

logical equivalences, using one of the equivalences in Table 6 at a time, starting with $\neg(p \rightarrow q)$ and ending with $\neg p \wedge \neg q$. We have the following equivalences.

$$\begin{aligned}\neg(p \rightarrow q) &\equiv \neg(\neg p \vee q) && \text{by Example 3} \\ &\equiv \neg(\neg p) \wedge \neg q && \text{by the second De Morgan law} \\ &\equiv p \wedge \neg q && \text{by the double negation law}\end{aligned}$$



EXAMPLE 7 Show that $\neg(p \vee (\neg p \wedge q))$ and $\neg p \wedge \neg q$ are logically equivalent by developing a series of logical equivalences.

Solution: We will use one of the equivalences in Table 6 at a time, starting with $\neg(p \vee (\neg p \wedge q))$ and ending with $\neg p \wedge \neg q$. (Note: we could also easily establish this equivalence using a truth table.) We have the following equivalences.

$$\begin{aligned}\neg(p \vee (\neg p \wedge q)) &\equiv \neg p \wedge \neg(\neg p \wedge q) && \text{by the second De Morgan law} \\ &\equiv \neg p \wedge [\neg(\neg p) \vee \neg q] && \text{by the first De Morgan law} \\ &\equiv \neg p \wedge (p \vee \neg q) && \text{by the double negation law} \\ &\equiv (\neg p \wedge p) \vee (\neg p \wedge \neg q) && \text{by the second distributive law} \\ &\equiv \mathbf{F} \vee (\neg p \wedge \neg q) && \text{because } \neg p \wedge p \equiv \mathbf{F} \\ &\equiv (\neg p \wedge \neg q) \vee \mathbf{F} && \text{by the commutative law for disjunction} \\ &\equiv \neg p \wedge \neg q && \text{by the identity law for } \mathbf{F}\end{aligned}$$

Consequently $\neg(p \vee (\neg p \wedge q))$ and $\neg p \wedge \neg q$ are logically equivalent.



EXAMPLE 8 Show that $(p \wedge q) \rightarrow (p \vee q)$ is a tautology.

Solution: To show that this statement is a tautology, we will use logical equivalences to demonstrate that it is logically equivalent to \mathbf{T} . (Note: This could also be done using a truth table.)

$$\begin{aligned}(p \wedge q) \rightarrow (p \vee q) &\equiv \neg(p \wedge q) \vee (p \vee q) && \text{by Example 3} \\ &\equiv (\neg p \vee \neg q) \vee (p \vee q) && \text{by the first De Morgan law} \\ &\equiv (\neg p \vee p) \vee (\neg q \vee q) && \text{by the associative and commutative laws for disjunction} \\ &\equiv \mathbf{T} \vee \mathbf{T} && \text{by Example 1 and the commutative law for disjunction} \\ &\equiv \mathbf{T} && \text{by the domination law}\end{aligned}$$



Propositional Satisfiability

A compound proposition is **satisfiable** if there is an assignment of truth values to its variables that makes it true. When no such assignments exists, that is, when the compound proposition is false for all assignments of truth values to its variables, the compound proposition is **unsatisfiable**.

Note that a compound proposition is unsatisfiable if and only if its negation is true for all assignments of truth values to the variables, that is, if and only if its negation is a tautology.

When we find a particular assignment of truth values that makes a compound proposition true, we have shown that it is satisfiable; such an assignment is called a **solution** of this particular

satisfiability problem. However, to show that a compound proposition is unsatisfiable, we need to show that *every* assignment of truth values to its variables makes it false. Although we can always use a truth table to determine whether a compound proposition is satisfiable, it is often more efficient not to, as Example 9 demonstrates.

EXAMPLE 9 Determine whether each of the compound propositions $(p \vee \neg q) \wedge (q \vee \neg r) \wedge (r \vee \neg p)$, $(p \vee q \vee r) \wedge (\neg p \vee \neg q \vee \neg r)$, and $(p \vee \neg q) \wedge (q \vee \neg r) \wedge (r \vee \neg p) \wedge (p \vee q \vee r) \wedge (\neg p \vee \neg q \vee \neg r)$ is satisfiable.

Solution: Instead of using truth table to solve this problem, we will reason about truth values. Note that $(p \vee \neg q) \wedge (q \vee \neg r) \wedge (r \vee \neg p)$ is true when the three variable p , q , and r have the same truth value (see Exercise 40 of Section 1.1). Hence, it is satisfiable as there is at least one assignment of truth values for p , q , and r that makes it true. Similarly, note that $(p \vee q \vee r) \wedge (\neg p \vee \neg q \vee \neg r)$ is true when at least one of p , q , and r is true and at least one is false (see Exercise 41 of Section 1.1). Hence, $(p \vee q \vee r) \wedge (\neg p \vee \neg q \vee \neg r)$ is satisfiable, as there is at least one assignment of truth values for p , q , and r that makes it true.

Finally, note that for $(p \vee \neg q) \wedge (q \vee \neg r) \wedge (r \vee \neg p) \wedge (p \vee q \vee r) \wedge (\neg p \vee \neg q \vee \neg r)$ to be true, $(p \vee \neg q) \wedge (q \vee \neg r) \wedge (r \vee \neg p)$ and $(p \vee q \vee r) \wedge (\neg p \vee \neg q \vee \neg r)$ must both be true. For the first to be true, the three variables must have the same truth values, and for the second to be true, at least one of three variables must be true and at least one must be false. However, these conditions are contradictory. From these observations we conclude that no assignment of truth values to p , q , and r makes $(p \vee \neg q) \wedge (q \vee \neg r) \wedge (r \vee \neg p) \wedge (p \vee q \vee r) \wedge (\neg p \vee \neg q \vee \neg r)$ true. Hence, it is unsatisfiable. ◀

Links



AUGUSTA ADA, COUNTESS OF LOVELACE (1815–1852) Augusta Ada was the only child from the marriage of the famous poet Lord Byron and Lady Byron, Annabella Millbanke, who separated when Ada was 1 month old, because of Lord Byron's scandalous affair with his half sister. The Lord Byron had quite a reputation, being described by one of his lovers as "mad, bad, and dangerous to know." Lady Byron was noted for her intellect and had a passion for mathematics; she was called by Lord Byron "The Princess of Parallelograms." Augusta was raised by her mother, who encouraged her intellectual talents especially in music and mathematics, to counter what Lady Byron considered dangerous poetic tendencies. At this time, women were not allowed to attend universities and could not join learned societies. Nevertheless, Augusta pursued her mathematical studies independently and with mathematicians, including William Frend. She was also encouraged by another female mathematician, Mary Somerville, and in 1834 at a dinner party hosted by Mary Somerville, she learned about Charles Babbage's ideas for a calculating machine, called the Analytic Engine. In 1838 Augusta Ada married Lord King, later elevated to Earl of Lovelace. Together they had three children.

Augusta Ada continued her mathematical studies after her marriage. Charles Babbage had continued work on his Analytic Engine and lectured on this in Europe. In 1842 Babbage asked Augusta Ada to translate an article in French describing Babbage's invention. When Babbage saw her translation, he suggested she add her own notes, and the resulting work was three times the length of the original. The most complete accounts of the Analytic Engine are found in Augusta Ada's notes. In her notes, she compared the working of the Analytic Engine to that of the Jacquard loom, with Babbage's punch cards analogous to the cards used to create patterns on the loom. Furthermore, she recognized the promise of the machine as a general purpose computer much better than Babbage did. She stated that the "engine is the material expression of any indefinite function of any degree of generality and complexity." Her notes on the Analytic Engine anticipate many future developments, including computer-generated music. Augusta Ada published her writings under her initials A.A.L. concealing her identity as a woman as did many women at a time when women were not considered to be the intellectual equals of men. After 1845 she and Babbage worked toward the development of a system to predict horse races. Unfortunately, their system did not work well, leaving Augusta Ada heavily in debt at the time of her death at an unfortunately young age from uterine cancer.

In 1953 Augusta Ada's notes on the Analytic Engine were republished more than 100 years after they were written, and after they had been long forgotten. In his work in the 1950s on the capacity of computers to think (and his famous Turing Test), Alan Turing responded to Augusta Ada's statement that "The Analytic Engine has no pretensions whatever to originate anything. It can do whatever we know how to order it to perform." This "dialogue" between Turing and Augusta Ada is still the subject of controversy. Because of her fundamental contributions to computing, the programming language Ada is named in honor of the Countess of Lovelace.

	2	9				4		
			5			1		
4								
			4	2				
6						7		
5								
7		3					5	
1		9						6

FIGURE 1 A 9×9 Sudoku puzzle.

Applications of Satisfiability

Many problems, in diverse areas such as robotics, software testing, computer-aided design, machine vision, integrated circuit design, computer networking, and genetics, can be modeled in terms of propositional satisfiability. Although most of these applications are beyond the scope of this book, we will study one application here. In particular, we will show how to use propositional satisfiability to model Sudoku puzzles.

SUDOKU A **Sudoku puzzle** is represented by a 9×9 grid made up of nine 3×3 subgrids, known as **blocks**, as shown in Figure 1. For each puzzle, some of the 81 cells, called **givens**, are assigned one of the numbers 1, 2, ..., 9, and the other cells are blank. The puzzle is solved by assigning a number to each blank cell so that every row, every column, and every one of the nine 3×3 blocks contains each of the nine possible numbers. Note that instead of using a 9×9 grid, Sudoku puzzles can be based on $n^2 \times n^2$ grids, for any positive integer n , with the $n^2 \times n^2$ grid made up of n^2 $n \times n$ subgrids.

The popularity of Sudoku dates back to the 1980s when it was introduced in Japan. It took 20 years for Sudoku to spread to rest of the world, but by 2005, Sudoku puzzles were a worldwide craze. The name Sudoku is short for the Japanese *suuji wa dokushin ni kagiru*, which means “the digits must remain single.” The modern game of Sudoku was apparently designed in the late 1970s by an American puzzle designer. The basic ideas of Sudoku date back even further; puzzles printed in French newspapers in the 1890s were quite similar, but not identical, to modern Sudoku.

Sudoku puzzles designed for entertainment have two additional important properties. First, they have exactly one solution. Second, they can be solved using reasoning alone, that is, without resorting to searching all possible assignments of numbers to the cells. As a Sudoku puzzle is solved, entries in blank cells are successively determined by already known values. For instance, in the grid in Figure 1, the number 4 must appear in exactly one cell in the second row. How can we determine which of the seven blank cells it must appear? First, we observe that 4 cannot appear in one of the first three cells or in one of the last three cells of this row, because it already appears in another cell in the block each of these cells is in. We can also see that 4 cannot appear in the fifth cell in this row, as it already appears in the fifth column in the fourth row. This means that 4 must appear in the sixth cell of the second row.

Many strategies based on logic and mathematics have been devised for solving Sudoku puzzles (see [Da10], for example). Here, we discuss one of the ways that have been developed for solving Sudoku puzzles with the aid of a computer, which depends on modeling the puzzle as a propositional satisfiability problem. Using the model we describe, particular Sudoku puzzles can be solved using software developed to solve satisfiability problems. Currently, Sudoku puzzles can be solved in less than 10 milliseconds this way. It should be noted that there are many other approaches for solving Sudoku puzzles via computers using other techniques.



To encode a Sudoku puzzle, let $p(i, j, n)$ denote the proposition that is true when the number n is in the cell in the i th row and j th column. There are $9 \times 9 \times 9 = 729$ such propositions, as i , j , and n all range from 1 to 9. For example, for the puzzle in Figure 1, the number 6 is given as the value in the fifth row and first column. Hence, we see that $p(5, 1, 6)$ is true, but $p(5, j, 6)$ is false for $j = 2, 3, \dots, 9$.

Given a particular Sudoku puzzle, we begin by encoding each of the given values. Then, we construct compound propositions that assert that every row contains every number, every column contains every number, every 3×3 block contains every number, and each cell contains no more than one number. It follows, as the reader should verify, that the Sudoku puzzle is solved by finding an assignment of truth values to the 729 propositions $p(i, j, n)$ with i , j , and n each ranging from 1 to 9 that makes the conjunction of all these compound propositions true. After listing these assertions, we will explain how to construct the assertion that every row contains every integer from 1 to 9. We will leave the construction of the other assertions that every column contains every number and each of the nine 3×3 blocks contains every number to the exercises.

- For each cell with a given value, we assert $p(i, j, n)$ when the cell in row i and column j has the given value n .
- We assert that every row contains every number:

$$\bigwedge_{i=1}^9 \bigwedge_{n=1}^9 \bigvee_{j=1}^9 p(i, j, n)$$

- We assert that every column contains every number:

$$\bigwedge_{j=1}^9 \bigwedge_{n=1}^9 \bigvee_{i=1}^9 p(i, j, n)$$

- We assert that each of the nine 3×3 blocks contains every number:

$$\bigwedge_{r=0}^2 \bigwedge_{s=0}^2 \bigwedge_{n=1}^9 \bigvee_{i=1}^3 \bigvee_{j=1}^3 p(3r + i, 3s + j, n)$$

- To assert that no cell contains more than one number, we take the conjunction over all values of n, n', i , and j where each variable ranges from 1 to 9 and $n \neq n'$ of $p(i, j, n) \rightarrow \neg p(i, j, n')$.

We now explain how to construct the assertion that every row contains every number. First, to assert that row i contains the number n , we form $\bigvee_{j=1}^9 p(i, j, n)$. To assert that row i contains all n numbers, we form the conjunction of these disjunctions over all nine possible values of n , giving us $\bigwedge_{n=1}^9 \bigvee_{j=1}^9 p(i, j, n)$. Finally, to assert that every row contains every number, we take the conjunction of $\bigwedge_{n=1}^9 \bigvee_{j=1}^9 p(i, j, n)$ over all nine rows. This gives us $\bigwedge_{i=1}^9 \bigwedge_{n=1}^9 \bigvee_{j=1}^9 p(i, j, n)$. (Exercises 65 and 66 ask for explanations of the assertions that every column contains every number and that each of the nine 3×3 blocks contains every number.)

Given a particular Sudoku puzzle, to solve this puzzle we can find a solution to the satisfiability problems that asks for a set of truth values for the 729 variables $p(i, j, n)$ that makes the conjunction of all the listed assertions true.

 It is tricky setting up the two inner indices so that all nine cells in each square block are examined.

Solving Satisfiability Problems

A truth table can be used to determine whether a compound proposition is satisfiable, or equivalently, whether its negation is a tautology (see Exercise 60). This can be done by hand for a compound proposition with a small number of variables, but when the number of variables grows, this becomes impractical. For instance, there are $2^{20} = 1,048,576$ rows in the truth table for a compound proposition with 20 variables. Clearly, you need a computer to help you determine, in this way, whether a compound proposition in 20 variables is satisfiable.

When many applications are modeled, questions concerning the satisfiability of compound propositions with hundreds, thousands, or millions of variables arise. Note, for example, that when there are 1000 variables, checking every one of the 2^{1000} (a number with more than 300 decimal digits) possible combinations of truth values of the variables in a compound proposition cannot be done by a computer in even trillions of years. No procedure is known that a computer can follow to determine in a reasonable amount of time whether an arbitrary compound proposition in such a large number of variables is satisfiable. However, progress has been made developing methods for solving the satisfiability problem for the particular types of compound propositions that arise in practical applications, such as for the solution of Sudoku puzzles. Many computer programs have been developed for solving satisfiability problems which have practical use. In our discussion of the subject of algorithms in Chapter 3, we will discuss this question further. In particular, we will explain the important role the propositional satisfiability problem plays in the study of the complexity of algorithms.



