Proof: Let X be the random variable equal to the number of successes in n trials. By Theorem 2 of Section 7.2 we see that $p(X = k) = C(n, k)p^k q^{n-k}$. Hence, we have

$$
\begin{aligned}
E(X) &= \sum_{k=1}^{n} kp(X = k) && \text{by Theorem 1}\\
&= \sum_{k=1}^{n} kC(n, k)p^k q^{n-k} && \text{by Theorem 2 in Section 7.2}\\
&= \sum_{k=1}^{n} nC(n-1, k-1)p^k q^{n-k} && \text{by Exercise 21 in Section 6.4}\\
&= np\sum_{k=1}^{n} C(n-1, k-1)p^{k-1} q^{n-k} && \text{factoring } np \text{ from each term}\\
&= np\sum_{j=0}^{n-1} C(n-1, j)p^{j} q^{n-1-j} && \text{shifting index of summation with } j = k-1\\
&= np(p+q)^{n-1} && \text{by the binomial theorem}\\
&= np. && \text{because } p + q = 1
\end{aligned}
$$

This completes the proof because it shows that the expected number of successes in n mutually independent Bernoulli trials is np. ◁

We will also show that the hypothesis that the Bernoulli trials are mutually independent in Theorem 2 is not necessary.

Linearity of Expectations

Theorem 3 tells us that expected values are linear. For example, the expected value of the sum of random variables is the sum of their expected values. We will find this property exceedingly useful.

THEOREM 3 If X_i, $i = 1, 2, \ldots, n$ with n a positive integer, are random variables on S, and if a and b are real numbers, then

(*i*) $E(X_1 + X_2 + \cdots + X_n) = E(X_1) + E(X_2) + \cdots + E(X_n)$
(*ii*) $E(aX + b) = aE(X) + b$.

Proof: Part *(i)* follows for $n = 2$ directly from the definition of expected value, because

$$
\begin{aligned}
E(X_1 + X_2) &= \sum_{s \in S} p(s)(X_1(s) + X_2(s))\\
&= \sum_{s \in S} p(s)X_1(s) + \sum_{s \in S} p(s)X_2(s)\\
&= E(X_1) + E(X_2).
\end{aligned}
$$

The case for n random variables follows easily by mathematical induction using the case of two random variables. (We leave it to the reader to complete the proof.)

To prove part *(ii)*, note that

$$
\begin{aligned}
E(aX + b) &= \textstyle\sum_{s\in S} p(s)(aX(s) + b) \\
&= a \textstyle\sum_{s\in S} p(s)X(s) + b \sum_{s\in S} p(s) \\
&= aE(X) + b \text{ because } \textstyle\sum_{s\in S} p(s) = 1.
\end{aligned}
$$

◁

Examples 4 and 5 illustrate how to use Theorem 3.

EXAMPLE 4 Use Theorem 3 to find the expected value of the sum of the numbers that appear when a pair of fair dice is rolled. (This was done in Example 3 without the benefit of this theorem.)

Solution: Let X_1 and X_2 be the random variables with $X_1((i, j)) = i$ and $X_2((i, j)) = j$, so that X_1 is the number appearing on the first die and X_2 is the number appearing on the second die. It is easy to see that $E(X_1) = E(X_2) = 7/2$ because both equal $(1 + 2 + 3 + 4 + 5 + 6)/6 = 21/6 = 7/2$. The sum of the two numbers that appear when the two dice are rolled is the sum $X_1 + X_2$. By Theorem 3, the expected value of the sum is $E(X_1 + X_2) = E(X_1) + E(X_2) = 7/2 + 7/2 = 7$. ◀

EXAMPLE 5 In the proof of Theorem 2 we found the expected value of the number of successes when n independent Bernoulli trials are performed, where p is the probability of success on each trial by direct computation. Show how Theorem 3 can be used to derive this result where the Bernoulli trials are not necessarily independent.

Solution: Let X_i be the random variable with $X_i((t_1, t_2, \ldots, t_n)) = 1$ if t_i is a success and $X_i((t_1, t_2, \ldots, t_n)) = 0$ if t_i is a failure. The expected value of X_i is $E(X_i) = 1 \cdot p + 0 \cdot (1 - p) = p$ for $i = 1, 2, \ldots, n$. Let $X = X_1 + X_2 + \cdots + X_n$, so that X counts the number of successes when these n Bernoulli trials are performed. Theorem 3, applied to the sum of n random variables, shows that $E(X) = E(X_1) + E(X_2) + \cdots + E(X_n) = np$. ◀

We can take advantage of the linearity of expectations to find the solutions of many seemingly difficult problems. The key step is to express a random variable whose expectation we wish to find as the sum of random variables whose expectations are easy to find. Examples 6 and 7 illustrate this technique.

EXAMPLE 6 **Expected Value in the Hatcheck Problem** A new employee checks the hats of n people at a restaurant, forgetting to put claim check numbers on the hats. When customers return for their hats, the checker gives them back hats chosen at random from the remaining hats. What is the expected number of hats that are returned correctly?

Solution: Let X be the random variable that equals the number of people who receive the correct hat from the checker. Let X_i be the random variable with $X_i = 1$ if the ith person receives the correct hat and $X_i = 0$ otherwise. It follows that

$$X = X_1 + X_2 + \cdots + X_n.$$

Because it is equally likely that the checker returns any of the hats to this person, it follows that the probability that the ith person receives the correct hat is $1/n$. Consequently, by Theorem 1, for all i we have

$$E(X_i) = 1 \cdot p(X_i = 1) + 0 \cdot p(X_i = 0) = 1 \cdot 1/n + 0 = 1/n.$$

By the linearity of expectations (Theorem 3), it follows that

$$E(X) = E(X_1) + E(X_2) + \cdots + E(X_n) = n \cdot 1/n = 1.$$

Consequently, the average number of people who receive the correct hat is exactly 1. Note that this answer is independent of the number of people who have checked their hats! (We will find an explicit formula for the probability that no one receives the correct hat in Example 4 of Section 8.6.) ◀

EXAMPLE 7 **Expected Number of Inversions in a Permutation** The ordered pair (i, j) is called an **inversion** in a permutation of the first n positive integers if $i < j$ but j precedes i in the permutation. For instance, there are six inversions in the permutation 3, 5, 1, 4, 2; these inversions are

$$(1, 3), (1, 5), (2, 3), (2, 4), (2, 5), (4, 5).$$

Let $I_{i,j}$ be the random variable on the set of all permutations of the first n positive integers with $I_{i,j} = 1$ if (i, j) is an inversion of the permutation and $I_{i,j} = 0$ otherwise. It follows that if X is the random variable equal to the number of inversions in the permutation, then

$$X = \sum_{1 \le i < j \le n} I_{i,j}.$$

Note that it is equally likely for i to precede j in a randomly chosen permutation as it is for j to precede i. (To see this, note that there are an equal number of permutations with each of these properties.) Consequently, for all pairs i and j we have

$$E(I_{i,j}) = 1 \cdot p(I_{i,j} = 1) + 0 \cdot p(I_{i,j} = 0) = 1 \cdot 1/2 + 0 = 1/2.$$

Because there are $\binom{n}{2}$ pairs i and j with $1 \le i < j \le n$ and by the linearity of expectations (Theorem 3), we have

$$E(X) = \sum_{1 \le i < j \le n} E(I_{i,j}) = \binom{n}{2} \cdot \frac{1}{2} = \frac{n(n-1)}{4}.$$

It follows that there are an average of $n(n-1)/4$ inversions in a permutation of the first n positive integers. ◀

Average-Case Computational Complexity

Links

Computing the average-case computational complexity of an algorithm can be interpreted as computing the expected value of a random variable. Let the sample space of an experiment be the set of possible inputs a_j, $j = 1, 2, \ldots, n$, and let X be the random variable that assigns to a_j the number of operations used by the algorithm when given a_j as input. Based on our knowledge of the input, we assign a probability $p(a_j)$ to each possible input value a_j. Then, the average-case complexity of the algorithm is

$$E(X) = \sum_{j=1}^{n} p(a_j)X(a_j).$$

This is the expected value of X.

Finding the average-case computational complexity of an algorithm is usually much more difficult than finding its worst-case computational complexity, and often involves the use of sophisticated methods. However, there are some algorithms for which the analysis required to find the average-case computational complexity is not difficult. For instance, in Example 8 we will illustrate how to find the average-case computational complexity of the linear search algorithm under different assumptions concerning the probability that the element for which we search is an element of the list.

EXAMPLE 8 **Average-Case Complexity of the Linear Search Algorithm** We are given a real number x and a list of n distinct real numbers. The linear search algorithm, described in Section 3.1, locates x by successively comparing it to each element in the list, terminating when x is located or when all the elements have been examined and it has been determined that x is not in the list. What is the average-case computational complexity of the linear search algorithm if the probability that x is in the list is p and it is equally likely that x is any of the n elements in the list? (There are $n+1$ possible types of input: one type for each of the n numbers in the list and a last type for numbers not in the list, which we treat as a single input.)

Solution: In Example 4 of Section 3.3 we showed that $2i+1$ comparisons are used if x equals the ith element of the list and, in Example 2 of Section 3.3, we showed that $2n+2$ comparisons are used if x is not in the list. The probability that x equals a_i, the ith element in the list, is p/n, and the probability that x is not in the list is $q = 1 - p$. It follows that the average-case computational complexity of the linear search algorithm is

$$\begin{aligned} E &= \frac{3p}{n} + \frac{5p}{n} + \cdots + \frac{(2n+1)p}{n} + (2n+2)q \\ &= \frac{p}{n}(3 + 5 + \cdots + (2n+1)) + (2n+2)q \\ &= \frac{p}{n}((n+1)^2 - 1) + (2n+2)q \\ &= p(n+2) + (2n+2)q. \end{aligned}$$

(The third equality follows from Example 2 of Section 5.1.) For instance, when x is guaranteed to be in the list, we have $p = 1$ (so the probability that $x = a_i$ is $1/n$ for each i) and $q = 0$. Then $E = n + 2$, as we showed in Example 4 in Section 3.3.

When p, the probability that x is in the list, is $1/2$, it follows that $q = 1 - p = 1/2$, so $E = (n+2)/2 + n + 1 = (3n+4)/2$. Similarly, if the probability that x is in the list is $3/4$, we have $p = 3/4$ and $q = 1/4$, so $E = 3(n+2)/4 + (n+1)/2 = (5n+8)/4$.

Finally, when x is guaranteed not to be in the list, we have $p = 0$ and $q = 1$. It follows that $E = 2n + 2$, which is not surprising because we have to search the entire list. ◀

Example 9 illustrates how the linearity of expectations can help us find the average-case complexity of a sorting algorithm, the insertion sort.

EXAMPLE 9 **Average-Case Complexity of the Insertion Sort** What is the average number of comparisons used by the insertion sort to sort n distinct elements?

Solution: We first suppose that X is the random variable equal to the number of comparisons used by the insertion sort (described in Section 3.1) to sort a list $a_1, a_2, \ldots, a_n$ of n distinct elements. Then $E(X)$ is the average number of comparisons used. (Recall that at step i for $i = 2, \ldots, n$, the insertion sort inserts the ith element in the original list into the correct position in the sorted list of the first $i - 1$ elements of the original list.)

We let X_i be the random variable equal to the number of comparisons used to insert a_i into the proper position after the first $i-1$ elements $a_1, a_2, \ldots, a_{i-1}$ have been sorted. Because

$$X = X_2 + X_3 + \cdots + X_n,$$

we can use the linearity of expectations to conclude that

$$E(X) = E(X_2 + X_3 + \cdots + X_n) = E(X_2) + E(X_3) + \cdots + E(X_n).$$

To find $E(X_i)$ for $i = 2, 3, \ldots, n$, let $p_j(k)$ denote the probability that the largest of the first j elements in the list occurs at the kth position, that is, that $\max(a_1, a_2, \ldots, a_j) = a_k$, where $1 \le k \le j$. Because the elements of the list are randomly distributed, it is equally likely for the largest element among the first j elements to occur at any position. Consequently, $p_j(k) = 1/j$. If $X_i(k)$ equals the number of comparisons used by the insertion sort if a_i is inserted into the kth position in the list once $a_1, a_2, \ldots, a_{i-1}$ have been sorted, it follows that $X_i(k) = k$. Because it is possible that a_i is inserted in any of the first i positions, we find that

$$E(X_i) = \sum_{k=1}^{i} p_i(k) \cdot X_i(k) = \sum_{k=1}^{i} \frac{1}{i} \cdot k = \frac{1}{i} \cdot \sum_{k=1}^{i} k = \frac{1}{i} \cdot \frac{i(i+1)}{2} = \frac{i+1}{2}.$$

It follows that

$$\begin{aligned} E(X) &= \sum_{i=2}^{n} E(X_i) = \sum_{i=2}^{n} \frac{i+1}{2} = \frac{1}{2} \sum_{j=3}^{n+1} j \\ &= \frac{1}{2} \frac{(n+1)(n+2)}{2} - \frac{1}{2}(1+2) = \frac{n^2+3n-4}{4}. \end{aligned}$$

To obtain the third of these equalities we shifted the index of summation, setting $j = i + 1$. To obtain the fourth equality, we used the formula $\sum_{k=1}^{m} k = m(m+1)/2$ (from Table 2 in Section 2.4) with $m = n + 1$, subtracting off the missing terms with $j = 1$ and $j = 2$. We conclude that the average number of comparisons used by the insertion sort to sort n elements equals $(n^2 + 3n - 4)/4$, which is $\Theta(n^2)$. ◀

The Geometric Distribution

We now turn our attention to a random variable with infinitely many possible outcomes.

EXAMPLE 10 Suppose that the probability that a coin comes up tails is p. This coin is flipped repeatedly until it comes up tails. What is the expected number of flips until this coin comes up tails?

Links

Solution: We first note that the sample space consists of all sequences that begin with any number of heads, denoted by H, followed by a tail, denoted by T. Therefore, the sample space is the set $\{T, HT, HHT, HHHT, HHHHT, \ldots\}$. Note that this is an infinite sample space. We can determine the probability of an element of the sample space by noting that the coin flips are independent and that the probability of a head is $1 - p$. Therefore, $p(T) = p$, $p(HT) = (1-p)p$, $p(HHT) = (1-p)^2 p$, and in general the probability that the coin is flipped n times before a tail comes up, that is, that $n - 1$ heads come up followed by a tail, is $(1-p)^{n-1} p$. (Exercise 14 asks for a verification that the sum of the probabilities of the points in the sample space is 1.)

Now let X be the random variable equal to the number of flips in an element in the sample space. That is, $X(T) = 1$, $X(HT) = 2$, $X(HHT) = 3$, and so on. Note that $p(X = j) = (1 - p)^{j-1}p$. The expected number of flips until the coin comes up tails equals $E(X)$.

Using Theorem 1, we find that

$$E(X) = \sum_{j=1}^{\infty} j \cdot p(X = j) = \sum_{j=1}^{\infty} j(1 - p)^{j-1}p = p\sum_{j=1}^{\infty} j(1 - p)^{j-1} = p \cdot \frac{1}{p^2} = \frac{1}{p}.$$

[The third equality in this chain follows from Table 2 in Section 2.4, which tells us that $\sum_{j=1}^{\infty} j(1 - p)^{j-1} = 1/(1 - (1 - p))^2 = 1/p^2$.] It follows that the expected number of times the coin is flipped until tails comes up is $1/p$. Note that when the coin is fair we have $p = 1/2$, so the expected number of flips until it comes up tails is $1/(1/2) = 2$. ◀

The random variable X that equals the number of flips expected before a coin comes up tails is an example of a random variable with a **geometric distribution**.

DEFINITION 2 A random variable X has a *geometric distribution with parameter* p if $p(X = k) = (1 - p)^{k-1}p$ for $k = 1, 2, 3, \ldots$, where p is a real number with $0 \le p \le 1$.

Geometric distributions arise in many applications because they are used to study the time required before a particular event happens, such as the time required before we find an object with a certain property, the number of attempts before an experiment succeeds, the number of times a product can be used before it fails, and so on.

When we computed the expected value of the number of flips required before a coin comes up tails, we proved Theorem 4.

THEOREM 4 If the random variable X has the geometric distribution with parameter p, then $E(X) = 1/p$.

Independent Random Variables

We have already discussed independent events. We will now define what it means for two random variables to be independent.

DEFINITION 3 The random variables X and Y on a sample space S are *independent* if

$$p(X = r_1 \text{ and } Y = r_2) = p(X = r_1) \cdot p(Y = r_2),$$

or in words, if the probability that $X = r_1$ and $Y = r_2$ equals the product of the probabilities that $X = r_1$ and $Y = r_2$, for all real numbers r_1 and r_2.

EXAMPLE 11 Are the random variables X_1 and X_2 from Example 4 independent?

Extra Examples

Solution: Let $S = \{1, 2, 3, 4, 5, 6\}$, and let $i \in S$ and $j \in S$. Because there are 36 possible outcomes when the pair of dice is rolled and each is equally likely, we have

$$p(X_1 = i \text{ and } X_2 = j) = 1/36.$$

Furthermore, $p(X_1 = i) = 1/6$ and $p(X_2 = j) = 1/6$, because the probability that i appears on the first die and the probability that j appears on the second die are both $1/6$. It follows that

$$p(X_1 = i \text{ and } X_2 = j) = \frac{1}{36} \quad \text{and} \quad p(X_1 = i)p(X_2 = j) = \frac{1}{6} \cdot \frac{1}{6} = \frac{1}{36},$$

so X_1 and X_2 are independent. ◀

EXAMPLE 12 Show that the random variables X_1 and $X = X_1 + X_2$, where X_1 and X_2 are as defined in Example 4, are not independent.

Solution: Note that $p(X_1 = 1 \text{ and } X = 12) = 0$, because $X_1 = 1$ means the number appearing on the first die is 1, which implies that the sum of the numbers appearing on the two dice cannot equal 12. On the other hand, $p(X_1 = 1) = 1/6$ and $p(X = 12) = 1/36$. Hence $p(X_1 = 1 \text{ and } X = 12) \neq p(X_1 = 1) \cdot p(X = 12)$. This counterexample shows that X_1 and X are not independent. ◀

The expected value of the product of two independent random variables is the product of their expected values, as Theorem 5 shows.

THEOREM 5 If X and Y are independent random variables on a sample space S, then $E(XY) = E(X)E(Y)$.

Proof: To prove this formula, we use the key observation that the event $XY = r$ is the disjoint union of the events $X = r_1$ and $Y = r_2$ over all $r_1 \in X(S)$ and $r_2 \in Y(S)$ with $r = r_1 r_2$. We have

$$\begin{aligned}
E(XY) &= \sum_{r \in XY(S)} r \cdot p(XY = r) && \text{by Theorem 1} \\
&= \sum_{r_1 \in X(S), r_2 \in Y(S)} r_1 r_2 \cdot p(X = r_1 \text{ and } Y = r_2) && \text{expressing } XY = r \text{ as a disjoint union} \\
&= \sum_{r_1 \in X(S)} \sum_{r_2 \in Y(S)} r_1 r_2 \cdot p(X = r_1 \text{ and } Y = r_2) && \text{using a double sum to order the terms} \\
&= \sum_{r_1 \in X(S)} \sum_{r_2 \in Y(S)} r_1 r_2 \cdot p(X = r_1) \cdot p(Y = r_2) && \text{by the independence of } X \text{ and } Y \\
&= \sum_{r_1 \in X(S)} \Big(r_1 \cdot p(X = r_1) \cdot \sum_{r_2 \in Y(S)} r_2 \cdot p(Y = r_2)\Big) && \text{by factoring out } r_1 \cdot p(X = r_1) \\
&= \sum_{r_1 \in X(S)} r_1 \cdot p(X = r_1) \cdot E(Y) && \text{by the definition of } E(Y) \\
&= E(Y)\Big(\sum_{r_1 \in X(S)} r_1 \cdot p(X = r_1)\Big) && \text{by factoring out } E(Y) \\
&= E(Y)E(X) && \text{by the definition of } E(X)
\end{aligned}$$

We complete the proof by noting that $E(Y)E(X) = E(X)E(Y)$, which is a consequence of the commutative law for multiplication. ◁

Note that when X and Y are random variables that are not independent, we cannot conclude that $E(XY) = E(X)E(Y)$, as Example 13 shows.

EXAMPLE 13 Let X and Y be random variables that count the number of heads and the number of tails when a coin is flipped twice. Because $p(X = 2) = 1/4$, $p(X = 1) = 1/2$, and $p(X = 0) = 1/4$, by Theorem 1 we have

$$E(X) = 2 \cdot \frac{1}{4} + 1 \cdot \frac{1}{2} + 0 \cdot \frac{1}{4} = 1.$$

A similar computation shows that $E(Y) = 1$. We note that $XY = 0$ when either two heads and no tails or two tails and no heads come up and that $XY = 1$ when one head and one tail come up. Hence,

$$E(XY) = 1 \cdot \frac{1}{2} + 0 \cdot \frac{1}{2} = \frac{1}{2}.$$

It follows that

$$E(XY) \neq E(X)E(Y).$$

This does not contradict Theorem 5 because X and Y are not independent, as the reader should verify (see Exercise 16). ◀

Variance

Links

The expected value of a random variable tells us its average value, but nothing about how widely its values are distributed. For example, if X and Y are the random variables on the set $S = \{1, 2, 3, 4, 5, 6\}$, with $X(s) = 0$ for all $s \in S$ and $Y(s) = -1$ if $s \in \{1, 2, 3\}$ and $Y(s) = 1$ if $s \in \{4, 5, 6\}$, then the expected values of X and Y are both zero. However, the random variable X never varies from 0, while the random variable Y always differs from 0 by 1. The variance of a random variable helps us characterize how widely a random variable is distributed. In particular, it provides a measure of how widely X is distributed about its expected value.

DEFINITION 4 Let X be a random variable on a sample space S. The *variance* of X, denoted by $V(X)$, is

$$V(X) = \sum_{s \in S} (X(s) - E(X))^2 p(s).$$

That is, $V(X)$ is the weighted average of the square of the deviation of X. The *standard deviation* of X, denoted $\sigma(X)$, is defined to be $\sqrt{V(X)}$.

Theorem 6 provides a useful simple expression for the variance of a random variable.

THEOREM 6 If X is a random variable on a sample space S, then $V(X) = E(X^2) - E(X)^2$.

