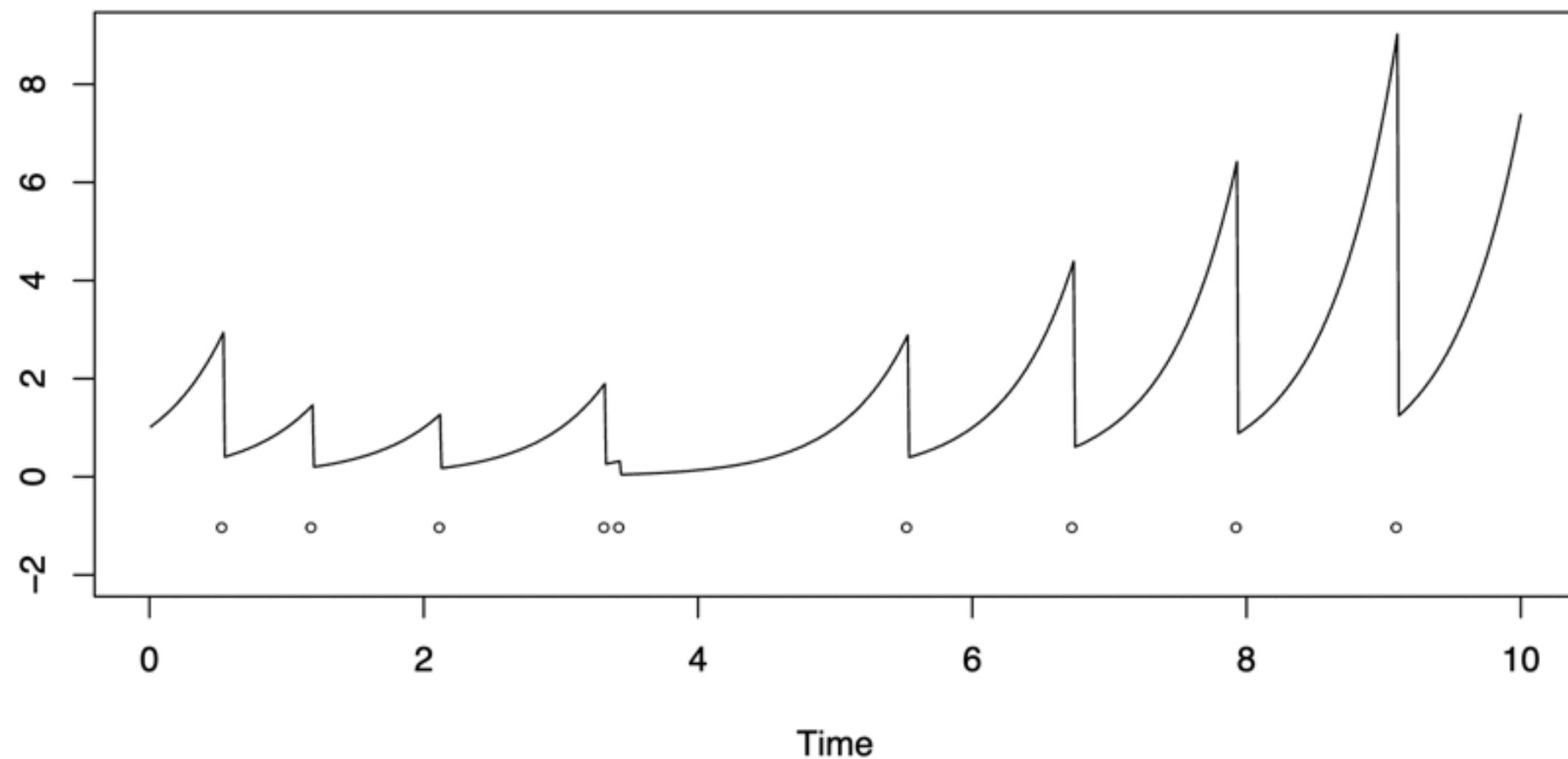


Damping:

Decreases intensity after each event.



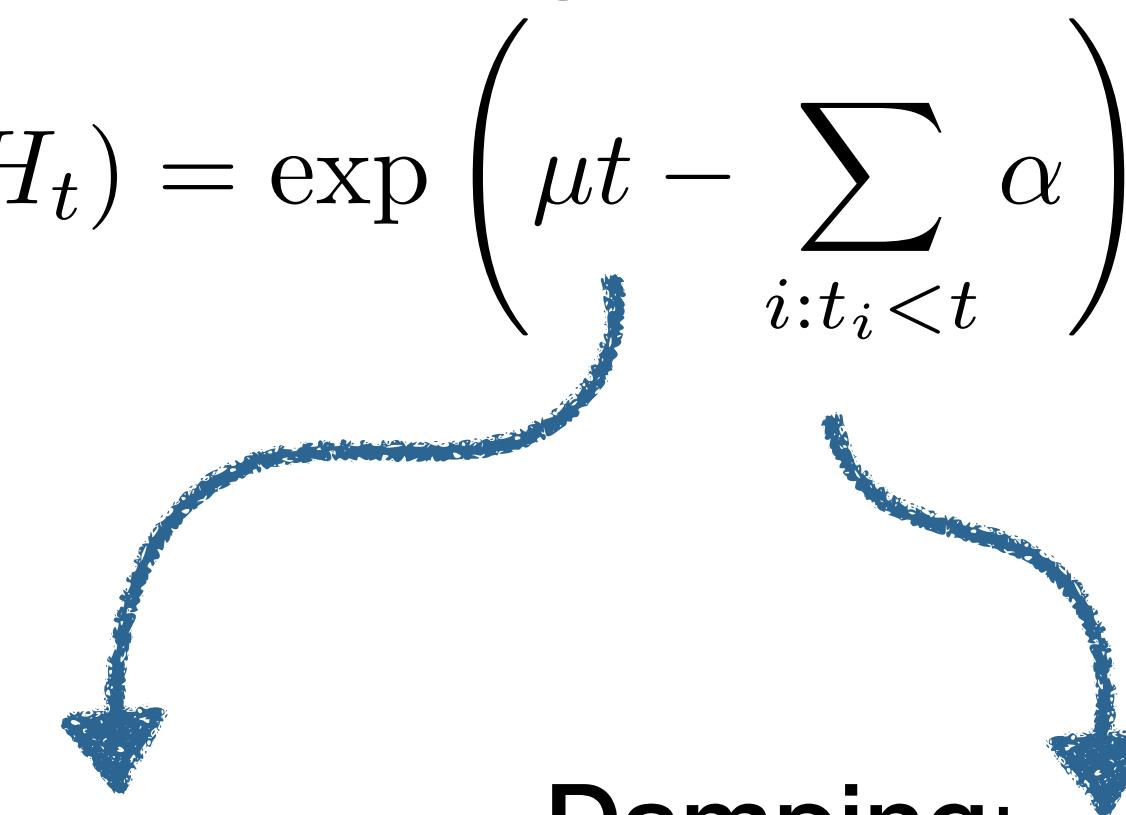
Self-Correcting Processes



Self-Correcting

$$\lambda(t, H_t) = \exp \left(\mu t - \sum_{i:t_i < t} \alpha \right)$$

Background rate:
Increases constantly.



Damping:
Decreases intensity after each event.

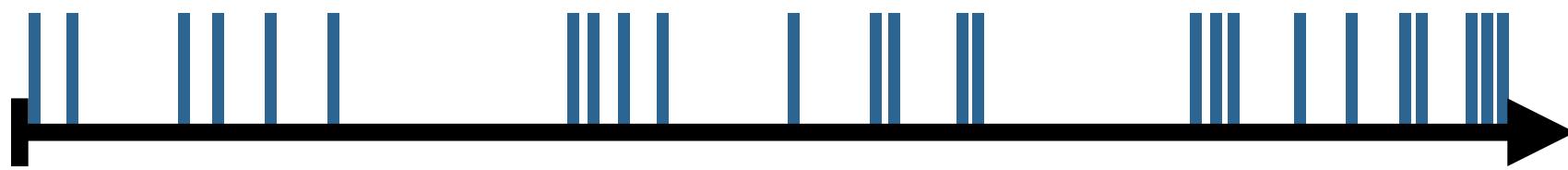
Maximum Likelihood

Self-Exciting

$$\lambda(t, H_t) = \mu + \sum_{i:t_i < t} \phi(t - t_i)$$

Self-Correcting

$$\lambda(t, H_t) = \exp\left(\mu t - \sum_{i:t_i < t} \alpha\right)$$



Assume you have a given event sequence and
want to infer μ, α, β , etc.

Closed form solution for homogeneous TPPs: $L(\lambda \mid H) = \lambda^n \exp(-\lambda T) \implies \hat{\lambda} = \frac{n}{T}$

The general likelihood can only be solved numerically: $L(\lambda(\cdot) \mid H) = (\prod_{i=1}^n \lambda(t_i \mid H_{t_i})) \exp\left(-\int_0^T \lambda(t \mid H_t) dt\right)$



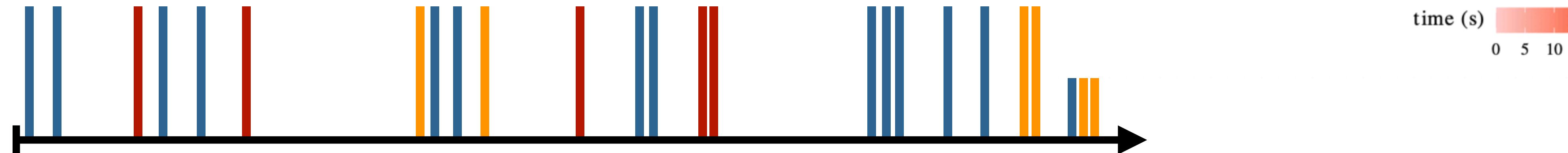
Part II

MARKED TEMPORAL POINT PROCESSES

Marked TPPs

- Each event time t_i is associated with a mark $m_i \in \mathcal{M}$.
- Event sequence: $H = [(t_0, m_0), (t_1, m_1), \dots, (t_n, m_n)]$.
- Intensity functions: $\{\lambda_m(t, H_t)\}_{m \in \mathcal{M}}$, where $\lambda_m(t, H_t)$ is the intensity for events with mark m , given history H_t .
- All intensity functions run in parallel to generate a sequence.

Three marks: **blue**, **red**, **yellow**:

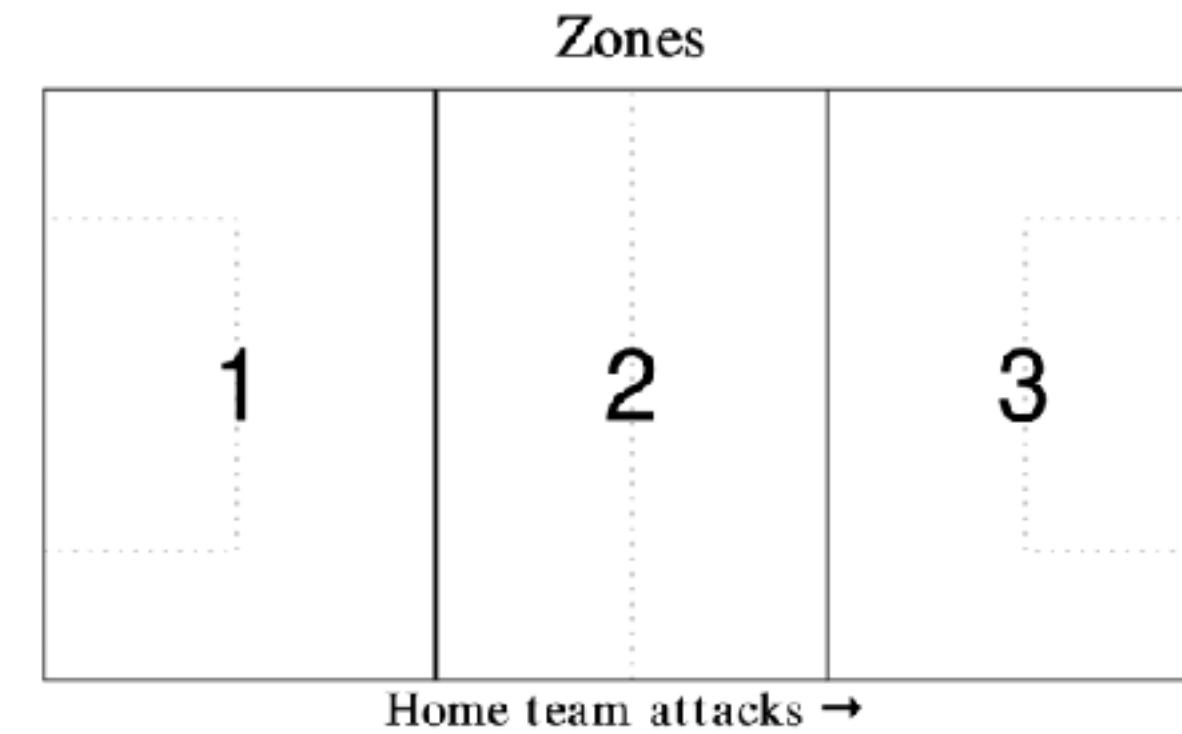
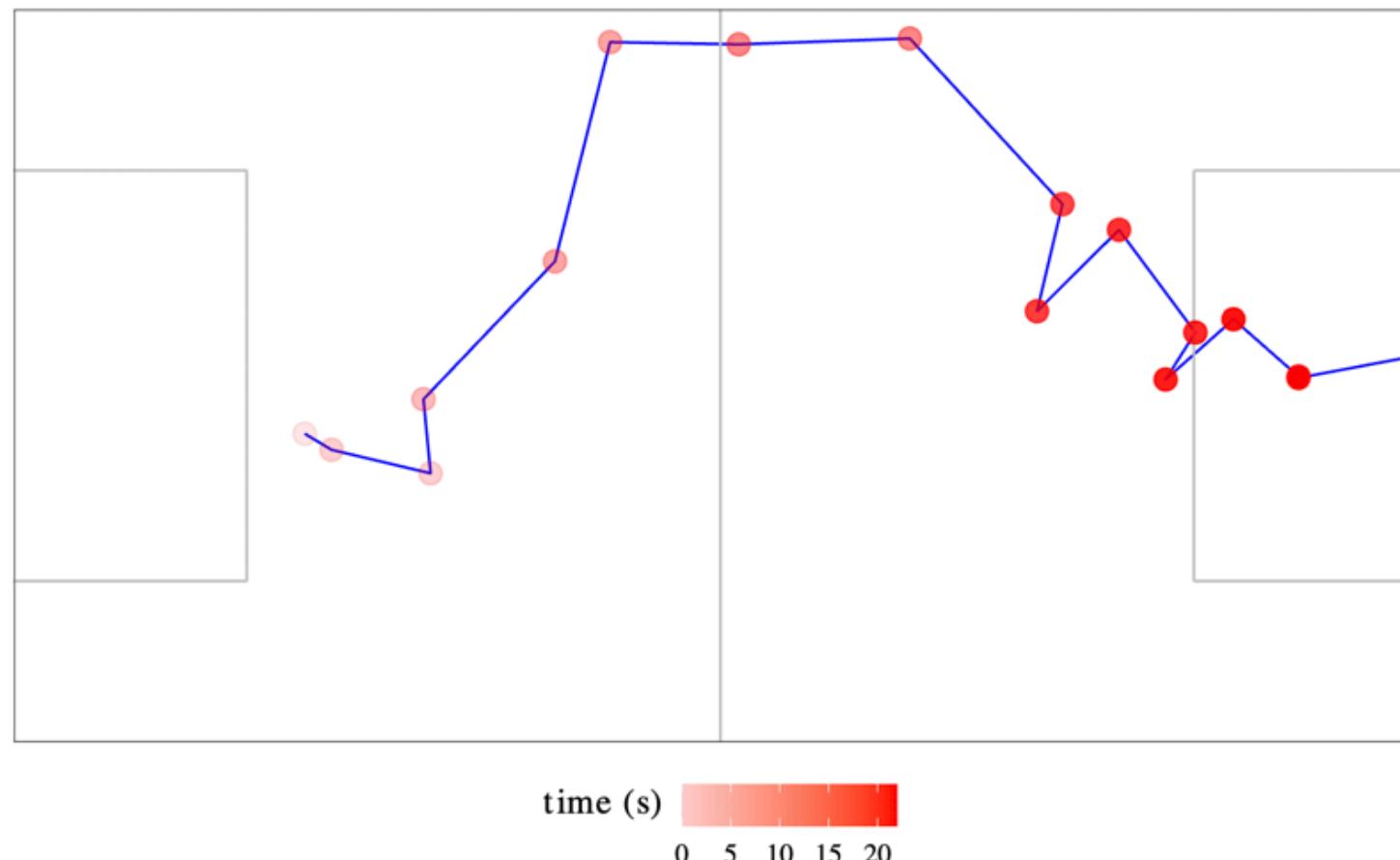


event type	frequency	event
Pass	376924	Saved
BallRecovery	36908	Save
Clearance	25462	Corne
Tackle	14581	Missee
TakeOn	13607	Offside
BallTouch	13517	Claim
Aerial	12871	Goal
Interception	10422	Punch
Dispossessed	8897	ShotO
Foul	8238	Smooth
KeeperPickup	5208	CrossL

