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Nonlinear dynamics, time series analysis, and machine learning

Chimera States

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1 Introduction

Systems of oscillators trivially show two different behaviours: Stable complete synchronisation or unstable anti-synchronisation of different populations within the system [1]. Interestingly these are not the only observable behaviours. There may also be symmetry splitting in the phase coherence of a system of oscillators. While not trivially expected to find such states they prove to be reproducible for carefully chosen configurations.

The goal of this study is the simulation of these states in a simplified system of two identical populations of identical Kuramoto style oscillators in the explicit model, i. e. solving the system for each oscillator, and in a reduced form as suggested by Ott and Antonsen [2]. Therefore it is central to choose initial conditions and parameters in a way that allow the evolution of chimera states.

These states are highly dependent on initial conditions, such that the bifurcation behaviour of the explicit model is not trivial, whereas the reduced system of Ott and Antonsen allows for analytical bifurcation analysis. Nevertheless even the latter case shows strong dependence of the probability of chimera states evolving on the parameters chosen [3]. Occurrence of these states is expected to reach beyond mathematical systems to complex biological systems, e. g. the brain or heart, adding significance to the understanding of chimera-like behaviour.

2 Kuramoto model

The Kuramoto model describes a continuous system of oscillators that can be characterised through the phase $\theta(x,t)$ of each given oscillator. Each oscillator has a natural frequency $\omega(x)$ and is linked to all other oscillators by a coupling term $C(x,t)$ [cf. 1, sec. 2]. Thus yielding

$$\frac{\partial}{\partial t}\theta(x,t) = \omega(x) + C(x,t), \tag{1}$$

where the coupling C is given as a sinusoidal term with phase lag α and a factor $G(x)$, resulting in

$$\frac{\partial}{\partial t}\theta(x,t) = \omega(x) + \int \mathrm{d}x'\, G(x-x') \sin\big(\theta(x',t) - \theta(x,t) - \alpha\big)\,, \tag{2}$$

here $\mathrm{d}x'$ indicates integration over all oscillators in the considered system.

2.1 Two point-like populations

In this case a simple system of two finite populations of identical homogeneous oscillators is considered [4]. Homogeneity means $\omega_i = \omega = \text{const.}, \forall\, i$, where i is the index of the oscillator. Assuming that the coupling is as in eq. (2), with $G = K_{\sigma\sigma'} = \text{const.}$ where σ and σ' are indices for the populations, eq. (1) becomes [4, p. 1]

$$\frac{\mathrm{d}\theta_i^\sigma}{\mathrm{d}t} = \omega + \sum_{\sigma'=1}^{2} \frac{K_{\sigma\sigma'}}{N_{\sigma'}} \sum_{j=1}^{N_{\sigma'}} \sin\left(\theta_j^{\sigma'} - \theta_i^\sigma - \alpha\right), \tag{3}$$

where N_σ is the number of oscillators in population σ and $\sigma \in \{1,2\}$.

The coupling within the population itself is stronger than the coupling to the other population and given a symmetric system it is easily concluded that [4, p. 1]

$$K_{\sigma\sigma} > K_{\sigma\sigma'}, \qquad \sigma, \sigma' \in \{1,2\}, \sigma \neq \sigma'. \tag{4}$$

For convenience these factors can be scaled so that

$$K_{ii} + K_{ij} = 1, \qquad i,j \in \{\sigma, \sigma'\}, i \neq j. \tag{5}$$

As indicated in [1, 4] is is further useful to introduce two parameters, A and β, that will be used to describe the system:

$$A = K_{ii} - K_{ij}, \qquad \text{(coupling disparity)}, \tag{6}$$

$$\beta = \frac{\pi}{2} - \alpha, \qquad \text{(conjugate phase lag)}. \tag{7}$$

Since this system has no spatial expansion or dependence, except for belonging to either of the two observed populations of oscillators, it is sometimes described as a *point-like* system [1].

2.2 Numerical solution

The explicit system of Kuramoto-oscillators, as described in section 2.1, can be solved in a straightforward manner. In this particular case explicit integration of the system of ordinary differential equations using the Runge-Kutta fourth-order integrator from the *Boost Odeint C++ Library* [5] is performed. As the differential equations are all ordinary and a fixed time step in the integration is desired this method is fully sufficient. Time is only considered in units of time steps, because no real time scale is connected to the model in this case. In total integration is always performed much longer than the transient, in order to reach a stable state.