Proof: Note that

$$\begin{aligned} V(X) &= \sum_{s \in S} (X(s) - E(X))^2 p(s) \\ &= \sum_{s \in S} X(s)^2 p(s) - 2E(X) \sum_{s \in S} X(s) p(s) + E(X)^2 \sum_{s \in S} p(s) \\ &= E(X^2) - 2E(X)E(X) + E(X)^2 \\ &= E(X^2) - E(X)^2. \end{aligned}$$

We have used the fact that $\sum_{s \in S} p(s) = 1$ in the next-to-last step. ◁

We can use Theorems 3 and 6 to derive an alternative formula for $V(X)$ that provides some insight into the meaning of the variance of a random variable.

COROLLARY 1 If X is a random variable on a sample space S and $E(X) = \mu$, then $V(X) = E((X - \mu)^2)$.

μ is the Greek letter mu.

Proof: If X is a random variable with $E(X) = \mu$, then

$$
\begin{aligned}
E((X - \mu)^2) &= E(X^2 - 2\mu X + \mu^2) && \text{expanding } (X-\mu)^2 \\
&= E(X^2) - E(2\mu X) + E(\mu^2) && \text{by part } (i) \text{ of Theorem 3} \\
&= E(X^2) - 2\mu E(X) + E(\mu^2) && \text{by part } (ii) \text{ of Theorem 3, noting that } \mu \text{ is a constant} \\
&= E(X^2) - 2\mu E(X) + \mu^2 && \text{as } E(\mu^2) = \mu^2 \text{, because } \mu^2 \text{ is a constant} \\
&= E(X^2) - 2\mu^2 + \mu^2 && \text{because } E(X) = \mu \\
&= E(X^2) - \mu^2 && \text{simplifying} \\
&= V(X) && \text{by Theorem 6 and noting that } E(X) = \mu.
\end{aligned}
$$

This completes the proof. ◁

Corollary 1 tells us that the variance of a random variable X is the expected value of the square of the difference between X and its own expected value. This is commonly expressed as saying that the variance of X is the mean of the square of its deviation. We also say that the standard deviation of X is the square root of the mean of the square of its deviation (often read as the "root mean square" of the deviation).

We now compute the variance of some random variables.

EXAMPLE 14 What is the variance of the random variable X with $X(t) = 1$ if a Bernoulli trial is a success and $X(t) = 0$ if it is a failure, where p is the probability of success and q is the probability of failure?

Extra Examples

Solution: Because X takes only the values 0 and 1, it follows that $X^2(t) = X(t)$. Hence,

$$V(X) = E(X^2) - E(X)^2 = p - p^2 = p(1 - p) = pq.$$ ◀

EXAMPLE 15 **Variance of the Value of a Die** What is the variance of the random variable X, where X is the number that comes up when a fair die is rolled?

Solution: We have $V(X) = E(X^2) - E(X)^2$. By Example 1 we know that $E(X) = 7/2$. To find $E(X^2)$ note that X^2 takes the values i^2, $i = 1, 2, \ldots, 6$, each with probability $1/6$. It follows that

$$E(X^2) = \frac{1}{6}(1^2 + 2^2 + 3^2 + 4^2 + 5^2 + 6^2) = \frac{91}{6}.$$

We conclude that

$$V(X) = \frac{91}{6} - \left(\frac{7}{2}\right)^2 = \frac{35}{12}.$$ ◀

EXAMPLE 16 What is the variance of the random variable $X((i, j)) = 2i$, where i is the number appearing on the first die and j is the number appearing on the second die, when two fair dice are rolled?

Solution: We will use Theorem 6 to find the variance of X. To do so, we need to find the expected values of X and X^2. Note that because $p(X = k)$ is $1/6$ for $k = 2, 4, 6, 8, 10, 12$ and is 0 otherwise,

$$E(X) = (2 + 4 + 6 + 8 + 10 + 12)/6 = 7,$$

and

$$E(X^2) = (2^2 + 4^2 + 6^2 + 8^2 + 10^2 + 12^2)/6 = 182/3.$$

It follows from Theorem 6 that

$$V(X) = E(X^2) - E(X)^2 = 182/3 - 49 = 35/3.$$ ◀

Another useful property is that the variance of the sum of two or more independent random variables is the sum of their variances. The formula that expresses this property is known as **Bienaymé's formula**, after Irenée-Jules Bienaymé, the French mathematician who discovered it in 1853. Bienaymé's formula is useful for computing the variance of the result of n independent Bernoulli trials, for instance.

THEOREM 7 **BIENAYMÉ'S FORMULA** If X and Y are two independent random variables on a sample space S, then $V(X + Y) = V(X) + V(Y)$. Furthermore, if $X_i, i = 1, 2, \ldots, n$, with n a positive integer, are pairwise independent random variables on S, then $V(X_1 + X_2 + \cdots + X_n) = V(X_1) + V(X_2) + \cdots + V(X_n)$.

Links

IRENÉE-JULES BIENAYMÉ (1796–1878) Bienaymé, born in Paris, moved with his family to Bruges in 1803 when his father became a government administrator. Bienaymé attended the Lycée impérial in Bruges, and when his family returned to Paris in 1811, the Lycée Louis-le-Grand. As a teenager, he helped defend Paris during the 1814 Napoleonic Wars; in 1815, he became a student at the École Polytechnique. In 1816 he joined the Ministry of Finances to help support his family. In 1819, he left the civil service, taking a job lecturing mathematics at the Académie militaire de Saint-Cyr. Unhappy with conditions there, he soon returned to the Ministry of Finances. He attained the position of inspector general, remaining until forced to retire in 1848 for political reasons. He was able to return as inspector general in 1850, but he retired a second time in 1852. In 1851 he briefly was professor at the Sorbonne and also served as an expert statistician for Napoleon III. Bienaymé was one of the founders of the Société Mathématique de France, and in 1875 was its president.

Bienaymé was noted for his ingenuity, but his papers frustrated readers by omitting important proofs. He published sparsely, often in obscure journals. However, he made important contributions to probability and statistics, and to their applications to the social sciences and to finance. Among his important contributions are the Bienaymé-Chebyshev inequality, which provides a simple proof of the law of large numbers, a generalization of Laplace's least square method, and Bienaymé's formula for the variance of a sum of random variables. He studied the extinction of aristocratic families, declining despite general population growth. Bienaymé was a skilled linguist; he translated the works of Chebyshev, a close friend, from Russian to French. It has been suggested that his relative obscurity results from his modesty, his lack of interest in asserting the priority of his discoveries, and the fact that his work was often ahead of its time. He and his brother married two sisters who were daughters of a family friend. Bienaymé and his wife had two sons and three daughters.

Proof: From Theorem 6, we have

$$V(X+Y) = E((X+Y)^2) - E(X+Y)^2.$$

It follows that

$$\begin{aligned} V(X+Y) &= E(X^2 + 2XY + Y^2) - (E(X) + E(Y))^2 \\ &= E(X^2) + 2E(XY) + E(Y^2) - E(X)^2 - 2E(X)E(Y) - E(Y)^2. \end{aligned}$$

Because X and Y are independent, by Theorem 5 we have $E(XY) = E(X)E(Y)$. It follows that

$$\begin{aligned} V(X+Y) &= (E(X^2) - E(X)^2) + (E(Y^2) - E(Y)^2) \\ &= V(X) + V(Y). \end{aligned}$$

We leave the proof of the case for n pairwise independent random variables to the reader (Exercise 34). Such a proof can be constructed by generalizing the proof we have given for the case for two random variables. Note that it is not possible to use mathematical induction in a straightforward way to prove the general case (see Exercise 33). ◁

EXAMPLE 17 Find the variance and standard deviation of the random variable X whose value when two fair dice are rolled is $X((i, j)) = i + j$, where i is the number appearing on the first die and j is the number appearing on the second die.

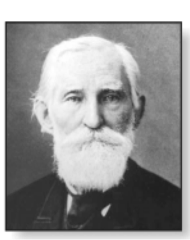
Solution: Let X_1 and X_2 be the random variables defined by $X_1((i, j)) = i$ and $X_2((i, j)) = j$ for a roll of the dice. Then $X = X_1 + X_2$, and X_1 and X_2 are independent, as Example 11 showed. From Theorem 7 it follows that $V(X) = V(X_1) + V(X_2)$. A simple computation as in Example 16, together with Exercise 29 in the Supplementary Exercises, tells us that $V(X_1) = V(X_2) = 35/12$. Hence, $V(X) = 35/12 + 35/12 = 35/6$ and $\sigma(X) = \sqrt{35/6}$. ◀

We will now find the variance of the random variable that counts the number of successes when n independent Bernoulli trials are carried out.

EXAMPLE 18 What is the variance of the number of successes when n independent Bernoulli trials are performed, where, on each trial, p is the probability of success and q is the probability of failure?

Solution: Let X_i be the random variable with $X_i((t_1, t_2, \ldots, t_n)) = 1$ if trial t_i is a success and $X_i((t_1, t_2, \ldots, t_n)) = 0$ if trial t_i is a failure. Let $X = X_1 + X_2 + \cdots + X_n$. Then X counts the number of successes in the n trials. From Theorem 7 it follows that $V(X) = V(X_1) + V(X_2) + \cdots + V(X_n)$. Using Example 14 we have $V(X_i) = pq$ for $i = 1, 2, \ldots, n$. It follows that $V(X) = npq$. ◀

Links



PAFNUTY LVOVICH CHEBYSHEV (1821–1894) Chebyshev was born into the gentry in Okatovo, Russia. His father was a retired army officer who had fought against Napoleon. In 1832 the family, with its nine children, moved to Moscow, where Pafnuty completed his high school education at home. He entered the Department of Physics and Mathematics at Moscow University. As a student, he developed a new method for approximating the roots of equations. He graduated from Moscow University in 1841 with a degree in mathematics, and he continued his studies, passing his master's exam in 1843 and completing his master's thesis in 1846.

Chebyshev was appointed in 1847 to a position as an assistant at the University of St. Petersburg. He wrote and defended a thesis in 1847. He became a professor at St. Petersburg in 1860, a position he held until 1882. His book on the theory of congruences written in 1849 was influential in the development of number theory. His work on the distribution of prime numbers was seminal. He proved Bertrand's conjecture that for every integer $n > 3$, there is a prime between n and $2n - 2$. Chebyshev helped develop ideas that were later used to prove the prime number theorem. Chebyshev's work on the approximation of functions using polynomials is used extensively when computers are used to find values of functions. Chebyshev was also interested in mechanics. He studied the conversion of rotary motion into rectilinear motion by mechanical coupling. The Chebyshev parallel motion is three linked bars approximating rectilinear motion.

Chebyshev's Inequality

How likely is it that a random variable takes a value far from its expected value? Theorem 8, called Chebyshev's inequality, helps answer this question by providing an upper bound on the probability that the value of a random variable differs from the expected value of the random variable by more than a specified amount.

THEOREM 8 CHEBYSHEV'S INEQUALITY Let X be a random variable on a sample space S with probability function p. If r is a positive real number, then

$$p(|X(s) - E(X)| \geq r) \leq V(X)/r^2.$$

Proof: Let A be the event

$$A = \{s \in S \mid |X(s) - E(X)| \geq r\}.$$

What we want to prove is that $p(A) \leq V(X)/r^2$. Note that

$$\begin{aligned} V(X) &= \sum_{s \in S} (X(s) - E(X))^2 p(s) \\ &= \sum_{s \in A} (X(s) - E(X))^2 p(s) + \sum_{s \notin A} (X(s) - E(X))^2 p(s). \end{aligned}$$

The second sum in this expression is nonnegative, because each of its summands is nonnegative. Also, because for each element s in A, $(X(s) - E(X))^2 \geq r^2$, the first sum in this expression is at least $\sum_{s \in A} r^2 p(s)$. Hence, $V(X) \geq \sum_{s \in A} r^2 p(s) = r^2 p(A)$. It follows that $V(X)/r^2 \geq p(A)$, so $p(A) \leq V(X)/r^2$, completing the proof. ◁

EXAMPLE 19 **Deviation from the Mean when Counting Tails** Suppose that X is the random variable that counts the number of tails when a fair coin is tossed n times. Note that X is the number of successes when n independent Bernoulli trials, each with probability of success $1/2$, are performed. It follows that $E(X) = n/2$ (by Theorem 2) and $V(X) = n/4$ (by Example 18). Applying Chebyshev's inequality with $r = \sqrt{n}$ shows that

$$p(|X(s) - n/2| \geq \sqrt{n}) \leq (n/4)/(\sqrt{n})^2 = 1/4.$$

Consequently, the probability is no more than $1/4$ that the number of tails that come up when a fair coin is tossed n times deviates from the mean by more than $\sqrt{n}$. ◀

Chebyshev's inequality, although applicable to any random variable, often fails to provide a practical estimate for the probability that the value of a random variable exceeds its mean by a large amount. This is illustrated by Example 20.

EXAMPLE 20 Let X be the random variable whose value is the number appearing when a fair die is rolled. We have $E(X) = 7/2$ (see Example 1) and $V(X) = 35/12$ (see Example 15). Because the only possible values of X are 1, 2, 3, 4, 5, and 6, X cannot take a value more than $5/2$ from its mean, $E(X) = 7/2$. Hence, $p(|X - 7/2| \geq r) = 0$ if $r > 5/2$. By Chebyshev's inequality we know that $p(|X - 7/2| \geq r) \leq (35/12)/r^2$.

For example, when $r = 3$, Chebyshev's inequality tells us that $p(|X - 7/2| \geq 3) \leq (35/12)/9 = 35/108 \approx 0.324$, which is a poor estimate, because $p(|X - 7/2| \geq 3) = 0$. ◀

Exercises

1. What is the expected number of heads that come up when a fair coin is flipped five times?
2. What is the expected number of heads that come up when a fair coin is flipped 10 times?
3. What is the expected number of times a 6 appears when a fair die is rolled 10 times?
4. A coin is biased so that the probability a head comes up when it is flipped is 0.6. What is the expected number of heads that come up when it is flipped 10 times?
5. What is the expected sum of the numbers that appear on two dice, each biased so that a 3 comes up twice as often as each other number?
6. What is the expected value when a \$1 lottery ticket is bought in which the purchaser wins exactly \$10 million if the ticket contains the six winning numbers chosen from the set $\{1, 2, 3, \ldots, 50\}$ and the purchaser wins nothing otherwise?
7. The final exam of a discrete mathematics course consists of 50 true/false questions, each worth two points, and 25 multiple-choice questions, each worth four points. The probability that Linda answers a true/false question correctly is 0.9, and the probability that she answers a multiple-choice question correctly is 0.8. What is her expected score on the final?
8. What is the expected sum of the numbers that appear when three fair dice are rolled?
9. Suppose that the probability that x is in a list of n distinct integers is $2/3$ and that it is equally likely that x equals any element in the list. Find the average number of comparisons used by the linear search algorithm to find x or to determine that it is not in the list.
10. Suppose that we flip a fair coin until either it comes up tails twice or we have flipped it six times. What is the expected number of times we flip the coin?
11. Suppose that we roll a fair die until a 6 comes up or we have rolled it 10 times. What is the expected number of times we roll the die?
12. Suppose that we roll a fair die until a 6 comes up.
    a) What is the probability that we roll the die n times?
    b) What is the expected number of times we roll the die?
13. Suppose that we roll a pair of fair dice until the sum of the numbers on the dice is seven. What is the expected number of times we roll the dice?
14. Show that the sum of the probabilities of a random variable with geometric distribution with parameter p, where $0 < p \leq 1$, equals 1.
15. Show that if the random variable X has the geometric distribution with parameter p, and j is a positive integer, then $p(X \geq j) = (1-p)^{j-1}$.
16. Let X and Y be the random variables that count the number of heads and the number of tails that come up when two fair coins are flipped. Show that X and Y are not independent.
17. Estimate the expected number of integers with 1000 digits that need to be selected at random to find a prime, if the probability a number with 1000 digits is prime is approximately $1/2302$.
18. Suppose that X and Y are random variables and that X and Y are nonnegative for all points in a sample space S. Let Z be the random variable defined by $Z(s) = \max(X(s), Y(s))$ for all elements $s \in S$. Show that $E(Z) \leq E(X) + E(Y)$.
19. Let X be the number appearing on the first die when two fair dice are rolled and let Y be the sum of the numbers appearing on the two dice. Show that $E(X)E(Y) \neq E(XY)$.

***20.** Show that if $X_1, X_2, \ldots, X_n$ are mutually independent random variables, then $E(\prod_{i=1}^{n} X_i) = \prod_{i=1}^{n} E(X_i)$.

The **conditional expectation** of the random variable X given the event A from the sample space S is $E(X|A) = \sum_{r \in X(S)} r \cdot P(X = r|A)$.

21. What is expected value of the sum of the numbers appearing on two fair dice when they are rolled given that the sum of these numbers is at least nine. That is, what is $E(X|A)$ where X is the sum of the numbers appearing on the two dice and A is the event that $X \geq 9$?

The **law of total expectation** states that if the sample space S is the disjoint union of the events $S_1, S_2, \ldots, S_n$ and X is a random variable, then $E(X) = \sum_{j=1}^{n} E(X|S_j)P(S_j)$.

22. Prove the law of total expectations.
23. Use the law of total expectation to find the average weight of a breeding elephant seal, given that 12% of the breeding elephant seals are male and the rest are female, and the expected weights of a breeding elephant seal is 4,200 pounds for a male and 1,100 pounds for a female.
24. Let A be an event. Then I_A, the **indicator random variable** of A, equals 1 if A occurs and equals 0 otherwise. Show that the expectation of the indicator random variable of A equals the probability of A, that is, $E(I_A) = p(A)$.
25. A **run** is a maximal sequence of successes in a sequence of Bernoulli trials. For example, in the sequence $S, S, S, F, S, S, F, F, S$, where S represents success and F represents failure, there are three runs consisting of three successes, two successes, and one success, respectively. Let R denote the random variable on the set of sequences of n independent Bernoulli trials that counts the number of runs in this sequence. Find $E(R)$. [*Hint:* Show that $R = \sum_{j=1}^{n} I_j$, where $I_j = 1$ if a run begins at the jth Bernoulli trial and $I_j = 0$ otherwise. Find $E(I_1)$ and then find $E(I_j)$, where $1 < j \leq n$.]
26. Let $X(s)$ be a random variable, where $X(s)$ is a nonnegative integer for all $s \in S$, and let A_k be the event that $X(s) \geq k$. Show that $E(X) = \sum_{k=1}^{\infty} p(A_k)$.
27. What is the variance of the number of heads that come up when a fair coin is flipped 10 times?

28. What is the variance of the number of times a 6 appears when a fair die is rolled 10 times?

29. Let X_n be the random variable that equals the number of tails minus the number of heads when n fair coins are flipped.

a) What is the expected value of X_n?
b) What is the variance of X_n?

30. Show that if X and Y are independent random variables, then $V(XY) = E(X)^2V(Y) + E(Y)^2V(X) + V(X)V(Y)$

31. Let $A(X) = E(|X - E(X)|)$, the expected value of the absolute value of the deviation of X, where X is a random variable. Prove or disprove that $A(X + Y) = A(X) + A(Y)$ for all random variables X and Y.

32. Provide an example that shows that the variance of the sum of two random variables is not necessarily equal to the sum of their variances when the random variables are not independent.

33. Suppose that X_1 and X_2 are independent Bernoulli trials each with probability $1/2$, and let $X_3 = (X_1 + X_2)$ **mod** 2.

a) Show that X_1, X_2, and X_3 are pairwise independent, but X_3 and $X_1 + X_2$ are not independent.
b) Show that $V(X_1 + X_2 + X_3) = V(X_1) + V(X_2) + V(X_3)$.
c) Explain why a proof by mathematical induction of Theorem 7 does not work by considering the random variables X_1, X_2, and X_3.

***34.** Prove the general case of Theorem 7. That is, show that if $X_1, X_2, \ldots, X_n$ are pairwise independent random variables on a sample space S, where n is a positive integer, then $V(X_1 + X_2 + \cdots + X_n) = V(X_1) + V(X_2) + \cdots + V(X_n)$. [*Hint:* Generalize the proof given in Theorem 7 for two random variables. Note that a proof using mathematical induction does not work; see Exercise 33.]

35. Use Chebyshev's inequality to find an upper bound on the probability that the number of tails that come up when a fair coin is tossed n times deviates from the mean by more than $5\sqrt{n}$.

36. Use Chebyshev's inequality to find an upper bound on the probability that the number of tails that come up when a biased coin with probability of heads equal to 0.6 is tossed n times deviates from the mean by more than $\sqrt{n}$.

37. Let X be a random variable on a sample space S such that $X(s) \geq 0$ for all $s \in S$. Show that $p(X(s) \geq a) \leq E(X)/a$ for every positive real number a. This inequality is called **Markov's inequality**.

38. Suppose that the number of cans of soda pop filled in a day at a bottling plant is a random variable with an expected value of 10,000 and a variance of 1000.

a) Use Markov's inequality (Exercise 37) to obtain an upper bound on the probability that the plant will fill more than 11,000 cans on a particular day.
b) Use Chebyshev's inequality to obtain a lower bound on the probability that the plant will fill between 9000 and 11,000 cans on a particular day.

39. Suppose that the number of tin cans recycled in a day at a recycling center is a random variable with an expected value of 50,000 and a variance of 10,000.

a) Use Markov's inequality (Exercise 37) to find an upper bound on the probability that the center will recycle more than 55,000 cans on a particular day.

b) Use Chebyshev's inequality to provide a lower bound on the probability that the center will recycle 40,000 to 60,000 cans on a certain day.

***40.** Suppose the probability that x is the ith element in a list of n distinct integers is $i/[n(n + 1)]$. Find the average number of comparisons used by the linear search algorithm to find x or to determine that it is not in the list.

***41.** In this exercise we derive an estimate of the average-case complexity of the variant of the bubble sort algorithm that terminates once a pass has been made with no interchanges. Let X be the random variable on the set of permutations of a set of n distinct integers $\{a_1, a_2, \ldots, a_n\}$ with $a_1 < a_2 < \cdots < a_n$ such that $X(P)$ equals the number of comparisons used by the bubble sort to put these integers into increasing order.

a) Show that, under the assumption that the input is equally likely to be any of the $n!$ permutations of these integers, the average number of comparisons used by the bubble sort equals $E(X)$.

b) Use Example 5 in Section 3.3 to show that $E(X) \leq n(n - 1)/2$.

c) Show that the sort makes at least one comparison for every inversion of two integers in the input.

d) Let $I(P)$ be the random variable that equals the number of inversions in the permutation P. Show that $E(X) \geq E(I)$.

e) Let $I_{j,k}$ be the random variable with $I_{j,k}(P) = 1$ if a_k precedes a_j in P and $I_{j,k} = 0$ otherwise. Show that $I(P) = \sum_k \sum_{j<k} I_{j,k}(P)$.

f) Show that $E(I) = \sum_k \sum_{j<k} E(I_{j,k})$.

g) Show that $E(I_{j,k}) = 1/2$. [*Hint:* Show that $E(I_{j,k})$ = probability that a_k precedes a_j in a permutation P. Then show it is equally likely for a_k to precede a_j as it is for a_j to precede a_k in a permutation.]

h) Use parts (f) and (g) to show that $E(I) = n(n - 1)/4$.

i) Conclude from parts (b), (d), and (h) that the average number of comparisons used to sort n integers is $\Theta(n^2)$.

