

DEFINITION 6

Let R be a relation from a set A to a set B and S a relation from B to a set C . The *composite* of R and S is the relation consisting of ordered pairs (a, c) , where $a \in A$, $c \in C$, and for which there exists an element $b \in B$ such that $(a, b) \in R$ and $(b, c) \in S$. We denote the composite of R and S by $S \circ R$.

Computing the composite of two relations requires that we find elements that are the second element of ordered pairs in the first relation and the first element of ordered pairs in the second relation, as Examples 20 and 21 illustrate.

EXAMPLE 20

What is the composite of the relations R and S , where R is the relation from $\{1, 2, 3\}$ to $\{1, 2, 3, 4\}$ with $R = \{(1, 1), (1, 4), (2, 3), (3, 1), (3, 4)\}$ and S is the relation from $\{1, 2, 3, 4\}$ to $\{0, 1, 2\}$ with $S = \{(1, 0), (2, 0), (3, 1), (3, 2), (4, 1)\}$?

Solution: $S \circ R$ is constructed using all ordered pairs in R and ordered pairs in S , where the second element of the ordered pair in R agrees with the first element of the ordered pair in S . For example, the ordered pairs $(2, 3)$ in R and $(3, 1)$ in S produce the ordered pair $(2, 1)$ in $S \circ R$. Computing all the ordered pairs in the composite, we find

$$S \circ R = \{(1, 0), (1, 1), (2, 1), (2, 2), (3, 0), (3, 1)\}.$$

EXAMPLE 21

Composing the Parent Relation with Itself Let R be the relation on the set of all people such that $(a, b) \in R$ if person a is a parent of person b . Then $(a, c) \in R \circ R$ if and only if there is a person b such that $(a, b) \in R$ and $(b, c) \in R$, that is, if and only if there is a person b such that a is a parent of b and b is a parent of c . In other words, $(a, c) \in R \circ R$ if and only if a is a grandparent of c .

The powers of a relation R can be recursively defined from the definition of a composite of two relations.

DEFINITION 7

Let R be a relation on the set A . The powers R^n , $n = 1, 2, 3, \dots$, are defined recursively by

$$R^1 = R \quad \text{and} \quad R^{n+1} = R^n \circ R.$$

The definition shows that $R^2 = R \circ R$, $R^3 = R^2 \circ R = (R \circ R) \circ R$, and so on.

EXAMPLE 22

Let $R = \{(1, 1), (2, 1), (3, 2), (4, 3)\}$. Find the powers R^n , $n = 2, 3, 4, \dots$

Solution: Because $R^2 = R \circ R$, we find that $R^2 = \{(1, 1), (2, 1), (3, 1), (4, 2)\}$. Furthermore, because $R^3 = R^2 \circ R$, $R^3 = \{(1, 1), (2, 1), (3, 1), (4, 1)\}$. Additional computation shows that R^4 is the same as R^3 , so $R^4 = \{(1, 1), (2, 1), (3, 1), (4, 1)\}$. It also follows that $R^n = R^3$ for $n = 5, 6, 7, \dots$. The reader should verify this.

The following theorem shows that the powers of a transitive relation are subsets of this relation. It will be used in Section 9.4.

THEOREM 1

The relation R on a set A is transitive if and only if $R^n \subseteq R$ for $n = 1, 2, 3, \dots$

Proof: We first prove the “if” part of the theorem. We suppose that $R^n \subseteq R$ for $n = 1, 2, 3, \dots$. In particular, $R^2 \subseteq R$. To see that this implies R is transitive, note that if $(a, b) \in R$ and $(b, c) \in R$, then by the definition of composition, $(a, c) \in R^2$. Because $R^2 \subseteq R$, this means that $(a, c) \in R$. Hence, R is transitive.



We will use mathematical induction to prove the only if part of the theorem. Note that this part of the theorem is trivially true for $n = 1$.

Assume that $R^n \subseteq R$, where n is a positive integer. This is the inductive hypothesis. To complete the inductive step we must show that this implies that R^{n+1} is also a subset of R . To show this, assume that $(a, b) \in R^{n+1}$. Then, because $R^{n+1} = R^n \circ R$, there is an element x with $x \in A$ such that $(a, x) \in R$ and $(x, b) \in R^n$. The inductive hypothesis, namely, that $R^n \subseteq R$, implies that $(x, b) \in R$. Furthermore, because R is transitive, and $(a, x) \in R$ and $(x, b) \in R$, it follows that $(a, b) \in R$. This shows that $R^{n+1} \subseteq R$, completing the proof. \triangleleft

Exercises

1. List the ordered pairs in the relation R from $A = \{0, 1, 2, 3, 4\}$ to $B = \{0, 1, 2, 3\}$, where $(a, b) \in R$ if and only if
 - a) $a = b$.
 - b) $a + b = 4$.
 - c) $a > b$.
 - d) $a \mid b$.
 - e) $\gcd(a, b) = 1$.
 - f) $\text{lcm}(a, b) = 2$.
 2. a) List all the ordered pairs in the relation $R = \{(a, b) \mid a \text{ divides } b\}$ on the set $\{1, 2, 3, 4, 5, 6\}$.
b) Display this relation graphically, as was done in Example 4.
c) Display this relation in tabular form, as was done in Example 4.
 3. For each of these relations on the set $\{1, 2, 3, 4\}$, decide whether it is reflexive, whether it is symmetric, whether it is antisymmetric, and whether it is transitive.
 - a) $\{(2, 2), (2, 3), (2, 4), (3, 2), (3, 3), (3, 4)\}$
 - b) $\{(1, 1), (1, 2), (2, 1), (2, 2), (3, 3), (4, 4)\}$
 - c) $\{(2, 4), (4, 2)\}$
 - d) $\{(1, 2), (2, 3), (3, 4)\}$
 - e) $\{(1, 1), (2, 2), (3, 3), (4, 4)\}$
 - f) $\{(1, 3), (1, 4), (2, 3), (2, 4), (3, 1), (3, 4)\}$
 4. Determine whether the relation R on the set of all people is reflexive, symmetric, antisymmetric, and/or transitive, where $(a, b) \in R$ if and only if
 - a) a is taller than b .
 - b) a and b were born on the same day.
 - c) a has the same first name as b .
 - d) a and b have a common grandparent.
 5. Determine whether the relation R on the set of all Web pages is reflexive, symmetric, antisymmetric, and/or transitive, where $(a, b) \in R$ if and only if
 - a) everyone who has visited Web page a has also visited Web page b .
 - b) there are no common links found on both Web page a and Web page b .
 - c) there is at least one common link on Web page a and Web page b .
 6. Determine whether the relation R on the set of all real numbers is reflexive, symmetric, antisymmetric, and/or transitive, where $(x, y) \in R$ if and only if
 - a) $x + y = 0$.
 - b) $x = \pm y$.
 - c) $x - y$ is a rational number.
 - d) $x = 2y$.
 - e) $xy \geq 0$.
 - f) $xy = 0$.
 - g) $x = 1$.
 - h) $x = 1$ or $y = 1$.
 7. Determine whether the relation R on the set of all integers is reflexive, symmetric, antisymmetric, and/or transitive, where $(x, y) \in R$ if and only if
 - a) $x \neq y$.
 - b) $xy \geq 1$.
 - c) $x = y + 1$ or $x = y - 1$.
 - d) $x \equiv y \pmod{7}$.
 - e) x is a multiple of y .
 - f) x and y are both negative or both nonnegative.
 - g) $x = y^2$.
 - h) $x \geq y^2$.
 8. Show that the relation $R = \emptyset$ on a nonempty set S is symmetric and transitive, but not reflexive.
 9. Show that the relation $R = \emptyset$ on the empty set $S = \emptyset$ is reflexive, symmetric, and transitive.
 10. Give an example of a relation on a set that is
 - a) both symmetric and antisymmetric.
 - b) neither symmetric nor antisymmetric.
- A relation R on the set A is **irreflexive** if for every $a \in A$, $(a, a) \notin R$. That is, R is irreflexive if no element in A is related to itself.
11. Which relations in Exercise 3 are irreflexive?
 12. Which relations in Exercise 4 are irreflexive?
 13. Which relations in Exercise 5 are irreflexive?
 14. Which relations in Exercise 6 are irreflexive?
 15. Can a relation on a set be neither reflexive nor irreflexive?
 16. Use quantifiers to express what it means for a relation to be irreflexive.
 17. Give an example of an irreflexive relation on the set of all people.

A relation R is called **asymmetric** if $(a, b) \in R$ implies that $(b, a) \notin R$. Exercises 18–24 explore the notion of an asymmetric relation. Exercise 22 focuses on the difference between asymmetry and antisymmetry.

18. Which relations in Exercise 3 are asymmetric?
19. Which relations in Exercise 4 are asymmetric?
20. Which relations in Exercise 5 are asymmetric?
21. Which relations in Exercise 6 are asymmetric?
22. Must an asymmetric relation also be antisymmetric? Must an antisymmetric relation be asymmetric? Give reasons for your answers.
23. Use quantifiers to express what it means for a relation to be asymmetric.
24. Give an example of an asymmetric relation on the set of all people.
25. How many different relations are there from a set with m elements to a set with n elements?
- Let R be a relation from a set A to a set B . The **inverse relation** from B to A , denoted by R^{-1} , is the set of ordered pairs $\{(b, a) \mid (a, b) \in R\}$. The **complementary relation** \bar{R} is the set of ordered pairs $\{(a, b) \mid (a, b) \notin R\}$.
26. Let R be the relation $R = \{(a, b) \mid a < b\}$ on the set of integers. Find
 - a) R^{-1} .
 - b) \bar{R} .
27. Let R be the relation $R = \{(a, b) \mid a \text{ divides } b\}$ on the set of positive integers. Find
 - a) R^{-1} .
 - b) \bar{R} .
28. Let R be the relation on the set of all states in the United States consisting of pairs (a, b) where state a borders state b . Find
 - a) R^{-1} .
 - b) \bar{R} .
29. Suppose that the function f from A to B is a one-to-one correspondence. Let R be the relation that equals the graph of f . That is, $R = \{(a, f(a)) \mid a \in A\}$. What is the inverse relation R^{-1} ?
30. Let $R_1 = \{(1, 2), (2, 3), (3, 4)\}$ and $R_2 = \{(1, 1), (1, 2), (2, 1), (2, 2), (2, 3), (3, 1), (3, 2), (3, 3), (3, 4)\}$ be relations from $\{1, 2, 3\}$ to $\{1, 2, 3, 4\}$. Find
 - a) $R_1 \cup R_2$.
 - b) $R_1 \cap R_2$.
 - c) $R_1 - R_2$.
 - d) $R_2 - R_1$.
31. Let A be the set of students at your school and B the set of books in the school library. Let R_1 and R_2 be the relations consisting of all ordered pairs (a, b) , where student a is required to read book b in a course, and where student a has read book b , respectively. Describe the ordered pairs in each of these relations.
 - a) $R_1 \cup R_2$
 - b) $R_1 \cap R_2$
 - c) $R_1 \oplus R_2$
 - d) $R_1 - R_2$
 - e) $R_2 - R_1$
32. Let R be the relation $\{(1, 2), (1, 3), (2, 3), (2, 4), (3, 1)\}$, and let S be the relation $\{(2, 1), (3, 1), (3, 2), (4, 2)\}$. Find $S \circ R$.

33. Let R be the relation on the set of people consisting of pairs (a, b) , where a is a parent of b . Let S be the relation on the set of people consisting of pairs (a, b) , where a and b are siblings (brothers or sisters). What are $S \circ R$ and $R \circ S$?

Exercises 34–37 deal with these relations on the set of real numbers:

$$\begin{aligned} R_1 &= \{(a, b) \in \mathbf{R}^2 \mid a > b\}, \text{ the “greater than” relation,} \\ R_2 &= \{(a, b) \in \mathbf{R}^2 \mid a \geq b\}, \text{ the “greater than or equal to” relation,} \\ R_3 &= \{(a, b) \in \mathbf{R}^2 \mid a < b\}, \text{ the “less than” relation,} \\ R_4 &= \{(a, b) \in \mathbf{R}^2 \mid a \leq b\}, \text{ the “less than or equal to” relation,} \\ R_5 &= \{(a, b) \in \mathbf{R}^2 \mid a = b\}, \text{ the “equal to” relation,} \\ R_6 &= \{(a, b) \in \mathbf{R}^2 \mid a \neq b\}, \text{ the “unequal to” relation.} \end{aligned}$$

34. Find

a) $R_1 \cup R_3$.	b) $R_1 \cup R_5$.
c) $R_2 \cap R_4$.	d) $R_3 \cap R_5$.
e) $R_1 - R_2$.	f) $R_2 - R_1$.
g) $R_1 \oplus R_3$.	h) $R_2 \oplus R_4$.

35. Find

a) $R_2 \cup R_4$.	b) $R_3 \cup R_6$.
c) $R_3 \cap R_6$.	d) $R_4 \cap R_6$.
e) $R_3 - R_6$.	f) $R_6 - R_3$.
g) $R_2 \oplus R_6$.	h) $R_3 \oplus R_5$.

36. Find

a) $R_1 \circ R_1$.	b) $R_1 \circ R_2$.
c) $R_1 \circ R_3$.	d) $R_1 \circ R_4$.
e) $R_1 \circ R_5$.	f) $R_1 \circ R_6$.
g) $R_2 \circ R_3$.	h) $R_3 \circ R_3$.

37. Find

a) $R_2 \circ R_1$.	b) $R_2 \circ R_2$.
c) $R_3 \circ R_5$.	d) $R_4 \circ R_1$.
e) $R_5 \circ R_3$.	f) $R_3 \circ R_6$.
g) $R_4 \circ R_6$.	h) $R_6 \circ R_6$.

38. Let R be the parent relation on the set of all people (see Example 21). When is an ordered pair in the relation R^3 ?
39. Let R be the relation on the set of people with doctorates such that $(a, b) \in R$ if and only if a was the thesis advisor of b . When is an ordered pair (a, b) in R^2 ? When is an ordered pair (a, b) in R^n , when n is a positive integer? (Assume that every person with a doctorate has a thesis advisor.)

40. Let R_1 and R_2 be the “divides” and “is a multiple of” relations on the set of all positive integers, respectively. That is, $R_1 = \{(a, b) \mid a \text{ divides } b\}$ and $R_2 = \{(a, b) \mid a \text{ is a multiple of } b\}$. Find

a) $R_1 \cup R_2$.	b) $R_1 \cap R_2$.
c) $R_1 - R_2$.	d) $R_2 - R_1$.
e) $R_1 \oplus R_2$.	

- 41.** Let R_1 and R_2 be the “congruent modulo 3” and the “congruent modulo 4” relations, respectively, on the set of integers. That is, $R_1 = \{(a, b) \mid a \equiv b \pmod{3}\}$ and $R_2 = \{(a, b) \mid a \equiv b \pmod{4}\}$. Find
- $R_1 \cup R_2$.
 - $R_1 \cap R_2$.
 - $R_1 - R_2$.
 - $R_2 - R_1$.
- 42.** List the 16 different relations on the set $\{0, 1\}$.
- 43.** How many of the 16 different relations on $\{0, 1\}$ contain the pair $(0, 1)$?
- 44.** Which of the 16 relations on $\{0, 1\}$, which you listed in Exercise 42, are
- reflexive?
 - irreflexive?
 - symmetric?
 - antisymmetric?
 - asymmetric?
 - transitive?
- 45.** a) How many relations are there on the set $\{a, b, c, d\}$?
b) How many relations are there on the set $\{a, b, c, d\}$ that contain the pair (a, a) ?
- 46.** Let S be a set with n elements and let a and b be distinct elements of S . How many relations R are there on S such that
- $(a, b) \in R$?
 - $(a, b) \notin R$?
 - no ordered pair in R has a as its first element?
 - at least one ordered pair in R has a as its first element?
 - no ordered pair in R has a as its first element or b as its second element?
 - at least one ordered pair in R either has a as its first element or has b as its second element?
- *47.** How many relations are there on a set with n elements that are
- symmetric?
 - antisymmetric?
 - asymmetric?
 - irreflexive?
 - reflexive and symmetric?
 - neither reflexive nor irreflexive?
- *48.** How many transitive relations are there on a set with n elements if
- $n = 1$?
 - $n = 2$?
 - $n = 3$?
- 49.** Find the error in the “proof” of the following “theorem.”
- “Theorem”: Let R be a relation on a set A that is symmetric and transitive. Then R is reflexive.
- “Proof”: Let $a \in A$. Take an element $b \in A$ such that $(a, b) \in R$. Because R is symmetric, we also have $(b, a) \in R$. Now using the transitive property, we can conclude that $(a, a) \in R$ because $(a, b) \in R$ and $(b, a) \in R$.
- 50.** Suppose that R and S are reflexive relations on a set A . Prove or disprove each of these statements.
- $R \cup S$ is reflexive.
 - $R \cap S$ is reflexive.
 - $R \oplus S$ is irreflexive.
 - $R - S$ is irreflexive.
 - $S \circ R$ is reflexive.
- 51.** Show that the relation R on a set A is symmetric if and only if $R = R^{-1}$, where R^{-1} is the inverse relation.
- 52.** Show that the relation R on a set A is antisymmetric if and only if $R \cap R^{-1}$ is a subset of the diagonal relation $\Delta = \{(a, a) \mid a \in A\}$.
- 53.** Show that the relation R on a set A is reflexive if and only if the inverse relation R^{-1} is reflexive.
- 54.** Show that the relation R on a set A is reflexive if and only if the complementary relation \bar{R} is irreflexive.
- 55.** Let R be a relation that is reflexive and transitive. Prove that $R^n = R$ for all positive integers n .
- 56.** Let R be the relation on the set $\{1, 2, 3, 4, 5\}$ containing the ordered pairs $(1, 1), (1, 2), (1, 3), (2, 3), (2, 4), (3, 1), (3, 4), (3, 5), (4, 2), (4, 5), (5, 1), (5, 2)$, and $(5, 4)$. Find
- R^2 .
 - R^3 .
 - R^4 .
 - R^5 .
- 57.** Let R be a reflexive relation on a set A . Show that R^n is reflexive for all positive integers n .
- *58.** Let R be a symmetric relation. Show that R^n is symmetric for all positive integers n .
- 59.** Suppose that the relation R is irreflexive. Is R^2 necessarily irreflexive? Give a reason for your answer.

9.2 *n*-ary Relations and Their Applications

Introduction

Relationships among elements of more than two sets often arise. For instance, there is a relationship involving the name of a student, the student’s major, and the student’s grade point average. Similarly, there is a relationship involving the airline, flight number, starting point, destination, departure time, and arrival time of a flight. An example of such a relationship in mathematics involves three integers, where the first integer is larger than the second integer, which is larger than the third. Another example is the betweenness relationship involving points on a line, such that three points are related when the second point is between the first and the third.

We will study relationships among elements from more than two sets in this section. These relationships are called ***n*-ary relations**. These relations are used to represent computer databases. These representations help us answer queries about the information stored in databases, such as: Which flights land at O’Hare Airport between 3 A.M. and 4 A.M.? Which students at your

school are sophomores majoring in mathematics or computer science and have greater than a 3.0 average? Which employees of a company have worked for the company less than 5 years and make more than \$50,000?

n-ary Relations

We begin with the basic definition on which the theory of relational databases rests.

DEFINITION 1

Let A_1, A_2, \dots, A_n be sets. An *n-ary relation* on these sets is a subset of $A_1 \times A_2 \times \dots \times A_n$. The sets A_1, A_2, \dots, A_n are called the *domains* of the relation, and n is called its *degree*.

EXAMPLE 1

Let R be the relation on $\mathbb{N} \times \mathbb{N} \times \mathbb{N}$ consisting of triples (a, b, c) , where a , b , and c are integers with $a < b < c$. Then $(1, 2, 3) \in R$, but $(2, 4, 3) \notin R$. The degree of this relation is 3. Its domains are all equal to the set of natural numbers. ◀

EXAMPLE 2

Let R be the relation on $\mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}$ consisting of all triples of integers (a, b, c) in which a , b , and c form an arithmetic progression. That is, $(a, b, c) \in R$ if and only if there is an integer k such that $b = a + k$ and $c = a + 2k$, or equivalently, such that $b - a = k$ and $c - b = k$. Note that $(1, 3, 5) \in R$ because $3 = 1 + 2$ and $5 = 1 + 2 \cdot 2$, but $(2, 5, 9) \notin R$ because $5 - 2 = 3$ while $9 - 5 = 4$. This relation has degree 3 and its domains are all equal to the set of integers. ◀

EXAMPLE 3

Let R be the relation on $\mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}^+$ consisting of triples (a, b, m) , where a , b , and m are integers with $m \geq 1$ and $a \equiv b \pmod{m}$. Then $(8, 2, 3)$, $(-1, 9, 5)$, and $(14, 0, 7)$ all belong to R , but $(7, 2, 3)$, $(-2, -8, 5)$, and $(11, 0, 6)$ do not belong to R because $8 \equiv 2 \pmod{3}$, $-1 \equiv 9 \pmod{5}$, and $14 \equiv 0 \pmod{7}$, but $7 \not\equiv 2 \pmod{3}$, $-2 \not\equiv -8 \pmod{5}$, and $11 \not\equiv 0 \pmod{6}$. This relation has degree 3 and its first two domains are the set of all integers and its third domain is the set of positive integers. ◀

EXAMPLE 4

Let R be the relation consisting of 5-tuples (A, N, S, D, T) representing airplane flights, where A is the airline, N is the flight number, S is the starting point, D is the destination, and T is the departure time. For instance, if Nadir Express Airlines has flight 963 from Newark to Bangor at 15:00, then $(\text{Nadir}, 963, \text{Newark}, \text{Bangor}, 15:00)$ belongs to R . The degree of this relation is 5, and its domains are the set of all airlines, the set of flight numbers, the set of cities, the set of cities (again), and the set of times. ◀

Databases and Relations



The time required to manipulate information in a database depends on how this information is stored. The operations of adding and deleting records, updating records, searching for records, and combining records from overlapping databases are performed millions of times each day in a large database. Because of the importance of these operations, various methods for representing databases have been developed. We will discuss one of these methods, called the **relational data model**, based on the concept of a relation.

A database consists of **records**, which are *n*-tuples, made up of **fields**. The fields are the entries of the *n*-tuples. For instance, a database of student records may be made up of fields containing the name, student number, major, and grade point average of the student. The relational data model represents a database of records as an *n*-ary relation. Thus, student records

TABLE 1 Students.

