n-combinations from a set with k elements when repetition is allowed and the ways to place n indistinguishable balls into k distinguishable boxes. To set up this correspondence, we put a ball in the ith bin each time the ith element of the set is included in the n-combination.

EXAMPLE 9 How many ways are there to place 10 indistinguishable balls into eight distinguishable bins?

Solution: The number of ways to place 10 indistinguishable balls into eight bins equals the number of 10-combinations from a set with eight elements when repetition is allowed. Consequently, there are

$$C(8 + 10 - 1, 10) = C(17, 10) = \frac{17!}{10!7!} = 19{,}448.$$

◀

This means that there are $C(n + r - 1, n - 1)$ ways to place r indistinguishable objects into n distinguishable boxes.

DISTINGUISHABLE OBJECTS AND INDISTINGUISHABLE BOXES Counting the ways to place n distinguishable objects into k indistinguishable boxes is more difficult than counting the ways to place objects, distinguishable or indistinguishable objects, into distinguishable boxes. We illustrate this with an example.

EXAMPLE 10 How many ways are there to put four different employees into three indistinguishable offices, when each office can contain any number of employees?

Solution: We will solve this problem by enumerating all the ways these employees can be placed into the offices. We represent the four employees by A, B, C, and D. First, we note that we can distribute employees so that all four are put into one office, three are put into one office and a fourth is put into a second office, two employees are put into one office and two put into a second office, and finally, two are put into one office, and one each put into the other two offices. Each way to distribute these employees to these offices can be represented by a way to partition the elements A, B, C, and D into disjoint subsets.

We can put all four employees into one office in exactly one way, represented by $\{\{A, B, C, D\}\}$. We can put three employees into one office and the fourth employee into a different office in exactly four ways, represented by $\{\{A, B, C\}, \{D\}\}$, $\{\{A, B, D\}, \{C\}\}$, $\{\{A, C, D\}, \{B\}\}$, and $\{\{B, C, D\}, \{A\}\}$. We can put two employees into one office and two into a second office in exactly three ways, represented by $\{\{A, B\}, \{C, D\}\}$, $\{\{A, C\}, \{B, D\}\}$, and $\{\{A, D\}, \{B, C\}\}$. Finally, we can put two employees into one office, and one each into each of the remaining two offices in six ways, represented by $\{\{A, B\}, \{C\}, \{D\}\}$, $\{\{A, C\}, \{B\}, \{D\}\}$, $\{\{A, D\}, \{B\}, \{C\}\}$, $\{\{B, C\}, \{A\}, \{D\}\}$, $\{\{B, D\}\}, \{A\}, \{C\}\}$, and $\{\{C, D\}, \{A\}, \{B\}\}$.

Counting all the possibilities, we find that there are 14 ways to put four different employees into three indistinguishable offices. Another way to look at this problem is to look at the number of offices into which we put employees. Note that there are six ways to put four different employees into three indistinguishable offices so that no office is empty, seven ways to put four different employees into two indistinguishable offices so that no office is empty, and one way to put four employees into one office so that it is not empty. ◀

There is no simple closed formula for the number of ways to distribute n distinguishable objects into j indistinguishable boxes. However, there is a formula involving a summation, which we will now describe. Let $S(n, j)$ denote the number of ways to distribute n distinguishable objects into j indistinguishable boxes so that no box is empty. The numbers $S(n, j)$ are called **Stirling numbers of the second kind**. For instance, Example 10 shows that $S(4, 3) = 6$, $S(4, 2) = 7$, and $S(4, 1) = 1$. We see that the number of ways to distribute n distinguishable objects into k indistinguishable boxes (where the number of boxes that are nonempty equals k, $k - 1, \ldots, 2$, or 1) equals $\sum_{j=1}^{k} S(n, j)$. For instance, following the reasoning in Example 10, the number of ways to distribute four distinguishable objects into three indistinguishable boxes

equals $S(4, 1) + S(4, 2) + S(4, 3) = 1 + 7 + 6 = 14$. Using the inclusion–exclusion principle (see Section 8.6) it can be shown that

$$S(n, j) = \frac{1}{j!}\sum_{i=0}^{j-1}(-1)^i\binom{j}{i}(j-i)^n.$$

Consequently, the number of ways to distribute n distinguishable objects into k indistinguishable boxes equals

$$\sum_{j=1}^{k} S(n, j) = \sum_{j=1}^{k}\frac{1}{j!}\sum_{i=0}^{j-1}(-1)^i\binom{j}{i}(j-i)^n.$$

Remark: The reader may be curious about the Stirling numbers of the first kind. A combinatorial definition of the **signless Stirling numbers of the first kind**, the absolute values of the Stirling numbers of the first kind, can be found in the preamble to Exercise 47 in the Supplementary Exercises. For the definition of Stirling numbers of the first kind, for more information about Stirling numbers of the second kind, and to learn more about Stirling numbers of the first kind and the relationship between Stirling numbers of the first and second kind, see combinatorics textbooks such as [Bó07], [Br99], and [RoTe05], and Chapter 6 in [MiRo91].

INDISTINGUISHABLE OBJECTS AND INDISTINGUISHABLE BOXES Some counting problems can be solved by determining the number of ways to distribute indistinguishable objects into indistinguishable boxes. We illustrate this principle with an example.

EXAMPLE 11 How many ways are there to pack six copies of the same book into four identical boxes, where a box can contain as many as six books?

Solution: We will enumerate all ways to pack the books. For each way to pack the books, we will list the number of books in the box with the largest number of books, followed by the numbers of books in each box containing at least one book, in order of decreasing number of books in a box. The ways we can pack the books are

6
5, 1
4, 2
4, 1, 1
3, 3
3, 2, 1
3, 1, 1, 1
2, 2, 2
2, 2, 1, 1.

For example, 4, 1, 1 indicates that one box contains four books, a second box contains a single book, and a third box contains a single book (and the fourth box is empty). We conclude that there are nine allowable ways to pack the books, because we have listed them all. ◀

Observe that distributing n indistinguishable objects into k indistinguishable boxes is the same as writing n as the sum of at most k positive integers in nonincreasing order. If $a_1 + a_2 + \cdots + a_j = n$, where $a_1, a_2, \ldots, a_j$ are positive integers with $a_1 \geq a_2 \geq \cdots \geq a_j$, we say that $a_1, a_2, \ldots, a_j$ is a **partition** of the positive integer n into j positive integers. We see that if $p_k(n)$ is the number of partitions of n into at most k positive integers, then there are $p_k(n)$ ways to distribute n indistinguishable objects into k indistinguishable boxes. No simple closed formula exists for this number. For more information about partitions of positive integers, see [Ro11].

Exercises

1. In how many different ways can five elements be selected in order from a set with three elements when repetition is allowed?
2. In how many different ways can five elements be selected in order from a set with five elements when repetition is allowed?
3. How many strings of six letters are there?
4. Every day a student randomly chooses a sandwich for lunch from a pile of wrapped sandwiches. If there are six kinds of sandwiches, how many different ways are there for the student to choose sandwiches for the seven days of a week if the order in which the sandwiches are chosen matters?
5. How many ways are there to assign three jobs to five employees if each employee can be given more than one job?
6. How many ways are there to select five unordered elements from a set with three elements when repetition is allowed?
7. How many ways are there to select three unordered elements from a set with five elements when repetition is allowed?
8. How many different ways are there to choose a dozen donuts from the 21 varieties at a donut shop?
9. A bagel shop has onion bagels, poppy seed bagels, egg bagels, salty bagels, pumpernickel bagels, sesame seed bagels, raisin bagels, and plain bagels. How many ways are there to choose
   a) six bagels?
   b) a dozen bagels?
   c) two dozen bagels?
   d) a dozen bagels with at least one of each kind?
   e) a dozen bagels with at least three egg bagels and no more than two salty bagels?
10. A croissant shop has plain croissants, cherry croissants, chocolate croissants, almond croissants, apple croissants, and broccoli croissants. How many ways are there to choose
    a) a dozen croissants?
    b) three dozen croissants?
    c) two dozen croissants with at least two of each kind?
    d) two dozen croissants with no more than two broccoli croissants?
    e) two dozen croissants with at least five chocolate croissants and at least three almond croissants?
    f) two dozen croissants with at least one plain croissant, at least two cherry croissants, at least three chocolate croissants, at least one almond croissant, at least two apple croissants, and no more than three broccoli croissants?
11. How many ways are there to choose eight coins from a piggy bank containing 100 identical pennies and 80 identical nickels?
12. How many different combinations of pennies, nickels, dimes, quarters, and half dollars can a piggy bank contain if it has 20 coins in it?
13. A book publisher has 3000 copies of a discrete mathematics book. How many ways are there to store these books in their three warehouses if the copies of the book are indistinguishable?
14. How many solutions are there to the equation
    $$x_1 + x_2 + x_3 + x_4 = 17,$$
    where x_1, x_2, x_3, and x_4 are nonnegative integers?
15. How many solutions are there to the equation
    $$x_1 + x_2 + x_3 + x_4 + x_5 = 21,$$
    where x_i, $i = 1, 2, 3, 4, 5$, is a nonnegative integer such that
    a) $x_1 \geq 1$?
    b) $x_i \geq 2$ for $i = 1, 2, 3, 4, 5$?
    c) $0 \leq x_1 \leq 10$?
    d) $0 \leq x_1 \leq 3$, $1 \leq x_2 < 4$, and $x_3 \geq 15$?
16. How many solutions are there to the equation
    $$x_1 + x_2 + x_3 + x_4 + x_5 + x_6 = 29,$$
    where x_i, $i = 1, 2, 3, 4, 5, 6$, is a nonnegative integer such that
    a) $x_i > 1$ for $i = 1, 2, 3, 4, 5, 6$?
    b) $x_1 \geq 1$, $x_2 \geq 2$, $x_3 \geq 3$, $x_4 \geq 4$, $x_5 > 5$, and $x_6 \geq 6$?
    c) $x_1 \leq 5$?
    d) $x_1 < 8$ and $x_2 > 8$?
17. How many strings of 10 ternary digits (0, 1, or 2) are there that contain exactly two 0s, three 1s, and five 2s?
18. How many strings of 20-decimal digits are there that contain two 0s, four 1s, three 2s, one 3, two 4s, three 5s, two 7s, and three 9s?
19. Suppose that a large family has 14 children, including two sets of identical triplets, three sets of identical twins, and two individual children. How many ways are there to seat these children in a row of chairs if the identical triplets or twins cannot be distinguished from one another?
20. How many solutions are there to the inequality
    $$x_1 + x_2 + x_3 \leq 11,$$
    where x_1, x_2, and x_3 are nonnegative integers? [*Hint:* Introduce an auxiliary variable x_4 such that $x_1 + x_2 + x_3 + x_4 = 11$.]
21. How many ways are there to distribute six indistinguishable balls into nine distinguishable bins?
22. How many ways are there to distribute 12 indistinguishable balls into six distinguishable bins?
23. How many ways are there to distribute 12 distinguishable objects into six distinguishable boxes so that two objects are placed in each box?
24. How many ways are there to distribute 15 distinguishable objects into five distinguishable boxes so that the boxes have one, two, three, four, and five objects in them, respectively.

25. How many positive integers less than 1,000,000 have the sum of their digits equal to 19?

26. How many positive integers less than 1,000,000 have exactly one digit equal to 9 and have a sum of digits equal to 13?

27. There are 10 questions on a discrete mathematics final exam. How many ways are there to assign scores to the problems if the sum of the scores is 100 and each question is worth at least 5 points?

28. Show that there are $C(n + r - q_1 - q_2 - \cdots - q_r - 1, n - q_1 - q_2 - \cdots - q_r)$ different unordered selections of n objects of r different types that include at least q_1 objects of type one, q_2 objects of type two, ..., and q_r objects of type r.

29. How many different bit strings can be transmitted if the string must begin with a 1 bit, must include three additional 1 bits (so that a total of four 1 bits is sent), must include a total of 12 0 bits, and must have at least two 0 bits following each 1 bit?

30. How many different strings can be made from the letters in *MISSISSIPPI*, using all the letters?

31. How many different strings can be made from the letters in *ABRACADABRA*, using all the letters?

32. How many different strings can be made from the letters in *AARDVARK*, using all the letters, if all three *A*s must be consecutive?

33. How many different strings can be made from the letters in *ORONO*, using some or all of the letters?

34. How many strings with five or more characters can be formed from the letters in *SEERESS*?

35. How many strings with seven or more characters can be formed from the letters in *EVERGREEN*?

36. How many different bit strings can be formed using six 1s and eight 0s?

37. A student has three mangos, two papayas, and two kiwi fruits. If the student eats one piece of fruit each day, and only the type of fruit matters, in how many different ways can these fruits be consumed?

38. A professor packs her collection of 40 issues of a mathematics journal in four boxes with 10 issues per box. How many ways can she distribute the journals if

a) each box is numbered, so that they are distinguishable?

b) the boxes are identical, so that they cannot be distinguished?

39. How many ways are there to travel in xyz space from the origin $(0, 0, 0)$ to the point $(4, 3, 5)$ by taking steps one unit in the positive x direction, one unit in the positive y direction, or one unit in the positive z direction? (Moving in the negative x, y, or z direction is prohibited, so that no backtracking is allowed.)

40. How many ways are there to travel in $xyzw$ space from the origin $(0, 0, 0, 0)$ to the point $(4, 3, 5, 4)$ by taking steps one unit in the positive x, positive y, positive z, or positive w direction?

41. How many ways are there to deal hands of seven cards to each of five players from a standard deck of 52 cards?

42. In bridge, the 52 cards of a standard deck are dealt to four players. How many different ways are there to deal bridge hands to four players?

43. How many ways are there to deal hands of five cards to each of six players from a deck containing 48 different cards?

44. In how many ways can a dozen books be placed on four distinguishable shelves

a) if the books are indistinguishable copies of the same title?

b) if no two books are the same, and the positions of the books on the shelves matter? [*Hint:* Break this into 12 tasks, placing each book separately. Start with the sequence 1, 2, 3, 4 to represent the shelves. Represent the books by $b_i, i = 1, 2, \ldots, 12$. Place b_1 to the right of one of the terms in 1, 2, 3, 4. Then successively place $b_2, b_3, \ldots$, and b_{12}.]

45. How many ways can n books be placed on k distinguishable shelves

a) if the books are indistinguishable copies of the same title?

b) if no two books are the same, and the positions of the books on the shelves matter?

46. A shelf holds 12 books in a row. How many ways are there to choose five books so that no two adjacent books are chosen? [*Hint:* Represent the books that are chosen by bars and the books not chosen by stars. Count the number of sequences of five bars and seven stars so that no two bars are adjacent.]

***47.** Use the product rule to prove Theorem 4, by first placing objects in the first box, then placing objects in the second box, and so on.

***48.** Prove Theorem 4 by first setting up a one-to-one correspondence between permutations of n objects with n_i indistinguishable objects of type $i, i = 1, 2, 3, \ldots, k$, and the distributions of n objects in k boxes such that n_i objects are placed in box $i, i = 1, 2, 3, \ldots, k$ and then applying Theorem 3.

***49.** In this exercise we will prove Theorem 2 by setting up a one-to-one correspondence between the set of r-combinations with repetition allowed of $S = \{1, 2, 3, \ldots, n\}$ and the set of r-combinations of the set $T = \{1, 2, 3, \ldots, n + r - 1\}$.

a) Arrange the elements in an r-combination, with repetition allowed, of S into an increasing sequence $x_1 \leq x_2 \leq \cdots \leq x_r$. Show that the sequence formed by adding $k - 1$ to the kth term is strictly increasing. Conclude that this sequence is made up of r distinct elements from T.

b) Show that the procedure described in (a) defines a one-to-one correspondence between the set of r-combinations, with repetition allowed, of S and the r-combinations of T. [*Hint:* Show the correspondence can be reversed by associating to the r-combination $\{x_1, x_2, \ldots, x_r\}$ of T, with $1 \leq x_1 < x_2 < \cdots < x_r \leq n + r - 1$, the r-combination with

repetition allowed from S, formed by subtracting $k-1$ from the kth element.]

c) Conclude that there are $C(n+r-1, r)$ r-combinations with repetition allowed from a set with n elements.

50. How many ways are there to distribute five distinguishable objects into three indistinguishable boxes?

51. How many ways are there to distribute six distinguishable objects into four indistinguishable boxes so that each of the boxes contains at least one object?

52. How many ways are there to put five temporary employees into four identical offices?

53. How many ways are there to put six temporary employees into four identical offices so that there is at least one temporary employee in each of these four offices?

54. How many ways are there to distribute five indistinguishable objects into three indistinguishable boxes?

55. How many ways are there to distribute six indistinguishable objects into four indistinguishable boxes so that each of the boxes contains at least one object?

56. How many ways are there to pack eight identical DVDs into five indistinguishable boxes so that each box contains at least one DVD?

57. How many ways are there to pack nine identical DVDs into three indistinguishable boxes so that each box contains at least two DVDs?

58. How many ways are there to distribute five balls into seven boxes if each box must have at most one ball in it if

a) both the balls and boxes are labeled?

b) the balls are labeled, but the boxes are unlabeled?

c) the balls are unlabeled, but the boxes are labeled?

d) both the balls and boxes are unlabeled?

59. How many ways are there to distribute five balls into three boxes if each box must have at least one ball in it if

a) both the balls and boxes are labeled?

b) the balls are labeled, but the boxes are unlabeled?

c) the balls are unlabeled, but the boxes are labeled?

d) both the balls and boxes are unlabeled?

60. Suppose that a basketball league has 32 teams, split into two conferences of 16 teams each. Each conference is split into three divisions. Suppose that the North Central Division has five teams. Each of the teams in the North Central Division plays four games against each of the other teams in this division, three games against each of the 11 remaining teams in the conference, and two games against each of the 16 teams in the other conference. In how many different orders can the games of one of the teams in the North Central Division be scheduled?

***61.** Suppose that a weapons inspector must inspect each of five different sites twice, visiting one site per day. The inspector is free to select the order in which to visit these sites, but cannot visit site X, the most suspicious site, on two consecutive days. In how many different orders can the inspector visit these sites?

62. How many different terms are there in the expansion of $(x_1 + x_2 + \cdots + x_m)^n$ after all terms with identical sets of exponents are added?

***63.** Prove the **Multinomial Theorem:** If n is a positive integer, then

$$(x_1 + x_2 + \cdots + x_m)^n = \sum_{n_1+n_2+\cdots+n_m=n} C(n; n_1, n_2, \ldots, n_m) x_1^{n_1} x_2^{n_2} \cdots x_m^{n_m},$$

where

$$C(n; n_1, n_2, \ldots, n_m) = \frac{n!}{n_1!\, n_2! \cdots n_m!}$$

is a **multinomial coefficient**.

64. Find the expansion of $(x + y + z)^4$.

65. Find the coefficient of $x^3y^2z^5$ in $(x + y + z)^{10}$.

66. How many terms are there in the expansion of

$$(x + y + z)^{100}?$$

6.6 Generating Permutations and Combinations

Introduction

Methods for counting various types of permutations and combinations were described in the previous sections of this chapter, but sometimes permutations or combinations need to be generated, not just counted. Consider the following three problems. First, suppose that a salesperson must visit six different cities. In which order should these cities be visited to minimize total travel time? One way to determine the best order is to determine the travel time for each of the $6! = 720$ different orders in which the cities can be visited and choose the one with the smallest travel time. Second, suppose we are given a set of six positive integers and wish to find a subset of them that has 100 as their sum, if such a subset exists. One way to find these numbers is to generate all $2^6 = 64$ subsets and check the sum of their elements. Third, suppose a laboratory has 95 employees. A group of 12 of these employees with a particular set of 25 skills is needed for a project. (Each employee can have one or more of these skills.) One way to find such a

set of employees is to generate all sets of 12 of these employees and check whether they have the desired skills. These examples show that it is often necessary to generate permutations and combinations to solve problems.

Generating Permutations

Links

Any set with n elements can be placed in one-to-one correspondence with the set $\{1, 2, 3, \ldots, n\}$. We can list the permutations of any set of n elements by generating the permutations of the n smallest positive integers and then replacing these integers with the corresponding elements. Many different algorithms have been developed to generate the $n!$ permutations of this set. We will describe one of these that is based on the **lexicographic** (or **dictionary**) **ordering** of the set of permutations of $\{1, 2, 3, \ldots, n\}$. In this ordering, the permutation $a_1a_2 \cdots a_n$ precedes the permutation of $b_1b_2 \cdots b_n$, if for some k, with $1 \leq k \leq n$, $a_1 = b_1, a_2 = b_2, \ldots, a_{k-1} = b_{k-1}$, and $a_k < b_k$. In other words, a permutation of the set of the n smallest positive integers precedes (in lexicographic order) a second permutation if the number in this permutation in the first position where the two permutations disagree is smaller than the number in that position in the second permutation.