***42.** In this exercise we find the average-case complexity of the quick sort algorithm, described in the preamble to Exercise 50 in Section 5.4, assuming a uniform distribution on the set of permutations.

a) Let X be the number of comparisons used by the quick sort algorithm to sort a list of n distinct integers. Show that the average number of comparisons used by the quick sort algorithm is $E(X)$ (where the sample space is the set of all $n!$ permutations of n integers).

b) Let $I_{j,k}$ denote the random variable that equals 1 if the jth smallest element and the kth smallest element of the initial list are ever compared as the quick sort algorithm sorts the list and equals 0 otherwise. Show that $X = \sum_{k=2}^{n} \sum_{j=1}^{k-1} I_{j,k}$.

c) Show that $E(X) = \sum_{k=2}^{n} \sum_{j=1}^{k-1} p(\text{the } j\text{th smallest element and the } k\text{th smallest element are compared})$.

d) Show that $p(\text{the } j\text{th smallest element and the } k\text{th smallest element are compared})$, where $k > j$, equals $2/(k - j + 1)$.

e) Use parts (c) and (d) to show that $E(X) = 2(n + 1)(\sum_{i=2}^{n} 1/i) - 2(n - 1)$.

f) Conclude from part (e) and the fact that $\sum_{j=1}^{n} 1/j \approx \ln n + \gamma$, where $\gamma = 0.57721\ldots$ is Euler's constant, that the average number of comparisons used by the quick sort algorithm is $\Theta(n \log n)$.

***43.** What is the variance of the number of **fixed elements**, that is, elements left in the same position, of a randomly selected permutation of n elements? [*Hint:* Let X denote the number of fixed points of a random permutation. Write $X = X_1 + X_2 + \cdots + X_n$, where $X_i = 1$ if the permutation fixes the ith element and $X_i = 0$ otherwise.]

The **covariance** of two random variables X and Y on a sample space S, denoted by $\text{Cov}(X, Y)$, is defined to be the expected value of the random variable $(X - E(X))(Y - E(Y))$. That is, $\text{Cov}(X, Y) = E((X - E(X))(Y - E(Y)))$.

44. Show that $\text{Cov}(X, Y) = E(XY) - E(X)E(Y)$, and use this result to conclude that $\text{Cov}(X, Y) = 0$ if X and Y are independent random variables.

45. Show that $V(X + Y) = V(X) + V(Y) + 2\,\text{Cov}(X, Y)$.

46. Find $\text{Cov}(X, Y)$ if X and Y are the random variables with $X((i, j)) = 2i$ and $Y((i, j)) = i + j$, where i and j are the numbers that appear on the first and second of two dice when they are rolled.

47. When m balls are distributed into n bins uniformly at random, what is the probability that the first bin remains empty?

48. What is the expected number of balls that fall into the first bin when m balls are distributed into n bins uniformly at random?

49. What is the expected number of bins that remain empty when m balls are distributed into n bins uniformly at random?

Key Terms and Results

TERMS

sample space: the set of possible outcomes of an experiment

event: a subset of the sample space of an experiment

probability of an event (Laplace's definition): the number of successful outcomes of this event divided by the number of possible outcomes

probability distribution: a function p from the set of all outcomes of a sample space S for which $0 \leq p(x_i) \leq 1$ for $i = 1, 2, \ldots, n$ and $\sum_{i=1}^{n} p(x_i) = 1$, where $x_1, \ldots, x_n$ are the possible outcomes

probability of an event E: the sum of the probabilities of the outcomes in E

$p(E|F)$ (conditional probability of E given F): the ratio $p(E \cap F)/p(F)$

independent events: events E and F such that $p(E \cap F) = p(E)p(F)$

pairwise independent events: events $E_1, E_2, \ldots, E_n$ such that $p(E_i \cap E_j) = p(E_i)p(E_j)$ for all pairs of integers i and j with $1 \leq j < k \leq n$

mutually independent events: events $E_1, E_2, \ldots, E_n$ such that $p(E_{i_1} \cap E_{i_2} \cap \cdots \cap E_{i_m}) = p(E_{i_1})p(E_{i_2}) \cdots p(E_{i_m})$ whenever $i_j, j = 1, 2, \ldots, m$, are integers with $1 \leq i_1 < i_2 < \cdots < i_m \leq n$ and $m \geq 2$

random variable: a function that assigns a real number to each possible outcome of an experiment

distribution of a random variable X: the set of pairs $(r, p(X = r))$ for $r \in X(S)$

uniform distribution: the assignment of equal probabilities to the elements of a finite set

expected value of a random variable: the weighted average of a random variable, with values of the random variable weighted by the probability of outcomes, that is, $E(X) = \sum_{s \in S} p(s)X(s)$

geometric distribution: the distribution of a random variable X such that $p(X = k) = (1 - p)^{k-1}p$ for $k = 1, 2, \ldots$ for some real number p with $0 \leq p \leq 1$.

independent random variables: random variables X and Y such that $p(X = r_1 \text{ and } Y = r_2) = p(X = r_1)p(Y = r_2)$ for all real numbers r_1 and r_2

variance of a random variable X: the weighted average of the square of the difference between the value of X and its expected value $E(X)$, with weights given by the probability of outcomes, that is, $V(X) = \sum_{s \in S}(X(s) - E(X))^2 p(s)$

standard deviation of a random variable X: the square root of the variance of X, that is, $\sigma(X) = \sqrt{V(X)}$

Bernoulli trial: an experiment with two possible outcomes

probabilistic (or Monte Carlo) algorithm: an algorithm in which random choices are made at one or more steps

probabilistic method: a technique for proving the existence of objects in a set with certain properties that proceeds by assigning probabilities to objects and showing that the probability that an object has these properties is positive

RESULTS

The probability of exactly k successes when n independent Bernoulli trials are carried out equals $C(n, k)p^k q^{n-k}$, where p is the probability of success and $q = 1 - p$ is the probability of failure.

Bayes' theorem: If E and F are events from a sample space S such that $p(E) \neq 0$ and $p(F) \neq 0$, then

$$p(F \mid E) = \frac{p(E \mid F)p(F)}{p(E \mid F)p(F) + p(E \mid \overline{F})p(\overline{F})}.$$

$E(X) = \sum_{r \in X(S)} p(X = r)r$.

linearity of expectations: $E(X_1 + X_2 + \cdots + X_n) = E(X_1) + E(X_2) + \cdots + E(X_n)$ if $X_1, X_2, \ldots, X_n$ are random variables

If X and Y are independent random variables, then $E(XY) = E(X)E(Y)$.

Bienaymé's formula: If $X_1, X_2, \ldots, X_n$ are independent random variables, then $V(X_1 + X_2 + \cdots + X_n) = V(X_1) + V(X_2) + \cdots + V(X_n)$.

Chebyshev's inequality: $p(|X(s) - E(X)| \geq r) \leq V(X)/r^2$, where X is a random variable with probability function p and r is a positive real number.

Review Questions

1. a) Define the probability of an event when all outcomes are equally likely.
b) What is the probability that you select the six winning numbers in a lottery if the six different winning numbers are selected from the first 50 positive integers?

2. a) What conditions should be met by the probabilities assigned to the outcomes from a finite sample space?
b) What probabilities should be assigned to the outcome of heads and the outcome of tails if heads comes up three times as often as tails?

3. a) Define the conditional probability of an event E given an event F.
b) Suppose E is the event that when a die is rolled it comes up an even number, and F is the event that when a die is rolled it comes up 1, 2, or 3. What is the probability of F given E?

4. a) When are two events E and F independent?
b) Suppose E is the event that an even number appears when a fair die is rolled, and F is the event that a 5 or 6 comes up. Are E and F independent?

5. a) What is a random variable?
b) What are the possible values assigned by the random variable X that assigns to a roll of two dice the larger number that appears on the two dice?

6. a) Define the expected value of a random variable X.
b) What is the expected value of the random variable X that assigns to a roll of two dice the larger number that appears on the two dice?

7. a) Explain how the average-case computational complexity of an algorithm, with finitely many possible input values, can be interpreted as an expected value.
b) What is the average-case computational complexity of the linear search algorithm, if the probability that the element for which we search is in the list is $1/3$, and it is equally likely that this element is any of the n elements in the list?

8. a) What is meant by a Bernoulli trial?
b) What is the probability of k successes in n independent Bernoulli trials?
c) What is the expected value of the number of successes in n independent Bernoulli trials?

9. a) What does the linearity of expectations of random variables mean?
b) How can the linearity of expectations help us find the expected number of people who receive the correct hat when a hatcheck person returns hats at random?

10. a) How can probability be used to solve a decision problem, if a small probability of error is acceptable?
b) How can we quickly determine whether a positive integer is prime, if we are willing to accept a small probability of making an error?

11. State Bayes' theorem and use it to find $p(F \mid E)$ if $p(E \mid F) = 1/3$, $p(E \mid \overline{F}) = 1/4$, and $p(F) = 2/3$, where E and F are events from a sample space S.

12. a) What does it mean to say that a random variable has a geometric distribution with parameter p?
b) What is the mean of a geometric distribution with parameter p?

13. a) What is the variance of a random variable?
b) What is the variance of a Bernoulli trial with probability p of success?

14. a) What is the variance of the sum of n independent random variables?
b) What is the variance of the number of successes when n independent Bernoulli trials, each with probability p of success, are carried out?

15. What does Chebyshev's inequality tell us about the probability that a random variable deviates from its mean by more than a specified amount?

Supplementary Exercises

1. What is the probability that six consecutive integers will be chosen as the winning numbers in a lottery where each number chosen is an integer between 1 and 40 (inclusive)?

2. A player in the Mega Millions lottery picks five different integers between 1 and 56, inclusive, and a sixth integer between 1 and 46, which may duplicate one of the earlier five integers. The player wins the jackpot if the first five numbers picked match the first five numbers drawn and the sixth number matches the sixth number drawn.

a) What is the probability that a player wins the jackpot?

b) What is the probability that a player wins \$250,000, which is the prize for matching the first five numbers, but not the sixth number, drawn?

c) What is the probability that a player wins \$150 by matching exactly three of the first five numbers and the sixth number or by matching four of the first five numbers but not the sixth number?

d) What is the probability that a player wins a prize, if a prize is given when the player matches at least three of the first five numbers or the last number.

3. A player in the Powerball lottery picks five different integers between 1 and 59, inclusive, and a sixth integer between 1 and 39, which may duplicate one of the earlier five integers. The player wins the jackpot if the first five numbers picked match the first five number drawn and the sixth number matches the sixth number drawn.

a) What is the probability that a player wins the jackpot?

b) What is the probability that a player wins \$200,000, which is the prize for matching the first five numbers, but not the sixth number, drawn?

c) What is the probability that a player wins \$100 by matching exactly three of the first five and the sixth numbers or four of the first five numbers but not the sixth number?

d) What is the probability that a player wins a prize, if a prize is given when the player matches at least three of the first five numbers or the last number.

4. What is the probability that a hand of 13 cards contains no pairs?

5. What is the probability that a 13-card bridge hand contains

a) all 13 hearts?

b) 13 cards of the same suit?

c) seven spades and six clubs?

d) seven cards of one suit and six cards of a second suit?

e) four diamonds, six hearts, two spades, and one club?

f) four cards of one suit, six cards of a second suit, two cards of a third suit, and one card of the fourth suit?

6. What is the probability that a seven-card poker hand contains

a) four cards of one kind and three cards of a second kind?

b) three cards of one kind and pairs of each of two different kinds?

c) pairs of each of three different kinds and a single card of a fourth kind?

d) pairs of each of two different kinds and three cards of a third, fourth, and fifth kind?

e) cards of seven different kinds?

f) a seven-card flush?

g) a seven-card straight?

h) a seven-card straight flush?

An **octahedral die** has eight faces that are numbered 1 through 8.

7. a) What is the expected value of the number that comes up when a fair octahedral die is rolled?

b) What is the variance of the number that comes up when a fair octahedral die is rolled?

A **dodecahedral die** has 12 faces that are numbered 1 through 12.

8. a) What is the expected value of the number that comes up when a fair dodecahedral die is rolled?

b) What is the variance of the number that comes up when a fair dodecahedral die is rolled?

9. Suppose that a pair of fair octahedral dice is rolled.

a) What is the expected value of the sum of the numbers that come up?

b) What is the variance of the sum of the numbers that come up?

10. Suppose that a pair of fair dodecahedral dice is rolled.

a) What is the expected value of the sum of the numbers that come up?

b) What is the variance of the sum of the numbers that come up?

11. Suppose that a fair standard (cubic) die and a fair octahedral die are rolled together.

a) What is the expected value of the sum of the numbers that come up?

b) What is the variance of the sum of the numbers that come up?

12. Suppose that a fair octahedral die and a fair dodecahedral die are rolled together.

a) What is the expected value of the sum of the numbers that come up?

b) What is the variance of the sum of the numbers that come up?

13. Suppose n people, $n \geq 3$, play "odd person out" to decide who will buy the next round of refreshments. The n people each flip a fair coin simultaneously. If all the coins but one come up the same, the person whose coin comes up different buys the refreshments. Otherwise, the people flip the coins again and continue until just one coin comes up different from all the others.

a) What is the probability that the odd person out is decided in just one coin flip?

b) What is the probability that the odd person out is decided with the kth flip?

c) What is the expected number of flips needed to decide odd person out with n people?

14. Suppose that p and q are primes and $n = pq$. What is the probability that a randomly chosen positive integer less than n is not divisible by either p or q?

***15.** Suppose that m and n are positive integers. What is the probability that a randomly chosen positive integer less than mn is not divisible by either m or n?

16. Suppose that $E_1, E_2, \ldots, E_n$ are n events with $p(E_i) > 0$ for $i = 1, 2, \ldots, n$. Show that

$$\begin{aligned} &p(E_1 \cap E_2 \cap \cdots \cap E_n) \\ &\quad = p(E_1)p(E_2 \mid E_1)p(E_3 \mid E_1 \cap E_2) \\ &\qquad \cdots p(E_n \mid E_1 \cap E_2 \cap \cdots \cap E_{n-1}). \end{aligned}$$

17. There are three cards in a box. Both sides of one card are black, both sides of one card are red, and the third card has one black side and one red side. We pick a card at random and observe only one side.

a) If the side is black, what is the probability that the other side is also black?

b) What is the probability that the opposite side is the same color as the one we observed?

18. What is the probability that when a fair coin is flipped n times an equal number of heads and tails appear?

19. What is the probability that a randomly selected bit string of length 10 is a palindrome?

20. What is the probability that a randomly selected bit string of length 11 is a palindrome?

21. Consider the following game. A person flips a coin repeatedly until a head comes up. This person receives a payment of 2^n dollars if the first head comes up at the nth flip.

a) Let X be a random variable equal to the amount of money the person wins. Show that the expected value of X does not exist (that is, it is infinite). Show that a rational gambler, that is, someone willing to pay to play the game as long as the price to play is not more than the expected payoff, should be willing to wager any amount of money to play this game. (This is known as the **St. Petersburg paradox**. Why do you suppose it is called a paradox?)

b) Suppose that the person receives 2^n dollars if the first head comes up on the nth flip where $n < 8$ and $2^8 = 256$ dollars if the first head comes up on or after the eighth flip. What is the expected value of the amount of money the person wins? How much money should a person be willing to pay to play this game?

22. Suppose that n balls are tossed into b bins so that each ball is equally likely to fall into any of the bins and that the tosses are independent.

a) Find the probability that a particular ball lands in a specified bin.

b) What is the expected number of balls that land in a particular bin?

c) What is the expected number of balls tossed until a particular bin contains a ball?

***d)** What is the expected number of balls tossed until all bins contain a ball? [*Hint:* Let X_i denote the number of tosses required to have a ball land in an ith bin once $i - 1$ bins contain a ball. Find $E(X_i)$ and use the linearity of expectations.]

23. Suppose that A and B are events with probabilities $p(A) = 3/4$ and $p(B) = 1/3$.

a) What is the largest $p(A \cap B)$ can be? What is the smallest it can be? Give examples to show that both extremes for $p(A \cap B)$ are possible.

b) What is the largest $p(A \cup B)$ can be? What is the smallest it can be? Give examples to show that both extremes for $p(A \cup B)$ are possible.

24. Suppose that A and B are events with probabilities $p(A) = 2/3$ and $p(B) = 1/2$.

a) What is the largest $p(A \cap B)$ can be? What is the smallest it can be? Give examples to show that both extremes for $p(A \cap B)$ are possible.

b) What is the largest $p(A \cup B)$ can be? What is the smallest it can be? Give examples to show that both extremes for $p(A \cup B)$ are possible.

25. Recall from Definition 5 in Section 7.2 that the events $E_1, E_2, \ldots, E_n$ are **mutually independent** if $p(E_{i_1} \cap E_{i_2} \cap \cdots \cap E_{i_m}) = p(E_{i_1})p(E_{i_2}) \cdots p(E_{i_m})$ whenever i_j, $j = 1, 2, \ldots, m$, are integers with $1 \le i_1 < i_2 < \cdots < i_m \le n$ and $m \ge 2$.

a) Write out the conditions required for three events E_1, E_2, and E_3 to be mutually independent.

b) Let E_1, E_2, and E_3 be the events that the first flip comes up heads, that the second flip comes up tails, and that the third flip comes up tails, respectively, when a fair coin is flipped three times. Are E_1, E_2, and E_3 mutually independent?

c) Let E_1, E_2, and E_3 be the events that the first flip comes up heads, that the third flip comes up heads, and that an even number of heads come up, respectively, when a fair coin is flipped three times. Are E_1, E_2, and E_3 pairwise independent? Are they mutually independent?

d) Let E_1, E_2, and E_3 be the events that the first flip comes up heads, that the third flip comes up heads, and that exactly one of the first flip and third flip come up heads, respectively, when a fair coin is flipped three times. Are E_1, E_2, and E_3 pairwise independent? Are they mutually independent?

e) How many conditions must be checked to show that n events are mutually independent?

26. Suppose that A and B are events from a sample space S such that $p(A) \neq 0$ and $p(B) \neq 0$. Show that if $p(B \mid A) < p(B)$, then $p(A \mid B) < p(A)$.

In Exercise 27 we consider the **two children problem**, introduced in 1959 by Martin Garnder is his Mathematical Games column in *Scientific American*. A version of the puzzle asks: "We meet Mr. Smith as he is walking down the street with a young child whom he introduces as his son. He also tells us that he has two children. What is the probability that his other child is a son?" We will show that this puzzle is ambiguous, leading to a paradox, by showing that there are two reasonable answers to this problem and we will describe how to make the puzzle unambiguous.

***27. a)** Solve this puzzle in two different ways. First, answer the problem by considering the probability of the gender of the second child. Then, determine the probability differently, by considering the four different possibilities for a family of two children.

b) Show that the answer to the puzzle becomes unambiguous if we also know that Mr. Smith chose his walking companion at random from his two children.

c) Another variation of this puzzle asks "When we meet Mr. Smith, he tells us that he has two children and at least one is a son. What is the probability that his other child is a son?" Solve this variation of the puzzle, explaining why it is unambiguous.

28. In 2010, the puzzle designer Gary Foshee posed this problem: "Mr. Smith has two children, one of whom is a son born on a Tuesday. What is the probability that Mr. Smith has two sons?" Show that there are two different answers to this puzzle, depending on whether Mr. Smith specifically mentioned his son because he was born on a Tuesday or whether he randomly chose a child and reported its gender and birth day of the week. [*Hint:* For the first possibility, enumerate all the equally likely possibilities for the gender and birth day of the week of the other child. To do, this consider first the cases where the older child is a boy born on a Tuesday and then the case where the older child is not a boy born on a Tuesday.]

29. Let X be a random variable on a sample space S. Show that $V(aX + b) = a^2V(X)$ whenever a and b are real numbers.

30. Use Chebyshev's inequality to show that the probability that more than 10 people get the correct hat back when a hatcheck person returns hats at random does not exceed $1/100$ no matter how many people check their hats. [*Hint:* Use Example 6 and Exercise 43 in Section 7.4.]

31. Suppose that at least one of the events E_j, $j = 1, 2, \ldots, m$, is guaranteed to occur and no more than two can occur. Show that if $p(E_j) = q$ for $j = 1, 2, \ldots, m$ and $p(E_j \cap E_k) = r$ for $1 \leq j < k \leq m$, then $q \geq 1/m$ and $r \leq 2/m$.

32. Show that if m is a positive integer, then the probability that the mth success occurs on the $(m + n)$th trial when independent Bernoulli trials, each with probability p of success, are run, is $\binom{n+m-1}{n} q^n p^m$.

33. There are n different types of collectible cards you can get as prizes when you buy a particular product. Suppose that every time you buy this product it is equally likely that you get any type of these cards. Let X be the random variable equal to the number of products that need to be purchased to obtain at least one of each type of card and let X_j be the random variable equal to the number of additional products that must be purchased after j different cards have been collected until a new card is obtained for $j = 0, 1, \ldots, n - 1$.

a) Show that $X = \sum_{j=0}^{n-1} X_j$.

b) Show that after j distinct types of cards have been obtained, the card obtained with the next purchase will be a card of a new type with probability $(n - j)/n$.

c) Show that X_j has a geometric distribution with parameter $(n - j)/n$.

d) Use parts (a) and (c) to show that $E(X) = n \sum_{j=1}^{n} 1/j$.

e) Use the approximation $\sum_{j=1}^{n} 1/j \approx \ln n + \gamma$, where $\gamma = 0.57721\ldots$ is Euler's constant, to find the expected number of products that you need to buy to get one card of each type if there are 50 different types of cards.