<i>Student_name</i>	<i>ID_number</i>	<i>Major</i>	<i>GPA</i>
Ackermann	231455	Computer Science	3.88
Adams	888323	Physics	3.45
Chou	102147	Computer Science	3.49
Goodfriend	453876	Mathematics	3.45
Rao	678543	Mathematics	3.90
Stevens	786576	Psychology	2.99

are represented as 4-tuples of the form (*Student_name*, *ID_number*, *Major*, *GPA*). A sample database of six such records is

- (Ackermann, 231455, Computer Science, 3.88)
- (Adams, 888323, Physics, 3.45)
- (Chou, 102147, Computer Science, 3.49)
- (Goodfriend, 453876, Mathematics, 3.45)
- (Rao, 678543, Mathematics, 3.90)
- (Stevens, 786576, Psychology, 2.99).

Relations used to represent databases are also called **tables**, because these relations are often displayed as tables. Each column of the table corresponds to an *attribute* of the database. For instance, the same database of students is displayed in Table 1. The attributes of this database are Student Name, ID Number, Major, and GPA.

A domain of an *n*-ary relation is called a **primary key** when the value of the *n*-tuple from this domain determines the *n*-tuple. That is, a domain is a primary key when no two *n*-tuples in the relation have the same value from this domain.

Records are often added to or deleted from databases. Because of this, the property that a domain is a primary key is time-dependent. Consequently, a primary key should be chosen that remains one whenever the database is changed. The current collection of *n*-tuples in a relation is called the **extension** of the relation. The more permanent part of a database, including the name and attributes of the database, is called its **intension**. When selecting a primary key, the goal should be to select a key that can serve as a primary key for all possible extensions of the database. To do this, it is necessary to examine the intension of the database to understand the set of possible *n*-tuples that can occur in an extension.

EXAMPLE 5 Which domains are primary keys for the *n*-ary relation displayed in Table 1, assuming that no *n*-tuples will be added in the future?

Solution: Because there is only one 4-tuple in this table for each student name, the domain of student names is a primary key. Similarly, the ID numbers in this table are unique, so the domain of ID numbers is also a primary key. However, the domain of major fields of study is not a primary key, because more than one 4-tuple contains the same major field of study. The domain of grade point averages is also not a primary key, because there are two 4-tuples containing the same GPA. ◀

Combinations of domains can also uniquely identify *n*-tuples in an *n*-ary relation. When the values of a set of domains determine an *n*-tuple in a relation, the Cartesian product of these domains is called a **composite key**.

EXAMPLE 6 Is the Cartesian product of the domain of major fields of study and the domain of GPAs a composite key for the n -ary relation from Table 1, assuming that no n -tuples are ever added?

Solution: Because no two 4-tuples from this table have both the same major and the same GPA, this Cartesian product is a composite key. ◀

Because primary and composite keys are used to identify records uniquely in a database, it is important that keys remain valid when new records are added to the database. Hence, checks should be made to ensure that every new record has values that are different in the appropriate field, or fields, from all other records in this table. For instance, it makes sense to use the student identification number as a key for student records if no two students ever have the same student identification number. A university should not use the name field as a key, because two students may have the same name (such as John Smith).

Operations on n -ary Relations

There are a variety of operations on n -ary relations that can be used to form new n -ary relations. Applied together, these operations can answer queries on databases that ask for all n -tuples that satisfy certain conditions.

The most basic operation on an n -ary relation is determining all n -tuples in the n -ary relation that satisfy certain conditions. For example, we may want to find all the records of all computer science majors in a database of student records. We may want to find all students who have a grade point average above 3.5. We may want to find the records of all computer science majors who have a grade point average above 3.5. To perform such tasks we use the selection operator.

DEFINITION 2

Let R be an n -ary relation and C a condition that elements in R may satisfy. Then the *selection operator* s_C maps the n -ary relation R to the n -ary relation of all n -tuples from R that satisfy the condition C .

EXAMPLE 7

To find the records of computer science majors in the n -ary relation R shown in Table 1, we use the operator s_{C_1} , where C_1 is the condition Major = “Computer Science.” The result is the two 4-tuples (Ackermann, 231455, Computer Science, 3.88) and (Chou, 102147, Computer Science, 3.49). Similarly, to find the records of students who have a grade point average above 3.5 in this database, we use the operator s_{C_2} , where C_2 is the condition GPA > 3.5. The result is the two 4-tuples (Ackermann, 231455, Computer Science, 3.88) and (Rao, 678543, Mathematics, 3.90). Finally, to find the records of computer science majors who have a GPA above 3.5, we use the operator s_{C_3} , where C_3 is the condition (Major = “Computer Science” \wedge GPA > 3.5). The result consists of the single 4-tuple (Ackermann, 231455, Computer Science, 3.88). ◀

Projections are used to form new n -ary relations by deleting the same fields in every record of the relation.

DEFINITION 3

The *projection* P_{i_1, i_2, \dots, i_m} where $i_1 < i_2 < \dots < i_m$, maps the n -tuple (a_1, a_2, \dots, a_n) to the m -tuple $(a_{i_1}, a_{i_2}, \dots, a_{i_m})$, where $m \leq n$.

In other words, the projection P_{i_1, i_2, \dots, i_m} deletes $n - m$ of the components of an n -tuple, leaving the i_1 th, i_2 th, ..., and i_m th components.

TABLE 2 GPAs.	
Student_name	GPA
Ackermann	3.88
Adams	3.45
Chou	3.49
Goodfriend	3.45
Rao	3.90
Stevens	2.99

TABLE 3 Enrollments.		
Student	Major	Course
Glauser	Biology	BI 290
Glauser	Biology	MS 475
Glauser	Biology	PY 410
Marcus	Mathematics	MS 511
Marcus	Mathematics	MS 603
Marcus	Mathematics	CS 322
Miller	Computer Science	MS 575
Miller	Computer Science	CS 455

TABLE 4 Majors.	
Student	Major
Glauser	Biology
Marcus	Mathematics
Miller	Computer Science

EXAMPLE 8 What results when the projection $P_{1,3}$ is applied to the 4-tuples $(2, 3, 0, 4)$, (Jane Doe, 234111001, Geography, 3.14), and (a_1, a_2, a_3, a_4) ?

Solution: The projection $P_{1,3}$ sends these 4-tuples to $(2, 0)$, (Jane Doe, Geography), and (a_1, a_3) , respectively. ◀

Example 9 illustrates how new relations are produced using projections.

EXAMPLE 9 What relation results when the projection $P_{1,4}$ is applied to the relation in Table 1?

Solution: When the projection $P_{1,4}$ is used, the second and third columns of the table are deleted, and pairs representing student names and grade point averages are obtained. Table 2 displays the results of this projection. ◀

Fewer rows may result when a projection is applied to the table for a relation. This happens when some of the n -tuples in the relation have identical values in each of the m components of the projection, and only disagree in components deleted by the projection. For instance, consider the following example.

EXAMPLE 10 What is the table obtained when the projection $P_{1,2}$ is applied to the relation in Table 3?

Solution: Table 4 displays the relation obtained when $P_{1,2}$ is applied to Table 3. Note that there are fewer rows after this projection is applied. ◀

The **join** operation is used to combine two tables into one when these tables share some identical fields. For instance, a table containing fields for airline, flight number, and gate, and another table containing fields for flight number, gate, and departure time can be combined into a table containing fields for airline, flight number, gate, and departure time.

DEFINITION 4

Let R be a relation of degree m and S a relation of degree n . The *join* $J_p(R, S)$, where $p \leq m$ and $p \leq n$, is a relation of degree $m + n - p$ that consists of all $(m + n - p)$ -tuples $(a_1, a_2, \dots, a_{m-p}, c_1, c_2, \dots, c_p, b_1, b_2, \dots, b_{n-p})$, where the m -tuple $(a_1, a_2, \dots, a_{m-p}, c_1, c_2, \dots, c_p)$ belongs to R and the n -tuple $(c_1, c_2, \dots, c_p, b_1, b_2, \dots, b_{n-p})$ belongs to S .

In other words, the join operator J_p produces a new relation from two relations by combining all m -tuples of the first relation with all n -tuples of the second relation, where the last p components of the m -tuples agree with the first p components of the n -tuples.

TABLE 5 Teaching_assignments.

<i>Professor</i>	<i>Department</i>	<i>Course_number</i>
Cruz	Zoology	335
Cruz	Zoology	412
Farber	Psychology	501
Farber	Psychology	617
Grammer	Physics	544
Grammer	Physics	551
Rosen	Computer Science	518
Rosen	Mathematics	575

TABLE 6 Class_schedule.

<i>Department</i>	<i>Course_number</i>	<i>Room</i>	<i>Time</i>
Computer Science	518	N521	2:00 P.M.
Mathematics	575	N502	3:00 P.M.
Mathematics	611	N521	4:00 P.M.
Physics	544	B505	4:00 P.M.
Psychology	501	A100	3:00 P.M.
Psychology	617	A110	11:00 A.M.
Zoology	335	A100	9:00 A.M.
Zoology	412	A100	8:00 A.M.

EXAMPLE 11 What relation results when the join operator J_2 is used to combine the relation displayed in Tables 5 and 6?

Solution: The join J_2 produces the relation shown in Table 7. ◀

There are other operators besides projections and joins that produce new relations from existing relations. A description of these operations can be found in books on database theory.

SQL



The database query language SQL (short for Structured Query Language) can be used to carry out the operations we have described in this section. Example 12 illustrates how SQL commands are related to operations on n -ary relations.

EXAMPLE 12 We will illustrate how SQL is used to express queries by showing how SQL can be employed to make a query about airline flights using Table 8. The SQL statement

```
SELECT Departure_time
FROM Flights
WHERE Destination='Detroit'
```

is used to find the projection P_5 (on the *Departure_time* attribute) of the selection of 5-tuples in the *Flights* database that satisfy the condition: *Destination* = 'Detroit'. The output would be a list containing the times of flights that have Detroit as their destination, namely, 08:10, 08:47,

TABLE 7 Teaching_schedule.

<i>Professor</i>	<i>Department</i>	<i>Course_number</i>	<i>Room</i>	<i>Time</i>
Cruz	Zoology	335	A100	9:00 A.M.
Cruz	Zoology	412	A100	8:00 A.M.
Farber	Psychology	501	A100	3:00 P.M.
Farber	Psychology	617	A110	11:00 A.M.
Grammer	Physics	544	B505	4:00 P.M.
Rosen	Computer Science	518	N521	2:00 P.M.
Rosen	Mathematics	575	N502	3:00 P.M.

TABLE 8 Flights.

Airline	Flight_number	Gate	Destination	Departure_time
Nadir	122	34	Detroit	08:10
Acme	221	22	Denver	08:17
Acme	122	33	Anchorage	08:22
Acme	323	34	Honolulu	08:30
Nadir	199	13	Detroit	08:47
Acme	222	22	Denver	09:10
Nadir	322	34	Detroit	09:44

and 09:44. SQL uses the FROM clause to identify the *n*-ary relation the query is applied to, the WHERE clause to specify the condition of the selection operation, and the SELECT clause to specify the projection operation that is to be applied. (*Beware*: SQL uses SELECT to represent a projection, rather than a selection operation. This is an unfortunate example of conflicting terminology.)

Example 13 shows how SQL queries can be made involving more than one table.

EXAMPLE 13 The SQL statement

```
SELECT Professor, Time
FROM Teaching_assignments, Class_schedule
WHERE Department='Mathematics'
```

is used to find the projection $P_{1,5}$ of the 5-tuples in the database (shown in Table 7), which is the join J_2 of the Teaching_assignments and Class_schedule databases in Tables 5 and 6, respectively, which satisfy the condition: Department = Mathematics. The output would consist of the single 2-tuple (Rosen, 3:00 P.M.). The SQL FROM clause is used here to find the join of two different databases.

We have only touched on the basic concepts of relational databases in this section. More information can be found in [AhU195].

Exercises

1. List the triples in the relation $\{(a, b, c) \mid a, b, \text{ and } c \text{ are integers with } 0 < a < b < c < 5\}$.
2. Which 4-tuples are in the relation $\{(a, b, c, d) \mid a, b, c, \text{ and } d \text{ are positive integers with } abcd = 6\}$?
3. List the 5-tuples in the relation in Table 8.
4. Assuming that no new *n*-tuples are added, find all the primary keys for the relations displayed in
 - a) Table 3.
 - b) Table 5.
 - c) Table 6.
 - d) Table 8.
5. Assuming that no new *n*-tuples are added, find a composite key with two fields containing the *Airline* field for the database in Table 8.
6. Assuming that no new *n*-tuples are added, find a composite key with two fields containing the *Professor* field for the database in Table 7.
7. The 3-tuples in a 3-ary relation represent the following attributes of a student database: student ID number, name, phone number.
 - a) Is student ID number likely to be a primary key?
 - b) Is name likely to be a primary key?
 - c) Is phone number likely to be a primary key?
8. The 4-tuples in a 4-ary relation represent these attributes of published books: title, ISBN, publication date, number of pages.
 - a) What is a likely primary key for this relation?
 - b) Under what conditions would (title, publication date) be a composite key?
 - c) Under what conditions would (title, number of pages) be a composite key?

9. The 5-tuples in a 5-ary relation represent these attributes of all people in the United States: name, Social Security number, street address, city, state.
- Determine a primary key for this relation.
 - Under what conditions would (name, street address) be a composite key?
 - Under what conditions would (name, street address, city) be a composite key?
10. What do you obtain when you apply the selection operator s_C , where C is the condition Room = A100, to the database in Table 7?
11. What do you obtain when you apply the selection operator s_C , where C is the condition Destination = Detroit, to the database in Table 8?
12. What do you obtain when you apply the selection operator s_C , where C is the condition (Project = 2) \wedge (Quantity \geq 50), to the database in Table 10?
13. What do you obtain when you apply the selection operator s_C , where C is the condition (Airline = Nadir) \vee (Destination = Denver), to the database in Table 8?
14. What do you obtain when you apply the projection $P_{2,3,5}$ to the 5-tuple (a, b, c, d, e) ?
15. Which projection mapping is used to delete the first, second, and fourth components of a 6-tuple?
16. Display the table produced by applying the projection $P_{1,2,4}$ to Table 8.
17. Display the table produced by applying the projection $P_{1,4}$ to Table 8.
18. How many components are there in the n -tuples in the table obtained by applying the join operator J_3 to two tables with 5-tuples and 8-tuples, respectively?
19. Construct the table obtained by applying the join operator J_2 to the relations in Tables 9 and 10.
20. Show that if C_1 and C_2 are conditions that elements of the n -ary relation R may satisfy, then $s_{C_1 \wedge C_2}(R) = s_{C_1}(s_{C_2}(R))$.
21. Show that if C_1 and C_2 are conditions that elements of the n -ary relation R may satisfy, then $s_{C_1}(s_{C_2}(R)) = s_{C_2}(s_{C_1}(R))$.
22. Show that if C is a condition that elements of the n -ary relations R and S may satisfy, then $s_C(R \cup S) = s_C(R) \cup s_C(S)$.
23. Show that if C is a condition that elements of the n -ary relations R and S may satisfy, then $s_C(R \cap S) = s_C(R) \cap s_C(S)$.
24. Show that if C is a condition that elements of the n -ary relations R and S may satisfy, then $s_C(R - S) = s_C(R) - s_C(S)$.
25. Show that if R and S are both n -ary relations, then $P_{i_1, i_2, \dots, i_m}(R \cup S) = P_{i_1, i_2, \dots, i_m}(R) \cup P_{i_1, i_2, \dots, i_m}(S)$.
26. Give an example to show that if R and S are both n -ary relations, then $P_{i_1, i_2, \dots, i_m}(R \cap S)$ may be different from $P_{i_1, i_2, \dots, i_m}(R) \cap P_{i_1, i_2, \dots, i_m}(S)$.
27. Give an example to show that if R and S are both n -ary relations, then $P_{i_1, i_2, \dots, i_m}(R - S)$ may be different from $P_{i_1, i_2, \dots, i_m}(R) - P_{i_1, i_2, \dots, i_m}(S)$.
28. a) What are the operations that correspond to the query expressed using this SQL statement?

```
SELECT Supplier
  FROM Part_needs
 WHERE 1000 ≤ Part_number ≤ 5000
```

b) What is the output of this query given the database in Table 9 as input?
29. a) What are the operations that correspond to the query expressed using this SQL statement?

```
SELECT Supplier, Project
  FROM Part_needs, Parts_inventory
 WHERE Quantity ≤ 10
```

b) What is the output of this query given the databases in Tables 9 and 10 as input?
30. Determine whether there is a primary key for the relation in Example 2.
31. Determine whether there is a primary key for the relation in Example 3.
32. Show that an n -ary relation with a primary key can be thought of as the graph of a function that maps values of the primary key to $(n - 1)$ -tuples formed from values of the other domains.

TABLE 9 Part_needs.

Supplier	Part_number	Project
23	1092	1
23	1101	3
23	9048	4
31	4975	3
31	3477	2
32	6984	4
32	9191	2
33	1001	1

TABLE 10 Parts_inventory.

Part_number	Project	Quantity	Color_code
1001	1	14	8
1092	1	2	2
1101	3	1	1
3477	2	25	2
4975	3	6	2
6984	4	10	1
9048	4	12	2
9191	2	80	4

9.3 Representing Relations

Introduction

In this section, and in the remainder of this chapter, all relations we study will be binary relations. Because of this, in this section and in the rest of this chapter, the word relation will always refer to a binary relation. There are many ways to represent a relation between finite sets. As we have seen in Section 9.1, one way is to list its ordered pairs. Another way to represent a relation is to use a table, as we did in Example 3 in Section 9.1. In this section we will discuss two alternative methods for representing relations. One method uses zero–one matrices. The other method uses pictorial representations called directed graphs, which we will discuss later in this section.

Generally, matrices are appropriate for the representation of relations in computer programs. On the other hand, people often find the representation of relations using directed graphs useful for understanding the properties of these relations.

Representing Relations Using Matrices

A relation between finite sets can be represented using a zero–one matrix. Suppose that R is a relation from $A = \{a_1, a_2, \dots, a_m\}$ to $B = \{b_1, b_2, \dots, b_n\}$. (Here the elements of the sets A and B have been listed in a particular, but arbitrary, order. Furthermore, when $A = B$ we use the same ordering for A and B .) The relation R can be represented by the matrix $\mathbf{M}_R = [m_{ij}]$, where

$$m_{ij} = \begin{cases} 1 & \text{if } (a_i, b_j) \in R, \\ 0 & \text{if } (a_i, b_j) \notin R. \end{cases}$$

In other words, the zero–one matrix representing R has a 1 as its (i, j) entry when a_i is related to b_j , and a 0 in this position if a_i is not related to b_j . (Such a representation depends on the orderings used for A and B .)

The use of matrices to represent relations is illustrated in Examples 1–6.

EXAMPLE 1 Suppose that $A = \{1, 2, 3\}$ and $B = \{1, 2\}$. Let R be the relation from A to B containing (a, b) if $a \in A$, $b \in B$, and $a > b$. What is the matrix representing R if $a_1 = 1$, $a_2 = 2$, and $a_3 = 3$, and $b_1 = 1$ and $b_2 = 2$?

Solution: Because $R = \{(2, 1), (3, 1), (3, 2)\}$, the matrix for R is

$$\mathbf{M}_R = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 1 & 1 \end{bmatrix}.$$

The 1s in \mathbf{M}_R show that the pairs $(2, 1)$, $(3, 1)$, and $(3, 2)$ belong to R . The 0s show that no other pairs belong to R . ◀

EXAMPLE 2 Let $A = \{a_1, a_2, a_3\}$ and $B = \{b_1, b_2, b_3, b_4, b_5\}$. Which ordered pairs are in the relation R represented by the matrix

$$\mathbf{M}_R = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \end{bmatrix} ?$$

Solution: Because R consists of those ordered pairs (a_i, b_j) with $m_{ij} = 1$, it follows that

$$R = \{(a_1, b_2), (a_2, b_1), (a_2, b_3), (a_2, b_4), (a_3, b_1), (a_3, b_3), (a_3, b_5)\}. \quad \blacktriangleleft$$

$$\begin{bmatrix} 1 & & & & \\ 1 & 1 & & & \\ & 1 & \ddots & & \\ & & \ddots & \ddots & \\ & & & 1 & 1 \end{bmatrix}$$

FIGURE 1 The Zero–One Matrix for a Reflexive Relation. (Off Diagonal Elements Can Be 0 or 1.)

The matrix of a relation on a set, which is a square matrix, can be used to determine whether the relation has certain properties. Recall that a relation R on A is reflexive if $(a, a) \in R$ whenever $a \in A$. Thus, R is reflexive if and only if $(a_i, a_i) \in R$ for $i = 1, 2, \dots, n$. Hence, R is reflexive if and only if $m_{ii} = 1$, for $i = 1, 2, \dots, n$. In other words, R is reflexive if all the elements on the main diagonal of \mathbf{M}_R are equal to 1, as shown in Figure 1. Note that the elements off the main diagonal can be either 0 or 1.

The relation R is symmetric if $(a, b) \in R$ implies that $(b, a) \in R$. Consequently, the relation R on the set $A = \{a_1, a_2, \dots, a_n\}$ is symmetric if and only if $(a_j, a_i) \in R$ whenever $(a_i, a_j) \in R$. In terms of the entries of \mathbf{M}_R , R is symmetric if and only if $m_{ji} = 1$ whenever $m_{ij} = 1$. This also means $m_{ji} = 0$ whenever $m_{ij} = 0$. Consequently, R is symmetric if and only if $m_{ij} = m_{ji}$, for all pairs of integers i and j with $i = 1, 2, \dots, n$ and $j = 1, 2, \dots, n$. Recalling the definition of the transpose of a matrix from Section 2.6, we see that R is symmetric if and only if

$$\mathbf{M}_R = (\mathbf{M}_R)^t,$$

that is, if \mathbf{M}_R is a symmetric matrix. The form of the matrix for a symmetric relation is illustrated in Figure 2(a).