Due to the explicit interaction of each oscillator with all other oscillators this computation is rather time expensive. Some tests have shown that computation time scales roughly with

$$t_{\text{comp}} \sim N^2, \tag{8}$$

where N is the total number of computed oscillators. Thus the evaluation of the explicit Kuramoto system was confined to systems no larger than $N = 256$, see figure 1.

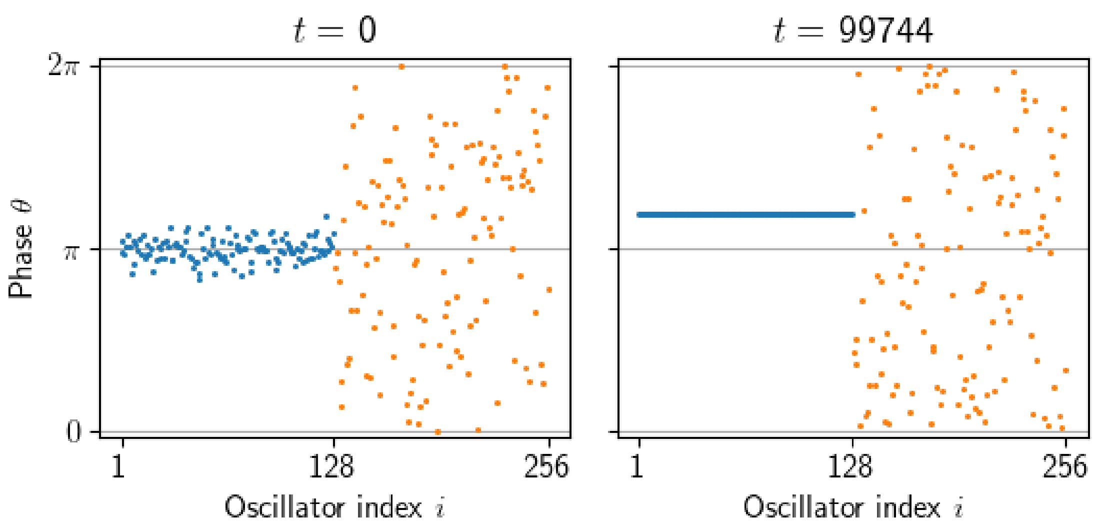

Figure 1. Example of the explicit Kuramoto model for a given initial condition close to a chimera state (left, $t = 0$) and after integration over a time scale more than five times as long as the transient (right). For the initial state of the system a normal distribution with a variance of 0.2 for the first population and π for the second population was chosen. Initial values outside of $[0, 2\pi]$ are omitted and regenerated. Each population consists of 128 oscillators and the phases are normalized to be within $[0, 2\pi)$. Further $A = 0.4$ and $\beta = 0.01$.

Without loss of generality $\omega = 0$ is chosen for all performed calculations. As further discussed in section 3.2 the choice of initial conditions is not obvious and highly influences the outcome of the numerical solution.

2.3 Introducing an order parameter

The phase coherence of a population σ can be described distinctly by a dynamic order parameter $r_\sigma(t)$ [4]

$$r_\sigma(t) = \left| \left\langle \mathrm{e}^{\mathrm{i}\theta_j(t)} \right\rangle_{j\in\sigma} \right|, \tag{9}$$

where $\langle\cdot\rangle_{j\in\sigma}$ denotes an average over all oscillators in population σ.

For finite numbers of oscillators and using the identities $\left|\mathrm{e}^{\mathrm{i}\phi}\right|^2 = \mathrm{e}^{\mathrm{i}\phi}\mathrm{e}^{-\mathrm{i}\phi}$ and $\mathrm{e}^{\mathrm{i}\phi} + \mathrm{e}^{-\mathrm{i}\phi} = 2\cos\phi$ this can be written as

$$r_\sigma(t) = \frac{1}{N_\sigma} \left(N_\sigma^2 + 2\sum_{i<j} \cos(\theta_i - \theta_j) \right)^{\frac{1}{2}}, \tag{10}$$

where i, j are oscillator indices within a population. By definition r satisfies $r \in [0, 1]$.