34. The **maximum satisfiability problem** asks for an assignment of truth values to the variables in a compound proposition in conjunctive normal form (which expresses a compound proposition as the conjunction of clauses where each clause is the disjunction of two or more variables or their negations) that makes as many of these clauses true as possible. For example, three but not four of the clauses in

$$(p \vee q) \wedge (p \vee \neg q) \wedge (\neg p \vee r) \wedge (\neg p \vee \neg r)$$

can be made true by an assignment of truth values to p, q, and r. We will show that probabilistic methods can provide a lower bound for the number of clauses that can be made true by an assignment of truth values to the variables.

a) Suppose that there are n variables in a compound proposition in conjunctive normal form. If we pick a truth value for each variable randomly by flipping a coin and assigning true to the variable if the coin comes up heads and false if it comes up tails, what is the probability of each possible assignment of truth values to the n variables?

b) Assuming that each clause is the disjunction of exactly two distinct variables or their negations, what is the probability that a given clause is true, given the random assignment of truth values from part (a)?

c) Suppose that there are D clauses in the compound proposition. What is the expected number of these clauses that are true, given the random assignment of truth values of the variables?

d) Use part (c) to show that for every compound proposition in conjunctive normal form there is an assignment of truth values to the variables that makes at least 3/4 of the clauses true.

35. What is the probability that each player has a hand containing an ace when the 52 cards of a standard deck are dealt to four players?

***36.** The following method can be used to generate a random permutation of a sequence of n terms. First, interchange the nth term and the $r(n)$th term where $r(n)$ is a randomly selected integer with $1 \leq r(n) \leq n$. Next, interchange the $(n-1)$st term of the resulting sequence with its $r(n-1)$st term where $r(n-1)$ is a randomly selected integer with $1 \leq r(n-1) \leq n-1$. Continue this process until $j = n$, where at the jth step you interchange the $(n-j+1)$st term of the resulting sequence with its $r(n-j+1)$st term, where $r(n-j+1)$ is a randomly selected integer with $1 \leq r(n-j+1) \leq n-j+1$. Show that when this method is followed, each of the $n!$ different permutations of the terms of the sequence is equally likely to be generated. [*Hint:* Use mathematical induction, assuming that the probability that each of the permutations of $n-1$ terms produced by this procedure for a sequence of $n-1$ terms is $1/(n-1)!$.]

Computer Projects

Write programs with these input and output.

1. Given a real number p with $0 \leq p \leq 1$, generate random numbers taken from a Bernoulli distribution with probability p.

2. Given a positive integer n, generate a random permutation of the set $\{1, 2, 3, \ldots, n\}$. (See Exercise 36 in the Supplementary Exercises.)

3. Given positive integers m and n, generate m random permutations of the first n positive integers. Find the number of inversions in each permutation and determine the average number of these inversions.

4. Given a positive integer n, simulate n repeated flips of a biased coin with probability p of heads and determine the number of heads that come up. Display the cumulative results.

5. Given positive integers n and m, generate m random permutations of the first n positive integers. Sort each permutation using the insertion sort, counting the number of comparisons used. Determine the average number of comparisons used over all m permutations.

6. Given positive integers n and m, generate m random permutations of the first n positive integers. Sort each permutation using the version of the bubble sort that terminates when a pass has been made with no interchanges, counting the number of comparisons used. Determine the average number of comparisons used over all m permutations.

7. Given a positive integer m, simulate the collection of cards that come with the purchase of products to find the number of products that must be purchased to obtain a full set of m different collector cards. (See Supplementary Exercise 33.)

8. Given positive integers m and n, simulate the placement of n keys, where a record with key k is placed at location $h(k) = k \mathbf{\ mod\ } m$ and determine whether there is at least one collision.

9. Given a positive integer n, find the probability of selecting the six integers from the set $\{1, 2, \ldots, n\}$ that were mechanically selected in a lottery.

10. Simulate repeated trials of the Monty Hall Three-Door problem (Example 10 in Section 7.1) to calculate the probability of winning with each strategy.

11. Given a list of words and the empirical probabilities they occur in spam e-mails and in e-mails that are not spam, determine the probability that a new e-mail message is spam.

Computations and Explorations

Use a computational program or programs you have written to do these exercises.

1. Find the probabilities of each type of hand in five-card poker and rank the types of hands by their probability.

2. Find some conditions such that the expected value of buying a \$1 lottery ticket in the New Jersey Pick-6 lottery has an expected value of more than \$1. To win you have to select the six numbers drawn, where order does not matter, from the positive integers 1 to 49, inclusive. The winnings are split evenly among holders of winning tickets. Be sure to consider the total size of the pot going into the drawing and the number of people buying tickets.

3. Estimate the probability that two integers selected at random are relatively prime by testing a large number of randomly selected pairs of integers. Look up the theorem that gives this probability and compare your results with the correct probability.

4. Determine the number of people needed to ensure that the probability at least two of them have the same day of the year as their birthday is at least 70%, at least 80%, at least 90%, at least 95%, at least 98%, and at least 99%.

5. Generate a list of 100 randomly selected permutations of the set of the first 100 positive integers. (See Exercise 36 in the Supplementary Exercises.)
6. Given a collection of e-mail messages, each determined to be spam or not to be spam, develop a Bayesian filter based on the appearance of particular words in these messages.
7. Simulate the odd-person-out procedure (described in Exercise 13 of the Supplementary Exercises) for n people with $3 \leq n \leq 10$. Run a large number of trials for each value of n and use the results to estimate the expected number of flips needed to find the odd person out. Does your result agree with that found in Exercise 29 in Section 7.2? Vary the problem by supposing that exactly one person has a biased coin with probability of heads $p \neq 0.5$.
8. Given a positive integer n, simulate a hatcheck person randomly giving hats back to people. Determine the number of people who get the correct hat back.

Writing Projects

Respond to these with essays using outside sources.

1. Describe the origins of probability theory and the first uses of this theory, including those by Cardano, Pascal, and Laplace.
2. Describe the different bets you can make when you play roulette. Find the probability of each of these bets in the American version where the wheel contains the numbers 0 and 00. Which is the best bet and which is the worst for you?
3. Discuss the probability of winning when you play the game of blackjack versus a casino. Is there a winning strategy for the person playing against the house?
4. Investigate the game of craps and discuss the probability that the shooter wins and how close to a fair game it is.
5. Discuss issues involved in developing successful spam filters and the current situation in the war between spammers and people trying to filter spam out.
6. Discuss the history and solution of what is known as the Newton–Pepys problem, which asks which is most likely: rolling at least one six when six dice are rolled, rolling at least two sixes when 12 dice are rolled, or rolling at least three sixes when 18 dice are rolled.
7. Explain how Erdős and Rényi first used the probabilistic method and describe some other applications of this method.
8. Discuss the different types of probabilistic algorithms and describe some examples of each type.

CHAPTER

8 Advanced Counting Techniques

Many counting problems cannot be solved easily using the methods discussed in Chapter 6. One such problem is: How many bit strings of length n do not contain two consecutive zeros? To solve this problem, let a_n be the number of such strings of length n. An argument can be given that shows that the sequence $\{a_n\}$ satisfies the recurrence relation $a_{n+1} = a_n + a_{n-1}$ and the initial conditions $a_1 = 2$ and $a_2 = 3$. This recurrence relation and the initial conditions determine the sequence $\{a_n\}$. Moreover, an explicit formula can be found for a_n from the equation relating the terms of the sequence. As we will see, a similar technique can be used to solve many different types of counting problems.

We will discuss two ways that recurrence relations play important roles in the study of algorithms. First, we will introduce an important algorithmic paradigm known as dynamic programming. Algorithms that follow this paradigm break down a problem into overlapping subproblems. The solution to the problem is then found from the solutions to the subproblems through the use of a recurrence relation. Second, we will study another important algorithmic paradigm, divide-and-conquer. Algorithms that follow this paradigm can be used to solve a problem by recursively breaking it into a fixed number of nonoverlapping subproblems until these problems can be solved directly. The complexity of such algorithms can be analyzed using a special type of recurrence relation. In this chapter we will discuss a variety of divide-and-conquer algorithms and analyze their complexity using recurrence relations.

We will also see that many counting problems can be solved using formal power series, called generating functions, where the coefficients of powers of x represent terms of the sequence we are interested in. Besides solving counting problems, we will also be able to use generating functions to solve recurrence relations and to prove combinatorial identities.

Many other kinds of counting problems cannot be solved using the techniques discussed in Chapter 6, such as: How many ways are there to assign seven jobs to three employees so that each employee is assigned at least one job? How many primes are there less than 1000? Both of these problems can be solved by counting the number of elements in the union of sets. We will develop a technique, called the principle of inclusion–exclusion, that counts the number of elements in a union of sets, and we will show how this principle can be used to solve counting problems.

The techniques studied in this chapter, together with the basic techniques of Chapter 6, can be used to solve many counting problems.

8.1 Applications of Recurrence Relations

Introduction

Recall from Chapter 2 that a recursive definition of a sequence specifies one or more initial terms and a rule for determining subsequent terms from those that precede them. Also, recall that a rule of the latter sort (whether or not it is part of a recursive definition) is called a **recurrence relation** and that a sequence is called a *solution* of a recurrence relation if its terms satisfy the recurrence relation.

In this section we will show that such relations can be used to study and to solve counting problems. For example, suppose that the number of bacteria in a colony doubles every hour. If a colony begins with five bacteria, how many will be present in n hours? To solve this problem, let a_n be the number of bacteria at the end of n hours. Because the number of bacteria doubles every hour, the relationship $a_n = 2a_{n-1}$ holds whenever n is a positive integer. This recurrence relation, together with the initial condition $a_0 = 5$, uniquely determines a_n for all nonnegative integers n. We can find a formula for a_n using the iterative approach followed in Chapter 2, namely that $a_n = 5 \cdot 2^n$ for all nonnegative integers n.

Some of the counting problems that cannot be solved using the techniques discussed in Chapter 6 can be solved by finding recurrence relations involving the terms of a sequence, as was done in the problem involving bacteria. In this section we will study a variety of counting problems that can be modeled using recurrence relations. In Chapter 2 we developed methods for solving certain recurrence relation. In Section 8.2 we will study methods for finding explicit formulae for the terms of sequences that satisfy certain types of recurrence relations.

We conclude this section by introducing the algorithmic paradigm of dynamic programming. After explaining how this paradigm works, we will illustrate its use with an example.

Modeling With Recurrence Relations

Assessment

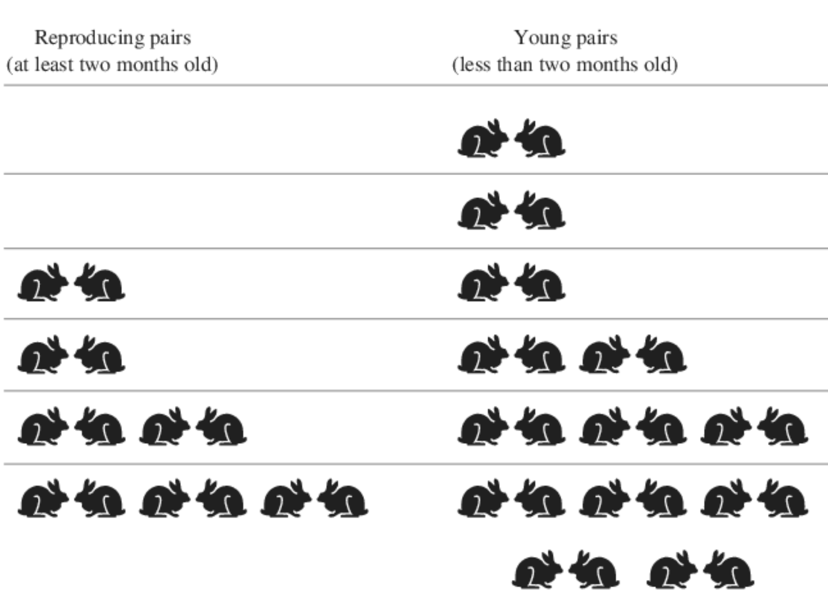
We can use recurrence relations to model a wide variety of problems, such as finding compound interest (see Example 11 in Section2.4), counting rabbits on an island, determining the number of moves in the Tower of Hanoi puzzle, and counting bit strings with certain properties.

Extra Examples

Example 1 shows how the population of rabbits on an island can be modeled using a recurrence relation.

EXAMPLE 1 **Rabbits and the Fibonacci Numbers** Consider this problem, which was originally posed by Leonardo Pisano, also known as Fibonacci, in the thirteenth century in his book *Liber abaci*. A young pair of rabbits (one of each sex) is placed on an island. A pair of rabbits does not breed until they are 2 months old. After they are 2 months old, each pair of rabbits produces another pair each month, as shown in Figure 1. Find a recurrence relation for the number of pairs of rabbits on the island after n months, assuming that no rabbits ever die.

Links



Month	Reproducing pairs	Young pairs	Total pairs
1	0	1	1
2	0	1	1
3	1	1	2
4	1	2	3
5	2	3	5
6	3	5	8

FIGURE 1 **Rabbits on an Island.**

Solution: Denote by f_n the number of pairs of rabbits after n months. We will show that f_n, $n = 1, 2, 3, \ldots$, are the terms of the Fibonacci sequence.

The rabbit population can be modeled using a recurrence relation. At the end of the first month, the number of pairs of rabbits on the island is $f_1 = 1$. Because this pair does not breed during the second month, $f_2 = 1$ also. To find the number of pairs after n months, add the number on the island the previous month, f_{n-1}, and the number of newborn pairs, which equals f_{n-2}, because each newborn pair comes from a pair at least 2 months old.

The Fibonacci numbers appear in many other places in nature, including the number of petals on flowers and the number of spirals on seedheads.

Consequently, the sequence $\{f_n\}$ satisfies the recurrence relation

$$f_n = f_{n-1} + f_{n-2}$$

for $n \geq 3$ together with the initial conditions $f_1 = 1$ and $f_2 = 1$. Because this recurrence relation and the initial conditions uniquely determine this sequence, the number of pairs of rabbits on the island after n months is given by the nth Fibonacci number. ◀

Demo

Example 2 involves a famous puzzle.

EXAMPLE 2 **The Tower of Hanoi** A popular puzzle of the late nineteenth century invented by the French mathematician Édouard Lucas, called the Tower of Hanoi, consists of three pegs mounted on a board together with disks of different sizes. Initially these disks are placed on the first peg in order of size, with the largest on the bottom (as shown in Figure 2). The rules of the puzzle allow disks to be moved one at a time from one peg to another as long as a disk is never placed on top of a smaller disk. The goal of the puzzle is to have all the disks on the second peg in order of size, with the largest on the bottom.

Links

Let H_n denote the number of moves needed to solve the Tower of Hanoi problem with n disks. Set up a recurrence relation for the sequence $\{H_n\}$.

Solution: Begin with n disks on peg 1. We can transfer the top $n-1$ disks, following the rules of the puzzle, to peg 3 using H_{n-1} moves (see Figure 3 for an illustration of the pegs and disks at this point). We keep the largest disk fixed during these moves. Then, we use one move to transfer the largest disk to the second peg. We can transfer the $n-1$ disks on peg 3 to peg 2 using H_{n-1} additional moves, placing them on top of the largest disk, which always stays fixed on the bottom of peg 2. Moreover, it is easy to see that the puzzle cannot be solved using fewer steps. This shows that

Schemes for efficiently backing up computer files on multiple tapes or other media are based on the moves used to solve the Tower of Hanoi puzzle.

$$H_n = 2H_{n-1} + 1.$$

The initial condition is $H_1 = 1$, because one disk can be transferred from peg 1 to peg 2, according to the rules of the puzzle, in one move.

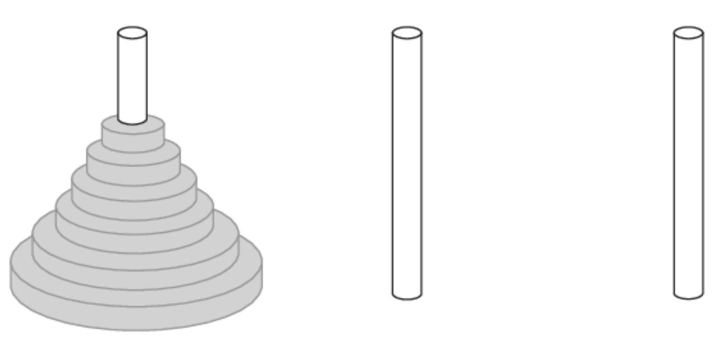


FIGURE 2 The Initial Position in the Tower of Hanoi.

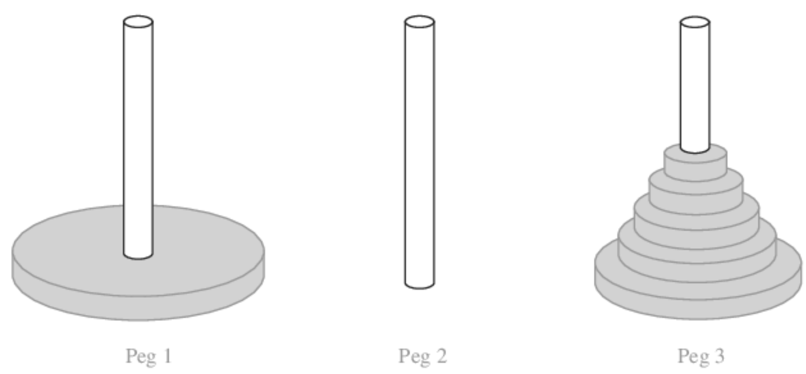


FIGURE 3 **An Intermediate Position in the Tower of Hanoi.**

We can use an iterative approach to solve this recurrence relation. Note that

$$\begin{aligned} H_n &= 2H_{n-1} + 1 \\ &= 2(2H_{n-2} + 1) + 1 = 2^2 H_{n-2} + 2 + 1 \\ &= 2^2(2H_{n-3} + 1) + 2 + 1 = 2^3 H_{n-3} + 2^2 + 2 + 1 \\ &\;\vdots \\ &= 2^{n-1} H_1 + 2^{n-2} + 2^{n-3} + \cdots + 2 + 1 \\ &= 2^{n-1} + 2^{n-2} + \cdots + 2 + 1 \\ &= 2^n - 1. \end{aligned}$$

We have used the recurrence relation repeatedly to express H_n in terms of previous terms of the sequence. In the next to last equality, the initial condition $H_1 = 1$ has been used. The last equality is based on the formula for the sum of the terms of a geometric series, which can be found in Theorem 1 in Section 2.4.

The iterative approach has produced the solution to the recurrence relation $H_n = 2H_{n-1} + 1$ with the initial condition $H_1 = 1$. This formula can be proved using mathematical induction. This is left for the reader as Exercise 1.

A myth created to accompany the puzzle tells of a tower in Hanoi where monks are transferring 64 gold disks from one peg to another, according to the rules of the puzzle. The myth says that the world will end when they finish the puzzle. How long after the monks started will the world end if the monks take one second to move a disk?

From the explicit formula, the monks require

$$2^{64} - 1 = 18{,}446{,}744{,}073{,}709{,}551{,}615$$

moves to transfer the disks. Making one move per second, it will take them more than 500 billion years to complete the transfer, so the world should survive a while longer than it already has. ◀

Remark: Many people have studied variations of the original Tower of Hanoi puzzle discussed in Example 2. Some variations use more pegs, some allow disks to be of the same size, and some restrict the types of allowable disk moves. One of the oldest and most interesting variations is the **Reve's puzzle**,* proposed in 1907 by Henry Dudeney in his book *The Canterbury Puzzles*. The Reve's puzzle involves pilgrims challenged by the Reve to move a stack of cheeses of varying sizes from the first of four stools to another stool without ever placing a cheese on one of smaller diameter. The Reve's puzzle, expressed in terms of pegs and disks, follows the same rules as the

***Reve*, more commonly spelled *reeve*, is an archaic word for *governor*.

Tower of Hanoi puzzle, except that four pegs are used. You may find it surprising that no one has been able to establish the minimum number of moves required to solve this puzzle for n disks. However, there is a conjecture, now more than 50 years old, that the minimum number of moves required equals the number of moves used by an algorithm invented by Frame and Stewart in 1939. (See Exercises 38–45 and [St94] for more information.)

Example 3 illustrates how recurrence relations can be used to count bit strings of a specified length that have a certain property.

EXAMPLE 3 Find a recurrence relation and give initial conditions for the number of bit strings of length n that do not have two consecutive 0s. How many such bit strings are there of length five?

Solution: Let a_n denote the number of bit strings of length n that do not have two consecutive 0s. To obtain a recurrence relation for $\{a_n\}$, note that by the sum rule, the number of bit strings of length n that do not have two consecutive 0s equals the number of such bit strings ending with a 0 plus the number of such bit strings ending with a 1. We will assume that $n \geq 3$, so that the bit string has at least three bits.

The bit strings of length n ending with 1 that do not have two consecutive 0s are precisely the bit strings of length $n-1$ with no two consecutive 0s with a 1 added at the end. Consequently, there are a_{n-1} such bit strings.

Bit strings of length n ending with a 0 that do not have two consecutive 0s must have 1 as their $(n-1)$st bit; otherwise they would end with a pair of 0s. It follows that the bit strings of length n ending with a 0 that have no two consecutive 0s are precisely the bit strings of length $n-2$ with no two consecutive 0s with 10 added at the end. Consequently, there are a_{n-2} such bit strings.

We conclude, as illustrated in Figure 4, that

$$a_n = a_{n-1} + a_{n-2}$$

for $n \geq 3$.

The initial conditions are $a_1 = 2$, because both bit strings of length one, 0 and 1 do not have consecutive 0s, and $a_2 = 3$, because the valid bit strings of length two are 01, 10, and 11. To obtain a_5, we use the recurrence relation three times to find that

$$a_3 = a_2 + a_1 = 3 + 2 = 5,$$
$$a_4 = a_3 + a_2 = 5 + 3 = 8,$$
$$a_5 = a_4 + a_3 = 8 + 5 = 13.$$

◀

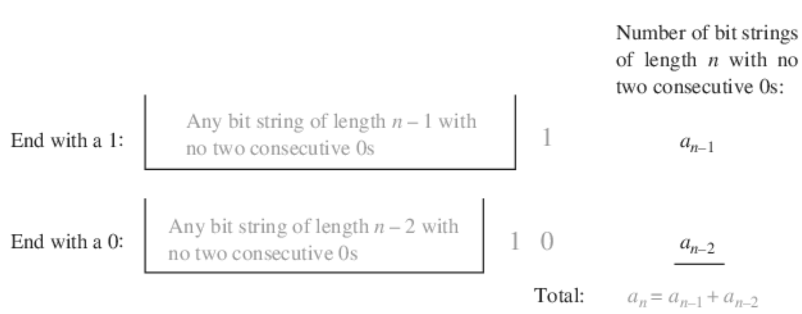


FIGURE 4 **Counting Bit Strings of Length n with No Two Consecutive 0s.**

Remark: Note that $\{a_n\}$ satisfies the same recurrence relation as the Fibonacci sequence. Because $a_1 = f_3$ and $a_2 = f_4$ it follows that $a_n = f_{n+2}$.