The relation R is antisymmetric if and only if $(a, b) \in R$ and $(b, a) \in R$ imply that $a = b$. Consequently, the matrix of an antisymmetric relation has the property that if $m_{ij} = 1$ with $i \neq j$, then $m_{ji} = 0$. Or, in other words, either $m_{ij} = 0$ or $m_{ji} = 0$ when $i \neq j$. The form of the matrix for an antisymmetric relation is illustrated in Figure 2(b).

$$\begin{array}{cc} \begin{bmatrix} & 1 & \\ 1 & & 0 \\ & 0 & \end{bmatrix} & \begin{bmatrix} & 1 & 0 & 0 \\ 0 & & & \\ 0 & & & \\ & 1 & & \end{bmatrix} \\ \text{(a) Symmetric} & \text{(b) Antisymmetric} \end{array}$$

FIGURE 2 The Zero–One Matrices for Symmetric and Antisymmetric Relations.

EXAMPLE 3 Suppose that the relation R on a set is represented by the matrix

$$\mathbf{M}_R = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}.$$

Is R reflexive, symmetric, and/or antisymmetric?

Solution: Because all the diagonal elements of this matrix are equal to 1, R is reflexive. Moreover, because \mathbf{M}_R is symmetric, it follows that R is symmetric. It is also easy to see that R is not antisymmetric. \blacktriangleleft

The Boolean operations join and meet (discussed in Section 2.6) can be used to find the matrices representing the union and the intersection of two relations. Suppose that R_1 and R_2 are relations on a set A represented by the matrices \mathbf{M}_{R_1} and \mathbf{M}_{R_2} , respectively. The matrix

representing the union of these relations has a 1 in the positions where either \mathbf{M}_{R_1} or \mathbf{M}_{R_2} has a 1. The matrix representing the intersection of these relations has a 1 in the positions where both \mathbf{M}_{R_1} and \mathbf{M}_{R_2} have a 1. Thus, the matrices representing the union and intersection of these relations are

$$\mathbf{M}_{R_1 \cup R_2} = \mathbf{M}_{R_1} \vee \mathbf{M}_{R_2} \quad \text{and} \quad \mathbf{M}_{R_1 \cap R_2} = \mathbf{M}_{R_1} \wedge \mathbf{M}_{R_2}.$$

EXAMPLE 4 Suppose that the relations R_1 and R_2 on a set A are represented by the matrices

$$\mathbf{M}_{R_1} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \quad \text{and} \quad \mathbf{M}_{R_2} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix}.$$

What are the matrices representing $R_1 \cup R_2$ and $R_1 \cap R_2$?

Solution: The matrices of these relations are

$$\mathbf{M}_{R_1 \cup R_2} = \mathbf{M}_{R_1} \vee \mathbf{M}_{R_2} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix},$$

$$\mathbf{M}_{R_1 \cap R_2} = \mathbf{M}_{R_1} \wedge \mathbf{M}_{R_2} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \quad \blacktriangleleft$$

We now turn our attention to determining the matrix for the composite of relations. This matrix can be found using the Boolean product of the matrices (discussed in Section 2.6) for these relations. In particular, suppose that R is a relation from A to B and S is a relation from B to C . Suppose that A , B , and C have m , n , and p elements, respectively. Let the zero-one matrices for $S \circ R$, R , and S be $\mathbf{M}_{S \circ R} = [t_{ij}]$, $\mathbf{M}_R = [r_{ij}]$, and $\mathbf{M}_S = [s_{ij}]$, respectively (these matrices have sizes $m \times p$, $m \times n$, and $n \times p$, respectively). The ordered pair (a_i, c_j) belongs to $S \circ R$ if and only if there is an element b_k such that (a_i, b_k) belongs to R and (b_k, c_j) belongs to S . It follows that $t_{ij} = 1$ if and only if $r_{ik} = s_{kj} = 1$ for some k . From the definition of the Boolean product, this means that

$$\mathbf{M}_{S \circ R} = \mathbf{M}_R \odot \mathbf{M}_S.$$

EXAMPLE 5 Find the matrix representing the relations $S \circ R$, where the matrices representing R and S are

$$\mathbf{M}_R = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad \mathbf{M}_S = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix}.$$

Solution: The matrix for $S \circ R$ is

$$\mathbf{M}_{S \circ R} = \mathbf{M}_R \odot \mathbf{M}_S = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}. \quad \blacktriangleleft$$

The matrix representing the composite of two relations can be used to find the matrix for \mathbf{M}_{R^n} . In particular,

$$\mathbf{M}_{R^n} = \mathbf{M}_R^{[n]},$$

from the definition of Boolean powers. Exercise 35 asks for a proof of this formula.

EXAMPLE 6 Find the matrix representing the relation R^2 , where the matrix representing R is

$$\mathbf{M}_R = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix}.$$

Solution: The matrix for R^2 is

$$\mathbf{M}_{R^2} = \mathbf{M}_R^{[2]} = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix}.$$



Representing Relations Using Digraphs

We have shown that a relation can be represented by listing all of its ordered pairs or by using a zero-one matrix. There is another important way of representing a relation using a pictorial representation. Each element of the set is represented by a point, and each ordered pair is represented using an arc with its direction indicated by an arrow. We use such pictorial representations when we think of relations on a finite set as **directed graphs**, or **digraphs**.

DEFINITION 1

A *directed graph*, or *digraph*, consists of a set V of *vertices* (or *nodes*) together with a set E of ordered pairs of elements of V called *edges* (or *arcs*). The vertex a is called the *initial vertex* of the edge (a, b) , and the vertex b is called the *terminal vertex* of this edge.

An edge of the form (a, a) is represented using an arc from the vertex a back to itself. Such an edge is called a **loop**.

EXAMPLE 7

The directed graph with vertices a , b , c , and d , and edges (a, b) , (a, d) , (b, b) , (b, d) , (c, a) , (c, b) , and (d, b) is displayed in Figure 3.

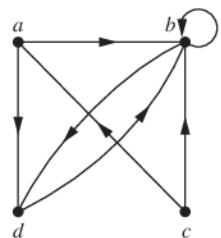


FIGURE 3
A Directed Graph.

The relation R on a set A is represented by the directed graph that has the elements of A as its vertices and the ordered pairs (a, b) , where $(a, b) \in R$, as edges. This assignment sets up a one-to-one correspondence between the relations on a set A and the directed graphs with A as their set of vertices. Thus, every statement about relations corresponds to a statement about directed graphs, and vice versa. Directed graphs give a visual display of information about relations. As such, they are often used to study relations and their properties. (Note that relations from a set A to a set B can be represented by a directed graph where there is a vertex for each element of A and a vertex for each element of B , as shown in Section 9.1. However, when $A = B$, such representation provides much less insight than the digraph representations described here.) The use of directed graphs to represent relations on a set is illustrated in Examples 8–10.

EXAMPLE 8 The directed graph of the relation

$$R = \{(1, 1), (1, 3), (2, 1), (2, 3), (2, 4), (3, 1), (3, 2), (4, 1)\}$$

on the set $\{1, 2, 3, 4\}$ is shown in Figure 4. ◀

EXAMPLE 9 What are the ordered pairs in the relation R represented by the directed graph shown in Figure 5?

Solution: The ordered pairs (x, y) in the relation are

$$R = \{(1, 3), (1, 4), (2, 1), (2, 2), (2, 3), (3, 1), (3, 3), (4, 1), (4, 3)\}.$$

Each of these pairs corresponds to an edge of the directed graph, with $(2, 2)$ and $(3, 3)$ corresponding to loops. ◀

We will study directed graphs extensively in Chapter 10.

The directed graph representing a relation can be used to determine whether the relation has various properties. For instance, a relation is reflexive if and only if there is a loop at every vertex of the directed graph, so that every ordered pair of the form (x, x) occurs in the relation. A relation is symmetric if and only if for every edge between distinct vertices in its digraph there is an edge in the opposite direction, so that (y, x) is in the relation whenever (x, y) is in the relation. Similarly, a relation is antisymmetric if and only if there are never two edges in opposite directions between distinct vertices. Finally, a relation is transitive if and only if whenever there is an edge from a vertex x to a vertex y and an edge from a vertex y to a vertex z , there is an edge from x to z (completing a triangle where each side is a directed edge with the correct direction).

Remark: Note that a symmetric relation can be represented by an undirected graph, which is a graph where edges do not have directions. We will study undirected graphs in Chapter 10.

EXAMPLE 10 Determine whether the relations for the directed graphs shown in Figure 6 are reflexive, symmetric, antisymmetric, and/or transitive.

Solution: Because there are loops at every vertex of the directed graph of R , it is reflexive. R is neither symmetric nor antisymmetric because there is an edge from a to b but not one from b to a , but there are edges in both directions connecting b and c . Finally, R is not transitive because there is an edge from a to b and an edge from b to c , but no edge from a to c .

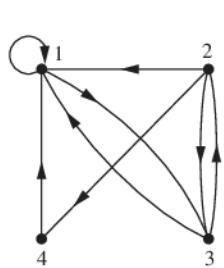


FIGURE 4 The Directed Graph of the Relation R .

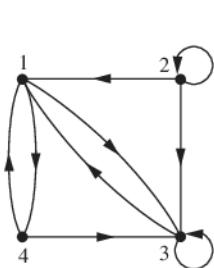


FIGURE 5 The Directed Graph of the Relation R .

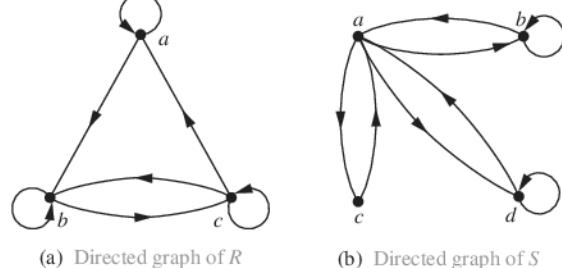


FIGURE 6(a) Directed graph of R .

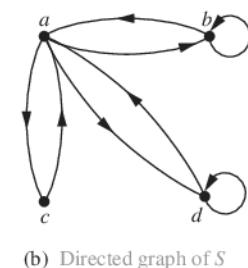


FIGURE 6(b) Directed graph of S .

Because loops are not present at all the vertices of the directed graph of S , this relation is not reflexive. It is symmetric and not antisymmetric, because every edge between distinct vertices is accompanied by an edge in the opposite direction. It is also not hard to see from the directed graph that S is not transitive, because (c, a) and (a, b) belong to S , but (c, b) does not belong to S . \blacktriangleleft

Exercises

1. Represent each of these relations on $\{1, 2, 3\}$ with a matrix (with the elements of this set listed in increasing order).

- a) $\{(1, 1), (1, 2), (1, 3)\}$
 - b) $\{(1, 2), (2, 1), (2, 2), (3, 3)\}$
 - c) $\{(1, 1), (1, 2), (1, 3), (2, 2), (2, 3), (3, 3)\}$
 - d) $\{(1, 3), (3, 1)\}$
2. Represent each of these relations on $\{1, 2, 3, 4\}$ with a matrix (with the elements of this set listed in increasing order).
- a) $\{(1, 2), (1, 3), (1, 4), (2, 3), (2, 4), (3, 4)\}$
 - b) $\{(1, 1), (1, 4), (2, 2), (3, 3), (4, 1)\}$
 - c) $\{(1, 2), (1, 3), (1, 4), (2, 1), (2, 3), (2, 4), (3, 1), (3, 2), (3, 4), (4, 1), (4, 2), (4, 3)\}$
 - d) $\{(2, 4), (3, 1), (3, 2), (3, 4)\}$
3. List the ordered pairs in the relations on $\{1, 2, 3\}$ corresponding to these matrices (where the rows and columns correspond to the integers listed in increasing order).

$$\begin{array}{ll} \text{a)} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} & \text{b)} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix} \\ \text{c)} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix} & \end{array}$$

4. List the ordered pairs in the relations on $\{1, 2, 3, 4\}$ corresponding to these matrices (where the rows and columns correspond to the integers listed in increasing order).

$$\begin{array}{ll} \text{a)} \begin{bmatrix} 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \end{bmatrix} & \text{b)} \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \end{bmatrix} \\ \text{c)} \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix} & \end{array}$$

5. How can the matrix representing a relation R on a set A be used to determine whether the relation is irreflexive?
 6. How can the matrix representing a relation R on a set A be used to determine whether the relation is asymmetric?
 7. Determine whether the relations represented by the matrices in Exercise 3 are reflexive, irreflexive, symmetric, antisymmetric, and/or transitive.
 8. Determine whether the relations represented by the matrices in Exercise 4 are reflexive, irreflexive, symmetric, antisymmetric, and/or transitive.

9. How many nonzero entries does the matrix representing the relation R on $A = \{1, 2, 3, \dots, 100\}$ consisting of the first 100 positive integers have if R is

- a) $\{(a, b) \mid a > b\}$? b) $\{(a, b) \mid a \neq b\}$?
- c) $\{(a, b) \mid a = b + 1\}$? d) $\{(a, b) \mid a = 1\}$?
- e) $\{(a, b) \mid ab = 1\}$?

10. How many nonzero entries does the matrix representing the relation R on $A = \{1, 2, 3, \dots, 1000\}$ consisting of the first 1000 positive integers have if R is

- a) $\{(a, b) \mid a \leq b\}$?
- b) $\{(a, b) \mid a = b \pm 1\}$?
- c) $\{(a, b) \mid a + b = 1000\}$?
- d) $\{(a, b) \mid a + b \leq 1001\}$?
- e) $\{(a, b) \mid a \neq 0\}$?

11. How can the matrix for \bar{R} , the complement of the relation R , be found from the matrix representing R , when R is a relation on a finite set A ?

12. How can the matrix for R^{-1} , the inverse of the relation R , be found from the matrix representing R , when R is a relation on a finite set A ?

13. Let R be the relation represented by the matrix

$$\mathbf{M}_R = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}.$$

Find the matrix representing

- a) R^{-1} . b) \bar{R} . c) R^2 .

14. Let R_1 and R_2 be relations on a set A represented by the matrices

$$\mathbf{M}_{R_1} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix} \quad \text{and} \quad \mathbf{M}_{R_2} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}.$$

Find the matrices that represent

- a) $R_1 \cup R_2$. b) $R_1 \cap R_2$. c) $R_2 \circ R_1$.
- d) $R_1 \circ R_1$. e) $R_1 \oplus R_2$.

15. Let R be the relation represented by the matrix

$$\mathbf{M}_R = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}.$$

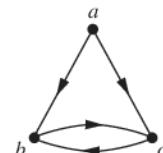
Find the matrices that represent

- a) R^2 . b) R^3 . c) R^4 .

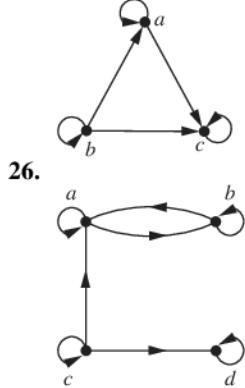
- 16.** Let R be a relation on a set A with n elements. If there are k nonzero entries in \mathbf{M}_R , the matrix representing R , how many nonzero entries are there in $\mathbf{M}_{R^{-1}}$, the matrix representing R^{-1} , the inverse of R ?
- 17.** Let R be a relation on a set A with n elements. If there are k nonzero entries in \mathbf{M}_R , the matrix representing R , how many nonzero entries are there in $\mathbf{M}_{\bar{R}}$, the matrix representing \bar{R} , the complement of R ?
- 18.** Draw the directed graphs representing each of the relations from Exercise 1.
- 19.** Draw the directed graphs representing each of the relations from Exercise 2.
- 20.** Draw the directed graph representing each of the relations from Exercise 3.
- 21.** Draw the directed graph representing each of the relations from Exercise 4.
- 22.** Draw the directed graph that represents the relation $\{(a, a), (a, b), (b, c), (c, b), (c, d), (d, a), (d, b)\}$.

In Exercises 23–28 list the ordered pairs in the relations represented by the directed graphs.

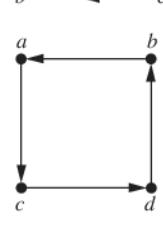
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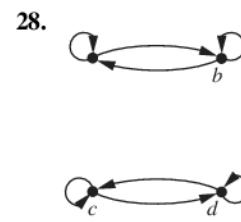
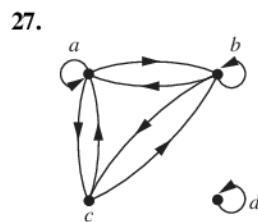
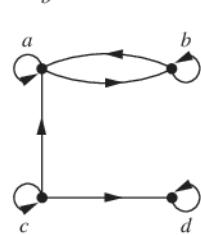
24.



25.



26.



- 29.** How can the directed graph of a relation R on a finite set A be used to determine whether a relation is asymmetric?
- 30.** How can the directed graph of a relation R on a finite set A be used to determine whether a relation is irreflexive?
- 31.** Determine whether the relations represented by the directed graphs shown in Exercises 23–25 are reflexive, irreflexive, symmetric, antisymmetric, and/or transitive.
- 32.** Determine whether the relations represented by the directed graphs shown in Exercises 26–28 are reflexive, irreflexive, symmetric, antisymmetric, asymmetric, and/or transitive.
- 33.** Let R be a relation on a set A . Explain how to use the directed graph representing R to obtain the directed graph representing the inverse relation R^{-1} .
- 34.** Let R be a relation on a set A . Explain how to use the directed graph representing R to obtain the directed graph representing the complementary relation \bar{R} .
- 35.** Show that if \mathbf{M}_R is the matrix representing the relation R , then $\mathbf{M}_R^{[n]}$ is the matrix representing the relation R^n .
- 36.** Given the directed graphs representing two relations, how can the directed graph of the union, intersection, symmetric difference, difference, and composition of these relations be found?

9.4 Closures of Relations

Introduction

A computer network has data centers in Boston, Chicago, Denver, Detroit, New York, and San Diego. There are direct, one-way telephone lines from Boston to Chicago, from Boston to Detroit, from Chicago to Detroit, from Detroit to Denver, and from New York to San Diego. Let R be the relation containing (a, b) if there is a telephone line from the data center in a to that in b . How can we determine if there is some (possibly indirect) link composed of one or more telephone lines from one center to another? Because not all links are direct, such as the link from Boston to Denver that goes through Detroit, R cannot be used directly to answer this. In the language of relations, R is not transitive, so it does not contain all the pairs that can be linked. As we will show in this section, we can find all pairs of data centers that have a link by constructing a transitive relation S containing R such that S is a subset of every transitive relation containing R . Here, S is the smallest transitive relation that contains R . This relation is called the **transitive closure** of R .

In general, let R be a relation on a set A . R may or may not have some property **P**, such as reflexivity, symmetry, or transitivity. If there is a relation S with property **P** containing R such that S is a subset of every relation with property **P** containing R , then S is called the **closure**

of R with respect to **P**. (Note that the closure of a relation with respect to a property may not exist; see Exercises 15 and 35.) We will show how reflexive, symmetric, and transitive closures of relations can be found.

Closures

The relation $R = \{(1, 1), (1, 2), (2, 1), (3, 2)\}$ on the set $A = \{1, 2, 3\}$ is not reflexive. How can we produce a reflexive relation containing R that is as small as possible? This can be done by adding $(2, 2)$ and $(3, 3)$ to R , because these are the only pairs of the form (a, a) that are not in R . Clearly, this new relation contains R . Furthermore, any reflexive relation that contains R must also contain $(2, 2)$ and $(3, 3)$. Because this relation contains R , is reflexive, and is contained within every reflexive relation that contains R , it is called the **reflexive closure** of R .

As this example illustrates, given a relation R on a set A , the reflexive closure of R can be formed by adding to R all pairs of the form (a, a) with $a \in A$, not already in R . The addition of these pairs produces a new relation that is reflexive, contains R , and is contained within any reflexive relation containing R . We see that the reflexive closure of R equals $R \cup \Delta$, where $\Delta = \{(a, a) \mid a \in A\}$ is the **diagonal relation** on A . (The reader should verify this.)

EXAMPLE 1 What is the reflexive closure of the relation $R = \{(a, b) \mid a < b\}$ on the set of integers?

Solution: The reflexive closure of R is

$$R \cup \Delta = \{(a, b) \mid a < b\} \cup \{(a, a) \mid a \in \mathbf{Z}\} = \{(a, b) \mid a \leq b\}. \quad \blacktriangleleft$$

The relation $\{(1, 1), (1, 2), (2, 2), (2, 3), (3, 1), (3, 2)\}$ on $\{1, 2, 3\}$ is not symmetric. How can we produce a symmetric relation that is as small as possible and contains R ? To do this, we need only add $(2, 1)$ and $(1, 3)$, because these are the only pairs of the form (b, a) with $(a, b) \in R$ that are not in R . This new relation is symmetric and contains R . Furthermore, any symmetric relation that contains R must contain this new relation, because a symmetric relation that contains R must contain $(2, 1)$ and $(1, 3)$. Consequently, this new relation is called the **symmetric closure** of R .

As this example illustrates, the symmetric closure of a relation R can be constructed by adding all ordered pairs of the form (b, a) , where (a, b) is in the relation, that are not already present in R . Adding these pairs produces a relation that is symmetric, that contains R , and that is contained in any symmetric relation that contains R . The symmetric closure of a relation can be constructed by taking the union of a relation with its inverse (defined in the preamble of Exercise 26 in Section 9.1); that is, $R \cup R^{-1}$ is the symmetric closure of R , where $R^{-1} = \{(b, a) \mid (a, b) \in R\}$. The reader should verify this statement.

EXAMPLE 2 What is the symmetric closure of the relation $R = \{(a, b) \mid a > b\}$ on the set of positive integers?



Solution: The symmetric closure of R is the relation

$$R \cup R^{-1} = \{(a, b) \mid a > b\} \cup \{(b, a) \mid a > b\} = \{(a, b) \mid a \neq b\}. \quad \blacktriangleleft$$

This last equality follows because R contains all ordered pairs of positive integers where the first element is greater than the second element and R^{-1} contains all ordered pairs of positive integers where the first element is less than the second. \blacktriangleleft

Suppose that a relation R is not transitive. How can we produce a transitive relation that contains R such that this new relation is contained within any transitive relation that contains R ? Can the transitive closure of a relation R be produced by adding all the pairs of the form (a, c) , where (a, b) and (b, c) are already in the relation? Consider the relation

$R = \{(1, 3), (1, 4), (2, 1), (3, 2)\}$ on the set $\{1, 2, 3, 4\}$. This relation is not transitive because it does not contain all pairs of the form (a, c) where (a, b) and (b, c) are in R . The pairs of this form not in R are $(1, 2), (2, 3), (2, 4)$, and $(3, 1)$. Adding these pairs does *not* produce a transitive relation, because the resulting relation contains $(3, 1)$ and $(1, 4)$ but does not contain $(3, 4)$. This shows that constructing the transitive closure of a relation is more complicated than constructing either the reflexive or symmetric closure. The rest of this section develops algorithms for constructing transitive closures. As will be shown later in this section, the transitive closure of a relation can be found by adding new ordered pairs that must be present and then repeating this process until no new ordered pairs are needed.