Marked TPPs

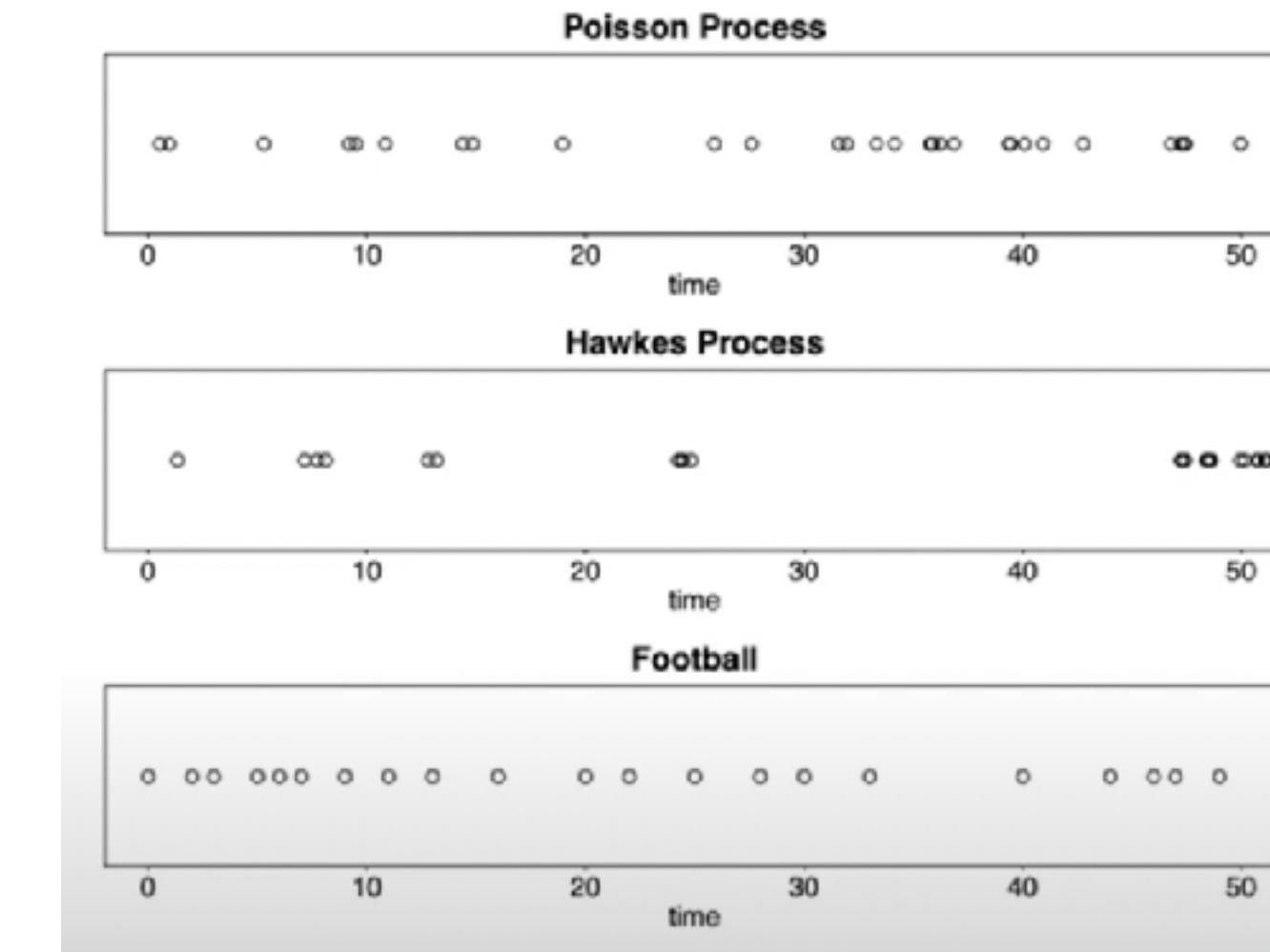
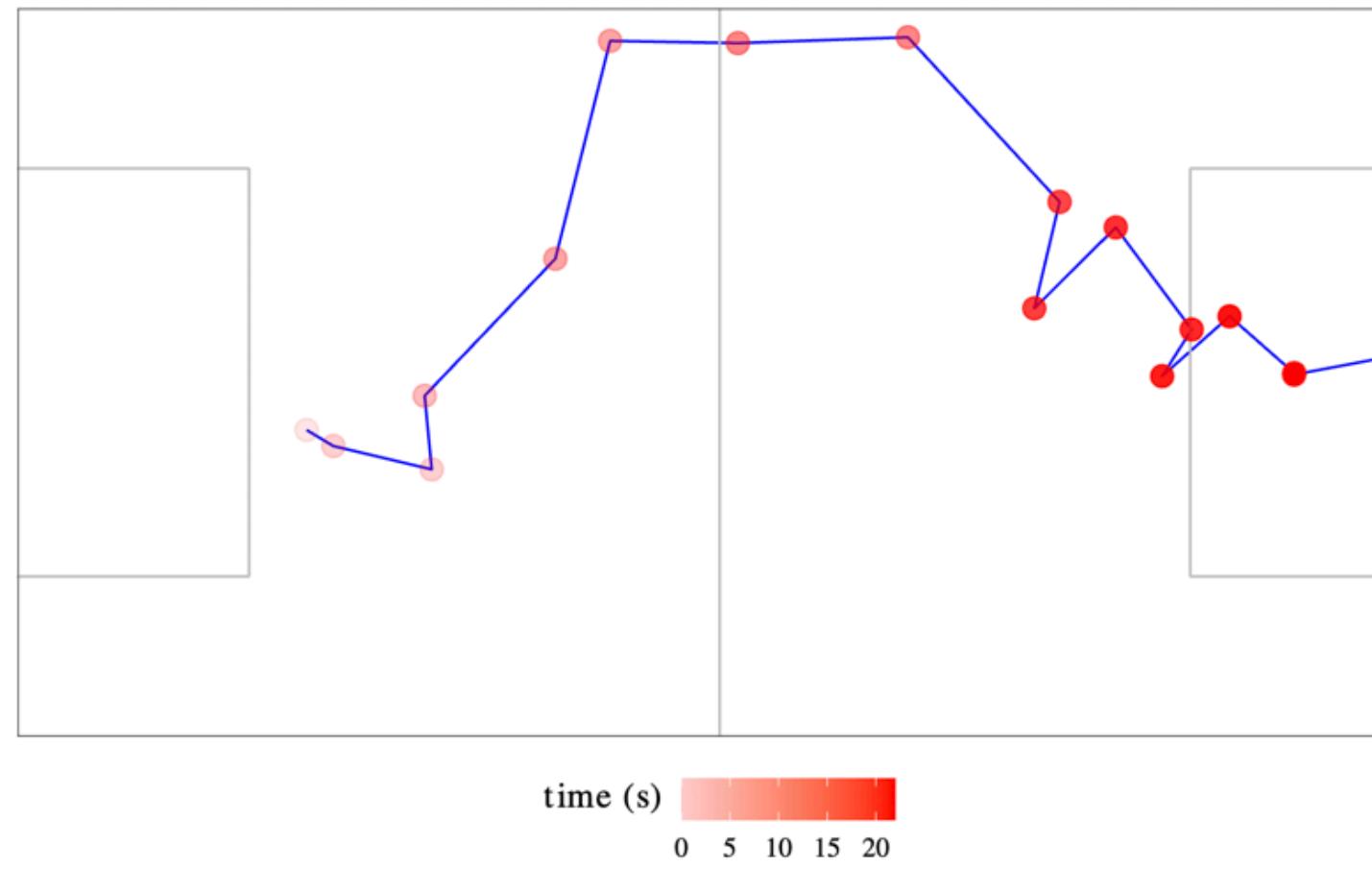
Table 2: Frequencies of the 22 distinct types of touch-ball events in the data.

event type	frequency	event type	frequency
Pass	376924	SavedShot	4971
BallRecovery	36908	Save	4910
Clearance	25462	CornerAwarded	4100
Tackle	14581	MissedShots	4076
TakeOn	13607	OffsidePass	1582
BallTouch	13517	Claim	1181
Aerial	12871	Goal	1052
Interception	10422	Punch	380
Dispossessed	8897	ShotOnPost	187
Foul	8238	Smother	122
KeeperPickup	5208	CrossNotClaimed	81

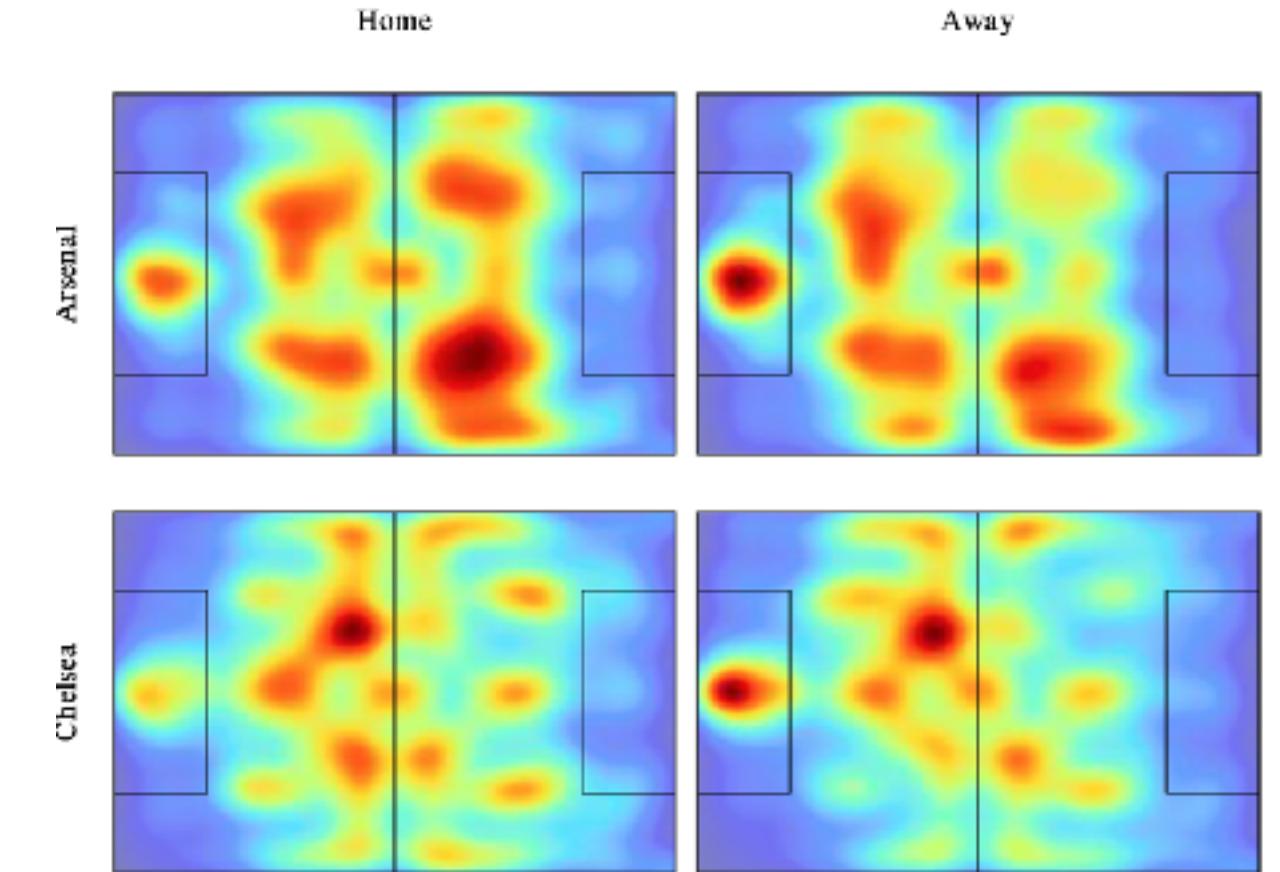


i	id	period	team_id	time (t_i)	zone (z_i)	mark (m_i)
1	101	1	1	0	2	18
2	101	1	1	1	2	19
3	101	1	2	3	1	8
4	101	1	1	6	3	16
5	101	1	1	8	3	18
6	101	1	1	15	2	18
7	101	1	1	16	1	19
8	101	1	2	19	1	12

Marked TPPs



[youtube.com/watch?v=TJTkj8Ym5d0&t=1639s&ab_channel=RoyalStatSoc](https://www.youtube.com/watch?v=TJTkj8Ym5d0&t=1639s&ab_channel=RoyalStatSoc)



Density of ball-touches is shifted to the right in their home games

- identify player and team strength
- predict event probabilities
- contribution of background process
- identify home advantage
- explore strategies



Part III

NEURAL TEMPORAL POINT PROCESSES

Neural Temporal Point Processes

Self-Exciting

$$\lambda(t, H_t) = \mu + \sum_{i:t_i < t} \phi(t - t_i)$$

Neural

?

Statistical models of TPPs only capture basic patterns. Neural network models can learn **arbitrary dynamics**.

Self-Correcting

$$\lambda(t, H_t) = \exp \left(\mu t - \sum_{i:t_i < t} \alpha \right)$$

Model **intensity** function with neural network:

- Numerical integration is expensive
- NN architecture needs to process event sequence of arbitrary length.

$$\lambda(t, H_t) = \text{NN}(t, H_t)$$

Model **predictive distribution** with neural network:

- Predict valid PDF
- PDF representation must support both efficient sampling and evaluation.
- NN architecture needs to process event sequence of arbitrary length.

$$f^{\text{pred}}(t, H_t) = \text{NN}(t, H_t)$$

Neural Temporal Point Processes

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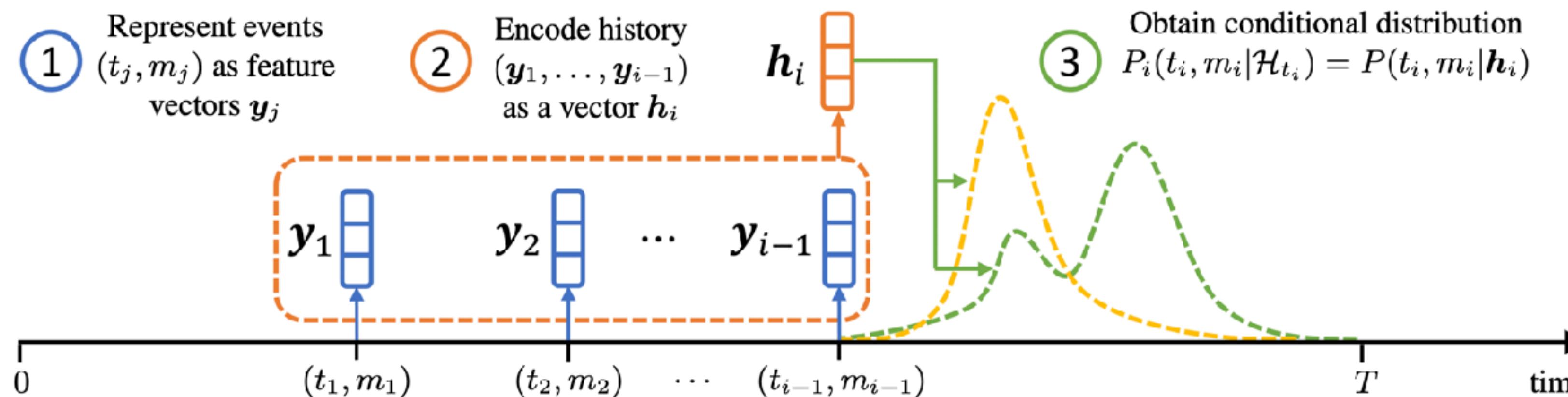
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$$f^{\text{pred}}(t, H_t) = \text{NN}(t, H_t)$$

Neural Temporal Point Processes



Neural Temporal Point Processes: A Review (Shchur et al.)

1. Use **MLP** to map event (t_i, m_i) to feature.
2. Use **RNN** to aggregate all event features into single history feature \mathbf{h}_i .
3. Use **MLP** to map \mathbf{h}_i to parameters θ of a parametric PDF $f_\theta(t)$.
 - Predict λ, k of a Weibull distribution
 - Predict α, d, p of a generalized gamma distribution