Example 4 shows how a recurrence relation can be used to model the number of codewords that are allowable using certain validity checks.

EXAMPLE 4 **Codeword Enumeration** A computer system considers a string of decimal digits a valid codeword if it contains an even number of 0 digits. For instance, 1230407869 is valid, whereas 120987045608 is not valid. Let a_n be the number of valid n-digit codewords. Find a recurrence relation for a_n.

Solution: Note that $a_1 = 9$ because there are 10 one-digit strings, and only one, namely, the string 0, is not valid. A recurrence relation can be derived for this sequence by considering how a valid n-digit string can be obtained from strings of $n - 1$ digits. There are two ways to form a valid string with n digits from a string with one fewer digit.

First, a valid string of n digits can be obtained by appending a valid string of $n - 1$ digits with a digit other than 0. This appending can be done in nine ways. Hence, a valid string with n digits can be formed in this manner in $9a_{n-1}$ ways.

Second, a valid string of n digits can be obtained by appending a 0 to a string of length $n - 1$ that is not valid. (This produces a string with an even number of 0 digits because the invalid string of length $n - 1$ has an odd number of 0 digits.) The number of ways that this can be done equals the number of invalid $(n - 1)$-digit strings. Because there are 10^{n-1} strings of length $n - 1$, and a_{n-1} are valid, there are $10^{n-1} - a_{n-1}$ valid n-digit strings obtained by appending an invalid string of length $n - 1$ with a 0.

Because all valid strings of length n are produced in one of these two ways, it follows that there are

$$\begin{aligned} a_n &= 9a_{n-1} + (10^{n-1} - a_{n-1}) \\ &= 8a_{n-1} + 10^{n-1} \end{aligned}$$

valid strings of length n. ◀

Example 5 establishes a recurrence relation that appears in many different contexts.

EXAMPLE 5 Find a recurrence relation for C_n, the number of ways to parenthesize the product of $n + 1$ numbers, $x_0 \cdot x_1 \cdot x_2 \cdot \cdots \cdot x_n$, to specify the order of multiplication. For example, $C_3 = 5$ because there are five ways to parenthesize $x_0 \cdot x_1 \cdot x_2 \cdot x_3$ to determine the order of multiplication:

$$\begin{array}{lll} ((x_0 \cdot x_1) \cdot x_2) \cdot x_3 & (x_0 \cdot (x_1 \cdot x_2)) \cdot x_3 & (x_0 \cdot x_1) \cdot (x_2 \cdot x_3) \\ x_0 \cdot ((x_1 \cdot x_2) \cdot x_3) & x_0 \cdot (x_1 \cdot (x_2 \cdot x_3)). & \end{array}$$

Solution: To develop a recurrence relation for C_n, we note that however we insert parentheses in the product $x_0 \cdot x_1 \cdot x_2 \cdot \cdots \cdot x_n$, one "$\cdot$" operator remains outside all parentheses, namely, the operator for the final multiplication to be performed. [For example, in $(x_0 \cdot (x_1 \cdot x_2)) \cdot x_3$, it is the final "$\cdot$", while in $(x_0 \cdot x_1) \cdot (x_2 \cdot x_3)$ it is the second "$\cdot$".] This final operator appears between two of the $n + 1$ numbers, say, x_k and x_{k+1}. There are $C_k C_{n-k-1}$ ways to insert parentheses to determine the order of the $n + 1$ numbers to be multiplied when the final operator appears between x_k and x_{k+1}, because there are C_k ways to insert parentheses in the product $x_0 \cdot x_1 \cdot \cdots \cdot x_k$ to determine the order in which these $k + 1$ numbers are to be multiplied and C_{n-k-1} ways to insert parentheses in the product $x_{k+1} \cdot x_{k+2} \cdot \cdots \cdot x_n$ to determine the order in which these $n-k$ numbers are to be multiplied. Because this final operator can appear between any two of the $n+1$ numbers, it follows that

$$\begin{aligned} C_n &= C_0C_{n-1} + C_1C_{n-2} + \cdots + C_{n-2}C_1 + C_{n-1}C_0 \\ &= \sum_{k=0}^{n-1} C_kC_{n-k-1}. \end{aligned}$$

Note that the initial conditions are $C_0 = 1$ and $C_1 = 1$. ◀

The recurrence relation in Example 5 can be solved using the method of generating functions, which will be discussed in Section 8.4. It can be shown that $C_n = C(2n, n)/(n+1)$ (see Exercise 41 in Section 8.4) and that $C_n \sim \frac{4^n}{n^{3/2}\sqrt{\pi}}$ (see [GrKnPa94]). The sequence $\{C_n\}$ is the sequence of **Catalan numbers**, named after Eugène Charles Catalan. This sequence appears as the solution of many different counting problems besides the one considered here (see the chapter on Catalan numbers in [MiRo91] or [Ro84a] for details).

Algorithms and Recurrence Relations

Recurrence relations play an important role in many aspects of the study of algorithms and their complexity. In Section 8.3, we will show how recurrence relations can be used to analyze the complexity of divide-and-conquer algorithms, such as the merge sort algorithm introduced in Section 5.4. As we will see in Section 8.3, divide-and-conquer algorithms recursively divide a problem into a fixed number of non-overlapping subproblems until they become simple enough to solve directly. We conclude this section by introducing another algorithmic paradigm known as **dynamic programming**, which can be used to solve many optimization problems efficiently.

An algorithm follows the dynamic programming paradigm when it recursively breaks down a problem into simpler overlapping subproblems, and computes the solution using the solutions of the subproblems. Generally, recurrence relations are used to find the overall solution from the solutions of the subproblems. Dynamic programming has been used to solve important problems in such diverse areas as economics, computer vision, speech recognition, artificial intelligence, computer graphics, and bioinformatics. In this section we will illustrate the use of dynamic programming by constructing an algorithm for solving a scheduling problem. Before doing so, we will relate the amusing origin of the name *dynamic programming*, which was

EUGÈNE CHARLES CATALAN (1814–1894) Eugène Catalan was born in Bruges, then part of France. His father became a successful architect in Paris while Eugène was a boy. Catalan attended a Parisian school for design hoping to follow in his father's footsteps. At 15, he won the job of teaching geometry to his design school classmates. After graduating, Catalan attended a school for the fine arts, but because of his mathematical aptitude his instructors recommended that he enter the École Polytechnique. He became a student there, but after his first year, he was expelled because of his politics. However, he was readmitted, and in 1835, he graduated and won a position at the Collège de Châlons sur Marne.

In 1838, Catalan returned to Paris where he founded a preparatory school with two other mathematicians, Sturm and Liouville. After teaching there for a short time, he was appointed to a position at the École Polytechnique. He received his doctorate from the École Polytechnique in 1841, but his political activity in favor of the French Republic hurt his career prospects. In 1846 Catalan held a position at the Collège de Charlemagne; he was appointed to the Lycée Saint Louis in 1849. However, when Catalan would not take a required oath of allegiance to the new Emperor Louis-Napoleon Bonaparte, he lost his job. For 13 years he held no permanent position. Finally, in 1865 he was appointed to a chair of mathematics at the University of Liège, Belgium, a position he held until his 1884 retirement.

Catalan made many contributions to number theory and to the related subject of continued fractions. He defined what are now known as the Catalan numbers when he solved the problem of dissecting a polygon into triangles using non-intersecting diagonals. Catalan is also well known for formulating what was known as the *Catalan conjecture*. This asserted that 8 and 9 are the only consecutive powers of integers, a conjecture not solved until 2003. Catalan wrote many textbooks, including several that became quite popular and appeared in as many as 12 editions. Perhaps this textbook will have a 12th edition someday!

introduced by the mathematician Richard Bellman in the 1950s. Bellman was working at the RAND Corporation on projects for the U.S. military, and at that time, the U.S. Secretary of Defense was hostile to mathematical research. Bellman decided that to ensure funding, he needed a name not containing the word mathematics for his method for solving scheduling and planning problems. He decided to use the adjective *dynamic* because, as he said "it's impossible to use the word dynamic in a pejorative sense" and he thought that dynamic programming was "something not even a Congressman could object to."

AN EXAMPLE OF DYNAMIC PROGRAMMING The problem we use to illustrate dynamic programming is related to the problem studied in Example 7 in Section 3.1. In that problem our goal was to schedule as many talks as possible in a single lecture hall. These talks have preset start and end times; once a talk starts, it continues until it ends; no two talks can proceed at the same time; and a talk can begin at the same time another one ends. We developed a greedy algorithm that always produces an optimal schedule, as we proved in Example 12 in Section 5.1. Now suppose that our goal is not to schedule the most talks possible, but rather to have the largest possible combined attendance of the scheduled talks.

We formalize this problem by supposing that we have n talks, where talk j begins at time t_j, ends at time e_j, and will be attended by w_j students. We want a schedule that maximizes the total number of student attendees. That is, we wish to schedule a subset of talks to maximize the sum of w_j over all scheduled talks. (Note that when a student attends more than one talk, this student is counted according to the number of talks attended.) We denote by $T(j)$ the maximum number of total attendees for an optimal schedule from the first j talks, so $T(n)$ is the maximal number of total attendees for an optimal schedule for all n talks.

We first sort the talks in order of increasing end time. After doing this, we renumber the talks so that $e_1 \leq e_2 \leq \cdots \leq e_n$. We say that two talks are **compatible** if they can be part of the same schedule, that is, if the times they are scheduled do not overlap (other than the possibility one ends and the other starts at the same time). We define $p(j)$ to be largest integer i, $i < j$, for which $e_i \leq s_j$, if such an integer exists, and $p(j) = 0$ otherwise. That is, talk $p(j)$ is the talk ending latest among talks compatible with talk j that end before talk j ends, if such a talk exists, and $p(j) = 0$ if there are no such talks.

Links

RICHARD BELLMAN (1920–1984) Richard Bellman, born in Brooklyn, where his father was a grocer, spent many hours in the museums and libraries of New York as a child. After graduating high school, he studied mathematics at Brooklyn College and graduated in 1941. He began postgraduate work at Johns Hopkins University, but because of the war, left to teach electronics at the University of Wisconsin. He was able to continue his mathematics studies at Wisconsin, and in 1943 he received his masters degree there. Later, Bellman entered Princeton University, teaching in a special U.S. Army program. In late 1944, he was drafted into the army. He was assigned to the Manhattan Project at Los Alamos where he worked in theoretical physics. After the war, he returned to Princeton and received his Ph.D. in 1946.

After briefly teaching at Princeton, he moved to Stanford University, where he attained tenure. At Stanford he pursued his fascination with number theory. However, Bellman decided to focus on mathematical questions arising from real-world problems. In 1952, he joined the RAND Corporation, working on multistage decision processes, operations research problems, and applications to the social sciences and medicine. He worked on many military projects while at RAND. In 1965 he left RAND to become professor of mathematics, electrical and biomedical engineering and medicine at the University of Southern California.

In the 1950s Bellman pioneered the use of dynamic programming, a technique invented earlier, in a wide range of settings. He is also known for his work on stochastic control processes, in which he introduced what is now called the Bellman equation. He coined the term *curse of dimensionality* to describe problems caused by the exponential increase in volume associated with adding extra dimensions to a space. He wrote an amazing number of books and research papers with many coauthors, including many on industrial production and economic systems. His work led to the application of computing techniques in a wide variety of areas ranging from the design of guidance systems for space vehicles, to network optimization, and even to pest control.

Tragically, in 1973 Bellman was diagnosed with a brain tumor. Although it was removed successfully, complications left him severely disabled. Fortunately, he managed to continue his research and writing during his remaining ten years of life. Bellman received many prizes and awards, including the first Norbert Wiener Prize in Applied Mathematics and the IEEE Gold Medal of Honor. He was elected to the National Academy of Sciences. He was held in high regard for his achievements, courage, and admirable qualities. Bellman was the father of two children.

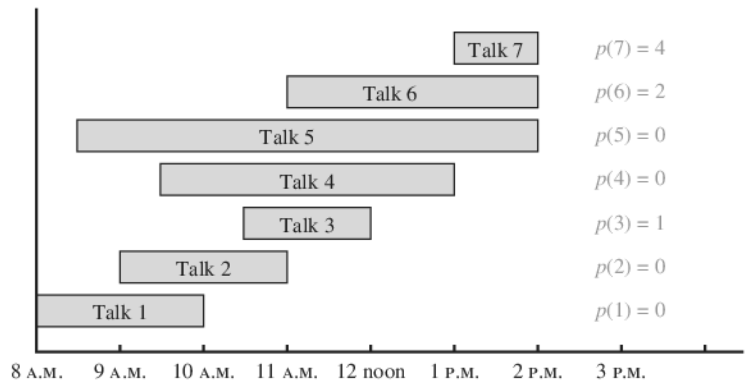


FIGURE 5 **A Schedule of Lectures with the Values of $p(n)$ Shown.**

EXAMPLE 6 Consider seven talks with these start times and end times, as illustrated in Figure 5.

Talk 1: start 8 A.M., end 10 A.M.
Talk 2: start 9 A.M., end 11 A.M.
Talk 3: start 10:30 A.M., end 12 noon
Talk 4: start 9:30 A.M., end 1 P.M.
Talk 5: start 8:30 A.M., end 2 P.M.
Talk 6: start 11 A.M., end 2 P.M.
Talk 7: start 1 P.M., end 2 P.M.

Find $p(j)$ for $j = 1, 2, \ldots, 7$.

Solution: We have $p(1) = 0$ and $p(2) = 0$, because no talks end before either of the first two talks begin. We have $p(3) = 1$ because talk 3 and talk 1 are compatible, but talk 3 and talk 2 are not compatible; $p(4) = 0$ because talk 4 is not compatible with any of talks 1, 2, and 3; $p(5) = 0$ because talk 5 is not compatible with any of talks 1, 2, 3, and 4; and $p(6) = 2$ because talk 6 and talk 2 are compatible, but talk 6 is not compatible with any of talks 3, 4, and 5. Finally, $p(7) = 4$, because talk 7 and talk 4 are compatible, but talk 7 is not compatible with either of talks 5 or 6. ◀

To develop a dynamic programming algorithm for this problem, we first develop a key recurrence relation. To do this, first note that if $j \leq n$, there are two possibilities for an optimal schedule of the first j talks (recall that we are assuming that the n talks are ordered by increasing end time): *(i)* talk j belongs to the optimal schedule or *(ii)* it does not.

Case (i): We know that talks $p(j) + 1, \ldots, j - 1$ do not belong to this schedule, for none of these other talks are compatible with talk j. Furthermore, the other talks in this optimal schedule must comprise an optimal schedule for talks $1, 2, \ldots, p(j)$. For if there were a better schedule for talks $1, 2, \ldots, p(j)$, by adding talk j, we will have a schedule better than the overall optimal schedule. Consequently, in case *(i)*, we have $T(j) = w_j + T(p(j))$.

Case (ii): When talk j does not belong to an optimal schedule, it follows that an optimal schedule from talks $1, 2, \ldots, j$ is the same as an optimal schedule from talks $1, 2, \ldots, j - 1$. Consequently, in case *(ii)*, we have $T(j) = T(j - 1)$. Combining cases *(i)* and *(ii)* leads us to the recurrence relation

$$T(j) = \max(w_j + T(p(j)), T(j - 1)).$$

Now that we have developed this recurrence relation, we can construct an efficient algorithm, Algorithm 1, for computing the maximum total number of attendees. We ensure that the algorithm is efficient by storing the value of each $T(j)$ after we compute it. This allows us to compute $T(j)$

only once. If we did not do this, the algorithm would have exponential worst-case complexity. The process of storing the values as each is computed is known as **memoization** and is an important technique for making recursive algorithms efficient.

ALGORITHM 1 Dynamic Programming Algorithm for Scheduling Talks.

procedure *Maximum Attendees* ($s_1, s_2, \ldots, s_n$: start times of talks;
$e_1, e_2, \ldots, e_n$: end times of talks; $w_1, w_2, \ldots, w_n$: number of attendees to talks)
sort talks by end time and relabel so that $e_1 \le e_2 \le \cdots \le e_n$
for $j := 1$ **to** n
    if no job i with $i < j$ is compatible with job j
        $p(j) = 0$
    else $p(j) := \max\{i \mid i < j$ and job i is compatible with job $j\}$
$T(0) := 0$
for $j := 1$ **to** n
    $T(j) := \max(w_j + T(p(j)), T(j-1))$
return $T(n)\{T(n)$ is the maximum number of attendees$\}$

In Algorithm 1 we determine the maximum number of attendees that can be achieved by a schedule of talks, but we do not find a schedule that achieves this maximum. To find talks we need to schedule, we use the fact that talk j belongs to an optimal solution for the first j talks if and only if $w_j + T(p(j)) \ge T(j-1)$. We leave it as Exercise 53 to construct an algorithm based on this observation that determines which talks should be scheduled to achieve the maximum total number of attendees.

Algorithm 1 is a good example of dynamic programming as the maximum total attendance is found using the optimal solutions of the overlapping subproblems, each of which determines the maximum total attendance of the first j talks for some j with $1 \le j \le n-1$. See Exercises 56 and 57 and Supplementary Exercises 14 and 17 for other examples of dynamic programming.

Exercises

1. Use mathematical induction to verify the formula derived in Example 2 for the number of moves required to complete the Tower of Hanoi puzzle.

2. a) Find a recurrence relation for the number of permutations of a set with n elements.

b) Use this recurrence relation to find the number of permutations of a set with n elements using iteration.

3. A vending machine dispensing books of stamps accepts only one-dollar coins, \$1 bills, and \$5 bills.

a) Find a recurrence relation for the number of ways to deposit n dollars in the vending machine, where the order in which the coins and bills are deposited matters.

b) What are the initial conditions?

c) How many ways are there to deposit \$10 for a book of stamps?

4. A country uses as currency coins with values of 1 peso, 2 pesos, 5 pesos, and 10 pesos and bills with values of 5 pesos, 10 pesos, 20 pesos, 50 pesos, and 100 pesos. Find a recurrence relation for the number of ways to pay a bill of n pesos if the order in which the coins and bills are paid matters.

5. How many ways are there to pay a bill of 17 pesos using the currency described in Exercise 4, where the order in which coins and bills are paid matters?

***6. a)** Find a recurrence relation for the number of strictly increasing sequences of positive integers that have 1 as their first term and n as their last term, where n is a positive integer. That is, sequences $a_1, a_2, \ldots, a_k$, where $a_1 = 1$, $a_k = n$, and $a_j < a_{j+1}$ for $j = 1, 2, \ldots, k-1$.

b) What are the initial conditions?

c) How many sequences of the type described in (a) are there when n is an integer with $n \ge 2$?

7. a) Find a recurrence relation for the number of bit strings of length n that contain a pair of consecutive 0s.

b) What are the initial conditions?

c) How many bit strings of length seven contain two consecutive 0s?

8. a) Find a recurrence relation for the number of bit strings of length n that contain three consecutive 0s.

b) What are the initial conditions?

c) How many bit strings of length seven contain three consecutive 0s?

9. a) Find a recurrence relation for the number of bit strings of length n that do not contain three consecutive 0s.

b) What are the initial conditions?

c) How many bit strings of length seven do not contain three consecutive 0s?

***10. a)** Find a recurrence relation for the number of bit strings of length n that contain the string 01.

b) What are the initial conditions?

c) How many bit strings of length seven contain the string 01?

11. a) Find a recurrence relation for the number of ways to climb n stairs if the person climbing the stairs can take one stair or two stairs at a time.

b) What are the initial conditions?

c) In how many ways can this person climb a flight of eight stairs?

12. a) Find a recurrence relation for the number of ways to climb n stairs if the person climbing the stairs can take one, two, or three stairs at a time.

b) What are the initial conditions?

c) In many ways can this person climb a flight of eight stairs?

A string that contains only 0s, 1s, and 2s is called a **ternary string**.

13. a) Find a recurrence relation for the number of ternary strings of length n that do not contain two consecutive 0s.

b) What are the initial conditions?

c) How many ternary strings of length six do not contain two consecutive 0s?

14. a) Find a recurrence relation for the number of ternary strings of length n that contain two consecutive 0s.

b) What are the initial conditions?

c) How many ternary strings of length six contain two consecutive 0s?

***15. a)** Find a recurrence relation for the number of ternary strings of length n that do not contain two consecutive 0s or two consecutive 1s.

b) What are the initial conditions?

c) How many ternary strings of length six do not contain two consecutive 0s or two consecutive 1s?

***16. a)** Find a recurrence relation for the number of ternary strings of length n that contain either two consecutive 0s or two consecutive 1s.

b) What are the initial conditions?

c) How many ternary strings of length six contain two consecutive 0s or two consecutive 1s?

***17. a)** Find a recurrence relation for the number of ternary strings of length n that do not contain consecutive symbols that are the same.

b) What are the initial conditions?

c) How many ternary strings of length six do not contain consecutive symbols that are the same?

****18. a)** Find a recurrence relation for the number of ternary strings of length n that contain two consecutive symbols that are the same.

b) What are the initial conditions?

c) How many ternary strings of length six contain consecutive symbols that are the same?

19. Messages are transmitted over a communications channel using two signals. The transmittal of one signal requires 1 microsecond, and the transmittal of the other signal requires 2 microseconds.

a) Find a recurrence relation for the number of different messages consisting of sequences of these two signals, where each signal in the message is immediately followed by the next signal, that can be sent in n microseconds.

b) What are the initial conditions?

c) How many different messages can be sent in 10 microseconds using these two signals?

20. A bus driver pays all tolls, using only nickels and dimes, by throwing one coin at a time into the mechanical toll collector.

a) Find a recurrence relation for the number of different ways the bus driver can pay a toll of n cents (where the order in which the coins are used matters).

b) In how many different ways can the driver pay a toll of 45 cents?

21. a) Find the recurrence relation satisfied by R_n, where R_n is the number of regions that a plane is divided into by n lines, if no two of the lines are parallel and no three of the lines go through the same point.

b) Find R_n using iteration.