EXAMPLE 1 The permutation 23415 of the set $\{1, 2, 3, 4, 5\}$ precedes the permutation 23514, because these permutations agree in the first two positions, but the number in the third position in the first permutation, 4, is smaller than the number in the third position in the second permutation, 5. Similarly, the permutation 41532 precedes 52143. ◀

An algorithm for generating the permutations of $\{1, 2, \ldots, n\}$ can be based on a procedure that constructs the next permutation in lexicographic order following a given permutation $a_1a_2 \cdots a_n$. We will show how this can be done. First, suppose that $a_{n-1} < a_n$. Interchange a_{n-1} and a_n to obtain a larger permutation. No other permutation is both larger than the original permutation and smaller than the permutation obtained by interchanging a_{n-1} and a_n. For instance, the next larger permutation after 234156 is 234165. On the other hand, if $a_{n-1} > a_n$, then a larger permutation cannot be obtained by interchanging these last two terms in the permutation. Look at the last three integers in the permutation. If $a_{n-2} < a_{n-1}$, then the last three integers in the permutation can be rearranged to obtain the next largest permutation. Put the smaller of the two integers a_{n-1} and a_n that is greater than a_{n-2} in position $n - 2$. Then, place the remaining integer and a_{n-2} into the last two positions in increasing order. For instance, the next larger permutation after 234165 is 234516.

On the other hand, if $a_{n-2} > a_{n-1}$ (and $a_{n-1} > a_n$), then a larger permutation cannot be obtained by permuting the last three terms in the permutation. Based on these observations, a general method can be described for producing the next larger permutation in increasing order following a given permutation $a_1a_2 \cdots a_n$. First, find the integers a_j and a_{j+1} with $a_j < a_{j+1}$ and

$$a_{j+1} > a_{j+2} > \cdots > a_n,$$

that is, the last pair of adjacent integers in the permutation where the first integer in the pair is smaller than the second. Then, the next larger permutation in lexicographic order is obtained by putting in the jth position the least integer among $a_{j+1}, a_{j+2}, \ldots,$ and a_n that is greater than a_j and listing in increasing order the rest of the integers $a_j, a_{j+1}, \ldots, a_n$ in positions $j + 1$ to n. It is easy to see that there is no other permutation larger than the permutation $a_1a_2 \cdots a_n$ but smaller than the new permutation produced. (The verification of this fact is left as an exercise for the reader.)

EXAMPLE 2 What is the next permutation in lexicographic order after 362541?

Extra Examples

Solution: The last pair of integers a_j and a_{j+1} where $a_j < a_{j+1}$ is $a_3 = 2$ and $a_4 = 5$. The least integer to the right of 2 that is greater than 2 in the permutation is $a_5 = 4$. Hence, 4 is placed in the third position. Then the integers 2, 5, and 1 are placed in order in the last three positions, giving 125 as the last three positions of the permutation. Hence, the next permutation is 364125. ◀

To produce the $n!$ permutations of the integers $1, 2, 3, \ldots, n$, begin with the smallest permutation in lexicographic order, namely, $123 \cdots n$, and successively apply the procedure described for producing the next larger permutation of $n! - 1$ times. This yields all the permutations of the n smallest integers in lexicographic order.

EXAMPLE 3 Generate the permutations of the integers 1, 2, 3 in lexicographic order.

Solution: Begin with 123. The next permutation is obtained by interchanging 3 and 2 to obtain 132. Next, because $3 > 2$ and $1 < 3$, permute the three integers in 132. Put the smaller of 3 and 2 in the first position, and then put 1 and 3 in increasing order in positions 2 and 3 to obtain 213. This is followed by 231, obtained by interchanging 1 and 3, because $1 < 3$. The next larger permutation has 3 in the first position, followed by 1 and 2 in increasing order, namely, 312. Finally, interchange 1 and 2 to obtain the last permutation, 321. We have generated the permutations of 1, 2, 3 in lexicographic order. They are 123, 132, 213, 231, 312, and 321. ◀

Algorithm 1 displays the procedure for finding the next permutation in lexicographic order after a permutation that is not $n\ n-1\ n-2\ \ldots\ 2\ 1$, which is the largest permutation.

ALGORITHM 1 Generating the Next Permutation in Lexicographic Order.

procedure *next permutation*($a_1 a_2 \ldots a_n$: permutation of
$\{1, 2, \ldots, n\}$ not equal to $n\ n-1\ \ldots\ 2\ 1$)
$j := n - 1$
while $a_j > a_{j+1}$
  $j := j - 1$
{j is the largest subscript with $a_j < a_{j+1}$}
$k := n$
while $a_j > a_k$
  $k := k - 1$
{a_k is the smallest integer greater than a_j to the right of a_j}
interchange a_j and a_k
$r := n$
$s := j + 1$
while $r > s$
  interchange a_r and a_s
  $r := r - 1$
  $s := s + 1$
{this puts the tail end of the permutation after the jth position in increasing order}
{$a_1 a_2 \ldots a_n$ is now the next permutation}

Generating Combinations

Links

How can we generate all the combinations of the elements of a finite set? Because a combination is just a subset, we can use the correspondence between subsets of $\{a_1, a_2, \ldots, a_n\}$ and bit strings of length n.

Recall that the bit string corresponding to a subset has a 1 in position k if a_k is in the subset, and has a 0 in this position if a_k is not in the subset. If all the bit strings of length n can be listed, then by the correspondence between subsets and bit strings, a list of all the subsets is obtained.

Recall that a bit string of length n is also the binary expansion of an integer between 0 and $2^n - 1$. The 2^n bit strings can be listed in order of their increasing size as integers in their binary expansions. To produce all binary expansions of length n, start with the bit string 000 . . . 00, with n zeros. Then, successively find the next expansion until the bit string 111 . . . 11 is obtained. At each stage the next binary expansion is found by locating the first position from the right that is not a 1, then changing all the 1s to the right of this position to 0s and making this first 0 (from the right) a 1.

EXAMPLE 4 Find the next bit string after 10 0010 0111.

Solution: The first bit from the right that is not a 1 is the fourth bit from the right. Change this bit to a 1 and change all the following bits to 0s. This produces the next larger bit string, 10 0010 1000. ◀

The procedure for producing the next larger bit string after $b_{n-1}b_{n-2}\ldots b_1b_0$ is given as Algorithm 2.

ALGORITHM 2 Generating the Next Larger Bit String.

procedure *next bit string*($b_{n-1}\, b_{n-2}\ldots b_1b_0$: bit string not equal to 11...11)
$i := 0$
while $b_i = 1$
  $b_i := 0$
  $i := i + 1$
$b_i := 1$
{$b_{n-1}\, b_{n-2}\ldots b_1b_0$ is now the next bit string}

Next, an algorithm for generating the r-combinations of the set $\{1, 2, 3, \ldots, n\}$ will be given. An r-combination can be represented by a sequence containing the elements in the subset in increasing order. The r-combinations can be listed using lexicographic order on these sequences. In this lexicographic ordering, the first r-combination is $\{1, 2, \ldots, r-1, r\}$ and the last r-combination is $\{n - r + 1, n - r + 2, \ldots, n - 1, n\}$. The next r-combination after $a_1a_2 \cdots a_r$ can be obtained in the following way: First, locate the last element a_i in the sequence such that $a_i \neq n - r + i$. Then, replace a_i with $a_i + 1$ and a_j with $a_i + j - i + 1$, for $j = i + 1, i + 2, \ldots, r$. It is left for the reader to show that this produces the next larger r-combination in lexicographic order. This procedure is illustrated with Example 5.

EXAMPLE 5 Find the next larger 4-combination of the set $\{1, 2, 3, 4, 5, 6\}$ after $\{1, 2, 5, 6\}$.

Solution: The last term among the terms a_i with $a_1 = 1$, $a_2 = 2$, $a_3 = 5$, and $a_4 = 6$ such that $a_i \neq 6 - 4 + i$ is $a_2 = 2$. To obtain the next larger 4-combination, increment a_2 by 1 to obtain $a_2 = 3$. Then set $a_3 = 3 + 1 = 4$ and $a_4 = 3 + 2 = 5$. Hence the next larger 4-combination is $\{1, 3, 4, 5\}$. ◀

Algorithm 3 displays pseudocode for this procedure.

ALGORITHM 3 Generating the Next r-Combination in Lexicographic Order.

procedure *next r-combination*($\{a_1, a_2, \ldots, a_r\}$: proper subset of $\{1, 2, \ldots, n\}$ not equal to $\{n - r + 1, \ldots, n\}$ with $a_1 < a_2 < \cdots < a_r$)
$i := r$
while $a_i = n - r + i$
  $i := i - 1$
$a_i := a_i + 1$
for $j := i + 1$ **to** r
  $a_j := a_i + j - i$
{$\{a_1, a_2, \ldots, a_r\}$ is now the next combination}

Exercises

1. Place these permutations of $\{1, 2, 3, 4, 5\}$ in lexicographic order: 43521, 15432, 45321, 23451, 23514, 14532, 21345, 45213, 31452, 31542.
2. Place these permutations of $\{1,2,3,4,5,6\}$ in lexicographic order: 234561, 231456, 165432, 156423, 543216, 541236, 231465, 314562, 432561, 654321, 654312, 435612.
3. The name of a file in a computer directory consists of three uppercase letters followed by a digit, where each letter is either A, B, or C, and each digit is either 1 or 2. List the name of these files in lexicographic order, where we order letters using the usual alphabetic order of letters.
4. Suppose that the name of a file in a computer directory consists of three digits followed by two lowercase letters and each digit is 0, 1, or 2, and each letter is either a or b. List the name of these files in lexicographic order, where we order letters using the usual alphabetic order of letters.
5. Find the next larger permutation in lexicographic order after each of these permutations.
   a) 1432 **b)** 54123 **c)** 12453
   d) 45231 **e)** 6714235 **f)** 31528764
6. Find the next larger permutation in lexicographic order after each of these permutations.
   a) 1342 **b)** 45321 **c)** 13245
   d) 612345 **e)** 1623547 **f)** 23587416
7. Use Algorithm 1 to generate the 24 permutations of the first four positive integers in lexicographic order.
8. Use Algorithm 2 to list all the subsets of the set $\{1, 2, 3, 4\}$.
9. Use Algorithm 3 to list all the 3-combinations of $\{1, 2, 3, 4, 5\}$.
10. Show that Algorithm 1 produces the next larger permutation in lexicographic order.
11. Show that Algorithm 3 produces the next larger r-combination in lexicographic order after a given r-combination.
12. Develop an algorithm for generating the r-permutations of a set of n elements.
13. List all 3-permutations of $\{1, 2, 3, 4, 5\}$.

The remaining exercises in this section develop another algorithm for generating the permutations of $\{1, 2, 3, \ldots, n\}$. This algorithm is based on Cantor expansions of integers. Every nonnegative integer less than $n!$ has a unique Cantor expansion

$$a_1 1! + a_2 2! + \cdots + a_{n-1}(n-1)!$$

where a_i is a nonnegative integer not exceeding i, for $i = 1, 2, \ldots, n-1$. The integers $a_1, a_2, \ldots, a_{n-1}$ are called the **Cantor digits** of this integer.

Given a permutation of $\{1, 2, \ldots, n\}$, let $a_{k-1}, k = 2, 3, \ldots, n$, be the number of integers less than k that follow k in the permutation. For instance, in the permutation 43215, a_1 is the number of integers less than 2 that follow 2, so $a_1 = 1$. Similarly, for this example $a_2 = 2$, $a_3 = 3$, and $a_4 = 0$. Consider the function from the set of permutations of $\{1, 2, 3, \ldots, n\}$ to the set of nonnegative integers less than $n!$ that sends a permutation to the integer that has $a_1, a_2, \ldots, a_{n-1}$, defined in this way, as its Cantor digits.

14. Find the Cantor digits $a_1, a_2, \ldots, a_{n-1}$ that correspond to these permutations.
    a) 246531 **b)** 12345 **c)** 654321

***15.** Show that the correspondence described in the preamble is a bijection between the set of permutations of $\{1, 2, 3, \ldots, n\}$ and the nonnegative integers less than $n!$.

16. Find the permutations of $\{1, 2, 3, 4, 5\}$ that correspond to these integers with respect to the correspondence between Cantor expansions and permutations as described in the preamble to Exercise 14.

a) 3 **b)** 89 **c)** 111

17. Develop an algorithm for producing all permutations of a set of n elements based on the correspondence described in the preamble to Exercise 14.

Key Terms and Results

TERMS

combinatorics: the study of arrangements of objects

enumeration: the counting of arrangements of objects

tree diagram: a diagram made up of a root, branches leaving the root, and other branches leaving some of the endpoints of branches

permutation: an ordered arrangement of the elements of a set

***r*-permutation:** an ordered arrangement of r elements of a set

***P*(*n*,*r*):** the number of r-permutations of a set with n elements

***r*-combination:** an unordered selection of r elements of a set

***C*(*n*,*r*):** the number of r-combinations of a set with n elements

binomial coefficient $\binom{n}{r}$: also the number of r-combinations of a set with n elements

combinatorial proof: a proof that uses counting arguments rather than algebraic manipulation to prove a result

Pascal's triangle: a representation of the binomial coefficients where the ith row of the triangle contains $\binom{i}{j}$ for $j = 0, 1, 2, \ldots, i$

***S*(*n*,*j*):** the Stirling number of the second kind denoting the number of ways to distribute n distinguishable objects into j indistinguishable boxes so that no box is empty

RESULTS

product rule for counting: The number of ways to do a procedure that consists of two tasks is the product of the number of ways to do the first task and the number of ways to do the second task after the first task has been done.

product rule for sets: The number of elements in the Cartesian product of finite sets is the product of the number of elements in each set.

sum rule for counting: The number of ways to do a task in one of two ways is the sum of the number of ways to do these tasks if they cannot be done simultaneously.

sum rule for sets: The number of elements in the union of pairwise disjoint finite sets is the sum of the numbers of elements in these sets.

subtraction rule for counting or **inclusion–exclusion for sets:** If a task can be done in either n_1 ways or n_2 ways, then the number of ways to do the task is $n_1 + n_2$ minus the number of ways to do the task that are common to the two different ways.

subtraction rule or **inclusion–exclusion for sets:** The number of elements in the union of two sets is the sum of the number of elements in these sets minus the number of elements in their intersection.

division rule for counting: There are n/d ways to do a task if it can be done using a procedure that can be carried out in n ways, and for every way w, exactly d of the n ways correspond to way w.

division rule for sets: Suppose that a finite set A is the union of n disjoint subsets each with d elements. Then $n = |A|/d$.

the pigeonhole principle: When more than k objects are placed in k boxes, there must be a box containing more than one object.

the generalized pigeonhole principle: When N objects are placed in k boxes, there must be a box containing at least $\lceil N/k \rceil$ objects.

$$P(n, r) = \frac{n!}{(n-r)!}$$

$$C(n, r) = \binom{n}{r} = \frac{n!}{r!(n-r)!}$$

Pascal's identity: $\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}$

the binomial theorem: $(x + y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k$

There are n^r r-permutations of a set with n elements when repetition is allowed.

There are $C(n + r - 1, r)$ r-combinations of a set with n elements when repetition is allowed.

There are $n!/(n_1!n_2! \cdots n_k!)$ permutations of n objects of k types where there are n_i indistinguishable objects of type i for $i = 1, 2, 3, \ldots, k$.

the algorithm for generating the permutations of the set $\{1, 2, \ldots, n\}$

Review Questions

1. Explain how the sum and product rules can be used to find the number of bit strings with a length not exceeding 10.

2. Explain how to find the number of bit strings of length not exceeding 10 that have at least one 0 bit.

3. a) How can the product rule be used to find the number of functions from a set with m elements to a set with n elements?

b) How many functions are there from a set with five elements to a set with 10 elements?

c) How can the product rule be used to find the number of one-to-one functions from a set with m elements to a set with n elements?
d) How many one-to-one functions are there from a set with five elements to a set with 10 elements?
e) How many onto functions are there from a set with five elements to a set with 10 elements?

4. How can you find the number of possible outcomes of a playoff between two teams where the first team that wins four games wins the playoff?

5. How can you find the number of bit strings of length ten that either begin with 101 or end with 010?

6. a) State the pigeonhole principle.
b) Explain how the pigeonhole principle can be used to show that among any 11 integers, at least two must have the same last digit.

7. a) State the generalized pigeonhole principle.
b) Explain how the generalized pigeonhole principle can be used to show that among any 91 integers, there are at least ten that end with the same digit.

8. a) What is the difference between an r-combination and an r-permutation of a set with n elements?
b) Derive an equation that relates the number of r-combinations and the number of r-permutations of a set with n elements.
c) How many ways are there to select six students from a class of 25 to serve on a committee?
d) How many ways are there to select six students from a class of 25 to hold six different executive positions on a committee?

9. a) What is Pascal's triangle?
b) How can a row of Pascal's triangle be produced from the one above it?

10. What is meant by a combinatorial proof of an identity? How is such a proof different from an algebraic one?

11. Explain how to prove Pascal's identity using a combinatorial argument.

12. a) State the binomial theorem.
b) Explain how to prove the binomial theorem using a combinatorial argument.
c) Find the coefficient of $x^{100}y^{101}$ in the expansion of $(2x + 5y)^{201}$.

13. a) Explain how to find a formula for the number of ways to select r objects from n objects when repetition is allowed and order does not matter.
b) How many ways are there to select a dozen objects from among objects of five different types if objects of the same type are indistinguishable?
c) How many ways are there to select a dozen objects from these five different types if there must be at least three objects of the first type?
d) How many ways are there to select a dozen objects from these five different types if there cannot be more than four objects of the first type?
e) How many ways are there to select a dozen objects from these five different types if there must be at least two objects of the first type, but no more than three objects of the second type?

14. a) Let n and r be positive integers. Explain why the number of solutions of the equation $x_1 + x_2 + \cdots + x_n = r$, where x_i is a nonnegative integer for $i = 1, 2, 3, \ldots, n$, equals the number of r-combinations of a set with n elements.
b) How many solutions in nonnegative integers are there to the equation $x_1 + x_2 + x_3 + x_4 = 17$?
c) How many solutions in positive integers are there to the equation in part (b)?

15. a) Derive a formula for the number of permutations of n objects of k different types, where there are n_1 indistinguishable objects of type one, n_2 indistinguishable objects of type two, $\ldots$, and n_k indistinguishable objects of type k.
b) How many ways are there to order the letters of the word *INDISCREETNESS*?

16. Describe an algorithm for generating all the permutations of the set of the n smallest positive integers.

17. a) How many ways are there to deal hands of five cards to six players from a standard 52-card deck?
b) How many ways are there to distribute n distinguishable objects into k distinguishable boxes so that n_i objects are placed in box i?

18. Describe an algorithm for generating all the combinations of the set of the n smallest positive integers.

Supplementary Exercises

1. How many ways are there to choose 6 items from 10 distinct items when
a) the items in the choices are ordered and repetition is not allowed?
b) the items in the choices are ordered and repetition is allowed?
c) the items in the choices are unordered and repetition is not allowed?
d) the items in the choices are unordered and repetition is allowed?

2. How many ways are there to choose 10 items from 6 distinct items when
a) the items in the choices are ordered and repetition is not allowed?
b) the items in the choices are ordered and repetition is allowed?
c) the items in the choices are unordered and repetition is not allowed?
d) the items in the choices are unordered and repetition is allowed?

3. A test contains 100 true/false questions. How many different ways can a student answer the questions on the test, if answers may be left blank?

4. How many strings of length 10 either start with 000 or end with 1111?

5. How many bit strings of length 10 over the alphabet $\{a, b, c\}$ have either exactly three as or exactly four bs?

6. The internal telephone numbers in the phone system on a campus consist of five digits, with the first digit not equal to zero. How many different numbers can be assigned in this system?

7. An ice cream parlor has 28 different flavors, 8 different kinds of sauce, and 12 toppings.

a) In how many different ways can a dish of three scoops of ice cream be made where each flavor can be used more than once and the order of the scoops does not matter?

b) How many different kinds of small sundaes are there if a small sundae contains one scoop of ice cream, a sauce, and a topping?

c) How many different kinds of large sundaes are there if a large sundae contains three scoops of ice cream, where each flavor can be used more than once and the order of the scoops does not matter; two kinds of sauce, where each sauce can be used only once and the order of the sauces does not matter; and three toppings, where each topping can be used only once and the order of the toppings does not matter?

8. How many positive integers less than 1000

a) have exactly three decimal digits?

b) have an odd number of decimal digits?

c) have at least one decimal digit equal to 9?

d) have no odd decimal digits?

e) have two consecutive decimal digits equal to 5?

f) are palindromes (that is, read the same forward and backward)?

9. When the numbers from 1 to 1000 are written out in decimal notation, how many of each of these digits are used?

a) 0 **b)** 1 **c)** 2 **d)** 9

10. There are 12 signs of the zodiac. How many people are needed to guarantee that at least six of these people have the same sign?

11. A fortune cookie company makes 213 different fortunes. A student eats at a restaurant that uses fortunes from this company and gives each customer one fortune cookie at the end of a meal. What is the largest possible number of times that the student can eat at the restaurant without getting the same fortune four times?

12. How many people are needed to guarantee that at least two were born on the same day of the week and in the same month (perhaps in different years)?

13. Show that given any set of 10 positive integers not exceeding 50 there exist at least two different five-element subsets of this set that have the same sum.