Paths in Directed Graphs

We will see that representing relations by directed graphs helps in the construction of transitive closures. We now introduce some terminology that we will use for this purpose.

A path in a directed graph is obtained by traversing along edges (in the same direction as indicated by the arrow on the edge).

DEFINITION 1

A *path* from a to b in the directed graph G is a sequence of edges $(x_0, x_1), (x_1, x_2), (x_2, x_3), \dots, (x_{n-1}, x_n)$ in G , where n is a nonnegative integer, and $x_0 = a$ and $x_n = b$, that is, a sequence of edges where the terminal vertex of an edge is the same as the initial vertex in the next edge in the path. This path is denoted by $x_0, x_1, x_2, \dots, x_{n-1}, x_n$ and has *length* n . We view the empty set of edges as a path of length zero from a to a . A path of length $n \geq 1$ that begins and ends at the same vertex is called a *circuit* or *cycle*.

A path in a directed graph can pass through a vertex more than once. Moreover, an edge in a directed graph can occur more than once in a path.

EXAMPLE 3 Which of the following are paths in the directed graph shown in Figure 1: $a, b, e, d; a, e, c, d, b; b, a, c, b, a, a, b; d, c; c, b, a; e, b, a, b, a, b, e$? What are the lengths of those that are paths? Which of the paths in this list are circuits?

Solution: Because each of $(a, b), (b, e)$, and (e, d) is an edge, a, b, e, d is a path of length three. Because (c, d) is not an edge, a, e, c, d, b is not a path. Also, b, a, c, b, a, a, b is a path of length six because $(b, a), (a, c), (c, b), (b, a), (a, a)$, and (a, b) are all edges. We see that d, c is a path of length one, because (d, c) is an edge. Also c, b, a is a path of length two, because (c, b) and (b, a) are edges. All of $(e, b), (b, a), (a, b), (b, a), (a, b)$, and (b, e) are edges, so e, b, a, b, a, b, e is a path of length six.

The two paths b, a, c, b, a, a, b and e, b, a, b, a, b, e are circuits because they begin and end at the same vertex. The paths $a, b, e, d; c, b, a$; and d, c are not circuits. ◀

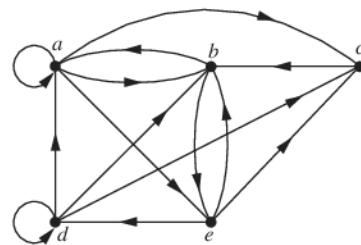


FIGURE 1 A Directed Graph.

The term *path* also applies to relations. Carrying over the definition from directed graphs to relations, there is a **path** from a to b in R if there is a sequence of elements $a, x_1, x_2, \dots, x_{n-1}, b$ with $(a, x_1) \in R$, $(x_1, x_2) \in R, \dots$, and $(x_{n-1}, b) \in R$. Theorem 1 can be obtained from the definition of a path in a relation.

THEOREM 1

Let R be a relation on a set A . There is a path of length n , where n is a positive integer, from a to b if and only if $(a, b) \in R^n$.

Proof: We will use mathematical induction. By definition, there is a path from a to b of length one if and only if $(a, b) \in R$, so the theorem is true when $n = 1$.

Assume that the theorem is true for the positive integer n . This is the inductive hypothesis. There is a path of length $n + 1$ from a to b if and only if there is an element $c \in A$ such that there is a path of length one from a to c , so $(a, c) \in R$, and a path of length n from c to b , that is, $(c, b) \in R^n$. Consequently, by the inductive hypothesis, there is a path of length $n + 1$ from a to b if and only if there is an element c with $(a, c) \in R$ and $(c, b) \in R^n$. But there is such an element if and only if $(a, b) \in R^{n+1}$. Therefore, there is a path of length $n + 1$ from a to b if and only if $(a, b) \in R^{n+1}$. This completes the proof. \triangleleft

Transitive Closures

We now show that finding the transitive closure of a relation is equivalent to determining which pairs of vertices in the associated directed graph are connected by a path. With this in mind, we define a new relation.

DEFINITION 2

Let R be a relation on a set A . The *connectivity relation* R^* consists of the pairs (a, b) such that there is a path of length at least one from a to b in R .

Because R^n consists of the pairs (a, b) such that there is a path of length n from a to b , it follows that R^* is the union of all the sets R^n . In other words,

$$R^* = \bigcup_{n=1}^{\infty} R^n.$$

The connectivity relation is useful in many models.

EXAMPLE 4 Let R be the relation on the set of all people in the world that contains (a, b) if a has met b . What is R^n , where n is a positive integer greater than one? What is R^* ?

Solution: The relation R^2 contains (a, b) if there is a person c such that $(a, c) \in R$ and $(c, b) \in R$, that is, if there is a person c such that a has met c and c has met b . Similarly, R^n consists of those pairs (a, b) such that there are people x_1, x_2, \dots, x_{n-1} such that a has met x_1 , x_1 has met x_2, \dots , and x_{n-1} has met b .

The relation R^* contains (a, b) if there is a sequence of people, starting with a and ending with b , such that each person in the sequence has met the next person in the sequence. (There are many interesting conjectures about R^* . Do you think that this connectivity relation includes the pair with you as the first element and the president of Mongolia as the second element? We will use graphs to model this application in Chapter 10.) \triangleleft

EXAMPLE 5 Let R be the relation on the set of all subway stops in New York City that contains (a, b) if it is possible to travel from stop a to stop b without changing trains. What is R^n when n is a positive integer? What is R^* ?

Solution: The relation R^n contains (a, b) if it is possible to travel from stop a to stop b by making at most $n - 1$ changes of trains. The relation R^* consists of the ordered pairs (a, b) where it is possible to travel from stop a to stop b making as many changes of trains as necessary. (The reader should verify these statements.) \blacktriangleleft

EXAMPLE 6 Let R be the relation on the set of all states in the United States that contains (a, b) if state a and state b have a common border. What is R^n , where n is a positive integer? What is R^* ?

Solution: The relation R^n consists of the pairs (a, b) , where it is possible to go from state a to state b by crossing exactly n state borders. R^* consists of the ordered pairs (a, b) , where it is possible to go from state a to state b crossing as many borders as necessary. (The reader should verify these statements.) The only ordered pairs not in R^* are those containing states that are not connected to the continental United States (i.e., those pairs containing Alaska or Hawaii). \blacktriangleleft

Theorem 2 shows that the transitive closure of a relation and the associated connectivity relation are the same.

THEOREM 2

The transitive closure of a relation R equals the connectivity relation R^* .

Proof: Note that R^* contains R by definition. To show that R^* is the transitive closure of R we must also show that R^* is transitive and that $R^* \subseteq S$ whenever S is a transitive relation that contains R .

First, we show that R^* is transitive. If $(a, b) \in R^*$ and $(b, c) \in R^*$, then there are paths from a to b and from b to c in R . We obtain a path from a to c by starting with the path from a to b and following it with the path from b to c . Hence, $(a, c) \in R^*$. It follows that R^* is transitive.

Now suppose that S is a transitive relation containing R . Because S is transitive, S^n also is transitive (the reader should verify this) and $S^n \subseteq S$ (by Theorem 1 of Section 9.1). Furthermore, because

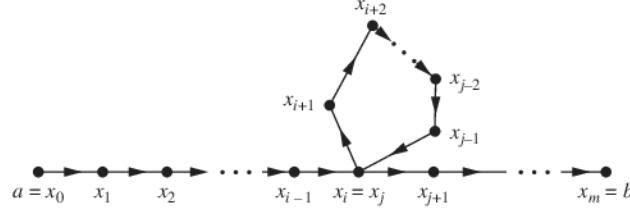
$$S^* = \bigcup_{k=1}^{\infty} S^k$$

and $S^k \subseteq S$, it follows that $S^* \subseteq S$. Now note that if $R \subseteq S$, then $R^* \subseteq S^*$, because any path in R is also a path in S . Consequently, $R^* \subseteq S^* \subseteq S$. Thus, any transitive relation that contains R must also contain R^* . Therefore, R^* is the transitive closure of R . \blacktriangleleft

Now that we know that the transitive closure equals the connectivity relation, we turn our attention to the problem of computing this relation. We do not need to examine arbitrarily long paths to determine whether there is a path between two vertices in a finite directed graph. As Lemma 1 shows, it is sufficient to examine paths containing no more than n edges, where n is the number of elements in the set.

LEMMA 1

Let A be a set with n elements, and let R be a relation on A . If there is a path of length at least one in R from a to b , then there is such a path with length not exceeding n . Moreover, when $a \neq b$, if there is a path of length at least one in R from a to b , then there is such a path with length not exceeding $n - 1$.

FIGURE 2 Producing a Path with Length Not Exceeding n .

Proof: Suppose there is a path from a to b in R . Let m be the length of the shortest such path. Suppose that $x_0, x_1, x_2, \dots, x_{m-1}, x_m$, where $x_0 = a$ and $x_m = b$, is such a path.

Suppose that $a = b$ and that $m > n$, so that $m \geq n + 1$. By the pigeonhole principle, because there are n vertices in A , among the m vertices x_0, x_1, \dots, x_{m-1} , at least two are equal (see Figure 2).

Suppose that $x_i = x_j$ with $0 \leq i < j \leq m - 1$. Then the path contains a circuit from x_i to itself. This circuit can be deleted from the path from a to b , leaving a path, namely, $x_0, x_1, \dots, x_i, x_{j+1}, \dots, x_{m-1}, x_m$, from a to b of shorter length. Hence, the path of shortest length must have length less than or equal to n .

The case where $a \neq b$ is left as an exercise for the reader. \square

From Lemma 1, we see that the transitive closure of R is the union of R , R^2 , R^3 , \dots , and R^n . This follows because there is a path in R^* between two vertices if and only if there is a path between these vertices in R^i , for some positive integer i with $i \leq n$. Because

$$R^* = R \cup R^2 \cup R^3 \cup \dots \cup R^n$$

and the zero-one matrix representing a union of relations is the join of the zero-one matrices of these relations, the zero-one matrix for the transitive closure is the join of the zero-one matrices of the first n powers of the zero-one matrix of R .

THEOREM 3

Let \mathbf{M}_R be the zero-one matrix of the relation R on a set with n elements. Then the zero-one matrix of the transitive closure R^* is

$$\mathbf{M}_{R^*} = \mathbf{M}_R \vee \mathbf{M}_R^{[2]} \vee \mathbf{M}_R^{[3]} \vee \dots \vee \mathbf{M}_R^{[n]}.$$

EXAMPLE 7 Find the zero-one matrix of the transitive closure of the relation R where

$$\mathbf{M}_R = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix}.$$

Solution: By Theorem 3, it follows that the zero-one matrix of R^* is

$$\mathbf{M}_{R^*} = \mathbf{M}_R \vee \mathbf{M}_R^{[2]} \vee \mathbf{M}_R^{[3]}.$$

Because

$$\mathbf{M}_R^{[2]} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \quad \text{and} \quad \mathbf{M}_R^{[3]} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix},$$

it follows that

$$\mathbf{M}_{R^*} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix} \vee \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \vee \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}. \quad \blacktriangleleft$$

Theorem 3 can be used as a basis for an algorithm for computing the matrix of the relation R^* . To find this matrix, the successive Boolean powers of \mathbf{M}_R , up to the n th power, are computed. As each power is calculated, its join with the join of all smaller powers is formed. When this is done with the n th power, the matrix for R^* has been found. This procedure is displayed as Algorithm 1.

ALGORITHM 1 A Procedure for Computing the Transitive Closure.

```

procedure transitive closure ( $\mathbf{M}_R$  : zero–one  $n \times n$  matrix)
   $\mathbf{A} := \mathbf{M}_R$ 
   $\mathbf{B} := \mathbf{A}$ 
  for  $i := 2$  to  $n$ 
     $\mathbf{A} := \mathbf{A} \odot \mathbf{M}_R$ 
     $\mathbf{B} := \mathbf{B} \vee \mathbf{A}$ 
  return  $\mathbf{B}$  { $\mathbf{B}$  is the zero–one matrix for  $R^*$ }
```

We can easily find the number of bit operations used by Algorithm 1 to determine the transitive closure of a relation. Computing the Boolean powers \mathbf{M}_R , $\mathbf{M}_R^{[2]}$, ..., $\mathbf{M}_R^{[n]}$ requires that $n - 1$ Boolean products of $n \times n$ zero–one matrices be found. Each of these Boolean products can be found using $n^2(2n - 1)$ bit operations. Hence, these products can be computed using $n^2(2n - 1)(n - 1)$ bit operations.

To find \mathbf{M}_{R^*} from the n Boolean powers of \mathbf{M}_R , $n - 1$ joins of zero–one matrices need to be found. Computing each of these joins uses n^2 bit operations. Hence, $(n - 1)n^2$ bit operations are used in this part of the computation. Therefore, when Algorithm 1 is used, the matrix of the transitive closure of a relation on a set with n elements can be found using $n^2(2n - 1)(n - 1) + (n - 1)n^2 = 2n^3(n - 1)$, which is $O(n^4)$ bit operations. The remainder of this section describes a more efficient algorithm for finding transitive closures.

Warshall's Algorithm

Warshall's algorithm, named after Stephen Warshall, who described this algorithm in 1960, is an efficient method for computing the transitive closure of a relation. Algorithm 1 can find the transitive closure of a relation on a set with n elements using $2n^3(n - 1)$ bit operations. However, the transitive closure can be found by Warshall's algorithm using only $2n^3$ bit operations.

Remark: Warshall's algorithm is sometimes called the Roy–Warshall algorithm, because Bernard Roy described this algorithm in 1959.

Suppose that R is a relation on a set with n elements. Let v_1, v_2, \dots, v_n be an arbitrary listing of these n elements. The concept of the **interior vertices** of a path is used in Warshall's algorithm. If $a, x_1, x_2, \dots, x_{m-1}, b$ is a path, its interior vertices are x_1, x_2, \dots, x_{m-1} , that is, all the vertices of the path that occur somewhere other than as the first and last vertices in the path. For instance, the interior vertices of a path a, c, d, f, g, h, b, j in a directed graph

are c, d, f, g, h , and b . The interior vertices of a, c, d, a, f, b are c, d, a , and f . (Note that the first vertex in the path is not an interior vertex unless it is visited again by the path, except as the last vertex. Similarly, the last vertex in the path is not an interior vertex unless it was visited previously by the path, except as the first vertex.)

Warshall's algorithm is based on the construction of a sequence of zero-one matrices. These matrices are $\mathbf{W}_0, \mathbf{W}_1, \dots, \mathbf{W}_n$, where $\mathbf{W}_0 = \mathbf{M}_R$ is the zero-one matrix of this relation, and $\mathbf{W}_k = [w_{ij}^{(k)}]$, where $w_{ij}^{(k)} = 1$ if there is a path from v_i to v_j such that all the interior vertices of this path are in the set $\{v_1, v_2, \dots, v_k\}$ (the first k vertices in the list) and is 0 otherwise. (The first and last vertices in the path may be outside the set of the first k vertices in the list.) Note that $\mathbf{W}_n = \mathbf{M}_{R^*}$, because the (i, j) th entry of \mathbf{M}_{R^*} is 1 if and only if there is a path from v_i to v_j , with all interior vertices in the set $\{v_1, v_2, \dots, v_n\}$ (but these are the only vertices in the directed graph). Example 8 illustrates what the matrix \mathbf{W}_k represents.

EXAMPLE 8

Let R be the relation with directed graph shown in Figure 3. Let a, b, c, d be a listing of the elements of the set. Find the matrices $\mathbf{W}_0, \mathbf{W}_1, \mathbf{W}_2, \mathbf{W}_3$, and \mathbf{W}_4 . The matrix \mathbf{W}_4 is the transitive closure of R .

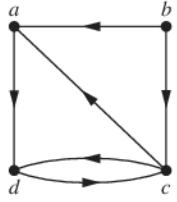


FIGURE 3
The Directed
Graph of the
Relation R .

Solution: Let $v_1 = a, v_2 = b, v_3 = c$, and $v_4 = d$. \mathbf{W}_0 is the matrix of the relation. Hence,

$$\mathbf{W}_0 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

\mathbf{W}_1 has 1 as its (i, j) th entry if there is a path from v_i to v_j that has only $v_1 = a$ as an interior vertex. Note that all paths of length one can still be used because they have no interior vertices. Also, there is now an allowable path from b to d , namely, b, a, d . Hence,

$$\mathbf{W}_1 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

\mathbf{W}_2 has 1 as its (i, j) th entry if there is a path from v_i to v_j that has only $v_1 = a$ and/or $v_2 = b$ as its interior vertices, if any. Because there are no edges that have b as a terminal vertex, no new paths are obtained when we permit b to be an interior vertex. Hence, $\mathbf{W}_2 = \mathbf{W}_1$.



STEPHEN WARSHALL (1935–2006) Stephen Warshall, born in New York City, went to public school in Brooklyn. He attended Harvard University, receiving his degree in mathematics in 1956. He never received an advanced degree, because at that time no programs were available in his areas of interest. However, he took graduate courses at several different universities and contributed to the development of computer science and software engineering.

After graduating from Harvard, Warshall worked at ORO (Operation Research Office), which was set up by Johns Hopkins to do research and development for the U.S. Army. In 1958 he left ORO to take a position at a company called Technical Operations, where he helped build a research and development laboratory for military software projects. In 1961 he left Technical Operations to found Massachusetts Computer Associates. Later, this company became part of Applied Data Research (ADR). After the merger, Warshall sat on the board of directors of ADR and managed a variety of projects and organizations. He retired from ADR in 1982.

During his career Warshall carried out research and development in operating systems, compiler design, language design, and operations research. In the 1971–1972 academic year he presented lectures on software engineering at French universities. There is an interesting anecdote about his proof that the transitive closure algorithm, now known as Warshall's algorithm, is correct. He and a colleague at Technical Operations bet a bottle of rum on who first could determine whether this algorithm always works. Warshall came up with his proof overnight, winning the bet and the rum, which he shared with the loser of the bet. Because Warshall did not like sitting at a desk, he did much of his creative work in unconventional places, such as on a sailboat in the Indian Ocean or in a Greek lemon orchard.

\mathbf{W}_3 has 1 as its (i, j) th entry if there is a path from v_i to v_j that has only $v_1 = a, v_2 = b$, and/or $v_3 = c$ as its interior vertices, if any. We now have paths from d to a , namely, d, c, a , and from d to d , namely, d, c, d . Hence,

$$\mathbf{W}_3 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 \end{bmatrix}.$$

Finally, \mathbf{W}_4 has 1 as its (i, j) th entry if there is a path from v_i to v_j that has $v_1 = a, v_2 = b, v_3 = c$, and/or $v_4 = d$ as interior vertices, if any. Because these are all the vertices of the graph, this entry is 1 if and only if there is a path from v_i to v_j . Hence,

$$\mathbf{W}_4 = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 \end{bmatrix}.$$

This last matrix, \mathbf{W}_4 , is the matrix of the transitive closure. \blacktriangleleft

Warshall's algorithm computes \mathbf{M}_{R^*} by efficiently computing $\mathbf{W}_0 = \mathbf{M}_R, \mathbf{W}_1, \mathbf{W}_2, \dots, \mathbf{W}_n = \mathbf{M}_{R^*}$. This observation shows that we can compute \mathbf{W}_k directly from \mathbf{W}_{k-1} : There is a path from v_i to v_j with no vertices other than v_1, v_2, \dots, v_k as interior vertices if and only if either there is a path from v_i to v_j with its interior vertices among the first $k - 1$ vertices in the list, or there are paths from v_i to v_k and from v_k to v_j that have interior vertices only among the first $k - 1$ vertices in the list. That is, either a path from v_i to v_j already existed before v_k was permitted as an interior vertex, or allowing v_k as an interior vertex produces a path that goes from v_i to v_k and then from v_k to v_j . These two cases are shown in Figure 4.

The first type of path exists if and only if $w_{ij}^{[k-1]} = 1$, and the second type of path exists if and only if both $w_{ik}^{[k-1]}$ and $w_{kj}^{[k-1]}$ are 1. Hence, $w_{ij}^{[k]}$ is 1 if and only if either $w_{ij}^{[k-1]}$ is 1 or both $w_{ik}^{[k-1]}$ and $w_{kj}^{[k-1]}$ are 1. This gives us Lemma 2.

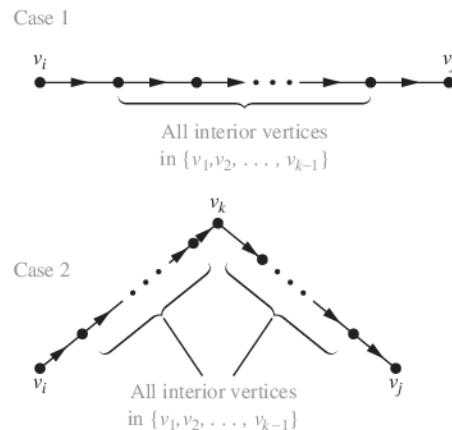


FIGURE 4 Adding v_k to the Set of Allowable Interior Vertices.

LEMMA 2

Let $\mathbf{W}_k = [w_{ij}^{[k]}]$ be the zero-one matrix that has a 1 in its (i, j) th position if and only if there is a path from v_i to v_j with interior vertices from the set $\{v_1, v_2, \dots, v_k\}$. Then

$$w_{ij}^{[k]} = w_{ij}^{[k-1]} \vee (w_{ik}^{[k-1]} \wedge w_{kj}^{[k-1]}),$$

whenever i , j , and k are positive integers not exceeding n .

Lemma 2 gives us the means to compute efficiently the matrices \mathbf{W}_k , $k = 1, 2, \dots, n$. We display the pseudocode for Marshall's algorithm, using Lemma 2, as Algorithm 2.