Neural Temporal Point Processes

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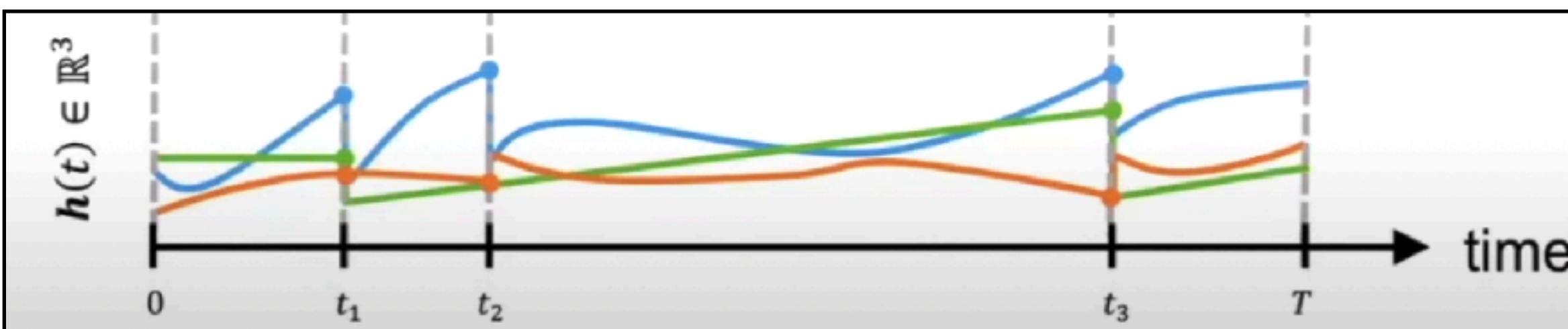
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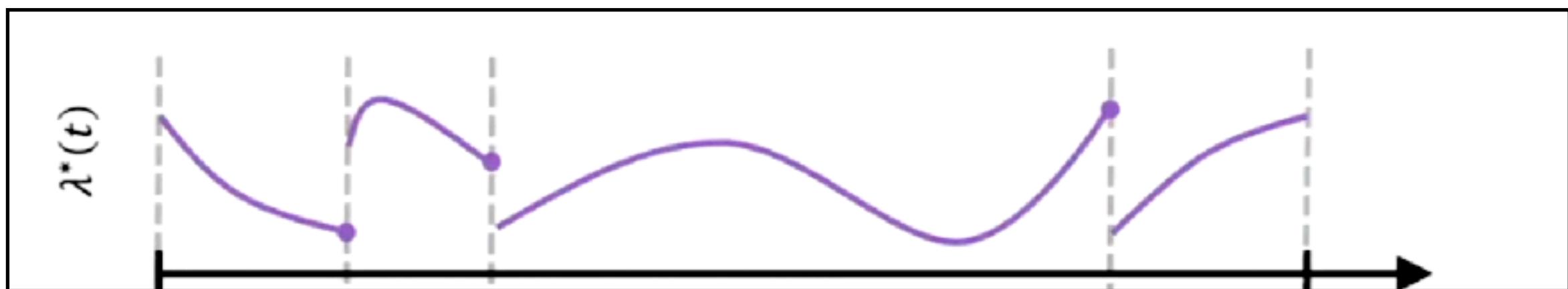
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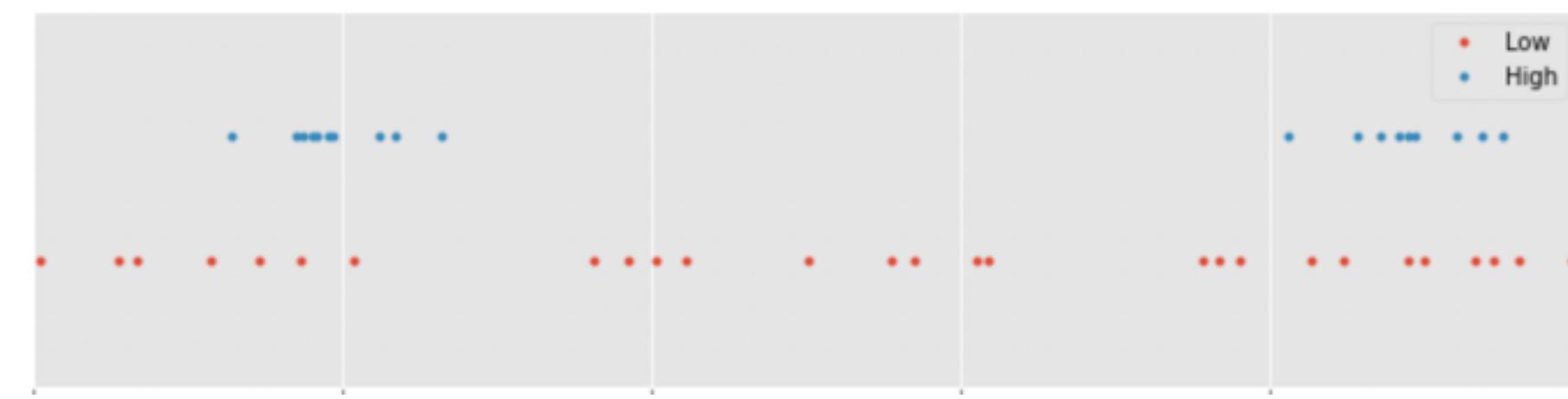
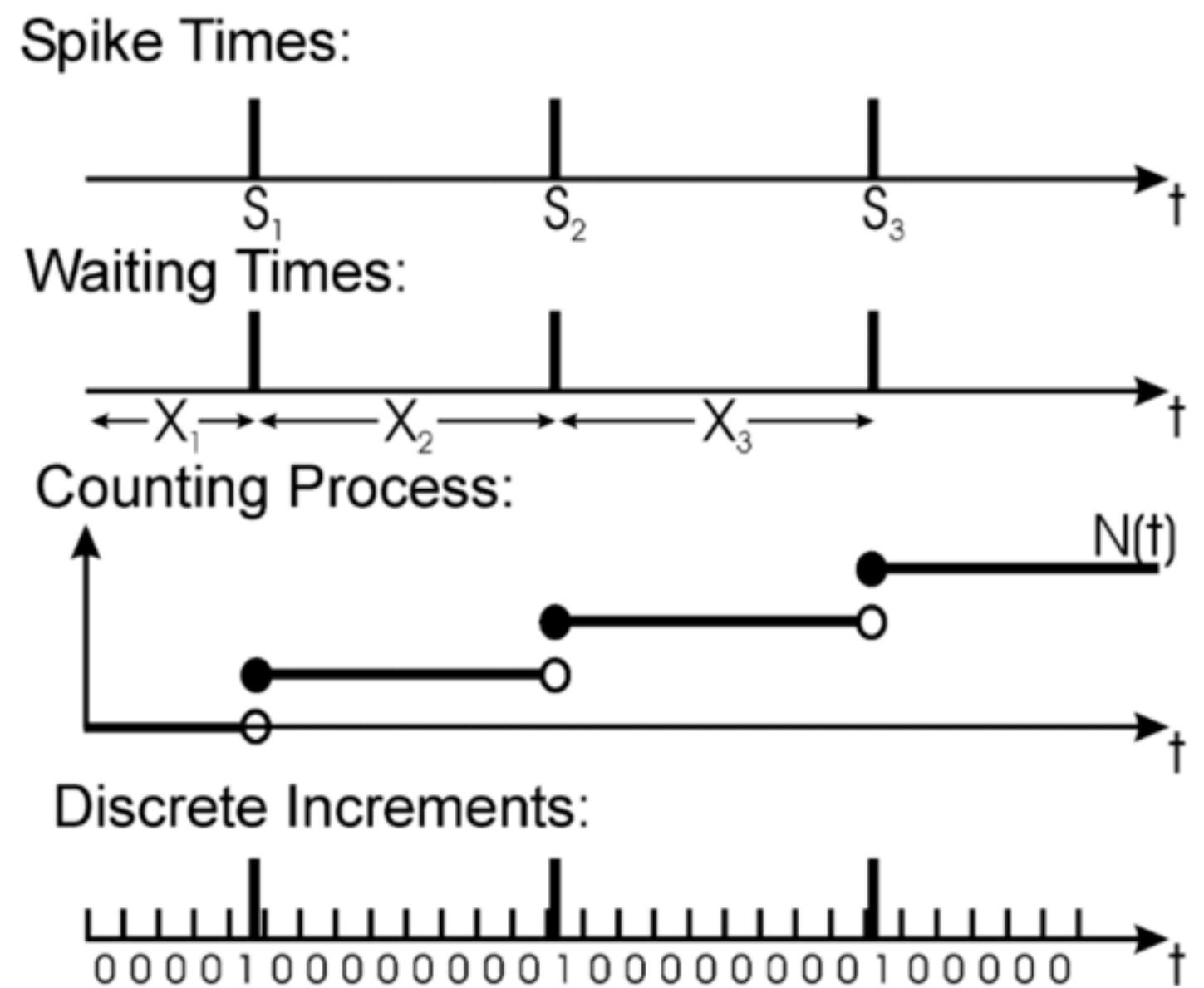
Neural Temporal Point Processes



1. Learn the latent feature $h(t)$ for each point in time.
Here: $h(t) \in \mathbb{R}^3$.
2. An **ODE** specifies how $h(t)$ evolves when no event is happening.
3. An **MLP** specifies how $h(t)$ jumps when an event occurs.
4. Another **MLP** converts $h(t)$ to the intensity value $\lambda(t, H_t)$.



Message Passing



<https://mark-kramer.github.io/Case-Studies-Python/08.html>

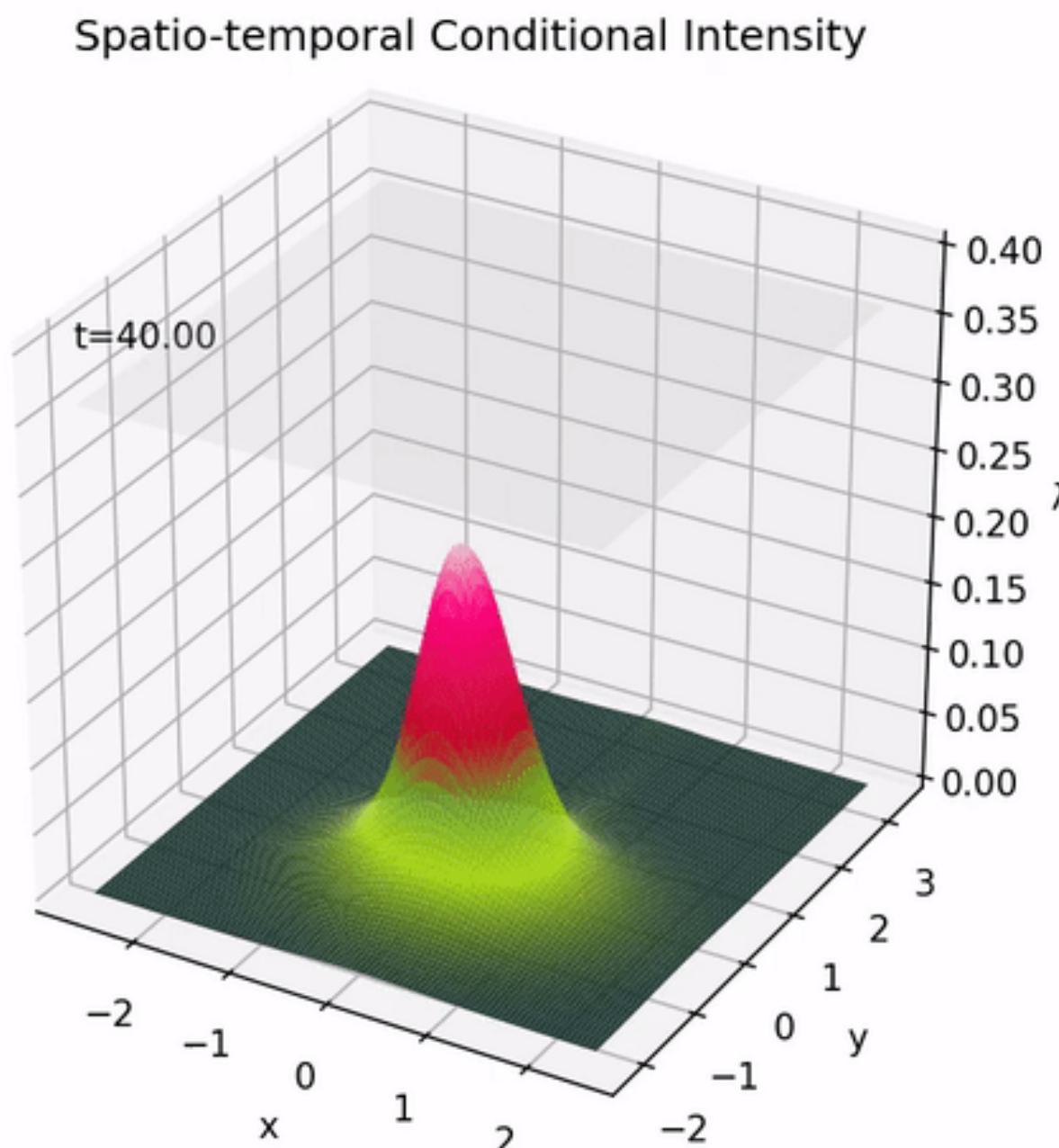
Figure 5. Multiple specifications for point process data.



Part IV

SPATIO-TEMPORAL POINT PROCESSES

STPP



- $H = [(t_1, s_1), (t_2, s_2), (t_3, s_3), \dots]$
- t_i denotes the event time and $s_i \in \mathcal{S} \subset \mathbb{R}^2$ the location.
- E.g., $H = [(1.3, (0.3, 0.8)), (4.2, (0.6, 0.2)), \dots]$

How does the intensity function look like?

Reminder **TPP**

$$\lambda(t, H_t) = \lim_{\Delta t \rightarrow 0} \frac{P(\text{Event in } [t, t + \Delta t) \mid H_t)}{\Delta t}$$

STPP

- $H = [(t_1, s_1), (t_2, s_2), (t_3, s_3), \dots]$
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Reminder **TPP**

$$\lambda(t, H_t) = \lim_{\Delta t \rightarrow 0} \frac{P(\text{Event in } [t, t + \Delta t) \mid H_t)}{\Delta t} \quad \lambda(s, t \mid H_t) := \lim_{\Delta s \rightarrow 0, \Delta t \rightarrow 0}$$

STPP

- $H = [(t_1, s_1), (t_2, s_2), (t_3, s_3), \dots]$
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Reminder **TPP**

$$\lambda(t, H_t) = \lim_{\Delta t \rightarrow 0} \frac{P(\text{Event in } [t, t + \Delta t) \mid H_t)}{\Delta t} \quad \lambda(s, t \mid H_t) := \lim_{\Delta s \rightarrow 0, \Delta t \rightarrow 0} \frac{P(\text{Event in } B(s, \Delta s) \times [t, t + \Delta t) \mid H_t)}{|B(s, \Delta s)| \Delta t} .$$

STPP

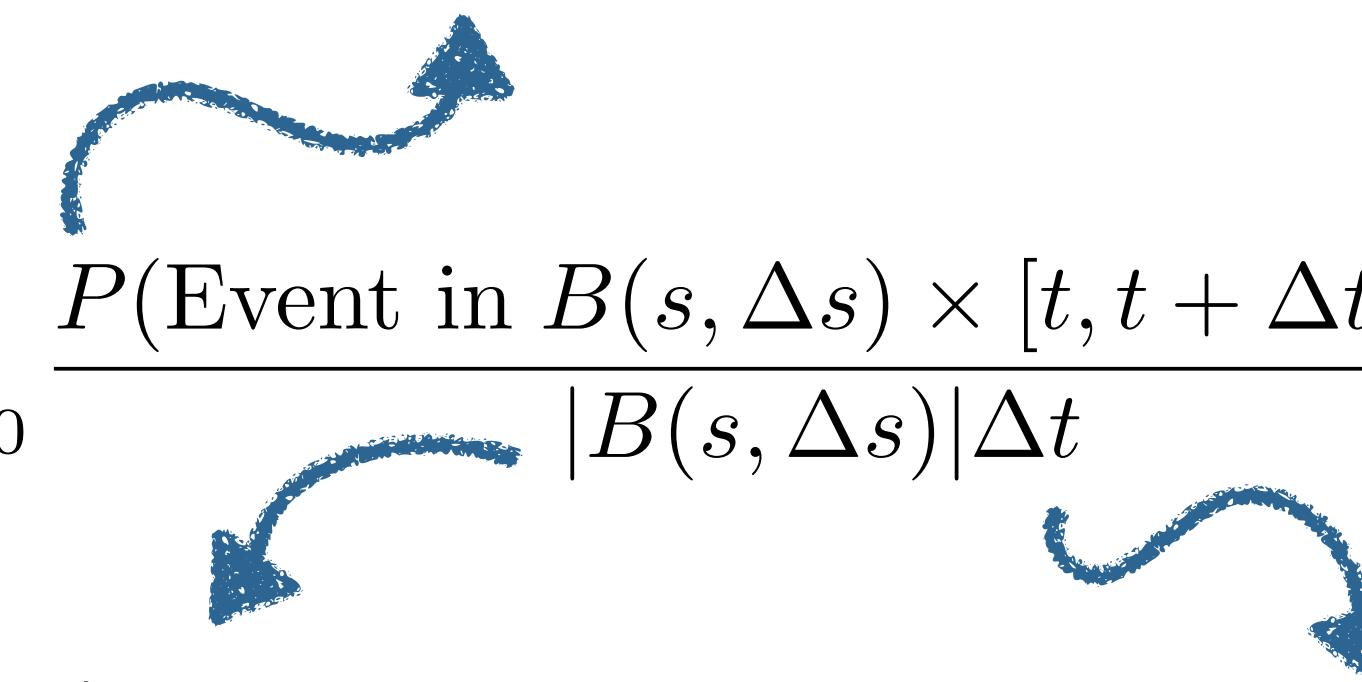
- $H = [(t_1, s_1), (t_2, s_2), (t_3, s_3), \dots]$
- t_i denotes the event time and $s_i \in \mathcal{S} \subset \mathbb{R}^2$ the location.
- E.g., $H = [(1.3, (0.3, 0.8)), (4.2, (0.6, 0.2)), \dots]$

Probability of an event in a very small unit of space and time

Reminder TPP

$$\lambda(t, H_t) = \lim_{\Delta t \rightarrow 0} \frac{P(\text{Event in } [t, t + \Delta t) \mid H_t)}{\Delta t}$$

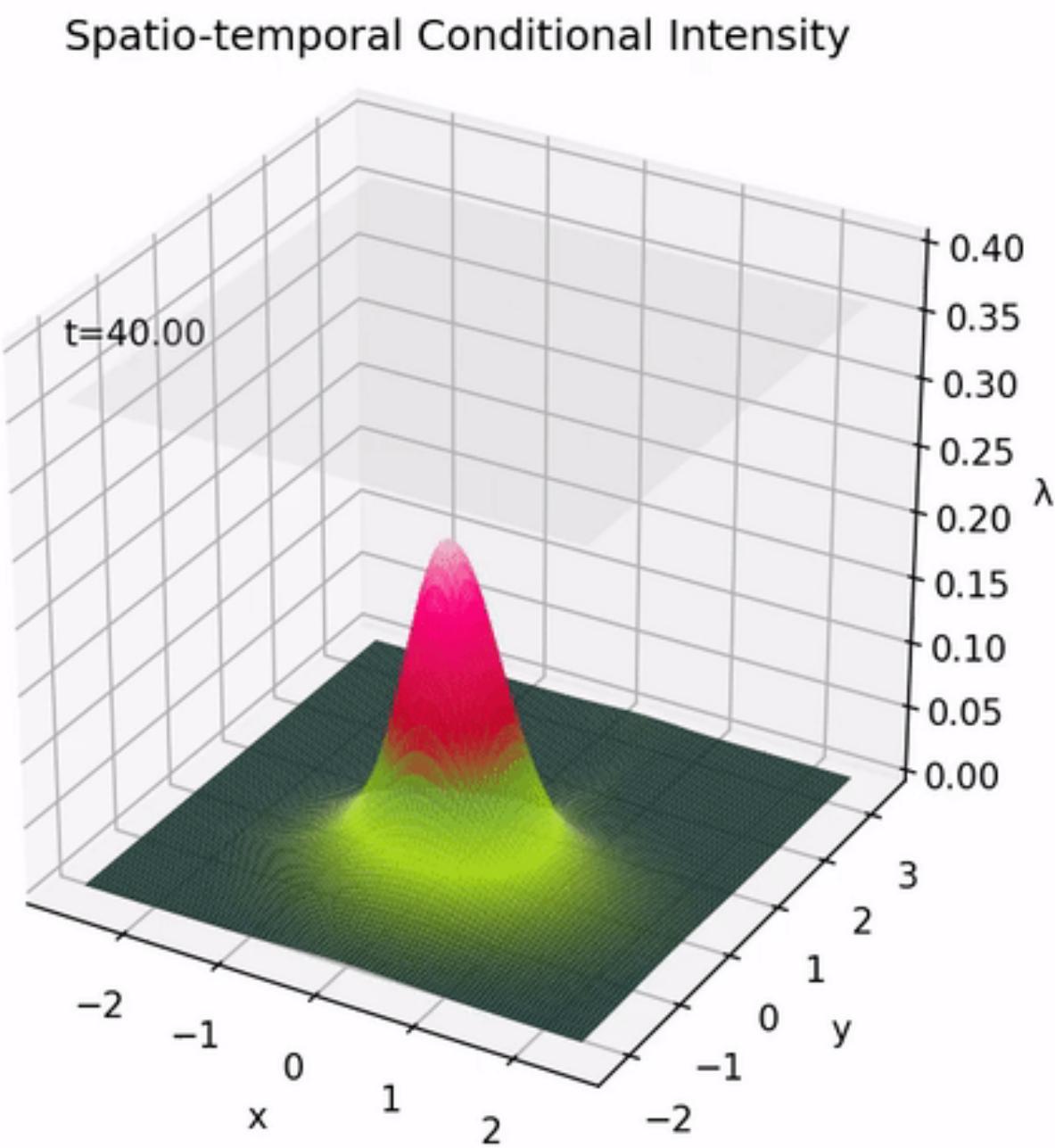
$$\lambda(s, t \mid H_t) := \lim_{\Delta s \rightarrow 0, \Delta t \rightarrow 0} \frac{P(\text{Event in } B(s, \Delta s) \times [t, t + \Delta t) \mid H_t)}{|B(s, \Delta s)| \Delta t} .$$