14. A package of baseball cards contains 20 cards. How many packages must be purchased to ensure that two cards in these packages are identical if there are a total of 550 different cards?

15. **a)** How many cards must be chosen from a standard deck of 52 cards to guarantee that at least two of the four aces are chosen?

b) How many cards must be chosen from a standard deck of 52 cards to guarantee that at least two of the four aces and at least two of the 13 kinds are chosen?

c) How many cards must be chosen from a standard deck of 52 cards to guarantee that there are at least two cards of the same kind?

d) How many cards must be chosen from a standard deck of 52 cards to guarantee that there are at least two cards of each of two different kinds?

***16.** Show that in any set of $n + 1$ positive integers not exceeding $2n$ there must be two that are relatively prime.

***17.** Show that in a sequence of m integers there exists one or more consecutive terms with a sum divisible by m.

18. Show that if five points are picked in the interior of a square with a side length of 2, then at least two of these points are no farther than $\sqrt{2}$ apart.

19. Show that the decimal expansion of a rational number must repeat itself from some point onward.

20. Once a computer worm infects a personal computer via an infected e-mail message, it sends a copy of itself to 100 e-mail addresses it finds in the electronic message mailbox on this personal computer. What is the maximum number of different computers this one computer can infect in the time it takes for the infected message to be forwarded five times?

21. How many ways are there to choose a dozen donuts from 20 varieties

a) if there are no two donuts of the same variety?

b) if all donuts are of the same variety?

c) if there are no restrictions?

d) if there are at least two varieties among the dozen donuts chosen?

e) if there must be at least six blueberry-filled donuts?

f) if there can be no more than six blueberry-filled donuts?

22. Find n if

a) $P(n, 2) = 110$. **b)** $P(n, n) = 5040$.

c) $P(n, 4) = 12P(n, 2)$.

23. Find n if

a) $C(n, 2) = 45$. **b)** $C(n, 3) = P(n, 2)$.

c) $C(n, 5) = C(n, 2)$.

24. Show that if n and r are nonnegative integers and $n \geq r$, then

$$P(n+1, r) = P(n, r)(n+1)/(n+1-r).$$

***25.** Suppose that S is a set with n elements. How many ordered pairs (A, B) are there such that A and B are subsets of S with $A \subseteq B$? [*Hint:* Show that each element of S belongs to A, $B - A$, or $S - B$.]

26. Give a combinatorial proof of Corollary 2 of Section 6.4 by setting up a correspondence between the subsets of a set with an even number of elements and the subsets of this set with an odd number of elements. [*Hint:* Take an element a in the set. Set up the correspondence by putting a in the subset if it is not already in it and taking it out if it is in the subset.]

27. Let n and r be integers with $1 \leq r < n$. Show that

$$C(n, r-1) = C(n+2, r+1) - 2C(n+1, r+1) + C(n, r+1).$$

28. Prove using mathematical induction that $\sum_{j=2}^{n} C(j, 2) = C(n+1, 3)$ whenever n is an integer greater than 1.

29. Show that if n is an integer then

$$\sum_{k=0}^{n} 3^k \binom{n}{k} = 4^n.$$

30. Show that $\sum_{i=1}^{n-1} \sum_{j=i+1}^{n} 1 = \binom{n}{2}$ if n is an integer with $n \geq 2$.

31. Show that $\sum_{i=1}^{n-2} \sum_{j=i+1}^{n-1} \sum_{k=j+1}^{n} 1 = \binom{n}{3}$ if n is an integer with $n \geq 3$.

32. In this exercise we will derive a formula for the sum of the squares of the n smallest positive integers. We will count the number of triples (i, j, k) where i, j, and k are integers such that $0 \leq i < k$, $0 \leq j < k$, and $1 \leq k \leq n$ in two ways.

a) Show that there are k^2 such triples with a fixed k. Deduce that there are $\sum_{k=1}^{n} k^2$ such triples.

b) Show that the number of such triples with $0 \leq i < j < k$ and the number of such triples with $0 \leq j < i < k$ both equal $C(n+1, 3)$.

c) Show that the number of such triples with $0 \leq i = j < k$ equals $C(n+1, 2)$.

d) Combining part (a) with parts (b) and (c), conclude that

$$\sum_{k=1}^{n} k^2 = 2C(n+1, 3) + C(n+1, 2) = n(n+1)(2n+1)/6.$$

***33.** How many bit strings of length n, where $n \geq 4$, contain exactly two occurrences of 01?

34. Let S be a set. We say that a collection of subsets $A_1, A_2, \ldots, A_n$ each containing d elements, where $d \geq 2$, is 2-*colorable* if it is possible to assign to each element of S one of two different colors so that in every subset A_i there are elements that have been assigned each color. Let $m(d)$ be the largest integer such that every collection of fewer than $m(d)$ sets each containing d elements is 2-colorable.

a) Show that the collection of all subsets with d elements of a set S with $2d - 1$ elements is not 2-colorable.

b) Show that $m(2) = 3$.

****c)** Show that $m(3) = 7$. [*Hint:* Show that the collection $\{1, 3, 5\}$, $\{1, 2, 6\}$, $\{1, 4, 7\}$, $\{2, 3, 4\}$, $\{2, 5, 7\}$, $\{3, 6, 7\}$, $\{4, 5, 6\}$ is not 2-colorable. Then show that all collections of six sets with three elements each are 2-colorable.]

35. A professor writes 20 multiple-choice questions, each with the possible answer a, b, c, or d, for a discrete mathematics test. If the number of questions with a, b, c, and d as their answer is 8, 3, 4, and 5, respectively, how many different answer keys are possible, if the questions can be placed in any order?

36. How many different arrangements are there of eight people seated at a round table, where two arrangements are considered the same if one can be obtained from the other by a rotation?

37. How many ways are there to assign 24 students to five faculty advisors?

38. How many ways are there to choose a dozen apples from a bushel containing 20 indistinguishable Delicious apples, 20 indistinguishable Macintosh apples, and 20 indistinguishable Granny Smith apples, if at least three of each kind must be chosen?

39. How many solutions are there to the equation $x_1 + x_2 + x_3 = 17$, where x_1, x_2, and x_3 are nonnegative integers with

a) $x_1 > 1$, $x_2 > 2$, and $x_3 > 3$?

b) $x_1 < 6$ and $x_3 > 5$?

c) $x_1 < 4$, $x_2 < 3$, and $x_3 > 5$?

40. a) How many different strings can be made from the word *PEPPERCORN* when all the letters are used?

b) How many of these strings start and end with the letter P?

c) In how many of these strings are the three letter Ps consecutive?

41. How many subsets of a set with ten elements

a) have fewer than five elements?

b) have more than seven elements?

c) have an odd number of elements?

42. A witness to a hit-and-run accident tells the police that the license plate of the car in the accident, which contains three letters followed by three digits, starts with the letters *AS* and contains both the digits 1 and 2. How many different license plates can fit this description?

43. How many ways are there to put n identical objects into m distinct containers so that no container is empty?

44. How many ways are there to seat six boys and eight girls in a row of chairs so that no two boys are seated next to each other?

45. How many ways are there to distribute six objects to five boxes if

a) both the objects and boxes are labeled?
b) the objects are labeled, but the boxes are unlabeled?
c) the objects are unlabeled, but the boxes are labeled?
d) both the objects and the boxes are unlabeled?

46. How many ways are there to distribute five objects into six boxes if

a) both the objects and boxes are labeled?
b) the objects are labeled, but the boxes are unlabeled?
c) the objects are unlabeled, but the boxes are labeled?
d) both the objects and the boxes are unlabeled?

The **signless Stirling number of the first kind** $c(n, k)$, where k and n are integers with $1 \leq k \leq n$, equals the number of ways to arrange n people around k circular tables with at least one person seated at each table, where two seatings of m people around a circular table are considered the same if everyone has the same left neighbor and the same right neighbor.

47. Find these signless Stirling numbers of the first kind.

a) c(3,2) **b)** c(4,2)
c) c(4,3) **d)** c(5,4)

48. Show that if n is a positive integer, then $\sum_{j=1}^{n} c(n, j) = n!$.

49. Show that if n is a positive integer with $n \geq 3$, then $c(n, n-2) = (3n-1)C(n, 3)/4$.

***50.** Show that if n and k are integers with $1 \leq k < n$, then $c(n+1, k) = c(n, k-1) + nc(n, k)$.

51. Give a combinatorial proof that 2^n divides $n!$ whenever n is an even positive integer. [*Hint:* Use Theorem 3 in Section 6.5 to count the number of permutations of $2n$ objects where there are two indistinguishable objects of n different types.

52. How many 11-element RNA sequences consist of 4 As, 3Cs, 2Us, and 2Gs, and end with CAA?

Exercises 53 and 54 are based on a discussion in [RoTe09]. A method used in the 1960s for sequencing RNA chains used enzymes to break chains after certain links. Some enzymes break RNA chains after each G link, while others break them after each C or U link. Using these enzymes it is sometimes possible to correctly sequence all the bases in an RNA chain.

***53.** Suppose that when an enzyme that breaks RNA chains after each G link is applied to a 12-link chain, the fragments obtained are G, CCG, AAAG, and UCCG, and when an enzyme that breaks RNA chains after each C or U link is applied, the fragments obtained are C, C, C, C, GGU, and GAAAG. Can you determine the entire 12-link RNA chain from these two sets of fragments? If so, what is this RNA chain?

***54.** Suppose that when an enzyme that breaks RNA chains after each G link is applied to a 12-link chain, the fragments obtained are AC, UG, and ACG and when an enzyme that breaks RNA chains after each C or U link is applied, the fragments obtained are U, GAC, and GAC. Can you determine the entire RNA chain from these two sets of fragments? If so, what is this RNA chain?

55. Devise an algorithm for generating all the r-permutations of a finite set when repetition is allowed.

56. Devise an algorithm for generating all the r-combinations of a finite set when repetition is allowed.

***57.** Show that if m and n are integers with $m \geq 3$ and $n \geq 3$, then $R(m, n) \leq R(m, n-1) + R(m-1, n)$.

***58.** Show that $R(3, 4) \geq 7$ by showing that in a group of six people, where any two people are friends or enemies, there are not necessarily three mutual friends or four mutual enemies.

Computer Projects

Write programs with these input and output.

1. Given a positive integer n and a nonnegative integer not exceeding n, find the number of r-permutations and r-combinations of a set with n elements.

2. Given positive integers n and r, find the number of r-permutations when repetition is allowed and r-combinations when repetition is allowed of a set with n elements.

3. Given a sequence of positive integers, find the longest increasing and the longest decreasing subsequence of the sequence.

***4.** Given an equation $x_1 + x_2 + \cdots + x_n = C$, where C is a constant, and $x_1, x_2, \ldots, x_n$ are nonnegative integers, list all the solutions.

5. Given a positive integer n, list all the permutations of the set $\{1, 2, 3, \ldots, n\}$ in lexicographic order.

6. Given a positive integer n and a nonnegative integer r not exceeding n, list all the r-combinations of the set $\{1, 2, 3, \ldots, n\}$ in lexicographic order.

7. Given a positive integer n and a nonnegative integer r not exceeding n, list all the r-permutations of the set $\{1, 2, 3, \ldots, n\}$ in lexicographic order.

8. Given a positive integer n, list all the combinations of the set $\{1, 2, 3, \ldots, n\}$.

9. Given positive integers n and r, list all the r-permutations, with repetition allowed, of the set $\{1, 2, 3, \ldots, n\}$.

10. Given positive integers n and r, list all the r-combinations, with repetition allowed, of the set $\{1, 2, 3, \ldots, n\}$.

Computations and Explorations

Use a computational program or programs you have written to do these exercises.

1. Find the number of possible outcomes in a two-team playoff when the winner is the first team to win 5 out of 9, 6 out of 11, 7 out of 13, and 8 out of 15.

2. Which binomial coefficients are odd? Can you formulate a conjecture based on numerical evidence?

3. Verify that $C(2n, n)$ is divisible by the square of a prime, when $n \neq 1, 2$, or 4, for as many positive integers n as you can. [The theorem that tells that $C(2n, n)$ is divisible by the square of a prime with $n \neq 1, 2$, or 4 was proved in 1996 by Andrew Granville and Olivier Ramaré. Their proof settled a conjecture made in 1980 by Paul Erdős and Ron Graham.]

4. Find as many odd integers n less than 200 as you can for which $C(n, \lfloor n/2 \rfloor)$ is not divisible by the square of a prime. Formulate a conjecture based on your evidence.

***5.** For each integer less than 100 determine whether $C(2n, n)$ is divisible by 3. Can you formulate a conjecture that tells us for which integers n the binomial coefficient $C(2n, n)$ is divisible by 3 based on the digits in the base three expansion of n?

6. Generate all the permutations of a set with eight elements.

7. Generate all the 6-permutations of a set with nine elements.

8. Generate all combinations of a set with eight elements.

9. Generate all 5-combinations with repetition allowed of a set with seven elements.

Writing Projects

Respond to these with essays using outside sources.

1. Describe some of the earliest uses of the pigeonhole principle by Dirichlet and other mathematicians.

2. Discuss ways in which the current telephone numbering plan can be extended to accommodate the rapid demand for more telephone numbers. (See if you can find some of the proposals coming from the telecommunications industry.) For each new numbering plan you discuss, show how to find the number of different telephone numbers it supports.

3. Discuss the importance of combinatorial reasoning in gene sequencing and related problems involving genomes.

4. Many combinatorial identities are described in this book. Find some sources of such identities and describe important combinatorial identities besides those already introduced in this book. Give some representative proofs, including combinatorial ones, of some of these identities.

5. Describe the different models used to model the distribution of particles in statistical mechanics, including Maxwell–Boltzmann, Bose–Einstein, and Fermi–Dirac statistics. In each case, describe the counting techniques used in the model.

6. Define the Stirling numbers of the first kind and describe some of their properties and the identities they satisfy.

7. Describe some of the properties and the identities that Stirling numbers of the second kind satisfy, including the connection between Stirling numbers of the first and second kinds.

8. Describe the latest discoveries of values and bounds for Ramsey numbers.

9. Describe additional ways to generate all the permutations of a set with n elements besides those found in Section 6.6. Compare these algorithms and the algorithms described in the text and exercises of Section 6.6 in terms of their computational complexity.

10. Describe at least one way to generate all the partitions of a positive integer n. (See Exercise 47 in Section 5.3.)

CHAPTER

7 Discrete Probability

Combinatorics and probability theory share common origins. The theory of probability was first developed more than 300 years ago, when certain gambling games were analyzed. Although probability theory was originally invented to study gambling, it now plays an essential role in a wide variety of disciplines. For example, probability theory is extensively applied in the study of genetics, where it can be used to help understand the inheritance of traits. Of course, probability still remains an extremely popular part of mathematics because of its applicability to gambling, which continues to be an extremely popular human endeavor.

In computer science, probability theory plays an important role in the study of the complexity of algorithms. In particular, ideas and techniques from probability theory are used to determine the average-case complexity of algorithms. Probabilistic algorithms can be used to solve many problems that cannot be easily or practically solved by deterministic algorithms. In a probabilistic algorithm, instead of always following the same steps when given the same input, as a deterministic algorithm does, the algorithm makes one or more random choices, which may lead to different output. In combinatorics, probability theory can even be used to show that objects with certain properties exist. The probabilistic method, a technique in combinatorics introduced by Paul Erdős and Alfréd Rényi, shows that an object with a specified property exists by showing that there is a positive probability that a randomly constructed object has this property. Probability theory can help us answer questions that involve uncertainty, such as determining whether we should reject an incoming mail message as spam based on the words that appear in the message.

7.1 An Introduction to Discrete Probability

Introduction

Probability theory dates back to 1526 when the Italian mathematician, physician, and gambler Girolamo Cardano wrote the first known systematic treatment of the subject in his book *Liber de Ludo Aleae* (*Book on Games of Chance*). (This book was not published until 1663, which may have held back the development of probability theory.) In the seventeenth century the French mathematician Blaise Pascal determined the odds of winning some popular bets based on the outcome when a pair of dice is repeatedly rolled. In the eighteenth century, the French mathematician Laplace, who also studied gambling, defined the probability of an event as the number of successful outcomes divided by the number of possible outcomes. For instance, the probability that a die comes up an odd number when it is rolled is the number of successful outcomes—namely, the number of ways it can come up odd—divided by the number of possible outcomes—namely, the number of different ways the die can come up. There are a total of six possible outcomes—namely, 1, 2, 3, 4, 5, and 6—and exactly three of these are successful outcomes—namely, 1, 3, and 5. Hence, the probability that the die comes up an odd number is $3/6 = 1/2$. (Note that it has been assumed that all possible outcomes are equally likely, or, in other words, that the die is fair.)

In this section we will restrict ourselves to experiments that have finitely many, equally likely, outcomes. This permits us to use Laplace's definition of the probability of an event. We will continue our study of probability in Section 7.2, where we will study experiments with finitely many outcomes that are not necessarily equally likely. In Section 7.2 we will also introduce

some key concepts in probability theory, including conditional probability, independence of events, and random variables. In Section 7.4 we will introduce the concepts of the expectation and variance of a random variable.

Finite Probability

An **experiment** is a procedure that yields one of a given set of possible outcomes. The **sample space** of the experiment is the set of possible outcomes. An **event** is a subset of the sample space. Laplace's definition of the probability of an event with finitely many possible outcomes will now be stated.

DEFINITION 1 If S is a finite nonempty sample space of equally likely outcomes, and E is an event, that is, a subset of S, then the *probability* of E is $p(E) = \dfrac{|E|}{|S|}$.

The probability of an event can never be negative or more than one!

According to Laplace's definition, the probability of an event is between 0 and 1. To see this, note that if E is an event from a finite sample space S, then $0 \leq |E| \leq |S|$, because $E \subseteq S$. Thus, $0 \leq p(E) = |E|/|S| \leq 1$.

Examples 1–7 illustrate how the probability of an event is found.

EXAMPLE 1 An urn contains four blue balls and five red balls. What is the probability that a ball chosen at random from the urn is blue?

Extra Examples

Solution: To calculate the probability, note that there are nine possible outcomes, and four of these possible outcomes produce a blue ball. Hence, the probability that a blue ball is chosen is $4/9$. ◀

EXAMPLE 2 What is the probability that when two dice are rolled, the sum of the numbers on the two dice is 7?

Solution: There are a total of 36 equally likely possible outcomes when two dice are rolled. (The product rule can be used to see this; because each die has six possible outcomes, the total

Links

GIROLAMO CARDANO (1501–1576) Cardano, born in Pavia, Italy, was the illegitimate child of Fazio Cardano, a lawyer, mathematician, and friend of Leonardo da Vinci, and Chiara Micheria, a young widow. In spite of illness and poverty, Cardano was able to study at the universities of Pavia and Padua, from where he received his medical degree. Cardano was not accepted into Milan's College of Physicians because of his illegitimate birth, as well as his eccentricity and confrontational style. Nevertheless, his medical skills were highly regarded. One of his main accomplishments as a physician is the first description of typhoid fever.

Cardano published more than 100 books on a diverse range of subjects, including medicine, the natural sciences, mathematics, gambling, physical inventions and experiments, and astrology. He also wrote a fascinating autobiography. In mathematics, Cardano's book *Ars Magna*, published in 1545, established the foundations of abstract algebra. This was the most comprehensive book on abstract algebra for more than a century; it presents many novel ideas of Cardano and of others, including methods for solving cubic and quartic equations from their coefficients. Cardano also made several important contributions to cryptography. Cardano was an advocate of education for the deaf, believing, unlike his contemporaries, that deaf people could learn to read and write before learning to speak, and could use their minds just as well as hearing people.

Cardano was often short of money. However, he kept himself solvent through gambling and winning money by beating others at chess. His book about games of chance, *Liber de Ludo Aleae*, written in 1526 (but published in 1663), offers the first systematic treatment of probability; it also describes effective ways to cheat. Cardano was considered to be a man of dubious moral character; he was often described as a liar, gambler, lecher, and heretic.

number of outcomes when two dice are rolled is $6^2 = 36$.) There are six successful outcomes, namely, (1, 6), (2, 5), (3, 4), (4, 3), (5, 2), and (6, 1), where the values of the first and second dice are represented by an ordered pair. Hence, the probability that a seven comes up when two fair dice are rolled is $6/36 = 1/6$. ◀

Links Lotteries are extremely popular throughout the world. We can easily compute the odds of winning different types of lotteries, as illustrated in Examples 3 and 4. (The odd of winning the popular Mega Millions and Powerball lotteries are studied in the supplementary exercises.)

EXAMPLE 3 In a lottery, players win a large prize when they pick four digits that match, in the correct order, four digits selected by a random mechanical process. A smaller prize is won if only three digits are matched. What is the probability that a player wins the large prize? What is the probability that a player wins the small prize?

Solution: There is only one way to choose all four digits correctly. By the product rule, there are $10^4 = 10{,}000$ ways to choose four digits. Hence, the probability that a player wins the large prize is $1/10{,}000 = 0.0001$.