ALGORITHM 2 Warshall Algorithm.

```

procedure Warshall ( $\mathbf{M}_R : n \times n$  zero-one matrix)
   $\mathbf{W} := \mathbf{M}_R$ 
  for  $k := 1$  to  $n$ 
    for  $i := 1$  to  $n$ 
      for  $j := 1$  to  $n$ 
         $w_{ij} := w_{ij} \vee (w_{ik} \wedge w_{kj})$ 
  return  $\mathbf{W}$  ( $\mathbf{W} = [w_{ij}]$  is  $\mathbf{M}_{R^*}$ )

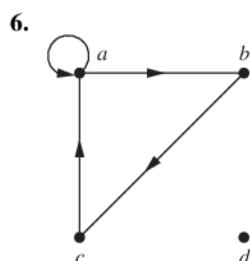
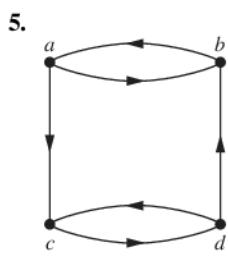
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The computational complexity of Warshall's algorithm can easily be computed in terms of bit operations. To find the entry $w_{ij}^{[k]}$ from the entries $w_{ij}^{[k-1]}$, $w_{ik}^{[k-1]}$, and $w_{kj}^{[k-1]}$ using Lemma 2 requires two bit operations. To find all n^2 entries of \mathbf{W}_k from those of \mathbf{W}_{k-1} requires $2n^2$ bit operations. Because Warshall's algorithm begins with $\mathbf{W}_0 = \mathbf{M}_R$ and computes the sequence of n zero-one matrices $\mathbf{W}_1, \mathbf{W}_2, \dots, \mathbf{W}_n = \mathbf{M}_{R^*}$, the total number of bit operations used is $n \cdot 2n^2 = 2n^3$.

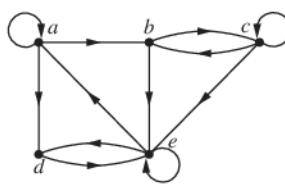
Exercises

1. Let R be the relation on the set $\{0, 1, 2, 3\}$ containing the ordered pairs $(0, 1)$, $(1, 1)$, $(1, 2)$, $(2, 0)$, $(2, 2)$, and $(3, 0)$. Find the
 - a) reflexive closure of R .
 - b) symmetric closure of R .
2. Let R be the relation $\{(a, b) \mid a \neq b\}$ on the set of integers. What is the reflexive closure of R ?
3. Let R be the relation $\{(a, b) \mid a \text{ divides } b\}$ on the set of integers. What is the symmetric closure of R ?
4. How can the directed graph representing the reflexive closure of a relation on a finite set be constructed from the directed graph of the relation?

In Exercises 5–7 draw the directed graph of the reflexive closure of the relations with the directed graph shown.



- 7.
8. How can the directed graph representing the symmetric closure of a relation on a finite set be constructed from the directed graph for this relation?
9. Find the directed graphs of the symmetric closures of the relations with directed graphs shown in Exercises 5–7.
10. Find the smallest relation containing the relation in Example 2 that is both reflexive and symmetric.
11. Find the directed graph of the smallest relation that is both reflexive and symmetric that contains each of the relations with directed graphs shown in Exercises 5–7.
12. Suppose that the relation R on the finite set A is represented by the matrix \mathbf{M}_R . Show that the matrix that represents the reflexive closure of R is $\mathbf{M}_R \vee \mathbf{I}_n$.

13. Suppose that the relation R on the finite set A is represented by the matrix \mathbf{M}_R . Show that the matrix that represents the symmetric closure of R is $\mathbf{M}_R \vee \mathbf{M}_R^T$.
14. Show that the closure of a relation R with respect to a property \mathbf{P} , if it exists, is the intersection of all the relations with property \mathbf{P} that contain R .
15. When is it possible to define the “irreflexive closure” of a relation R , that is, a relation that contains R , is irreflexive, and is contained in every irreflexive relation that contains R ?
16. Determine whether these sequences of vertices are paths in this directed graph.
- a) a, b, c, e
b) b, e, c, b, e
c) a, a, b, e, d, e
d) b, c, e, d, a, a, b
e) b, c, c, b, e, d, e, d
f) a, a, b, b, c, b, e, d
- 
17. Find all circuits of length three in the directed graph in Exercise 16.
18. Determine whether there is a path in the directed graph in Exercise 16 beginning at the first vertex given and ending at the second vertex given.
- a) a, b b) b, a c) b, b
d) a, e e) b, d f) c, d
g) d, d h) e, a i) e, c
19. Let R be the relation on the set $\{1, 2, 3, 4, 5\}$ containing the ordered pairs $(1, 3), (2, 4), (3, 1), (3, 5), (4, 3), (5, 1), (5, 2)$, and $(5, 4)$. Find
- a) R^2 . b) R^3 . c) R^4 .
d) R^5 . e) R^6 . f) R^* .
20. Let R be the relation that contains the pair (a, b) if a and b are cities such that there is a direct non-stop airline flight from a to b . When is (a, b) in
- a) R^2 ? b) R^3 ? c) R^* ?
21. Let R be the relation on the set of all students containing the ordered pair (a, b) if a and b are in at least one common class and $a \neq b$. When is (a, b) in
- a) R^2 ? b) R^3 ? c) R^* ?
22. Suppose that the relation R is reflexive. Show that R^* is reflexive.
23. Suppose that the relation R is symmetric. Show that R^* is symmetric.
24. Suppose that the relation R is irreflexive. Is the relation R^2 necessarily irreflexive?
25. Use Algorithm 1 to find the transitive closures of these relations on $\{1, 2, 3, 4\}$.
- a) $\{(1, 2), (2, 1), (2, 3), (3, 4), (4, 1)\}$
b) $\{(2, 1), (2, 3), (3, 1), (3, 4), (4, 1), (4, 3)\}$
c) $\{(1, 2), (1, 3), (1, 4), (2, 3), (2, 4), (3, 4)\}$
d) $\{(1, 1), (1, 4), (2, 1), (2, 3), (3, 1), (3, 2), (3, 4), (4, 2)\}$
26. Use Algorithm 1 to find the transitive closures of these relations on $\{a, b, c, d, e\}$.
- a) $\{(a, c), (b, d), (c, a), (d, b), (e, d)\}$
b) $\{(b, c), (b, e), (c, e), (d, a), (e, b), (e, c)\}$
c) $\{(a, b), (a, c), (a, e), (b, a), (b, c), (c, a), (c, b), (d, a), (e, d)\}$
d) $\{(a, e), (b, a), (b, d), (c, d), (d, a), (d, c), (e, a), (e, b), (e, c), (e, e)\}$
27. Use Warshall's algorithm to find the transitive closures of the relations in Exercise 25.
28. Use Warshall's algorithm to find the transitive closures of the relations in Exercise 26.
29. Find the smallest relation containing the relation $\{(1, 2), (1, 4), (3, 3), (4, 1)\}$ that is
- a) reflexive and transitive.
b) symmetric and transitive.
c) reflexive, symmetric, and transitive.
30. Finish the proof of the case when $a \neq b$ in Lemma 1.
31. Algorithms have been devised that use $O(n^{2.8})$ bit operations to compute the Boolean product of two $n \times n$ zero-one matrices. Assuming that these algorithms can be used, give big- O estimates for the number of bit operations using Algorithm 1 and using Warshall's algorithm to find the transitive closure of a relation on a set with n elements.
- *32. Devise an algorithm using the concept of interior vertices in a path to find the length of the shortest path between two vertices in a directed graph, if such a path exists.
33. Adapt Algorithm 1 to find the reflexive closure of the transitive closure of a relation on a set with n elements.
34. Adapt Warshall's algorithm to find the reflexive closure of the transitive closure of a relation on a set with n elements.
35. Show that the closure with respect to the property \mathbf{P} of the relation $R = \{(0, 0), (0, 1), (1, 1), (2, 2)\}$ on the set $\{0, 1, 2\}$ does not exist if \mathbf{P} is the property
- a) “is not reflexive.”
b) “has an odd number of elements.”

9.5 Equivalence Relations

Introduction

In some programming languages the names of variables can contain an unlimited number of characters. However, there is a limit on the number of characters that are checked when a compiler determines whether two variables are equal. For instance, in traditional C, only the first eight characters of a variable name are checked by the compiler. (These characters are

uppercase or lowercase letters, digits, or underscores.) Consequently, the compiler considers strings longer than eight characters that agree in their first eight characters the same. Let R be the relation on the set of strings of characters such that sRt , where s and t are two strings, if s and t are at least eight characters long and the first eight characters of s and t agree, or $s = t$. It is easy to see that R is reflexive, symmetric, and transitive. Moreover, R divides the set of all strings into classes, where all strings in a particular class are considered the same by a compiler for traditional C.

The integers a and b are related by the “congruence modulo 4” relation when 4 divides $a - b$. We will show later that this relation is reflexive, symmetric, and transitive. It is not hard to see that a is related to b if and only if a and b have the same remainder when divided by 4. It follows that this relation splits the set of integers into four different classes. When we care only what remainder an integer leaves when it is divided by 4, we need only know which class it is in, not its particular value.

These two relations, R and congruence modulo 4, are examples of equivalence relations, namely, relations that are reflexive, symmetric, and transitive. In this section we will show that such relations split sets into disjoint classes of equivalent elements. Equivalence relations arise whenever we care only whether an element of a set is in a certain class of elements, instead of caring about its particular identity.

Equivalence Relations



In this section we will study relations with a particular combination of properties that allows them to be used to relate objects that are similar in some way.

DEFINITION 1

Equivalence relations are important in every branch of mathematics!

A relation on a set A is called an *equivalence relation* if it is reflexive, symmetric, and transitive.

Equivalence relations are important throughout mathematics and computer science. One reason for this is that in an equivalence relation, when two elements are related it makes sense to say they are equivalent.

DEFINITION 2

Two elements a and b that are related by an equivalence relation are called *equivalent*. The notation $a \sim b$ is often used to denote that a and b are equivalent elements with respect to a particular equivalence relation.

For the notion of equivalent elements to make sense, every element should be equivalent to itself, as the reflexive property guarantees for an equivalence relation. It makes sense to say that a and b are related (not just that a is related to b) by an equivalence relation, because when a is related to b , by the symmetric property, b is related to a . Furthermore, because an equivalence relation is transitive, if a and b are equivalent and b and c are equivalent, it follows that a and c are equivalent.

Examples 1–5 illustrate the notion of an equivalence relation.

EXAMPLE 1

Let R be the relation on the set of integers such that aRb if and only if $a = b$ or $a = -b$. In Section 9.1 we showed that R is reflexive, symmetric, and transitive. It follows that R is an equivalence relation. ◀

EXAMPLE 2

Let R be the relation on the set of real numbers such that aRb if and only if $a - b$ is an integer. Is R an equivalence relation?



Solution: Because $a - a = 0$ is an integer for all real numbers a , aRa for all real numbers a . Hence, R is reflexive. Now suppose that aRb . Then $a - b$ is an integer, so $b - a$ is also an integer. Hence, bRa . It follows that R is symmetric. If aRb and bRc , then $a - b$ and $b - c$ are integers. Therefore, $a - c = (a - b) + (b - c)$ is also an integer. Hence, aRc . Thus, R is transitive. Consequently, R is an equivalence relation. ◀

One of the most widely used equivalence relations is congruence modulo m , where m is an integer greater than 1.

EXAMPLE 3 Congruence Modulo m Let m be an integer with $m > 1$. Show that the relation

$$R = \{(a, b) \mid a \equiv b \pmod{m}\}$$

is an equivalence relation on the set of integers.

Solution: Recall from Section 4.1 that $a \equiv b \pmod{m}$ if and only if m divides $a - b$. Note that $a - a = 0$ is divisible by m , because $0 = 0 \cdot m$. Hence, $a \equiv a \pmod{m}$, so congruence modulo m is reflexive. Now suppose that $a \equiv b \pmod{m}$. Then $a - b$ is divisible by m , so $a - b = km$, where k is an integer. It follows that $b - a = (-k)m$, so $b \equiv a \pmod{m}$. Hence, congruence modulo m is symmetric. Next, suppose that $a \equiv b \pmod{m}$ and $b \equiv c \pmod{m}$. Then m divides both $a - b$ and $b - c$. Therefore, there are integers k and l with $a - b = km$ and $b - c = lm$. Adding these two equations shows that $a - c = (a - b) + (b - c) = km + lm = (k + l)m$. Thus, $a \equiv c \pmod{m}$. Therefore, congruence modulo m is transitive. It follows that congruence modulo m is an equivalence relation. ◀

EXAMPLE 4 Suppose that R is the relation on the set of strings of English letters such that aRb if and only if $l(a) = l(b)$, where $l(x)$ is the length of the string x . Is R an equivalence relation?

Solution: Because $l(a) = l(a)$, it follows that aRa whenever a is a string, so that R is reflexive. Next, suppose that aRb , so that $l(a) = l(b)$. Then bRa , because $l(b) = l(a)$. Hence, R is symmetric. Finally, suppose that aRb and bRc . Then $l(a) = l(b)$ and $l(b) = l(c)$. Hence, $l(a) = l(c)$, so aRc . Consequently, R is transitive. Because R is reflexive, symmetric, and transitive, it is an equivalence relation. ◀

EXAMPLE 5 Let n be a positive integer and S a set of strings. Suppose that R_n is the relation on S such that $sR_n t$ if and only if $s = t$, or both s and t have at least n characters and the first n characters of s and t are the same. That is, a string of fewer than n characters is related only to itself; a string s with at least n characters is related to a string t if and only if t has at least n characters and t begins with the n characters at the start of s . For example, let $n = 3$ and let S be the set of all bit strings. Then $sR_3 t$ either when $s = t$ or both s and t are bit strings of length 3 or more that begin with the same three bits. For instance, $01R_3 01$ and $00111R_3 00101$, but $01R_3 010$ and $01011R_3 01110$.

Show that for every set S of strings and every positive integer n , R_n is an equivalence relation on S .

Solution: The relation R_n is reflexive because $s = s$, so that $sR_n s$ whenever s is a string in S . If $sR_n t$, then either $s = t$ or s and t are both at least n characters long that begin with the same n characters. This means that $tR_n s$. We conclude that R_n is symmetric.

Now suppose that $sR_n t$ and $tR_n u$. Then either $s = t$ or s and t are at least n characters long and s and t begin with the same n characters, and either $t = u$ or t and u are at least n characters long and t and u begin with the same n characters. From this, we can deduce that either $s = u$ or both s and u are n characters long and s and u begin with the same n characters (because in this case we know that s , t , and u are all at least n characters long and both s and u begin with the same n characters as t does). Consequently, R_n is transitive. It follows that R_n is an equivalence relation. ◀

In Examples 6 and 7 we look at two relations that are not equivalence relations.

EXAMPLE 6 Show that the “divides” relation is the set of positive integers is not an equivalence relation.

Solution: By Examples 9 and 15 in Section 9.1, we know that the “divides” relation is reflexive and transitive. However, by Example 12 in Section 9.1, we know that this relation is not symmetric (for instance, $2 \mid 4$ but $4 \nmid 2$). We conclude that the “divides” relation on the set of positive integers is not an equivalence relation. ◀

EXAMPLE 7 Let R be the relation on the set of real numbers such that $x R y$ if and only if x and y are real numbers that differ by less than 1, that is $|x - y| < 1$. Show that R is not an equivalence relation.

Solution: R is reflexive because $|x - x| = 0 < 1$ whenever $x \in \mathbf{R}$. R is symmetric, for if $x R y$, where x and y are real numbers, then $|x - y| < 1$, which tells us that $|y - x| = |x - y| < 1$, so that $y Rx$. However, R is not an equivalence relation because it is not transitive. Take $x = 2.8$, $y = 1.9$, and $z = 1.1$, so that $|x - y| = |2.8 - 1.9| = 0.9 < 1$, $|y - z| = |1.9 - 1.1| = 0.8 < 1$, but $|x - z| = |2.8 - 1.1| = 1.7 > 1$. That is, $2.8 R 1.9$, $1.9 R 1.1$, but $2.8 \not R 1.1$. ◀

Equivalence Classes

Let A be the set of all students in your school who graduated from high school. Consider the relation R on A that consists of all pairs (x, y) , where x and y graduated from the same high school. Given a student x , we can form the set of all students equivalent to x with respect to R . This set consists of all students who graduated from the same high school as x did. This subset of A is called an equivalence class of the relation.

DEFINITION 3

Let R be an equivalence relation on a set A . The set of all elements that are related to an element a of A is called the *equivalence class* of a . The equivalence class of a with respect to R is denoted by $[a]_R$. When only one relation is under consideration, we can delete the subscript R and write $[a]$ for this equivalence class.

In other words, if R is an equivalence relation on a set A , the equivalence class of the element a is

$$[a]_R = \{s \mid (a, s) \in R\}.$$

If $b \in [a]_R$, then b is called a **representative** of this equivalence class. Any element of a class can be used as a representative of this class. That is, there is nothing special about the particular element chosen as the representative of the class.

EXAMPLE 8 What is the equivalence class of an integer for the equivalence relation of Example 1?

Solution: Because an integer is equivalent to itself and its negative in this equivalence relation, it follows that $[a] = \{-a, a\}$. This set contains two distinct integers unless $a = 0$. For instance, $[7] = \{-7, 7\}$, $[-5] = \{-5, 5\}$, and $[0] = \{0\}$. ◀

EXAMPLE 9 What are the equivalence classes of 0 and 1 for congruence modulo 4?

Solution: The equivalence class of 0 contains all integers a such that $a \equiv 0 \pmod{4}$. The integers in this class are those divisible by 4. Hence, the equivalence class of 0 for this relation is

$$[0] = \{\dots, -8, -4, 0, 4, 8, \dots\}.$$

The equivalence class of 1 contains all the integers a such that $a \equiv 1 \pmod{4}$. The integers in this class are those that have a remainder of 1 when divided by 4. Hence, the equivalence class of 1 for this relation is

$$[1] = \{\dots, -7, -3, 1, 5, 9, \dots\}.$$

In Example 9 the equivalence classes of 0 and 1 with respect to congruence modulo 4 were found. Example 9 can easily be generalized, replacing 4 with any positive integer m . The equivalence classes of the relation congruence modulo m are called the **congruence classes modulo m** . The congruence class of an integer a modulo m is denoted by $[a]_m$, so $[a]_m = \{\dots, a - 2m, a - m, a, a + m, a + 2m, \dots\}$. For instance, from Example 9 it follows that $[0]_4 = \{\dots, -8, -4, 0, 4, 8, \dots\}$ and $[1]_4 = \{\dots, -7, -3, 1, 5, 9, \dots\}$.

EXAMPLE 10 What is the equivalence class of the string 0111 with respect to the equivalence relation R_3 from Example 5 on the set of all bit strings? (Recall that $s R_3 t$ if and only if s and t are bit strings with $s = t$ or s and t are strings of at least three bits that start with the same three bits.)

Solution: The bit strings equivalent to 0111 are the bit strings with at least three bits that begin with 011. These are the bit strings 011, 0110, 0111, 01100, 01101, 01110, 01111, and so on. Consequently,

$$[011]_{R_3} = \{011, 0110, 0111, 01100, 01101, 01110, 01111, \dots\}.$$

EXAMPLE 11 Identifiers in the C Programming Language In the C programming language, an **identifier** is the name of a variable, a function, or another type of entity. Each identifier is a nonempty string of characters where each character is a lowercase or an uppercase English letter, a digit, or an underscore, and the first character is a lowercase or an uppercase English letter. Identifiers can be any length. This allows developers to use as many characters as they want to name an entity, such as a variable. However, for compilers for some versions of C, there is a limit on the number of characters checked when two names are compared to see whether they refer to the same thing. For example, Standard C compilers consider two identifiers the same when they agree in their first 31 characters. Consequently, developers must be careful not to use identifiers with the same initial 31 characters for different things. We see that two identifiers are considered the same when they are related by the relation R_{31} in Example 5. Using Example 5, we know that R_{31} , on the set of all identifiers in Standard C, is an equivalence relation.

What are the equivalence classes of each of the identifiers Number_of_tropical_storms, Number_of_named_tropical_storms, and Number_of_named_tropical_storms_in_the_Atlantic_in_2005?

Solution: Note that when an identifier is less than 31 characters long, by the definition of R_{31} , its equivalence class contains only itself. Because the identifier Number_of_tropical_storms is 25 characters long, its equivalence class contains exactly one element, namely, itself.

The identifier Number_of_named_tropical_storms is exactly 31 characters long. An identifier is equivalent to it when it starts with these same 31 characters. Consequently, every identifier at least 31 characters long that starts with Number_of_named_tropical_storms is equivalent to this identifier. It follows that the equivalence class of Number_of_named_tropical_storms is the set of all identifiers that begin with the 31 characters Number_of_named_tropical_storms.

An identifier is equivalent to the Number_of_named_tropical_storms_in_the_Atlantic_in_2005 if and only if it begins with its first 31 characters. Because these characters are Number_of_named_tropical_storms, we see that an identifier is equivalent to Number_of_named_tropical_storms_in_the_Atlantic_in_2005 if and only if it is equivalent to Number_of_named_tropical_storms. It follows that these last two identifiers have the same equivalence class.

Equivalence Classes and Partitions

Let A be the set of students at your school who are majoring in exactly one subject, and let R be the relation on A consisting of pairs (x, y) , where x and y are students with the same major. Then R is an equivalence relation, as the reader should verify. We can see that R splits all students in A into a collection of disjoint subsets, where each subset contains students with a specified major. For instance, one subset contains all students majoring (just) in computer science, and a second subset contains all students majoring in history. Furthermore, these subsets are equivalence classes of R . This example illustrates how the equivalence classes of an equivalence relation partition a set into disjoint, nonempty subsets. We will make these notions more precise in the following discussion.

Let R be a relation on the set A . Theorem 1 shows that the equivalence classes of two elements of A are either identical or disjoint.

THEOREM 1

Let R be an equivalence relation on a set A . These statements for elements a and b of A are equivalent:

- (i) aRb
- (ii) $[a] = [b]$
- (iii) $[a] \cap [b] \neq \emptyset$

Proof: We first show that (i) implies (ii). Assume that aRb . We will prove that $[a] = [b]$ by showing $[a] \subseteq [b]$ and $[b] \subseteq [a]$. Suppose $c \in [a]$. Then aRc . Because aRb and R is symmetric, we know that bRa . Furthermore, because R is transitive and bRa and aRc , it follows that bRc . Hence, $c \in [b]$. This shows that $[a] \subseteq [b]$. The proof that $[b] \subseteq [a]$ is similar; it is left as an exercise for the reader.

Second, we will show that (ii) implies (iii). Assume that $[a] = [b]$. It follows that $[a] \cap [b] \neq \emptyset$ because $[a]$ is nonempty (because $a \in [a]$ because R is reflexive).

Next, we will show that (iii) implies (i). Suppose that $[a] \cap [b] \neq \emptyset$. Then there is an element c with $c \in [a]$ and $c \in [b]$. In other words, aRc and bRc . By the symmetric property, cRb . Then by transitivity, because aRc and cRb , we have aRb .

Because (i) implies (ii), (ii) implies (iii), and (iii) implies (i), the three statements, (i), (ii), and (iii), are equivalent. \square

We are now in a position to show how an equivalence relation *partitions* a set. Let R be an equivalence relation on a set A . The union of the equivalence classes of R is all of A , because an element a of A is in its own equivalence class, namely, $[a]_R$. In other words,

$$\bigcup_{a \in A} [a]_R = A.$$

In addition, from Theorem 1, it follows that these equivalence classes are either equal or disjoint, so

$$[a]_R \cap [b]_R = \emptyset,$$

when $[a]_R \neq [b]_R$.

These two observations show that the equivalence classes form a partition of A , because they split A into disjoint subsets. More precisely, a **partition** of a set S is a collection of disjoint nonempty subsets of S that have S as their union. In other words, the collection of subsets A_i , $i \in I$ (where I is an index set) forms a partition of S if and only if

$$A_i \neq \emptyset \text{ for } i \in I,$$

$$A_i \cap A_j = \emptyset \text{ when } i \neq j,$$

Recall that an *index set* is a set whose members label, or index, the elements of a set.

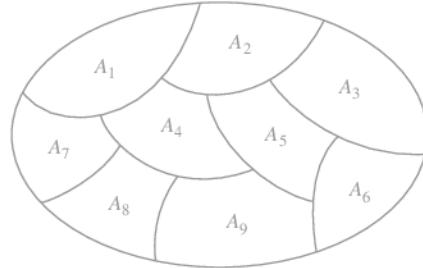


FIGURE 1 A Partition of a Set.

and

$$\bigcup_{i \in I} A_i = S.$$

(Here the notation $\bigcup_{i \in I} A_i$ represents the union of the sets A_i for all $i \in I$.) Figure 1 illustrates the concept of a partition of a set.

EXAMPLE 12 Suppose that $S = \{1, 2, 3, 4, 5, 6\}$. The collection of sets $A_1 = \{1, 2, 3\}$, $A_2 = \{4, 5\}$, and $A_3 = \{6\}$ forms a partition of S , because these sets are disjoint and their union is S . \blacktriangleleft

We have seen that the equivalence classes of an equivalence relation on a set form a partition of the set. The subsets in this partition are the equivalence classes. Conversely, every partition of a set can be used to form an equivalence relation. Two elements are equivalent with respect to this relation if and only if they are in the same subset of the partition.

To see this, assume that $\{A_i \mid i \in I\}$ is a partition on S . Let R be the relation on S consisting of the pairs (x, y) , where x and y belong to the same subset A_i in the partition. To show that R is an equivalence relation we must show that R is reflexive, symmetric, and transitive.

We see that $(a, a) \in R$ for every $a \in S$, because a is in the same subset as itself. Hence, R is reflexive. If $(a, b) \in R$, then a and b are in the same subset of the partition, so that $(b, a) \in R$ as well. Hence, R is symmetric. If $(a, b) \in R$ and $(b, c) \in R$, then a and b are in the same subset X in the partition, and b and c are in the same subset Y of the partition. Because the subsets of the partition are disjoint and b belongs to X and Y , it follows that $X = Y$. Consequently, a and c belong to the same subset of the partition, so $(a, c) \in R$. Thus, R is transitive.

It follows that R is an equivalence relation. The equivalence classes of R consist of subsets of S containing related elements, and by the definition of R , these are the subsets of the partition. Theorem 2 summarizes the connections we have established between equivalence relations and partitions.

THEOREM 2

Let R be an equivalence relation on a set S . Then the equivalence classes of R form a partition of S . Conversely, given a partition $\{A_i \mid i \in I\}$ of the set S , there is an equivalence relation R that has the sets A_i , $i \in I$, as its equivalence classes.