A 2D-ball (disk) around s with radius Δs

Very small time interval

STPP



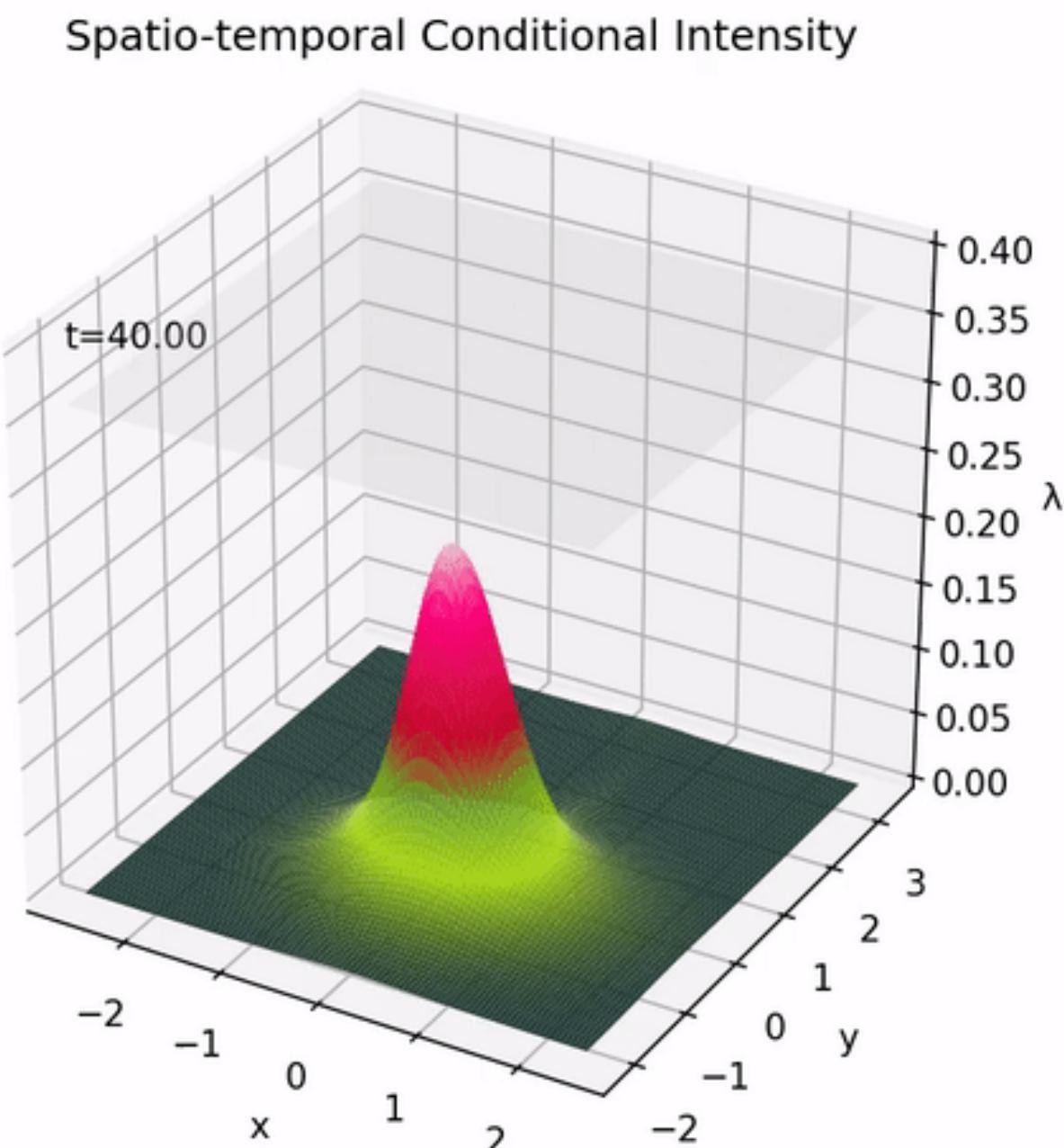
- $H = [(t_1, s_1), (t_2, s_2), (t_3, s_3), \dots]$
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- E.g., $H = [(1.3, (0.3, 0.8)), (4.2, (0.6, 0.2)), \dots]$

How do we specify an intensity?

Reminder TPP

$$\lambda(t, H_t) = \mu + \sum_{i: t_i < t} \phi(t - t_i)$$

STPP



- $H = [(t_1, s_1), (t_2, s_2), (t_3, s_3), \dots]$
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- E.g., $H = [(1.3, (0.3, 0.8)), (4.2, (0.6, 0.2)), \dots]$

How do we specify an intensity?

$$\lambda(s, t, H_t) = \mu g_0(s) + \sum_{(t_i, s_i) \in H_t} g_1(t, t_i) g_2(s, s_i)$$

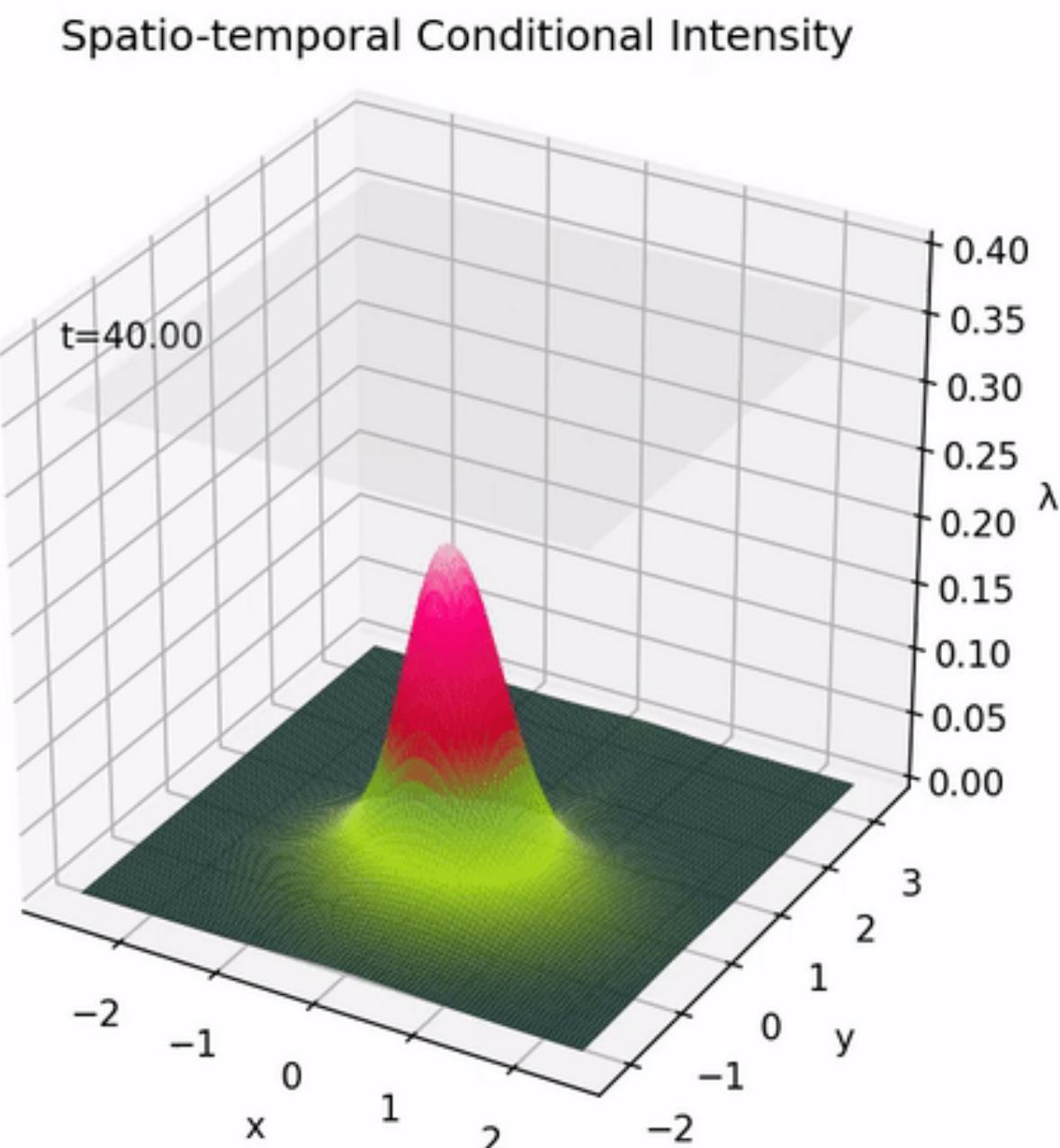
with

$$g_1(t, t_i) = \alpha \exp(-\beta(t - t_i))$$

and

$g_2(s, s_i)$ being a Normal distribution.

STPP



- $H = [(t_1, s_1), (t_2, s_2), (t_3, s_3), \dots]$
- t_i denotes the event time and $s_i \in \mathcal{S} \subset \mathbb{R}^2$ the location.
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- $\lambda(s, t, H_t) = \mu g_0(s) + \sum_{(t_i, s_i) \in H_t} g_1(t, t_i)g_2(s, s_i)$

How do we simulate STPPs?

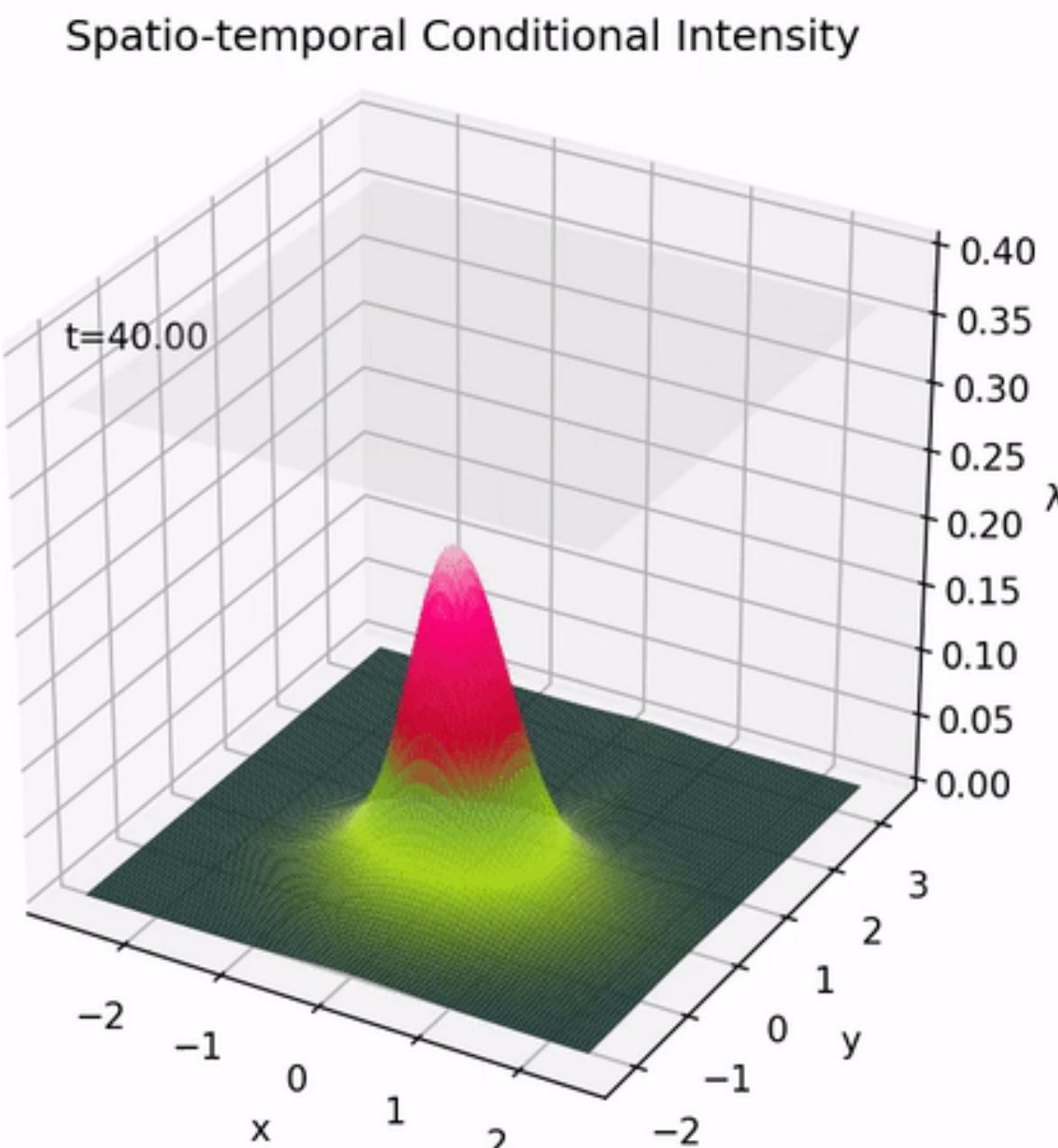
Reminder TPP

Initialize $H = []$

Until horizon is reached:

- Sample next event time t_i using $\lambda(\cdot | H)$ ($t_i > \max(H)$)
- Add t_i to H

STPP



- $H = [(t_1, s_1), (t_2, s_2), (t_3, s_3), \dots]$
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- $\lambda(s, t, H_t) = \mu g_0(s) + \sum_{(t_i, s_i) \in H_t} g_1(t, t_i)g_2(s, s_i)$

How do we simulate STPPs?

Reminder TPP

But how?

Initialize $H = []$

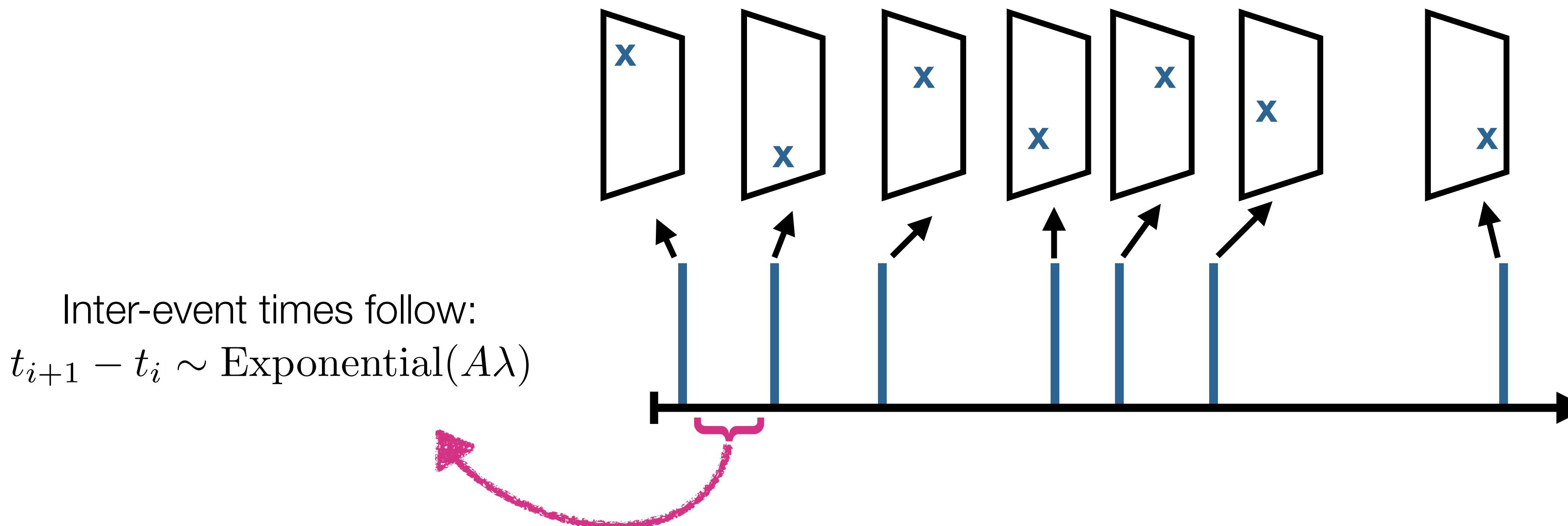
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Simulating Homogeneous STPPs