Players win the smaller prize when they correctly choose exactly three of the four digits. Exactly one digit must be wrong to get three digits correct, but not all four correct. By the sum rule, to find the number of ways to choose exactly three digits correctly, we add the number of ways to choose four digits matching the digits picked in all but the ith position, for $i = 1, 2, 3, 4$.

To count the number of successes with the first digit incorrect, note that there are nine possible choices for the first digit (all but the one correct digit), and one choice for each of the other digits, namely, the correct digits for these slots. Hence, there are nine ways to choose four digits where the first digit is incorrect, but the last three are correct. Similarly, there are nine ways to choose four digits where the second digit is incorrect, nine with the third digit incorrect, and nine with the fourth digit incorrect. Hence, there is a total of 36 ways to choose four digits with exactly three of the four digits correct. Thus, the probability that a player wins the smaller prize is $36/10{,}000 = 9/2500 = 0.0036$. ◀

EXAMPLE 4 There are many lotteries now that award enormous prizes to people who correctly choose a set of six numbers out of the first n positive integers, where n is usually between 30 and 60. What is the probability that a person picks the correct six numbers out of 40?

Solution: There is only one winning combination. The total number of ways to choose six numbers out of 40 is

$$C(40, 6) = \frac{40!}{34!\,6!} = 3{,}838{,}380.$$

Consequently, the probability of picking a winning combination is $1/3{,}838{,}380 \approx 0.00000026$. (Here the symbol $\approx$ means approximately equal to.) ◀

Links

PIERRE-SIMON LAPLACE (1749–1827) Pierre-Simon Laplace came from humble origins in Normandy. In his childhood he was educated in a school run by the Benedictines. At 16 he entered the University of Caen intending to study theology. However, he soon realized his true interests were in mathematics. After completing his studies, he was named a provisional professor at Caen, and in 1769 he became professor of mathematics at the Paris Military School.

Laplace is best known for his contributions to celestial mechanics, the study of the motions of heavenly bodies. His *Traité de Mécanique Céleste* is considered one of the greatest scientific works of the early nineteenth century. Laplace was one of the founders of probability theory and made many contributions to mathematical statistics. His work in this area is documented in his book *Théorie Analytique des Probabilités*, in which he defined the probability of an event as the ratio of the number of favorable outcomes to the total number of outcomes of an experiment.

Laplace was famous for his political flexibility. He was loyal, in succession, to the French Republic, Napoleon, and King Louis XVIII. This flexibility permitted him to be productive before, during, and after the French Revolution.

Links

Poker, and other card games, are growing in popularity. To win at these games it helps to know the probability of different hands. We can find the probability of specific hands that arise in card games using the techniques developed so far. A deck of cards contains 52 cards. There are 13 different kinds of cards, with four cards of each kind. (Among the terms commonly used instead of "kind" are "rank," "face value," "denomination," and "value.") These kinds are twos, threes, fours, fives, sixes, sevens, eights, nines, tens, jacks, queens, kings, and aces. There are also four suits: spades, clubs, hearts, and diamonds, each containing 13 cards, with one card of each kind in a suit. In many poker games, a hand consists of five cards.

EXAMPLE 5 Find the probability that a hand of five cards in poker contains four cards of one kind.

Solution: By the product rule, the number of hands of five cards with four cards of one kind is the product of the number of ways to pick one kind, the number of ways to pick the four of this kind out of the four in the deck of this kind, and the number of ways to pick the fifth card. This is

$$C(13, 1)C(4, 4)C(48, 1).$$

By Example 11 in Section 6.3 there are $C(52, 5)$ different hands of five cards. Hence, the probability that a hand contains four cards of one kind is

$$\frac{C(13, 1)C(4, 4)C(48, 1)}{C(52, 5)} = \frac{13 \cdot 1 \cdot 48}{2{,}598{,}960} \approx 0.00024.$$ ◀

EXAMPLE 6 What is the probability that a poker hand contains a full house, that is, three of one kind and two of another kind?

Solution: By the product rule, the number of hands containing a full house is the product of the number of ways to pick two kinds in order, the number of ways to pick three out of four for the first kind, and the number of ways to pick two out of four for the second kind. (Note that the order of the two kinds matters, because, for instance, three queens and two aces is different from three aces and two queens.) We see that the number of hands containing a full house is

$$P(13, 2)C(4, 3)C(4, 2) = 13 \cdot 12 \cdot 4 \cdot 6 = 3744.$$

Because there are $C(52, 5) = 2{,}598{,}960$ poker hands, the probability of a full house is

$$\frac{3744}{2{,}598{,}960} \approx 0.0014.$$ ◀

EXAMPLE 7 What is the probability that the numbers 11, 4, 17, 39, and 23 are drawn in that order from a bin containing 50 balls labeled with the numbers $1, 2, \ldots, 50$ if (a) the ball selected is not returned to the bin before the next ball is selected and (b) the ball selected is returned to the bin before the next ball is selected?

Solution: (a) By the product rule, there are $50 \cdot 49 \cdot 48 \cdot 47 \cdot 46 = 254{,}251{,}200$ ways to select the balls because each time a ball is drawn there is one fewer ball to choose from. Consequently, the probability that 11, 4, 17, 39, and 23 are drawn in that order is $1/254{,}251{,}200$. This is an example of **sampling without replacement**.

(b) By the product rule, there are $50^5 = 312{,}500{,}000$ ways to select the balls because there are 50 possible balls to choose from each time a ball is drawn. Consequently, the probability that 11, 4, 17, 39, and 23 are drawn in that order is $1/312{,}500{,}000$. This is an example of **sampling with replacement**. ◀

Probabilities of Complements and Unions of Events

We can use counting techniques to find the probability of events derived from other events.

THEOREM 1 Let E be an event in a sample space S. The probability of the event $\overline{E} = S - E$, the complementary event of E, is given by

$$p(\overline{E}) = 1 - p(E).$$

Proof: To find the probability of the event $\overline{E} = S - E$, note that $|\overline{E}| = |S| - |E|$. Hence,

$$p(\overline{E}) = \frac{|S| - |E|}{|S|} = 1 - \frac{|E|}{|S|} = 1 - p(E).$$

◁

There is an alternative strategy for finding the probability of an event when a direct approach does not work well. Instead of determining the probability of the event, the probability of its complement can be found. This is often easier to do, as Example 8 shows.

EXAMPLE 8 A sequence of 10 bits is randomly generated. What is the probability that at least one of these bits is 0?

Solution: Let E be the event that at least one of the 10 bits is 0. Then $\overline{E}$ is the event that all the bits are 1s. Because the sample space S is the set of all bit strings of length 10, it follows that

$$p(E) = 1 - p(\overline{E}) = 1 - \frac{|\overline{E}|}{|S|} = 1 - \frac{1}{2^{10}}$$
$$= 1 - \frac{1}{1024} = \frac{1023}{1024}.$$

Hence, the probability that the bit string will contain at least one 0 bit is $1023/1024$. It is quite difficult to find this probability directly without using Theorem 1. ◀

We can also find the probability of the union of two events.

THEOREM 2 Let E_1 and E_2 be events in the sample space S. Then

$$p(E_1 \cup E_2) = p(E_1) + p(E_2) - p(E_1 \cap E_2).$$

Proof: Using the formula given in Section 2.2 for the number of elements in the union of two sets, it follows that

$$|E_1 \cup E_2| = |E_1| + |E_2| - |E_1 \cap E_2|.$$

Hence,

$$\begin{aligned} p(E_1 \cup E_2) &= \frac{|E_1 \cup E_2|}{|S|} \\ &= \frac{|E_1| + |E_2| - |E_1 \cap E_2|}{|S|} \\ &= \frac{|E_1|}{|S|} + \frac{|E_2|}{|S|} - \frac{|E_1 \cap E_2|}{|S|} \\ &= p(E_1) + p(E_2) - p(E_1 \cap E_2). \end{aligned}$$

◁

EXAMPLE 9 What is the probability that a positive integer selected at random from the set of positive integers not exceeding 100 is divisible by either 2 or 5?

Extra Examples

Solution: Let E_1 be the event that the integer selected at random is divisible by 2, and let E_2 be the event that it is divisible by 5. Then $E_1 \cup E_2$ is the event that it is divisible by either 2 or 5. Also, $E_1 \cap E_2$ is the event that it is divisible by both 2 and 5, or equivalently, that it is divisible by 10. Because $|E_1| = 50$, $|E_2| = 20$, and $|E_1 \cap E_2| = 10$, it follows that

$$\begin{aligned} p(E_1 \cup E_2) &= p(E_1) + p(E_2) - p(E_1 \cap E_2) \\ &= \frac{50}{100} + \frac{20}{100} - \frac{10}{100} = \frac{3}{5}. \end{aligned}$$

◀

Probabilistic Reasoning

A common problem is determining which of two events is more likely. Analyzing the probabilities of such events can be tricky. Example 10 describes a problem of this type. It discusses a famous problem originating with the television game show *Let's Make a Deal* and named after the host of the show, Monty Hall.

EXAMPLE 10 **The Monty Hall Three-Door Puzzle** Suppose you are a game show contestant. You have a chance to win a large prize. You are asked to select one of three doors to open; the large prize is behind one of the three doors and the other two doors are losers. Once you select a door, the game show host, who knows what is behind each door, does the following. First, whether or not you selected the winning door, he opens one of the other two doors that he knows is a losing door (selecting at random if both are losing doors). Then he asks you whether you would like to switch doors. Which strategy should you use? Should you change doors or keep your original selection, or does it not matter?

Links

Solution: The probability you select the correct door (before the host opens a door and asks you whether you want to change) is $1/3$, because the three doors are equally likely to be the correct door. The probability this is the correct door does not change once the game show host opens one of the other doors, because he will always open a door that the prize is not behind.

The probability that you selected incorrectly is the probability the prize is behind one of the two doors you did not select. Consequently, the probability that you selected incorrectly is $2/3$. If you selected incorrectly, when the game show host opens a door to show you that the prize is not behind it, the prize is behind the other door. You will always win if your initial choice was incorrect and you change doors. So, by changing doors, the probability you win is $2/3$. In other words, you should always change doors when given the chance to do so by the game show host. This doubles the probability that you will win. (A more rigorous treatment of this puzzle can be found in Exercise 15 of Section 7.3. For much more on this famous puzzle and its variations, see [Ro09].) ◀

Exercises

1. What is the probability that a card selected at random from a standard deck of 52 cards is an ace?

2. What is the probability that a fair die comes up six when it is rolled?

3. What is the probability that a randomly selected integer chosen from the first 100 positive integers is odd?

4. What is the probability that a randomly selected day of a leap year (with 366 possible days) is in April?

5. What is the probability that the sum of the numbers on two dice is even when they are rolled?

6. What is the probability that a card selected at random from a standard deck of 52 cards is an ace or a heart?

7. What is the probability that when a coin is flipped six times in a row, it lands heads up every time?

8. What is the probability that a five-card poker hand contains the ace of hearts?

9. What is the probability that a five-card poker hand does not contain the queen of hearts?

10. What is the probability that a five-card poker hand contains the two of diamonds and the three of spades?

11. What is the probability that a five-card poker hand contains the two of diamonds, the three of spades, the six of hearts, the ten of clubs, and the king of hearts?

12. What is the probability that a five-card poker hand contains exactly one ace?

13. What is the probability that a five-card poker hand contains at least one ace?

14. What is the probability that a five-card poker hand contains cards of five different kinds?

15. What is the probability that a five-card poker hand contains two pairs (that is, two of each of two different kinds and a fifth card of a third kind)?

16. What is the probability that a five-card poker hand contains a flush, that is, five cards of the same suit?

17. What is the probability that a five-card poker hand contains a straight, that is, five cards that have consecutive kinds? (Note that an ace can be considered either the lowest card of an A-2-3-4-5 straight or the highest card of a 10-J-Q-K-A straight.)

18. What is the probability that a five-card poker hand contains a straight flush, that is, five cards of the same suit of consecutive kinds?

***19.** What is the probability that a five-card poker hand contains cards of five different kinds and does not contain a flush or a straight?

20. What is the probability that a five-card poker hand contains a royal flush, that is, the 10, jack, queen, king, and ace of one suit?

21. What is the probability that a fair die never comes up an even number when it is rolled six times?

22. What is the probability that a positive integer not exceeding 100 selected at random is divisible by 3?

23. What is the probability that a positive integer not exceeding 100 selected at random is divisible by 5 or 7?

24. Find the probability of winning a lottery by selecting the correct six integers, where the order in which these integers are selected does not matter, from the positive integers not exceeding

a) 30. **b)** 36. **c)** 42. **d)** 48.

25. Find the probability of winning a lottery by selecting the correct six integers, where the order in which these integers are selected does not matter, from the positive integers not exceeding

a) 50. **b)** 52. **c)** 56. **d)** 60.

26. Find the probability of selecting none of the correct six integers in a lottery, where the order in which these integers are selected does not matter, from the positive integers not exceeding

a) 40. **b)** 48. **c)** 56. **d)** 64.

27. Find the probability of selecting exactly one of the correct six integers in a lottery, where the order in which these integers are selected does not matter, from the positive integers not exceeding

a) 40. **b)** 48. **c)** 56. **d)** 64.

28. In a superlottery, a player selects 7 numbers out of the first 80 positive integers. What is the probability that a person wins the grand prize by picking 7 numbers that are among the 11 numbers selected at random by a computer.

29. In a superlottery, players win a fortune if they choose the eight numbers selected by a computer from the positive integers not exceeding 100. What is the probability that a player wins this superlottery?

30. What is the probability that a player of a lottery wins the prize offered for correctly choosing five (but not six) numbers out of six integers chosen at random from the integers between 1 and 40, inclusive?

31. Suppose that 100 people enter a contest and that different winners are selected at random for first, second, and third prizes. What is the probability that Michelle wins one of these prizes if she is one of the contestants?

32. Suppose that 100 people enter a contest and that different winners are selected at random for first, second, and third prizes. What is the probability that Kumar, Janice, and Pedro each win a prize if each has entered the contest?

33. What is the probability that Abby, Barry, and Sylvia win the first, second, and third prizes, respectively, in a drawing if 200 people enter a contest and

a) no one can win more than one prize.
b) winning more than one prize is allowed.

34. What is the probability that Bo, Colleen, Jeff, and Rohini win the first, second, third, and fourth prizes, respectively, in a drawing if 50 people enter a contest and

a) no one can win more than one prize.
b) winning more than one prize is allowed.

35. In roulette, a wheel with 38 numbers is spun. Of these, 18 are red, and 18 are black. The other two numbers, which are neither black nor red, are 0 and 00. The probability that when the wheel is spun it lands on any particular number is 1/38.

a) What is the probability that the wheel lands on a red number?

b) What is the probability that the wheel lands on a black number twice in a row?

c) What is the probability that the wheel lands on 0 or 00?

d) What is the probability that in five spins the wheel never lands on either 0 or 00?

e) What is the probability that the wheel lands on one of the first six integers on one spin, but does not land on any of them on the next spin?

36. Which is more likely: rolling a total of 8 when two dice are rolled or rolling a total of 8 when three dice are rolled?

37. Which is more likely: rolling a total of 9 when two dice are rolled or rolling a total of 9 when three dice are rolled?

38. Two events E_1 and E_2 are called **independent** if $p(E_1 \cap E_2) = p(E_1)p(E_2)$. For each of the following pairs of events, which are subsets of the set of all possible outcomes when a coin is tossed three times, determine whether or not they are independent.

a) E_1: tails comes up with the coin is tossed the first time; E_2: heads comes up when the coin is tossed the second time.

b) E_1: the first coin comes up tails; E_2: two, and not three, heads come up in a row.

c) E_1: the second coin comes up tails; E_2: two, and not three, heads come up in a row.

(We will study independence of events in more depth in Section 7.2.)

39. Explain what is wrong with the statement that in the Monty Hall Three-Door Puzzle the probability that the prize is behind the first door you select and the probability that the prize is behind the other of the two doors that Monty does not open are both 1/2, because there are two doors left.

40. Suppose that instead of three doors, there are four doors in the Monty Hall puzzle. What is the probability that you win by not changing once the host, who knows what is behind each door, opens a losing door and gives you the chance to change doors? What is the probability that you win by changing the door you select to one of the two remaining doors among the three that you did not select?

41. This problem was posed by the Chevalier de Méré and was solved by Blaise Pascal and Pierre de Fermat.

a) Find the probability of rolling at least one six when a fair die is rolled four times.

b) Find the probability that a double six comes up at least once when a pair of dice is rolled 24 times. Answer the query the Chevalier de Méré made to Pascal asking whether this probability was greater than 1/2.

c) Is it more likely that a six comes up at least once when a fair die is rolled four times or that a double six comes up at least once when a pair of dice is rolled 24 times?

7.2 Probability Theory

Introduction

Links

In Section 7.1 we introduced the notion of the probability of an event. (Recall that an event is a subset of the possible outcomes of an experiment.) We defined the probability of an event E as Laplace did, that is,

$$p(E) = \frac{|E|}{|S|},$$

the number of outcomes in E divided by the total number of outcomes. This definition assumes that all outcomes are equally likely. However, many experiments have outcomes that are not equally likely. For instance, a coin may be biased so that it comes up heads twice as often as tails. Similarly, the likelihood that the input of a linear search is a particular element in a list, or is not in the list, depends on how the input is generated. How can we model the likelihood of events in such situations? In this section we will show how to define probabilities of outcomes to study probabilities of experiments where outcomes may not be equally likely.

Suppose that a fair coin is flipped four times, and the first time it comes up heads. Given this information, what is the probability that heads comes up three times? To answer this and

similar questions, we will introduce the concept of *conditional probability*. Does knowing that the first flip comes up heads change the probability that heads comes up three times? If not, these two events are called *independent*, a concept studied later in this section.

Many questions address a particular numerical value associated with the outcome of an experiment. For instance, when we flip a coin 100 times, what is the probability that exactly 40 heads appear? How many heads should we expect to appear? In this section we will introduce *random variables*, which are functions that associate numerical values to the outcomes of experiments.

Assigning Probabilities

Let S be the sample space of an experiment with a finite or countable number of outcomes. We assign a probability $p(s)$ to each outcome s. We require that two conditions be met:

(i) $0 \leq p(s) \leq 1$ for each $s \in S$

and

(ii) $\sum_{s \in S} p(s) = 1.$

Condition (i) states that the probability of each outcome is a nonnegative real number no greater than 1. Condition (ii) states that the sum of the probabilities of all possible outcomes should be 1; that is, when we do the experiment, it is a certainty that one of these outcomes occurs. (Note that when the sample space is infinite, $\sum_{s \in S} p(s)$ is a convergent infinite series.) This is a generalization of Laplace's definition in which each of n outcomes is assigned a probability of $1/n$. Indeed, conditions (i) and (ii) are met when Laplace's definition of probabilities of equally likely outcomes is used and S is finite. (See Exercise 4.)

Note that when there are n possible outcomes, $x_1, x_2, \ldots, x_n$, the two conditions to be met are

(i) $0 \leq p(x_i) \leq 1$ for $i = 1, 2, \ldots, n$

and

(ii) $\sum_{i=1}^{n} p(x_i) = 1.$

The function p from the set of all outcomes of the sample space S is called a **probability distribution**.

To model an experiment, the probability $p(s)$ assigned to an outcome s should equal the limit of the number of times s occurs divided by the number of times the experiment is performed, as this number grows without bound. (We will assume that all experiments discussed have outcomes that are predictable on the average, so that this limit exists. We also assume that the outcomes of successive trials of an experiment do not depend on past results.)

Links HISTORICAL NOTE The Chevalier de Méré was a French nobleman, a famous gambler, and a bon vivant. He was successful at making bets with odds slightly greater than 1/2 (such as having at least one six come up in four tosses of a fair die). His correspondence with Pascal asking about the probability of having at least one double six come up when a pair of dice is rolled 24 times led to the development of probability theory. According to one account, Pascal wrote to Fermat about the Chevalier saying something like "He's a good guy but, alas, he's no mathematician."

Remark: We will not discuss probabilities of events when the set of outcomes is not finite or countable, such as when the outcome of an experiment can be any real number. In such cases, integral calculus is usually required for the study of the probabilities of events.

We can model experiments in which outcomes are either equally likely or not equally likely by choosing the appropriate function $p(s)$, as Example 1 illustrates.

EXAMPLE 1 What probabilities should we assign to the outcomes H (heads) and T (tails) when a fair coin is flipped? What probabilities should be assigned to these outcomes when the coin is biased so that heads comes up twice as often as tails?

Solution: For a fair coin, the probability that heads comes up when the coin is flipped equals the probability that tails comes up, so the outcomes are equally likely. Consequently, we assign the probability $1/2$ to each of the two possible outcomes, that is, $p(H) = p(T) = 1/2$.

For the biased coin we have

$$p(H) = 2p(T).$$

Because

$$p(H) + p(T) = 1,$$

it follows that

$$2p(T) + p(T) = 3p(T) = 1.$$

We conclude that $p(T) = 1/3$ and $p(H) = 2/3$. ◀

DEFINITION 1 Suppose that S is a set with n elements. The *uniform distribution* assigns the probability $1/n$ to each element of S.

We now define the probability of an event as the sum of the probabilities of the outcomes in this event.