Exercises

1. Use truth tables to verify these equivalences.

a) $p \wedge \mathbf{T} \equiv p$	b) $p \vee \mathbf{F} \equiv p$
c) $p \wedge \mathbf{F} \equiv \mathbf{F}$	d) $p \vee \mathbf{T} \equiv \mathbf{T}$
e) $p \vee p \equiv p$	f) $p \wedge p \equiv p$
2. Show that $\neg(\neg p)$ and p are logically equivalent.
3. Use truth tables to verify the commutative laws

a) $p \vee q \equiv q \vee p$.	b) $p \wedge q \equiv q \wedge p$.
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4. Use truth tables to verify the associative laws

a) $(p \vee q) \vee r \equiv p \vee (q \vee r)$.	b) $(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$.
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5. Use a truth table to verify the distributive law

$$p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r).$$
6. Use a truth table to verify the first De Morgan law

$$\neg(p \wedge q) \equiv \neg p \vee \neg q.$$
7. Use De Morgan's laws to find the negation of each of the following statements.

a) Jan is rich and happy.	b) Carlos will bicycle or run tomorrow.
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HENRY MAURICE SHEFFER (1883–1964) Henry Maurice Sheffer, born to Jewish parents in the western Ukraine, emigrated to the United States in 1892 with his parents and six siblings. He studied at the Boston Latin School before entering Harvard, where he completed his undergraduate degree in 1905, his master's in 1907, and his Ph.D. in philosophy in 1908. After holding a postdoctoral position at Harvard, Henry traveled to Europe on a fellowship. Upon returning to the United States, he became an academic nomad, spending one year each at the University of Washington, Cornell, the University of Minnesota, the University of Missouri, and City College in New York. In 1916 he returned to Harvard as a faculty member in the philosophy department. He remained at Harvard until his retirement in 1952.

Sheffer introduced what is now known as the Sheffer stroke in 1913; it became well known only after its use in the 1925 edition of Whitehead and Russell's *Principia Mathematica*. In this same edition Russell wrote that Sheffer had invented a powerful method that could be used to simplify the *Principia*. Because of this comment, Sheffer was something of a mystery man to logicians, especially because Sheffer, who published little in his career, never published the details of this method, only describing it in mimeographed notes and in a brief published abstract.

Sheffer was a dedicated teacher of mathematical logic. He liked his classes to be small and did not like auditors. When strangers appeared in his classroom, Sheffer would order them to leave, even his colleagues or distinguished guests visiting Harvard. Sheffer was barely five feet tall; he was noted for his wit and vigor, as well as for his nervousness and irritability. Although widely liked, he was quite lonely. He is noted for a quip he spoke at his retirement: "Old professors never die, they just become emeriti." Sheffer is also credited with coining the term "Boolean algebra" (the subject of Chapter 12 of this text). Sheffer was briefly married and lived most of his later life in small rooms at a hotel packed with his logic books and vast files of slips of paper he used to jot down his ideas. Unfortunately, Sheffer suffered from severe depression during the last two decades of his life.



- c) Mei walks or takes the bus to class.
 d) Ibrahim is smart and hard working.
8. Use De Morgan's laws to find the negation of each of the following statements.
- Kwame will take a job in industry or go to graduate school.
 - Yoshiko knows Java and calculus.
 - James is young and strong.
 - Rita will move to Oregon or Washington.
9. Show that each of these conditional statements is a tautology by using truth tables.
- $(p \wedge q) \rightarrow p$
 - $p \rightarrow (p \vee q)$
 - $\neg p \rightarrow (p \rightarrow q)$
 - $(p \wedge q) \rightarrow (p \rightarrow q)$
 - $\neg(p \rightarrow q) \rightarrow p$
 - $\neg(p \rightarrow q) \rightarrow \neg q$
10. Show that each of these conditional statements is a tautology by using truth tables.
- $[\neg p \wedge (p \vee q)] \rightarrow q$
 - $[(p \rightarrow q) \wedge (q \rightarrow r)] \rightarrow (p \rightarrow r)$
 - $[p \wedge (p \rightarrow q)] \rightarrow q$
 - $[(p \vee q) \wedge (p \rightarrow r) \wedge (q \rightarrow r)] \rightarrow r$
11. Show that each conditional statement in Exercise 9 is a tautology without using truth tables.
12. Show that each conditional statement in Exercise 10 is a tautology without using truth tables.
13. Use truth tables to verify the absorption laws.
- $p \vee (p \wedge q) \equiv p$
 - $p \wedge (p \vee q) \equiv p$
14. Determine whether $(\neg p \wedge (p \rightarrow q)) \rightarrow \neg q$ is a tautology.
15. Determine whether $(\neg q \wedge (p \rightarrow q)) \rightarrow \neg p$ is a tautology.
- Each of Exercises 16–28 asks you to show that two compound propositions are logically equivalent. To do this, either show that both sides are true, or that both sides are false, for exactly the same combinations of truth values of the propositional variables in these expressions (whichever is easier).
- Show that $p \leftrightarrow q$ and $(p \wedge q) \vee (\neg p \wedge \neg q)$ are logically equivalent.
 - Show that $\neg(p \leftrightarrow q)$ and $p \leftrightarrow \neg q$ are logically equivalent.
 - Show that $p \rightarrow q$ and $\neg q \rightarrow \neg p$ are logically equivalent.
 - Show that $\neg p \leftrightarrow q$ and $p \leftrightarrow \neg q$ are logically equivalent.
 - Show that $\neg(p \oplus q)$ and $p \leftrightarrow q$ are logically equivalent.
 - Show that $\neg(p \leftrightarrow q)$ and $\neg p \leftrightarrow q$ are logically equivalent.
 - Show that $(p \rightarrow q) \wedge (p \rightarrow r)$ and $p \rightarrow (q \wedge r)$ are logically equivalent.
 - Show that $(p \rightarrow r) \wedge (q \rightarrow r)$ and $(p \vee q) \rightarrow r$ are logically equivalent.
 - Show that $(p \rightarrow q) \vee (p \rightarrow r)$ and $p \rightarrow (q \vee r)$ are logically equivalent.
 - Show that $(p \rightarrow r) \vee (q \rightarrow r)$ and $(p \wedge q) \rightarrow r$ are logically equivalent.
 - Show that $\neg p \rightarrow (q \rightarrow r)$ and $q \rightarrow (p \vee r)$ are logically equivalent.
 - Show that $p \leftrightarrow q$ and $(p \rightarrow q) \wedge (q \rightarrow p)$ are logically equivalent.
 - Show that $p \leftrightarrow q$ and $\neg p \leftrightarrow \neg q$ are logically equivalent.
29. Show that $(p \rightarrow q) \wedge (q \rightarrow r) \rightarrow (p \rightarrow r)$ is a tautology.
30. Show that $(p \vee q) \wedge (\neg p \vee r) \rightarrow (q \vee r)$ is a tautology.
31. Show that $(p \rightarrow q) \rightarrow r$ and $p \rightarrow (q \rightarrow r)$ are not logically equivalent.
32. Show that $(p \wedge q) \rightarrow r$ and $(p \rightarrow r) \wedge (q \rightarrow r)$ are not logically equivalent.
33. Show that $(p \rightarrow q) \rightarrow (r \rightarrow s)$ and $(p \rightarrow r) \rightarrow (q \rightarrow s)$ are not logically equivalent.
- The **dual** of a compound proposition that contains only the logical operators \vee , \wedge , and \neg is the compound proposition obtained by replacing each \vee by \wedge , each \wedge by \vee , each **T** by **F**, and each **F** by **T**. The dual of s is denoted by s^* .
34. Find the dual of each of these compound propositions.
- $p \vee \neg q$
 - $p \wedge (q \vee (r \wedge T))$
 - $(p \wedge \neg q) \vee (q \wedge F)$
35. Find the dual of each of these compound propositions.
- $p \wedge \neg q \wedge \neg r$
 - $(p \wedge q \wedge r) \vee s$
 - $(p \vee F) \wedge (q \vee T)$
36. When does $s^* = s$, where s is a compound proposition?
37. Show that $(s^*)^* = s$ when s is a compound proposition.
38. Show that the logical equivalences in Table 6, except for the double negation law, come in pairs, where each pair contains compound propositions that are duals of each other.
- **39. Why are the duals of two equivalent compound propositions also equivalent, where these compound propositions contain only the operators \wedge , \vee , and \neg ?
40. Find a compound proposition involving the propositional variables p , q , and r that is true when p and q are true and r is false, but is false otherwise. [Hint: Use a conjunction of each propositional variable or its negation.]
41. Find a compound proposition involving the propositional variables p , q , and r that is true when exactly two of p , q , and r are true and is false otherwise. [Hint: Form a disjunction of conjunctions. Include a conjunction for each combination of values for which the compound proposition is true. Each conjunction should include each of the three propositional variables or its negations.]
42. Suppose that a truth table in n propositional variables is specified. Show that a compound proposition with this truth table can be formed by taking the disjunction of conjunctions of the variables or their negations, with one conjunction included for each combination of values for which the compound proposition is true. The resulting compound proposition is said to be in **disjunctive normal form**.
- A collection of logical operators is called **functionally complete** if every compound proposition is logically equivalent to a compound proposition involving only these logical operators.
43. Show that \neg , \wedge , and \vee form a functionally complete collection of logical operators. [Hint: Use the fact that every compound proposition is logically equivalent to one in disjunctive normal form, as shown in Exercise 42.]

- *44. Show that \neg and \wedge form a functionally complete collection of logical operators. [Hint: First use a De Morgan law to show that $p \vee q$ is logically equivalent to $\neg(\neg p \wedge \neg q)$.]
- *45. Show that \neg and \vee form a functionally complete collection of logical operators.
- The following exercises involve the logical operators *NAND* and *NOR*. The proposition $p \text{ NAND } q$ is true when either p or q , or both, are false; and it is false when both p and q are true. The proposition $p \text{ NOR } q$ is true when both p and q are false, and it is false otherwise. The propositions $p \text{ NAND } q$ and $p \text{ NOR } q$ are denoted by $p \mid q$ and $p \downarrow q$, respectively. (The operators \mid and \downarrow are called the **Sheffer stroke** and the **Peirce arrow** after H. M. Sheffer and C. S. Peirce, respectively.)
46. Construct a truth table for the logical operator *NAND*.
47. Show that $p \mid q$ is logically equivalent to $\neg(p \wedge q)$.
48. Construct a truth table for the logical operator *NOR*.
49. Show that $p \downarrow q$ is logically equivalent to $\neg(p \vee q)$.
50. In this exercise we will show that $\{\downarrow\}$ is a functionally complete collection of logical operators.
- Show that $p \downarrow p$ is logically equivalent to $\neg p$.
 - Show that $(p \downarrow q) \downarrow (p \downarrow q)$ is logically equivalent to $p \vee q$.
 - Conclude from parts (a) and (b), and Exercise 49, that $\{\downarrow\}$ is a functionally complete collection of logical operators.
- *51. Find a compound proposition logically equivalent to $p \rightarrow q$ using only the logical operator \downarrow .
52. Show that $\{\mid\}$ is a functionally complete collection of logical operators.
53. Show that $p \mid q$ and $q \mid p$ are equivalent.
54. Show that $p \mid (q \mid r)$ and $(p \mid q) \mid r$ are not equivalent, so that the logical operator \mid is not associative.
- *55. How many different truth tables of compound propositions are there that involve the propositional variables p and q ?
56. Show that if p , q , and r are compound propositions such that p and q are logically equivalent and q and r are logically equivalent, then p and r are logically equivalent.
57. The following sentence is taken from the specification of a telephone system: “If the directory database is opened, then the monitor is put in a closed state, if the system is not in its initial state.” This specification is hard to under-

stand because it involves two conditional statements. Find an equivalent, easier-to-understand specification that involves disjunctions and negations but not conditional statements.

58. How many of the disjunctions $p \vee \neg q$, $\neg p \vee q$, $q \vee r$, $q \vee \neg r$, and $\neg q \vee \neg r$ can be made simultaneously true by an assignment of truth values to p , q , and r ?
59. How many of the disjunctions $p \vee \neg q \vee s$, $\neg p \vee \neg r \vee s$, $\neg p \vee q \vee \neg s$, $q \vee r \vee \neg s$, $q \vee \neg r \vee \neg s$, $\neg p \vee \neg q \vee \neg s$, $p \vee r \vee s$, and $p \vee r \vee \neg s$ can be made simultaneously true by an assignment of truth values to p , q , r , and s ?
60. Show that the negation of an unsatisfiable compound proposition is a tautology and the negation of a compound proposition that is a tautology is unsatisfiable.
61. Determine whether each of these compound propositions is satisfiable.
- $(p \vee \neg q) \wedge (\neg p \vee q) \wedge (\neg p \vee \neg q)$
 - $(p \rightarrow q) \wedge (p \rightarrow \neg q) \wedge (\neg p \rightarrow q) \wedge (\neg p \rightarrow \neg q)$
 - $(p \leftrightarrow q) \wedge (\neg p \leftrightarrow q)$
62. Determine whether each of these compound propositions is satisfiable.
- $(p \vee q \vee \neg r) \wedge (p \vee \neg q \vee \neg s) \wedge (p \vee \neg r \vee \neg s) \wedge (\neg p \vee \neg q \vee \neg s) \wedge (p \vee q \vee \neg s)$
 - $(\neg p \vee \neg q \vee r) \wedge (\neg p \vee q \vee \neg s) \wedge (p \vee \neg q \vee \neg s) \wedge (\neg p \vee \neg r \vee \neg s) \wedge (p \vee q \vee \neg r) \wedge (p \vee \neg r \vee \neg s)$
 - $(p \vee q \vee r) \wedge (p \vee \neg q \vee \neg s) \wedge (q \vee \neg r \vee s) \wedge (\neg p \vee r \vee s) \wedge (\neg p \vee q \vee \neg s) \wedge (p \vee \neg q \vee \neg r) \wedge (\neg p \vee \neg q \vee s) \wedge (\neg p \vee \neg r \vee \neg s)$
63. Show how the solution of a given 4×4 Sudoku puzzle can be found by solving a satisfiability problem.
64. Construct a compound proposition that asserts that every cell of a 9×9 Sudoku puzzle contains at least one number.
65. Explain the steps in the construction of the compound proposition given in the text that asserts that every column of a 9×9 Sudoku puzzle contains every number.
- *66. Explain the steps in the construction of the compound proposition given in the text that asserts that each of the nine 3×3 blocks of a 9×9 Sudoku puzzle contains every number.

1.4 Predicates and Quantifiers

Introduction

Propositional logic, studied in Sections 1.1–1.3, cannot adequately express the meaning of all statements in mathematics and in natural language. For example, suppose that we know that

“Every computer connected to the university network is functioning properly.”

No rules of propositional logic allow us to conclude the truth of the statement

“MATH3 is functioning properly,”

where MATH3 is one of the computers connected to the university network. Likewise, we cannot use the rules of propositional logic to conclude from the statement

“CS2 is under attack by an intruder,”

where CS2 is a computer on the university network, to conclude the truth of

“There is a computer on the university network that is under attack by an intruder.”

In this section we will introduce a more powerful type of logic called **predicate logic**. We will see how predicate logic can be used to express the meaning of a wide range of statements in mathematics and computer science in ways that permit us to reason and explore relationships between objects. To understand predicate logic, we first need to introduce the concept of a predicate. Afterward, we will introduce the notion of quantifiers, which enable us to reason with statements that assert that a certain property holds for all objects of a certain type and with statements that assert the existence of an object with a particular property.

Predicates

Statements involving variables, such as

“ $x > 3$,” “ $x = y + 3$,” “ $x + y = z$,”

and

“computer x is under attack by an intruder,”

and

“computer x is functioning properly,”

are often found in mathematical assertions, in computer programs, and in system specifications. These statements are neither true nor false when the values of the variables are not specified. In this section, we will discuss the ways that propositions can be produced from such statements.

The statement “ x is greater than 3” has two parts. The first part, the variable x , is the subject of the statement. The second part—the **predicate**, “is greater than 3”—refers to a property that the subject of the statement can have. We can denote the statement “ x is greater than 3” by $P(x)$, where P denotes the predicate “is greater than 3” and x is the variable. The statement $P(x)$ is also said to be the value of the **propositional function** P at x . Once a value has been assigned to the variable x , the statement $P(x)$ becomes a proposition and has a truth value. Consider Examples 1 and 2.

EXAMPLE 1 Let $P(x)$ denote the statement “ $x > 3$.” What are the truth values of $P(4)$ and $P(2)$?

Solution: We obtain the statement $P(4)$ by setting $x = 4$ in the statement “ $x > 3$.” Hence, $P(4)$, which is the statement “ $4 > 3$,” is true. However, $P(2)$, which is the statement “ $2 > 3$,” is false. ◀

EXAMPLE 2 Let $A(x)$ denote the statement “Computer x is under attack by an intruder.” Suppose that of the computers on campus, only CS2 and MATH1 are currently under attack by intruders. What are truth values of $A(\text{CS1})$, $A(\text{CS2})$, and $A(\text{MATH1})$?