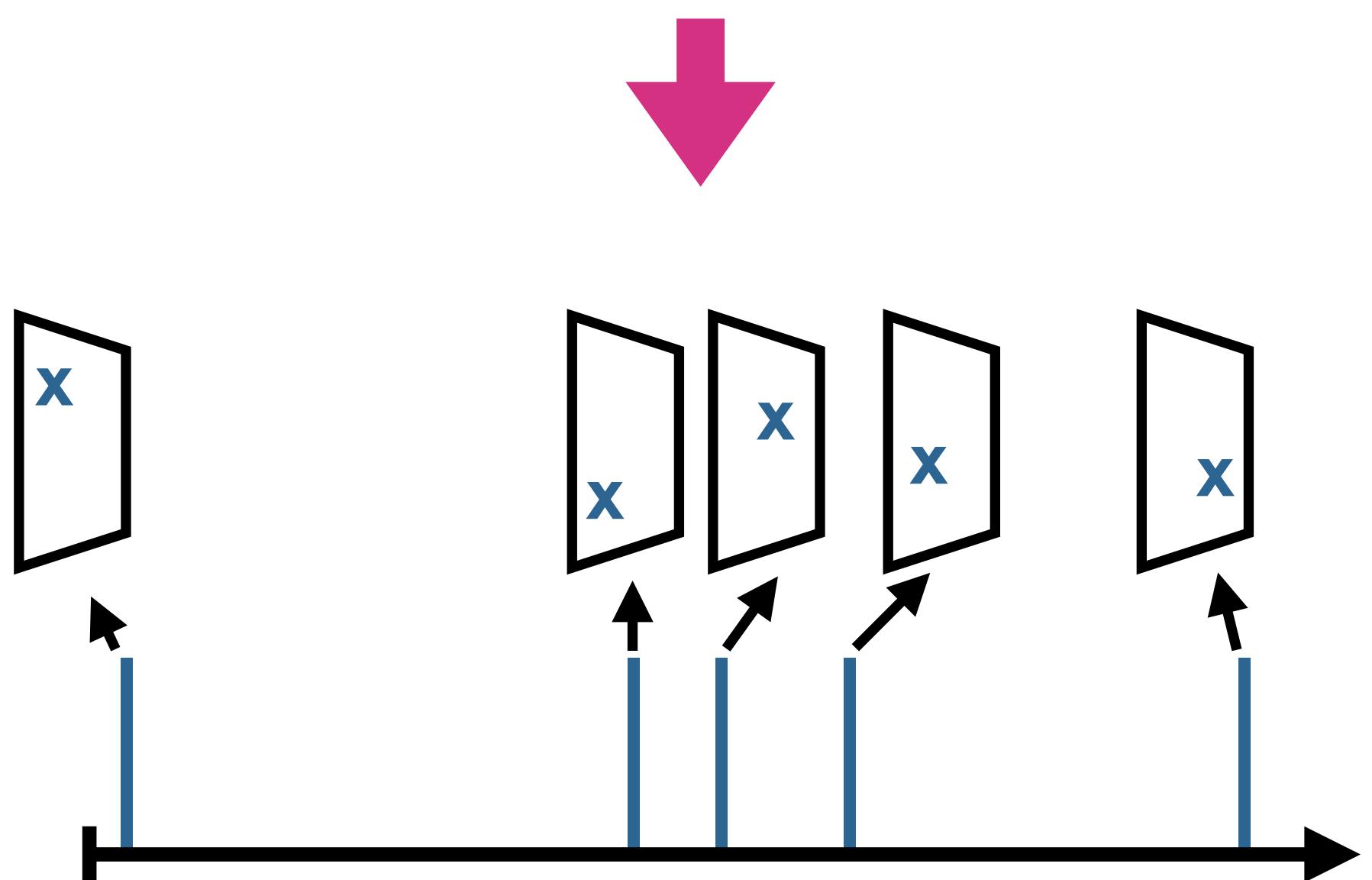
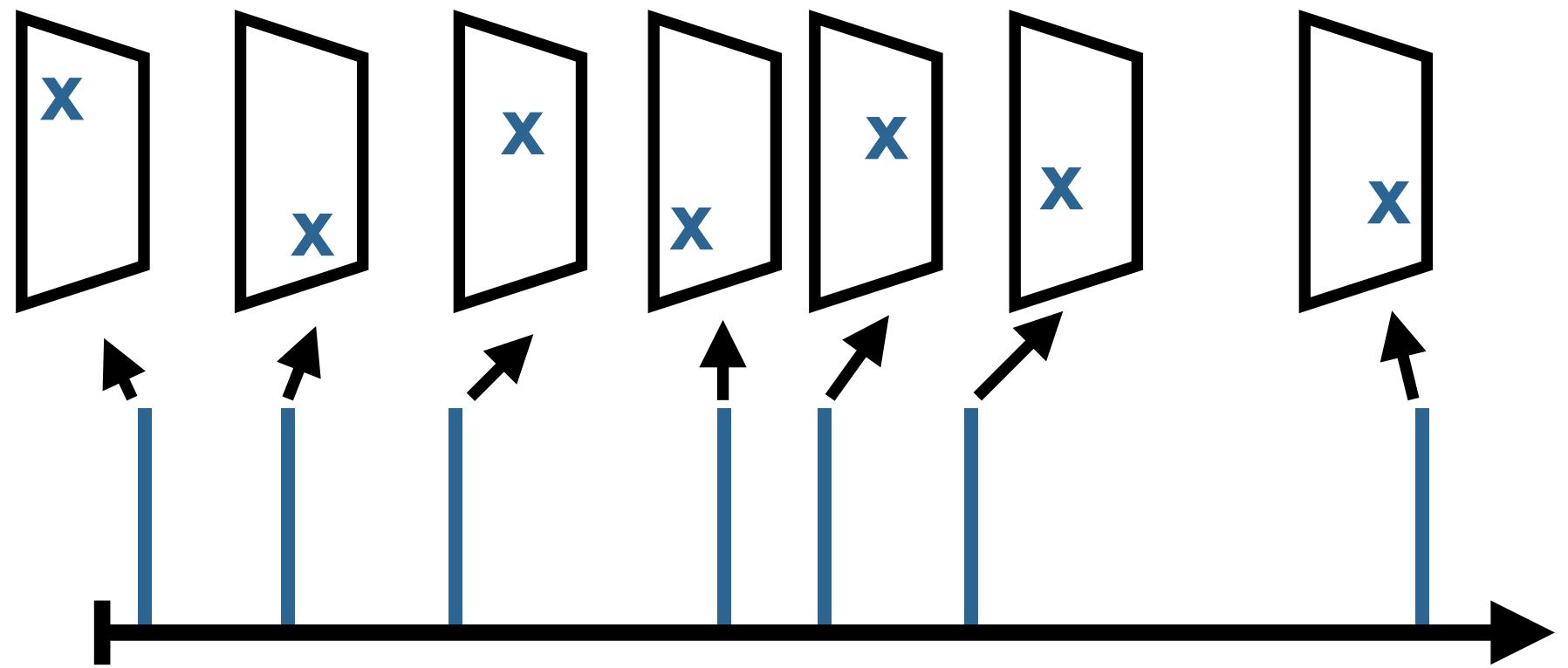
Homogeneous STPP Assume we want to simulate on a space $S = [0, 1] \times [0, 1]$ with area $A = 1$ from time $t = 0$ to 1

1. Simulate event times with rate $\lambda \cdot A$, where λ is the rate and A is the area of the space (typically also 1).
2. For each event time, place the event uniformly over the space.

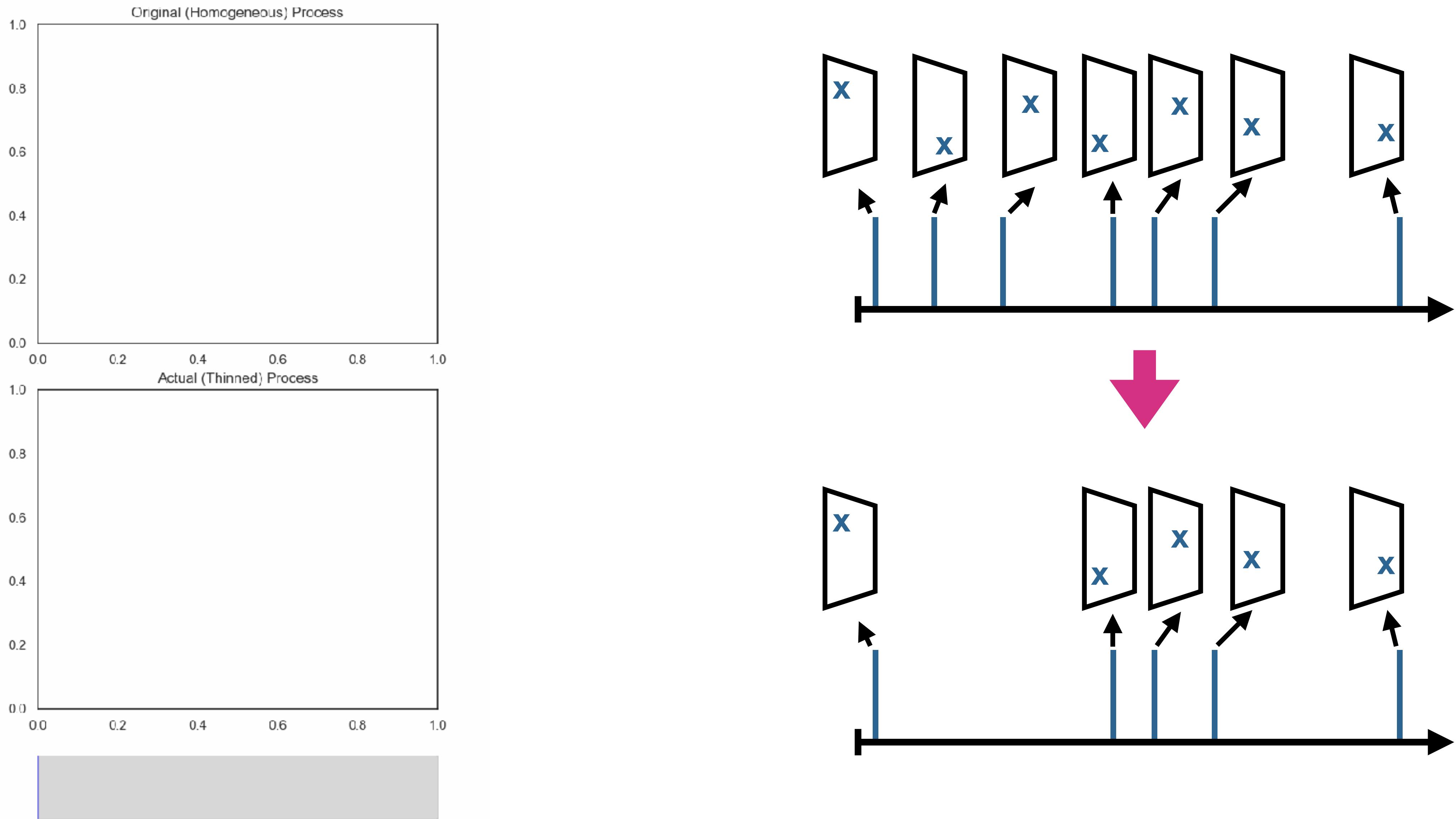


Simulating Heterogeneous STPPs

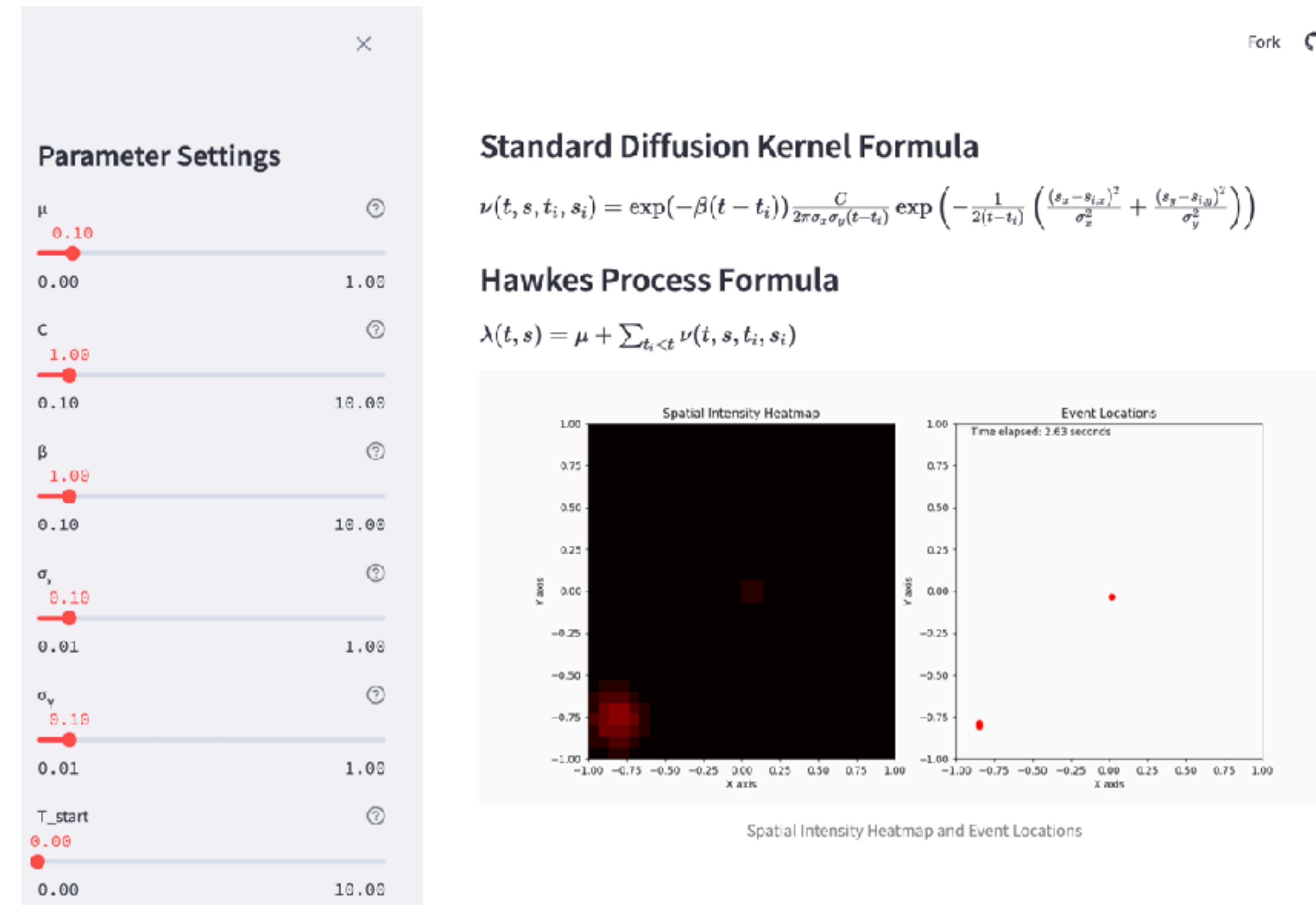
1. Generate a homogeneous STPP with a rate λ_{\max} that upper bounds the actual rate.
2. For each event (from left to right):
 - (a) Compute the actual intensity $\lambda(s, t | H_t)$.
 - (b) Reject the event with probability $1 - \frac{\lambda(s, t | H_t)}{\lambda_{\max}}$.