DEFINITION 2 The *probability* of the event E is the sum of the probabilities of the outcomes in E. That is,

$$p(E) = \sum_{s \in E} p(s).$$

(Note that when E is an infinite set, $\sum_{s \in E} p(s)$ is a convergent infinite series.)

Note that when there are n outcomes in the event E, that is, if $E = \{a_1, a_2, \ldots, a_n\}$, then $p(E) = \sum_{i=1}^{n} p(a_i)$. Note also that the uniform distribution assigns the same probability to an event that Laplace's original definition of probability assigns to this event. The experiment of selecting an element from a sample space with a uniform distribution is called selecting an element of S **at random**.

EXAMPLE 2 Suppose that a die is biased (or loaded) so that 3 appears twice as often as each other number but that the other five outcomes are equally likely. What is the probability that an odd number appears when we roll this die?

Solution: We want to find the probability of the event $E = \{1, 3, 5\}$. By Exercise 2, we have

$$p(1) = p(2) = p(4) = p(5) = p(6) = 1/7;\ p(3) = 2/7.$$

It follows that

$$p(E) = p(1) + p(3) + p(5) = 1/7 + 2/7 + 1/7 = 4/7. \quad \blacktriangleleft$$

When possible outcomes are equally likely and there are a finite number of possible outcomes, the definition of the probability of an event given in this section (Definition 2) agrees with Laplace's definition (Definition 1 of Section 7.1). To see this, suppose that there are n equally likely outcomes; each possible outcome has probability $1/n$, because the sum of their probabilities is 1. Suppose the event E contains m outcomes. According to Definition 2,

$$p(E) = \sum_{i=1}^{m} \frac{1}{n} = \frac{m}{n}.$$

Because $|E| = m$ and $|S| = n$, it follows that

$$p(E) = \frac{m}{n} = \frac{|E|}{|S|}.$$

This is Laplace's definition of the probability of the event E.

Probabilities of Complements and Unions of Events

The formulae for probabilities of combinations of events in Section 7.1 continue to hold when we use Definition 2 to define the probability of an event. For example, Theorem 1 of Section 7.1 asserts that

$$p(\overline{E}) = 1 - p(E),$$

where $\overline{E}$ is the complementary event of the event E. This equality also holds when Definition 2 is used. To see this, note that because the sum of the probabilities of the n possible outcomes is 1, and each outcome is either in E or in $\overline{E}$, but not in both, we have

$$\sum_{s \in S} p(s) = 1 = p(E) + p(\overline{E}).$$

Hence, $p(\overline{E}) = 1 - p(E)$.

Under Laplace's definition, by Theorem 2 in Section 7.1, we have

$$p(E_1 \cup E_2) = p(E_1) + p(E_2) - p(E_1 \cap E_2)$$

whenever E_1 and E_2 are events in a sample space S. This also holds when we define the probability of an event as we do in this section. To see this, note that $p(E_1 \cup E_2)$ is the sum of the probabilities of the outcomes in $E_1 \cup E_2$. When an outcome x is in one, but not both, of E_1 and E_2, $p(x)$ occurs in exactly one of the sums for $p(E_1)$ and $p(E_2)$. When an outcome x is in both E_1 and E_2, $p(x)$ occurs in the sum for $p(E_1)$, in the sum for $p(E_2)$, and in the sum for $p(E_1 \cap E_2)$, so it occurs $1 + 1 - 1 = 1$ time on the right-hand side. Consequently, the left-hand side and right-hand side are equal.

Also, note that if the events E_1 and E_2 are disjoint, then $p(E_1 \cap E_2) = 0$, which implies that

$$p(E_1 \cup E_2) = p(E_1) + p(E_2) - p(E_1 \cap E_2) = p(E_1) + p(E_2).$$

Theorem 1 generalizes this last formula by providing a formula for the probability of the union of pairwise disjoint events.

THEOREM 1

If $E_1, E_2, \ldots$ is a sequence of pairwise disjoint events in a sample space S, then

$$p\left(\bigcup_i E_i\right) = \sum_i p(E_i).$$

(Note that this theorem applies when the sequence $E_1, E_2, \ldots$ consists of a finite number or a countably infinite number of pairwise disjoint events.)

We leave the proof of Theorem 1 to the reader (see Exercises 36 and 37).

Conditional Probability

Links

Suppose that we flip a coin three times, and all eight possibilities are equally likely. Moreover, suppose we know that the event F, that the first flip comes up tails, occurs. Given this information, what is the probability of the event E, that an odd number of tails appears? Because the first flip comes up tails, there are only four possible outcomes: *TTT*, *TTH*, *THT*, and *THH*, where *H* and *T* represent heads and tails, respectively. An odd number of tails appears only for the outcomes *TTT* and *THH*. Because the eight outcomes have equal probability, each of the four possible outcomes, given that F occurs, should also have an equal probability of $1/4$. This suggests that we should assign the probability of $2/4 = 1/2$ to E, given that F occurs. This probability is called the **conditional probability** of E given F.

In general, to find the conditional probability of E given F, we use F as the sample space. For an outcome from E to occur, this outcome must also belong to $E \cap F$. With this motivation, we make Definition 3.

DEFINITION 3

Let E and F be events with $p(F) > 0$. The *conditional probability* of E given F, denoted by $p(E \mid F)$, is defined as

$$p(E \mid F) = \frac{p(E \cap F)}{p(F)}.$$

EXAMPLE 3

Extra Examples

A bit string of length four is generated at random so that each of the 16 bit strings of length four is equally likely. What is the probability that it contains at least two consecutive 0s, given that its first bit is a 0? (We assume that 0 bits and 1 bits are equally likely.)

Solution: Let E be the event that a bit string of length four contains at least two consecutive 0s, and let F be the event that the first bit of a bit string of length four is a 0. The probability that a bit string of length four has at least two consecutive 0s, given that its first bit is a 0, equals

$$p(E \mid F) = \frac{p(E \cap F)}{p(F)}.$$

Because $E \cap F = \{0000, 0001, 0010, 0011, 0100\}$, we see that $p(E \cap F) = 5/16$. Because there are eight bit strings of length four that start with a 0, we have $p(F) = 8/16 = 1/2$. Consequently,

$$p(E \mid F) = \frac{5/16}{1/2} = \frac{5}{8}.$$

◀

EXAMPLE 4 What is the conditional probability that a family with two children has two boys, given they have at least one boy? Assume that each of the possibilities *BB*, *BG*, *GB*, and *GG* is equally likely, where *B* represents a boy and *G* represents a girl. (Note that *BG* represents a family with an older boy and a younger girl while *GB* represents a family with an older girl and a younger boy.)

Solution: Let E be the event that a family with two children has two boys, and let F be the event that a family with two children has at least one boy. It follows that $E = \{BB\}$, $F = \{BB,\ BG,\ GB\}$, and $E \cap F = \{BB\}$. Because the four possibilities are equally likely, it follows that $p(F) = 3/4$ and $p(E \cap F) = 1/4$. We conclude that

$$p(E \mid F) = \frac{p(E \cap F)}{p(F)} = \frac{1/4}{3/4} = \frac{1}{3}.$$

◀

Independence

Links

Suppose a coin is flipped three times, as described in the introduction to our discussion of conditional probability. Does knowing that the first flip comes up tails (event F) alter the probability that tails comes up an odd number of times (event E)? In other words, is it the case that $p(E \mid F) = p(E)$? This equality is valid for the events E and F, because $p(E \mid F) = 1/2$ and $p(E) = 1/2$. Because this equality holds, we say that E and F are **independent events**. When two events are independent, the occurrence of one of the events gives no information about the probability that the other event occurs.

Because $p(E \mid F) = p(E \cap F)/p(F)$, asking whether $p(E \mid F) = p(E)$ is the same as asking whether $p(E \cap F) = p(E)p(F)$. This leads to Definition 4.

DEFINITION 4 The events E and F are *independent* if and only if $p(E \cap F) = p(E)p(F)$.

EXAMPLE 5 Suppose E is the event that a randomly generated bit string of length four begins with a 1 and F is the event that this bit string contains an even number of 1s. Are E and F independent, if the 16 bit strings of length four are equally likely?

Extra Examples

Solution: There are eight bit strings of length four that begin with a one: 1000, 1001, 1010, 1011, 1100, 1101, 1110, and 1111. There are also eight bit strings of length four that contain an even number of ones: 0000, 0011, 0101, 0110, 1001, 1010, 1100, 1111. Because there are 16 bit strings of length four, it follows that

$$p(E) = p(F) = 8/16 = 1/2.$$

Because $E \cap F = \{1111, 1100, 1010, 1001\}$, we see that

$$p(E \cap F) = 4/16 = 1/4.$$

Because

$$p(E \cap F) = 1/4 = (1/2)(1/2) = p(E)p(F),$$

we conclude that E and F are independent. ◀

Probability has many applications to genetics, as Examples 6 and 7 illustrate.

EXAMPLE 6 Assume, as in Example 4, that each of the four ways a family can have two children is equally likely. Are the events E, that a family with two children has two boys, and F, that a family with two children has at least one boy, independent?

Solution: Because $E = \{BB\}$, we have $p(E) = 1/4$. In Example 4 we showed that $p(F) = 3/4$ and that $p(E \cap F) = 1/4$. But $p(E)p(F) = \frac{1}{4} \cdot \frac{3}{4} = \frac{3}{16}$. Therefore $p(E \cap F) \neq p(E)p(F)$, so the events E and F are not independent. ◀

EXAMPLE 7 Are the events E, that a family with three children has children of both sexes, and F, that this family has at most one boy, independent? Assume that the eight ways a family can have three children are equally likely.

Solution: By assumption, each of the eight ways a family can have three children, BBB, BBG, BGB, BGG, GBB, GBG, GGB, and GGG, has a probability of $1/8$. Because $E = \{BBG,\ BGB,\ BGG,\ GBB,\ GBG,\ GGB\}$, $F = \{BGG,\ GBG,\ GGB,\ GGG\}$, and $E \cap F = \{BGG,\ GBG,\ GGB\}$, it follows that $p(E) = 6/8 = 3/4$, $p(F) = 4/8 = 1/2$, and $p(E \cap F) = 3/8$. Because

$$p(E)p(F) = \frac{3}{4} \cdot \frac{1}{2} = \frac{3}{8},$$

it follows that $p(E \cap F) = p(E)p(F)$, so E and F are independent. (This conclusion may seem surprising. Indeed, if we change the number of children, the conclusion may no longer hold. See Exercise 27.) ◀

PAIRWISE AND MUTUAL INDEPENDENCE We can also define the independence of more than two events. However, there are two different types of independence, given in Definition 5.

DEFINITION 5 The events $E_1, E_2, \ldots, E_n$ are *pairwise independent* if and only if $p(E_i \cap E_j) = p(E_i)p(E_j)$ for all pairs of integers i and j with $1 \leq i < j \leq n$. These events are *mutually independent* if $p(E_{i_1} \cap E_{i_2} \cap \cdots \cap E_{i_m}) = p(E_{i_1})p(E_{i_2}) \cdots p(E_{i_m})$ whenever i_j, $j = 1, 2, \ldots, m$, are integers with $1 \leq i_1 < i_2 < \cdots < i_m \leq n$ and $m \geq 2$.

From Definition 5, we see that every set of n mutually independent events is also pairwise independent. However, n pairwise independent events are not necessarily mutually independent, as we see in Exercise 25 in the Supplementary Exercises. Many theorems about n events include the hypothesis that these events are mutually independent, and not just pairwise independent. We will introduce several such theorems later in this chapter.

Bernoulli Trials and the Binomial Distribution

Links

Suppose that an experiment can have only two possible outcomes. For instance, when a bit is generated at random, the possible outcomes are 0 and 1. When a coin is flipped, the possible outcomes are heads and tails. Each performance of an experiment with two possible outcomes is called a **Bernoulli trial**, after James Bernoulli, who made important contributions to probability theory. In general, a possible outcome of a Bernoulli trial is called a **success** or a **failure**. If p is the probability of a success and q is the probability of a failure, it follows that $p + q = 1$.

Many problems can be solved by determining the probability of k successes when an experiment consists of n mutually independent Bernoulli trials. (Bernoulli trials are **mutually independent** if the conditional probability of success on any given trial is p, given any information whatsoever about the outcomes of the other trials.) Consider Example 8.

EXAMPLE 8 A coin is biased so that the probability of heads is 2/3. What is the probability that exactly four heads come up when the coin is flipped seven times, assuming that the flips are independent?

Solution: There are $2^7 = 128$ possible outcomes when a coin is flipped seven times. The number of ways four of the seven flips can be heads is $C(7, 4)$. Because the seven flips are independent, the probability of each of these outcomes (four heads and three tails) is $(2/3)^4(1/3)^3$. Consequently, the probability that exactly four heads appear is

$$C(7, 4)(2/3)^4(1/3)^3 = \frac{35 \cdot 16}{3^7} = \frac{560}{2187}.$$

◀

Following the same reasoning as was used in Example 8, we can find the probability of k successes in n independent Bernoulli trials.

THEOREM 2 The probability of exactly k successes in n independent Bernoulli trials, with probability of success p and probability of failure $q = 1 - p$, is

$$C(n, k)p^k q^{n-k}.$$

Proof: When n Bernoulli trials are carried out, the outcome is an n-tuple $(t_1, t_2, \ldots, t_n)$, where $t_i = S$ (for success) or $t_i = F$ (for failure) for $i = 1, 2, \ldots, n$. Because the n trials are independent, the probability of each outcome of n trials consisting of k successes and $n - k$ failures (in any order) is $p^k q^{n-k}$. Because there are $C(n, k)$ n-tuples of S's and F's that contain exactly k S's, the probability of exactly k successes is

$$C(n, k)p^k q^{n-k}.$$

◁

We denote by $b(k; n, p)$ the probability of k successes in n independent Bernoulli trials with probability of success p and probability of failure $q = 1 - p$. Considered as a function of k, we call this function the **binomial distribution**. Theorem 2 tells us that $b(k; n, p) = C(n, k)p^k q^{n-k}$.

EXAMPLE 9 Extra Examples Suppose that the probability that a 0 bit is generated is 0.9, that the probability that a 1 bit is generated is 0.1, and that bits are generated independently. What is the probability that exactly eight 0 bits are generated when 10 bits are generated?

Solution: By Theorem 2, the probability that exactly eight 0 bits are generated is

$$b(8; 10, 0.9) = C(10, 8)(0.9)^8(0.1)^2 = 0.1937102445.$$

◀

Links

JAMES BERNOULLI (1654–1705) James Bernoulli (also known as Jacob I), was born in Basel, Switzerland. He is one of the eight prominent mathematicians in the Bernoulli family (see Section 10.1 for the Bernoulli family tree of mathematicians). Following his father's wish, James studied theology and entered the ministry. But contrary to the desires of his parents, he also studied mathematics and astronomy. He traveled throughout Europe from 1676 to 1682, learning about the latest discoveries in mathematics and the sciences. Upon returning to Basel in 1682, he founded a school for mathematics and the sciences. He was appointed professor of mathematics at the University of Basel in 1687, remaining in this position for the rest of his life.

James Bernoulli is best known for the work *Ars Conjectandi*, published eight years after his death. In this work, he described the known results in probability theory and in enumeration, often providing alternative proofs of known results. This work also includes the application of probability theory to games of chance and his introduction of the theorem known as the **law of large numbers**. This law states that if $\epsilon > 0$, as n becomes arbitrarily large the probability approaches 1 that the fraction of times an event E occurs during n trials is within ϵ of $p(E)$.

Note that the sum of the probabilities that there are k successes when n independent Bernoulli trials are carried out, for $k = 0, 1, 2, \ldots, n$, equals

$$\sum_{k=0}^{n} C(n, k)p^k q^{n-k} = (p + q)^n = 1,$$

as should be the case. The first equality in this string of equalities is a consequence of the binomial theorem (see Section 6.4). The second equality follows because $q = 1 - p$.

Random Variables

Many problems are concerned with a numerical value associated with the outcome of an experiment. For instance, we may be interested in the total number of one bits in a randomly generated string of 10 bits; or in the number of times tails come up when a coin is flipped 20 times. To study problems of this type we introduce the concept of a random variable.

DEFINITION 6 A *random variable* is a function from the sample space of an experiment to the set of real numbers. That is, a random variable assigns a real number to each possible outcome.

Remark: Note that a random variable is a function. It is not a variable, and it is not random! The name *random variable* (the translation of *variabile casuale*) was introduced by the Italian mathematician F. P. Cantelli in 1916. In the late 1940s, the mathematicians, W. Feller and J. L. Doob flipped a coin to see whether both would use "random variable" or the more fitting term "chance variable." Feller won; unfortunately "random varible" was used in both books and ever since.

EXAMPLE 10 Suppose that a coin is flipped three times. Let $X(t)$ be the random variable that equals the number of heads that appear when t is the outcome. Then $X(t)$ takes on the following values:

$$\begin{aligned} X(HHH) &= 3, \\ X(HHT) &= X(HTH) = X(THH) = 2, \\ X(TTH) &= X(THT) = X(HTT) = 1, \\ X(TTT) &= 0. \end{aligned}$$

◀

DEFINITION 7 The *distribution* of a random variable X on a sample space S is the set of pairs $(r, p(X = r))$ for all $r \in X(S)$, where $p(X = r)$ is the probability that X takes the value r. (The set of pairs in this distribution is determined by the probabilities $p(X = r)$ for $r \in X(S)$.)

EXAMPLE 11 Each of the eight possible outcomes when a fair coin is flipped three times has probability $1/8$. So, the distribution of the random variable $X(t)$ in Example 10 is determined by the probabilities $P(X = 3) = 1/8$, $P(X = 2) = 3/8$, $P(X = 1) = 3/8$, and $P(X = 0) = 1/8$. Consequently, the distribution of $X(t)$ in Example 10 is the set of pairs $(3, 1/8)$, $(2, 3/8)$, $(1, 3/8)$, and $(0, 1/8)$. ◀

EXAMPLE 12 Let X be the sum of the numbers that appear when a pair of dice is rolled. What are the values of this random variable for the 36 possible outcomes (i, j), where i and j are the numbers that appear on the first die and the second die, respectively, when these two dice are rolled?

Solution: The random variable X takes on the following values:

$$
\begin{aligned}
&X((1,1)) = 2,\\
&X((1,2)) = X((2,1)) = 3,\\
&X((1,3)) = X((2,2)) = X((3,1)) = 4,\\
&X((1,4)) = X((2,3)) = X((3,2)) = X((4,1)) = 5,\\
&X((1,5)) = X((2,4)) = X((3,3)) = X((4,2)) = X((5,1)) = 6,\\
&X((1,6)) = X((2,5)) = X((3,4)) = X((4,3)) = X((5,2)) = X((6,1)) = 7,\\
&X((2,6)) = X((3,5)) = X((4,4)) = X((5,3)) = X((6,2)) = 8,\\
&X((3,6)) = X((4,5)) = X((5,4)) = X((6,3)) = 9,\\
&X((4,6)) = X((5,5)) = X((6,4)) = 10,\\
&X((5,6)) = X((6,5)) = 11,\\
&X((6,6)) = 12.
\end{aligned}
$$

We will continue our study of random variables in Section 7.4, where we will show how they can be used in a variety of applications.

The Birthday Problem

A famous puzzle asks for the smallest number of people needed in a room so that it is more likely than not that at least two of them have the same day of the year as their birthday. Most people find the answer, which we determine in Example 13, to be surprisingly small. After we solve this famous problem, we will show how similar reasoning can be adapted to solve a question about hashing functions.

EXAMPLE 13 Links

The Birthday Problem What is the minimum number of people who need to be in a room so that the probability that at least two of them have the same birthday is greater than $1/2$?

Solution: First, we state some assumptions. We assume that the birthdays of the people in the room are independent. Furthermore, we assume that each birthday is equally likely and that there are 366 days in the year. (In reality, more people are born on some days of the year than others, such as days nine months after some holidays including New Year's Eve, and only leap years have 366 days.)

To find the probability that at least two of n people in a room have the same birthday, we first calculate the probability p_n that these people all have different birthdays. Then, the probability that at least two people have the same birthday is $1 - p_n$. To compute p_n, we consider the birthdays of the n people in some fixed order. Imagine them entering the room one at a time; we will compute the probability that each successive person entering the room has a birthday different from those of the people already in the room.

The birthday of the first person certainly does not match the birthday of someone already in the room. The probability that the birthday of the second person is different from that of the first person is $365/366$ because the second person has a different birthday when he or she was born on one of the 365 days of the year other than the day the first person was born. (The assumption that it is equally likely for someone to be born on any of the 366 days of the year enters into this and subsequent steps.)