***22. a)** Find the recurrence relation satisfied by R_n, where R_n is the number of regions into which the surface of a sphere is divided by n great circles (which are the intersections of the sphere and planes passing through the center of the sphere), if no three of the great circles go through the same point.

b) Find R_n using iteration.

***23. a)** Find the recurrence relation satisfied by S_n, where S_n is the number of regions into which three-dimensional space is divided by n planes if every three of the planes meet in one point, but no four of the planes go through the same point.

b) Find S_n using iteration.

24. Find a recurrence relation for the number of bit sequences of length n with an even number of 0s.

25. How many bit sequences of length seven contain an even number of 0s?

26. a) Find a recurrence relation for the number of ways to completely cover a $2 \times n$ checkerboard with 1×2 dominoes. [*Hint:* Consider separately the coverings where the position in the top right corner of the checkerboard is covered by a domino positioned horizontally and where it is covered by a domino positioned vertically.]

b) What are the initial conditions for the recurrence relation in part (a)?

c) How many ways are there to completely cover a 2×17 checkerboard with 1×2 dominoes?

27. a) Find a recurrence relation for the number of ways to lay out a walkway with slate tiles if the tiles are red, green, or gray, so that no two red tiles are adjacent and tiles of the same color are considered indistinguishable.

b) What are the initial conditions for the recurrence relation in part (a)?

c) How many ways are there to lay out a path of seven tiles as described in part (a)?

28. Show that the Fibonacci numbers satisfy the recurrence relation $f_n = 5f_{n-4} + 3f_{n-5}$ for $n = 5, 6, 7, \ldots,$ together with the initial conditions $f_0 = 0$, $f_1 = 1$, $f_2 = 1$, $f_3 = 2$, and $f_4 = 3$. Use this recurrence relation to show that f_{5n} is divisible by 5, for $n = 1, 2, 3, \ldots$.

***29.** Let $S(m, n)$ denote the number of onto functions from a set with m elements to a set with n elements. Show that $S(m, n)$ satisfies the recurrence relation

$$S(m, n) = n^m - \sum_{k=1}^{n-1} C(n, k)S(m, k)$$

whenever $m \geq n$ and $n > 1$, with the initial condition $S(m, 1) = 1$.

30. a) Write out all the ways the product $x_0 \cdot x_1 \cdot x_2 \cdot x_3 \cdot x_4$ can be parenthesized to determine the order of multiplication.

b) Use the recurrence relation developed in Example 5 to calculate C_4, the number of ways to parenthesize the product of five numbers so as to determine the order of multiplication. Verify that you listed the correct number of ways in part (a).

c) Check your result in part (b) by finding C_4, using the closed formula for C_n mentioned in the solution of Example 5.

31. a) Use the recurrence relation developed in Example 5 to determine C_5, the number of ways to parenthesize the product of six numbers so as to determine the order of multiplication.

b) Check your result with the closed formula for C_5 mentioned in the solution of Example 5.

***32.** In the Tower of Hanoi puzzle, suppose our goal is to transfer all n disks from peg 1 to peg 3, but we cannot move a disk directly between pegs 1 and 3. Each move of a disk must be a move involving peg 2. As usual, we cannot place a disk on top of a smaller disk.

a) Find a recurrence relation for the number of moves required to solve the puzzle for n disks with this added restriction.

b) Solve this recurrence relation to find a formula for the number of moves required to solve the puzzle for n disks.

c) How many different arrangements are there of the n disks on three pegs so that no disk is on top of a smaller disk?

d) Show that every allowable arrangement of the n disks occurs in the solution of this variation of the puzzle.

Exercises 33–37 deal with a variation of the **Josephus problem** described by Graham, Knuth, and Patashnik in [GrKnPa94]. This problem is based on an account by the historian Flavius Josephus, who was part of a band of 41 Jewish rebels trapped in a cave by the Romans during the Jewish-Roman war of the first century. The rebels preferred suicide to capture; they decided to form a circle and to repeatedly count off around the circle, killing every third rebel left alive. However, Josephus and another rebel did not want to be killed this way; they determined the positions where they should stand to be the last two rebels remaining alive. The variation we consider begins with n people, numbered 1 to n, standing around a circle. In each stage, every second person still left alive is eliminated until only one survives. We denote the number of the survivor by $J(n)$.

33. Determine the value of $J(n)$ for each integer n with $1 \leq n \leq 16$.

34. Use the values you found in Exercise 33 to conjecture a formula for $J(n)$. [*Hint:* Write $n = 2^m + k$, where m is a nonnegative integer and k is a nonnegative integer less than 2^m.]

35. Show that $J(n)$ satisfies the recurrence relation $J(2n) = 2J(n) - 1$ and $J(2n + 1) = 2J(n) + 1$, for $n \geq 1$, and $J(1) = 1$.

36. Use mathematical induction to prove the formula you conjectured in Exercise 34, making use of the recurrence relation from Exercise 35.

37. Determine $J(100)$, $J(1000)$, and $J(10{,}000)$ from your formula for $J(n)$.

Exercises 38–45 involve the Reve's puzzle, the variation of the Tower of Hanoi puzzle with four pegs and n disks. Before presenting these exercises, we describe the Frame–Stewart algorithm for moving the disks from peg 1 to peg 4 so that no disk is ever on top of a smaller one. This algorithm, given the number of disks n as input, depends on a choice of an integer k with $1 \leq k \leq n$. When there is only one disk, move it from peg 1 to peg 4 and stop. For $n > 1$, the algorithm proceeds recursively, using these three steps. Recursively move the stack of the $n - k$ smallest disks from peg 1 to peg 2, using all four pegs. Next move the stack of the k largest disks from peg 1 to peg 4, using the three-peg algorithm from the Tower of Hanoi puzzle without using the peg holding the $n - k$ smallest disks. Finally, recursively move the smallest $n - k$ disks to peg 4, using all four pegs. Frame and Stewart showed that to produce the fewest moves using their algorithm, k should be chosen to be the smallest integer such that n does not exceed $t_k = k(k+1)/2$, the kth triangular number, that is, $t_{k-1} < n \leq t_k$. The unsettled conjecture, known as **Frame's conjecture**, is that this algorithm uses the fewest number of moves required to solve the puzzle, no matter how the disks are moved.

38. Show that the Reve's puzzle with three disks can be solved using five, and no fewer, moves.

39. Show that the Reve's puzzle with four disks can be solved using nine, and no fewer, moves.

40. Describe the moves made by the Frame–Stewart algorithm, with k chosen so that the fewest moves are required, for

a) 5 disks. **b)** 6 disks. **c)** 7 disks. **d)** 8 disks.

***41.** Show that if $R(n)$ is the number of moves used by the Frame–Stewart algorithm to solve the Reve's puzzle with n disks, where k is chosen to be the smallest integer with $n \leq k(k+1)/2$, then $R(n)$ satisfies the recurrence relation $R(n) = 2R(n-k) + 2^k - 1$, with $R(0) = 0$ and $R(1) = 1$.

***42.** Show that if k is as chosen in Exercise 41, then $R(n) - R(n-1) = 2^{k-1}$.

***43.** Show that if k is as chosen in Exercise 41, then $R(n) = \sum_{i=1}^{k} i2^{i-1} - (t_k - n)2^{k-1}$.

***44.** Use Exercise 43 to give an upper bound on the number of moves required to solve the Reve's puzzle for all integers n with $1 \leq n \leq 25$.

***45.** Show that $R(n)$ is $O(\sqrt{n}2^{\sqrt{2n}})$.

Let $\{a_n\}$ be a sequence of real numbers. The **backward differences** of this sequence are defined recursively as shown next. The **first difference** ∇a_n is

$$\nabla a_n = a_n - a_{n-1}.$$

The **$(k+1)$st difference** $\nabla^{k+1} a_n$ is obtained from $\nabla^k a_n$ by

$$\nabla^{k+1} a_n = \nabla^k a_n - \nabla^k a_{n-1}.$$

46. Find ∇a_n for the sequence $\{a_n\}$, where

a) $a_n = 4$. **b)** $a_n = 2n$.

c) $a_n = n^2$. **d)** $a_n = 2^n$.

47. Find $\nabla^2 a_n$ for the sequences in Exercise 46.

48. Show that $a_{n-1} = a_n - \nabla a_n$.

49. Show that $a_{n-2} = a_n - 2\nabla a_n + \nabla^2 a_n$.

***50.** Prove that a_{n-k} can be expressed in terms of a_n, ∇a_n, $\nabla^2 a_n, \ldots, \nabla^k a_n$.

51. Express the recurrence relation $a_n = a_{n-1} + a_{n-2}$ in terms of a_n, ∇a_n, and $\nabla^2 a_n$.

52. Show that any recurrence relation for the sequence $\{a_n\}$ can be written in terms of a_n, ∇a_n, $\nabla^2 a_n, \ldots$. The resulting equation involving the sequences and its differences is called a **difference equation**.

***53.** Construct the algorithm described in the text after Algorithm 1 for determining which talks should be scheduled to maximize the total number of attendees and not just the maximum total number of attendees determined by Algorithm 1.

54. Use Algorithm 1 to determine the maximum number of total attendees in the talks in Example 6 if w_i, the number of attendees of talk i, $i = 1, 2, \ldots, 7$, is

a) 20, 10, 50, 30, 15, 25, 40.

b) 100, 5, 10, 20, 25, 40, 30.

c) 2, 3, 8, 5, 4, 7, 10.

d) 10, 8, 7, 25, 20, 30, 5.

55. For each part of Exercise 54, use your algorithm from Exercise 53 to find the optimal schedule for talks so that the total number of attendees is maximized.

56. In this exercise we will develop a dynamic programming algorithm for finding the maximum sum of consecutive terms of a sequence of real numbers. That is, given a sequence of real numbers $a_1, a_2, \ldots, a_n$, the algorithm computes the maximum sum $\sum_{i=j}^{k} a_i$ where $1 \leq j \leq k \leq n$.

a) Show that if all terms of the sequence are nonnegative, this problem is solved by taking the sum of all terms. Then, give an example where the maximum sum of consecutive terms is not the sum of all terms.

b) Let $M(k)$ be the maximum of the sums of consecutive terms of the sequence ending at a_k. That is, $M(k) = \max_{1\leq j\leq k} \sum_{i=j}^{k} a_i$. Explain why the recurrence relation $M(k) = \max(M(k-1) + a_k, a_k)$ holds for $k = 2, \ldots, n$.

c) Use part (b) to develop a dynamic programming algorithm for solving this problem.

d) Show each step your algorithm from part (c) uses to find the maximum sum of consecutive terms of the sequence $2, -3, 4, 1, -2, 3$.

e) Show that the worst-case complexity in terms of the number of additions and comparisons of your algorithm from part (c) is linear.

***57.** Dynamic programming can be used to develop an algorithm for solving the matrix-chain multiplication problem introduced in Section 3.3. This is the problem of determining how the product $\mathbf{A}_1\mathbf{A}_2 \cdots \mathbf{A}_n$ can be computed using the fewest integer multiplications, where $\mathbf{A}_1, \mathbf{A}_2, \ldots, \mathbf{A}_n$ are $m_1 \times m_2, m_2 \times m_3, \ldots, m_n \times m_{n+1}$ matrices, respectively, and each matrix has integer entries. Recall that by the associative law, the product does not depend on the order in which the matrices are multiplied.

a) Show that the brute-force method of determining the minimum number of integer multiplications needed to solve a matrix-chain multiplication problem has exponential worst-case complexity. [*Hint:* Do this by first showing that the order of multiplication of matrices is specified by parenthesizing the product. Then, use Example 5 and the result of part (c) of Exercise 41 in Section 8.4.]

b) Denote by $\mathbf{A}_{ij}$ the product $\mathbf{A}_i\mathbf{A}_{i+1}\ldots,\mathbf{A}_j$, and $M(i, j)$ the minimum number of integer multiplications required to find $\mathbf{A}_{ij}$. Show that if the least number of integer multiplications are used to compute $\mathbf{A}_{ij}$, where $i < j$, by splitting the product into the product of $\mathbf{A}_i$ through $\mathbf{A}_k$ and the product of $\mathbf{A}_{k+1}$ through $\mathbf{A}_j$, then the first k terms must be parenthesized so that $\mathbf{A}_{ik}$ is computed in the optimal way using $M(i, k)$ integer multiplications and $\mathbf{A}_{k+1,j}$ must be parenthesized so that $\mathbf{A}_{k+1,j}$ is computed in the optimal way using $M(k + 1, j)$ integer multiplications.

c) Explain why part (b) leads to the recurrence relation $M(i, j) = \min_{i\le k<j}(M(i, k) + M(k + 1, j) + m_i m_{k+1} m_{j+1})$ if $1 \le i \le j < j \le n$.

d) Use the recurrence relation in part (c) to construct an efficient algorithm for determining the order the n matrices should be multiplied to use the minimum number of integer multiplications. Store the partial results $M(i, j)$ as you find them so that your algorithm will not have exponential complexity.

e) Show that your algorithm from part (d) has $O(n^3)$ worst-case complexity in terms of multiplications of integers.

8.2 Solving Linear Recurrence Relations

Introduction

Links

A wide variety of recurrence relations occur in models. Some of these recurrence relations can be solved using iteration or some other ad hoc technique. However, one important class of recurrence relations can be explicitly solved in a systematic way. These are recurrence relations that express the terms of a sequence as linear combinations of previous terms.

DEFINITION 1 A *linear homogeneous recurrence relation of degree k with constant coefficients* is a recurrence relation of the form

$$a_n = c_1a_{n-1} + c_2a_{n-2} + \cdots + c_ka_{n-k},$$

where $c_1, c_2, \ldots, c_k$ are real numbers, and $c_k \neq 0$.

The recurrence relation in the definition is **linear** because the right-hand side is a sum of previous terms of the sequence each multiplied by a function of n. The recurrence relation is **homogeneous** because no terms occur that are not multiples of the a_js. The coefficients of the terms of the sequence are all **constants**, rather than functions that depend on n. The **degree** is k because a_n is expressed in terms of the previous k terms of the sequence.

A consequence of the second principle of mathematical induction is that a sequence satisfying the recurrence relation in the definition is uniquely determined by this recurrence relation and the k initial conditions

$$a_0 = C_0, a_1 = C_1, \ldots, a_{k-1} = C_{k-1}.$$

EXAMPLE 1 The recurrence relation $P_n = (1.11)P_{n-1}$ is a linear homogeneous recurrence relation of degree one. The recurrence relation $f_n = f_{n-1} + f_{n-2}$ is a linear homogeneous recurrence relation of degree two. The recurrence relation $a_n = a_{n-5}$ is a linear homogeneous recurrence relation of degree five. ◀

Example 2 presents some examples of recurrence relations that are not linear homogeneous recurrence relations with constant coefficients.

EXAMPLE 2 The recurrence relation $a_n = a_{n-1} + a_{n-2}^2$ is not linear. The recurrence relation $H_n = 2H_{n-1} + 1$ is not homogeneous. The recurrence relation $B_n = nB_{n-1}$ does not have constant coefficients. ◀

Linear homogeneous recurrence relations are studied for two reasons. First, they often occur in modeling of problems. Second, they can be systematically solved.

Solving Linear Homogeneous Recurrence Relations with Constant Coefficients

The basic approach for solving linear homogeneous recurrence relations is to look for solutions of the form $a_n = r^n$, where r is a constant. Note that $a_n = r^n$ is a solution of the recurrence relation $a_n = c_1a_{n-1} + c_2a_{n-2} + \cdots + c_ka_{n-k}$ if and only if

$$r^n = c_1r^{n-1} + c_2r^{n-2} + \cdots + c_kr^{n-k}.$$

When both sides of this equation are divided by r^{n-k} and the right-hand side is subtracted from the left, we obtain the equation

$$r^k - c_1r^{k-1} - c_2r^{k-2} - \cdots - c_{k-1}r - c_k = 0.$$

Consequently, the sequence $\{a_n\}$ with $a_n = r^n$ is a solution if and only if r is a solution of this last equation. We call this the **characteristic equation** of the recurrence relation. The solutions of this equation are called the **characteristic roots** of the recurrence relation. As we will see, these characteristic roots can be used to give an explicit formula for all the solutions of the recurrence relation.

We will first develop results that deal with linear homogeneous recurrence relations with constant coefficients of degree two. Then corresponding general results when the degree may be greater than two will be stated. Because the proofs needed to establish the results in the general case are more complicated, they will not be given here.

We now turn our attention to linear homogeneous recurrence relations of degree two. First, consider the case when there are two distinct characteristic roots.

THEOREM 1 Let c_1 and c_2 be real numbers. Suppose that $r^2 - c_1r - c_2 = 0$ has two distinct roots r_1 and r_2. Then the sequence $\{a_n\}$ is a solution of the recurrence relation $a_n = c_1a_{n-1} + c_2a_{n-2}$ if and only if $a_n = \alpha_1r_1^n + \alpha_2r_2^n$ for $n = 0, 1, 2, \ldots$, where α_1 and α_2 are constants.

Proof: We must do two things to prove the theorem. First, it must be shown that if r_1 and r_2 are the roots of the characteristic equation, and α_1 and α_2 are constants, then the sequence $\{a_n\}$ with $a_n = \alpha_1r_1^n + \alpha_2r_2^n$ is a solution of the recurrence relation. Second, it must be shown that if the sequence $\{a_n\}$ is a solution, then $a_n = \alpha_1r_1^n + \alpha_2r_2^n$ for some constants α_1 and α_2.

Now we will show that if $a_n = \alpha_1r_1^n + \alpha_2r_2^n$, then the sequence $\{a_n\}$ is a solution of the recurrence relation. Because r_1 and r_2 are roots of $r^2 - c_1r - c_2 = 0$, it follows that $r_1^2 = c_1r_1 + c_2$, $r_2^2 = c_1r_2 + c_2$.

From these equations, we see that

$$\begin{aligned}
c_1a_{n-1} + c_2a_{n-2} &= c_1(\alpha_1r_1^{n-1} + \alpha_2r_2^{n-1}) + c_2(\alpha_1r_1^{n-2} + \alpha_2r_2^{n-2})\\
&= \alpha_1r_1^{n-2}(c_1r_1 + c_2) + \alpha_2r_2^{n-2}(c_1r_2 + c_2)\\
&= \alpha_1r_1^{n-2}r_1^2 + \alpha_2r_2^{n-2}r_2^2\\
&= \alpha_1r_1^n + \alpha_2r_2^n\\
&= a_n.
\end{aligned}$$

This shows that the sequence $\{a_n\}$ with $a_n = \alpha_1r_1^n + \alpha_2r_2^n$ is a solution of the recurrence relation.

To show that every solution $\{a_n\}$ of the recurrence relation $a_n = c_1a_{n-1} + c_2a_{n-2}$ has $a_n = \alpha_1 r_1^n + \alpha_2 r_2^n$ for $n = 0, 1, 2, \ldots$, for some constants α_1 and α_2, suppose that $\{a_n\}$ is a solution of the recurrence relation, and the initial conditions $a_0 = C_0$ and $a_1 = C_1$ hold. It will be shown that there are constants α_1 and α_2 such that the sequence $\{a_n\}$ with $a_n = \alpha_1 r_1^n + \alpha_2 r_2^n$ satisfies these same initial conditions. This requires that

$$a_0 = C_0 = \alpha_1 + \alpha_2,$$
$$a_1 = C_1 = \alpha_1 r_1 + \alpha_2 r_2.$$

We can solve these two equations for α_1 and α_2. From the first equation it follows that $\alpha_2 = C_0 - \alpha_1$. Inserting this expression into the second equation gives

$$C_1 = \alpha_1 r_1 + (C_0 - \alpha_1) r_2.$$

Hence,

$$C_1 = \alpha_1(r_1 - r_2) + C_0 r_2.$$

This shows that

$$\alpha_1 = \frac{C_1 - C_0 r_2}{r_1 - r_2}$$

and

$$\alpha_2 = C_0 - \alpha_1 = C_0 - \frac{C_1 - C_0 r_2}{r_1 - r_2} = \frac{C_0 r_1 - C_1}{r_1 - r_2},$$

where these expressions for α_1 and α_2 depend on the fact that $r_1 \neq r_2$. (When $r_1 = r_2$, this theorem is not true.) Hence, with these values for α_1 and α_2, the sequence $\{a_n\}$ with $\alpha_1 r_1^n + \alpha_2 r_2^n$ satisfies the two initial conditions.

We know that $\{a_n\}$ and $\{\alpha_1 r_1^n + \alpha_2 r_2^n\}$ are both solutions of the recurrence relation $a_n = c_1a_{n-1} + c_2a_{n-2}$ and both satisfy the initial conditions when $n = 0$ and $n = 1$. Because there is a unique solution of a linear homogeneous recurrence relation of degree two with two initial conditions, it follows that the two solutions are the same, that is, $a_n = \alpha_1 r_1^n + \alpha_2 r_2^n$ for all nonnegative integers n. We have completed the proof by showing that a solution of the linear homogeneous recurrence relation with constant coefficients of degree two must be of the form $a_n = \alpha_1 r_1^n + \alpha_2 r_2^n$, where α_1 and α_2 are constants. ◁

The characteristic roots of a linear homogeneous recurrence relation with constant coefficients may be complex numbers. Theorem 1 (and also subsequent theorems in this section) still applies in this case. Recurrence relations with complex characteristic roots will not be discussed in the text. Readers familiar with complex numbers may wish to solve Exercises 38 and 39.

Examples 3 and 4 show how to use Theorem 1 to solve recurrence relations.

EXAMPLE 3 What is the solution of the recurrence relation

$$a_n = a_{n-1} + 2a_{n-2}$$

with $a_0 = 2$ and $a_1 = 7$?

Extra Examples

Solution: Theorem 1 can be used to solve this problem. The characteristic equation of the recurrence relation is $r^2 - r - 2 = 0$. Its roots are $r = 2$ and $r = -1$. Hence, the sequence $\{a_n\}$ is a solution to the recurrence relation if and only if

$$a_n = \alpha_1 2^n + \alpha_2(-1)^n,$$

for some constants α_1 and α_2. From the initial conditions, it follows that

$$a_0 = 2 = \alpha_1 + \alpha_2,$$
$$a_1 = 7 = \alpha_1 \cdot 2 + \alpha_2 \cdot (-1).$$

Solving these two equations shows that $\alpha_1 = 3$ and $\alpha_2 = -1$. Hence, the solution to the recurrence relation and initial conditions is the sequence $\{a_n\}$ with

$$a_n = 3 \cdot 2^n - (-1)^n.$$

◀

EXAMPLE 4 Find an explicit formula for the Fibonacci numbers.