Further r is simply referred to as the *order parameter* of a population. It is essentially an indicator for the current state of the population and $r_\sigma(t) = 1$ means that population σ is fully synchronised.

2.4 Ott-Antonsen reduction

Considering the thermodynamic limit ($N_\sigma \to \infty, \forall\, \sigma$) allows a more analytical approach to the system.

In the thermodynamic limit the continuous case of the Kuramoto model, as in eq. (2), can be considered. Thus the system can be described by continuous probability density functions $f_\sigma(\theta_\sigma, t)$ [2, 3]. This statistical system can be fully described by mean field order parameters r_σ, ϕ_σ, where r_σ is the phase coherence of population σ, as in eq. (9), and ϕ_σ is the mean phase of population σ.

This case further gives rise to a continuity equation [2, p. 5], due to conservation of the number of oscillators,

$$\frac{\partial f_\sigma}{\partial t} + \frac{\partial f_\sigma v_\sigma}{\partial \theta_\sigma} = 0, \tag{11}$$

where v_σ is the phase velocity of population σ given as [3, p. 11]

$$v_\sigma(\theta_\sigma, t) = \omega + \sum_{\sigma=1}^{2} K_{\sigma\sigma'} \int \mathrm{d}\theta'_\sigma\, \mathrm{e}^{\mathrm{i}\theta'} f_{\sigma'}(\theta'_\sigma, t). \tag{12}$$

Expanding the probability density function f in a Fourier series [2, p. 6] leads to

$$f_\sigma(\theta_\sigma, t) = \frac{1}{2\pi} \sum_{n=0}^{\infty} \left(a_\sigma^*(t)\, \mathrm{e}^{\mathrm{i}\theta_\sigma} \right)^n + \text{const.}, \tag{13}$$

where $a_\sigma^*(t)$ are the complex conjugated Fourier coefficients, given as the inverse Fourier transform [3, p. 11]

$$a_\sigma(t) = \int \mathrm{d}\theta_\sigma\, \mathrm{e}^{\mathrm{i}\theta_\sigma} f_\sigma(\theta_\sigma, t). \tag{14}$$

Defining the Fourier coefficients as follows [2, 3]

$$a_\sigma(t) = r_\sigma(t)\mathrm{e}^{\mathrm{i}\phi_\sigma(t)}, \tag{15}$$

with r_σ, ϕ_σ as mentioned above, and inserting all above equations into the continuity equation (11) yields

$$\frac{\mathrm{d}r_\sigma}{\mathrm{d}t} = \frac{1-r_\sigma^2}{2}\sum_{\sigma'=1}^{2} K_{\sigma\sigma'} r_{\sigma'} \sin\left(\phi_{\sigma'} - \phi_\sigma + \beta\right), \tag{16a}$$

$$\frac{\mathrm{d}\phi_\sigma}{\mathrm{d}t} = \omega - \frac{1+r_\sigma^2}{2r_\sigma}\sum_{\sigma'=1}^{2} K_{\sigma\sigma'} r_{\sigma'} \cos\left(\phi_{\sigma'} - \phi_\sigma + \beta\right). \tag{16b}$$

Thus the statistical dynamics of the system can be described through two ordinary differential equations per population, one for each mean field order parameter.

In the case of just two populations, $\sigma = \{1, 2\}$, the system can be further reduced by considering only the mean phase difference ψ between the populations

$$\psi = |\phi_1 - \phi_2|. \tag{17}$$

This special case therefore being able to fully describe the system with the three quantities $\{r_1, r_2, \psi\}$.

3 Chimera states

A chimera state is a space- and time-dependent pattern in a system of identical oscillators which divides into populations of synchronised (coherent) and asynchronised (incoherent) oscillation. As this is mathematically a hybrid-state Strogatz and Abrams first called it a "chimera" [1], referring to the hybrid creature originating from Greek mythology.

The system of two point-like populations of oscillators, as discussed in section 2, is one of the simplest systems in which chimera states can be observed.

Characteristically one of the two populations in this case will synchronise after a transient time while the other will stay asynchronised, see figure 1. This behaviour is easily identifiable by observing the order parameters r_σ in a given system.