Solution: We obtain the statement $A(\text{CS1})$ by setting $x = \text{CS1}$ in the statement “Computer x is under attack by an intruder.” Because CS1 is not on the list of computers currently under attack, we conclude that $A(\text{CS1})$ is false. Similarly, because CS2 and MATH1 are on the list of computers under attack, we know that $A(\text{CS2})$ and $A(\text{MATH1})$ are true. ◀

We can also have statements that involve more than one variable. For instance, consider the statement “ $x = y + 3$.” We can denote this statement by $Q(x, y)$, where x and y are variables and Q is the predicate. When values are assigned to the variables x and y , the statement $Q(x, y)$ has a truth value.

EXAMPLE 3 Let $Q(x, y)$ denote the statement “ $x = y + 3$.” What are the truth values of the propositions $Q(1, 2)$ and $Q(3, 0)$?



Solution: To obtain $Q(1, 2)$, set $x = 1$ and $y = 2$ in the statement $Q(x, y)$. Hence, $Q(1, 2)$ is the statement “ $1 = 2 + 3$,” which is false. The statement $Q(3, 0)$ is the proposition “ $3 = 0 + 3$,” which is true. ◀



CHARLES SANDERS PEIRCE (1839–1914) Many consider Charles Peirce, born in Cambridge, Massachusetts, to be the most original and versatile American intellect. He made important contributions to an amazing number of disciplines, including mathematics, astronomy, chemistry, geodesy, metrology, engineering, psychology, philology, the history of science, and economics. Peirce was also an inventor, a lifelong student of medicine, a book reviewer, a dramatist and an actor, a short story writer, a phenomenologist, a logician, and a metaphysician. He is noted as the preeminent system-building philosopher competent and productive in logic, mathematics, and a wide range of sciences. He was encouraged by his father, Benjamin Peirce, a professor of mathematics and natural philosophy at Harvard, to pursue a career in science. Instead, he decided to study logic and scientific methodology. Peirce attended Harvard (1855–1859) and received a Harvard master of arts degree (1862) and an advanced degree in chemistry from the Lawrence Scientific School (1863).

In 1861, Peirce became an aide in the U.S. Coast Survey, with the goal of better understanding scientific methodology. His service for the Survey exempted him from military service during the Civil War. While working for the Survey, Peirce did astronomical and geodesic work. He made fundamental contributions to the design of pendulums and to map projections, applying new mathematical developments in the theory of elliptic functions. He was the first person to use the wavelength of light as a unit of measurement. Peirce rose to the position of Assistant for the Survey, a position he held until forced to resign in 1891 when he disagreed with the direction taken by the Survey’s new administration.

While making his living from work in the physical sciences, Peirce developed a hierarchy of sciences, with mathematics at the top rung, in which the methods of one science could be adapted for use by those sciences under it in the hierarchy. During this time, he also founded the American philosophical theory of pragmatism.

The only academic position Peirce ever held was lecturer in logic at Johns Hopkins University in Baltimore (1879–1884). His mathematical work during this time included contributions to logic, set theory, abstract algebra, and the philosophy of mathematics. His work is still relevant today, with recent applications of this work on logic to artificial intelligence. Peirce believed that the study of mathematics could develop the mind’s powers of imagination, abstraction, and generalization. His diverse activities after retiring from the Survey included writing for periodicals, contributing to scholarly dictionaries, translating scientific papers, guest lecturing, and textbook writing. Unfortunately, his income from these pursuits was insufficient to protect him and his second wife from abject poverty. He was supported in his later years by a fund created by his many admirers and administered by the philosopher William James, his lifelong friend. Although Peirce wrote and published voluminously in a vast range of subjects, he left more than 100,000 pages of unpublished manuscripts. Because of the difficulty of studying his unpublished writings, scholars have only recently started to understand some of his varied contributions. A group of people is devoted to making his work available over the Internet to bring a better appreciation of Peirce’s accomplishments to the world.

EXAMPLE 4 Let $A(c, n)$ denote the statement “Computer c is connected to network n ,” where c is a variable representing a computer and n is a variable representing a network. Suppose that the computer MATH1 is connected to network CAMPUS2, but not to network CAMPUS1. What are the values of $A(\text{MATH1}, \text{CAMPUS1})$ and $A(\text{MATH1}, \text{CAMPUS2})$?

Solution: Because MATH1 is not connected to the CAMPUS1 network, we see that $A(\text{MATH1}, \text{CAMPUS1})$ is false. However, because MATH1 is connected to the CAMPUS2 network, we see that $A(\text{MATH1}, \text{CAMPUS2})$ is true. ◀

Similarly, we can let $R(x, y, z)$ denote the statement “ $x + y = z$.” When values are assigned to the variables x , y , and z , this statement has a truth value.

EXAMPLE 5 What are the truth values of the propositions $R(1, 2, 3)$ and $R(0, 0, 1)$?

Solution: The proposition $R(1, 2, 3)$ is obtained by setting $x = 1$, $y = 2$, and $z = 3$ in the statement $R(x, y, z)$. We see that $R(1, 2, 3)$ is the statement “ $1 + 2 = 3$,” which is true. Also note that $R(0, 0, 1)$, which is the statement “ $0 + 0 = 1$,” is false. ◀

In general, a statement involving the n variables x_1, x_2, \dots, x_n can be denoted by

$$P(x_1, x_2, \dots, x_n).$$

A statement of the form $P(x_1, x_2, \dots, x_n)$ is the value of the **propositional function** P at the n -tuple (x_1, x_2, \dots, x_n) , and P is also called an **n -place predicate** or a **n -ary predicate**. Propositional functions occur in computer programs, as Example 6 demonstrates.

EXAMPLE 6 Consider the statement

if $x > 0$ **then** $x := x + 1$.

When this statement is encountered in a program, the value of the variable x at that point in the execution of the program is inserted into $P(x)$, which is “ $x > 0$.” If $P(x)$ is true for this value of x , the assignment statement $x := x + 1$ is executed, so the value of x is increased by 1. If $P(x)$ is false for this value of x , the assignment statement is not executed, so the value of x is not changed. ◀

PRECONDITIONS AND POSTCONDITIONS Predicates are also used to establish the correctness of computer programs, that is, to show that computer programs always produce the desired output when given valid input. (Note that unless the correctness of a computer program is established, no amount of testing can show that it produces the desired output for all input values, unless every input value is tested.) The statements that describe valid input are known as **preconditions** and the conditions that the output should satisfy when the program has run are known as **postconditions**. As Example 7 illustrates, we use predicates to describe both preconditions and postconditions. We will study this process in greater detail in Section 5.5.

EXAMPLE 7 Consider the following program, designed to interchange the values of two variables x and y .

```
temp := x
x := y
y := temp
```

Find predicates that we can use as the precondition and the postcondition to verify the correctness of this program. Then explain how to use them to verify that for all valid input the program does what is intended.

Solution: For the precondition, we need to express that x and y have particular values before we run the program. So, for this precondition we can use the predicate $P(x, y)$, where $P(x, y)$ is the statement “ $x = a$ and $y = b$,” where a and b are the values of x and y before we run the program. Because we want to verify that the program swaps the values of x and y for all input values, for the postcondition we can use $Q(x, y)$, where $Q(x, y)$ is the statement “ $x = b$ and $y = a$.”

To verify that the program always does what it is supposed to do, suppose that the precondition $P(x, y)$ holds. That is, we suppose that the statement “ $x = a$ and $y = b$ ” is true. This means that $x = a$ and $y = b$. The first step of the program, $\text{temp} := x$, assigns the value of x to the variable temp , so after this step we know that $x = a$, $\text{temp} = a$, and $y = b$. After the second step of the program, $x := y$, we know that $x = b$, $\text{temp} = a$, and $y = b$. Finally, after the third step, we know that $x = b$, $\text{temp} = a$, and $y = a$. Consequently, after this program is run, the postcondition $Q(x, y)$ holds, that is, the statement “ $x = b$ and $y = a$ ” is true. ◀

Quantifiers



When the variables in a propositional function are assigned values, the resulting statement becomes a proposition with a certain truth value. However, there is another important way, called **quantification**, to create a proposition from a propositional function. Quantification expresses the extent to which a predicate is true over a range of elements. In English, the words *all*, *some*, *many*, *none*, and *few* are used in quantifications. We will focus on two types of quantification here: universal quantification, which tells us that a predicate is true for every element under consideration, and existential quantification, which tells us that there is one or more element under consideration for which the predicate is true. The area of logic that deals with predicates and quantifiers is called the **predicate calculus**.



THE UNIVERSAL QUANTIFIER Many mathematical statements assert that a property is true for all values of a variable in a particular domain, called the **domain of discourse** (or the **universe of discourse**), often just referred to as the **domain**. Such a statement is expressed using universal quantification. The universal quantification of $P(x)$ for a particular domain is the proposition that asserts that $P(x)$ is true for all values of x in this domain. Note that the domain specifies the possible values of the variable x . The meaning of the universal quantification of $P(x)$ changes when we change the domain. The domain must always be specified when a universal quantifier is used; without it, the universal quantification of a statement is not defined.

DEFINITION 1

The *universal quantification* of $P(x)$ is the statement

“ $P(x)$ for all values of x in the domain.”

The notation $\forall x P(x)$ denotes the universal quantification of $P(x)$. Here \forall is called the **universal quantifier**. We read $\forall x P(x)$ as “for all $x P(x)$ ” or “for every $x P(x)$.” An element for which $P(x)$ is false is called a **counterexample** of $\forall x P(x)$.

The meaning of the universal quantifier is summarized in the first row of Table 1. We illustrate the use of the universal quantifier in Examples 8–13.

TABLE 1 Quantifiers.

<i>Statement</i>	<i>When True?</i>	<i>When False?</i>
$\forall x P(x)$	$P(x)$ is true for every x .	There is an x for which $P(x)$ is false.
$\exists x P(x)$	There is an x for which $P(x)$ is true.	$P(x)$ is false for every x .

EXAMPLE 8 Let $P(x)$ be the statement “ $x + 1 > x$.” What is the truth value of the quantification $\forall x P(x)$, where the domain consists of all real numbers?



Solution: Because $P(x)$ is true for all real numbers x , the quantification

$$\forall x P(x)$$

is true. ◀

Remark: Generally, an implicit assumption is made that all domains of discourse for quantifiers are nonempty. Note that if the domain is empty, then $\forall x P(x)$ is true for any propositional function $P(x)$ because there are no elements x in the domain for which $P(x)$ is false.

Remember that the truth value of $\forall x P(x)$ depends on the domain!

Besides “for all” and “for every,” universal quantification can be expressed in many other ways, including “all of,” “for each,” “given any,” “for arbitrary,” “for each,” and “for any.”

Remark: It is best to avoid using “for any x ” because it is often ambiguous as to whether “any” means “every” or “some.” In some cases, “any” is unambiguous, such as when it is used in negatives, for example, “there is not any reason to avoid studying.”

A statement $\forall x P(x)$ is false, where $P(x)$ is a propositional function, if and only if $P(x)$ is not always true when x is in the domain. One way to show that $P(x)$ is not always true when x is in the domain is to find a counterexample to the statement $\forall x P(x)$. Note that a single counterexample is all we need to establish that $\forall x P(x)$ is false. Example 9 illustrates how counterexamples are used.

EXAMPLE 9 Let $Q(x)$ be the statement “ $x < 2$.” What is the truth value of the quantification $\forall x Q(x)$, where the domain consists of all real numbers?

Solution: $Q(x)$ is not true for every real number x , because, for instance, $Q(3)$ is false. That is, $x = 3$ is a counterexample for the statement $\forall x Q(x)$. Thus

$$\forall x Q(x)$$

is false. ◀

EXAMPLE 10 Suppose that $P(x)$ is “ $x^2 > 0$.” To show that the statement $\forall x P(x)$ is false where the universe of discourse consists of all integers, we give a counterexample. We see that $x = 0$ is a counterexample because $x^2 = 0$ when $x = 0$, so that x^2 is not greater than 0 when $x = 0$. ◀

Looking for counterexamples to universally quantified statements is an important activity in the study of mathematics, as we will see in subsequent sections of this book.

When all the elements in the domain can be listed—say, x_1, x_2, \dots, x_n —it follows that the universal quantification $\forall x P(x)$ is the same as the conjunction

$$P(x_1) \wedge P(x_2) \wedge \cdots \wedge P(x_n),$$

because this conjunction is true if and only if $P(x_1), P(x_2), \dots, P(x_n)$ are all true.

EXAMPLE 11 What is the truth value of $\forall x P(x)$, where $P(x)$ is the statement “ $x^2 < 10$ ” and the domain consists of the positive integers not exceeding 4?

Solution: The statement $\forall x P(x)$ is the same as the conjunction

$$P(1) \wedge P(2) \wedge P(3) \wedge P(4),$$

because the domain consists of the integers 1, 2, 3, and 4. Because $P(4)$, which is the statement “ $4^2 < 10$,” is false, it follows that $\forall x P(x)$ is false. ◀

EXAMPLE 12 What does the statement $\forall x N(x)$ mean if $N(x)$ is “Computer x is connected to the network” and the domain consists of all computers on campus?

Solution: The statement $\forall x N(x)$ means that for every computer x on campus, that computer x is connected to the network. This statement can be expressed in English as “Every computer on campus is connected to the network.” ◀

As we have pointed out, specifying the domain is mandatory when quantifiers are used. The truth value of a quantified statement often depends on which elements are in this domain, as Example 13 shows.

EXAMPLE 13 What is the truth value of $\forall x (x^2 \geq x)$ if the domain consists of all real numbers? What is the truth value of this statement if the domain consists of all integers?

Solution: The universal quantification $\forall x (x^2 \geq x)$, where the domain consists of all real numbers, is false. For example, $(\frac{1}{2})^2 \not\geq \frac{1}{2}$. Note that $x^2 \geq x$ if and only if $x^2 - x = x(x - 1) \geq 0$. Consequently, $x^2 \geq x$ if and only if $x \leq 0$ or $x \geq 1$. It follows that $\forall x (x^2 \geq x)$ is false if the domain consists of all real numbers (because the inequality is false for all real numbers x with $0 < x < 1$). However, if the domain consists of the integers, $\forall x (x^2 \geq x)$ is true, because there are no integers x with $0 < x < 1$. ◀

THE EXISTENTIAL QUANTIFIER Many mathematical statements assert that there is an element with a certain property. Such statements are expressed using existential quantification. With existential quantification, we form a proposition that is true if and only if $P(x)$ is true for at least one value of x in the domain.

DEFINITION 2

The *existential quantification* of $P(x)$ is the proposition

“There exists an element x in the domain such that $P(x)$.”

We use the notation $\exists x P(x)$ for the existential quantification of $P(x)$. Here \exists is called the *existential quantifier*.

A domain must always be specified when a statement $\exists x P(x)$ is used. Furthermore, the meaning of $\exists x P(x)$ changes when the domain changes. Without specifying the domain, the statement $\exists x P(x)$ has no meaning.

Besides the phrase “there exists,” we can also express existential quantification in many other ways, such as by using the words “for some,” “for at least one,” or “there is.” The existential quantification $\exists x P(x)$ is read as

“There is an x such that $P(x)$,”

“There is at least one x such that $P(x)$,”

or

“For some $x P(x)$.”

The meaning of the existential quantifier is summarized in the second row of Table 1. We illustrate the use of the existential quantifier in Examples 14–16.

EXAMPLE 14 Let $P(x)$ denote the statement “ $x > 3$.” What is the truth value of the quantification $\exists x P(x)$, where the domain consists of all real numbers?



Solution: Because “ $x > 3$ ” is sometimes true—for instance, when $x = 4$ —the existential quantification of $P(x)$, which is $\exists x P(x)$, is true. ◀

Observe that the statement $\exists x P(x)$ is false if and only if there is no element x in the domain for which $P(x)$ is true. That is, $\exists x P(x)$ is false if and only if $P(x)$ is false for every element of the domain. We illustrate this observation in Example 15.

EXAMPLE 15 Let $Q(x)$ denote the statement “ $x = x + 1$.” What is the truth value of the quantification $\exists x Q(x)$, where the domain consists of all real numbers?

Solution: Because $Q(x)$ is false for every real number x , the existential quantification of $Q(x)$, which is $\exists x Q(x)$, is false. ◀

Remember that the truth value of $\exists x P(x)$ depends on the domain!