Solution: Recall that the sequence of Fibonacci numbers satisfies the recurrence relation $f_n = f_{n-1} + f_{n-2}$ and also satisfies the initial conditions $f_0 = 0$ and $f_1 = 1$. The roots of the characteristic equation $r^2 - r - 1 = 0$ are $r_1 = (1 + \sqrt{5})/2$ and $r_2 = (1 - \sqrt{5})/2$. Therefore, from Theorem 1 it follows that the Fibonacci numbers are given by

$$f_n = \alpha_1 \left(\frac{1+\sqrt{5}}{2}\right)^n + \alpha_2 \left(\frac{1-\sqrt{5}}{2}\right)^n,$$

for some constants α_1 and α_2. The initial conditions $f_0 = 0$ and $f_1 = 1$ can be used to find these constants. We have

$$f_0 = \alpha_1 + \alpha_2 = 0,$$
$$f_1 = \alpha_1 \left(\frac{1+\sqrt{5}}{2}\right) + \alpha_2 \left(\frac{1-\sqrt{5}}{2}\right) = 1.$$

The solution to these simultaneous equations for α_1 and α_2 is

$$\alpha_1 = 1/\sqrt{5}, \qquad \alpha_2 = -1/\sqrt{5}.$$

Consequently, the Fibonacci numbers are given by

$$f_n = \frac{1}{\sqrt{5}} \left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}} \left(\frac{1-\sqrt{5}}{2}\right)^n.$$

◀

Theorem 1 does not apply when there is one characteristic root of multiplicity two. If this happens, then $a_n = nr_0^n$ is another solution of the recurrence relation when r_0 is a root of multiplicity two of the characteristic equation. Theorem 2 shows how to handle this case.

THEOREM 2 Let c_1 and c_2 be real numbers with $c_2 \neq 0$. Suppose that $r^2 - c_1 r - c_2 = 0$ has only one root r_0. A sequence $\{a_n\}$ is a solution of the recurrence relation $a_n = c_1 a_{n-1} + c_2 a_{n-2}$ if and only if $a_n = \alpha_1 r_0^n + \alpha_2 n r_0^n$, for $n = 0, 1, 2, \ldots$, where α_1 and α_2 are constants.

The proof of Theorem 2 is left as Exercise 10. Example 5 illustrates the use of this theorem.

EXAMPLE 5 What is the solution of the recurrence relation

$$a_n = 6a_{n-1} - 9a_{n-2}$$

with initial conditions $a_0 = 1$ and $a_1 = 6$?

Solution: The only root of $r^2 - 6r + 9 = 0$ is $r = 3$. Hence, the solution to this recurrence relation is

$$a_n = \alpha_1 3^n + \alpha_2 n 3^n$$

for some constants α_1 and α_2. Using the initial conditions, it follows that

$$a_0 = 1 = \alpha_1,$$
$$a_1 = 6 = \alpha_1 \cdot 3 + \alpha_2 \cdot 3.$$

Solving these two equations shows that $\alpha_1 = 1$ and $\alpha_2 = 1$. Consequently, the solution to this recurrence relation and the initial conditions is

$$a_n = 3^n + n3^n.$$

◀

We will now state the general result about the solution of linear homogeneous recurrence relations with constant coefficients, where the degree may be greater than two, under the assumption that the characteristic equation has distinct roots. The proof of this result will be left as Exercise 16.

THEOREM 3 Let $c_1, c_2, \ldots, c_k$ be real numbers. Suppose that the characteristic equation

$$r^k - c_1 r^{k-1} - \cdots - c_k = 0$$

has k distinct roots $r_1, r_2, \ldots, r_k$. Then a sequence $\{a_n\}$ is a solution of the recurrence relation

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k}$$

if and only if

$$a_n = \alpha_1 r_1^n + \alpha_2 r_2^n + \cdots + \alpha_k r_k^n$$

for $n = 0, 1, 2, \ldots$, where $\alpha_1, \alpha_2, \ldots, \alpha_k$ are constants.

We illustrate the use of the theorem with Example 6.

EXAMPLE 6 Find the solution to the recurrence relation

$$a_n = 6a_{n-1} - 11a_{n-2} + 6a_{n-3}$$

with the initial conditions $a_0 = 2$, $a_1 = 5$, and $a_2 = 15$.

Solution: The characteristic polynomial of this recurrence relation is

$$r^3 - 6r^2 + 11r - 6.$$

The characteristic roots are $r = 1$, $r = 2$, and $r = 3$, because $r^3 - 6r^2 + 11r - 6 = (r - 1)(r - 2)(r - 3)$. Hence, the solutions to this recurrence relation are of the form

$$a_n = \alpha_1 \cdot 1^n + \alpha_2 \cdot 2^n + \alpha_3 \cdot 3^n.$$

To find the constants α_1, α_2, and α_3, use the initial conditions. This gives

$$\begin{aligned} a_0 &= 2 = \alpha_1 + \alpha_2 + \alpha_3, \\ a_1 &= 5 = \alpha_1 + \alpha_2 \cdot 2 + \alpha_3 \cdot 3, \\ a_2 &= 15 = \alpha_1 + \alpha_2 \cdot 4 + \alpha_3 \cdot 9. \end{aligned}$$

When these three simultaneous equations are solved for α_1, α_2, and α_3, we find that $\alpha_1 = 1$, $\alpha_2 = -1$, and $\alpha_3 = 2$. Hence, the unique solution to this recurrence relation and the given initial conditions is the sequence $\{a_n\}$ with

$$a_n = 1 - 2^n + 2 \cdot 3^n.$$ ◀

We now state the most general result about linear homogeneous recurrence relations with constant coefficients, allowing the characteristic equation to have multiple roots. The key point is that for each root r of the characteristic equation, the general solution has a summand of the form $P(n)r^n$, where $P(n)$ is a polynomial of degree $m - 1$, with m the multiplicity of this root. We leave the proof of this result as Exercise 51.

THEOREM 4 Let $c_1, c_2, \ldots, c_k$ be real numbers. Suppose that the characteristic equation

$$r^k - c_1 r^{k-1} - \cdots - c_k = 0$$

has t distinct roots $r_1, r_2, \ldots, r_t$ with multiplicities $m_1, m_2, \ldots, m_t$, respectively, so that $m_i \geq 1$ for $i = 1, 2, \ldots, t$ and $m_1 + m_2 + \cdots + m_t = k$. Then a sequence $\{a_n\}$ is a solution of the recurrence relation

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k}$$

if and only if

$$\begin{aligned} a_n = \ & (\alpha_{1,0} + \alpha_{1,1} n + \cdots + \alpha_{1,m_1-1} n^{m_1-1}) r_1^n \\ & + (\alpha_{2,0} + \alpha_{2,1} n + \cdots + \alpha_{2,m_2-1} n^{m_2-1}) r_2^n \\ & + \cdots + (\alpha_{t,0} + \alpha_{t,1} n + \cdots + \alpha_{t,m_t-1} n^{m_t-1}) r_t^n \end{aligned}$$

for $n = 0, 1, 2, \ldots$, where $\alpha_{i,j}$ are constants for $1 \leq i \leq t$ and $0 \leq j \leq m_i - 1$.

Example 7 illustrates how Theorem 4 is used to find the general form of a solution of a linear homogeneous recurrence relation when the characteristic equation has several repeated roots.

EXAMPLE 7 Suppose that the roots of the characteristic equation of a linear homogeneous recurrence relation are 2, 2, 2, 5, 5, and 9 (that is, there are three roots, the root 2 with multiplicity three, the root 5 with multiplicity two, and the root 9 with multiplicity one). What is the form of the general solution?

Solution: By Theorem 4, the general form of the solution is

$$(\alpha_{1,0} + \alpha_{1,1}n + \alpha_{1,2}n^2)2^n + (\alpha_{2,0} + \alpha_{2,1}n)5^n + \alpha_{3,0}9^n.$$

◀

We now illustrate the use of Theorem 4 to solve a linear homogeneous recurrence relation with constant coefficients when the characteristic equation has a root of multiplicity three.

EXAMPLE 8 Find the solution to the recurrence relation

$$a_n = -3a_{n-1} - 3a_{n-2} - a_{n-3}$$

with initial conditions $a_0 = 1$, $a_1 = -2$, and $a_2 = -1$.

Solution: The characteristic equation of this recurrence relation is

$$r^3 + 3r^2 + 3r + 1 = 0.$$

Because $r^3 + 3r^2 + 3r + 1 = (r + 1)^3$, there is a single root $r = -1$ of multiplicity three of the characteristic equation. By Theorem 4 the solutions of this recurrence relation are of the form

$$a_n = \alpha_{1,0}(-1)^n + \alpha_{1,1}n(-1)^n + \alpha_{1,2}n^2(-1)^n.$$

To find the constants $\alpha_{1,0}$, $\alpha_{1,1}$, and $\alpha_{1,2}$, use the initial conditions. This gives

$$a_0 = 1 = \alpha_{1,0},$$
$$a_1 = -2 = -\alpha_{1,0} - \alpha_{1,1} - \alpha_{1,2},$$
$$a_2 = -1 = \alpha_{1,0} + 2\alpha_{1,1} + 4\alpha_{1,2}.$$

The simultaneous solution of these three equations is $\alpha_{1,0} = 1$, $\alpha_{1,1} = 3$, and $\alpha_{1,2} = -2$. Hence, the unique solution to this recurrence relation and the given initial conditions is the sequence $\{a_n\}$ with

$$a_n = (1 + 3n - 2n^2)(-1)^n.$$

◀

Linear Nonhomogeneous Recurrence Relations with Constant Coefficients

We have seen how to solve linear homogeneous recurrence relations with constant coefficients. Is there a relatively simple technique for solving a linear, but not homogeneous, recurrence relation with constant coefficients, such as $a_n = 3a_{n-1} + 2n$? We will see that the answer is yes for certain families of such recurrence relations.

The recurrence relation $a_n = 3a_{n-1} + 2n$ is an example of a **linear nonhomogeneous recurrence relation with constant coefficients**, that is, a recurrence relation of the form

$$a_n = c_1a_{n-1} + c_2a_{n-2} + \cdots + c_ka_{n-k} + F(n),$$

where $c_1, c_2, \ldots, c_k$ are real numbers and $F(n)$ is a function not identically zero depending only on n. The recurrence relation

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k}$$

is called the **associated homogeneous recurrence relation**. It plays an important role in the solution of the nonhomogeneous recurrence relation.

EXAMPLE 9 Each of the recurrence relations $a_n = a_{n-1} + 2^n$, $a_n = a_{n-1} + a_{n-2} + n^2 + n + 1$, $a_n = 3a_{n-1} + n3^n$, and $a_n = a_{n-1} + a_{n-2} + a_{n-3} + n!$ is a linear nonhomogeneous recurrence relation with constant coefficients. The associated linear homogeneous recurrence relations are $a_n = a_{n-1}$, $a_n = a_{n-1} + a_{n-2}$, $a_n = 3a_{n-1}$, and $a_n = a_{n-1} + a_{n-2} + a_{n-3}$, respectively. ◀

The key fact about linear nonhomogeneous recurrence relations with constant coefficients is that every solution is the sum of a particular solution and a solution of the associated linear homogeneous recurrence relation, as Theorem 5 shows.

THEOREM 5 If $\{a_n^{(p)}\}$ is a particular solution of the nonhomogeneous linear recurrence relation with constant coefficients

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k} + F(n),$$

then every solution is of the form $\{a_n^{(p)} + a_n^{(h)}\}$, where $\{a_n^{(h)}\}$ is a solution of the associated homogeneous recurrence relation

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k}.$$

Proof: Because $\{a_n^{(p)}\}$ is a particular solution of the nonhomogeneous recurrence relation, we know that

$$a_n^{(p)} = c_1 a_{n-1}^{(p)} + c_2 a_{n-2}^{(p)} + \cdots + c_k a_{n-k}^{(p)} + F(n).$$

Now suppose that $\{b_n\}$ is a second solution of the nonhomogeneous recurrence relation, so that

$$b_n = c_1 b_{n-1} + c_2 b_{n-2} + \cdots + c_k b_{n-k} + F(n).$$

Subtracting the first of these two equations from the second shows that

$$b_n - a_n^{(p)} = c_1(b_{n-1} - a_{n-1}^{(p)}) + c_2(b_{n-2} - a_{n-2}^{(p)}) + \cdots + c_k(b_{n-k} - a_{n-k}^{(p)}).$$

It follows that $\{b_n - a_n^p\}$ is a solution of the associated homogeneous linear recurrence, say, $\{a_n^{(h)}\}$. Consequently, $b_n = a_n^{(p)} + a_n^{(h)}$ for all n. ◁

By Theorem 5, we see that the key to solving nonhomogeneous recurrence relations with constant coefficients is finding a particular solution. Then every solution is a sum of this solution and a solution of the associated homogeneous recurrence relation. Although there is no general method for finding such a solution that works for every function $F(n)$, there are techniques that work for certain types of functions $F(n)$, such as polynomials and powers of constants. This is illustrated in Examples 10 and 11.

EXAMPLE 10 Find all solutions of the recurrence relation $a_n = 3a_{n-1} + 2n$. What is the solution with $a_1 = 3$?

Solution: To solve this linear nonhomogeneous recurrence relation with constant coefficients, we need to solve its associated linear homogeneous equation and to find a particular solution for the given nonhomogeneous equation. The associated linear homogeneous equation is $a_n = 3a_{n-1}$. Its solutions are $a_n^{(h)} = \alpha 3^n$, where α is a constant.

We now find a particular solution. Because $F(n) = 2n$ is a polynomial in n of degree one, a reasonable trial solution is a linear function in n, say, $p_n = cn + d$, where c and d are constants. To determine whether there are any solutions of this form, suppose that $p_n = cn + d$ is such a solution. Then the equation $a_n = 3a_{n-1} + 2n$ becomes $cn + d = 3(c(n-1) + d) + 2n$. Simplifying and combining like terms gives $(2 + 2c)n + (2d - 3c) = 0$. It follows that $cn + d$ is a solution if and only if $2 + 2c = 0$ and $2d - 3c = 0$. This shows that $cn + d$ is a solution if and only if $c = -1$ and $d = -3/2$. Consequently, $a_n^{(p)} = -n - 3/2$ is a particular solution.

By Theorem 5 all solutions are of the form

$$a_n = a_n^{(p)} + a_n^{(h)} = -n - \frac{3}{2} + \alpha \cdot 3^n,$$

where α is a constant.

To find the solution with $a_1 = 3$, let $n = 1$ in the formula we obtained for the general solution. We find that $3 = -1 - 3/2 + 3\alpha$, which implies that $\alpha = 11/6$. The solution we seek is $a_n = -n - 3/2 + (11/6)3^n$. ◀

EXAMPLE 11 Find all solutions of the recurrence relation

Extra Examples

$$a_n = 5a_{n-1} - 6a_{n-2} + 7^n.$$

Solution: This is a linear nonhomogeneous recurrence relation. The solutions of its associated homogeneous recurrence relation

$$a_n = 5a_{n-1} - 6a_{n-2}$$

are $a_n^{(h)} = \alpha_1 \cdot 3^n + \alpha_2 \cdot 2^n$, where α_1 and α_2 are constants. Because $F(n) = 7^n$, a reasonable trial solution is $a_n^{(p)} = C \cdot 7^n$, where C is a constant. Substituting the terms of this sequence into the recurrence relation implies that $C \cdot 7^n = 5C \cdot 7^{n-1} - 6C \cdot 7^{n-2} + 7^n$. Factoring out 7^{n-2}, this equation becomes $49C = 35C - 6C + 49$, which implies that $20C = 49$, or that $C = 49/20$. Hence, $a_n^{(p)} = (49/20)7^n$ is a particular solution. By Theorem 5, all solutions are of the form

$$a_n = \alpha_1 \cdot 3^n + \alpha_2 \cdot 2^n + (49/20)7^n.$$ ◀

In Examples 10 and 11, we made an educated guess that there are solutions of a particular form. In both cases we were able to find particular solutions. This was not an accident. Whenever $F(n)$ is the product of a polynomial in n and the nth power of a constant, we know exactly what form a particular solution has, as stated in Theorem 6. We leave the proof of Theorem 6 as Exercise 52.

THEOREM 6 Suppose that $\{a_n\}$ satisfies the linear nonhomogeneous recurrence relation

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k} + F(n),$$

where $c_1, c_2, \ldots, c_k$ are real numbers, and

$$F(n) = (b_t n^t + b_{t-1} n^{t-1} + \cdots + b_1 n + b_0) s^n,$$

where $b_0, b_1, \ldots, b_t$ and s are real numbers. When s is not a root of the characteristic equation of the associated linear homogeneous recurrence relation, there is a particular solution of the form

$$(p_t n^t + p_{t-1} n^{t-1} + \cdots + p_1 n + p_0) s^n.$$

When s is a root of this characteristic equation and its multiplicity is m, there is a particular solution of the form

$$n^m (p_t n^t + p_{t-1} n^{t-1} + \cdots + p_1 n + p_0) s^n.$$

Note that in the case when s is a root of multiplicity m of the characteristic equation of the associated linear homogeneous recurrence relation, the factor n^m ensures that the proposed particular solution will not already be a solution of the associated linear homogeneous recurrence relation. We next provide Example 12 to illustrate the form of a particular solution provided by Theorem 6.

EXAMPLE 12 What form does a particular solution of the linear nonhomogeneous recurrence relation $a_n = 6a_{n-1} - 9a_{n-2} + F(n)$ have when $F(n) = 3^n$, $F(n) = n3^n$, $F(n) = n^2 2^n$, and $F(n) = (n^2 + 1)3^n$?

Solution: The associated linear homogeneous recurrence relation is $a_n = 6a_{n-1} - 9a_{n-2}$. Its characteristic equation, $r^2 - 6r + 9 = (r - 3)^2 = 0$, has a single root, 3, of multiplicity two. To apply Theorem 6, with $F(n)$ of the form $P(n)s^n$, where $P(n)$ is a polynomial and s is a constant, we need to ask whether s is a root of this characteristic equation.

Because $s = 3$ is a root with multiplicity $m = 2$ but $s = 2$ is not a root, Theorem 6 tells us that a particular solution has the form $p_0 n^2 3^n$ if $F(n) = 3^n$, the form $n^2(p_1 n + p_0)3^n$ if $F(n) = n3^n$, the form $(p_2 n^2 + p_1 n + p_0)2^n$ if $F(n) = n^2 2^n$, and the form $n^2(p_2 n^2 + p_1 n + p_0)3^n$ if $F(n) = (n^2 + 1)3^n$. ◀

Care must be taken when $s = 1$ when solving recurrence relations of the type covered by Theorem 6. In particular, to apply this theorem with $F(n) = b_t n_t + b_{t-1} n_{t-1} + \cdots + b_1 n + b_0$, the parameter s takes the value $s = 1$ (even though the term 1^n does not explicitly appear). By the theorem, the form of the solution then depends on whether 1 is a root of the characteristic equation of the associated linear homogeneous recurrence relation. This is illustrated in Example 13, which shows how Theorem 6 can be used to find a formula for the sum of the first n positive integers.

EXAMPLE 13 Let a_n be the sum of the first n positive integers, so that

$$a_n = \sum_{k=1}^{n} k.$$

Note that a_n satisfies the linear nonhomogeneous recurrence relation

$$a_n = a_{n-1} + n.$$

(To obtain a_n, the sum of the first n positive integers, from a_{n-1}, the sum of the first $n-1$ positive integers, we add n.) Note that the initial condition is $a_1 = 1$.

The associated linear homogeneous recurrence relation for a_n is

$$a_n = a_{n-1}.$$

The solutions of this homogeneous recurrence relation are given by $a_n^{(h)} = c(1)^n = c$, where c is a constant. To find all solutions of $a_n = a_{n-1} + n$, we need find only a single particular solution. By Theorem 6, because $F(n) = n = n \cdot (1)^n$ and $s = 1$ is a root of degree one of the characteristic equation of the associated linear homogeneous recurrence relation, there is a particular solution of the form $n(p_1 n + p_0) = p_1 n^2 + p_0 n$.

Inserting this into the recurrence relation gives $p_1 n^2 + p_0 n = p_1(n-1)^2 + p_0(n-1) + n$. Simplifying, we see that $n(2p_1 - 1) + (p_0 - p_1) = 0$, which means that $2p_1 - 1 = 0$ and $p_0 - p_1 = 0$, so $p_0 = p_1 = 1/2$. Hence,

$$a_n^{(p)} = \frac{n^2}{2} + \frac{n}{2} = \frac{n(n+1)}{2}$$

is a particular solution. Hence, all solutions of the original recurrence relation $a_n = a_{n-1} + n$ are given by $a_n = a_n^{(h)} + a_n^{(p)} = c + n(n+1)/2$. Because $a_1 = 1$, we have $1 = a_1 = c + 1 \cdot 2/2 = c + 1$, so $c = 0$. It follows that $a_n = n(n+1)/2$. (This is the same formula given in Table 2 in Section 2.4 and derived previously.) ◀

Exercises

1. Determine which of these are linear homogeneous recurrence relations with constant coefficients. Also, find the degree of those that are.

a) $a_n = 3a_{n-1} + 4a_{n-2} + 5a_{n-3}$
b) $a_n = 2na_{n-1} + a_{n-2}$
c) $a_n = a_{n-1} + a_{n-4}$
d) $a_n = a_{n-1} + 2$
e) $a_n = a_{n-1}^2 + a_{n-2}$
f) $a_n = a_{n-2}$
g) $a_n = a_{n-1} + n$

2. Determine which of these are linear homogeneous recurrence relations with constant coefficients. Also, find the degree of those that are.

a) $a_n = 3a_{n-2}$
b) $a_n = 3$
c) $a_n = a_{n-1}^2$
d) $a_n = a_{n-1} + 2a_{n-3}$
e) $a_n = a_{n-1}/n$
f) $a_n = a_{n-1} + a_{n-2} + n + 3$
g) $a_n = 4a_{n-2} + 5a_{n-4} + 9a_{n-7}$

3. Solve these recurrence relations together with the initial conditions given.

a) $a_n = 2a_{n-1}$ for $n \geq 1$, $a_0 = 3$
b) $a_n = a_{n-1}$ for $n \geq 1$, $a_0 = 2$
c) $a_n = 5a_{n-1} - 6a_{n-2}$ for $n \geq 2$, $a_0 = 1$, $a_1 = 0$
d) $a_n = 4a_{n-1} - 4a_{n-2}$ for $n \geq 2$, $a_0 = 6$, $a_1 = 8$
e) $a_n = -4a_{n-1} - 4a_{n-2}$ for $n \geq 2$, $a_0 = 0$, $a_1 = 1$
f) $a_n = 4a_{n-2}$ for $n \geq 2$, $a_0 = 0$, $a_1 = 4$
g) $a_n = a_{n-2}/4$ for $n \geq 2$, $a_0 = 1$, $a_1 = 0$

4. Solve these recurrence relations together with the initial conditions given.

a) $a_n = a_{n-1} + 6a_{n-2}$ for $n \geq 2$, $a_0 = 3$, $a_1 = 6$
b) $a_n = 7a_{n-1} - 10a_{n-2}$ for $n \geq 2$, $a_0 = 2$, $a_1 = 1$
c) $a_n = 6a_{n-1} - 8a_{n-2}$ for $n \geq 2$, $a_0 = 4$, $a_1 = 10$
d) $a_n = 2a_{n-1} - a_{n-2}$ for $n \geq 2$, $a_0 = 4$, $a_1 = 1$
e) $a_n = a_{n-2}$ for $n \geq 2$, $a_0 = 5$, $a_1 = -1$
f) $a_n = -6a_{n-1} - 9a_{n-2}$ for $n \geq 2$, $a_0 = 3$, $a_1 = -3$
g) $a_{n+2} = -4a_{n+1} + 5a_n$ for $n \geq 0$, $a_0 = 2$, $a_1 = 8$

5. How many different messages can be transmitted in n microseconds using the two signals described in Exercise 19 in Section 8.1?