Simulating Heterogeneous STPPs

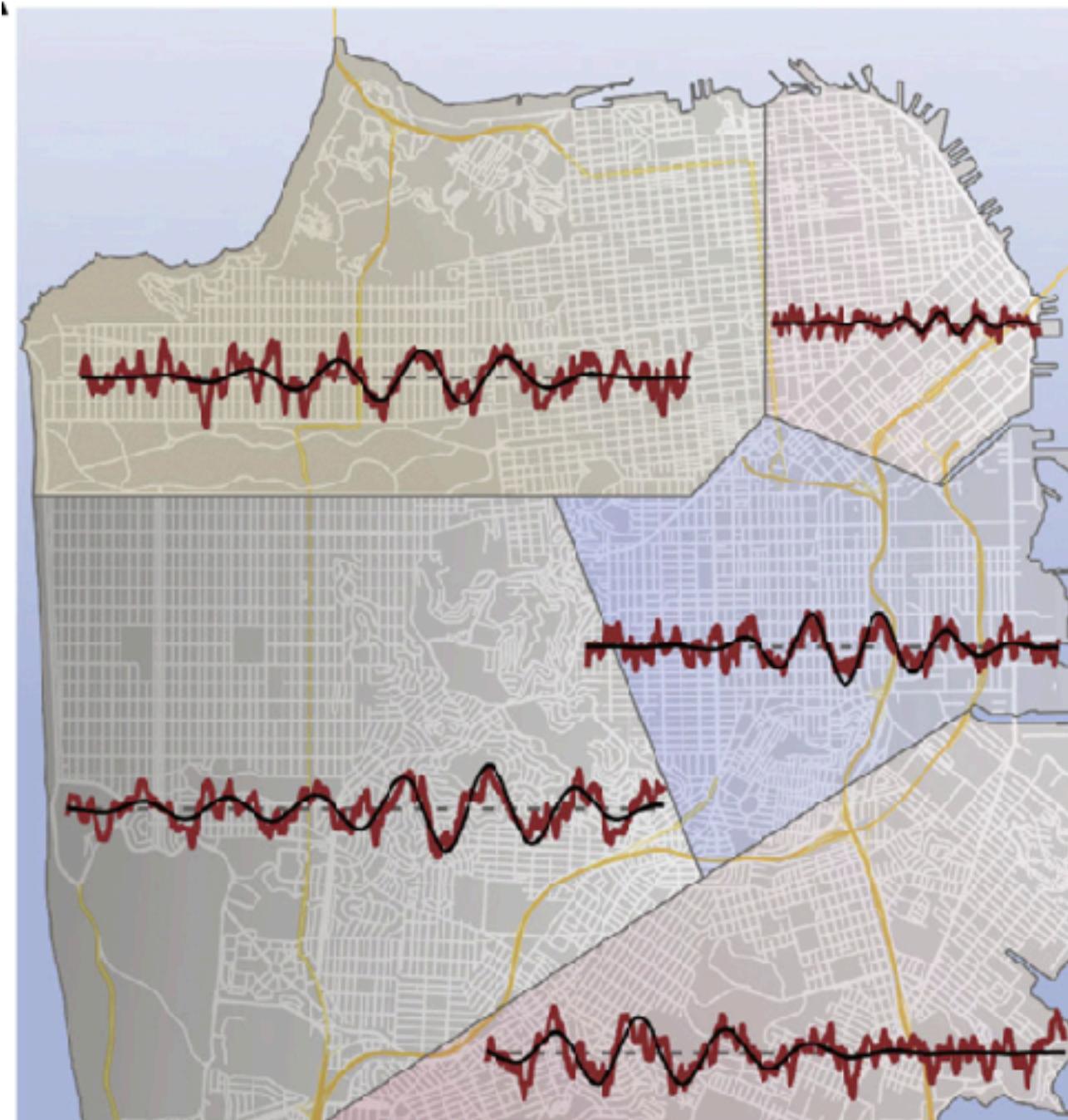


Simulating Heterogeneous STPPs

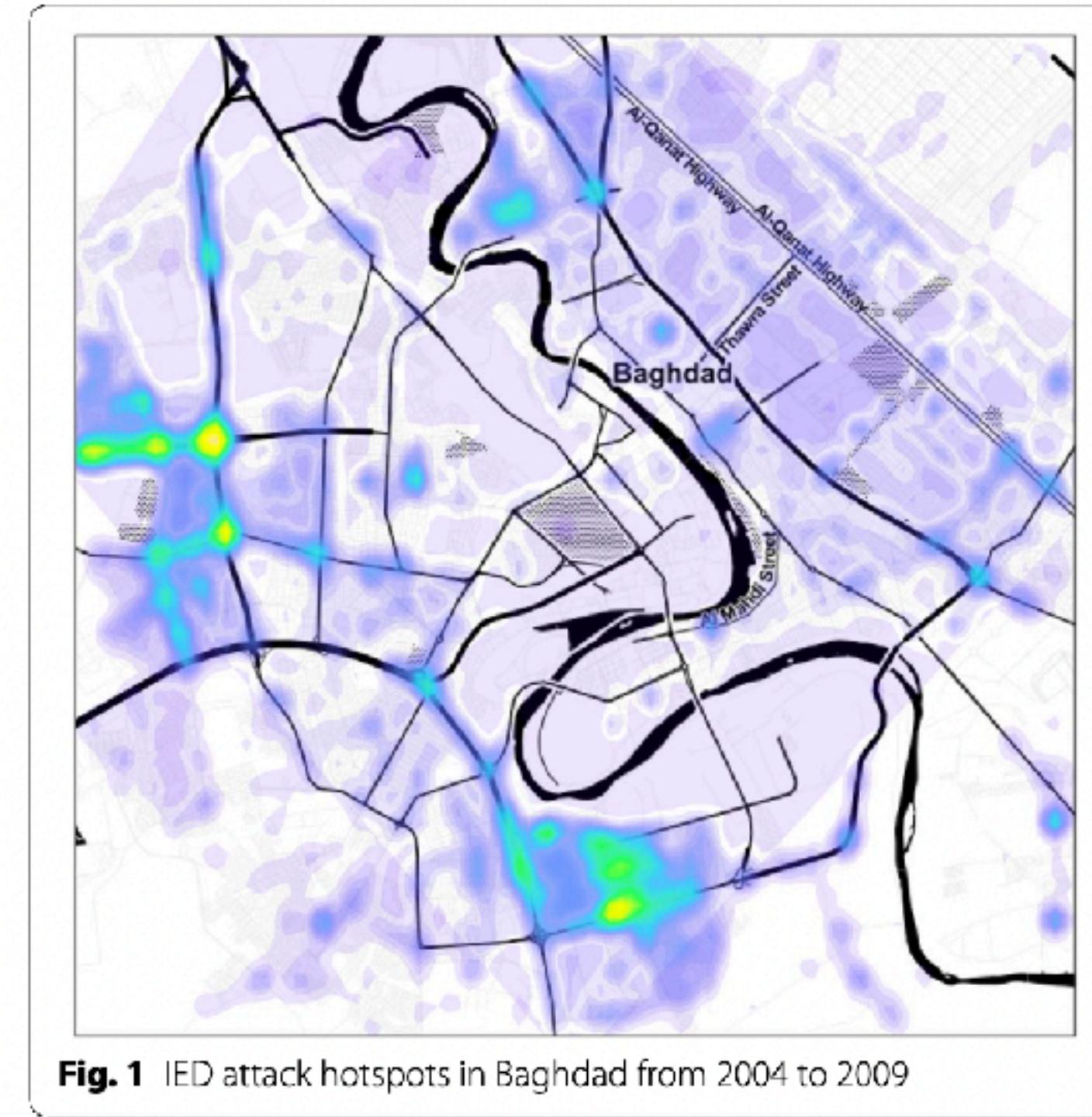


stpp-simulator.streamlit.app

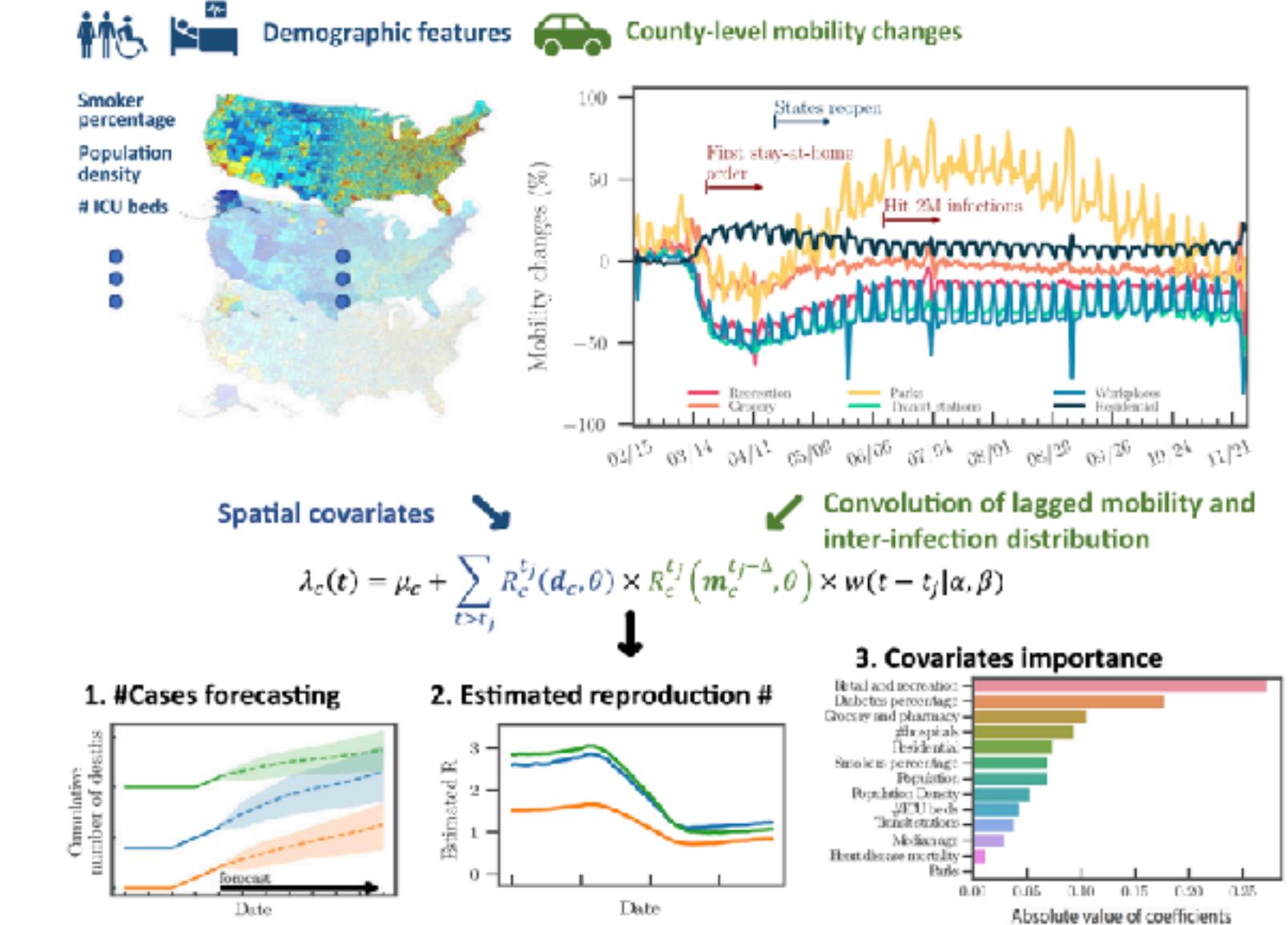
Applications



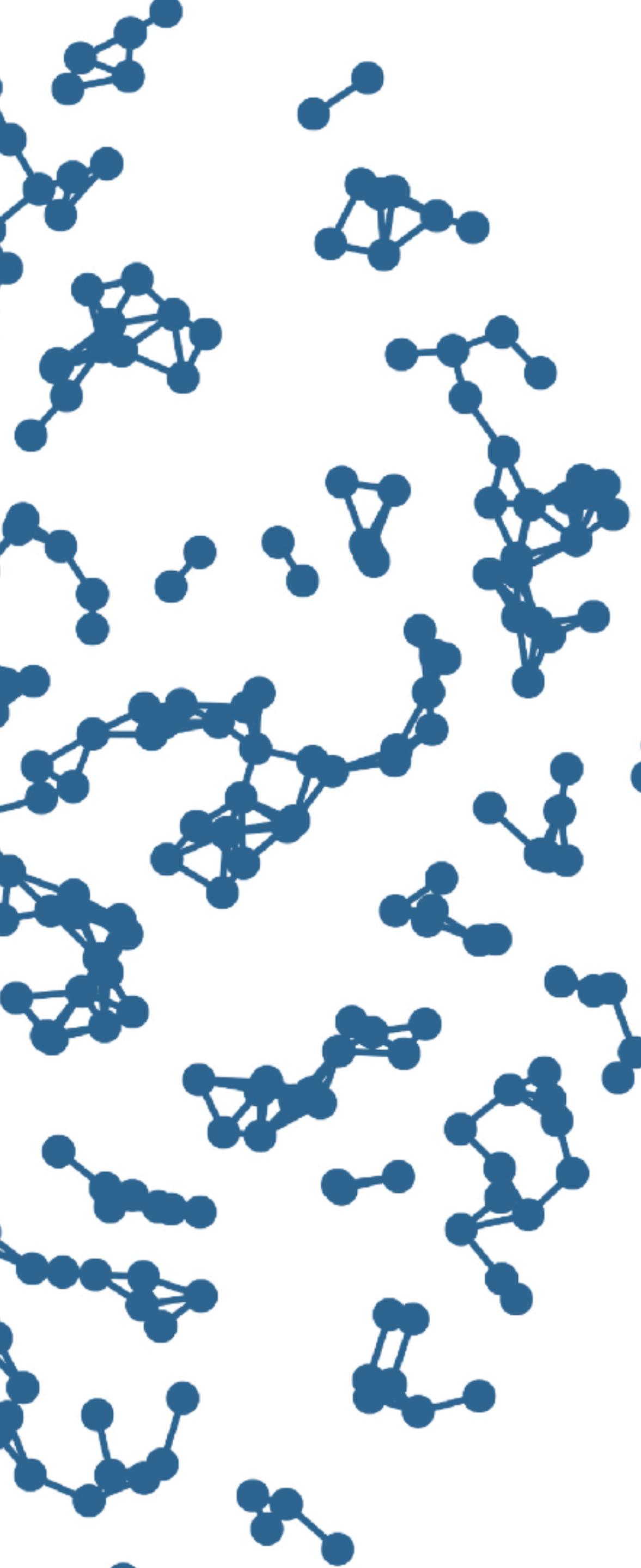
Spatio-temporal variations in the urban rhythm: the travelling waves of crime (Oliveira)



Learning to rank spatio-temporal event hotspots (Mohler et al.)



Hawkes process modeling of COVID-19 with mobility leading indicators and spatial covariates (Chiang et al.)



Part V

NEURAL STTPS

Neural STTPs

How do we generalize this towards STPPs?

Model **intensity** function with neural network:

$$\lambda(t, H_t) = \text{NN}(t, H_t)$$

- Numerical integration is expensive
- NN architecture needs to process event sequence of arbitrary length.

Model **predictive distribution** with neural network:

$$f^{\text{pred}}(t, H_t) = \text{NN}(t, H_t)$$

- Predict valid PDF
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Neural STTPs

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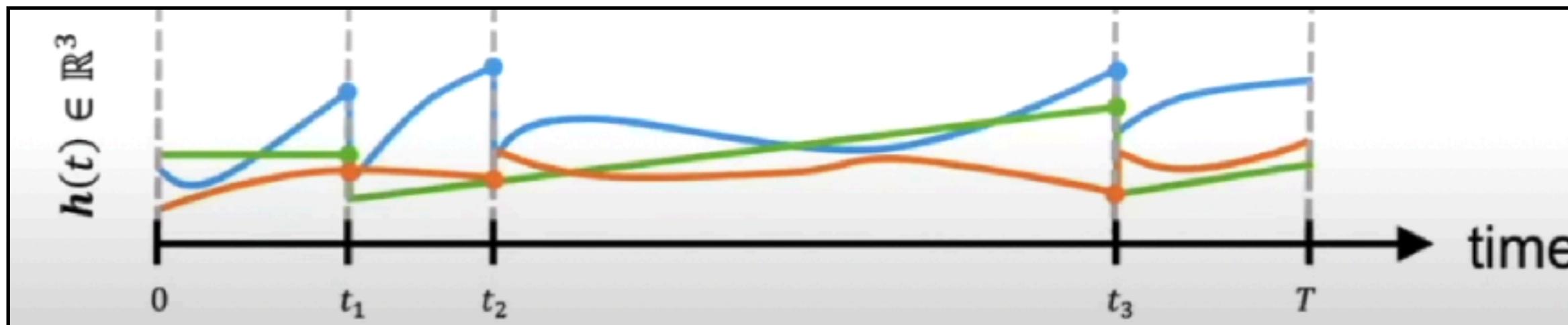
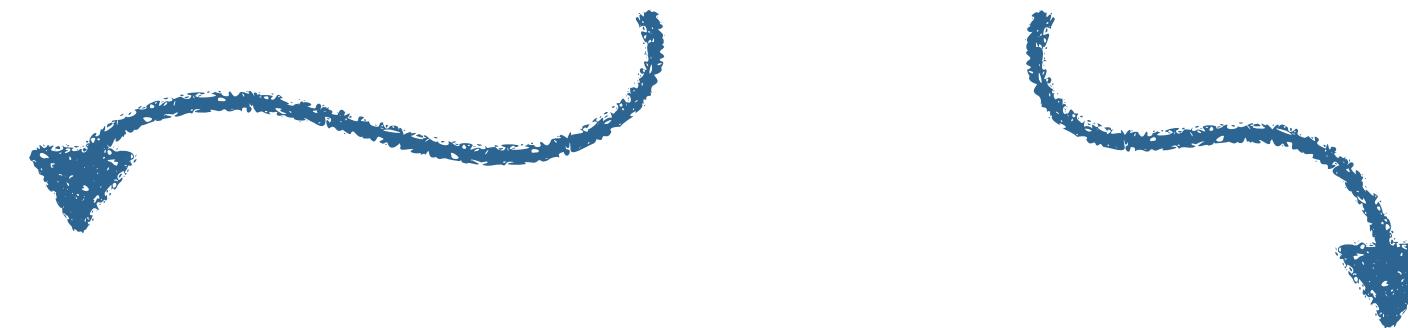
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Neural STTPs

$$\lambda(t, \mathbf{s} \mid H_t) = \lambda(t \mid H_t) \cdot p(\mathbf{s} \mid H_t)$$



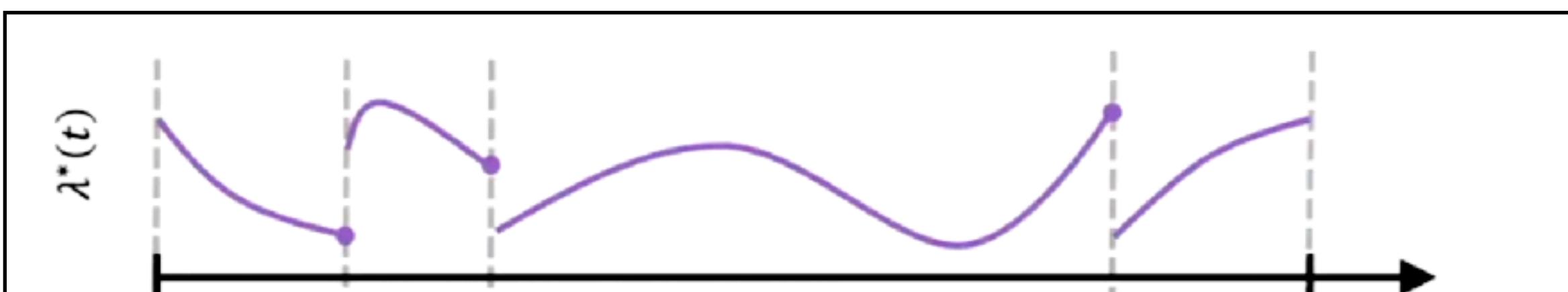
youtube.com/watch?v=J7qH7i0EyfU&t=749s&ab_channel=TUM-DAML