3.1 Stable and breathing cases

Chimera states can be categorised according to their stability. Generally stable and oscillating (or "breathing") states can be observed. When considering the Ott-Antonsen reduced system, chimera states undergo a Hopf-bifurcation in the A-β parameter space [3, 4], see figure 4. Abrams et al. [4] suggest that the explicit Kuramoto model behaves likewise. This study of the system, however, was not able to reproduce stable chimera states in the explicit model, see also section 3.1.1.

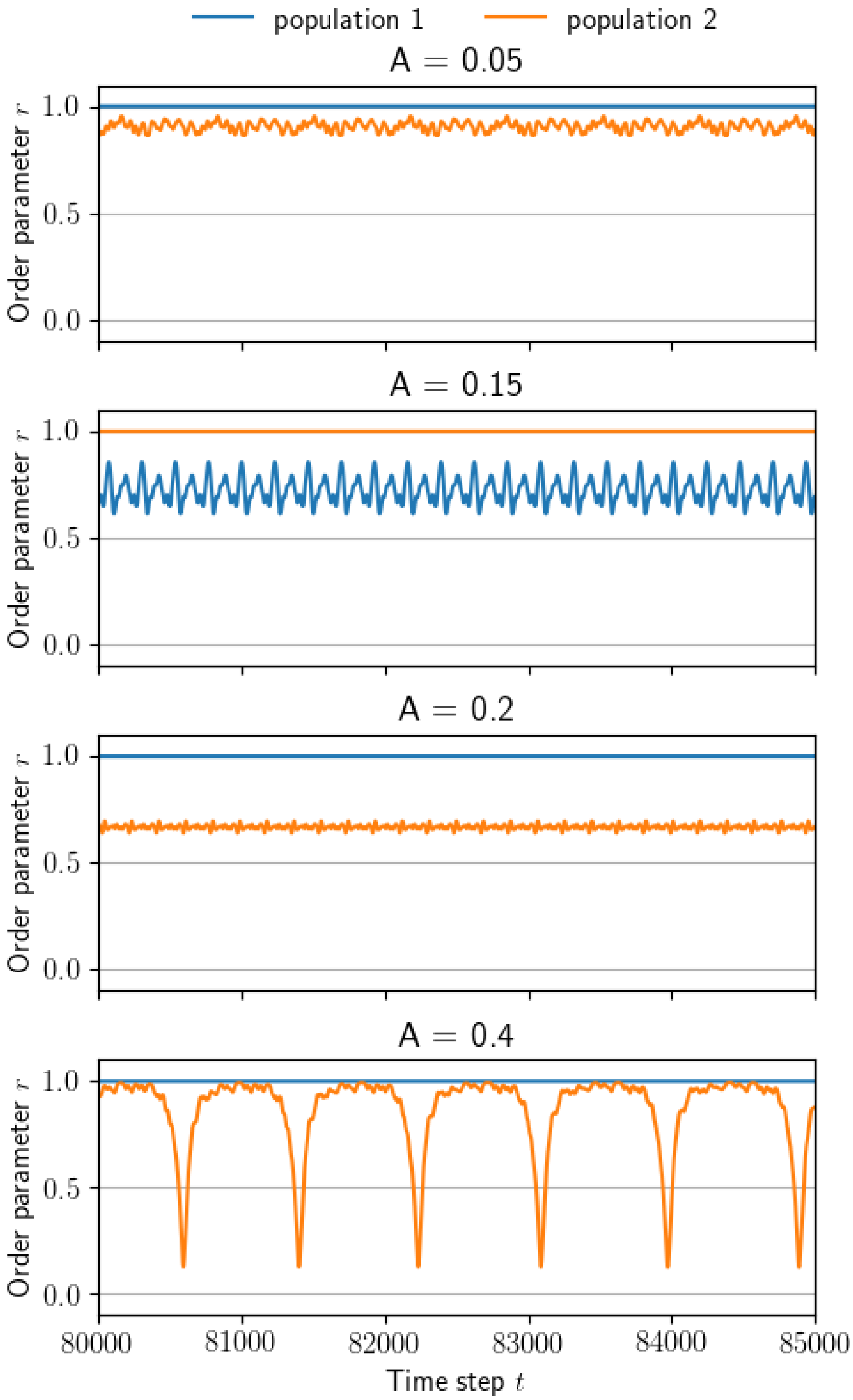

Figure 2. Long-term behaviour of the order parameters of the explicitly integrated Kuramoto system with $N_\sigma = 128, \forall\,\sigma$. A time more than twice as long as the transient is omitted. In all cases $\beta = 0.01$ and A is varied. The integration is started from Gaussian distributed initial conditions with variances of 0.2 for the first population and π for the second population, just as in figure 1. As before, only initial values within $[0, 2\pi]$ are considered valid and are regenerated as needed.