The probability that the third person has a birthday different from both the birthdays of the first and second people given that these two people have different birthdays is $364/366$. In general, the probability that the jth person, with $2 \leq j \leq 366$, has a birthday different from the

birthdays of the $j-1$ people already in the room given that these $j-1$ people have different birthdays is

$$\frac{366-(j-1)}{366} = \frac{367-j}{366}.$$

Because we have assumed that the birthdays of the people in the room are independent, we can conclude that the probability that the n people in the room have different birthdays is

$$p_n = \frac{365}{366}\frac{364}{366}\frac{363}{366}\cdots\frac{367-n}{366}.$$

It follows that the probability that among n people there are at least two people with the same birthday is

$$1-p_n = 1-\frac{365}{366}\frac{364}{366}\frac{363}{366}\cdots\frac{367-n}{366}.$$

To determine the minimum number of people in the room so that the probability that at least two of them have the same birthday is greater than $1/2$, we use the formula we have found for $1-p_n$ to compute it for increasing values of n until it becomes greater than $1/2$. (There are more sophisticated approaches using calculus that can eliminate this computation, but we will not use them here.) After considerable computation we find that for $n = 22$, $1-p_n \approx 0.475$, while for $n = 23$, $1-p_n \approx 0.506$. Consequently, the minimum number of people needed so that the probability that at least two people have the same birthday is greater than $1/2$ is 23. ◀

The solution to the birthday problem leads to the solution of the question in Example 14 about hashing functions.

EXAMPLE 14 **Probability of a Collision in Hashing Functions** Recall from Section 4.5 that a hashing function $h(k)$ is a mapping of the keys (of the records that are to be stored in a database) to storage locations. Hashing functions map a large universe of keys (such as the approximately 300 million Social Security numbers in the United States) to a much smaller set of storage locations. A good hashing function yields few **collisions**, which are mappings of two different keys to the same memory location, when relatively few of the records are in play in a given application. What is the probability that no two keys are mapped to the same location by a hashing function, or, in other words, that there are no collisions?

Solution: To calculate this probability, we assume that the probability that a randomly selected key is mapped to a location is $1/m$, where m is the number of available locations, that is, the hashing function distributes keys uniformly. (In practice, hashing functions may not satisfy this assumption. However, for a good hashing function, this assumption should be close to correct.) Furthermore, we assume that the keys of the records selected have an equal probability to be any of the elements of the key universe and that these keys are independently selected.

Suppose that the keys are $k_1, k_2, \ldots, k_n$. When we add the second record, the probability that it is mapped to a location different from the location of the first record, that $h(k_2) \neq h(k_1)$, is $(m-1)/m$ because there are $m-1$ free locations after the first record has been placed. The probability that the third record is mapped to a free location after the first and second records have been placed without a collision is $(m-2)/m$. In general, the probability that the jth record is mapped to a free location after the first $j-1$ records have been mapped to locations $h(k_1)$, $h(k_2), \ldots, h(k_{j-1})$ without collisions is $(m-(j-1))/m$ because $j-1$ of the m locations are taken.

Because the keys are independent, the probability that all n keys are mapped to different locations is

$$p_n = \frac{m-1}{m}\cdot\frac{m-2}{m}\cdot\cdots\cdot\frac{m-n+1}{m}.$$

It follows that the probability that there is at least one collision, that is, at least two keys are mapped to the same location, is

$$1 - p_n = 1 - \frac{m-1}{m} \cdot \frac{m-2}{m} \cdot \cdots \cdot \frac{m-n+1}{m}.$$

Techniques from calculus can be used to find the smallest value of n given a value of m such that the probability of a collision is greater than a particular threshold. It can be shown that the smallest integer n such that the probability of a collision is greater than $1/2$ is approximately $n = 1.177\sqrt{m}$. For example, when $m = 1{,}000{,}000$, the smallest integer n such that the probability of a collision is greater than $1/2$ is 1178. ◀

Monte Carlo Algorithms

The algorithms discussed so far in this book are all deterministic. That is, each algorithm always proceeds in the same way whenever given the same input. However, there are many situations where we would like an algorithm to make a random choice at one or more steps. Such a situation arises when a deterministic algorithm would have to go through a huge number, or even an unknown number, of possible cases. Algorithms that make random choices at one or more steps are called **probabilistic algorithms**. We will discuss a particular class of probabilistic algorithms in this section, namely, **Monte Carlo algorithms**, for decision problems. Monte Carlo algorithms always produce answers to problems, but a small probability remains that these answers may be incorrect. However, the probability that the answer is incorrect decreases rapidly when the algorithm carries out sufficient computation. Decision problems have either "true" or "false" as their answer. The designation "Monte Carlo" is a reference to the famous casino in Monaco; the use of randomness and the repetitive processes in these algorithms make them similar to some gambling games. This name was introduced by the inventors of Monte Carlo methods, including Stan Ulam, Enrico Fermi, and John von Neumann.

Monte Carlo methods were invented to help develop the first nuclear weapons.

A Monte Carlo algorithm for a decision problem uses a sequence of tests. The probability that the algorithm answers the decision problem correctly increases as more tests are carried out. At each step of the algorithm, possible responses are "true," which means that the answer is "true" and no additional iterations are needed, or "unknown," which means that the answer could be either "true" or "false." After running all the iterations in such an algorithm, the final answer produced is "true" if at least one iteration yields the answer "true," and the answer is "false" if every iteration yields the answer "unknown." If the correct answer is "false," then the algorithm answers "false," because every iteration will yield "unknown." However, if the correct answer is "true," then the algorithm could answer either "true" or "false," because it may be possible that each iteration produced the response "unknown" even though the correct response was "true." We will show that this possibility becomes extremely unlikely as the number of tests increases.

Suppose that p is the probability that the response of a test is "true," given that the answer is "true." It follows that $1-p$ is the probability that the response is "unknown," given that the answer is "true." Because the algorithm answers "false" when all n iterations yield the answer "unknown" and the iterations perform independent tests, the probability of error is $(1-p)^n$. When $p \neq 0$, this probability approaches 0 as the number of tests increases. Consequently, the probability that the algorithm answers "true" when the answer is "true" approaches 1.

EXAMPLE 15 **Quality Control** (This example is adapted from [AhUl95].) Suppose that a manufacturer orders processor chips in batches of size n, where n is a positive integer. The chip maker has tested only some of these batches to make sure that all the chips in the batch are good (replacing any bad chips found during testing with good ones). In previously untested batches, the probability that a particular chip is bad has been observed to be 0.1 when random testing is done. The PC manufacturer wants to decide whether all the chips in a batch are good. To

do this, the PC manufacturer can test each chip in a batch to see whether it is good. However, this requires n tests. Assuming that each test can be carried out in constant time, these tests require $O(n)$ seconds. Can the PC manufacturer determine whether a batch of chips has been tested by the chip maker using less time?

Solution: We can use a Monte Carlo algorithm to determine whether a batch of chips has been tested by the chip maker as long as we are willing to accept some probability of error. The algorithm is set up to answer the question: "Has this batch of chips not been tested by the chip maker?" It proceeds by successively selecting chips at random from the batch and testing them one by one. When a bad chip is encountered, the algorithm answers "true" and stops. If a tested chip is good, the algorithm answers "unknown" and goes on to the next chip. After the algorithm has tested a specified number of chips, say k chips, without getting an answer of "true," the algorithm terminates with the answer "false"; that is, the algorithm concludes that the batch is good, that is, that the chip maker has tested all the chips in the batch.

The only way for this algorithm to answer incorrectly is for it to conclude that an untested batch of chips has been tested by the chip maker. The probability that a chip is good, but that it came from an untested batch, is $1 - 0.1 = 0.9$. Because the events of testing different chips from a batch are independent, the probability that all k steps of the algorithm produce the answer "unknown," given that the batch of chips is untested, is 0.9^k.

By taking k large enough, we can make this probability as small as we like. For example, by testing 66 chips, the probability that the algorithm decides a batch has been tested by the chip maker is 0.9^{66}, which is less than 0.001. That is, the probability is less than 1 in 1000 that the algorithm has answered incorrectly. Note that this probability is independent of n, the number of chips in a batch. That is, the Monte Carlo algorithm uses a constant number, or $O(1)$, tests and requires $O(1)$ seconds, no matter how many chips are in a batch. As long as the PC manufacturer can live with an error rate of less than 1 in 1000, the Monte Carlo algorithm will save the PC manufacturer a lot of testing. If a smaller error rate is needed, the PC manufacturer can test more chips in each batch; the reader can verify that 132 tests lower the error rate to less than 1 in 1,000,000. ◀

EXAMPLE 16 **Probabilistic Primality Testing** In Chapter 4 we remarked that a composite integer, that is, an integer greater than one that is not prime, passes Miller's test (see the preamble to Exercise 44 in Section 4.4) for fewer than $n/4$ bases b with $1 < b < n$. This observation is the basis for a Monte Carlo algorithm to determine whether an integer greater than one is prime. Because large primes play an essential role in public-key cryptography (see Section 4.6), being able to generate large primes quickly has become extremely important.

The goal of the algorithm is to decide the question "Is n composite?" Given an integer n greater than one, we select an integer b at random with $1 < b < n$ and determine whether n passes Miller's test to the base b. If n fails the test, the answer is "true" because n must be composite, and the algorithm ends. Otherwise, we perform the test k times, where k is a positive integer. Each time we select a random integer b and determine whether n passes Miller's test to the base b. If the answer is "unknown" at each step, the algorithm answers "false," that is, it says that n is not composite, so that it is prime. The only possibility for the algorithm to return an incorrect answer occurs when n is composite, and the answer "unknown" is the output at each of the k iterations. The probability that a composite integer n passes Miller's test for a randomly selected base b is less than $1/4$. Because the integer b with $1 < b < n$ is selected at random at each iteration and these iterations are independent, the probability that n is composite but the algorithm responds that n is prime is less than $(1/4)^k$. By taking k to be sufficiently large, we can make this probability extremely small. For example, with 10 iterations, the probability that the algorithm decides that n is prime when it really is composite is less than 1 in 1,000,000. With 30 iterations, this probability drops to less than 1 in 10^{18}, an extremely unlikely event.

A number that passes many iterations of a probabilistic primality test is called an *industrial strength prime*, even though it may be composite.

To generate large primes, say with 200 digits, we randomly choose an integer n with 200 digits and run this algorithm, with 30 iterations. If the algorithm decides that n is prime, we

can use it as one of the two primes used in an encryption key for the RSA cryptosystem. If n is actually composite and is used as part of the key, the procedures used to decrypt messages will not produce the original encrypted message. The key is then discarded and two new possible primes are used. ◀

The Probabilistic Method

We discussed existence proofs in Chapter 1 and illustrated the difference between constructive existence proofs and nonconstructive existence proofs. The probabilistic method, introduced by Paul Erdős and Alfréd Rényi, is a powerful technique that can be used to create nonconstructive existence proofs. To use the probabilistic method to prove results about a set S, such as the existence of an element in S with a specified property, we assign probabilities to the elements of S. We then use methods from probability theory to prove results about the elements of S. In particular, we can show that an element with a specified property exists by showing that the probability an element $x \in S$ has this property is positive. The probabilistic method is based on the equivalent statement in Theorem 3.

THEOREM 3 **THE PROBABILISTIC METHOD** If the probability that an element chosen at random from a S does not have a particular property is less than 1, there exists an element in S with this property.

An existence proof based on the probabilistic method is nonconstructive because it does not find a particular element with the desired property.

We illustrate the power of the probabilistic method by finding a lower bound for the Ramsey number $R(k, k)$. Recall from Section 6.2 that $R(k, k)$ equals the minimum number of people at a party needed to ensure that there are at least k mutual friends or k mutual enemies (assuming that any two people are friends or enemies).

THEOREM 4 If k is an integer with $k \geq 2$, then $R(k, k) \geq 2^{k/2}$.

Proof: We note that the theorem holds for $k = 2$ and $k = 3$ because $R(2, 2) = 2$ and $R(3, 3) = 6$, as was shown in Section 6.2. Now suppose that $k \geq 4$. We will use the probabilistic method to show that if there are fewer than $2^{k/2}$ people at a party, it is possible that no k of them are mutual friends or mutual enemies. This will show that $R(k, k)$ is at least $2^{k/2}$.

To use the probabilistic method, we assume that it is equally likely for two people to be friends or enemies. (Note that this assumption does not have to be realistic.) Suppose there are n people at the party. It follows that there are $\binom{n}{k}$ different sets of k people at this party, which we list as $S_1, S_2, \ldots, S_{\binom{n}{k}}$. Let E_i be the event that all k people in S_i are either mutual friends or mutual enemies. The probability that there are either k mutual friends or k mutual enemies among the n people equals $p(\bigcup_{i=1}^{\binom{n}{k}} E_i)$.

According to our assumption it is equally likely for two people to be friends or enemies. The probability that two people are friends equals the probability that they are enemies; both probabilities equal $1/2$. Furthermore, there are $\binom{k}{2} = k(k-1)/2$ pairs of people in S_i because there are k people in S_i. Hence, the probability that all k people in S_i are mutual friends and the probability that all k people in S_i are mutual enemies both equal $(1/2)^{k(k-1)/2}$. It follows that $p(E_i) = 2(1/2)^{k(k-1)/2}$.

The probability that there are either k mutual friends or k mutual enemies in the group of n people equals $p(\bigcup_{i=1}^{\binom{n}{k}} E_i)$. Using Boole's inequality (Exercise 15), it follows that

$$p\left(\bigcup_{i=1}^{\binom{n}{k}} E_i\right) \leq \sum_{i=1}^{\binom{n}{k}} p(E_i) = \binom{n}{k} \cdot 2\left(\frac{1}{2}\right)^{k(k-1)/2}.$$

By Exercise 17 in Section 6.4, we have $\binom{n}{k} \leq n^k/2^{k-1}$. Hence,

$$\binom{n}{k} 2\left(\frac{1}{2}\right)^{k(k-1)/2} \leq \frac{n^k}{2^{k-1}} 2\left(\frac{1}{2}\right)^{k(k-1)/2}.$$

Now if $n < 2^{k/2}$, we have

$$\frac{n^k}{2^{k-1}} 2\left(\frac{1}{2}\right)^{k(k-1)/2} < \frac{2^{k(k/2)}}{2^{k-1}} 2\left(\frac{1}{2}\right)^{k(k-1)/2} = 2^{2-(k/2)} \leq 1,$$

where the last step follows because $k \geq 4$.

We can now conclude that $p(\bigcup_{i=1}^{\binom{n}{k}} E_i) < 1$ when $k \geq 4$. Hence, the probability of the complementary event, that there is no set of either k mutual friends or mutual enemies at the party, is greater than 0. It follows that if $n < 2^{k/2}$, there is at least one set such that no subset of k people are mutual friends or mutual enemies. ◁

Exercises

1. What probability should be assigned to the outcome of heads when a biased coin is tossed, if heads is three times as likely to come up as tails? What probability should be assigned to the outcome of tails?
2. Find the probability of each outcome when a loaded die is rolled, if a 3 is twice as likely to appear as each of the other five numbers on the die.
3. Find the probability of each outcome when a biased die is rolled, if rolling a 2 or rolling a 4 is three times as likely as rolling each of the other four numbers on the die and it is equally likely to roll a 2 or a 4.
4. Show that conditions (*i*) and (*ii*) are met under Laplace's definition of probability, when outcomes are equally likely.
5. A pair of dice is loaded. The probability that a 4 appears on the first die is $2/7$, and the probability that a 3 appears on the second die is $2/7$. Other outcomes for each die appear with probability $1/7$. What is the probability of 7 appearing as the sum of the numbers when the two dice are rolled?
6. What is the probability of these events when we randomly select a permutation of $\{1, 2, 3\}$?
   a) 1 precedes 3.
   b) 3 precedes 1.
   c) 3 precedes 1 and 3 precedes 2.
7. What is the probability of these events when we randomly select a permutation of $\{1, 2, 3, 4\}$?
   a) 1 precedes 4.
   b) 4 precedes 1.
   c) 4 precedes 1 and 4 precedes 2.
   d) 4 precedes 1, 4 precedes 2, and 4 precedes 3.
   e) 4 precedes 3 and 2 precedes 1.
8. What is the probability of these events when we randomly select a permutation of $\{1, 2, \ldots, n\}$ where $n \geq 4$?
   a) 1 precedes 2.
   b) 2 precedes 1.
   c) 1 immediately precedes 2.
   d) n precedes 1 and $n - 1$ precedes 2.
   e) n precedes 1 and n precedes 2.
9. What is the probability of these events when we randomly select a permutation of the 26 lowercase letters of the English alphabet?
   a) The permutation consists of the letters in reverse alphabetic order.
   b) z is the first letter of the permutation.
   c) z precedes a in the permutation.
   d) a immediately precedes z in the permutation.
   e) a immediately precedes m, which immediately precedes z in the permutation.
   f) m, n, and o are in their original places in the permutation.

10. What is the probability of these events when we randomly select a permutation of the 26 lowercase letters of the English alphabet?
   a) The first 13 letters of the permutation are in alphabetical order.
   b) a is the first letter of the permutation and z is the last letter.
   c) a and z are next to each other in the permutation.
   d) a and b are not next to each other in the permutation.
   e) a and z are separated by at least 23 letters in the permutation.
   f) z precedes both a and b in the permutation.

11. Suppose that E and F are events such that $p(E) = 0.7$ and $p(F) = 0.5$. Show that $p(E \cup F) \geq 0.7$ and $p(E \cap F) \geq 0.2$.

12. Suppose that E and F are events such that $p(E) = 0.8$ and $p(F) = 0.6$. Show that $p(E \cup F) \geq 0.8$ and $p(E \cap F) \geq 0.4$.

13. Show that if E and F are events, then $p(E \cap F) \geq p(E) + p(F) - 1$. This is known as **Bonferroni's inequality**.

14. Use mathematical induction to prove the following generalization of Bonferroni's inequality:

$$p(E_1 \cap E_2 \cap \cdots \cap E_n) \geq p(E_1) + p(E_2) + \cdots + p(E_n) - (n-1),$$

where $E_1, E_2, \ldots, E_n$ are n events.

☞ **15.** Show that if $E_1, E_2, \ldots, E_n$ are events from a finite sample space, then

$$p(E_1 \cup E_2 \cup \cdots \cup E_n) \leq p(E_1) + p(E_2) + \cdots + p(E_n).$$

This is known as **Boole's inequality**.

16. Show that if E and F are independent events, then $\overline{E}$ and $\overline{F}$ are also independent events.

17. If E and F are independent events, prove or disprove that $\overline{E}$ and F are necessarily independent events.

In Exercises 18, 20, and 21 assume that the year has 366 days and all birthdays are equally likely. In Exercise 19 assume it is equally likely that a person is born in any given month of the year.

18. a) What is the probability that two people chosen at random were born on the same day of the week?
   b) What is the probability that in a group of n people chosen at random, there are at least two born on the same day of the week?
   c) How many people chosen at random are needed to make the probability greater than $1/2$ that there are at least two people born on the same day of the week?

19. a) What is the probability that two people chosen at random were born during the same month of the year?
   b) What is the probability that in a group of n people chosen at random, there are at least two born in the same month of the year?
   c) How many people chosen at random are needed to make the probability greater than $1/2$ that there are at least two people born in the same month of the year?

20. Find the smallest number of people you need to choose at random so that the probability that at least one of them has a birthday today exceeds $1/2$.

21. Find the smallest number of people you need to choose at random so that the probability that at least two of them were both born on April 1 exceeds $1/2$.

***22.** February 29 occurs only in leap years. Years divisible by 4, but not by 100, are always leap years. Years divisible by 100, but not by 400, are not leap years, but years divisible by 400 are leap years.
   a) What probability distribution for birthdays should be used to reflect how often February 29 occurs?
   b) Using the probability distribution from part (a), what is the probability that in a group of n people at least two have the same birthday?

23. What is the conditional probability that exactly four heads appear when a fair coin is flipped five times, given that the first flip came up heads?

24. What is the conditional probability that exactly four heads appear when a fair coin is flipped five times, given that the first flip came up tails?

25. What is the conditional probability that a randomly generated bit string of length four contains at least two consecutive 0s, given that the first bit is a 1? (Assume the probabilities of a 0 and a 1 are the same.)

26. Let E be the event that a randomly generated bit string of length three contains an odd number of 1s, and let F be the event that the string starts with 1. Are E and F independent?

27. Let E and F be the events that a family of n children has children of both sexes and has at most one boy, respectively. Are E and F independent if
   a) $n = 2$? **b)** $n = 4$? **c)** $n = 5$?

28. Assume that the probability a child is a boy is 0.51 and that the sexes of children born into a family are independent. What is the probability that a family of five children has
   a) exactly three boys?
   b) at least one boy?
   c) at least one girl?
   d) all children of the same sex?

29. A group of six people play the game of "odd person out" to determine who will buy refreshments. Each person flips a fair coin. If there is a person whose outcome is not the same as that of any other member of the group, this person has to buy the refreshments. What is the probability that there is an odd person out after the coins are flipped once?

30. Find the probability that a randomly generated bit string of length 10 does not contain a 0 if bits are independent and if
   a) a 0 bit and a 1 bit are equally likely.
   b) the probability that a bit is a 1 is 0.6.
   c) the probability that the ith bit is a 1 is $1/2^i$ for $i = 1, 2, 3, \ldots, 10$.

31. Find the probability that a family with five children does not have a boy, if the sexes of children are independent and if

a) a boy and a girl are equally likely.

b) the probability of a boy is 0.51.

c) the probability that the ith child is a boy is $0.51 - (i/100)$.