Specify **ODE** that moves current position

$$\mathbf{x}_0 \sim p(\mathbf{x}_0)$$

$$\frac{d\mathbf{x}_t}{dt} = f_x(t, \mathbf{x}_t, \mathbf{h}_t)$$

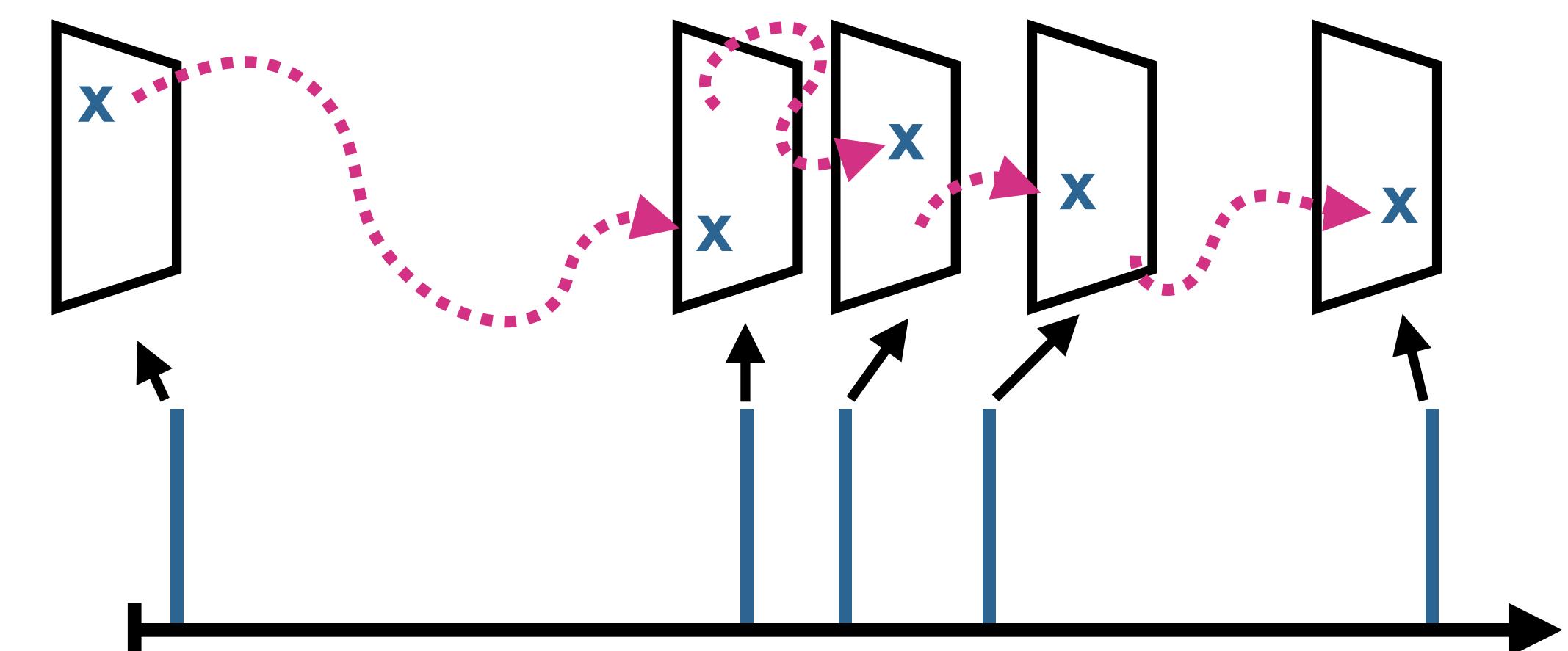
$$\lim_{\varepsilon \rightarrow 0} \mathbf{x}_{t_i + \varepsilon} = g_x(t_i, \mathbf{x}_{t_i}, \mathbf{h}_{t_i})$$



NEURAL SPATIO-TEMPORAL POINT PROCESSES

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Facebook AI Research
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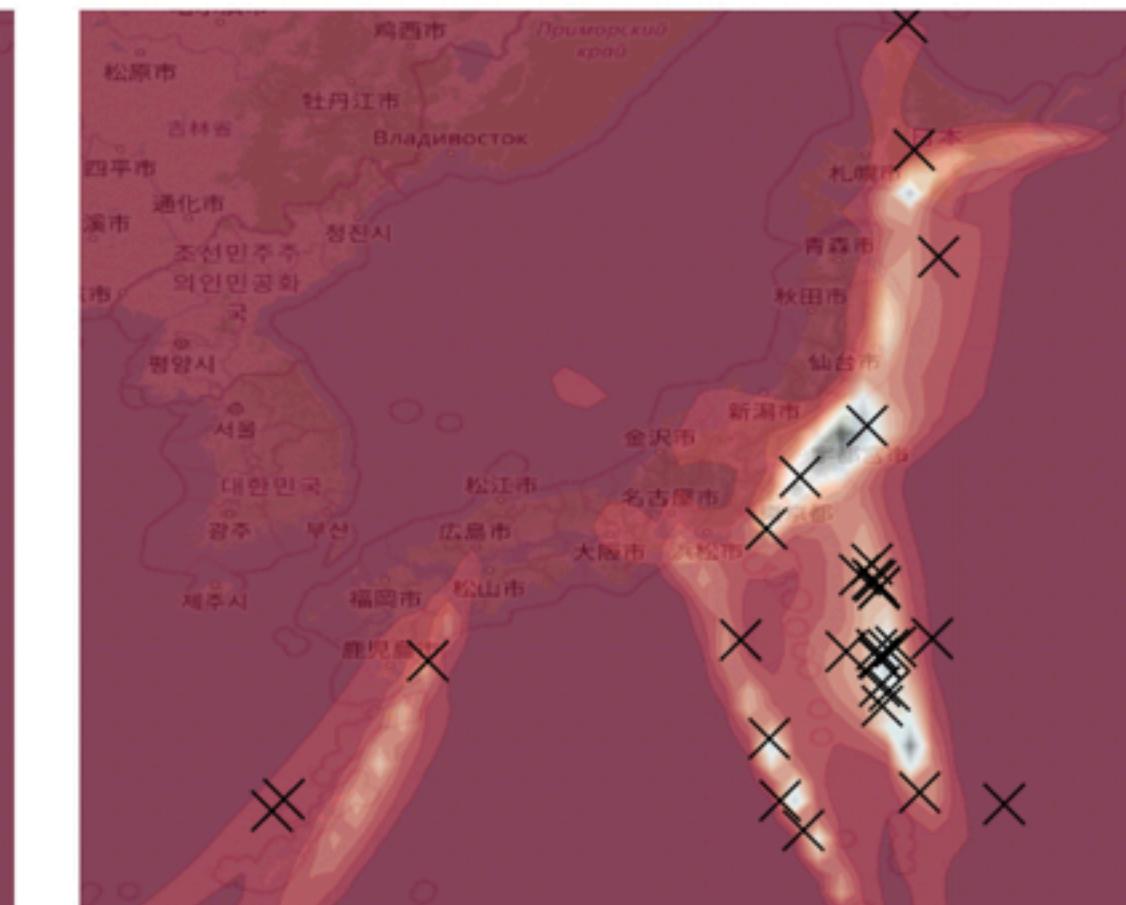
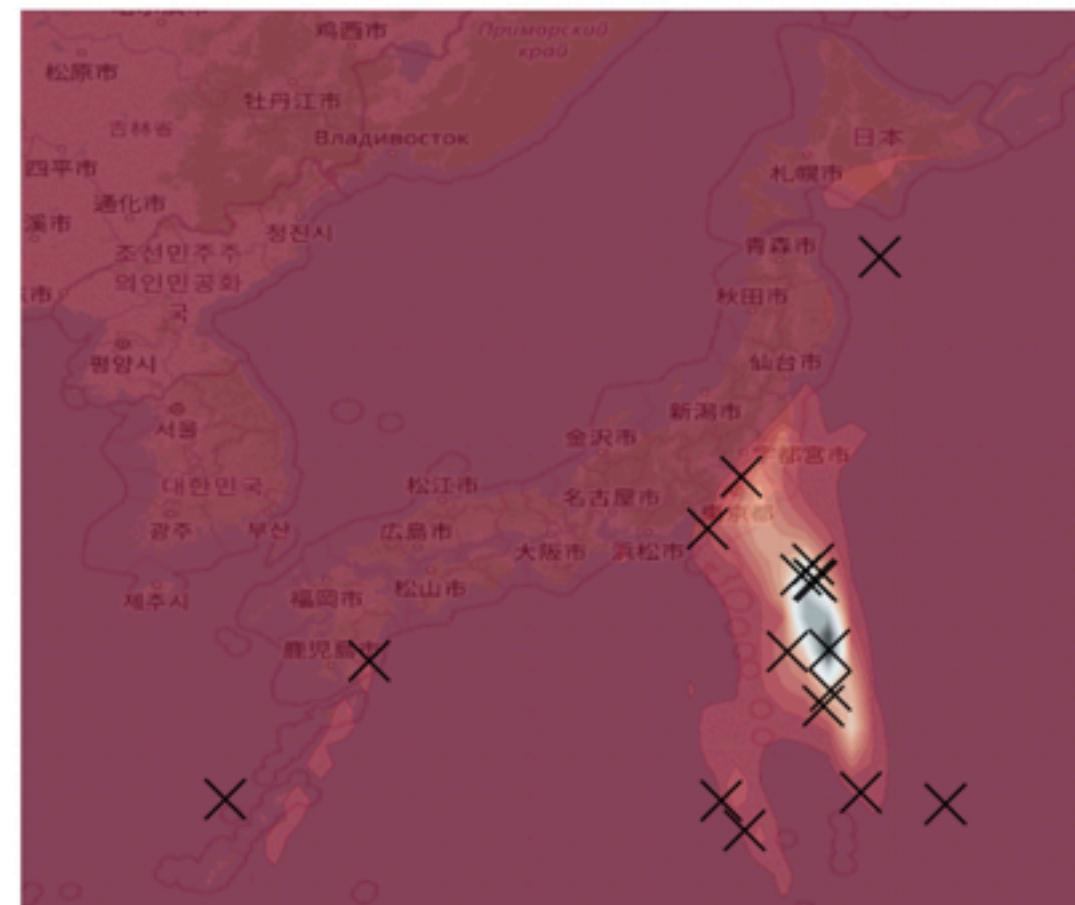
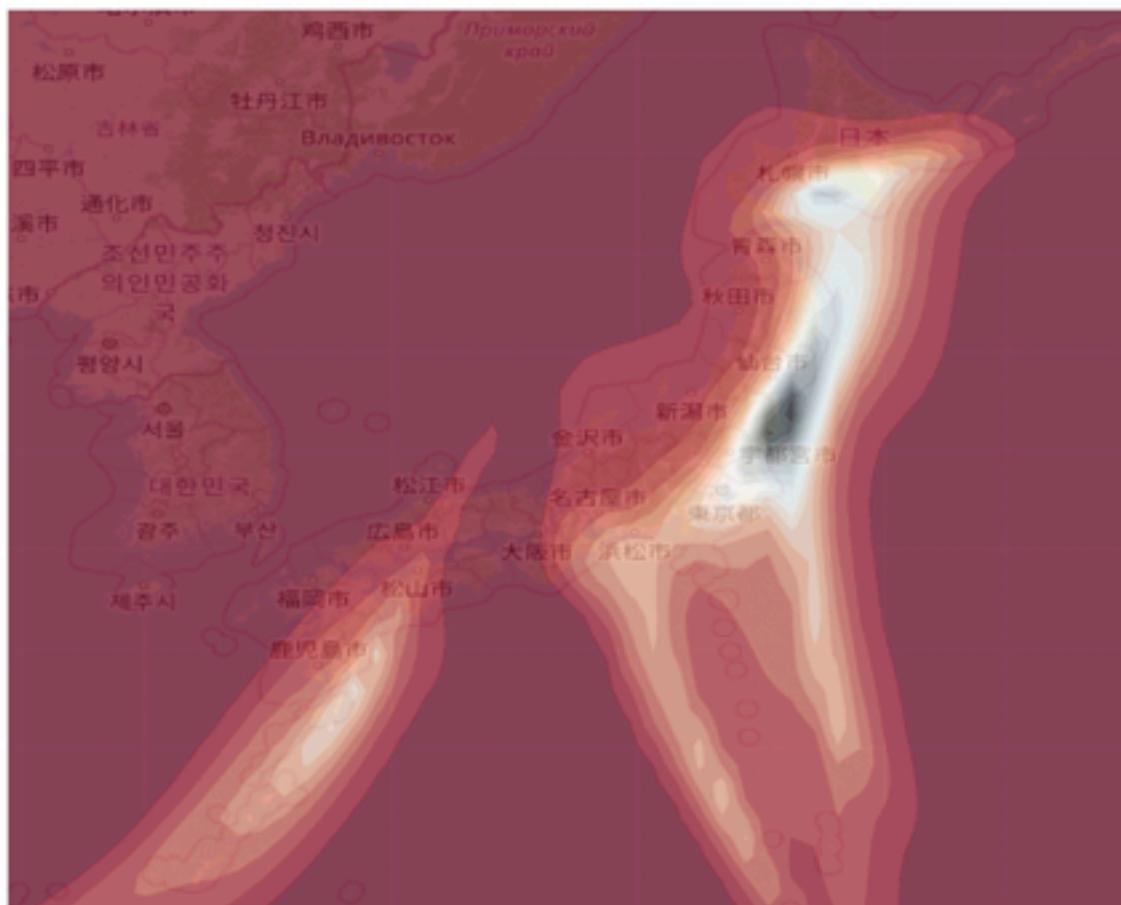


Neural STTPs

NEURAL SPATIO-TEMPORAL POINT PROCESSES

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Neural STTPs

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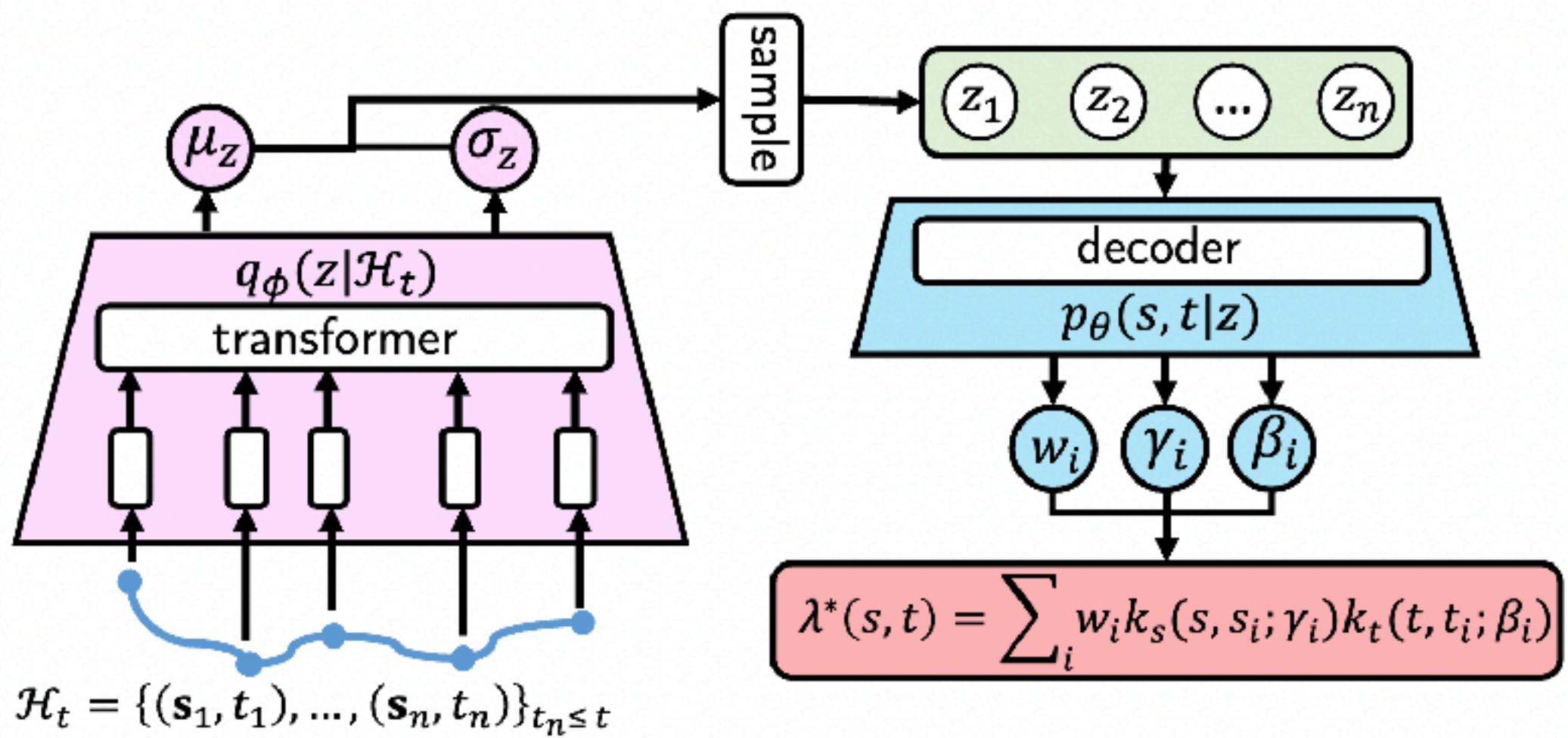
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Model **predictive distribution** with neural network:

$$f^{\text{pred}}(t, H_t) = \text{NN}(t, H_t)$$

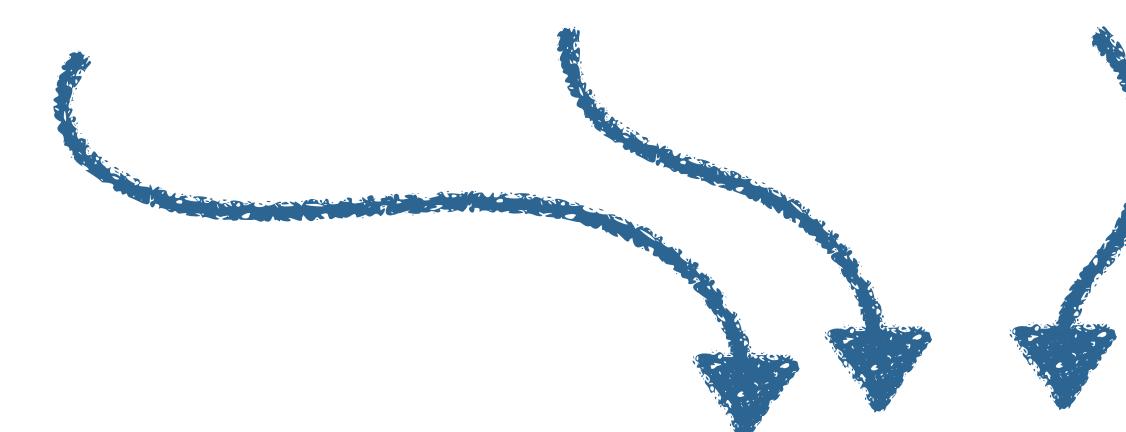
- Predict valid PDF
- PDF representation must support both efficient sampling and evaluation.
- NN architecture needs to process event sequence of arbitrary length.