Remark: Generally, an implicit assumption is made that all domains of discourse for quantifiers are nonempty. If the domain is empty, then $\exists x Q(x)$ is false whenever $Q(x)$ is a propositional function because when the domain is empty, there can be no element x in the domain for which $Q(x)$ is true.

When all elements in the domain can be listed—say, x_1, x_2, \dots, x_n —the existential quantification $\exists x P(x)$ is the same as the disjunction

$$P(x_1) \vee P(x_2) \vee \cdots \vee P(x_n),$$

because this disjunction is true if and only if at least one of $P(x_1), P(x_2), \dots, P(x_n)$ is true.

EXAMPLE 16 What is the truth value of $\exists x P(x)$, where $P(x)$ is the statement “ $x^2 > 10$ ” and the universe of discourse consists of the positive integers not exceeding 4?

Solution: Because the domain is $\{1, 2, 3, 4\}$, the proposition $\exists x P(x)$ is the same as the disjunction

$$P(1) \vee P(2) \vee P(3) \vee P(4).$$

Because $P(4)$, which is the statement “ $4^2 > 10$,” is true, it follows that $\exists x P(x)$ is true. ◀

It is sometimes helpful to think in terms of looping and searching when determining the truth value of a quantification. Suppose that there are n objects in the domain for the variable x . To determine whether $\forall x P(x)$ is true, we can loop through all n values of x to see whether $P(x)$ is always true. If we encounter a value x for which $P(x)$ is false, then we have shown that $\forall x P(x)$ is false. Otherwise, $\forall x P(x)$ is true. To see whether $\exists x P(x)$ is true, we loop through the n values of x searching for a value for which $P(x)$ is true. If we find one, then $\exists x P(x)$ is true. If we never find such an x , then we have determined that $\exists x P(x)$ is false. (Note that this searching procedure does not apply if there are infinitely many values in the domain. However, it is still a useful way of thinking about the truth values of quantifications.)

THE UNIQUENESS QUANTIFIER We have now introduced universal and existential quantifiers. These are the most important quantifiers in mathematics and computer science. However, there is no limitation on the number of different quantifiers we can define, such as “there are exactly two,” “there are no more than three,” “there are at least 100,” and so on. Of these other quantifiers, the one that is most often seen is the **uniqueness quantifier**, denoted by $\exists!$ or \exists_1 . The notation $\exists!x P(x)$ [or $\exists_1 x P(x)$] states “There exists a unique x such that $P(x)$ is true.” (Other phrases for uniqueness quantification include “there is exactly one” and “there is one and only one.”) For instance, $\exists!x(x - 1 = 0)$, where the domain is the set of real numbers, states that there is a unique real number x such that $x - 1 = 0$. This is a true statement, as $x = 1$ is the unique real number such that $x - 1 = 0$. Observe that we can use quantifiers and propositional logic to express uniqueness (see Exercise 52 in Section 1.5), so the uniqueness quantifier can be avoided. Generally, it is best to stick with existential and universal quantifiers so that rules of inference for these quantifiers can be used.

Quantifiers with Restricted Domains

An abbreviated notation is often used to restrict the domain of a quantifier. In this notation, a condition a variable must satisfy is included after the quantifier. This is illustrated in Example 17. We will also describe other forms of this notation involving set membership in Section 2.1.

EXAMPLE 17 What do the statements $\forall x < 0 (x^2 > 0)$, $\forall y \neq 0 (y^3 \neq 0)$, and $\exists z > 0 (z^2 = 2)$ mean, where the domain in each case consists of the real numbers?

Solution: The statement $\forall x < 0 (x^2 > 0)$ states that for every real number x with $x < 0$, $x^2 > 0$. That is, it states “The square of a negative real number is positive.” This statement is the same as $\forall x(x < 0 \rightarrow x^2 > 0)$.

The statement $\forall y \neq 0 (y^3 \neq 0)$ states that for every real number y with $y \neq 0$, we have $y^3 \neq 0$. That is, it states “The cube of every nonzero real number is nonzero.” Note that this statement is equivalent to $\forall y(y \neq 0 \rightarrow y^3 \neq 0)$.

Finally, the statement $\exists z > 0 (z^2 = 2)$ states that there exists a real number z with $z > 0$ such that $z^2 = 2$. That is, it states “There is a positive square root of 2.” This statement is equivalent to $\exists z(z > 0 \wedge z^2 = 2)$. ◀

Note that the restriction of a universal quantification is the same as the universal quantification of a conditional statement. For instance, $\forall x < 0 (x^2 > 0)$ is another way of expressing $\forall x(x < 0 \rightarrow x^2 > 0)$. On the other hand, the restriction of an existential quantification is the same as the existential quantification of a conjunction. For instance, $\exists z > 0 (z^2 = 2)$ is another way of expressing $\exists z(z > 0 \wedge z^2 = 2)$.

Precedence of Quantifiers

The quantifiers \forall and \exists have higher precedence than all logical operators from propositional calculus. For example, $\forall x P(x) \vee Q(x)$ is the disjunction of $\forall x P(x)$ and $Q(x)$. In other words, it means $(\forall x P(x)) \vee Q(x)$ rather than $\forall x(P(x) \vee Q(x))$.

Binding Variables

When a quantifier is used on the variable x , we say that this occurrence of the variable is **bound**. An occurrence of a variable that is not bound by a quantifier or set equal to a particular value is said to be **free**. All the variables that occur in a propositional function must be bound or set equal to a particular value to turn it into a proposition. This can be done using a combination of universal quantifiers, existential quantifiers, and value assignments.

The part of a logical expression to which a quantifier is applied is called the **scope** of this quantifier. Consequently, a variable is free if it is outside the scope of all quantifiers in the formula that specify this variable.

EXAMPLE 18 In the statement $\exists x(x + y = 1)$, the variable x is bound by the existential quantification $\exists x$, but the variable y is free because it is not bound by a quantifier and no value is assigned to this variable. This illustrates that in the statement $\exists x(x + y = 1)$, x is bound, but y is free.

In the statement $\exists x(P(x) \wedge Q(x)) \vee \forall x R(x)$, all variables are bound. The scope of the first quantifier, $\exists x$, is the expression $P(x) \wedge Q(x)$ because $\exists x$ is applied only to $P(x) \wedge Q(x)$, and not to the rest of the statement. Similarly, the scope of the second quantifier, $\forall x$, is the expression $R(x)$. That is, the existential quantifier binds the variable x in $P(x) \wedge Q(x)$ and the universal quantifier $\forall x$ binds the variable x in $R(x)$. Observe that we could have written our statement using two different variables x and y , as $\exists x(P(x) \wedge Q(x)) \vee \forall y R(y)$, because the scopes of the two quantifiers do not overlap. The reader should be aware that in common usage, the same letter is often used to represent variables bound by different quantifiers with scopes that do not overlap. ◀

Logical Equivalences Involving Quantifiers

In Section 1.3 we introduced the notion of logical equivalences of compound propositions. We can extend this notion to expressions involving predicates and quantifiers.

DEFINITION 3

Statements involving predicates and quantifiers are *logically equivalent* if and only if they have the same truth value no matter which predicates are substituted into these statements and which domain of discourse is used for the variables in these propositional functions. We use the notation $S \equiv T$ to indicate that two statements S and T involving predicates and quantifiers are logically equivalent.

Example 19 illustrates how to show that two statements involving predicates and quantifiers are logically equivalent.

EXAMPLE 19 Show that $\forall x(P(x) \wedge Q(x))$ and $\forall x P(x) \wedge \forall x Q(x)$ are logically equivalent (where the same domain is used throughout). This logical equivalence shows that we can distribute a universal quantifier over a conjunction. Furthermore, we can also distribute an existential quantifier over a disjunction. However, we cannot distribute a universal quantifier over a disjunction, nor can we distribute an existential quantifier over a conjunction. (See Exercises 50 and 51.)

Solution: To show that these statements are logically equivalent, we must show that they always take the same truth value, no matter what the predicates P and Q are, and no matter which domain of discourse is used. Suppose we have particular predicates P and Q , with a common domain. We can show that $\forall x(P(x) \wedge Q(x))$ and $\forall x P(x) \wedge \forall x Q(x)$ are logically equivalent by doing two things. First, we show that if $\forall x(P(x) \wedge Q(x))$ is true, then $\forall x P(x) \wedge \forall x Q(x)$ is true. Second, we show that if $\forall x P(x) \wedge \forall x Q(x)$ is true, then $\forall x(P(x) \wedge Q(x))$ is true.

So, suppose that $\forall x(P(x) \wedge Q(x))$ is true. This means that if a is in the domain, then $P(a) \wedge Q(a)$ is true. Hence, $P(a)$ is true and $Q(a)$ is true. Because $P(a)$ is true and $Q(a)$ is true for every element in the domain, we can conclude that $\forall x P(x)$ and $\forall x Q(x)$ are both true. This means that $\forall x P(x) \wedge \forall x Q(x)$ is true.

Next, suppose that $\forall x P(x) \wedge \forall x Q(x)$ is true. It follows that $\forall x P(x)$ is true and $\forall x Q(x)$ is true. Hence, if a is in the domain, then $P(a)$ is true and $Q(a)$ is true [because $P(x)$ and $Q(x)$ are both true for all elements in the domain, there is no conflict using the same value of a here].

It follows that for all a , $P(a) \wedge Q(a)$ is true. It follows that $\forall x(P(x) \wedge Q(x))$ is true. We can now conclude that

$$\forall x(P(x) \wedge Q(x)) \equiv \forall xP(x) \wedge \forall xQ(x).$$



Negating Quantified Expressions

We will often want to consider the negation of a quantified expression. For instance, consider the negation of the statement

“Every student in your class has taken a course in calculus.”

This statement is a universal quantification, namely,

$$\forall xP(x),$$



where $P(x)$ is the statement “ x has taken a course in calculus” and the domain consists of the students in your class. The negation of this statement is “It is not the case that every student in your class has taken a course in calculus.” This is equivalent to “There is a student in your class who has not taken a course in calculus.” And this is simply the existential quantification of the negation of the original propositional function, namely,

$$\exists x \neg P(x).$$

This example illustrates the following logical equivalence:

$$\neg \forall xP(x) \equiv \exists x \neg P(x).$$

To show that $\neg \forall xP(x)$ and $\exists x \neg P(x)$ are logically equivalent no matter what the propositional function $P(x)$ is and what the domain is, first note that $\neg \forall xP(x)$ is true if and only if $\forall xP(x)$ is false. Next, note that $\forall xP(x)$ is false if and only if there is an element x in the domain for which $P(x)$ is false. This holds if and only if there is an element x in the domain for which $\neg P(x)$ is true. Finally, note that there is an element x in the domain for which $\neg P(x)$ is true if and only if $\exists x \neg P(x)$ is true. Putting these steps together, we can conclude that $\neg \forall xP(x)$ is true if and only if $\exists x \neg P(x)$ is true. It follows that $\neg \forall xP(x)$ and $\exists x \neg P(x)$ are logically equivalent.

Suppose we wish to negate an existential quantification. For instance, consider the proposition “There is a student in this class who has taken a course in calculus.” This is the existential quantification

$$\exists xQ(x),$$

where $Q(x)$ is the statement “ x has taken a course in calculus.” The negation of this statement is the proposition “It is not the case that there is a student in this class who has taken a course in calculus.” This is equivalent to “Every student in this class has not taken calculus,” which is just the universal quantification of the negation of the original propositional function, or, phrased in the language of quantifiers,

$$\forall x \neg Q(x).$$

This example illustrates the equivalence

$$\neg \exists xQ(x) \equiv \forall x \neg Q(x).$$

To show that $\neg \exists xQ(x)$ and $\forall x \neg Q(x)$ are logically equivalent no matter what $Q(x)$ is and what the domain is, first note that $\neg \exists xQ(x)$ is true if and only if $\exists xQ(x)$ is false. This is true if and

TABLE 2 De Morgan's Laws for Quantifiers.

<i>Negation</i>	<i>Equivalent Statement</i>	<i>When Is Negation True?</i>	<i>When False?</i>
$\neg\exists x P(x)$	$\forall x \neg P(x)$	For every x , $P(x)$ is false.	There is an x for which $P(x)$ is true.
$\neg\forall x P(x)$	$\exists x \neg P(x)$	There is an x for which $P(x)$ is false.	$P(x)$ is true for every x .

only if no x exists in the domain for which $Q(x)$ is true. Next, note that no x exists in the domain for which $Q(x)$ is true if and only if $Q(x)$ is false for every x in the domain. Finally, note that $Q(x)$ is false for every x in the domain if and only if $\neg Q(x)$ is true for all x in the domain, which holds if and only if $\forall x \neg Q(x)$ is true. Putting these steps together, we see that $\neg\exists x Q(x)$ is true if and only if $\forall x \neg Q(x)$ is true. We conclude that $\neg\exists x Q(x)$ and $\forall x \neg Q(x)$ are logically equivalent.

The rules for negations for quantifiers are called **De Morgan's laws for quantifiers**. These rules are summarized in Table 2.

Remark: When the domain of a predicate $P(x)$ consists of n elements, where n is a positive integer greater than one, the rules for negating quantified statements are exactly the same as De Morgan's laws discussed in Section 1.3. This is why these rules are called De Morgan's laws for quantifiers. When the domain has n elements x_1, x_2, \dots, x_n , it follows that $\neg\forall x P(x)$ is the same as $\neg(P(x_1) \wedge P(x_2) \wedge \dots \wedge P(x_n))$, which is equivalent to $\neg P(x_1) \vee \neg P(x_2) \vee \dots \vee \neg P(x_n)$ by De Morgan's laws, and this is the same as $\exists x \neg P(x)$. Similarly, $\neg\exists x P(x)$ is the same as $\neg(P(x_1) \vee P(x_2) \vee \dots \vee P(x_n))$, which by De Morgan's laws is equivalent to $\neg P(x_1) \wedge \neg P(x_2) \wedge \dots \wedge \neg P(x_n)$, and this is the same as $\forall x \neg P(x)$.

We illustrate the negation of quantified statements in Examples 20 and 21.

EXAMPLE 20 What are the negations of the statements “There is an honest politician” and “All Americans eat cheeseburgers”?

Solution: Let $H(x)$ denote “ x is honest.” Then the statement “There is an honest politician” is represented by $\exists x H(x)$, where the domain consists of all politicians. The negation of this statement is $\neg\exists x H(x)$, which is equivalent to $\forall x \neg H(x)$. This negation can be expressed as “Every politician is dishonest.” (Note: In English, the statement “All politicians are not honest” is ambiguous. In common usage, this statement often means “Not all politicians are honest.” Consequently, we do not use this statement to express this negation.)



Let $C(x)$ denote “ x eats cheeseburgers.” Then the statement “All Americans eat cheeseburgers” is represented by $\forall x C(x)$, where the domain consists of all Americans. The negation of this statement is $\neg\forall x C(x)$, which is equivalent to $\exists x \neg C(x)$. This negation can be expressed in several different ways, including “Some American does not eat cheeseburgers” and “There is an American who does not eat cheeseburgers.” ◀

EXAMPLE 21 What are the negations of the statements $\forall x (x^2 > x)$ and $\exists x (x^2 = 2)$?

Solution: The negation of $\forall x (x^2 > x)$ is the statement $\neg\forall x (x^2 > x)$, which is equivalent to $\exists x \neg(x^2 > x)$. This can be rewritten as $\exists x (x^2 \leq x)$. The negation of $\exists x (x^2 = 2)$ is the statement $\neg\exists x (x^2 = 2)$, which is equivalent to $\forall x \neg(x^2 = 2)$. This can be rewritten as $\forall x (x^2 \neq 2)$. The truth values of these statements depend on the domain. ◀

We use De Morgan's laws for quantifiers in Example 22.

EXAMPLE 22 Show that $\neg\forall x(P(x) \rightarrow Q(x))$ and $\exists x(P(x) \wedge \neg Q(x))$ are logically equivalent.