Example 13 shows how to construct an equivalence relation from a partition.

EXAMPLE 13 List the ordered pairs in the equivalence relation R produced by the partition $A_1 = \{1, 2, 3\}$, $A_2 = \{4, 5\}$, and $A_3 = \{6\}$ of $S = \{1, 2, 3, 4, 5, 6\}$, given in Example 12.

Solution: The subsets in the partition are the equivalence classes of R . The pair $(a, b) \in R$ if and only if a and b are in the same subset of the partition. The pairs $(1, 1), (1, 2), (1, 3), (2, 1), (2, 2), (2, 3), (3, 1), (3, 2)$, and $(3, 3)$ belong to R because $A_1 = \{1, 2, 3\}$ is an equivalence class; the pairs $(4, 4), (4, 5), (5, 4)$, and $(5, 5)$ belong to R because $A_2 = \{4, 5\}$ is an equivalence class; and finally the pair $(6, 6)$ belongs to R because $\{6\}$ is an equivalence class. No pair other than those listed belongs to R . \blacktriangleleft

The congruence classes modulo m provide a useful illustration of Theorem 2. There are m different congruence classes modulo m , corresponding to the m different remainders possible when an integer is divided by m . These m congruence classes are denoted by $[0]_m, [1]_m, \dots, [m - 1]_m$. They form a partition of the set of integers.

EXAMPLE 14 What are the sets in the partition of the integers arising from congruence modulo 4?

Solution: There are four congruence classes, corresponding to $[0]_4, [1]_4, [2]_4$, and $[3]_4$. They are the sets

$$\begin{aligned}[0]_4 &= \{\dots, -8, -4, 0, 4, 8, \dots\}, \\ [1]_4 &= \{\dots, -7, -3, 1, 5, 9, \dots\}, \\ [2]_4 &= \{\dots, -6, -2, 2, 6, 10, \dots\}, \\ [3]_4 &= \{\dots, -5, -1, 3, 7, 11, \dots\}.\end{aligned}$$

These congruence classes are disjoint, and every integer is in exactly one of them. In other words, as Theorem 2 says, these congruence classes form a partition. \blacktriangleleft

We now provide an example of a partition of the set of all strings arising from an equivalence relation on this set.

EXAMPLE 15 Let R_3 be the relation from Example 5. What are the sets in the partition of the set of all bit strings arising from the relation R_3 on the set of all bit strings? (Recall that $s R_3 t$, where s and t are bit strings, if $s = t$ or s and t are bit strings with at least three bits that agree in their first three bits.)

Solution: Note that every bit string of length less than three is equivalent only to itself. Hence $[\lambda]_{R_3} = \{\lambda\}, [0]_{R_3} = \{0\}, [1]_{R_3} = \{1\}, [00]_{R_3} = \{00\}, [01]_{R_3} = \{01\}, [10]_{R_3} = \{10\}$, and $[11]_{R_3} = \{11\}$. Note that every bit string of length three or more is equivalent to one of the eight bit strings 000, 001, 010, 011, 100, 101, 110, and 111. We have

$$\begin{aligned}[000]_{R_3} &= \{000, 0000, 0001, 00000, 00001, 00010, 00011, \dots\}, \\ [001]_{R_3} &= \{001, 0010, 0011, 00100, 00101, 00110, 00111, \dots\}, \\ [010]_{R_3} &= \{010, 0100, 0101, 01000, 01001, 01010, 01011, \dots\}, \\ [011]_{R_3} &= \{011, 0110, 0111, 01100, 01101, 01110, 01111, \dots\}, \\ [100]_{R_3} &= \{100, 1000, 1001, 10000, 10001, 10010, 10011, \dots\}, \\ [101]_{R_3} &= \{101, 1010, 1011, 10100, 10101, 10110, 10111, \dots\}, \\ [110]_{R_3} &= \{110, 1100, 1101, 11000, 11001, 11010, 11011, \dots\}, \\ [111]_{R_3} &= \{111, 1110, 1111, 11100, 11101, 11110, 11111, \dots\}.\end{aligned}$$

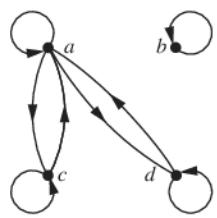
These 15 equivalence classes are disjoint and every bit string is in exactly one of them. As Theorem 2 tells us, these equivalence classes partition the set of all bit strings. \blacktriangleleft

Exercises

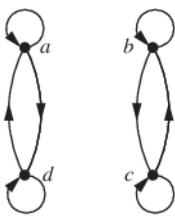
1. Which of these relations on $\{0, 1, 2, 3\}$ are equivalence relations? Determine the properties of an equivalence relation that the others lack.
 - a) $\{(0, 0), (1, 1), (2, 2), (3, 3)\}$
 - b) $\{(0, 0), (0, 2), (2, 0), (2, 2), (2, 3), (3, 2), (3, 3)\}$
 - c) $\{(0, 0), (1, 1), (1, 2), (2, 1), (2, 2), (3, 3)\}$
 - d) $\{(0, 0), (1, 1), (1, 3), (2, 2), (2, 3), (3, 1), (3, 2), (3, 3)\}$
 - e) $\{(0, 0), (0, 1), (0, 2), (1, 0), (1, 1), (1, 2), (2, 0), (2, 2), (3, 3)\}$
2. Which of these relations on the set of all people are equivalence relations? Determine the properties of an equivalence relation that the others lack.
 - a) $\{(a, b) \mid a \text{ and } b \text{ are the same age}\}$
 - b) $\{(a, b) \mid a \text{ and } b \text{ have the same parents}\}$
 - c) $\{(a, b) \mid a \text{ and } b \text{ share a common parent}\}$
 - d) $\{(a, b) \mid a \text{ and } b \text{ have met}\}$
 - e) $\{(a, b) \mid a \text{ and } b \text{ speak a common language}\}$
3. Which of these relations on the set of all functions from \mathbf{Z} to \mathbf{Z} are equivalence relations? Determine the properties of an equivalence relation that the others lack.
 - a) $\{(f, g) \mid f(1) = g(1)\}$
 - b) $\{(f, g) \mid f(0) = g(0) \text{ or } f(1) = g(1)\}$
 - c) $\{(f, g) \mid f(x) - g(x) = 1 \text{ for all } x \in \mathbf{Z}\}$
 - d) $\{(f, g) \mid \text{for some } C \in \mathbf{Z}, \text{ for all } x \in \mathbf{Z}, f(x) - g(x) = C\}$
 - e) $\{(f, g) \mid f(0) = g(1) \text{ and } f(1) = g(0)\}$
4. Define three equivalence relations on the set of students in your discrete mathematics class different from the relations discussed in the text. Determine the equivalence classes for each of these equivalence relations.
5. Define three equivalence relations on the set of buildings on a college campus. Determine the equivalence classes for each of these equivalence relations.
6. Define three equivalence relations on the set of classes offered at your school. Determine the equivalence classes for each of these equivalence relations.
7. Show that the relation of logical equivalence on the set of all compound propositions is an equivalence relation. What are the equivalence classes of \mathbf{F} and of \mathbf{T} ?
8. Let R be the relation on the set of all sets of real numbers such that $S R T$ if and only if S and T have the same cardinality. Show that R is an equivalence relation. What are the equivalence classes of the sets $\{0, 1, 2\}$ and \mathbf{Z} ?
9. Suppose that A is a nonempty set, and f is a function that has A as its domain. Let R be the relation on A consisting of all ordered pairs (x, y) such that $f(x) = f(y)$.
 - a) Show that R is an equivalence relation on A .
 - b) What are the equivalence classes of R ?
10. Suppose that A is a nonempty set and R is an equivalence relation on A . Show that there is a function f with A as its domain such that $(x, y) \in R$ if and only if $f(x) = f(y)$.
11. Show that the relation R consisting of all pairs (x, y) such that x and y are bit strings of length three or more that agree in their first three bits is an equivalence relation on the set of all bit strings of length three or more.
12. Show that the relation R consisting of all pairs (x, y) such that x and y are bit strings of length three or more that agree except perhaps in their first three bits is an equivalence relation on the set of all bit strings of length three or more.
13. Show that the relation R consisting of all pairs (x, y) such that x and y are bit strings that agree in their first and third bits is an equivalence relation on the set of all bit strings of length three or more.
14. Let R be the relation consisting of all pairs (x, y) such that x and y are strings of uppercase and lowercase English letters with the property that for every positive integer n , the n th characters in x and y are the same letter, either uppercase or lowercase. Show that R is an equivalence relation.
15. Let R be the relation on the set of ordered pairs of positive integers such that $((a, b), (c, d)) \in R$ if and only if $a + d = b + c$. Show that R is an equivalence relation.
16. Let R be the relation on the set of ordered pairs of positive integers such that $((a, b), (c, d)) \in R$ if and only if $ad = bc$. Show that R is an equivalence relation.
17. (Requires calculus)
 - a) Show that the relation R on the set of all differentiable functions from \mathbf{R} to \mathbf{R} consisting of all pairs (f, g) such that $f'(x) = g'(x)$ for all real numbers x is an equivalence relation.
 - b) Which functions are in the same equivalence class as the function $f(x) = x^2$?
18. (Requires calculus)
 - a) Let n be a positive integer. Show that the relation R on the set of all polynomials with real-valued coefficients consisting of all pairs (f, g) such that $f^{(n)}(x) = g^{(n)}(x)$ is an equivalence relation. [Here $f^{(n)}(x)$ is the n th derivative of $f(x)$.]
 - b) Which functions are in the same equivalence class as the function $f(x) = x^4$, where $n = 3$?
19. Let R be the relation on the set of all URLs (or Web addresses) such that $x R y$ if and only if the Web page at x is the same as the Web page at y . Show that R is an equivalence relation.
20. Let R be the relation on the set of all people who have visited a particular Web page such that $x R y$ if and only if person x and person y have followed the same set of links starting at this Web page (going from Web page to Web page until they stop using the Web). Show that R is an equivalence relation.

In Exercises 21–23 determine whether the relation with the directed graph shown is an equivalence relation.

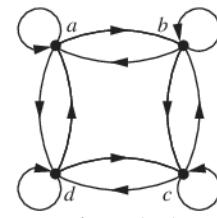
21.



22.



23.



24. Determine whether the relations represented by these zero-one matrices are equivalence relations.

a) $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$

b) $\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$

c) $\begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

25. Show that the relation R on the set of all bit strings such that $s R t$ if and only if s and t contain the same number of 1s is an equivalence relation.

26. What are the equivalence classes of the equivalence relations in Exercise 1?

27. What are the equivalence classes of the equivalence relations in Exercise 2?

28. What are the equivalence classes of the equivalence relations in Exercise 3?

29. What is the equivalence class of the bit string 011 for the equivalence relation in Exercise 25?

30. What are the equivalence classes of these bit strings for the equivalence relation in Exercise 11?

- a) 010 b) 1011 c) 11111 d) 01010101

31. What are the equivalence classes of the bit strings in Exercise 30 for the equivalence relation from Exercise 12?

32. What are the equivalence classes of the bit strings in Exercise 30 for the equivalence relation from Exercise 13?

33. What are the equivalence classes of the bit strings in Exercise 30 for the equivalence relation R_4 from Example 5 on the set of all bit strings? (Recall that bit strings s and t are equivalent under R_4 if and only if they are equal or they are both at least four bits long and agree in their first four bits.)

34. What are the equivalence classes of the bit strings in Exercise 30 for the equivalence relation R_5 from Example 5 on the set of all bit strings? (Recall that bit strings s and t are equivalent under R_5 if and only if they are equal or they are both at least five bits long and agree in their first five bits.)

35. What is the congruence class $[n]_5$ (that is, the equivalence class of n with respect to congruence modulo 5) when n is

- a) 2? b) 3? c) 6? d) -3?

36. What is the congruence class $[4]_m$ when m is

- a) 2? b) 3? c) 6? d) 8?

37. Give a description of each of the congruence classes modulo 6.

38. What is the equivalence class of each of these strings with respect to the equivalence relation in Exercise 14?

- a) No b) Yes c) Help

39. a) What is the equivalence class of (1, 2) with respect to the equivalence relation in Exercise 15?

b) Give an interpretation of the equivalence classes for the equivalence relation R in Exercise 15. [Hint: Look at the difference $a - b$ corresponding to (a, b) .]

40. a) What is the equivalence class of (1, 2) with respect to the equivalence relation in Exercise 16?

b) Give an interpretation of the equivalence classes for the equivalence relation R in Exercise 16. [Hint: Look at the ratio a/b corresponding to (a, b) .]

41. Which of these collections of subsets are partitions of $\{1, 2, 3, 4, 5, 6\}$?

- a) $\{1, 2\}, \{2, 3, 4\}, \{4, 5, 6\}$ b) $\{1\}, \{2, 3, 6\}, \{4\}, \{5\}$
c) $\{2, 4, 6\}, \{1, 3, 5\}$ d) $\{1, 4, 5\}, \{2, 6\}$

42. Which of these collections of subsets are partitions of $\{-3, -2, -1, 0, 1, 2, 3\}$?

- a) $\{-3, -1, 1, 3\}, \{-2, 0, 2\}$
b) $\{-3, -2, -1, 0\}, \{0, 1, 2, 3\}$
c) $\{-3, 3\}, \{-2, 2\}, \{-1, 1\}, \{0\}$
d) $\{-3, -2, 2, 3\}, \{-1, 1\}$

43. Which of these collections of subsets are partitions of the set of bit strings of length 8?

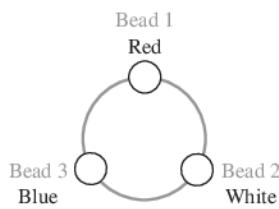
- a) the set of bit strings that begin with 1, the set of bit strings that begin with 00, and the set of bit strings that begin with 01
b) the set of bit strings that contain the string 00, the set of bit strings that contain the string 01, the set of bit strings that contain the string 10, and the set of bit strings that contain the string 11
c) the set of bit strings that end with 00, the set of bit strings that end with 01, the set of bit strings that end with 10, and the set of bit strings that end with 11
d) the set of bit strings that end with 111, the set of bit strings that end with 011, and the set of bit strings that end with 00
e) the set of bit strings that contain $3k$ ones for some nonnegative integer k ; the set of bit strings that contain $3k + 1$ ones for some nonnegative integer k ; and the set of bit strings that contain $3k + 2$ ones for some nonnegative integer k .

44. Which of these collections of subsets are partitions of the set of integers?

- a) the set of even integers and the set of odd integers
b) the set of positive integers and the set of negative integers

- c) the set of integers divisible by 3, the set of integers leaving a remainder of 1 when divided by 3, and the set of integers leaving a remainder of 2 when divided by 3
- d) the set of integers less than -100 , the set of integers with absolute value not exceeding 100, and the set of integers greater than 100
- e) the set of integers not divisible by 3, the set of even integers, and the set of integers that leave a remainder of 3 when divided by 6
- 45.** Which of these are partitions of the set $\mathbf{Z} \times \mathbf{Z}$ of ordered pairs of integers?
- the set of pairs (x, y) , where x or y is odd; the set of pairs (x, y) , where x is even; and the set of pairs (x, y) , where y is even
 - the set of pairs (x, y) , where both x and y are odd; the set of pairs (x, y) , where exactly one of x and y is odd; and the set of pairs (x, y) , where both x and y are even
 - the set of pairs (x, y) , where x is positive; the set of pairs (x, y) , where y is positive; and the set of pairs (x, y) , where both x and y are negative
 - the set of pairs (x, y) , where $3 \mid x$ and $3 \mid y$; the set of pairs (x, y) , where $3 \mid x$ and $3 \nmid y$; the set of pairs (x, y) , where $3 \nmid x$ and $3 \mid y$; and the set of pairs (x, y) , where $3 \nmid x$ and $3 \nmid y$
 - the set of pairs (x, y) , where $x > 0$ and $y > 0$; the set of pairs (x, y) , where $x > 0$ and $y \leq 0$; the set of pairs (x, y) , where $x \leq 0$ and $y > 0$; and the set of pairs (x, y) , where $x \leq 0$ and $y \leq 0$
 - the set of pairs (x, y) , where $x \neq 0$ and $y \neq 0$; the set of pairs (x, y) , where $x = 0$ and $y \neq 0$; and the set of pairs (x, y) , where $x \neq 0$ and $y = 0$
- 46.** Which of these are partitions of the set of real numbers?
- the negative real numbers, $\{0\}$, the positive real numbers
 - the set of irrational numbers, the set of rational numbers
 - the set of intervals $[k, k+1]$, $k = \dots, -2, -1, 0, 1, 2, \dots$
 - the set of intervals $(k, k+1)$, $k = \dots, -2, -1, 0, 1, 2, \dots$
 - the set of intervals $(k, k+1]$, $k = \dots, -2, -1, 0, 1, 2, \dots$
 - the sets $\{x + n \mid n \in \mathbf{Z}\}$ for all $x \in [0, 1]$
- 47.** List the ordered pairs in the equivalence relations produced by these partitions of $\{0, 1, 2, 3, 4, 5\}$.
- $\{0\}, \{1, 2\}, \{3, 4, 5\}$
 - $\{0, 1\}, \{2, 3\}, \{4, 5\}$
 - $\{0, 1, 2\}, \{3, 4, 5\}$
 - $\{0\}, \{1\}, \{2\}, \{3\}, \{4\}, \{5\}$
- 48.** List the ordered pairs in the equivalence relations produced by these partitions of $\{a, b, c, d, e, f, g\}$.
- $\{a, b\}, \{c, d\}, \{e, f, g\}$
- b) $\{a\}, \{b\}, \{c, d\}, \{e, f\}, \{g\}$
 c) $\{a, b, c, d\}, \{e, f, g\}$
 d) $\{a, c, e, g\}, \{b, d\}, \{f\}$
- A partition P_1 is called a **refinement** of the partition P_2 if every set in P_1 is a subset of one of the sets in P_2 .
- 49.** Show that the partition formed from congruence classes modulo 6 is a refinement of the partition formed from congruence classes modulo 3.
- 50.** Show that the partition of the set of people living in the United States consisting of subsets of people living in the same county (or parish) and same state is a refinement of the partition consisting of subsets of people living in the same state.
- 51.** Show that the partition of the set of bit strings of length 16 formed by equivalence classes of bit strings that agree on the last eight bits is a refinement of the partition formed from the equivalence classes of bit strings that agree on the last four bits.
- In Exercises 52 and 53, R_n refers to the family of equivalence relations defined in Example 5. Recall that $s R_n t$, where s and t are two strings if $s = t$ or s and t are strings with at least n characters that agree in their first n characters.
- 52.** Show that the partition of the set of all bit strings formed by equivalence classes of bit strings with respect to the equivalence relation R_4 is a refinement of the partition formed by equivalence classes of bit strings with respect to the equivalence relation R_3 .
- 53.** Show that the partition of the set of all identifiers in C formed by the equivalence classes of identifiers with respect to the equivalence relation R_{31} is a refinement of the partition formed by equivalence classes of identifiers with respect to the equivalence relation R_8 . (Compilers for “old” C consider identifiers the same when their names agree in their first eight characters, while compilers in standard C consider identifiers the same when their names agree in their first 31 characters.)
- 54.** Suppose that R_1 and R_2 are equivalence relations on a set A . Let P_1 and P_2 be the partitions that correspond to R_1 and R_2 , respectively. Show that $R_1 \subseteq R_2$ if and only if P_1 is a refinement of P_2 .
- 55.** Find the smallest equivalence relation on the set $\{a, b, c, d, e\}$ containing the relation $\{(a, b), (a, c), (d, e)\}$.
- 56.** Suppose that R_1 and R_2 are equivalence relations on the set S . Determine whether each of these combinations of R_1 and R_2 must be an equivalence relation.
- $R_1 \cup R_2$
 - $R_1 \cap R_2$
 - $R_1 \oplus R_2$
- 57.** Consider the equivalence relation from Example 2, namely, $R = \{(x, y) \mid x - y \text{ is an integer}\}$.
- What is the equivalence class of 1 for this equivalence relation?
 - What is the equivalence class of $1/2$ for this equivalence relation?

- *58. Each bead on a bracelet with three beads is either red, white, or blue, as illustrated in the figure shown.



Define the relation R between bracelets as: (B_1, B_2) , where B_1 and B_2 are bracelets, belongs to R if and only if B_2 can be obtained from B_1 by rotating it or rotating it and then reflecting it.

- a) Show that R is an equivalence relation.
 - b) What are the equivalence classes of R ?
- *59. Let R be the relation on the set of all colorings of the 2×2 checkerboard where each of the four squares is colored either red or blue so that (C_1, C_2) , where C_1 and C_2 are 2×2 checkerboards with each of their four squares colored blue or red, belongs to R if and only if C_2 can be obtained from C_1 either by rotating the checkerboard or by rotating it and then reflecting it.
- a) Show that R is an equivalence relation.
 - b) What are the equivalence classes of R ?
60. a) Let R be the relation on the set of functions from \mathbf{Z}^+ to \mathbf{Z}^+ such that (f, g) belongs to R if and only if f is $\Theta(g)$ (see Section 3.2). Show that R is an equivalence relation.
- b) Describe the equivalence class containing $f(n) = n^2$ for the equivalence relation of part (a).

61. Determine the number of different equivalence relations on a set with three elements by listing them.
62. Determine the number of different equivalence relations on a set with four elements by listing them.
- *63. Do we necessarily get an equivalence relation when we form the transitive closure of the symmetric closure of the reflexive closure of a relation?
- *64. Do we necessarily get an equivalence relation when we form the symmetric closure of the reflexive closure of the transitive closure of a relation?
65. Suppose we use Theorem 2 to form a partition P from an equivalence relation R . What is the equivalence relation R' that results if we use Theorem 2 again to form an equivalence relation from P ?
66. Suppose we use Theorem 2 to form an equivalence relation R from a partition P . What is the partition P' that results if we use Theorem 2 again to form a partition from R ?
67. Devise an algorithm to find the smallest equivalence relation containing a given relation.
- *68. Let $p(n)$ denote the number of different equivalence relations on a set with n elements (and by Theorem 2 the number of partitions of a set with n elements). Show that $p(n)$ satisfies the recurrence relation $p(n) = \sum_{j=0}^{n-1} C(n-1, j)p(n-j-1)$ and the initial condition $p(0) = 1$. (Note: The numbers $p(n)$ are called **Bell numbers** after the American mathematician E. T. Bell.)
69. Use Exercise 68 to find the number of different equivalence relations on a set with n elements, where n is a positive integer not exceeding 10.

9.6 Partial Orderings

Introduction

We often use relations to order some or all of the elements of sets. For instance, we order words using the relation containing pairs of words (x, y) , where x comes before y in the dictionary. We schedule projects using the relation consisting of pairs (x, y) , where x and y are tasks in a project such that x must be completed before y begins. We order the set of integers using the relation containing the pairs (x, y) , where x is less than y . When we add all of the pairs of the form (x, x) to these relations, we obtain a relation that is reflexive, antisymmetric, and transitive. These are properties that characterize relations used to order the elements of sets.



DEFINITION 1

A relation R on a set S is called a *partial ordering* or *partial order* if it is reflexive, antisymmetric, and transitive. A set S together with a partial ordering R is called a *partially ordered set*, or *poset*, and is denoted by (S, R) . Members of S are called *elements* of the poset.

We give examples of posets in Examples 1–3.

EXAMPLE 1 Show that the “greater than or equal” relation (\geq) is a partial ordering on the set of integers.



Solution: Because $a \geq a$ for every integer a , \geq is reflexive. If $a \geq b$ and $b \geq a$, then $a = b$. Hence, \geq is antisymmetric. Finally, \geq is transitive because $a \geq b$ and $b \geq c$ imply that $a \geq c$. It follows that \geq is a partial ordering on the set of integers and (\mathbb{Z}, \geq) is a poset. ◀

EXAMPLE 2 The divisibility relation $|$ is a partial ordering on the set of positive integers, because it is reflexive, antisymmetric, and transitive, as was shown in Section 9.1. We see that $(\mathbb{Z}^+, |)$ is a poset. Recall that $(\mathbb{Z}^+$ denotes the set of positive integers.) ◀

EXAMPLE 3 Show that the inclusion relation \subseteq is a partial ordering on the power set of a set S .

Solution: Because $A \subseteq A$ whenever A is a subset of S , \subseteq is reflexive. It is antisymmetric because $A \subseteq B$ and $B \subseteq A$ imply that $A = B$. Finally, \subseteq is transitive, because $A \subseteq B$ and $B \subseteq C$ imply that $A \subseteq C$. Hence, \subseteq is a partial ordering on $P(S)$, and $(P(S), \subseteq)$ is a poset. ◀

Example 4 illustrates a relation that is not a partial ordering.

EXAMPLE 4 Let R be the relation on the set of people such that $x R y$ if x and y are people and x is older than y . Show that R is not a partial ordering.



Solution: Note that R is antisymmetric because if a person x is older than a person y , then y is not older than x . That is, if $x R y$, then $y \not R x$. The relation R is transitive because if person x is older than person y and y is older than person z , then x is older than z . That is, if $x R y$ and $y R z$, then $x R z$. However, R is not reflexive, because no person is older than himself or herself. That is, $x \not R x$ for all people x . It follows that R is not a partial ordering. ◀

In different posets different symbols such as \leq , \subseteq , and $|$, are used for a partial ordering. However, we need a symbol that we can use when we discuss the ordering relation in an arbitrary poset. Customarily, the notation $a \preccurlyeq b$ is used to denote that $(a, b) \in R$ in an arbitrary poset (S, R) . This notation is used because the “less than or equal to” relation on the set of real numbers is the most familiar example of a partial ordering and the symbol \preccurlyeq is similar to the \leq symbol. (Note that the symbol \preccurlyeq is used to denote the relation in *any* poset, not just the “less than or equals” relation.) The notation $a \prec b$ denotes that $a \preccurlyeq b$, but $a \neq b$. Also, we say “ a is less than b ” or “ b is greater than a ” if $a \prec b$.

When a and b are elements of the poset (S, \preccurlyeq) , it is not necessary that either $a \preccurlyeq b$ or $b \preccurlyeq a$. For instance, in $(P(\mathbb{Z}), \subseteq)$, $\{1, 2\}$ is not related to $\{1, 3\}$, and vice versa, because neither set is contained within the other. Similarly, in $(\mathbb{Z}^+, |)$, 2 is not related to 3 and 3 is not related to 2, because $2 \nmid 3$ and $3 \nmid 2$. This leads to Definition 2.

DEFINITION 2

The elements a and b of a poset (S, \preccurlyeq) are called *comparable* if either $a \preccurlyeq b$ or $b \preccurlyeq a$. When a and b are elements of S such that neither $a \preccurlyeq b$ nor $b \preccurlyeq a$, a and b are called *incomparable*.

EXAMPLE 5 In the poset $(\mathbb{Z}^+, |)$, are the integers 3 and 9 comparable? Are 5 and 7 comparable?

Solution: The integers 3 and 9 are comparable, because $3 | 9$. The integers 5 and 7 are incomparable, because $5 \nmid 7$ and $7 \nmid 5$. ◀

The adjective “partial” is used to describe partial orderings because pairs of elements may be incomparable. When every two elements in the set are comparable, the relation is called a **total ordering**.

DEFINITION 3