Neural STTPs

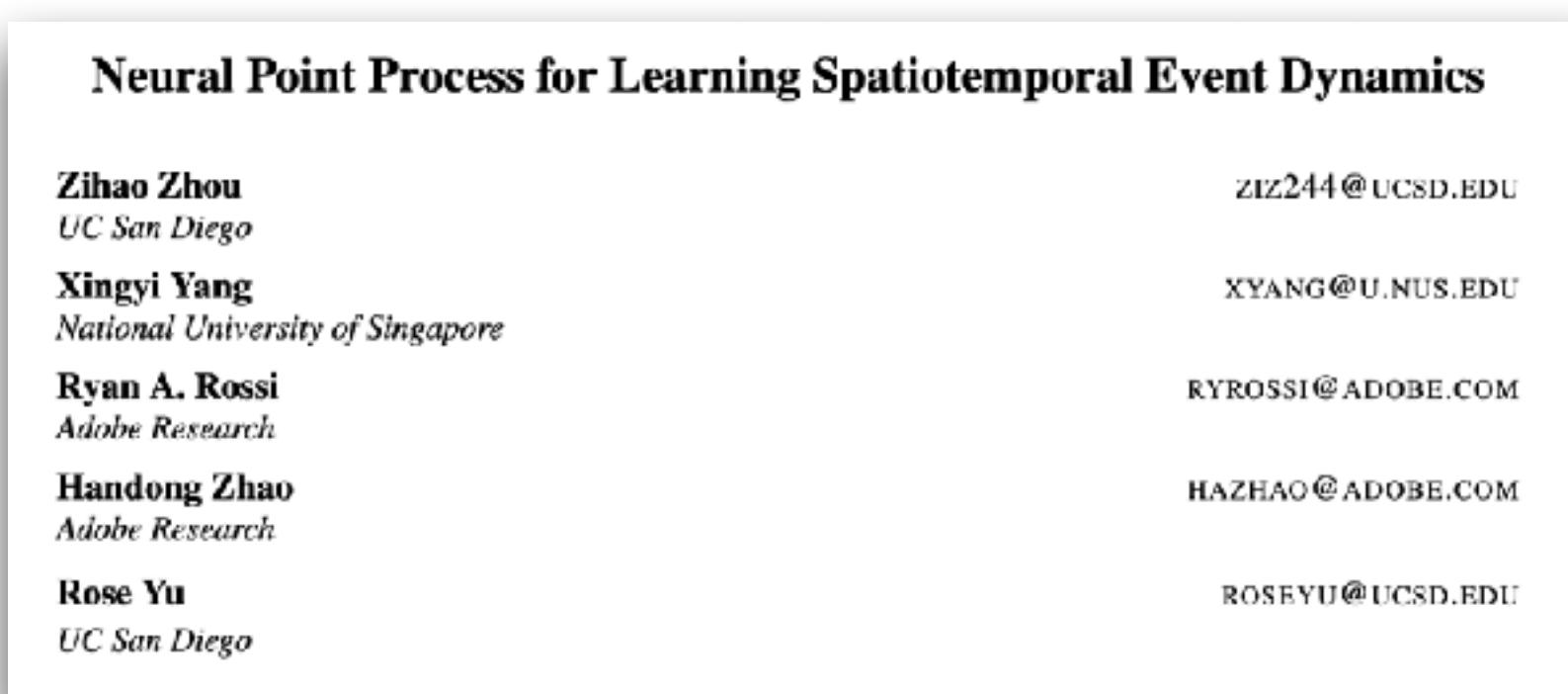


Find parametric form of the intensity function:

$$\lambda(s, t | H_t) = \sum_{i=1}^n w_i k_s(s, s_i; \gamma_i) k_t(t, t_i; \beta_i)$$



For each event i in H_t we learn w_i, γ_i, β_i



Define kernels:

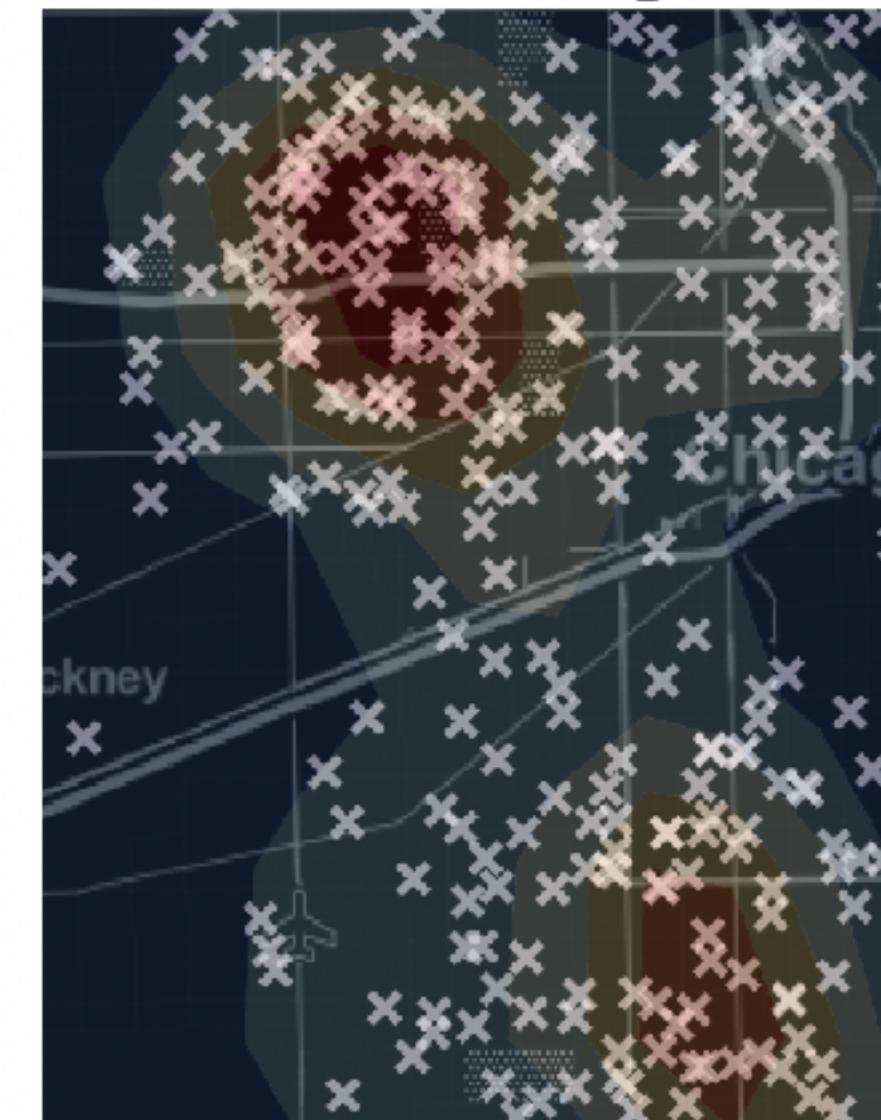
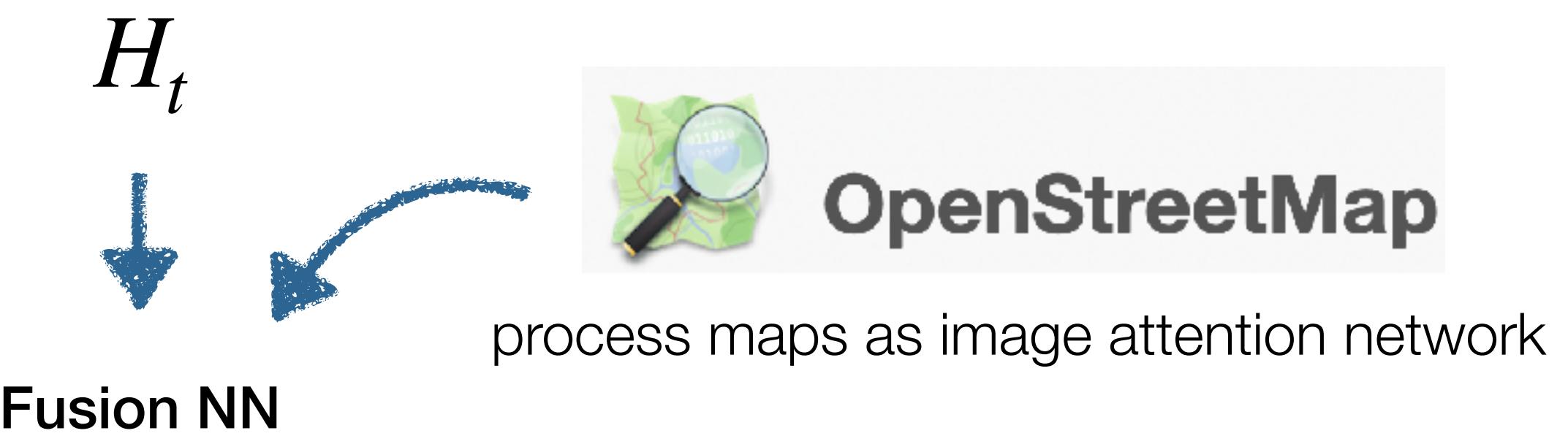
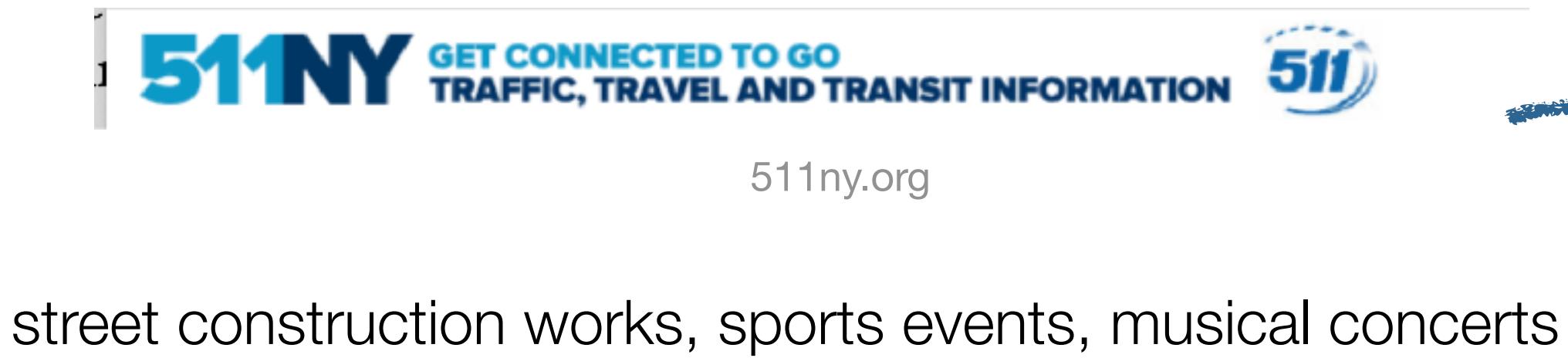
$$k_s(s, s_i) = \alpha^{-1} \exp(-\gamma_i \|s - s_i\|)$$

$$k_t(t, t_i) = \exp(-\beta_i \|t - t_i\|)$$

normalization constant

Neural STTPs with Context

"Human activities are largely influenced by **environmental features**, i.e., weather, geographical characteristics and traffic conditions. These features must be considered to accurately predict future events."

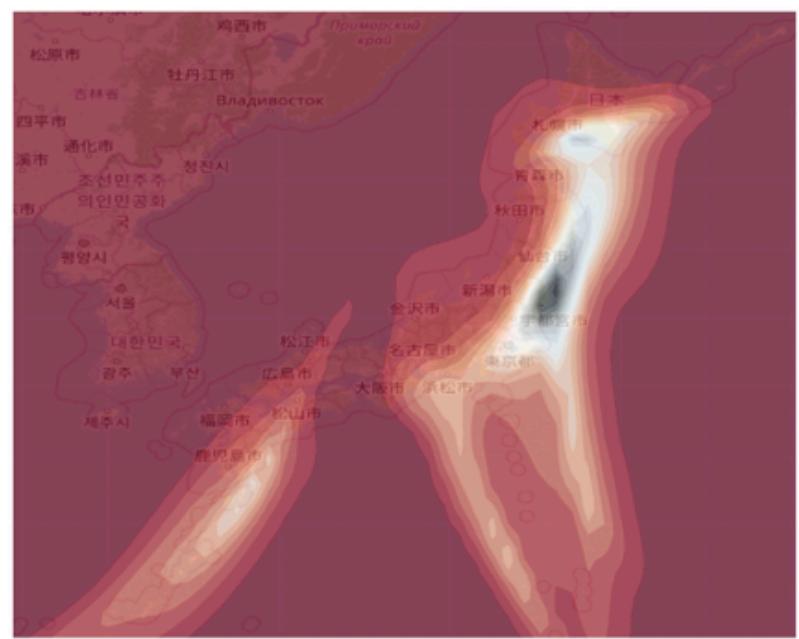


Deep Mixture Point Processes: Spatio-temporal Event Prediction with Rich Contextual Information

Maya Okawa¹, Tomoharu Iwata², Takeshi Kurashima¹, Yusuke Tanaka¹, Hiroyuki Toda¹ and Naonori Ueda²

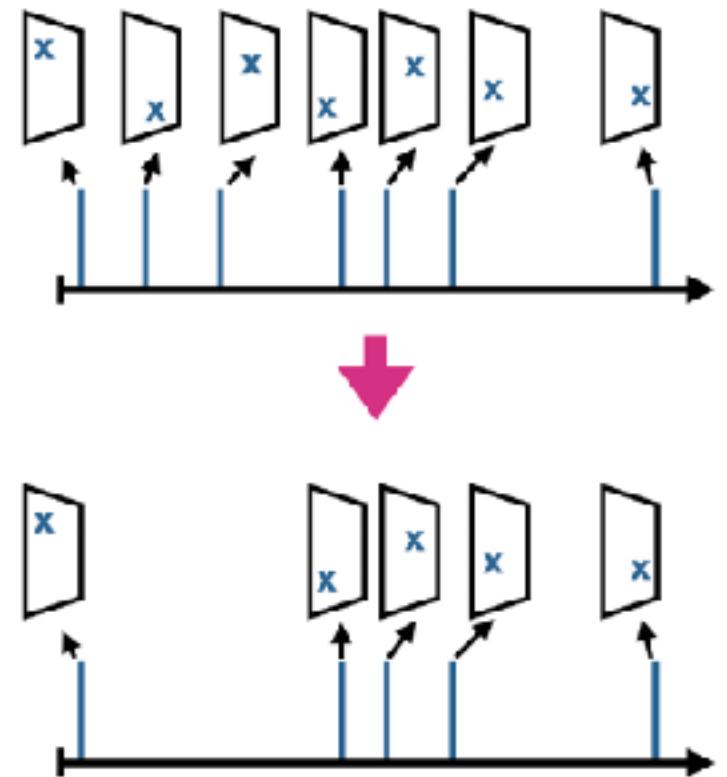
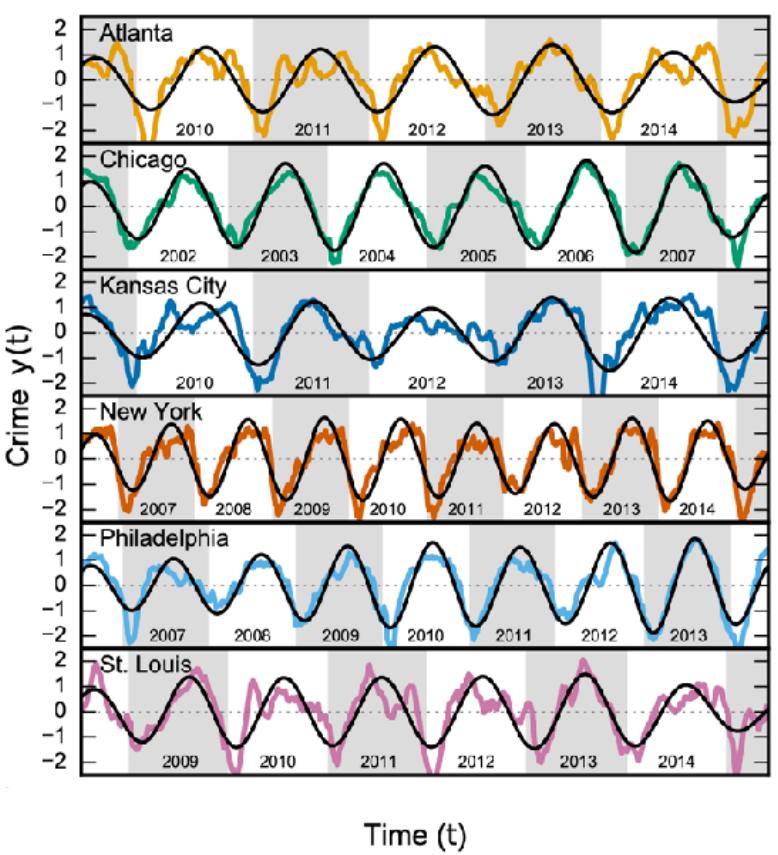


Recap

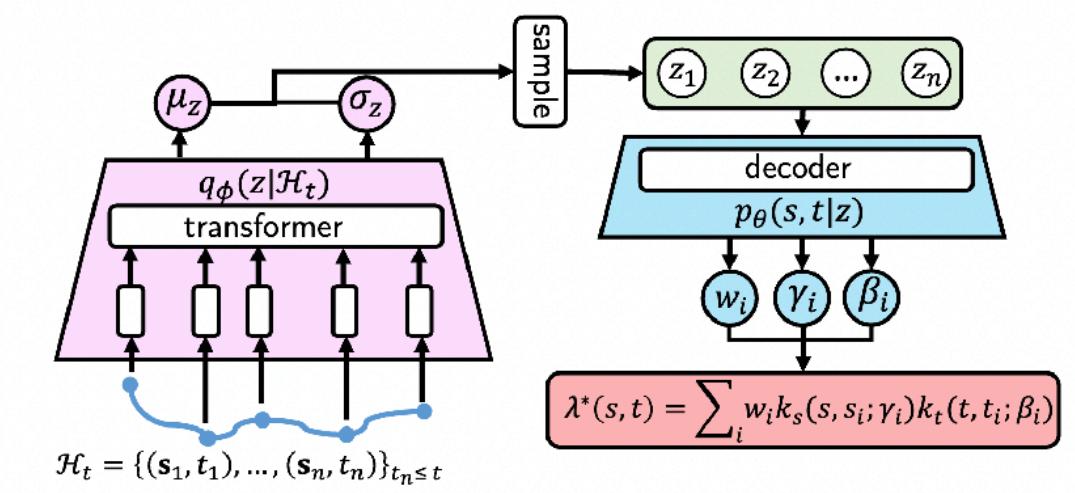


Spatio-temporal data is everywhere.

Statistical models capture patterns like bustiness, self-excitation, and self-correction



Simulation with the thinning method is easy and fast.



Neural models capture arbitrary dynamics and include contextual information.

Slides, code, and lecture notes available at github.com/gerritgr/spatio-temporal-lecture