Solution: By De Morgan's law for universal quantifiers, we know that $\neg\forall x(P(x) \rightarrow Q(x))$ and $\exists x(\neg(P(x) \rightarrow Q(x)))$ are logically equivalent. By the fifth logical equivalence in Table 7 in Section 1.3, we know that $\neg(P(x) \rightarrow Q(x))$ and $P(x) \wedge \neg Q(x)$ are logically equivalent for every x . Because we can substitute one logically equivalent expression for another in a logical equivalence, it follows that $\neg\forall x(P(x) \rightarrow Q(x))$ and $\exists x(P(x) \wedge \neg Q(x))$ are logically equivalent. \blacktriangleleft

Translating from English into Logical Expressions

Translating sentences in English (or other natural languages) into logical expressions is a crucial task in mathematics, logic programming, artificial intelligence, software engineering, and many other disciplines. We began studying this topic in Section 1.1, where we used propositions to express sentences in logical expressions. In that discussion, we purposely avoided sentences whose translations required predicates and quantifiers. Translating from English to logical expressions becomes even more complex when quantifiers are needed. Furthermore, there can be many ways to translate a particular sentence. (As a consequence, there is no "cookbook" approach that can be followed step by step.) We will use some examples to illustrate how to translate sentences from English into logical expressions. The goal in this translation is to produce simple and useful logical expressions. In this section, we restrict ourselves to sentences that can be translated into logical expressions using a single quantifier; in the next section, we will look at more complicated sentences that require multiple quantifiers.

EXAMPLE 23 Express the statement "Every student in this class has studied calculus" using predicates and quantifiers.

Solution: First, we rewrite the statement so that we can clearly identify the appropriate quantifiers to use. Doing so, we obtain:

"For every student in this class, that student has studied calculus."



Next, we introduce a variable x so that our statement becomes

"For every student x in this class, x has studied calculus."

Continuing, we introduce $C(x)$, which is the statement " x has studied calculus." Consequently, if the domain for x consists of the students in the class, we can translate our statement as $\forall x C(x)$.

However, there are other correct approaches; different domains of discourse and other predicates can be used. The approach we select depends on the subsequent reasoning we want to carry out. For example, we may be interested in a wider group of people than only those in this class. If we change the domain to consist of all people, we will need to express our statement as

"For every person x , if person x is a student in this class then x has studied calculus."



If $S(x)$ represents the statement that person x is in this class, we see that our statement can be expressed as $\forall x(S(x) \rightarrow C(x))$. [Caution! Our statement *cannot* be expressed as $\forall x(S(x) \wedge C(x))$ because this statement says that all people are students in this class and have studied calculus!]

Finally, when we are interested in the background of people in subjects besides calculus, we may prefer to use the two-variable quantifier $Q(x, y)$ for the statement "student x has studied subject y ." Then we would replace $C(x)$ by $Q(x, \text{calculus})$ in both approaches to obtain $\forall x Q(x, \text{calculus})$ or $\forall x(S(x) \rightarrow Q(x, \text{calculus}))$. \blacktriangleleft

In Example 23 we displayed different approaches for expressing the same statement using predicates and quantifiers. However, we should always adopt the simplest approach that is adequate for use in subsequent reasoning.

EXAMPLE 24 Express the statements “Some student in this class has visited Mexico” and “Every student in this class has visited either Canada or Mexico” using predicates and quantifiers.

Solution: The statement “Some student in this class has visited Mexico” means that

“There is a student in this class with the property that the student has visited Mexico.”

We can introduce a variable x , so that our statement becomes

“There is a student x in this class having the property that x has visited Mexico.”

We introduce $M(x)$, which is the statement “ x has visited Mexico.” If the domain for x consists of the students in this class, we can translate this first statement as $\exists x M(x)$.

However, if we are interested in people other than those in this class, we look at the statement a little differently. Our statement can be expressed as

“There is a person x having the properties that x is a student in this class and x has visited Mexico.”



In this case, the domain for the variable x consists of all people. We introduce $S(x)$ to represent “ x is a student in this class.” Our solution becomes $\exists x(S(x) \wedge M(x))$ because the statement is that there is a person x who is a student in this class and who has visited Mexico. [Caution! Our statement cannot be expressed as $\exists x(S(x) \rightarrow M(x))$, which is true when there is someone not in the class because, in that case, for such a person x , $S(x) \rightarrow M(x)$ becomes either $F \rightarrow T$ or $F \rightarrow F$, both of which are true.]

Similarly, the second statement can be expressed as

“For every x in this class, x has the property that x has visited Mexico or x has visited Canada.”

(Note that we are assuming the inclusive, rather than the exclusive, or here.) We let $C(x)$ be “ x has visited Canada.” Following our earlier reasoning, we see that if the domain for x consists of the students in this class, this second statement can be expressed as $\forall x(C(x) \vee M(x))$. However, if the domain for x consists of all people, our statement can be expressed as

“For every person x , if x is a student in this class, then x has visited Mexico or x has visited Canada.”

In this case, the statement can be expressed as $\forall x(S(x) \rightarrow (C(x) \vee M(x)))$.

Instead of using $M(x)$ and $C(x)$ to represent that x has visited Mexico and x has visited Canada, respectively, we could use a two-place predicate $V(x, y)$ to represent “ x has visited country y .” In this case, $V(x, \text{Mexico})$ and $V(x, \text{Canada})$ would have the same meaning as $M(x)$ and $C(x)$ and could replace them in our answers. If we are working with many statements that involve people visiting different countries, we might prefer to use this two-variable approach. Otherwise, for simplicity, we would stick with the one-variable predicates $M(x)$ and $C(x)$. ◀

Using Quantifiers in System Specifications

In Section 1.2 we used propositions to represent system specifications. However, many system specifications involve predicates and quantifications. This is illustrated in Example 25.

EXAMPLE 25

Use predicates and quantifiers to express the system specifications “Every mail message larger than one megabyte will be compressed” and “If a user is active, at least one network link will be available.”



Remember the rules of precedence for quantifiers and logical connectives!

Solution: Let $S(m, y)$ be “Mail message m is larger than y megabytes,” where the variable x has the domain of all mail messages and the variable y is a positive real number, and let $C(m)$ denote “Mail message m will be compressed.” Then the specification “Every mail message larger than one megabyte will be compressed” can be represented as $\forall m(S(m, 1) \rightarrow C(m))$.

Let $A(u)$ represent “User u is active,” where the variable u has the domain of all users, let $S(n, x)$ denote “Network link n is in state x ,” where n has the domain of all network links and x has the domain of all possible states for a network link. Then the specification “If a user is active, at least one network link will be available” can be represented by $\exists u A(u) \rightarrow \exists n S(n, \text{available})$. ◀

Examples from Lewis Carroll

Lewis Carroll (really C. L. Dodgson writing under a pseudonym), the author of *Alice in Wonderland*, is also the author of several works on symbolic logic. His books contain many examples of reasoning using quantifiers. Examples 26 and 27 come from his book *Symbolic Logic*; other examples from that book are given in the exercises at the end of this section. These examples illustrate how quantifiers are used to express various types of statements.

EXAMPLE 26

Consider these statements. The first two are called *premises* and the third is called the *conclusion*. The entire set is called an *argument*.

- “All lions are fierce.”
- “Some lions do not drink coffee.”
- “Some fierce creatures do not drink coffee.”

(In Section 1.6 we will discuss the issue of determining whether the conclusion is a valid consequence of the premises. In this example, it is.) Let $P(x)$, $Q(x)$, and $R(x)$ be the statements “ x is a lion,” “ x is fierce,” and “ x drinks coffee,” respectively. Assuming that the domain consists of all creatures, express the statements in the argument using quantifiers and $P(x)$, $Q(x)$, and $R(x)$.



CHARLES LUTWIDGE DODGSON (1832–1898) We know Charles Dodgson as Lewis Carroll—the pseudonym he used in his literary works. Dodgson, the son of a clergyman, was the third of 11 children, all of whom stuttered. He was uncomfortable in the company of adults and is said to have spoken without stuttering only to young girls, many of whom he entertained, corresponded with, and photographed (sometimes in poses that today would be considered inappropriate). Although attracted to young girls, he was extremely puritanical and religious. His friendship with the three young daughters of Dean Liddell led to his writing *Alice in Wonderland*, which brought him money and fame.

Dodgson graduated from Oxford in 1854 and obtained his master of arts degree in 1857. He was appointed lecturer in mathematics at Christ Church College, Oxford, in 1855. He was ordained in the Church of England in 1861 but never practiced his ministry. His writings published under this real name include articles and books on geometry, determinants, and the mathematics of tournaments and elections. (He also used the pseudonym Lewis Carroll for his many works on recreational logic.)

Solution: We can express these statements as:

$$\begin{aligned}\forall x(P(x) \rightarrow Q(x)). \\ \exists x(P(x) \wedge \neg R(x)). \\ \exists x(Q(x) \wedge \neg R(x)).\end{aligned}$$

Notice that the second statement cannot be written as $\exists x(P(x) \rightarrow \neg R(x))$. The reason is that $P(x) \rightarrow \neg R(x)$ is true whenever x is not a lion, so that $\exists x(P(x) \rightarrow \neg R(x))$ is true as long as there is at least one creature that is not a lion, even if every lion drinks coffee. Similarly, the third statement cannot be written as

$$\exists x(Q(x) \rightarrow \neg R(x)). \quad \blacktriangleleft$$

EXAMPLE 27 Consider these statements, of which the first three are premises and the fourth is a valid conclusion.

- “All hummingbirds are richly colored.”
- “No large birds live on honey.”
- “Birds that do not live on honey are dull in color.”
- “Hummingbirds are small.”

Let $P(x)$, $Q(x)$, $R(x)$, and $S(x)$ be the statements “ x is a hummingbird,” “ x is large,” “ x lives on honey,” and “ x is richly colored,” respectively. Assuming that the domain consists of all birds, express the statements in the argument using quantifiers and $P(x)$, $Q(x)$, $R(x)$, and $S(x)$.

Solution: We can express the statements in the argument as

$$\begin{aligned}\forall x(P(x) \rightarrow S(x)). \\ \neg \exists x(Q(x) \wedge R(x)). \\ \forall x(\neg R(x) \rightarrow \neg S(x)). \\ \forall x(P(x) \rightarrow \neg Q(x)).\end{aligned}$$

(Note we have assumed that “small” is the same as “not large” and that “dull in color” is the same as “not richly colored.” To show that the fourth statement is a valid conclusion of the first three, we need to use rules of inference that will be discussed in Section 1.6.) \blacktriangleleft

Logic Programming



An important type of programming language is designed to reason using the rules of predicate logic. Prolog (from *Programming in Logic*), developed in the 1970s by computer scientists working in the area of artificial intelligence, is an example of such a language. Prolog programs include a set of declarations consisting of two types of statements, **Prolog facts** and **Prolog rules**. Prolog facts define predicates by specifying the elements that satisfy these predicates. Prolog rules are used to define new predicates using those already defined by Prolog facts. Example 28 illustrates these notions.

EXAMPLE 28 Consider a Prolog program given facts telling it the instructor of each class and in which classes students are enrolled. The program uses these facts to answer queries concerning the professors who teach particular students. Such a program could use the predicates *instructor(p, c)* and

enrolled(s, c) to represent that professor *p* is the instructor of course *c* and that student *s* is enrolled in course *c*, respectively. For example, the Prolog facts in such a program might include:

```
instructor(chan,math273)
instructor(patel,ee222)
instructor(grossman,cs301)
enrolled(kevin,math273)
enrolled(juana,ee222)
enrolled(juana,cs301)
enrolled(kiko,math273)
enrolled(kiko,cs301)
```

(Lowercase letters have been used for entries because Prolog considers names beginning with an uppercase letter to be variables.)

A new predicate *teaches(p, s)*, representing that professor *p* teaches student *s*, can be defined using the Prolog rule

```
teaches(P,S) :- instructor(P,C), enrolled(S,C)
```

which means that *teaches(p, s)* is true if there exists a class *c* such that professor *p* is the instructor of class *c* and student *s* is enrolled in class *c*. (Note that a comma is used to represent a conjunction of predicates in Prolog. Similarly, a semicolon is used to represent a disjunction of predicates.)

Prolog answers queries using the facts and rules it is given. For example, using the facts and rules listed, the query

```
?enrolled(kevin,math273)
```

produces the response

```
yes
```

because the fact *enrolled(kevin, math273)* was provided as input. The query

```
?enrolled(X,math273)
```

produces the response

```
kevin
kiko
```

To produce this response, Prolog determines all possible values of *X* for which *enrolled(X, math273)* has been included as a Prolog fact. Similarly, to find all the professors who are instructors in classes being taken by Juana, we use the query

```
?teaches(X,juana)
```

This query returns

```
patel
grossman
```



Exercises

1. Let $P(x)$ denote the statement " $x \leq 4$." What are these truth values?
 - a) $P(0)$
 - b) $P(4)$
 - c) $P(6)$
2. Let $P(x)$ be the statement "the word x contains the letter a ." What are these truth values?
 - a) $P(\text{orange})$
 - b) $P(\text{lemon})$
 - c) $P(\text{true})$
 - d) $P(\text{false})$
3. Let $Q(x, y)$ denote the statement " x is the capital of y ." What are these truth values?
 - a) $Q(\text{Denver}, \text{Colorado})$
 - b) $Q(\text{Detroit}, \text{Michigan})$
 - c) $Q(\text{Massachusetts}, \text{Boston})$
 - d) $Q(\text{New York}, \text{New York})$
4. State the value of x after the statement **if** $P(x)$ **then** $x := 1$ is executed, where $P(x)$ is the statement " $x > 1$," if the value of x when this statement is reached is
 - a) $x = 0$.
 - b) $x = 1$.
 - c) $x = 2$.
5. Let $P(x)$ be the statement " x spends more than five hours every weekday in class," where the domain for x consists of all students. Express each of these quantifications in English.
 - a) $\exists x P(x)$
 - b) $\forall x P(x)$
 - c) $\exists x \neg P(x)$
 - d) $\forall x \neg P(x)$
6. Let $N(x)$ be the statement " x has visited North Dakota," where the domain consists of the students in your school. Express each of these quantifications in English.
 - a) $\exists x N(x)$
 - b) $\forall x N(x)$
 - c) $\neg \exists x N(x)$
 - d) $\exists x \neg N(x)$
 - e) $\neg \forall x N(x)$
 - f) $\forall x \neg N(x)$
7. Translate these statements into English, where $C(x)$ is " x is a comedian" and $F(x)$ is " x is funny" and the domain consists of all people.
 - a) $\forall x(C(x) \rightarrow F(x))$
 - b) $\forall x(C(x) \wedge F(x))$
 - c) $\exists x(C(x) \rightarrow F(x))$
 - d) $\exists x(C(x) \wedge F(x))$
8. Translate these statements into English, where $R(x)$ is " x is a rabbit" and $H(x)$ is " x hops" and the domain consists of all animals.
 - a) $\forall x(R(x) \rightarrow H(x))$
 - b) $\forall x(R(x) \wedge H(x))$
 - c) $\exists x(R(x) \rightarrow H(x))$
 - d) $\exists x(R(x) \wedge H(x))$
9. Let $P(x)$ be the statement " x can speak Russian" and let $Q(x)$ be the statement " x knows the computer language C++." Express each of these sentences in terms of $P(x)$, $Q(x)$, quantifiers, and logical connectives. The domain for quantifiers consists of all students at your school.
 - a) There is a student at your school who can speak Russian and who knows C++.
 - b) There is a student at your school who can speak Russian but who doesn't know C++.
 - c) Every student at your school either can speak Russian or knows C++.
 - d) No student at your school can speak Russian or knows C++.
10. Let $C(x)$ be the statement " x has a cat," let $D(x)$ be the statement " x has a dog," and let $F(x)$ be the statement " x has a ferret." Express each of these statements in terms of $C(x)$, $D(x)$, $F(x)$, quantifiers, and logical connectives. Let the domain consist of all students in your class.
 - a) A student in your class has a cat, a dog, and a ferret.
 - b) All students in your class have a cat, a dog, or a ferret.
 - c) Some student in your class has a cat and a ferret, but not a dog.
 - d) No student in your class has a cat, a dog, and a ferret.
 - e) For each of the three animals, cats, dogs, and ferrets, there is a student in your class who has this animal as a pet.
11. Let $P(x)$ be the statement " $x = x^2$." If the domain consists of the integers, what are these truth values?
 - a) $P(0)$
 - b) $P(1)$
 - c) $P(2)$
 - d) $P(-1)$
 - e) $\exists x P(x)$
 - f) $\forall x P(x)$
12. Let $Q(x)$ be the statement " $x + 1 > 2x$." If the domain consists of all integers, what are these truth values?
 - a) $Q(0)$
 - b) $Q(-1)$
 - c) $Q(1)$
 - d) $\exists x Q(x)$
 - e) $\forall x Q(x)$
 - f) $\exists x \neg Q(x)$
 - g) $\forall x \neg Q(x)$
13. Determine the truth value of each of these statements if the domain consists of all integers.
 - a) $\forall n(n + 1 > n)$
 - b) $\exists n(2n = 3n)$
 - c) $\exists n(n = -n)$
 - d) $\forall n(3n \leq 4n)$
14. Determine the truth value of each of these statements if the domain consists of all real numbers.
 - a) $\exists x(x^3 = -1)$
 - b) $\exists x(x^4 < x^2)$
 - c) $\forall x((-x)^2 = x^2)$
 - d) $\forall x(2x > x)$
15. Determine the truth value of each of these statements if the domain for all variables consists of all integers.
 - a) $\forall n(n^2 \geq 0)$
 - b) $\exists n(n^2 = 2)$
 - c) $\forall n(n^2 \geq n)$
 - d) $\exists n(n^2 < 0)$
16. Determine the truth value of each of these statements if the domain of each variable consists of all real numbers.
 - a) $\exists x(x^2 = 2)$
 - b) $\exists x(x^2 = -1)$
 - c) $\forall x(x^2 + 2 \geq 1)$
 - d) $\forall x(x^2 \neq x)$
17. Suppose that the domain of the propositional function $P(x)$ consists of the integers 0, 1, 2, 3, and 4. Write out each of these propositions using disjunctions, conjunctions, and negations.
 - a) $\exists x P(x)$
 - b) $\forall x P(x)$
 - c) $\exists x \neg P(x)$
 - d) $\forall x \neg P(x)$
 - e) $\neg \exists x P(x)$
 - f) $\neg \forall x P(x)$
18. Suppose that the domain of the propositional function $P(x)$ consists of the integers -2, -1, 0, 1, and 2. Write out each of these propositions using disjunctions, conjunctions, and negations.
 - a) $\exists x P(x)$
 - b) $\forall x P(x)$
 - c) $\exists x \neg P(x)$
 - d) $\forall x \neg P(x)$
 - e) $\neg \exists x P(x)$
 - f) $\neg \forall x P(x)$