If (S, \preccurlyeq) is a poset and every two elements of S are comparable, S is called a *totally ordered* or *linearly ordered set*, and \preccurlyeq is called a *total order* or a *linear order*. A totally ordered set is also called a *chain*.

EXAMPLE 6 The poset (\mathbf{Z}, \leq) is totally ordered, because $a \leq b$ or $b \leq a$ whenever a and b are integers. ◀

EXAMPLE 7 The poset $(\mathbf{Z}^+, |)$ is not totally ordered because it contains elements that are incomparable, such as 5 and 7. ◀

In Chapter 6 we noted that (\mathbf{Z}^+, \leq) is well-ordered, where \leq is the usual “less than or equal to” relation. We now define well-ordered sets.

DEFINITION 4

(S, \preceq) is a *well-ordered set* if it is a poset such that \preceq is a total ordering and every nonempty subset of S has a least element.

EXAMPLE 8 The set of ordered pairs of positive integers, $\mathbf{Z}^+ \times \mathbf{Z}^+$, with $(a_1, a_2) \preceq (b_1, b_2)$ if $a_1 < b_1$, or if $a_1 = b_1$ and $a_2 \leq b_2$ (the lexicographic ordering), is a well-ordered set. The verification of this is left as Exercise 53. The set \mathbf{Z} , with the usual \leq ordering, is not well-ordered because the set of negative integers, which is a subset of \mathbf{Z} , has no least element. ◀

At the end of Section 5.3 we showed how to use the principle of well-ordered induction (here called generalized induction) to prove results about a well-ordered set. We now state and prove that this proof technique is valid.

THEOREM 1

THE PRINCIPLE OF WELL-ORDERED INDUCTION Suppose that S is a well-ordered set. Then $P(x)$ is true for all $x \in S$, if

INDUCTIVE STEP: For every $y \in S$, if $P(x)$ is true for all $x \in S$ with $x \prec y$, then $P(y)$ is true.

Proof: Suppose it is not the case that $P(x)$ is true for all $x \in S$. Then there is an element $y \in S$ such that $P(y)$ is false. Consequently, the set $A = \{x \in S \mid P(x) \text{ is false}\}$ is nonempty. Because S is well ordered, A has a least element a . By the choice of a as a least element of A , we know that $P(x)$ is true for all $x \in S$ with $x \prec a$. This implies by the inductive step $P(a)$ is true. This contradiction shows that $P(x)$ must be true for all $x \in S$. ◀

Remark: We do not need a basis step in a proof using the principle of well-ordered induction because if x_0 is the least element of a well ordered set, the inductive step tells us that $P(x_0)$ is true. This follows because there are no elements $x \in S$ with $x \prec x_0$, so we know (using a vacuous proof) that $P(x)$ is true for all $x \in S$ with $x \prec x_0$.

The principle of well-ordered induction is a versatile technique for proving results about well-ordered sets. Even when it is possible to use mathematical induction for the set of positive integers to prove a theorem, it may be simpler to use the principle of well-ordered induction, as we saw in Examples 5 and 6 in Section 6.2, where we proved a result about the well-ordered set $(\mathbf{N} \times \mathbf{N}, \preceq)$ where \preceq is lexicographic ordering on $\mathbf{N} \times \mathbf{N}$.

Lexicographic Order

The words in a dictionary are listed in alphabetic, or lexicographic, order, which is based on the ordering of the letters in the alphabet. This is a special case of an ordering of strings on a set

constructed from a partial ordering on the set. We will show how this construction works in any poset.

First, we will show how to construct a partial ordering on the Cartesian product of two posets, (A_1, \preceq_1) and (A_2, \preceq_2) . The **lexicographic ordering** \preceq on $A_1 \times A_2$ is defined by specifying that one pair is less than a second pair if the first entry of the first pair is less than (in A_1) the first entry of the second pair, or if the first entries are equal, but the second entry of this pair is less than (in A_2) the second entry of the second pair. In other words, (a_1, a_2) is less than (b_1, b_2) , that is,

$$(a_1, a_2) \prec (b_1, b_2),$$

either if $a_1 \prec_1 b_1$ or if both $a_1 = b_1$ and $a_2 \prec_2 b_2$.

We obtain a partial ordering \preceq by adding equality to the ordering \prec on $A_1 \times A_2$. The verification of this is left as an exercise.

EXAMPLE 9 Determine whether $(3, 5) \prec (4, 8)$, whether $(3, 8) \prec (4, 5)$, and whether $(4, 9) \prec (4, 11)$ in the poset $(\mathbf{Z} \times \mathbf{Z}, \preceq)$, where \preceq is the lexicographic ordering constructed from the usual \leq relation on \mathbf{Z} .

Solution: Because $3 < 4$, it follows that $(3, 5) \prec (4, 8)$ and that $(3, 8) \prec (4, 5)$. We have $(4, 9) \prec (4, 11)$, because the first entries of $(4, 9)$ and $(4, 11)$ are the same but $9 < 11$. \blacktriangleleft

In Figure 1 the ordered pairs in $\mathbf{Z}^+ \times \mathbf{Z}^+$ that are less than $(3, 4)$ are highlighted. A lexicographic ordering can be defined on the Cartesian product of n posets (A_1, \preceq_1) , $(A_2, \preceq_2), \dots, (A_n, \preceq_n)$. Define the partial ordering \preceq on $A_1 \times A_2 \times \dots \times A_n$ by

$$(a_1, a_2, \dots, a_n) \prec (b_1, b_2, \dots, b_n)$$

if $a_1 \prec_1 b_1$, or if there is an integer $i > 0$ such that $a_1 = b_1, \dots, a_i = b_i$, and $a_{i+1} \prec_{i+1} b_{i+1}$. In other words, one n -tuple is less than a second n -tuple if the entry of the first n -tuple in the first position where the two n -tuples disagree is less than the entry in that position in the second n -tuple.

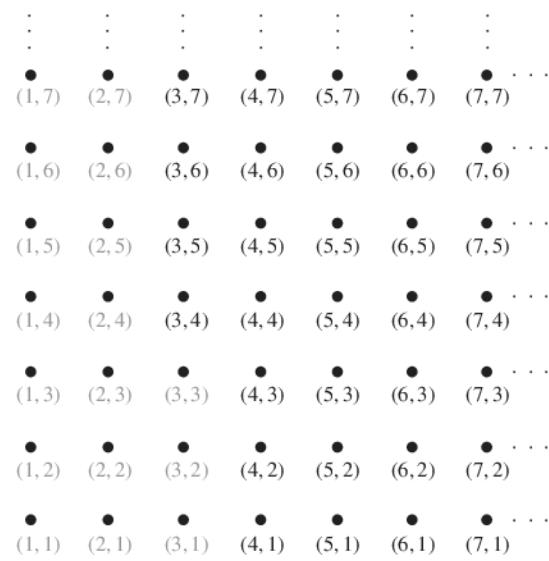


FIGURE 1 The Ordered Pairs Less Than $(3, 4)$ in Lexicographic Order.

EXAMPLE 10 Note that $(1, 2, 3, 5) \prec (1, 2, 4, 3)$, because the entries in the first two positions of these 4-tuples agree, but in the third position the entry in the first 4-tuple, 3, is less than that in the second 4-tuple, 4. (Here the ordering on 4-tuples is the lexicographic ordering that comes from the usual “less than or equals” relation on the set of integers.) \blacktriangleleft

We can now define lexicographic ordering of strings. Consider the strings $a_1a_2\dots a_m$ and $b_1b_2\dots b_n$ on a partially ordered set S . Suppose these strings are not equal. Let t be the minimum of m and n . The definition of lexicographic ordering is that the string $a_1a_2\dots a_m$ is less than $b_1b_2\dots b_n$ if and only if

$$(a_1, a_2, \dots, a_t) \prec (b_1, b_2, \dots, b_t), \text{ or} \\ (a_1, a_2, \dots, a_t) = (b_1, b_2, \dots, b_t) \text{ and } m < n,$$

where \prec in this inequality represents the lexicographic ordering of S^t . In other words, to determine the ordering of two different strings, the longer string is truncated to the length of the shorter string, namely, to $t = \min(m, n)$ terms. Then the t -tuples made up of the first t terms of each string are compared using the lexicographic ordering on S^t . One string is less than another string if the t -tuple corresponding to the first string is less than the t -tuple of the second string, or if these two t -tuples are the same, but the second string is longer. The verification that this is a partial ordering is left as Exercise 38 for the reader.

EXAMPLE 11 Consider the set of strings of lowercase English letters. Using the ordering of letters in the alphabet, a lexicographic ordering on the set of strings can be constructed. A string is less than a second string if the letter in the first string in the first position where the strings differ comes before the letter in the second string in this position, or if the first string and the second string agree in all positions, but the second string has more letters. This ordering is the same as that used in dictionaries. For example,

$$\textit{discreet} \prec \textit{discrete},$$

because these strings differ first in the seventh position, and $e \prec t$. Also,

$$\textit{discreet} \prec \textit{discreteness},$$

because the first eight letters agree, but the second string is longer. Furthermore,

$$\textit{discrete} \prec \textit{discretion},$$

because

$$\textit{discrete} \prec \textit{discreti}.$$



Hasse Diagrams

Many edges in the directed graph for a finite poset do not have to be shown because they must be present. For instance, consider the directed graph for the partial ordering $\{(a, b) \mid a \leq b\}$ on the set $\{1, 2, 3, 4\}$, shown in Figure 2(a). Because this relation is a partial ordering, it is reflexive, and its directed graph has loops at all vertices. Consequently, we do not have to show these loops because they must be present; in Figure 2(b) loops are not shown. Because a partial ordering is transitive, we do not have to show those edges that must be present because of transitivity. For example, in Figure 2(c) the edges $(1, 3)$, $(1, 4)$, and $(2, 4)$ are not shown because they must be present. If we assume that all edges are pointed “upward” (as they are drawn in the figure), we do not have to show the directions of the edges; Figure 2(c) does not show directions.

In general, we can represent a finite poset (S, \preceq) using this procedure: Start with the directed graph for this relation. Because a partial ordering is reflexive, a loop (a, a) is present at every vertex a . Remove these loops. Next, remove all edges that must be in the partial ordering because of the presence of other edges and transitivity. That is, remove all edges (x, y) for which there is an element $z \in S$ such that $x \prec z$ and $z \prec y$. Finally, arrange each edge so that

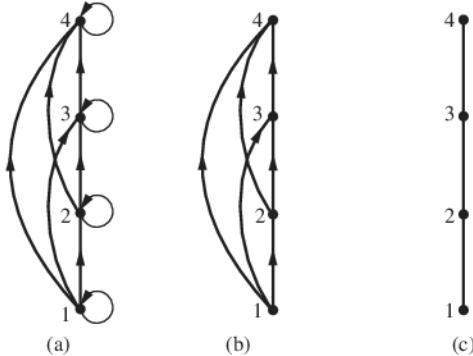


FIGURE 2 Constructing the Hasse Diagram for $\{1, 2, 3, 4\}$, \leq .



its initial vertex is below its terminal vertex (as it is drawn on paper). Remove all the arrows on the directed edges, because all edges point “upward” toward their terminal vertex.

These steps are well defined, and only a finite number of steps need to be carried out for a finite poset. When all the steps have been taken, the resulting diagram contains sufficient information to find the partial ordering, as we will explain later. The resulting diagram is called the **Hasse diagram** of (S, \preccurlyeq) , named after the twentieth-century German mathematician Helmut Hasse who made extensive use of them.

Let (S, \preccurlyeq) be a poset. We say that an element $y \in S$ **covers** an element $x \in S$ if $x \prec y$ and there is no element $z \in S$ such that $x \prec z \prec y$. The set of pairs (x, y) such that y covers x is called the **covering relation** of (S, \preccurlyeq) . From the description of the Hasse diagram of a poset, we see that the edges in the Hasse diagram of (S, \preccurlyeq) are upwardly pointing edges corresponding to the pairs in the covering relation of (S, \preccurlyeq) . Furthermore, we can recover a poset from its covering relation, because it is the reflexive transitive closure of its covering relation. (Exercise 31 asks for a proof of this fact.) This tells us that we can construct a partial ordering from its Hasse diagram.

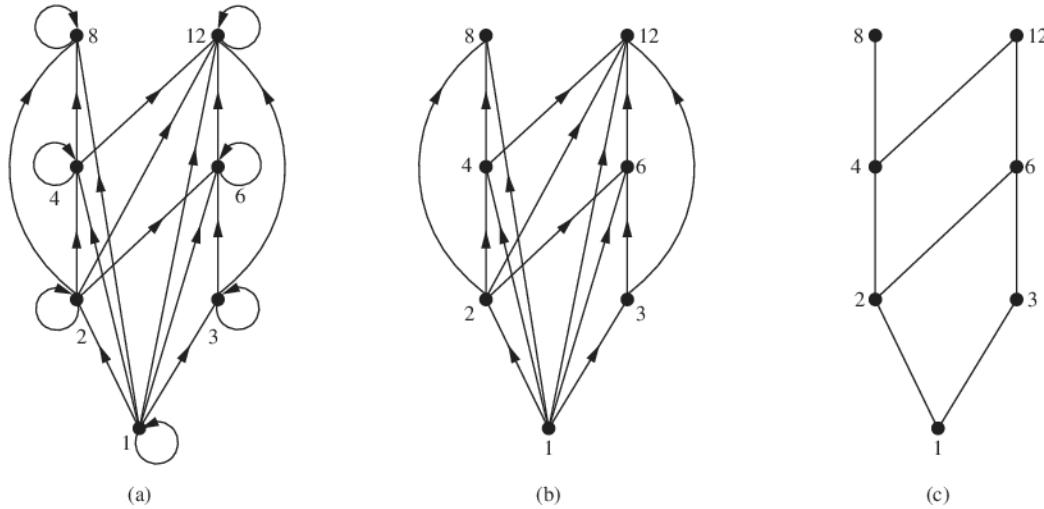
EXAMPLE 12 Draw the Hasse diagram representing the partial ordering $\{(a, b) | a \text{ divides } b\}$ on $\{1, 2, 3, 4, 6, 8, 12\}$.

Solution: Begin with the digraph for this partial order, as shown in Figure 3(a). Remove all loops, as shown in Figure 3(b). Then delete all the edges implied by the transitive property. These are $(1, 4)$, $(1, 6)$, $(1, 8)$, $(1, 12)$, $(2, 8)$, $(2, 12)$, and $(3, 12)$. Arrange all edges to point upward, and delete all arrows to obtain the Hasse diagram. The resulting Hasse diagram is shown in Figure 3(c). ◀

EXAMPLE 13 Draw the Hasse diagram for the partial ordering $\{(A, B) | A \subseteq B\}$ on the power set $P(S)$ where $S = \{a, b, c\}$.



HELmut HASSE (1898–1979) Helmut Hasse was born in Kassel, Germany. He served in the German navy after high school. He began his university studies at Göttingen University in 1918, moving in 1920 to Marburg University to study under the number theorist Kurt Hensel. During this time, Hasse made fundamental contributions to algebraic number theory. He became Hensel’s successor at Marburg, later becoming director of the famous mathematical institute at Göttingen in 1934, and took a position at Hamburg University in 1950. Hasse served for 50 years as an editor of *Crelle’s Journal*, a famous German mathematics periodical, taking over the job of chief editor in 1936 when the Nazis forced Hensel to resign. During World War II Hasse worked on applied mathematics research for the German navy. He was noted for the clarity and personal style of his lectures and was devoted both to number theory and to his students. (Hasse has been controversial for connections with the Nazi party. Investigations have shown he was a strong German nationalist but not an ardent Nazi.)

FIGURE 3 Constructing the Hasse Diagram of $(\{1, 2, 3, 4, 6, 8, 12\}, |)$.

Solution: The Hasse diagram for this partial ordering is obtained from the associated digraph by deleting all the loops and all the edges that occur from transitivity, namely, $(\emptyset, \{a, b\})$, $(\emptyset, \{a, c\})$, $(\emptyset, \{b, c\})$, $(\emptyset, \{a, b, c\})$, $(\{a\}, \{a, b, c\})$, $(\{b\}, \{a, b, c\})$, and $(\{c\}, \{a, b, c\})$. Finally all edges point upward, and arrows are deleted. The resulting Hasse diagram is illustrated in Figure 4. ◀

Maximal and Minimal Elements

Elements of posets that have certain extremal properties are important for many applications. An element of a poset is called maximal if it is not less than any element of the poset. That is, a is **maximal** in the poset (S, \preccurlyeq) if there is no $b \in S$ such that $a \prec b$. Similarly, an element of a poset is called minimal if it is not greater than any element of the poset. That is, a is **minimal** if there is no element $b \in S$ such that $b \prec a$. Maximal and minimal elements are easy to spot using a Hasse diagram. They are the “top” and “bottom” elements in the diagram.

EXAMPLE 14 Which elements of the poset $(\{2, 4, 5, 10, 12, 20, 25\}, |)$ are maximal, and which are minimal?

Solution: The Hasse diagram in Figure 5 for this poset shows that the maximal elements are 12, 20, and 25, and the minimal elements are 2 and 5. As this example shows, a poset can have more than one maximal element and more than one minimal element. ◀

Sometimes there is an element in a poset that is greater than every other element. Such an element is called the greatest element. That is, a is the **greatest element** of the poset (S, \preccurlyeq)

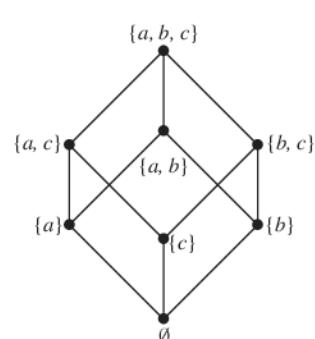
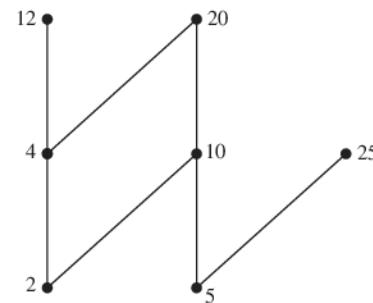
FIGURE 4 The Hasse Diagram of $(P(\{a, b, c\}), \subseteq)$.

FIGURE 5 The Hasse Diagram of a Poset.

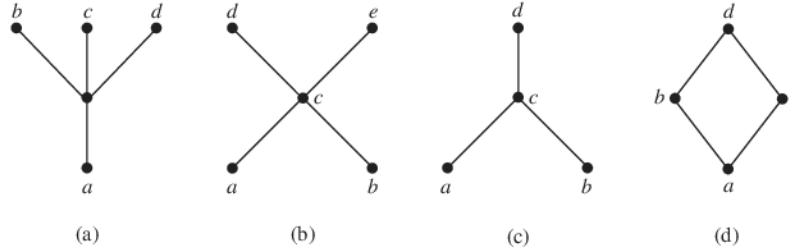


FIGURE 6 Hasse Diagrams of Four Posets.

if $b \preceq a$ for all $b \in S$. The greatest element is unique when it exists [see Exercise 40(a)]. Likewise, an element is called the least element if it is less than all the other elements in the poset. That is, a is the **least element** of (S, \preceq) if $a \preceq b$ for all $b \in S$. The least element is unique when it exists [see Exercise 40(b)].

EXAMPLE 15 Determine whether the posets represented by each of the Hasse diagrams in Figure 6 have a greatest element and a least element.

Solution: The least element of the poset with Hasse diagram (a) is a . This poset has no greatest element. The poset with Hasse diagram (b) has neither a least nor a greatest element. The poset with Hasse diagram (c) has no least element. Its greatest element is d . The poset with Hasse diagram (d) has least element a and greatest element d . ◀

EXAMPLE 16 Let S be a set. Determine whether there is a greatest element and a least element in the poset $(P(S), \subseteq)$.

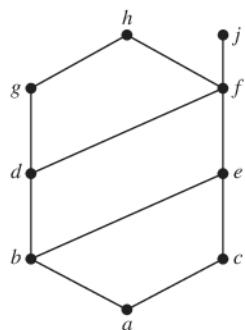
Solution: The least element is the empty set, because $\emptyset \subseteq T$ for any subset T of S . The set S is the greatest element in this poset, because $T \subseteq S$ whenever T is a subset of S . ◀

EXAMPLE 17 Is there a greatest element and a least element in the poset $(\mathbb{Z}^+, |)$?

Solution: The integer 1 is the least element because $1|n$ whenever n is a positive integer. Because there is no integer that is divisible by all positive integers, there is no greatest element. ◀

Sometimes it is possible to find an element that is greater than or equal to all the elements in a subset A of a poset (S, \preceq) . If u is an element of S such that $a \preceq u$ for all elements $a \in A$, then u is called an **upper bound** of A . Likewise, there may be an element less than or equal to all the elements in A . If l is an element of S such that $l \preceq a$ for all elements $a \in A$, then l is called a **lower bound** of A .

EXAMPLE 18 Find the lower and upper bounds of the subsets $\{a, b, c\}$, $\{j, h\}$, and $\{a, c, d, f\}$ in the poset with the Hasse diagram shown in Figure 7.



Solution: The upper bounds of $\{a, b, c\}$ are e, f, j , and h , and its only lower bound is a . There are no upper bounds of $\{j, h\}$, and its lower bounds are a, b, c, d, e , and f . The upper bounds of $\{a, c, d, f\}$ are f, h , and j , and its lower bound is a . ◀

The element x is called the **least upper bound** of the subset A if x is an upper bound that is less than every other upper bound of A . Because there is only one such element, if it exists, it makes sense to call this element *the* least upper bound [see Exercise 42(a)]. That is, x is the least upper bound of A if $a \preceq x$ whenever $a \in A$, and $x \preceq z$ whenever z is an upper bound of A . Similarly, the element y is called the **greatest lower bound** of A if y is a lower bound of A and $z \preceq y$ whenever z is a lower bound of A . The greatest lower bound of A is unique if it exists [see Exercise 42(b)]. The greatest lower bound and least upper bound of a subset A are denoted by $\text{glb}(A)$ and $\text{lub}(A)$, respectively.

FIGURE 7 The Hasse Diagram of a Poset.

EXAMPLE 19 Find the greatest lower bound and the least upper bound of $\{b, d, g\}$, if they exist, in the poset shown in Figure 7.

Solution: The upper bounds of $\{b, d, g\}$ are g and h . Because $g \prec h$, g is the least upper bound. The lower bounds of $\{b, d, g\}$ are a and b . Because $a \prec b$, b is the greatest lower bound. ◀

EXAMPLE 20 Find the greatest lower bound and the least upper bound of the sets $\{3, 9, 12\}$ and $\{1, 2, 4, 5, 10\}$, if they exist, in the poset $(\mathbb{Z}^+, |)$.



Solution: An integer is a lower bound of $\{3, 9, 12\}$ if 3, 9, and 12 are divisible by this integer. The only such integers are 1 and 3. Because $1 \mid 3$, 3 is the greatest lower bound of $\{3, 9, 12\}$. The only lower bound for the set $\{1, 2, 4, 5, 10\}$ with respect to $|$ is the element 1. Hence, 1 is the greatest lower bound for $\{1, 2, 4, 5, 10\}$.

An integer is an upper bound for $\{3, 9, 12\}$ if and only if it is divisible by 3, 9, and 12. The integers with this property are those divisible by the least common multiple of 3, 9, and 12, which is 36. Hence, 36 is the least upper bound of $\{3, 9, 12\}$. A positive integer is an upper bound for the set $\{1, 2, 4, 5, 10\}$ if and only if it is divisible by 1, 2, 4, 5, and 10. The integers with this property are those integers divisible by the least common multiple of these integers, which is 20. Hence, 20 is the least upper bound of $\{1, 2, 4, 5, 10\}$. ◀

Lattices

A partially ordered set in which every pair of elements has both a least upper bound and a greatest lower bound is called a **lattice**. Lattices have many special properties. Furthermore, lattices are used in many different applications such as models of information flow and play an important role in Boolean algebra.

EXAMPLE 21 Determine whether the posets represented by each of the Hasse diagrams in Figure 8 are lattices.

Solution: The posets represented by the Hasse diagrams in (a) and (c) are both lattices because in each poset every pair of elements has both a least upper bound and a greatest lower bound, as the reader should verify. On the other hand, the poset with the Hasse diagram shown in (b) is not a lattice, because the elements b and c have no least upper bound. To see this, note that each of the elements d , e , and f is an upper bound, but none of these three elements precedes the other two with respect to the ordering of this poset. ◀

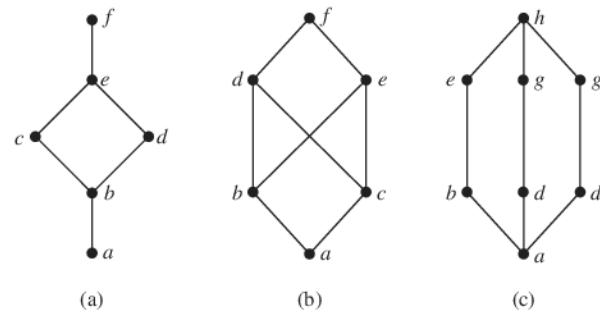


FIGURE 8 Hasse Diagrams of Three Posets.

EXAMPLE 22 Is the poset $(\mathbb{Z}^+, |)$ a lattice?

Solution: Let a and b be two positive integers. The least upper bound and greatest lower bound of these two integers are the least common multiple and the greatest common divisor of these integers, respectively, as the reader should verify. It follows that this poset is a lattice. ◀

EXAMPLE 23 Determine whether the posets $(\{1, 2, 3, 4, 5\}, |)$ and $(\{1, 2, 4, 8, 16\}, |)$ are lattices.

Solution: Because 2 and 3 have no upper bounds in $(\{1, 2, 3, 4, 5\}, |)$, they certainly do not have a least upper bound. Hence, the first poset is not a lattice.

Every two elements of the second poset have both a least upper bound and a greatest lower bound. The least upper bound of two elements in this poset is the larger of the elements and the greatest lower bound of two elements is the smaller of the elements, as the reader should verify. Hence, this second poset is a lattice. ◀

EXAMPLE 24 Determine whether $(P(S), \subseteq)$ is a lattice where S is a set.

Solution: Let A and B be two subsets of S . The least upper bound and the greatest lower bound of A and B are $A \cup B$ and $A \cap B$, respectively, as the reader can show. Hence, $(P(S), \subseteq)$ is a lattice. ◀

EXAMPLE 25



There are billions of pages of classified U.S. government documents.

The Lattice Model of Information Flow In many settings the flow of information from one person or computer program to another is restricted via security clearances. We can use a lattice model to represent different information flow policies. For example, one common information flow policy is the *multilevel security policy* used in government and military systems. Each piece of information is assigned to a security class, and each security class is represented by a pair (A, C) where A is an *authority level* and C is a *category*. People and computer programs are then allowed access to information from a specific restricted set of security classes.

The typical authority levels used in the U.S. government are unclassified (0), confidential (1), secret (2), and top secret (3). (Information is said to be classified if it is confidential, secret, or top secret.) Categories used in security classes are the subsets of a set of all *compartments* relevant to a particular area of interest. Each compartment represents a particular subject area. For example, if the set of compartments is $\{\text{spies, moles, double agents}\}$, then there are eight different categories, one for each of the eight subsets of the set of compartments, such as $\{\text{spies, moles}\}$.

We can order security classes by specifying that $(A_1, C_1) \preccurlyeq (A_2, C_2)$ if and only if $A_1 \leq A_2$ and $C_1 \subseteq C_2$. Information is permitted to flow from security class (A_1, C_1) into security class (A_2, C_2) if and only if $(A_1, C_1) \preccurlyeq (A_2, C_2)$. For example, information is permitted to flow from the security class $(\text{secret, } \{\text{spies, moles}\})$ into the security class $(\text{top secret, } \{\text{spies, moles, double agents}\})$, whereas information is not allowed to flow from the security class $(\text{top secret, } \{\text{spies, moles}\})$ into either of the security classes $(\text{secret, } \{\text{spies, moles, double agents}\})$ or $(\text{top secret, } \{\text{spies}\})$.

We leave it to the reader (see Exercise 48) to show that the set of all security classes with the ordering defined in this example forms a lattice. ◀

Topological Sorting

Suppose that a project is made up of 20 different tasks. Some tasks can be completed only after others have been finished. How can an order be found for these tasks? To model this problem we set up a partial order on the set of tasks so that $a \prec b$ if and only if a and b are tasks where b



cannot be started until a has been completed. To produce a schedule for the project, we need to produce an order for all 20 tasks that is compatible with this partial order. We will show how this can be done.

We begin with a definition. A total ordering \preccurlyeq is said to be **compatible** with the partial ordering R if $a \preccurlyeq b$ whenever aRb . Constructing a compatible total ordering from a partial ordering is called **topological sorting**.^{*} We will need to use Lemma 1.

LEMMA 1

Every finite nonempty poset (S, \preccurlyeq) has at least one minimal element.

Proof: Choose an element a_0 of S . If a_0 is not minimal, then there is an element a_1 with $a_1 \prec a_0$. If a_1 is not minimal, there is an element a_2 with $a_2 \prec a_1$. Continue this process, so that if a_n is not minimal, there is an element a_{n+1} with $a_{n+1} \prec a_n$. Because there are only a finite number of elements in the poset, this process must end with a minimal element a_n . \triangleleft

The topological sorting algorithm we will describe works for any finite nonempty poset. To define a total ordering on the poset (A, \preccurlyeq) , first choose a minimal element a_1 ; such an element exists by Lemma 1. Next, note that $(A - \{a_1\}, \preccurlyeq)$ is also a poset, as the reader should verify. (Here by \preccurlyeq we mean the restriction of the original relation \preccurlyeq on A to $A - \{a_1\}$.) If it is nonempty, choose a minimal element a_2 of this poset. Then remove a_2 as well, and if there are additional elements left, choose a minimal element a_3 in $A - \{a_1, a_2\}$. Continue this process by choosing a_{k+1} to be a minimal element in $A - \{a_1, a_2, \dots, a_k\}$, as long as elements remain.

Because A is a finite set, this process must terminate. The end product is a sequence of elements a_1, a_2, \dots, a_n . The desired total ordering \preccurlyeq_t is defined by

$$a_1 \prec_t a_2 \prec_t \cdots \prec_t a_n.$$

This total ordering is compatible with the original partial ordering. To see this, note that if $b \prec c$ in the original partial ordering, c is chosen as the minimal element at a phase of the algorithm where b has already been removed, for otherwise c would not be a minimal element. Pseudocode for this topological sorting algorithm is shown in Algorithm 1.

ALGORITHM 1 Topological Sorting.

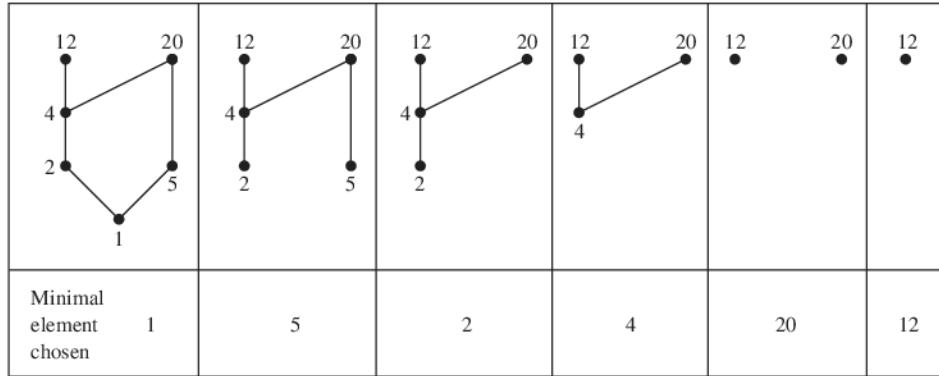
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procedure topological sort (( $S$ ,  $\preccurlyeq$ ): finite poset)
   $k := 1$ 
  while  $S \neq \emptyset$ 
     $a_k :=$  a minimal element of  $S$  {such an element exists by Lemma 1}
     $S := S - \{a_k\}$ 
     $k := k + 1$ 
  return  $a_1, a_2, \dots, a_n$  { $a_1, a_2, \dots, a_n$  is a compatible total ordering of  $S$ }
```