3.1.1 Explicit Kuramoto model

Integration of the full system given by eq. (3) shows regularly oscillating and long-term breathing chimera behaviour, see figure 2. As discussed in sec. 3.2, β must be chosen very close to 0 in order to ensure a high rate of chimera state evolution.

Further the oscillation displays noisy behaviour. This may in part be due to the relatively small system size $N_\sigma = 128$ for each population. For the same system size Abrams et al. [4] showed very smooth data and also stable chimera states, as in this case they can only be observed in the reduced model. Assuming they smoothed their data then small regular oscillations as in figure 2 for $A = 0.05$ could correspond to stable chimera states. Breathing chimera states can be observed as expected for parameters between the Hopf and homoclinic bifurcations (figure 4).

Panaggio and Abrams [1] discuss that for finite sized systems chimera states are likely not stable but rather transient states of the system. However, for $N_\sigma = 128$ as well as $N_\sigma = 50$ no such transient states are observed. Nevertheless as A approaches the point of Hopf bifurcation it is apparent that the transient leading up to a stable state grows, see figure 3. Due to computational expensiveness no deeper investigations as to the exact A-dependence of the transient time is undertaken. Nevertheless this behaviour is found in all considered cases.

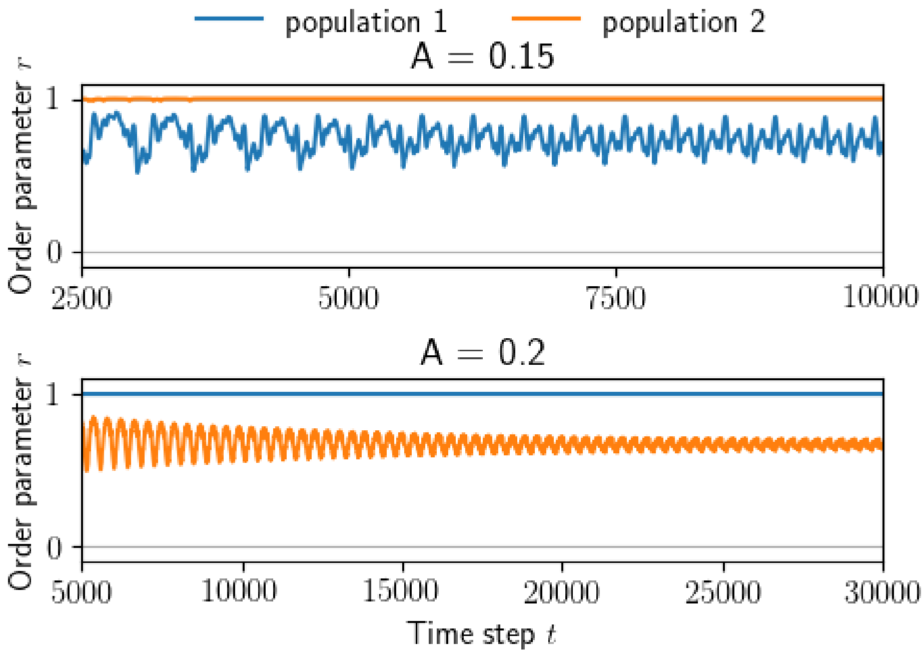

Figure 3. Order parameters of the explicit Kuramoto model, similar to figure 2. Notice the different time scales which clearly show varying transient times for different values of A. This behaviour is generally observed for A that approach the Hopf bifurcation.

3.1.2 Ott-Antonsen reduction

The according to Ott and Antonsen reduced system, eq. (16), considers infinitely large populations and is assumed to be the limit that the explicit system converges to for $N_\sigma \to \infty$.