32. Find the probability that a randomly generated bit string of length 10 begins with a 1 or ends with a 00 for the same conditions as in parts (a), (b), and (c) of Exercise 30, if bits are generated independently.

33. Find the probability that the first child of a family with five children is a boy or that the last two children of the family are girls, for the same conditions as in parts (a), (b), and (c) of Exercise 31.

34. Find each of the following probabilities when n independent Bernoulli trials are carried out with probability of success p.

a) the probability of no successes

b) the probability of at least one success

c) the probability of at most one success

d) the probability of at least two successes

35. Find each of the following probabilities when n independent Bernoulli trials are carried out with probability of success p.

a) the probability of no failures

b) the probability of at least one failure

c) the probability of at most one failure

d) the probability of at least two failures

36. Use mathematical induction to prove that if $E_1, E_2, \ldots, E_n$ is a sequence of n pairwise disjoint events in a sample space S, where n is a positive integer, then $p(\bigcup_{i=1}^{n} E_i) = \sum_{i=1}^{n} p(E_i)$.

***37.** (*Requires calculus*) Show that if $E_1, E_2, \ldots$ is an infinite sequence of pairwise disjoint events in a sample space S, then $p(\bigcup_{i=1}^{\infty} E_i) = \sum_{i=1}^{\infty} p(E_i)$. [*Hint:* Use Exercise 36 and take limits.]

38. A pair of dice is rolled in a remote location and when you ask an honest observer whether at least one die came up six, this honest observer answers in the affirmative.

a) What is the probability that the sum of the numbers that came up on the two dice is seven, given the information provided by the honest observer?

b) Suppose that the honest observer tells us that at least one die came up five. What is the probability the sum of the numbers that came up on the dice is seven, given this information?

****39.** This exercise employs the probabilistic method to prove a result about round-robin tournaments. In a **round-robin tournament** with m players, every two players play one game in which one player wins and the other loses.

We want to find conditions on positive integers m and k with $k < m$ such that it is possible for the outcomes of the tournament to have the property that for every set of k players, there is a player who beats every member in this set. So that we can use probabilistic reasoning to draw conclusions about round-robin tournaments, we assume that when two players compete it is equally likely that either player wins the game and we assume that the outcomes of different games are independent. Let E be the event that for every set S with k players, where k is a positive integer less than m, there is a player who has beaten all k players in S.

a) Show that $p(\overline{E}) \leq \sum_{j=1}^{\binom{m}{k}} p(F_j)$, where F_j is the event that there is no player who beats all k players from the jth set in a list of the $\binom{m}{k}$ sets of k players.

b) Show that the probability of F_j is $(1-2^{-k})^{m-k}$.

c) Conclude from parts (a) and (b) that $p(\overline{E}) \leq \binom{m}{k}(1-2^{-k})^{m-k}$ and, therefore, that there must be a tournament with the described property if $\binom{m}{k}(1-2^{-k})^{m-k} < 1$.

d) Use part (c) to find values of m such that there is a tournament with m players such that for every set S of two players, there is a player who has beaten both players in S. Repeat for sets of three players.

***40.** Devise a Monte Carlo algorithm that determines whether a permutation of the integers 1 through n has already been sorted (that is, it is in increasing order), or instead, is a random permutation. A step of the algorithm should answer "true" if it determines the list is not sorted and "unknown" otherwise. After k steps, the algorithm decides that the integers are sorted if the answer is "unknown" in each step. Show that as the number of steps increases, the probability that the algorithm produces an incorrect answer is extremely small. [*Hint:* For each step, test whether certain elements are in the correct order. Make sure these tests are independent.]

41. Use pseudocode to write out the probabilistic primality test described in Example 16.

7.3 Bayes' Theorem

Introduction

There are many times when we want to assess the probability that a particular event occurs on the basis of partial evidence. For example, suppose we know the percentage of people who have a particular disease for which there is a very accurate diagnostic test. People who test positive for

this disease would like to know the likelihood that they actually have the disease. In this section we introduce a result that can be used to determine this probability, namely, the probability that a person has the disease given that this person tests positive for it. To use this result, we will need to know the percentage of people who do not have the disease but test positive for it and the percentage of people who have the disease but test negative for it.

Similarly, suppose we know the percentage of incoming e-mail messages that are spam. We will see that we can determine the likelihood that an incoming e-mail message is spam using the occurrence of words in the message. To determine this likelihood, we need to know the percentage of incoming messages that are spam, the percentage of spam messages in which each of these words occurs, and the percentage of messages that are not spam in which each of these words occurs.

The result that we can use to answer questions such as these is called Bayes' theorem and dates back to the eighteenth century. In the past two decades, Bayes' theorem has been extensively applied to estimate probabilities based on partial evidence in areas as diverse as medicine, law, machine learning, engineering, and software development.

Bayes' Theorem

We illustrate the idea behind Bayes' theorem with an example that shows that when extra information is available, we can derive a more realistic estimate that a particular event occurs. That is, suppose we know $p(F)$, the probability that an event F occurs, but we have knowledge that an event E occurs. Then the conditional probability that F occurs given that E occurs, $p(F \mid E)$, is a more realistic estimate than $p(F)$ that F occurs. In Example 1 we will see that we can find $p(F \mid E)$ when we know $p(F)$, $p(E \mid F)$, and $p(E \mid \overline{F})$.

EXAMPLE 1 We have two boxes. The first contains two green balls and seven red balls; the second contains four green balls and three red balls. Bob selects a ball by first choosing one of the two boxes at random. He then selects one of the balls in this box at random. If Bob has selected a red ball, what is the probability that he selected a ball from the first box?

Extra Examples

Solution: Let E be the event that Bob has chosen a red ball; $\overline{E}$ is the event that Bob has chosen a green ball. Let F be the event that Bob has chosen a ball from the first box; $\overline{F}$ is the event that Bob has chosen a ball from the second box. We want to find $p(F \mid E)$, the probability that the ball Bob selected came from the first box, given that it is red. By the definition of conditional probability, we have $p(F \mid E) = p(F \cap E)/p(E)$. Can we use the information provided to determine both $p(F \cap E)$ and $p(E)$ so that we can find $p(F \mid E)$?

First, note that because the first box contains seven red balls out of a total of nine balls, we know that $p(E \mid F) = 7/9$. Similarly, because the second box contains three red balls out of a total of seven balls, we know that $p(E \mid \overline{F}) = 3/7$. We assumed that Bob selects a box at random, so $p(F) = p(\overline{F}) = 1/2$. Because $p(E \mid F) = p(E \cap F)/p(F)$, it follows that $p(E \cap F) = p(E \mid F)p(F) = \frac{7}{9} \cdot \frac{1}{2} = \frac{7}{18}$ [as we remarked earlier, this is one of the quantities we need to find to determine $p(F \mid E)$]. Similarly, because $p(E \mid \overline{F}) = p(E \cap \overline{F})/p(\overline{F})$, it follows that $p(E \cap \overline{F}) = p(E \mid \overline{F})p(\overline{F}) = \frac{3}{7} \cdot \frac{1}{2} = \frac{3}{14}$.

We can now find $p(E)$. Note that $E = (E \cap F) \cup (E \cap \overline{F})$, where $E \cap F$ and $E \cap \overline{F}$ are disjoint sets. (If x belongs to both $E \cap F$ and $E \cap \overline{F}$, then x belongs to both F and $\overline{F}$, which is impossible.) It follows that

$$p(E) = p(E \cap F) + p(E \cap \overline{F}) = \frac{7}{18} + \frac{3}{14} = \frac{49}{126} + \frac{27}{126} = \frac{76}{126} = \frac{38}{63}.$$

We have now found both $p(F \cap E) = 7/18$ and $p(E) = 38/63$. We conclude that

$$p(F \mid E) = \frac{p(F \cap E)}{p(E)} = \frac{7/18}{38/63} = \frac{49}{76} \approx 0.645.$$

Before we had any extra information, we assumed that the probability that Bob selected the first box was $1/2$. However, with the extra information that the ball selected at random is red, this probability has increased to approximately 0.645. That is, the probability that Bob selected a ball from the first box increased from $1/2$, when no extra information was available, to 0.645 once we knew that the ball selected was red. ◀

Using the same type of reasoning as in Example 1, we can find the conditional probability that an event F occurs, given that an event E has occurred, when we know $p(E \mid F)$, $p(E \mid \overline{F})$, and $p(F)$. The result we can obtain is called **Bayes' theorem**; it is named after Thomas Bayes, an eighteenth-century British mathematician and minister who introduced this result.

THEOREM 1 **BAYES' THEOREM** Suppose that E and F are events from a sample space S such that $p(E) \neq 0$ and $p(F) \neq 0$. Then

$$p(F \mid E) = \frac{p(E \mid F)p(F)}{p(E \mid F)p(F) + p(E \mid \overline{F})p(\overline{F})}.$$

Proof: The definition of conditional probability tells us that $p(F \mid E) = p(E \cap F)/p(E)$ and $p(E \mid F) = p(E \cap F)/p(F)$. Therefore, $p(E \cap F) = p(F \mid E)p(E)$ and $p(E \cap F) = p(E \mid F)p(F)$. Equating these two expressions for $p(E \cap F)$ shows that

$$p(F \mid E)p(E) = p(E \mid F)p(F).$$

Dividing both sides by $p(E)$, we find that

$$p(F \mid E) = \frac{p(E \mid F)p(F)}{p(E)}.$$

Next, we show that $p(E) = p(E \mid F)p(F) + p(E \mid \overline{F})p(\overline{F})$. To see this, first note that $E = E \cap S = E \cap (F \cup \overline{F}) = (E \cap F) \cup (E \cap \overline{F})$. Furthermore, $E \cap F$ and $E \cap \overline{F}$ are disjoint, because if $x \in E \cap F$ and $x \in E \cap \overline{F}$, then $x \in F \cap \overline{F} = \emptyset$. Consequently, $p(E) = p(E \cap F) + p(E \cap \overline{F})$. We have already shown that $p(E \cap F) = p(E \mid F)p(F)$. Moreover, we have $p(E \mid \overline{F}) = p(E \cap \overline{F})/p(\overline{F})$, which shows that $p(E \cap \overline{F}) = p(E \mid \overline{F})p(\overline{F})$. It now follows that

$$p(E) = p(E \cap F) + p(E \cap \overline{F}) = p(E \mid F)p(F) + p(E \mid \overline{F})p(\overline{F}).$$

To complete the proof we insert this expression for $p(E)$ into the equation $p(F \mid E) = p(E \mid F)p(F)/p(E)$. We have proved that

Links

$$p(F \mid E) = \frac{p(E \mid F)p(F)}{p(E \mid F)p(F) + p(E \mid \overline{F})p(\overline{F})}.$$

◁

APPLYING BAYES' THEOREM Bayes' theorem can be used to solve problems that arise in many disciplines. Next, we will discuss an application of Bayes' theorem to medicine. In particular, we will illustrate how Bayes' theorem can be used to assess the probability that someone testing positive for a disease actually has this disease. The results obtained from Bayes' theorem are often somewhat surprising, as Example 2 shows.

EXAMPLE 2 Suppose that one person in 100,000 has a particular rare disease for which there is a fairly accurate diagnostic test. This test is correct 99.0% of the time when given to a person selected at random who has the disease; it is correct 99.5% of the time when given to a person selected at random who does not have the disease. Given this information can we find

(a) the probability that a person who tests positive for the disease has the disease?

(b) the probability that a person who tests negative for the disease does not have the disease?

Should a person who tests positive be very concerned that he or she has the disease?

Solution: (a) Let F be the event that a person selected at random has the disease, and let E be the event that a person selected at random tests positive for the disease. We want to compute $p(F \mid E)$. To use Bayes' theorem to compute $p(F \mid E)$ we need to find $p(E \mid F)$, $p(E \mid \overline{F})$, $p(F)$, and $p(\overline{F})$.

We know that one person in 100,000 has this disease, so $p(F) = 1/100,000 = 0.00001$ and $p(\overline{F}) = 1 - 0.00001 = 0.99999$. Because a person who has the disease tests positive 99% of the time, we know that $p(E \mid F) = 0.99$; this is the probability of a true positive, that a person with the disease tests positive. It follows that $p(\overline{E} \mid F) = 1 - p(E \mid F) = 1 - 0.99 = 0.01$; this is the probability of a false negative, that a person who has the disease tests negative.

Furthermore, because a person who does not have the disease tests negative 99.5% of the time, we know that $p(\overline{E} \mid \overline{F}) = 0.995$. This is the probability of a true negative, that a person without the disease tests negative. Finally, we see that $p(E \mid \overline{F}) = 1 - p(\overline{E} \mid \overline{F}) = 1 - 0.995 = 0.005$; this is the probability of a false positive, that a person without the disease tests positive.

The probability that a person who tests positive for the disease actually has the disease is $p(F \mid E)$. By Bayes' theorem, we know that

$$p(F \mid E) = \frac{p(E \mid F)p(F)}{p(E \mid F)p(F) + p(E \mid \overline{F})p(\overline{F})}$$

$$= \frac{(0.99)(0.00001)}{(0.99)(0.00001) + (0.005)(0.99999)} \approx 0.002.$$

(b) The probability that someone who tests negative for the disease does not have the disease is $p(\overline{F} \mid \overline{E})$. By Bayes' theorem, we know that

$$p(\overline{F} \mid \overline{E}) = \frac{p(\overline{E} \mid \overline{F})p(\overline{F})}{p(\overline{E} \mid \overline{F})p(\overline{F}) + p(\overline{E} \mid F)p(F)}$$

$$= \frac{(0.995)(0.99999)}{(0.995)(0.99999) + (0.01)(0.00001)} \approx 0.9999999.$$

Consequently, 99.99999% of the people who test negative really do not have the disease.

In part (a) we showed that only 0.2% of people who test positive for the disease actually have the disease. Because the disease is extremely rare, the number of false positives on the diagnostic test is far greater than the number of true positives, making the percentage of people who test positive who actually have the disease extremely small. People who test positive for the diseases should not be overly concerned that they actually have the disease. ◀

GENERALIZING BAYES' THEOREM Note that in the statement of Bayes' theorem, the events F and $\overline{F}$ are mutually exclusive and cover the entire sample space S (that is, $F \cup \overline{F} = S$). We can extend Bayes' theorem to any collection of mutually exclusive events that cover the entire sample space S, in the following way.

THEOREM 2 **GENERALIZED BAYES' THEOREM** Suppose that E is an event from a sample space S and that $F_1, F_2, \ldots, F_n$ are mutually exclusive events such that $\bigcup_{i=1}^{n} F_i = S$. Assume that $p(E) \neq 0$ and $p(F_i) \neq 0$ for $i = 1, 2, \ldots, n$. Then

$$p(F_j \mid E) = \frac{p(E \mid F_j)p(F_j)}{\sum_{i=1}^{n} p(E \mid F_i)p(F_i)}.$$

We leave the proof of this generalized version of Bayes' theorem as Exercise 17.

Bayesian Spam Filters

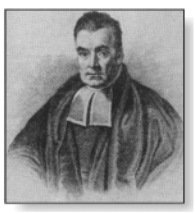
Most electronic mailboxes receive a flood of unwanted and unsolicited messages, known as **spam**. Because spam threatens to overwhelm electronic mail systems, a tremendous amount of work has been devoted to filtering it out. Some of the first tools developed for eliminating spam were based on Bayes' theorem, such as **Bayesian spam filters**.

Links

The use of the word *spam* for unsolicited e-mail comes from a Monty Python comedy sketch about a cafe where the food product Spam comes with everything regardless of whether customers want it.

A Bayesian spam filter uses information about previously seen e-mail messages to guess whether an incoming e-mail message is spam. Bayesian spam filters look for occurrences of particular words in messages. For a particular word w, the probability that w appears in a spam e-mail message is estimated by determining the number of times w appears in a message from a large set of messages known to be spam and the number of times it appears in a large set of messages known not to be spam. When we examine e-mail messages to determine whether they might be spam, we look at words that might be indicators of spam, such as "offer," "special," or "opportunity," as well as words that might indicate that a message is not spam, such as "mom," "lunch," or "Jan" (where Jan is one of your friends). Unfortunately, spam filters sometimes fail to identify a spam message as spam; this is called a false negative. And they sometimes identify a message that is not spam as spam; this is called a false positive. When testing for spam, it is important to minimize false positives, because filtering out wanted e-mail is much worse than letting some spam through.

Links



THOMAS BAYES (1702–1761) Thomas Bayes was the son a minister in a religious sect known as the Nonconformists. This sect was considered heretical in eighteenth-century Great Britain. Because of the secrecy of the Nonconformists, little is known of Thomas Bayes' life. When Thomas was young, his family moved to London. Thomas was likely educated privately; Nonconformist children generally did not attend school. In 1719 Bayes entered the University of Edinburgh, where he studied logic and theology. He was ordained as a Nonconformist minister like his father and began his work as a minister assisting his father. In 1733 he became minister of the Presbyterian Chapel in Tunbridge Wells, southeast of London, where he remained minister until 1752.

Bayes is best known for his essay on probability published in 1764, three years after his death. This essay was sent to the Royal Society by a friend who found it in the papers left behind when Bayes died. In the introduction to this essay, Bayes stated that his goal was to find a method that could measure the probability that an event happens, assuming that we know nothing about it, but that, under the same circumstances, it has happened a certain proportion of times. Bayes' conclusions were accepted by the great French mathematician Laplace but were later challenged by Boole, who questioned them in his book *Laws of Thought*. Since then Bayes' techniques have been subject to controversy.

Bayes also wrote an article that was published posthumously: "An Introduction to the Doctrine of Fluxions, and a Defense of the Mathematicians Against the Objections of the Author of The Analyst," which supported the logical foundations of calculus. Bayes was elected a Fellow of the Royal Society in 1742, with the support of important members of the Society, even though at that time he had no published mathematical works. Bayes' sole known publication during his lifetime was allegedly a mystical book entitled *Divine Benevolence*, discussing the original causation and ultimate purpose of the universe. Although the book is commonly attributed to Bayes, no author's name appeared on the title page, and the entire work is thought to be of dubious provenance. Evidence for Bayes' mathematical talents comes from a notebook that was almost certainly written by Bayes, which contains much mathematical work, including discussions of probability, trigonometry, geometry, solutions of equations, series, and differential calculus. There are also sections on natural philosophy, in which Bayes looks at topics that include electricity, optics, and celestial mechanics. Bayes is also the author of a mathematical publication on asymptotic series, which appeared after his death.

We will develop some basic Bayesian spam filters. First, suppose we have a set B of messages known to be spam and a set G of messages known not to be spam. (For example, users could classify messages as spam when they examine them in their inboxes.) We next identify the words that occur in B and in G. We count the number of messages in the set containing each word to find $n_B(w)$ and $n_G(w)$, the number of messages containing the word w in the sets B and G, respectively. Then, the empirical probability that a spam message contains the word w is $p(w) = n_B(w)/|B|$, and the empirical probability that a message that is not spam contains the word w is $q(w) = n_G(w)/|G|$. We note that $p(w)$ and $q(w)$ estimate the probabilities that an incoming spam message, and an incoming message that is not spam, contain the word w, respectively.

Now suppose we receive a new e-mail message containing the word w. Let S be the event that the message is spam. Let E be the event that the message contains the word w. The events S, that the message is spam, and $\overline{S}$, that the message is not spam, partition the set of all messages. Hence, by Bayes' theorem, the probability that the message is spam, given that it contains the word w, is

$$p(S \mid E) = \frac{p(E \mid S)p(S)}{p(E \mid S)p(S) + p(E \mid \overline{S})p(\overline{S})}.$$

To apply this formula, we first estimate $p(S)$, the probability that an incoming message is spam, as well as $p(\overline{S})$, the probability that the incoming message is not spam. Without prior knowledge about the likelihood that an incoming message is spam, for simplicity we assume that the message is equally likely to be spam as it is not to be spam. That is, we assume that $p(S) = p(\overline{S}) = 1/2$. Using this assumption, we find that the probability that a message is spam, given that it contains the word w, is

$$p(S \mid E) = \frac{p(E \mid S)}{p(E \mid S) + p(E \mid \overline{S})}.$$

(Note that if we have some empirical data about the ratio of spam messages to messages that are not spam, we can change this assumption to produce a better estimate for $p(S)$ and for $p(\overline{S})$; see Exercise 22.)

Next, we estimate $p(E \mid S)$, the conditional probability that the message contains the word w given that the message is spam, by $p(w)$. Similarly, we estimate $p(E \mid \overline{S})$, the conditional probability that the message contains the word w, given that the message is not spam, by $q(w)$. Inserting these estimates for $p(E \mid S)$ and $p(E \mid \overline{S})$ tells us that $p(S \mid E)$ can be estimated by

$$r(w) = \frac{p(w)}{p(w) + q(w)};$$

that is, $r(w)$ estimates the probability that the message is spam, given that it contains the word w. If $r(w)$ is greater than a threshold that we set, such as 0.9, then we classify the message as spam.

EXAMPLE 3 Suppose that we have found that the word "Rolex" occurs in 250 of 2000 messages known to be spam and in 5 of 1000 messages known not to be spam. Estimate the probability that an incoming message containing the word "Rolex" is spam, assuming that it is equally likely that an incoming message is spam or not spam. If our threshold for rejecting a message as spam is 0.9, will we reject such messages?