- 19.** Suppose that the domain of the propositional function $P(x)$ consists of the integers 1, 2, 3, 4, and 5. Express these statements without using quantifiers, instead using only negations, disjunctions, and conjunctions.
- $\exists x P(x)$
 - $\forall x P(x)$
 - $\neg \exists x P(x)$
 - $\neg \forall x P(x)$
 - $\forall x ((x \neq 3) \rightarrow P(x)) \vee \exists x \neg P(x)$
- 20.** Suppose that the domain of the propositional function $P(x)$ consists of $-5, -3, -1, 1, 3$, and 5. Express these statements without using quantifiers, instead using only negations, disjunctions, and conjunctions.
- $\exists x P(x)$
 - $\forall x P(x)$
 - $\forall x ((x \neq 1) \rightarrow P(x))$
 - $\exists x ((x \geq 0) \wedge P(x))$
 - $\exists x (\neg P(x)) \wedge \forall x ((x < 0) \rightarrow P(x))$
- 21.** For each of these statements find a domain for which the statement is true and a domain for which the statement is false.
- Everyone is studying discrete mathematics.
 - Everyone is older than 21 years.
 - Every two people have the same mother.
 - No two different people have the same grandmother.
- 22.** For each of these statements find a domain for which the statement is true and a domain for which the statement is false.
- Everyone speaks Hindi.
 - There is someone older than 21 years.
 - Every two people have the same first name.
 - Someone knows more than two other people.
- 23.** Translate in two ways each of these statements into logical expressions using predicates, quantifiers, and logical connectives. First, let the domain consist of the students in your class and second, let it consist of all people.
- Someone in your class can speak Hindi.
 - Everyone in your class is friendly.
 - There is a person in your class who was not born in California.
 - A student in your class has been in a movie.
 - No student in your class has taken a course in logic programming.
- 24.** Translate in two ways each of these statements into logical expressions using predicates, quantifiers, and logical connectives. First, let the domain consist of the students in your class and second, let it consist of all people.
- Everyone in your class has a cellular phone.
 - Somebody in your class has seen a foreign movie.
 - There is a person in your class who cannot swim.
 - All students in your class can solve quadratic equations.
 - Some student in your class does not want to be rich.
- 25.** Translate each of these statements into logical expressions using predicates, quantifiers, and logical connectives.
- No one is perfect.
 - Not everyone is perfect.
 - All your friends are perfect.
 - At least one of your friends is perfect.
- 26.** Translate each of these statements into logical expressions in three different ways by varying the domain and by using predicates with one and with two variables.
- Someone in your school has visited Uzbekistan.
 - Everyone in your class has studied calculus and C++.
 - No one in your school owns both a bicycle and a motorcycle.
 - There is a person in your school who is not happy.
 - Everyone in your school was born in the twentieth century.
- 27.** Translate each of these statements into logical expressions in three different ways by varying the domain and by using predicates with one and with two variables.
- A student in your school has lived in Vietnam.
 - There is a student in your school who cannot speak Hindi.
 - A student in your school knows Java, Prolog, and C++.
 - Everyone in your class enjoys Thai food.
 - Someone in your class does not play hockey.
- 28.** Translate each of these statements into logical expressions using predicates, quantifiers, and logical connectives.
- Something is not in the correct place.
 - All tools are in the correct place and are in excellent condition.
 - Everything is in the correct place and in excellent condition.
 - Nothing is in the correct place and is in excellent condition.
 - One of your tools is not in the correct place, but it is in excellent condition.
- 29.** Express each of these statements using logical operators, predicates, and quantifiers.
- Some propositions are tautologies.
 - The negation of a contradiction is a tautology.
 - The disjunction of two contingencies can be a tautology.
 - The conjunction of two tautologies is a tautology.
- 30.** Suppose the domain of the propositional function $P(x, y)$ consists of pairs x and y , where x is 1, 2, or 3 and y is 1, 2, or 3. Write out these propositions using disjunctions and conjunctions.
- $\exists x P(x, 3)$
 - $\forall y P(1, y)$
 - $\exists y \neg P(2, y)$
 - $\forall x \neg P(x, 2)$
- 31.** Suppose that the domain of $Q(x, y, z)$ consists of triples x, y, z , where $x = 0, 1$, or 2, $y = 0$ or 1, and $z = 0$ or 1. Write out these propositions using disjunctions and conjunctions.
- $\forall y Q(0, y, 0)$
 - $\exists x Q(x, 1, 1)$
 - $\exists z \neg Q(0, 0, z)$
 - $\exists x \neg Q(x, 0, 1)$

- 32.** Express each of these statements using quantifiers. Then form the negation of the statement so that no negation is to the left of a quantifier. Next, express the negation in simple English. (Do not simply use the phrase “It is not the case that.”)
- All dogs have fleas.
 - There is a horse that can add.
 - Every koala can climb.
 - No monkey can speak French.
 - There exists a pig that can swim and catch fish.
- 33.** Express each of these statements using quantifiers. Then form the negation of the statement, so that no negation is to the left of a quantifier. Next, express the negation in simple English. (Do not simply use the phrase “It is not the case that.”)
- Some old dogs can learn new tricks.
 - No rabbit knows calculus.
 - Every bird can fly.
 - There is no dog that can talk.
 - There is no one in this class who knows French and Russian.
- 34.** Express the negation of these propositions using quantifiers, and then express the negation in English.
- Some drivers do not obey the speed limit.
 - All Swedish movies are serious.
 - No one can keep a secret.
 - There is someone in this class who does not have a good attitude.
- 35.** Find a counterexample, if possible, to these universally quantified statements, where the domain for all variables consists of all integers.
- $\forall x(x^2 \geq x)$
 - $\forall x(x > 0 \vee x < 0)$
 - $\forall x(x = 1)$
- 36.** Find a counterexample, if possible, to these universally quantified statements, where the domain for all variables consists of all real numbers.
- $\forall x(x^2 \neq x)$
 - $\forall x(x^2 \neq 2)$
 - $\forall x(|x| > 0)$
- 37.** Express each of these statements using predicates and quantifiers.
- A passenger on an airline qualifies as an elite flyer if the passenger flies more than 25,000 miles in a year or takes more than 25 flights during that year.
 - A man qualifies for the marathon if his best previous time is less than 3 hours and a woman qualifies for the marathon if her best previous time is less than 3.5 hours.
 - A student must take at least 60 course hours, or at least 45 course hours and write a master’s thesis, and receive a grade no lower than a B in all required courses, to receive a master’s degree.
 - There is a student who has taken more than 21 credit hours in a semester and received all A’s.
- Exercises 38–42 deal with the translation between system specification and logical expressions involving quantifiers.
- 38.** Translate these system specifications into English where the predicate $S(x, y)$ is “ x is in state y ” and where the domain for x and y consists of all systems and all possible states, respectively.
- $\exists x S(x, \text{open})$
 - $\forall x (S(x, \text{malfunctioning}) \vee S(x, \text{diagnostic}))$
 - $\exists x S(x, \text{open}) \vee \exists x S(x, \text{diagnostic})$
 - $\exists x \neg S(x, \text{available})$
 - $\forall x \neg S(x, \text{working})$
- 39.** Translate these specifications into English where $F(p)$ is “Printer p is out of service,” $B(p)$ is “Printer p is busy,” $L(j)$ is “Print job j is lost,” and $Q(j)$ is “Print job j is queued.”
- $\exists p (F(p) \wedge B(p)) \rightarrow \exists j L(j)$
 - $\forall p B(p) \rightarrow \exists j Q(j)$
 - $\exists j (Q(j) \wedge L(j)) \rightarrow \exists p F(p)$
 - $(\forall p B(p) \wedge \forall j Q(j)) \rightarrow \exists j L(j)$
- 40.** Express each of these system specifications using predicates, quantifiers, and logical connectives.
- When there is less than 30 megabytes free on the hard disk, a warning message is sent to all users.
 - No directories in the file system can be opened and no files can be closed when system errors have been detected.
 - The file system cannot be backed up if there is a user currently logged on.
 - Video on demand can be delivered when there are at least 8 megabytes of memory available and the connection speed is at least 56 kilobits per second.
- 41.** Express each of these system specifications using predicates, quantifiers, and logical connectives.
- At least one mail message, among the nonempty set of messages, can be saved if there is a disk with more than 10 kilobytes of free space.
 - Whenever there is an active alert, all queued messages are transmitted.
 - The diagnostic monitor tracks the status of all systems except the main console.
 - Each participant on the conference call whom the host of the call did not put on a special list was billed.
- 42.** Express each of these system specifications using predicates, quantifiers, and logical connectives.
- Every user has access to an electronic mailbox.
 - The system mailbox can be accessed by everyone in the group if the file system is locked.
 - The firewall is in a diagnostic state only if the proxy server is in a diagnostic state.
 - At least one router is functioning normally if the throughput is between 100 kbps and 500 kbps and the proxy server is not in diagnostic mode.

- 43.** Determine whether $\forall x(P(x) \rightarrow Q(x))$ and $\forall x P(x) \rightarrow \forall x Q(x)$ are logically equivalent. Justify your answer.
- 44.** Determine whether $\forall x(P(x) \leftrightarrow Q(x))$ and $\forall x P(x) \leftrightarrow \forall x Q(x)$ are logically equivalent. Justify your answer.
- 45.** Show that $\exists x(P(x) \vee Q(x))$ and $\exists x P(x) \vee \exists x Q(x)$ are logically equivalent.

Exercises 46–49 establish rules for **null quantification** that we can use when a quantified variable does not appear in part of a statement.

- 46.** Establish these logical equivalences, where x does not occur as a free variable in A . Assume that the domain is nonempty.
- $(\forall x P(x)) \vee A \equiv \forall x(P(x) \vee A)$
 - $(\exists x P(x)) \vee A \equiv \exists x(P(x) \vee A)$
- 47.** Establish these logical equivalences, where x does not occur as a free variable in A . Assume that the domain is nonempty.
- $(\forall x P(x)) \wedge A \equiv \forall x(P(x) \wedge A)$
 - $(\exists x P(x)) \wedge A \equiv \exists x(P(x) \wedge A)$
- 48.** Establish these logical equivalences, where x does not occur as a free variable in A . Assume that the domain is nonempty.
- $\forall x(A \rightarrow P(x)) \equiv A \rightarrow \forall x P(x)$
 - $\exists x(A \rightarrow P(x)) \equiv A \rightarrow \exists x P(x)$
- 49.** Establish these logical equivalences, where x does not occur as a free variable in A . Assume that the domain is nonempty.
- $\forall x(P(x) \rightarrow A) \equiv \exists x P(x) \rightarrow A$
 - $\exists x(P(x) \rightarrow A) \equiv \forall x P(x) \rightarrow A$
- 50.** Show that $\forall x P(x) \vee \forall x Q(x)$ and $\forall x(P(x) \vee Q(x))$ are not logically equivalent.
- 51.** Show that $\exists x P(x) \wedge \exists x Q(x)$ and $\exists x(P(x) \wedge Q(x))$ are not logically equivalent.

- 52.** As mentioned in the text, the notation $\exists!x P(x)$ denotes “There exists a unique x such that $P(x)$ is true.” If the domain consists of all integers, what are the truth values of these statements?
- $\exists!x(x > 1)$
 - $\exists!x(x^2 = 1)$
 - $\exists!x(x + 3 = 2x)$
 - $\exists!x(x = x + 1)$
- 53.** What are the truth values of these statements?
- $\exists!x P(x) \rightarrow \exists x P(x)$
 - $\forall x P(x) \rightarrow \exists!x P(x)$
 - $\exists!x \neg P(x) \rightarrow \neg \forall x P(x)$
- 54.** Write out $\exists!x P(x)$, where the domain consists of the integers 1, 2, and 3, in terms of negations, conjunctions, and disjunctions.
- 55.** Given the Prolog facts in Example 28, what would Prolog return given these queries?
- ?instructor(chan,math273)
 - ?instructor(patel,cs301)
 - ?enrolled(X,cs301)
 - ?enrolled(kiko,Y)
 - ?teaches(grossman,Y)

- 56.** Given the Prolog facts in Example 28, what would Prolog return when given these queries?

- ?enrolled(kevin,ee222)
- ?enrolled(kiko,math273)
- ?instructor(grossman,X)
- ?instructor(X,cs301)
- ?teaches(X,kevin)

- 57.** Suppose that Prolog facts are used to define the predicates $mother(M, Y)$ and $father(F, X)$, which represent that M is the mother of Y and F is the father of X , respectively. Give a Prolog rule to define the predicate $sibling(X, Y)$, which represents that X and Y are siblings (that is, have the same mother and the same father).

- 58.** Suppose that Prolog facts are used to define the predicates $mother(M, Y)$ and $father(F, X)$, which represent that M is the mother of Y and F is the father of X , respectively. Give a Prolog rule to define the predicate $grandfather(X, Y)$, which represents that X is the grandfather of Y . [Hint: You can write a disjunction in Prolog either by using a semicolon to separate predicates or by putting these predicates on separate lines.]

Exercises 59–62 are based on questions found in the book *Symbolic Logic* by Lewis Carroll.

- 59.** Let $P(x)$, $Q(x)$, and $R(x)$ be the statements “ x is a professor,” “ x is ignorant,” and “ x is vain,” respectively. Express each of these statements using quantifiers; logical connectives; and $P(x)$, $Q(x)$, and $R(x)$, where the domain consists of all people.

- No professors are ignorant.
- All ignorant people are vain.
- No professors are vain.
- Does (c) follow from (a) and (b)?

- 60.** Let $P(x)$, $Q(x)$, and $R(x)$ be the statements “ x is a clear explanation,” “ x is satisfactory,” and “ x is an excuse,” respectively. Suppose that the domain for x consists of all English text. Express each of these statements using quantifiers, logical connectives, and $P(x)$, $Q(x)$, and $R(x)$.

- All clear explanations are satisfactory.
- Some excuses are unsatisfactory.
- Some excuses are not clear explanations.
- Does (c) follow from (a) and (b)?