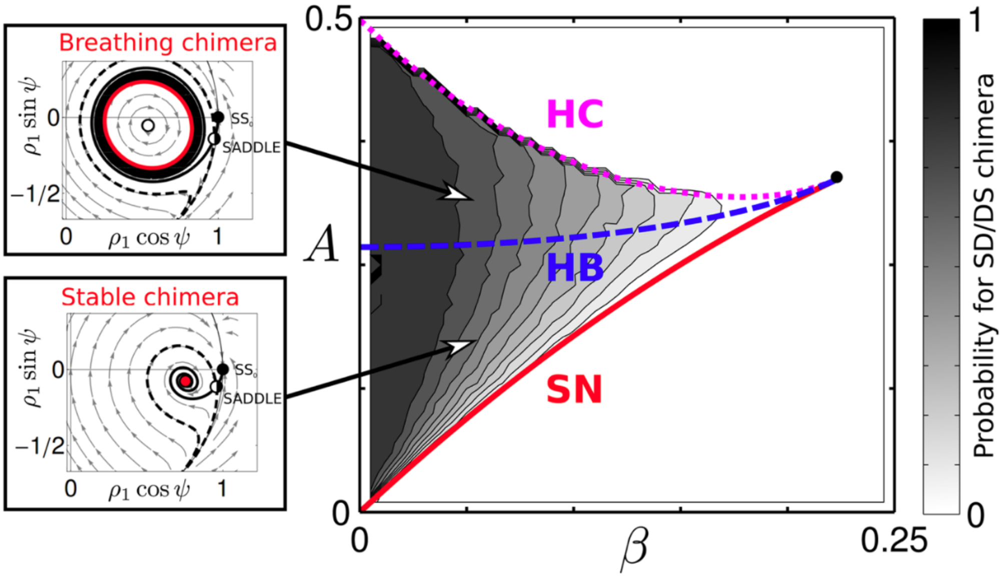

Figure 4. Probabilities to observe a chimera state in the Ott-Antonsen reduction for uniformly distributed initial values of $\{r_1, r_2, \psi\}$. Each given set of parameters was used for 1000 random initial conditions [3]. Chimeras only exist within the area between the saddle-node bifurcation (SN) and the homoclinic bifurcation (HC). For intermediate values of A the chimera state performs a Hopf bifurcation (HB) [4]. Analytically computed phase portraits are shown for the two given cases, compare figure 6. Note the stable synchronised state at $(0, 1)$ and the stable fixed point turning into a stable limit cycle through the Hopf bifurcation. Source: [3, edited by author].

Since only two ordinary differential equations have to be solved per population this system can be analysed much faster than the explicit Kuramoto model with N_σ equations per population. Thus for small β chimera states are easily observable for the domain of A. As in the explicit case the system may either fully synchronise or evolve to a chimera state. For A smaller than the Hopf bifurcation stable chimera states are observed while for A larger than the Hopf bifurcation the chimera states show breathing behaviour, see figures 4 and 5.

Since the reduced case delivers equations for the order parameters $\{r_1, r_2, \psi\}$ bifurcation analysis in the corresponding feature space can be performed, see figure 4. The stable synchronised state corresponds to the point $(0, 1)$, whereas chimera states correspond to another stable fixed point (stable chimera states) that eventually turns into a limit cycle (oscillating / breathing chimera states) [3, 4].

All trajectories of breathing chimera states show an oscillation of the stable synchronised population on a unit circle segment in feature space and a limit cycle of the breathing population, figure 6. As the breathing population's mean phase fluctuates the mean phase difference ψ must also fluctuate, explaining the curve in feature space.

Abrams et al. [4] additionally show stable oscillating chimera states that display sinusoidal fluctuation instead of the long breathing shown in figure 5. Search at $A = 0.28$, as suggested by them, however is not able to find such configurations for 100 random initial conditions. As no such result is shown in all integrations of the reduced system, search of this particular state is likely not useful and it does not display usual characteristic behaviour of chimera states.