Solution: We use the counts that the word "Rolex" appears in spam messages and messages that are not spam to find that $p(\text{Rolex}) = 250/2000 = 0.125$ and $q(\text{Rolex}) = 5/1000 = 0.005$.

Because we are assuming that it is equally likely for an incoming message to be spam as it is not to be spam, we can estimate the probability that an incoming message containing the word "Rolex" is spam by

$$r(\text{Rolex}) = \frac{p(\text{Rolex})}{p(\text{Rolex}) + q(\text{Rolex})} = \frac{0.125}{0.125 + 0.005} = \frac{0.125}{0.130} \approx 0.962.$$

Because $r(\text{Rolex})$ is greater than the threshold 0.9, we reject such messages as spam. ◀

Detecting spam based on the presence of a single word can lead to excessive false positives and false negatives. Consequently, spam filters look at the presence of multiple words. For example, suppose that the message contains the words w_1 and w_2. Let E_1 and E_2 denote the events that the message contains the words w_1 and w_2, respectively. To make our computations simpler, we assume that E_1 and E_2 are independent events and that $E_1 \mid S$ and $E_2 \mid S$ are independent events and that we have no prior knowledge regarding whether or not the message is spam. (The assumptions that E_1 and E_2 are independent and that $E_1 \mid S$ and $E_2 \mid S$ are independent may introduce some error into our computations; we assume that this error is small.) Using Bayes' theorem and our assumptions, we can show (see Exercise 23) that $p(S \mid E_1 \cap E_2)$, the probability that the message is spam given that it contains both w_1 and w_2, is

$$p(S \mid E_1 \cap E_2) = \frac{p(E_1 \mid S)p(E_2 \mid S)}{p(E_1 \mid S)p(E_2 \mid S) + p(E_1 \mid \overline{S})p(E_2 \mid \overline{S})}.$$

We estimate the probability $p(S \mid E_1 \cap E_2)$ by

$$r(w_1, w_2) = \frac{p(w_1)\,p(w_2)}{p(w_1)\,p(w_2) + q(w_1)q(w_2)}.$$

That is, $r(w_1, w_2)$ estimates the probability that the message is spam, given that it contains the words w_1 and w_2. When $r(w_1, w_2)$ is greater than a preset threshold, such as 0.9, we determine that the message is likely spam.

EXAMPLE 4 Suppose that we train a Bayesian spam filter on a set of 2000 spam messages and 1000 messages that are not spam. The word "stock" appears in 400 spam messages and 60 messages that are not spam, and the word "undervalued" appears in 200 spam messages and 25 messages that are not spam. Estimate the probability that an incoming message containing both the words "stock" and "undervalued" is spam, assuming that we have no prior knowledge about whether it is spam. Will we reject such messages as spam when we set the threshold at 0.9?

Solution: Using the counts of each of these two words in messages known to be spam or known not to be spam, we obtain the following estimates: $p(\text{stock}) = 400/2000 = 0.2$, $q(\text{stock}) = 60/1000 = 0.06$, $p(\text{undervalued}) = 200/2000 = 0.1$, and $q(\text{undervalued}) = 25/1000 = 0.025$. Using these probabilities, we can estimate the probability that the message is spam by

$$\begin{aligned} r(\text{stock}, \text{undervalued}) &= \frac{p(\text{stock})\,p(\text{undervalued})}{p(\text{stock})\,p(\text{undervalued}) + q(\text{stock})\,q(\text{undervalued})} \\ &= \frac{(0.2)(0.1)}{(0.2)(0.1) + (0.06)(0.025)} \approx 0.930. \end{aligned}$$

Because we have set the threshold for rejecting messages at 0.9, such messages will be rejected by the filter. ◀

The more words we use to estimate the probability that an incoming mail message is spam, the better is our chance that we correctly determine whether it is spam. In general, if E_i is the

event that the message contains word w_i, assuming that the number of incoming spam messages is approximately the same as the number of incoming messages that are not spam, and that the events $E_i \mid S$ are independent, then by Bayes' theorem the probability that a message containing all the words $w_1, w_2, \ldots, w_k$ is spam is

$$p(S \mid \bigcap_{i=1}^{k} E_i) = \frac{\prod_{i=1}^{k} p(E_i \mid S)}{\prod_{i=1}^{k} p(E_i \mid S) + \prod_{i=1}^{k} p(E_i \mid \overline{S})}.$$

We can estimate this probability by

$$r(w_1, w_2, \ldots, w_k) = \frac{\prod_{i=1}^{k} p(w_i)}{\prod_{i=1}^{k} p(w_i) + \prod_{i=1}^{k} q(w_i)}.$$

For the most effective spam filter, we choose words for which the probability that each of these words appears in spam is either very high or very low. When we compute this value for a particular message, we reject the message as spam if $r(w_1, w_2, \ldots, w_k)$ exceeds a preset threshold, such as 0.9.

Bayesian poisoning, the insertion of extra words to defeat spam filters, can use random or purposefully selected words.

Another way to improve the performance of a Bayesian spam filter is to look at the probabilities that particular pairs of words appear in spam and in messages that are not spam. We then treat appearances of these pairs of words as appearance of a single block, rather than as the appearance of two separate words. For example, the pair of words "enhance performance" most likely indicates spam, while "operatic performance" indicates a message that is not spam. Similarly, we can assess the likelihood that a message is spam by examining the structure of a message to determine where words appear in it. Also, spam filters look at appearances of certain types of strings of characters rather than just words. For example, a message with the valid e-mail address of one of your friends is less likely to be spam (if not sent by a worm) than one containing an e-mail address that came from a country known to originate a lot of spam. There is an ongoing war between people who create spam and those trying to filter their messages out. This leads to the introduction of many new techniques to defeat spam filters, including inserting into spam messages long strings of words that appear in messages that are not spam, as well as including words inside pictures. The techniques we have discussed here are only the first steps in fighting this war on spam.

Exercises

1. Suppose that E and F are events in a sample space and $p(E) = 1/3$, $p(F) = 1/2$, and $p(E \mid F) = 2/5$. Find $p(F \mid E)$.
2. Suppose that E and F are events in a sample space and $p(E) = 2/3$, $p(F) = 3/4$, and $p(F \mid E) = 5/8$. Find $p(E \mid F)$.
3. Suppose that Frida selects a ball by first picking one of two boxes at random and then selecting a ball from this box at random. The first box contains two white balls and three blue balls, and the second box contains four white balls and one blue ball. What is the probability that Frida picked a ball from the first box if she has selected a blue ball?
4. Suppose that Ann selects a ball by first picking one of two boxes at random and then selecting a ball from this box. The first box contains three orange balls and four black balls, and the second box contains five orange balls and six black balls. What is the probability that Ann picked a ball from the second box if she has selected an orange ball?
5. Suppose that 8% of all bicycle racers use steroids, that a bicyclist who uses steroids tests positive for steroids 96% of the time, and that a bicyclist who does not use steroids tests positive for steroids 9% of the time. What is the probability that a randomly selected bicyclist who tests positive for steroids actually uses steroids?
6. When a test for steroids is given to soccer players, 98% of the players taking steroids test positive and 12% of the players not taking steroids test positive. Suppose that 5% of soccer players take steroids. What is the probability that a soccer player who tests positive takes steroids?
7. Suppose that a test for opium use has a 2% false positive rate and a 5% false negative rate. That is, 2% of people who do not use opium test positive for opium, and

5% of opium users test negative for opium. Furthermore, suppose that 1% of people actually use opium.

a) Find the probability that someone who tests negative for opium use does not use opium.

b) Find the probability that someone who tests positive for opium use actually uses opium.

8. Suppose that one person in 10,000 people has a rare genetic disease. There is an excellent test for the disease; 99.9% of people with the disease test positive and only 0.02% who do not have the disease test positive.

a) What is the probability that someone who tests positive has the genetic disease?

b) What is the probability that someone who tests negative does not have the disease?

9. Suppose that 8% of the patients tested in a clinic are infected with HIV. Furthermore, suppose that when a blood test for HIV is given, 98% of the patients infected with HIV test positive and that 3% of the patients not infected with HIV test positive. What is the probability that

a) a patient testing positive for HIV with this test is infected with it?

b) a patient testing positive for HIV with this test is not infected with it?

c) a patient testing negative for HIV with this test is infected with it?

d) a patient testing negative for HIV with this test is not infected with it?

10. Suppose that 4% of the patients tested in a clinic are infected with avian influenza. Furthermore, suppose that when a blood test for avian influenza is given, 97% of the patients infected with avian influenza test positive and that 2% of the patients not infected with avian influenza test positive. What is the probability that

a) a patient testing positive for avian influenza with this test is infected with it?

b) a patient testing positive for avian influenza with this test is not infected with it?

c) a patient testing negative for avian influenza with this test is infected with it?

d) a patient testing negative for avian influenza with this test is not infected with it?

11. An electronics company is planning to introduce a new camera phone. The company commissions a marketing report for each new product that predicts either the success or the failure of the product. Of new products introduced by the company, 60% have been successes. Furthermore, 70% of their successful products were predicted to be successes, while 40% of failed products were predicted to be successes. Find the probability that this new camera phone will be successful if its success has been predicted.

***12.** A space probe near Neptune communicates with Earth using bit strings. Suppose that in its transmissions it sends a 1 one-third of the time and a 0 two-thirds of the time. When a 0 is sent, the probability that it is received correctly is 0.9, and the probability that it is received incorrectly (as a 1) is 0.1. When a 1 is sent, the probability that it is received correctly is 0.8, and the probability that it is received incorrectly (as a 0) is 0.2.

a) Find the probability that a 0 is received.

b) Use Bayes' theorem to find the probability that a 0 was transmitted, given that a 0 was received.

13. Suppose that E, F_1, F_2, and F_3 are events from a sample space S and that F_1, F_2, and F_3 are pairwise disjoint and their union is S. Find $p(F_1 \mid E)$ if $p(E \mid F_1) = 1/8$, $p(E \mid F_2) = 1/4$, $p(E \mid F_3) = 1/6$, $p(F_1) = 1/4$, $p(F_2) = 1/4$, and $p(F_3) = 1/2$.

14. Suppose that E, F_1, F_2, and F_3 are events from a sample space S and that F_1, F_2, and F_3 are pairwise disjoint and their union is S. Find $p(F_2 \mid E)$ if $p(E \mid F_1) = 2/7$, $p(E \mid F_2) = 3/8$, $p(E \mid F_3) = 1/2$, $p(F_1) = 1/6$, $p(F_2) = 1/2$, and $p(F_3) = 1/3$.

15. In this exercise we will use Bayes' theorem to solve the Monty Hall puzzle (Example 10 in Section 7.1). Recall that in this puzzle you are asked to select one of three doors to open. There is a large prize behind one of the three doors and the other two doors are losers. After you select a door, Monty Hall opens one of the two doors you did not select that he knows is a losing door, selecting at random if both are losing doors. Monty asks you whether you would like to switch doors. Suppose that the three doors in the puzzle are labeled 1, 2, and 3. Let W be the random variable whose value is the number of the winning door; assume that $p(W = k) = 1/3$ for $k = 1, 2, 3$. Let M denote the random variable whose value is the number of the door that Monty opens. Suppose you choose door i.

a) What is the probability that you will win the prize if the game ends without Monty asking you whether you want to change doors?

b) Find $p(M = j \mid W = k)$ for $j = 1, 2, 3$ and $k = 1, 2, 3$.

c) Use Bayes' theorem to find $p(W = j \mid M = k)$ where i and j and k are distinct values.

d) Explain why the answer to part (c) tells you whether you should change doors when Monty gives you the chance to do so.

16. Ramesh can get to work in three different ways: by bicycle, by car, or by bus. Because of commuter traffic, there is a 50% chance that he will be late when he drives his car. When he takes the bus, which uses a special lane reserved for buses, there is a 20% chance that he will be late. The probability that he is late when he rides his bicycle is only 5%. Ramesh arrives late one day. His boss wants to estimate the probability that he drove his car to work that day.

a) Suppose the boss assumes that there is a 1/3 chance that Ramesh takes each of the three ways he can get to work. What estimate for the probability that Ramesh drove his car does the boss obtain from Bayes' theorem under this assumption?

b) Suppose the boss knows that Ramesh drives 30% of the time, takes the bus only 10% of the time, and takes his bicycle 60% of the time. What estimate for the probability that Ramesh drove his car does the boss obtain from Bayes' theorem using this information?

***17.** Prove Theorem 2, the extended form of Bayes' theorem. That is, suppose that E is an event from a sample space S and that $F_1, F_2, \ldots, F_n$ are mutually exclusive events such that $\bigcup_{i=1}^{n} F_i = S$. Assume that $p(E) \neq 0$ and $p(F_i) \neq 0$ for $i = 1, 2, \ldots, n$. Show that

$$p(F_j \mid E) = \frac{p(E \mid F_j)p(F_j)}{\sum_{i=1}^{n} p(E \mid F_i)p(F_i)}.$$

[*Hint:* Use the fact that $E = \bigcup_{i=1}^{n}(E \cap F_i)$.]

18. Suppose that a Bayesian spam filter is trained on a set of 500 spam messages and 200 messages that are not spam. The word "exciting" appears in 40 spam messages and in 25 messages that are not spam. Would an incoming message be rejected as spam if it contains the word "exciting" and the threshold for rejecting spam is 0.9?

19. Suppose that a Bayesian spam filter is trained on a set of 1000 spam messages and 400 messages that are not spam. The word "opportunity" appears in 175 spam messages and 20 messages that are not spam. Would an incoming message be rejected as spam if it contains the word "opportunity" and the threshold for rejecting a message is 0.9?

20. Would we reject a message as spam in Example 4

a) using just the fact that the word "undervalued" occurs in the message?

b) using just the fact that the word "stock" occurs in the message?

21. Suppose that a Bayesian spam filter is trained on a set of 10,000 spam messages and 5000 messages that are not spam. The word "enhancement" appears in 1500 spam messages and 20 messages that are not spam, while the word "herbal" appears in 800 spam messages and 200 messages that are not spam. Estimate the probability that a received message containing both the words "enhancement" and "herbal" is spam. Will the message be rejected as spam if the threshold for rejecting spam is 0.9?

22. Suppose that we have prior information concerning whether a random incoming message is spam. In particular, suppose that over a time period, we find that s spam messages arrive and h messages arrive that are not spam.

a) Use this information to estimate $p(S)$, the probability that an incoming message is spam, and $p(\overline{S})$, the probability an incoming message is not spam.

b) Use Bayes' theorem and part (a) to estimate the probability that an incoming message containing the word w is spam, where $p(w)$ is the probability that w occurs in a spam message and $q(w)$ is the probability that w occurs in a message that is not spam.

23. Suppose that E_1 and E_2 are the events that an incoming mail message contains the words w_1 and w_2, respectively. Assuming that E_1 and E_2 are independent events and that $E_1 \mid S$ and $E_2 \mid S$ are independent events, where S is the event that an incoming message is spam, and that we have no prior knowledge regarding whether or not the message is spam, show that

$$p(S \mid E_1 \cap E_2) = \frac{p(E_1 \mid S)p(E_2 \mid S)}{p(E_1 \mid S)p(E_2 \mid S) + p(E_1 \mid \overline{S})p(E_2 \mid \overline{S})}.$$

7.4 Expected Value and Variance

Introduction

The **expected value** of a random variable is the sum over all elements in a sample space of the product of the probability of the element and the value of the random variable at this element. Consequently, the expected value is a weighted average of the values of a random variable. The expected value of a random variable provides a central point for the distribution of values of this random variable. We can solve many problems using the notion of the expected value of a random variable, such as determining who has an advantage in gambling games and computing the average-case complexity of algorithms. Another useful measure of a random variable is its **variance**, which tells us how spread out the values of this random variable are. We can use the variance of a random variable to help us estimate the probability that a random variable takes values far removed from its expected value.

Expected Values

Links

Many questions can be formulated in terms of the value we expect a random variable to take, or more precisely, the average value of a random variable when an experiment is performed a large number of times. Questions of this kind include: How many heads are expected to appear

when a coin is flipped 100 times? What is the expected number of comparisons used to find an element in a list using a linear search? To study such questions we introduce the concept of the expected value of a random variable.

DEFINITION 1

The *expected value*, also called the *expectation* or *mean*, of the random variable X on the sample space S is equal to

$$E(X) = \sum_{s \in S} p(s)X(s).$$

The *deviation* of X at $s \in S$ is $X(s) - E(X)$, the difference between the value of X and the mean of X.

Note that when the sample space S has n elements $S = \{x_1, x_2, \ldots, x_n\}$, $E(X) = \sum_{i=1}^{n} p(x_i)X(x_i)$.

Remark: When there are infinitely many elements of the sample space, the expectation is defined only when the infinite series in the definition is absolutely convergent. In particular, the expectation of a random variable on an infinite sample space is finite if it exists.

EXAMPLE 1 **Expected Value of a Die** Let X be the number that comes up when a fair die is rolled. What is the expected value of X?

Solution: The random variable X takes the values 1, 2, 3, 4, 5, or 6, each with probability 1/6. It follows that

$$E(X) = \frac{1}{6} \cdot 1 + \frac{1}{6} \cdot 2 + \frac{1}{6} \cdot 3 + \frac{1}{6} \cdot 4 + \frac{1}{6} \cdot 5 + \frac{1}{6} \cdot 6 = \frac{21}{6} = \frac{7}{2}.$$

◀

EXAMPLE 2 A fair coin is flipped three times. Let S be the sample space of the eight possible outcomes, and let X be the random variable that assigns to an outcome the number of heads in this outcome. What is the expected value of X?

Extra Examples

Solution: In Example 10 of Section 7.2 we listed the values of X for the eight possible outcomes when a coin is flipped three times. Because the coin is fair and the flips are independent, the probability of each outcome is 1/8. Consequently,

$$\begin{aligned} E(X) &= \frac{1}{8}[X(HHH) + X(HHT) + X(HTH) + X(THH) + X(TTH) \\ &\quad + X(THT) + X(HTT) + X(TTT)] \\ &= \frac{1}{8}(3 + 2 + 2 + 2 + 1 + 1 + 1 + 0) = \frac{12}{8} \\ &= \frac{3}{2}. \end{aligned}$$

Consequently, the expected number of heads that come up when a fair coin is flipped three times is 3/2. ◀

When an experiment has relatively few outcomes, we can compute the expected value of a random variable directly from its definition, as was done in Example 2. However, when an experiment has a large number of outcomes, it may be inconvenient to compute the expected value of a random variable directly from its definition. Instead, we can find the expected value

of a random variable by grouping together all outcomes assigned the same value by the random variable, as Theorem 1 shows.

THEOREM 1

If X is a random variable and $p(X = r)$ is the probability that $X = r$, so that $p(X = r) = \sum_{s \in S, X(s)=r} p(s)$, then

$$E(X) = \sum_{r \in X(S)} p(X = r)r.$$

Proof: Suppose that X is a random variable with range $X(S)$, and let $p(X = r)$ be the probability that the random variable X takes the value r. Consequently, $p(X = r)$ is the sum of the probabilities of the outcomes s such that $X(s) = r$. It follows that

$$E(X) = \sum_{r \in X(S)} p(X = r)r.$$

◁

Example 3 and the proof of Theorem 2 will illustrate the use of this formula. In Example 3 we will find the expected value of the sum of the numbers that appear on two fair dice when they are rolled. In Theorem 2 we will find the expected value of the number of successes when n Bernoulli trials are performed.

EXAMPLE 3 What is the expected value of the sum of the numbers that appear when a pair of fair dice is rolled?

Solution: Let X be the random variable equal to the sum of the numbers that appear when a pair of dice is rolled. In Example 12 of Section 7.2 we listed the value of X for the 36 outcomes of this experiment. The range of X is $\{2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$. By Example 12 of Section 7.2 we see that

$$\begin{aligned}
&p(X = 2) = p(X = 12) = 1/36,\\
&p(X = 3) = p(X = 11) = 2/36 = 1/18,\\
&p(X = 4) = p(X = 10) = 3/36 = 1/12,\\
&p(X = 5) = p(X = 9) = 4/36 = 1/9,\\
&p(X = 6) = p(X = 8) = 5/36,\\
&p(X = 7) = 6/36 = 1/6.
\end{aligned}$$

Substituting these values in the formula, we have

$$\begin{aligned}
E(X) &= 2 \cdot \frac{1}{36} + 3 \cdot \frac{1}{18} + 4 \cdot \frac{1}{12} + 5 \cdot \frac{1}{9} + 6 \cdot \frac{5}{36} + 7 \cdot \frac{1}{6}\\
&\quad + 8 \cdot \frac{5}{36} + 9 \cdot \frac{1}{9} + 10 \cdot \frac{1}{12} + 11 \cdot \frac{1}{18} + 12 \cdot \frac{1}{36}\\
&= 7.
\end{aligned}$$

◀

THEOREM 2

The expected number of successes when n mutually independent Bernoulli trials are performed, where p is the probability of success on each trial, is np.