- 61.** Let $P(x)$, $Q(x)$, $R(x)$, and $S(x)$ be the statements “ x is a baby,” “ x is logical,” “ x is able to manage a crocodile,” and “ x is despised,” respectively. Suppose that the domain consists of all people. Express each of these statements using quantifiers; logical connectives; and $P(x)$, $Q(x)$, $R(x)$, and $S(x)$.

- Babies are illogical.
- Nobody is despised who can manage a crocodile.
- Illogical persons are despised.
- Babies cannot manage crocodiles.

- ***e**) Does (d) follow from (a), (b), and (c)? If not, is there a correct conclusion?

- 62.** Let $P(x)$, $Q(x)$, $R(x)$, and $S(x)$ be the statements “ x is a duck,” “ x is one of my poultry,” “ x is an officer,” and “ x is willing to waltz,” respectively. Express each of these statements using quantifiers; logical connectives; and $P(x)$, $Q(x)$, $R(x)$, and $S(x)$.
- a) No ducks are willing to waltz.
- b) No officers ever decline to waltz.
- c) All my poultry are ducks.
- d) My poultry are not officers.
- *e) Does (d) follow from (a), (b), and (c)? If not, is there a correct conclusion?

1.5 Nested Quantifiers

Introduction

In Section 1.4 we defined the existential and universal quantifiers and showed how they can be used to represent mathematical statements. We also explained how they can be used to translate English sentences into logical expressions. However, in Section 1.4 we avoided **nested quantifiers**, where one quantifier is within the scope of another, such as

$$\forall x \exists y (x + y = 0).$$

Note that everything within the scope of a quantifier can be thought of as a propositional function. For example,

$$\forall x \exists y (x + y = 0)$$

is the same thing as $\forall x Q(x)$, where $Q(x)$ is $\exists y P(x, y)$, where $P(x, y)$ is $x + y = 0$.

Nested quantifiers commonly occur in mathematics and computer science. Although nested quantifiers can sometimes be difficult to understand, the rules we have already studied in Section 1.4 can help us use them. In this section we will gain experience working with nested quantifiers. We will see how to use nested quantifiers to express mathematical statements such as “The sum of two positive integers is always positive.” We will show how nested quantifiers can be used to translate English sentences such as “Everyone has exactly one best friend” into logical statements. Moreover, we will gain experience working with the negations of statements involving nested quantifiers.

Understanding Statements Involving Nested Quantifiers

To understand statements involving nested quantifiers, we need to unravel what the quantifiers and predicates that appear mean. This is illustrated in Examples 1 and 2.

EXAMPLE 1 Assume that the domain for the variables x and y consists of all real numbers. The statement

$$\forall x \forall y (x + y = y + x)$$



says that $x + y = y + x$ for all real numbers x and y . This is the commutative law for addition of real numbers. Likewise, the statement

$$\forall x \exists y (x + y = 0)$$

says that for every real number x there is a real number y such that $x + y = 0$. This states that every real number has an additive inverse. Similarly, the statement

$$\forall x \forall y \forall z (x + (y + z) = (x + y) + z)$$

is the associative law for addition of real numbers. ◀

EXAMPLE 2 Translate into English the statement

$$\forall x \forall y ((x > 0) \wedge (y < 0) \rightarrow (xy < 0)),$$

where the domain for both variables consists of all real numbers.

Solution: This statement says that for every real number x and for every real number y , if $x > 0$ and $y < 0$, then $xy < 0$. That is, this statement says that for real numbers x and y , if x is positive and y is negative, then xy is negative. This can be stated more succinctly as “The product of a positive real number and a negative real number is always a negative real number.” ◀

THINKING OF QUANTIFICATION AS LOOPS In working with quantifications of more than one variable, it is sometimes helpful to think in terms of nested loops. (Of course, if there are infinitely many elements in the domain of some variable, we cannot actually loop through all values. Nevertheless, this way of thinking is helpful in understanding nested quantifiers.) For example, to see whether $\forall x \forall y P(x, y)$ is true, we loop through the values for x , and for each x we loop through the values for y . If we find that $P(x, y)$ is true for all values for x and y , we have determined that $\forall x \forall y P(x, y)$ is true. If we ever hit a value x for which we hit a value y for which $P(x, y)$ is false, we have shown that $\forall x \forall y P(x, y)$ is false.

Similarly, to determine whether $\forall x \exists y P(x, y)$ is true, we loop through the values for x . For each x we loop through the values for y until we find a y for which $P(x, y)$ is true. If for every x we hit such a y , then $\forall x \exists y P(x, y)$ is true; if for some x we never hit such a y , then $\forall x \exists y P(x, y)$ is false.

To see whether $\exists x \forall y P(x, y)$ is true, we loop through the values for x until we find an x for which $P(x, y)$ is always true when we loop through all values for y . Once we find such an x , we know that $\exists x \forall y P(x, y)$ is true. If we never hit such an x , then we know that $\exists x \forall y P(x, y)$ is false.

Finally, to see whether $\exists x \exists y P(x, y)$ is true, we loop through the values for x , where for each x we loop through the values for y until we hit an x for which we hit a y for which $P(x, y)$ is true. The statement $\exists x \exists y P(x, y)$ is false only if we never hit an x for which we hit a y such that $P(x, y)$ is true.

The Order of Quantifiers

Many mathematical statements involve multiple quantifications of propositional functions involving more than one variable. It is important to note that the order of the quantifiers is important, unless all the quantifiers are universal quantifiers or all are existential quantifiers.

These remarks are illustrated by Examples 3–5.

EXAMPLE 3 Let $P(x, y)$ be the statement “ $x + y = y + x$.” What are the truth values of the quantifications $\forall x \forall y P(x, y)$ and $\forall y \forall x P(x, y)$ where the domain for all variables consists of all real numbers?

Solution: The quantification

$$\forall x \forall y P(x, y)$$



denotes the proposition

“For all real numbers x , for all real numbers y , $x + y = y + x$.”

Because $P(x, y)$ is true for all real numbers x and y (it is the commutative law for addition, which is an axiom for the real numbers—see Appendix 1), the proposition $\forall x \forall y P(x, y)$ is true. Note that the statement $\forall y \forall x P(x, y)$ says “For all real numbers y , for all real numbers x , $x + y = y + x$.” This has the same meaning as the statement “For all real numbers x , for all real numbers y , $x + y = y + x$.” That is, $\forall x \forall y P(x, y)$ and $\forall y \forall x P(x, y)$ have the same meaning,

and both are true. This illustrates the principle that the order of nested universal quantifiers in a statement without other quantifiers can be changed without changing the meaning of the quantified statement. ◀

EXAMPLE 4 Let $Q(x, y)$ denote “ $x + y = 0$.” What are the truth values of the quantifications $\exists y \forall x Q(x, y)$ and $\forall x \exists y Q(x, y)$, where the domain for all variables consists of all real numbers?

Solution: The quantification

$$\exists y \forall x Q(x, y)$$

denotes the proposition

“There is a real number y such that for every real number x , $Q(x, y)$.”

No matter what value of y is chosen, there is only one value of x for which $x + y = 0$. Because there is no real number y such that $x + y = 0$ for all real numbers x , the statement $\exists y \forall x Q(x, y)$ is false.

The quantification

$$\forall x \exists y Q(x, y)$$

denotes the proposition

“For every real number x there is a real number y such that $Q(x, y)$.”

Given a real number x , there is a real number y such that $x + y = 0$; namely, $y = -x$. Hence, the statement $\forall x \exists y Q(x, y)$ is true. ◀

Example 4 illustrates that the order in which quantifiers appear makes a difference. The statements $\exists y \forall x P(x, y)$ and $\forall x \exists y P(x, y)$ are not logically equivalent. The statement $\exists y \forall x P(x, y)$ is true if and only if there is a y that makes $P(x, y)$ true for every x . So, for this statement to be true, there must be a particular value of y for which $P(x, y)$ is true regardless of the choice of x . On the other hand, $\forall x \exists y P(x, y)$ is true if and only if for every value of x there is a value of y for which $P(x, y)$ is true. So, for this statement to be true, no matter which x you choose, there must be a value of y (possibly depending on the x you choose) for which $P(x, y)$ is true. In other words, in the second case, y can depend on x , whereas in the first case, y is a constant independent of x .

From these observations, it follows that if $\exists y \forall x P(x, y)$ is true, then $\forall x \exists y P(x, y)$ must also be true. However, if $\forall x \exists y P(x, y)$ is true, it is not necessary for $\exists y \forall x P(x, y)$ to be true. (See Supplementary Exercises 30 and 31.)

Table 1 summarizes the meanings of the different possible quantifications involving two variables.

Quantifications of more than two variables are also common, as Example 5 illustrates.

EXAMPLE 5 Let $Q(x, y, z)$ be the statement “ $x + y = z$.” What are the truth values of the statements $\forall x \forall y \exists z Q(x, y, z)$ and $\exists z \forall x \forall y Q(x, y, z)$, where the domain of all variables consists of all real numbers?

Solution: Suppose that x and y are assigned values. Then, there exists a real number z such that $x + y = z$. Consequently, the quantification

$$\forall x \forall y \exists z Q(x, y, z),$$

which is the statement

“For all real numbers x and for all real numbers y there is a real number z such that $x + y = z$,”

Be careful with the order of existential and universal quantifiers!

TABLE 1 Quantifications of Two Variables.

<i>Statement</i>	<i>When True?</i>	<i>When False?</i>
$\forall x \forall y P(x, y)$ $\forall y \forall x P(x, y)$	$P(x, y)$ is true for every pair x, y .	There is a pair x, y for which $P(x, y)$ is false.
$\forall x \exists y P(x, y)$	For every x there is a y for which $P(x, y)$ is true.	There is an x such that $P(x, y)$ is false for every y .
$\exists x \forall y P(x, y)$	There is an x for which $P(x, y)$ is true for every y .	For every x there is a y for which $P(x, y)$ is false.
$\exists x \exists y P(x, y)$ $\exists y \exists x P(x, y)$	There is a pair x, y for which $P(x, y)$ is true.	$P(x, y)$ is false for every pair x, y .

is true. The order of the quantification here is important, because the quantification

$$\exists z \forall x \forall y Q(x, y, z),$$

which is the statement

“There is a real number z such that for all real numbers x and for all real numbers y it is true that $x + y = z$,”

is false, because there is no value of z that satisfies the equation $x + y = z$ for all values of x and y . ◀

Translating Mathematical Statements into Statements Involving Nested Quantifiers

Mathematical statements expressed in English can be translated into logical expressions, as Examples 6–8 show.

EXAMPLE 6 Translate the statement “The sum of two positive integers is always positive” into a logical expression.

Solution: To translate this statement into a logical expression, we first rewrite it so that the implied quantifiers and a domain are shown: “For every two integers, if these integers are both positive, then the sum of these integers is positive.” Next, we introduce the variables x and y to obtain “For all positive integers x and y , $x + y$ is positive.” Consequently, we can express this statement as

$$\forall x \forall y ((x > 0) \wedge (y > 0) \rightarrow (x + y > 0)),$$

where the domain for both variables consists of all integers. Note that we could also translate this using the positive integers as the domain. Then the statement “The sum of two positive integers is always positive” becomes “For every two positive integers, the sum of these integers is positive. We can express this as

$$\forall x \forall y (x + y > 0),$$

where the domain for both variables consists of all positive integers. ◀

EXAMPLE 7 Translate the statement “Every real number except zero has a multiplicative inverse.” (A **multiplicative inverse** of a real number x is a real number y such that $xy = 1$.)



Solution: We first rewrite this as “For every real number x except zero, x has a multiplicative inverse.” We can rewrite this as “For every real number x , if $x \neq 0$, then there exists a real number y such that $xy = 1$.” This can be rewritten as

$$\forall x((x \neq 0) \rightarrow \exists y(xy = 1)).$$

One example that you may be familiar with is the concept of limit, which is important in calculus.

EXAMPLE 8 (*Requires calculus*) Use quantifiers to express the definition of the limit of a real-valued function $f(x)$ of a real variable x at a point a in its domain.

Solution: Recall that the definition of the statement

$$\lim_{x \rightarrow a} f(x) = L$$

is: For every real number $\epsilon > 0$ there exists a real number $\delta > 0$ such that $|f(x) - L| < \epsilon$ whenever $0 < |x - a| < \delta$. This definition of a limit can be phrased in terms of quantifiers by

$$\forall \epsilon \exists \delta \forall x (0 < |x - a| < \delta \rightarrow |f(x) - L| < \epsilon),$$

where the domain for the variables δ and ϵ consists of all positive real numbers and for x consists of all real numbers.

This definition can also be expressed as

$$\forall \epsilon > 0 \exists \delta > 0 \forall x (0 < |x - a| < \delta \rightarrow |f(x) - L| < \epsilon)$$

when the domain for the variables ϵ and δ consists of all real numbers, rather than just the positive real numbers. [Here, restricted quantifiers have been used. Recall that $\forall x > 0 P(x)$ means that for all x with $x > 0$, $P(x)$ is true.]

Translating from Nested Quantifiers into English

Expressions with nested quantifiers expressing statements in English can be quite complicated. The first step in translating such an expression is to write out what the quantifiers and predicates in the expression mean. The next step is to express this meaning in a simpler sentence. This process is illustrated in Examples 9 and 10.

EXAMPLE 9 Translate the statement

$$\forall x(C(x) \vee \exists y(C(y) \wedge F(x, y)))$$

into English, where $C(x)$ is “ x has a computer,” $F(x, y)$ is “ x and y are friends,” and the domain for both x and y consists of all students in your school.

Solution: The statement says that for every student x in your school, x has a computer or there is a student y such that y has a computer and x and y are friends. In other words, every student in your school has a computer or has a friend who has a computer.

EXAMPLE 10 Translate the statement

$$\exists x \forall y \forall z ((F(x, y) \wedge F(x, z) \wedge (y \neq z)) \rightarrow \neg F(y, z))$$

into English, where $F(a,b)$ means a and b are friends and the domain for x , y , and z consists of all students in your school.

Solution: We first examine the expression $(F(x,y) \wedge F(x,z) \wedge (y \neq z)) \rightarrow \neg F(y,z)$. This expression says that if students x and y are friends, and students x and z are friends, and furthermore, if y and z are not the same student, then y and z are not friends. It follows that the original statement, which is triply quantified, says that there is a student x such that for all students y and all students z other than y , if x and y are friends and x and z are friends, then y and z are not friends. In other words, there is a student none of whose friends are also friends with each other. ◀

Translating English Sentences into Logical Expressions

In Section 1.4 we showed how quantifiers can be used to translate sentences into logical expressions. However, we avoided sentences whose translation into logical expressions required the use of nested quantifiers. We now address the translation of such sentences.

EXAMPLE 11 Express the statement “If a person is female and is a parent, then this person is someone’s mother” as a logical expression involving predicates, quantifiers with a domain consisting of all people, and logical connectives.

Solution: The statement “If a person is female and is a parent, then this person is someone’s mother” can be expressed as “For every person x , if person x is female and person x is a parent, then there exists a person y such that person x is the mother of person y .” We introduce the propositional functions $F(x)$ to represent “ x is female,” $P(x)$ to represent “ x is a parent,” and $M(x, y)$ to represent “ x is the mother of y .” The original statement can be represented as

$$\forall x((F(x) \wedge P(x)) \rightarrow \exists y M(x, y)).$$

Using the null quantification rule in part (b) of Exercise 47 in Section 1.4, we can move $\exists y$ to the left so that it appears just after $\forall x$, because y does not appear in $F(x) \wedge P(x)$. We obtain the logically equivalent expression

$$\forall x \exists y ((F(x) \wedge P(x)) \rightarrow M(x, y)). \quad \blacktriangleleft$$

EXAMPLE 12 Express the statement “Everyone has exactly one best friend” as a logical expression involving predicates, quantifiers with a domain consisting of all people, and logical connectives.

Solution: The statement “Everyone has exactly one best friend” can be expressed as “For every person x , person x has exactly one best friend.” Introducing the universal quantifier, we see that this statement is the same as “ $\forall x$ (person x has exactly one best friend),” where the domain consists of all people.