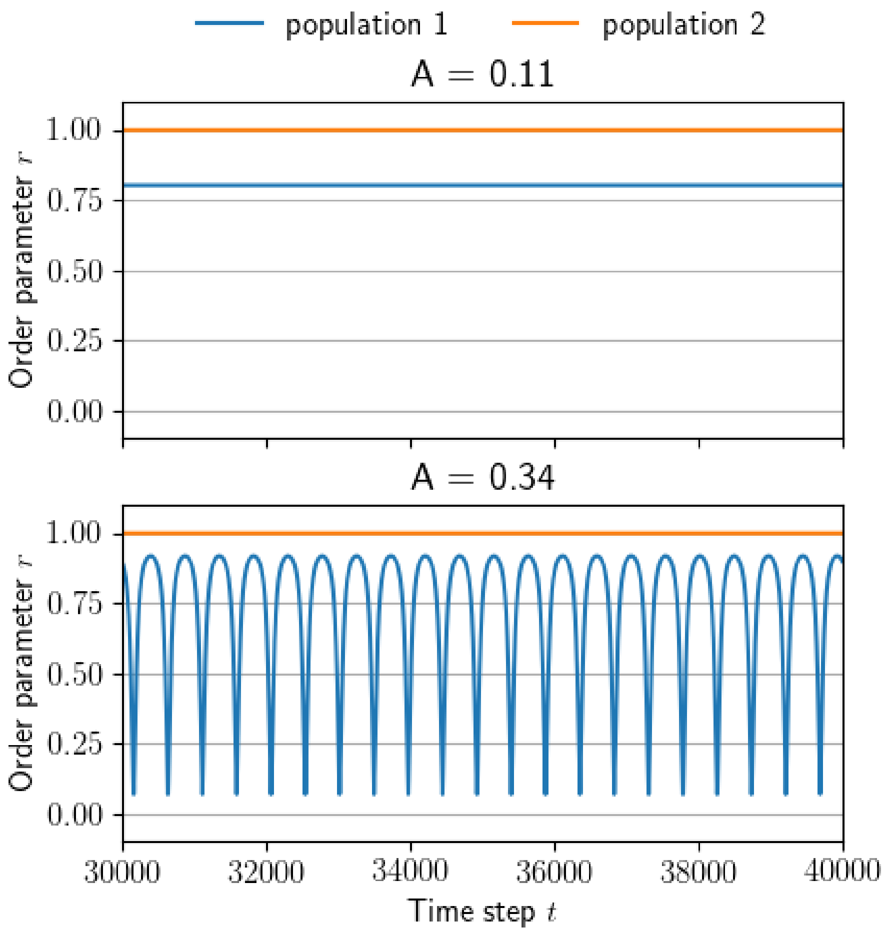

Figure 5. Long-term behaviour of the order parameters in the reduced Ott-Antonsen system. Initial conditions for $\{r_1, r_2, \psi\}$ are uniformly distributed random values within $[0, 1]$ and $[0, 2\pi]$ respectively. In both cases $\beta = 0.01$ and A as indicated in the figure. In order to show the stable state of the system the first 30000 timesteps are omitted as transient.

3.2 Sensitive dependence on initial conditions

The initial conditions chosen to start integration are crucial for the evolution of the system. In both the explicit and the reduced oscillator systems this shows to hold true. A study of the reduced system by Martens et al. [3], see figure 4, has investigated the explicit parameter dependence of chimera state emergence. Using the reduced system they show that the probability of the system evolving to a chimera state decreases rapidly for $\beta > 0.05$.

Some parameter search suggests that in the given case the probability of chimera state emergence mainly depends on the choice of β, as also suggested by Martens et al. [3]. As small values lead to the highest rates of chimera states $\beta = 0.01$ proves to be a good choice and is used in both systems. Nevertheless no set of parameters shows independence of initial conditions.