EXAMPLE 26

Find a compatible total ordering for the poset $(\{1, 2, 4, 5, 12, 20\}, |)$.

*“Topological sorting” is terminology used by computer scientists; mathematicians use the terminology “linearization of a partial ordering” for the same thing. In mathematics, topology is the branch of geometry dealing with properties of geometric figures that hold for all figures that can be transformed into one another by continuous bijections. In computer science, a topology is any arrangement of objects that can be connected with edges.

FIGURE 9 A Topological Sort of $\{1, 2, 4, 5, 12, 20\}$, $|$.

Solution: The first step is to choose a minimal element. This must be 1, because it is the only minimal element. Next, select a minimal element of $(\{2, 4, 5, 12, 20\}, |)$. There are two minimal elements in this poset, namely, 2 and 5. We select 5. The remaining elements are $\{2, 4, 12, 20\}$. The only minimal element at this stage is 2. Next, 4 is chosen because it is the only minimal element of $(\{4, 12, 20\}, |)$. Because both 12 and 20 are minimal elements of $(\{12, 20\}, |)$, either can be chosen next. We select 20, which leaves 12 as the last element left. This produces the total ordering

$$1 \prec 5 \prec 2 \prec 4 \prec 20 \prec 12.$$

The steps used by this sorting algorithm are displayed in Figure 9. ◀

Topological sorting has an application to the scheduling of projects.

EXAMPLE 27

A development project at a computer company requires the completion of seven tasks. Some of these tasks can be started only after other tasks are finished. A partial ordering on tasks is set up by considering task $X \prec$ task Y if task Y cannot be started until task X has been completed. The Hasse diagram for the seven tasks, with respect to this partial ordering, is shown in Figure 10. Find an order in which these tasks can be carried out to complete the project.

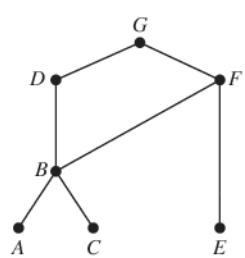


FIGURE 10 The Hasse Diagram for Seven Tasks.

Solution: An ordering of the seven tasks can be obtained by performing a topological sort. The steps of a sort are illustrated in Figure 11. The result of this sort, $A \prec C \prec B \prec E \prec F \prec D \prec G$, gives one possible order for the tasks. ◀

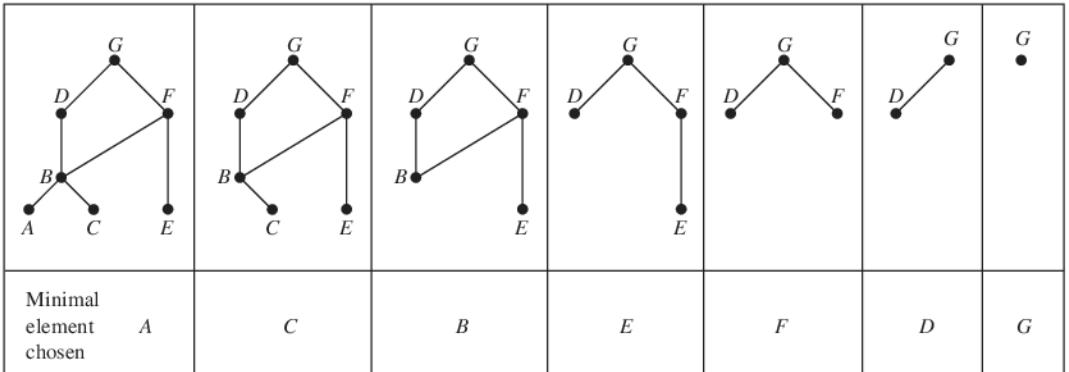


FIGURE 11 A Topological Sort of the Tasks.