6. How many different messages can be transmitted in n microseconds using three different signals if one signal requires 1 microsecond for transmittal, the other two signals require 2 microseconds each for transmittal, and a signal in a message is followed immediately by the next signal?

7. In how many ways can a $2 \times n$ rectangular checkerboard be tiled using 1×2 and 2×2 pieces?

8. A model for the number of lobsters caught per year is based on the assumption that the number of lobsters caught in a year is the average of the number caught in the two previous years.

a) Find a recurrence relation for $\{L_n\}$, where L_n is the number of lobsters caught in year n, under the assumption for this model.

b) Find L_n if 100,000 lobsters were caught in year 1 and 300,000 were caught in year 2.

9. A deposit of \$100,000 is made to an investment fund at the beginning of a year. On the last day of each year two dividends are awarded. The first dividend is 20% of the amount in the account during that year. The second dividend is 45% of the amount in the account in the previous year.

a) Find a recurrence relation for $\{P_n\}$, where P_n is the amount in the account at the end of n years if no money is ever withdrawn.

b) How much is in the account after n years if no money has been withdrawn?

***10.** Prove Theorem 2.

11. The **Lucas numbers** satisfy the recurrence relation

$$L_n = L_{n-1} + L_{n-2},$$

and the initial conditions $L_0 = 2$ and $L_1 = 1$.

a) Show that $L_n = f_{n-1} + f_{n+1}$ for $n = 2, 3, \ldots$, where f_n is the nth Fibonacci number.

b) Find an explicit formula for the Lucas numbers.

12. Find the solution to $a_n = 2a_{n-1} + a_{n-2} - 2a_{n-3}$ for $n = 3, 4, 5, \ldots$, with $a_0 = 3$, $a_1 = 6$, and $a_2 = 0$.

13. Find the solution to $a_n = 7a_{n-2} + 6a_{n-3}$ with $a_0 = 9$, $a_1 = 10$, and $a_2 = 32$.

14. Find the solution to $a_n = 5a_{n-2} - 4a_{n-4}$ with $a_0 = 3$, $a_1 = 2$, $a_2 = 6$, and $a_3 = 8$.

15. Find the solution to $a_n = 2a_{n-1} + 5a_{n-2} - 6a_{n-3}$ with $a_0 = 7$, $a_1 = -4$, and $a_2 = 8$.

***16.** Prove Theorem 3.

17. Prove this identity relating the Fibonacci numbers and the binomial coefficients:

$$f_{n+1} = C(n, 0) + C(n-1, 1) + \cdots + C(n-k, k),$$

where n is a positive integer and $k = \lfloor n/2 \rfloor$. [*Hint:* Let $a_n = C(n, 0) + C(n-1, 1) + \cdots + C(n-k, k)$. Show that the sequence $\{a_n\}$ satisfies the same recurrence relation and initial conditions satisfied by the sequence of Fibonacci numbers.]

18. Solve the recurrence relation $a_n = 6a_{n-1} - 12a_{n-2} + 8a_{n-3}$ with $a_0 = -5$, $a_1 = 4$, and $a_2 = 88$.

19. Solve the recurrence relation $a_n = -3a_{n-1} - 3a_{n-2} - a_{n-3}$ with $a_0 = 5$, $a_1 = -9$, and $a_2 = 15$.

20. Find the general form of the solutions of the recurrence relation $a_n = 8a_{n-2} - 16a_{n-4}$.

21. What is the general form of the solutions of a linear homogeneous recurrence relation if its characteristic equation has roots $1, 1, 1, 1, -2, -2, -2, 3, 3, -4$?

22. What is the general form of the solutions of a linear homogeneous recurrence relation if its characteristic equation has the roots $-1, -1, -1, 2, 2, 5, 5, 7$?

23. Consider the nonhomogeneous linear recurrence relation $a_n = 3a_{n-1} + 2^n$.

a) Show that $a_n = -2^{n+1}$ is a solution of this recurrence relation.

b) Use Theorem 5 to find all solutions of this recurrence relation.

c) Find the solution with $a_0 = 1$.

24. Consider the nonhomogeneous linear recurrence relation $a_n = 2a_{n-1} + 2^n$.

a) Show that $a_n = n2^n$ is a solution of this recurrence relation.

b) Use Theorem 5 to find all solutions of this recurrence relation.

c) Find the solution with $a_0 = 2$.

25. a) Determine values of the constants A and B such that $a_n = An + B$ is a solution of recurrence relation $a_n = 2a_{n-1} + n + 5$.

b) Use Theorem 5 to find all solutions of this recurrence relation.

c) Find the solution of this recurrence relation with $a_0 = 4$.

26. What is the general form of the particular solution guaranteed to exist by Theorem 6 of the linear nonhomogeneous recurrence relation $a_n = 6a_{n-1} - 12a_{n-2} + 8a_{n-3} + F(n)$ if

a) $F(n) = n^2$? **b)** $F(n) = 2^n$?
c) $F(n) = n2^n$? **d)** $F(n) = (-2)^n$?
e) $F(n) = n^2 2^n$? **f)** $F(n) = n^3(-2)^n$?
g) $F(n) = 3$?

27. What is the general form of the particular solution guaranteed to exist by Theorem 6 of the linear nonhomogeneous recurrence relation $a_n = 8a_{n-2} - 16a_{n-4} + F(n)$ if

a) $F(n) = n^3$? **b)** $F(n) = (-2)^n$?
c) $F(n) = n2^n$? **d)** $F(n) = n^2 4^n$?
e) $F(n) = (n^2 - 2)(-2)^n$? **f)** $F(n) = n^4 2^n$?
g) $F(n) = 2$?

28. a) Find all solutions of the recurrence relation $a_n = 2a_{n-1} + 2n^2$.

b) Find the solution of the recurrence relation in part (a) with initial condition $a_1 = 4$.

29. a) Find all solutions of the recurrence relation $a_n = 2a_{n-1} + 3^n$.

b) Find the solution of the recurrence relation in part (a) with initial condition $a_1 = 5$.

30. a) Find all solutions of the recurrence relation $a_n = -5a_{n-1} - 6a_{n-2} + 42 \cdot 4^n$.

b) Find the solution of this recurrence relation with $a_1 = 56$ and $a_2 = 278$.

31. Find all solutions of the recurrence relation $a_n = 5a_{n-1} - 6a_{n-2} + 2^n + 3n$. [*Hint:* Look for a particular solution of the form $qn2^n + p_1 n + p_2$, where q, p_1, and p_2 are constants.]

32. Find the solution of the recurrence relation $a_n = 2a_{n-1} + 3 \cdot 2^n$.

33. Find all solutions of the recurrence relation $a_n = 4a_{n-1} - 4a_{n-2} + (n+1)2^n$.

34. Find all solutions of the recurrence relation $a_n = 7a_{n-1} - 16a_{n-2} + 12a_{n-3} + n4^n$ with $a_0 = -2$, $a_1 = 0$, and $a_2 = 5$.

35. Find the solution of the recurrence relation $a_n = 4a_{n-1} - 3a_{n-2} + 2^n + n + 3$ with $a_0 = 1$ and $a_1 = 4$.

36. Let a_n be the sum of the first n perfect squares, that is, $a_n = \sum_{k=1}^{n} k^2$. Show that the sequence $\{a_n\}$ satisfies the linear nonhomogeneous recurrence relation $a_n = a_{n-1} + n^2$ and the initial condition $a_1 = 1$. Use Theorem 6 to determine a formula for a_n by solving this recurrence relation.

37. Let a_n be the sum of the first n triangular numbers, that is, $a_n = \sum_{k=1}^{n} t_k$, where $t_k = k(k+1)/2$. Show that $\{a_n\}$ satisfies the linear nonhomogeneous recurrence relation $a_n = a_{n-1} + n(n+1)/2$ and the initial condition $a_1 = 1$. Use Theorem 6 to determine a formula for a_n by solving this recurrence relation.

38. a) Find the characteristic roots of the linear homogeneous recurrence relation $a_n = 2a_{n-1} - 2a_{n-2}$. [*Note:* These are complex numbers.]

b) Find the solution of the recurrence relation in part (a) with $a_0 = 1$ and $a_1 = 2$.

***39. a)** Find the characteristic roots of the linear homogeneous recurrence relation $a_n = a_{n-4}$. [*Note:* These include complex numbers.]

b) Find the solution of the recurrence relation in part (a) with $a_0 = 1$, $a_1 = 0$, $a_2 = -1$, and $a_3 = 1$.

***40.** Solve the simultaneous recurrence relations

$$a_n = 3a_{n-1} + 2b_{n-1}$$
$$b_n = a_{n-1} + 2b_{n-1}$$

with $a_0 = 1$ and $b_0 = 2$.

***41. a)** Use the formula found in Example 4 for f_n, the nth Fibonacci number, to show that f_n is the integer closest to

$$\frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n.$$

b) Determine for which n f_n is greater than

$$\frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n$$

and for which n f_n is less than

$$\frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n.$$

42. Show that if $a_n = a_{n-1} + a_{n-2}$, $a_0 = s$ and $a_1 = t$, where s and t are constants, then $a_n = sf_{n-1} + tf_n$ for all positive integers n.

43. Express the solution of the linear nonhomogenous recurrence relation $a_n = a_{n-1} + a_{n-2} + 1$ for $n \geq 2$ where $a_0 = 0$ and $a_1 = 1$ in terms of the Fibonacci numbers. [*Hint:* Let $b_n = a_n + 1$ and apply Exercise 42 to the sequence b_n.]

***44.** (*Linear algebra required*) Let $\mathbf{A}_n$ be the $n \times n$ matrix with 2s on its main diagonal, 1s in all positions next to a diagonal element, and 0s everywhere else. Find a recurrence relation for d_n, the determinant of $\mathbf{A}_n$. Solve this recurrence relation to find a formula for d_n.

45. Suppose that each pair of a genetically engineered species of rabbits left on an island produces two new pairs of rabbits at the age of 1 month and six new pairs of rabbits at the age of 2 months and every month afterward. None of the rabbits ever die or leave the island.

a) Find a recurrence relation for the number of pairs of rabbits on the island n months after one newborn pair is left on the island.

b) By solving the recurrence relation in (a) determine the number of pairs of rabbits on the island n months after one pair is left on the island.

46. Suppose that there are two goats on an island initially. The number of goats on the island doubles every year by natural reproduction, and some goats are either added or removed each year.

a) Construct a recurrence relation for the number of goats on the island at the start of the nth year, assuming that during each year an extra 100 goats are put on the island.

b) Solve the recurrence relation from part (a) to find the number of goats on the island at the start of the nth year.

c) Construct a recurrence relation for the number of goats on the island at the start of the nth year, assuming that n goats are removed during the nth year for each $n \geq 3$.

d) Solve the recurrence relation in part (c) for the number of goats on the island at the start of the nth year.

47. A new employee at an exciting new software company starts with a salary of \$50,000 and is promised that at the end of each year her salary will be double her salary of the previous year, with an extra increment of \$10,000 for each year she has been with the company.

a) Construct a recurrence relation for her salary for her nth year of employment.

b) Solve this recurrence relation to find her salary for her nth year of employment.

Some linear recurrence relations that do not have constant coefficients can be systematically solved. This is the case for recurrence relations of the form $f(n)a_n = g(n)a_{n-1} + h(n)$. Exercises 48–50 illustrate this.

***48. a)** Show that the recurrence relation

$$f(n)a_n = g(n)a_{n-1} + h(n),$$

for $n \geq 1$, and with $a_0 = C$, can be reduced to a recurrence relation of the form

$$b_n = b_{n-1} + Q(n)h(n),$$

where $b_n = g(n+1)Q(n+1)a_n$, with

$Q(n) = (f(1)f(2)\cdots f(n-1))/(g(1)g(2)\cdots g(n))$.

b) Use part (a) to solve the original recurrence relation to obtain

$$a_n = \frac{C + \sum_{i=1}^{n} Q(i)h(i)}{g(n+1)Q(n+1)}.$$

***49.** Use Exercise 48 to solve the recurrence relation $(n+1)a_n = (n+3)a_{n-1} + n$, for $n \geq 1$, with $a_0 = 1$.

50. It can be shown that C_n, the average number of comparisons made by the quick sort algorithm (described in preamble to Exercise 50 in Section 5.4), when sorting n elements in random order, satisfies the recurrence relation

$$C_n = n + 1 + \frac{2}{n}\sum_{k=0}^{n-1} C_k$$

for $n = 1, 2, \ldots,$ with initial condition $C_0 = 0$.

a) Show that $\{C_n\}$ also satisfies the recurrence relation $nC_n = (n+1)C_{n-1} + 2n$ for $n = 1, 2, \ldots$.

b) Use Exercise 48 to solve the recurrence relation in part (a) to find an explicit formula for C_n.

****51.** Prove Theorem 4.

****52.** Prove Theorem 6.

53. Solve the recurrence relation $T(n) = nT^2(n/2)$ with initial condition $T(1) = 6$ when $n = 2^k$ for some integer k. [*Hint:* Let $n = 2^k$ and then make the substitution $a_k = \log T(2^k)$ to obtain a linear nonhomogeneous recurrence relation.]

8.3 Divide-and-Conquer Algorithms and Recurrence Relations

Introduction

Many recursive algorithms take a problem with a given input and divide it into one or more smaller problems. This reduction is successively applied until the solutions of the smaller problems can be found quickly. For instance, we perform a binary search by reducing the search for an element in a list to the search for this element in a list half as long. We successively apply this reduction until one element is left. When we sort a list of integers using the merge sort, we split the list into two halves of equal size and sort each half separately. We then merge the two sorted halves. Another example of this type of recursive algorithm is a procedure for multiplying integers that reduces the problem of the multiplication of two integers to three multiplications of pairs of integers with half as many bits. This reduction is successively applied until integers with one bit are obtained. These procedures follow an important algorithmic paradigm known as **divide-and-conquer**, and are called **divide-and-conquer algorithms**, because they *divide* a problem into one or more instances of the same problem of smaller size and they *conquer* the problem by using the solutions of the smaller problems to find a solution of the original problem, perhaps with some additional work.

"Divide et impera" (translation: *"Divide and conquer"* - Julius Caesar

In this section we will show how recurrence relations can be used to analyze the computational complexity of divide-and-conquer algorithms. We will use these recurrence relations to estimate the number of operations used by many different divide-and-conquer algorithms, including several that we introduce in this section.

Divide-and-Conquer Recurrence Relations

Suppose that a recursive algorithm divides a problem of size n into a subproblems, where each subproblem is of size n/b (for simplicity, assume that n is a multiple of b; in reality, the smaller problems are often of size equal to the nearest integers either less than or equal to, or greater than or equal to, n/b). Also, suppose that a total of $g(n)$ extra operations are required in the conquer step of the algorithm to combine the solutions of the subproblems into a solution of the original problem. Then, if $f(n)$ represents the number of operations required to solve the problem of size n, it follows that f satisfies the recurrence relation

$$f(n) = af(n/b) + g(n).$$

This is called a **divide-and-conquer recurrence relation**.

We will first set up the divide-and-conquer recurrence relations that can be used to study the complexity of some important algorithms. Then we will show how to use these divide-and-conquer recurrence relations to estimate the complexity of these algorithms.

EXAMPLE 1 Extra Examples **Binary Search** We introduced a binary search algorithm in Section 3.1. This binary search algorithm reduces the search for an element in a search sequence of size n to the binary search for this element in a search sequence of size $n/2$, when n is even. (Hence, the problem of size n has been reduced to *one* problem of size $n/2$.) Two comparisons are needed to implement this reduction (one to determine which half of the list to use and the other to determine whether any terms of the list remain). Hence, if $f(n)$ is the number of comparisons required to search for an element in a search sequence of size n, then

$$f(n) = f(n/2) + 2$$

when n is even. ◀

EXAMPLE 2 **Finding the Maximum and Minimum of a Sequence** Consider the following algorithm for locating the maximum and minimum elements of a sequence $a_1, a_2, \ldots, a_n$. If $n = 1$, then a_1 is the maximum and the minimum. If $n > 1$, split the sequence into two sequences, either where both have the same number of elements or where one of the sequences has one more element than the other. The problem is reduced to finding the maximum and minimum of each of the two smaller sequences. The solution to the original problem results from the comparison of the separate maxima and minima of the two smaller sequences to obtain the overall maximum and minimum.

Let $f(n)$ be the total number of comparisons needed to find the maximum and minimum elements of the sequence with n elements. We have shown that a problem of size n can be reduced into two problems of size $n/2$, when n is even, using two comparisons, one to compare the maxima of the two sequences and the other to compare the minima of the two sequences. This gives the recurrence relation

$$f(n) = 2f(n/2) + 2$$

when n is even. ◀

EXAMPLE 3 **Merge Sort** The merge sort algorithm (introduced in Section 5.4) splits a list to be sorted with n items, where n is even, into two lists with $n/2$ elements each, and uses fewer than n comparisons to merge the two sorted lists of $n/2$ items each into one sorted list. Consequently, the number of comparisons used by the merge sort to sort a list of n elements is less than $M(n)$, where the function $M(n)$ satisfies the divide-and-conquer recurrence relation

$$M(n) = 2M(n/2) + n.$$ ◀

EXAMPLE 4 Links **Fast Multiplication of Integers** Surprisingly, there are more efficient algorithms than the conventional algorithm (described in Section 4.2) for multiplying integers. One of these algorithms, which uses a divide-and-conquer technique, will be described here. This fast multiplication algorithm proceeds by splitting each of two $2n$-bit integers into two blocks, each with n bits. Then, the original multiplication is reduced from the multiplication of two $2n$-bit integers to three multiplications of n-bit integers, plus shifts and additions.

Suppose that a and b are integers with binary expansions of length $2n$ (add initial bits of zero in these expansions if necessary to make them the same length). Let

$$a = (a_{2n-1}a_{2n-2}\cdots a_1a_0)_2 \quad \text{and} \quad b = (b_{2n-1}b_{2n-2}\cdots b_1b_0)_2.$$

Let

$$a = 2^n A_1 + A_0, \quad b = 2^n B_1 + B_0,$$

where

$$A_1 = (a_{2n-1} \cdots a_{n+1} a_n)_2, \qquad A_0 = (a_{n-1} \cdots a_1 a_0)_2,$$
$$B_1 = (b_{2n-1} \cdots b_{n+1} b_n)_2, \qquad B_0 = (b_{n-1} \cdots b_1 b_0)_2.$$

The algorithm for fast multiplication of integers is based on the fact that ab can be rewritten as

$$ab = (2^{2n} + 2^n) A_1 B_1 + 2^n (A_1 - A_0)(B_0 - B_1) + (2^n + 1) A_0 B_0.$$

The important fact about this identity is that it shows that the multiplication of two $2n$-bit integers can be carried out using three multiplications of n-bit integers, together with additions, subtractions, and shifts. This shows that if $f(n)$ is the total number of bit operations needed to multiply two n-bit integers, then

$$f(2n) = 3f(n) + Cn.$$

The reasoning behind this equation is as follows. The three multiplications of n-bit integers are carried out using $3f(n)$-bit operations. Each of the additions, subtractions, and shifts uses a constant multiple of n-bit operations, and Cn represents the total number of bit operations used by these operations. ◀

EXAMPLE 5 **Fast Matrix Multiplication** In Example 7 of Section 3.3 we showed that multiplying two $n \times n$ matrices using the definition of matrix multiplication required n^3 multiplications and $n^2(n-1)$ additions. Consequently, computing the product of two $n \times n$ matrices in this way requires $O(n^3)$ operations (multiplications and additions). Surprisingly, there are more efficient divide-and-conquer algorithms for multiplying two $n \times n$ matrices. Such an algorithm, invented by Volker Strassen in 1969, reduces the multiplication of two $n \times n$ matrices, when n is even, to seven multiplications of two $(n/2) \times (n/2)$ matrices and 15 additions of $(n/2) \times (n/2)$ matrices. (See [CoLeRiSt09] for the details of this algorithm.) Hence, if $f(n)$ is the number of operations (multiplications and additions) used, it follows that

$$f(n) = 7f(n/2) + 15n^2/4$$

when n is even. ◀

As Examples 1–5 show, recurrence relations of the form $f(n) = af(n/b) + g(n)$ arise in many different situations. It is possible to derive estimates of the size of functions that satisfy such recurrence relations. Suppose that f satisfies this recurrence relation whenever n is divisible by b. Let $n = b^k$, where k is a positive integer. Then

$$\begin{aligned} f(n) &= af(n/b) + g(n) \\ &= a^2 f(n/b^2) + ag(n/b) + g(n) \\ &= a^3 f(n/b^3) + a^2 g(n/b^2) + ag(n/b) + g(n) \\ &\ \ \vdots \\ &= a^k f(n/b^k) + \sum_{j=0}^{k-1} a^j g(n/b^j). \end{aligned}$$