In case of the explicit Kuramoto system initial conditions close to a chimera state are suggested to be helpful [4]. Therefore a normal distribution with different variances for each population are chosen. In contrary, Martens et al. [3] states that dependence on initial conditions is more

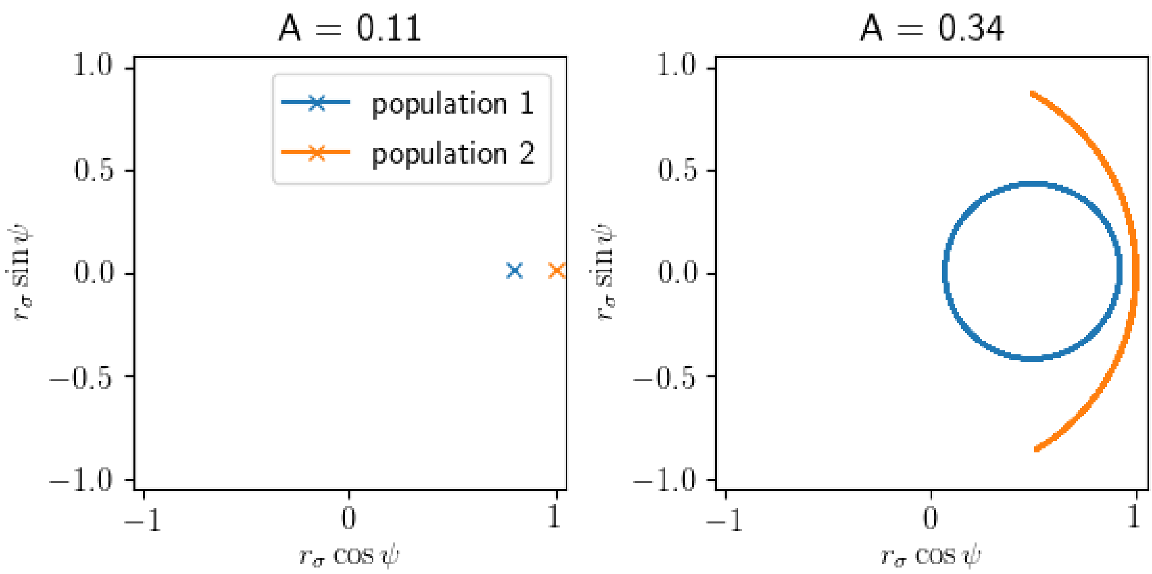

Figure 6. Trajectories in feature space corresponding to the time series in figure 5. The stable chimera (left) converges to two fixed points, whereas the breathing chimera state (right) displays a limit cycle for the breathing population and an oscillation for the synchronised population.

sensitive. This is supported by the finding that the population with smaller variance does not always synchronize but may also evolve to a chimera state while the less synchronised population fully synchronises, see figure 2.

These findings already indicate, that the basins of attraction for chimera states are not trivial. This was conclusively shown by Martens et al. [3]. It seems often suggested that the explicit case behaves similar to the reduced case but this is not easily provable, due to the system's dependence on many factors, e. g. initial conditions, parameters, system size, and so forth. Nevertheless, by choosing β close to 0 chimera states can be found with large probabilities for almost any $A \in [0, 0.5]$, enabling simulation of systems that will show evolution of chimera states for many initial conditions.

4 Conclusion

This short study of chimera states in systems of Kuramoto oscillators has shown the complexity of their dependence on initial conditions. As shown in other studies of the reduction of these systems [3] the basins of attraction separating chimera states from fully synchronised systems are not trivial. Further parameters have to be carefully chosen to reach high probabilities for evolution of chimera states. In the explicit system parameter dependence additionally depends on a given system size.

Unlike suggested by some articles [1, 4] chimera states are not as easily constructed as might seem at first. Furthermore the stability of chimera states is not yet conclusively proven [1]. For small N they might be of transient nature whereas for large N they might converge to the results of the Ott-Antonsen reduction. As the system is yet nonlinear, showing these properties is computationally very expensive. As it is not obvious that reduced and explicit case behave alike, the bifurcation behaviour of the explicit case can not be assumed to resemble figure 4 exactly. This would have to also be studied for different configurations of the system.

The specific case considered here regards only symmetric coupling between populations, i. e. $K_{\sigma\sigma'}$ is symmetric. According to Tian et al. [6] asymmetric coupling increases the probability of encounter of chimera states significantly. Thus, on top of an already complicated configuration, adding another variable to the system that can be explored.

Even in simple systems of oscillators the emergence of chimera states is tightly bound to the initial conditions and configuration of the system. Simulating them is not trivial and may in more complex systems be even more specific. As they are however expected to occur in biological systems [1], such as the brain or heart, studying the initial conditions that lead to a splitting of symmetry may lead to important advances in medicine and neural sciences.

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