To say that x has exactly one best friend means that there is a person y who is the best friend of x , and furthermore, that for every person z , if person z is not person y , then z is not the best friend of x . When we introduce the predicate $B(x, y)$ to be the statement “ y is the best friend of x ,” the statement that x has exactly one best friend can be represented as

$$\exists y(B(x, y) \wedge \forall z((z \neq y) \rightarrow \neg B(x, z))).$$

Consequently, our original statement can be expressed as

$$\forall x \exists y(B(x, y) \wedge \forall z((z \neq y) \rightarrow \neg B(x, z))).$$

[Note that we can write this statement as $\forall x \exists! y B(x, y)$, where $\exists!$ is the “uniqueness quantifier” defined in Section 1.4.] ▶

EXAMPLE 13 Use quantifiers to express the statement “There is a woman who has taken a flight on every airline in the world.”

Solution: Let $P(w, f)$ be “ w has taken f ” and $Q(f, a)$ be “ f is a flight on a .” We can express the statement as

$$\exists w \forall a \exists f (P(w, f) \wedge Q(f, a)),$$

where the domains of discourse for w , f , and a consist of all the women in the world, all airplane flights, and all airlines, respectively.

The statement could also be expressed as

$$\exists w \forall a \exists f R(w, f, a),$$

where $R(w, f, a)$ is “ w has taken f on a .” Although this is more compact, it somewhat obscures the relationships among the variables. Consequently, the first solution is usually preferable. ▶

Negating Nested Quantifiers



Statements involving nested quantifiers can be negated by successively applying the rules for negating statements involving a single quantifier. This is illustrated in Examples 14–16.

EXAMPLE 14 Express the negation of the statement $\forall x \exists y (xy = 1)$ so that no negation precedes a quantifier.



Solution: By successively applying De Morgan’s laws for quantifiers in Table 2 of Section 1.4, we can move the negation in $\neg \forall x \exists y (xy = 1)$ inside all the quantifiers. We find that $\neg \forall x \exists y (xy = 1)$ is equivalent to $\exists x \neg \exists y (xy = 1)$, which is equivalent to $\exists x \forall y \neg (xy = 1)$. Because $\neg(xy = 1)$ can be expressed more simply as $xy \neq 1$, we conclude that our negated statement can be expressed as $\exists x \forall y (xy \neq 1)$. ▶

EXAMPLE 15 Use quantifiers to express the statement that “There does not exist a woman who has taken a flight on every airline in the world.”

Solution: This statement is the negation of the statement “There is a woman who has taken a flight on every airline in the world” from Example 13. By Example 13, our statement can be expressed as $\neg \exists w \forall a \exists f (P(w, f) \wedge Q(f, a))$, where $P(w, f)$ is “ w has taken f ” and $Q(f, a)$ is “ f is a flight on a .” By successively applying De Morgan’s laws for quantifiers in Table 2 of Section 1.4 to move the negation inside successive quantifiers and by applying De Morgan’s law for negating a conjunction in the last step, we find that our statement is equivalent to each of this sequence of statements:

$$\begin{aligned} \forall w \neg \forall a \exists f (P(w, f) \wedge Q(f, a)) &\equiv \forall w \exists a \neg \exists f (P(w, f) \wedge Q(f, a)) \\ &\equiv \forall w \exists a \forall f \neg (P(w, f) \wedge Q(f, a)) \\ &\equiv \forall w \exists a \forall f (\neg P(w, f) \vee \neg Q(f, a)). \end{aligned}$$

This last statement states “For every woman there is an airline such that for all flights, this woman has not taken that flight or that flight is not on this airline.” ▶

EXAMPLE 16 (*Requires calculus*) Use quantifiers and predicates to express the fact that $\lim_{x \rightarrow a} f(x)$ does not exist where $f(x)$ is a real-valued function of a real variable x and a belongs to the domain of f .

Solution: To say that $\lim_{x \rightarrow a} f(x)$ does not exist means that for all real numbers L , $\lim_{x \rightarrow a} f(x) \neq L$. By using Example 8, the statement $\lim_{x \rightarrow a} f(x) \neq L$ can be expressed as

$$\neg \forall \epsilon > 0 \exists \delta > 0 \forall x (0 < |x - a| < \delta \rightarrow |f(x) - L| < \epsilon).$$

Successively applying the rules for negating quantified expressions, we construct this sequence of equivalent statements

$$\begin{aligned} &\neg \forall \epsilon > 0 \exists \delta > 0 \forall x (0 < |x - a| < \delta \rightarrow |f(x) - L| < \epsilon) \\ &\equiv \exists \epsilon > 0 \neg \exists \delta > 0 \forall x (0 < |x - a| < \delta \rightarrow |f(x) - L| < \epsilon) \\ &\equiv \exists \epsilon > 0 \forall \delta > 0 \neg \forall x (0 < |x - a| < \delta \rightarrow |f(x) - L| < \epsilon) \\ &\equiv \exists \epsilon > 0 \forall \delta > 0 \exists x \neg (0 < |x - a| < \delta \rightarrow |f(x) - L| < \epsilon) \\ &\equiv \exists \epsilon > 0 \forall \delta > 0 \exists x (0 < |x - a| < \delta \wedge |f(x) - L| \geq \epsilon). \end{aligned}$$

In the last step we used the equivalence $\neg(p \rightarrow q) \equiv p \wedge \neg q$, which follows from the fifth equivalence in Table 7 of Section 1.3.

Because the statement “ $\lim_{x \rightarrow a} f(x)$ does not exist” means for all real numbers L , $\lim_{x \rightarrow a} f(x) \neq L$, this can be expressed as

$$\forall L \exists \epsilon > 0 \forall \delta > 0 \exists x (0 < |x - a| < \delta \wedge |f(x) - L| \geq \epsilon).$$

This last statement says that for every real number L there is a real number $\epsilon > 0$ such that for every real number $\delta > 0$, there exists a real number x such that $0 < |x - a| < \delta$ and $|f(x) - L| \geq \epsilon$. ◀

Exercises

1. Translate these statements into English, where the domain for each variable consists of all real numbers.

- a) $\forall x \exists y (x < y)$
- b) $\forall x \forall y ((x \geq 0) \wedge (y \geq 0)) \rightarrow (xy \geq 0)$
- c) $\forall x \forall y \exists z (xy = z)$

2. Translate these statements into English, where the domain for each variable consists of all real numbers.

- a) $\exists x \forall y (xy = y)$
- b) $\forall x \forall y (((x \geq 0) \wedge (y < 0)) \rightarrow (x - y > 0))$
- c) $\forall x \forall y \exists z (x = y + z)$

3. Let $Q(x, y)$ be the statement “ x has sent an e-mail message to y ,” where the domain for both x and y consists of all students in your class. Express each of these quantifications in English.

- | | |
|----------------------------------|----------------------------------|
| a) $\exists x \exists y Q(x, y)$ | b) $\exists x \forall y Q(x, y)$ |
| c) $\forall x \exists y Q(x, y)$ | d) $\exists y \forall x Q(x, y)$ |
| e) $\forall y \exists x Q(x, y)$ | f) $\forall x \forall y Q(x, y)$ |

4. Let $P(x, y)$ be the statement “Student x has taken class y ,” where the domain for x consists of all students in your class and for y consists of all computer science courses

at your school. Express each of these quantifications in English.

- | | |
|----------------------------------|----------------------------------|
| a) $\exists x \exists y P(x, y)$ | b) $\exists x \forall y P(x, y)$ |
| c) $\forall x \exists y P(x, y)$ | d) $\exists y \forall x P(x, y)$ |
| e) $\forall y \exists x P(x, y)$ | f) $\forall x \forall y P(x, y)$ |

5. Let $W(x, y)$ mean that student x has visited website y , where the domain for x consists of all students in your school and the domain for y consists of all websites. Express each of these statements by a simple English sentence.

- a) $W(\text{Sarah Smith}, \text{www.att.com})$
- b) $\exists x W(x, \text{www.imdb.org})$
- c) $\exists y W(\text{José Orez}, y)$
- d) $\exists y (W(\text{Ashok Puri}, y) \wedge W(\text{Cindy Yoon}, y))$
- e) $\exists y \forall z (y \neq (\text{David Belcher}) \wedge (W(\text{David Belcher}, z) \rightarrow W(y, z)))$
- f) $\exists x \exists y \forall z ((x \neq y) \wedge (W(x, z) \leftrightarrow W(y, z)))$

6. Let $C(x, y)$ mean that student x is enrolled in class y , where the domain for x consists of all students in your school and the domain for y consists of all classes being

- given at your school. Express each of these statements by a simple English sentence.
- $C(\text{Randy Goldberg}, \text{CS 252})$
 - $\exists x C(x, \text{Math 695})$
 - $\exists y C(\text{Carol Sitea}, y)$
 - $\exists x(C(x, \text{Math 222}) \wedge C(x, \text{CS 252}))$
 - $\exists x \exists y \forall z((x \neq y) \wedge (C(x, z) \rightarrow C(y, z)))$
 - $\exists x \exists y \forall z((x \neq y) \wedge (C(x, z) \leftrightarrow C(y, z)))$
7. Let $T(x, y)$ mean that student x likes cuisine y , where the domain for x consists of all students at your school and the domain for y consists of all cuisines. Express each of these statements by a simple English sentence.
- $\neg T(\text{Abdallah Hussein}, \text{Japanese})$
 - $\exists x T(x, \text{Korean}) \wedge \forall x T(x, \text{Mexican})$
 - $\exists y(T(\text{Monique Arsenault}, y) \vee T(\text{Jay Johnson}, y))$
 - $\forall x \forall z \exists y((x \neq z) \rightarrow \neg(T(x, y) \wedge T(z, y)))$
 - $\exists x \exists z \forall y(T(x, y) \leftrightarrow T(z, y))$
 - $\forall x \forall z \exists y(T(x, y) \leftrightarrow T(z, y))$
8. Let $Q(x, y)$ be the statement “student x has been a contestant on quiz show y .” Express each of these sentences in terms of $Q(x, y)$, quantifiers, and logical connectives, where the domain for x consists of all students at your school and for y consists of all quiz shows on television.
- There is a student at your school who has been a contestant on a television quiz show.
 - No student at your school has ever been a contestant on a television quiz show.
 - There is a student at your school who has been a contestant on *Jeopardy* and on *Wheel of Fortune*.
 - Every television quiz show has had a student from your school as a contestant.
 - At least two students from your school have been contestants on *Jeopardy*.
9. Let $L(x, y)$ be the statement “ x loves y ,” where the domain for both x and y consists of all people in the world. Use quantifiers to express each of these statements.
- Everybody loves Jerry.
 - Everybody loves somebody.
 - There is somebody whom everybody loves.
 - Nobody loves everybody.
 - There is somebody whom Lydia does not love.
 - There is somebody whom no one loves.
 - There is exactly one person whom everybody loves.
 - There are exactly two people whom Lynn loves.
 - Everyone loves himself or herself.
 - There is someone who loves no one besides himself or herself.
10. Let $F(x, y)$ be the statement “ x can fool y ,” where the domain consists of all people in the world. Use quantifiers to express each of these statements.
- Everybody can fool Fred.
 - Evelyn can fool everybody.
 - Everybody can fool somebody.
 - There is no one who can fool everybody.
 - Everyone can be fooled by somebody.
 - No one can fool both Fred and Jerry.
 - Nancy can fool exactly two people.
- h) There is exactly one person whom everybody can fool.
- No one can fool himself or herself.
 - There is someone who can fool exactly one person besides himself or herself.
11. Let $S(x)$ be the predicate “ x is a student,” $F(x)$ the predicate “ x is a faculty member,” and $A(x, y)$ the predicate “ x has asked y a question,” where the domain consists of all people associated with your school. Use quantifiers to express each of these statements.
- Lois has asked Professor Michaels a question.
 - Every student has asked Professor Gross a question.
 - Every faculty member has either asked Professor Miller a question or been asked a question by Professor Miller.
 - Some student has not asked any faculty member a question.
 - There is a faculty member who has never been asked a question by a student.
 - Some student has asked every faculty member a question.
 - There is a faculty member who has asked every other faculty member a question.
 - Some student has never been asked a question by a faculty member.
12. Let $I(x)$ be the statement “ x has an Internet connection” and $C(x, y)$ be the statement “ x and y have chatted over the Internet,” where the domain for the variables x and y consists of all students in your class. Use quantifiers to express each of these statements.
- Jerry does not have an Internet connection.
 - Rachel has not chatted over the Internet with Chelsea.
 - Jan and Sharon have never chatted over the Internet.
 - No one in the class has chatted with Bob.
 - Sanjay has chatted with everyone except Joseph.
 - Someone in your class does not have an Internet connection.
 - Not everyone in your class has an Internet connection.
 - Exactly one student in your class has an Internet connection.
 - Everyone except one student in your class has an Internet connection.
 - Everyone in your class with an Internet connection has chatted over the Internet with at least one other student in your class.
 - Someone in your class has an Internet connection but has not chatted with anyone else in your class.
 - There are two students in your class who have not chatted with each other over the Internet.
 - There is a student in your class who has chatted with everyone in your class over the Internet.
 - There are at least two students in your class who have not chatted with the same person in your class.
 - There are two students in the class who between them have chatted with everyone else in the class.

- 13.** Let $M(x, y)$ be “ x has sent y an e-mail message” and $T(x, y)$ be “ x has telephoned y ,” where the domain consists of all students in your class. Use quantifiers to express each of these statements. (Assume that all e-mail messages that were sent are received, which is not the way things often work.)
- a) Chou has never sent an e-mail message to Koko.
 - b) Arlene has never sent an e-mail message to or telephoned Sarah.
 - c) José has never received an e-mail message from Deborah.
 - d) Every student in your class has sent an e-mail message to Ken.
 - e) No one in your class has telephoned Nina.
 - f) Everyone in your class has either telephoned Avi or sent him an e-mail message.
 - g) There is a student in your class who has sent everyone else in your class an e-mail message.
 - h) There is someone in your class who has either sent an e-mail message or telephoned everyone else in your class.
 - i) There are two different students in your class who have sent each other e-mail messages.
 - j) There is a student who has sent himself or herself an e-mail message.
 - k) There is a student in your class who has not received an e-mail message from anyone else in the class and who has not been called by any other student in the class.
 - l) Every student in the class has either received an e-mail message or received a telephone call from another student in the class.
 - m) There are at least two students in your class such that one student has sent the other e-mail and the second student has telephoned the first student.
 - n) There are two different students in your class who between them have sent an e-mail message to or telephoned everyone else in the class.
- 14.** Use quantifiers and predicates with more than one variable to express these statements.
- a) There is a student in this class who can speak Hindi.
 - b) Every student in this class plays some sport.
 - c) Some student in this class has visited Alaska but has not visited Hawaii.
 - d) All students in this class have learned at least one programming language.
 - e) There is a student in this class who has taken every course offered by one of the departments in this school.
 - f) Some student in this class grew up in the same town as exactly one other student in this class.
 - g) Every student in this class has chatted with at least one other student in at least one chat group.
- 15.** Use quantifiers and predicates with more than one variable to express these statements.
- a) Every computer science student needs a course in discrete mathematics.
- b)** There is a student in this class who owns a personal computer.
- c)** Every student in this class has taken at least one computer science course.
- d)** There is a student in this class who has taken at least one course in computer science.
- e)** Every student in this class has been in every building on campus.
- f)** There is a student in this class who has been in every room of at least one building on campus.
- g)** Every student in this class has been in at least one room of every building on campus.
- 16.** A discrete mathematics class contains 1 mathematics major who is a freshman, 12 mathematics majors who are sophomores, 15 computer science majors who are sophomores, 2 mathematics majors who are juniors, 2 computer science majors who are juniors, and 1 computer science major who is a senior. Express each of these statements in terms of quantifiers and then determine its truth value.
- a) There is a student in the class who is a junior.
 - b) Every student in the class is a computer science major.
 - c) There is a student in the class who is neither a mathematics major nor a junior.
 - d) Every student in the class is either a sophomore or a computer science major.
 - e) There is a major such that there is a student in the class in every year of study with that major.
- 17.** Express each of these system specifications using predicates, quantifiers, and logical connectives, if necessary.
- a) Every user has access to exactly one mailbox.
 - b) There is a process that continues to run during all error conditions only if the kernel is working correctly.
 - c) All users on the campus network can access all websites whose url has a .edu extension.
 - *d) There are exactly two systems that monitor every remote server.
- 18.** Express each of these system specifications using predicates, quantifiers, and logical connectives, if necessary.
- a) At least one console must be accessible during every fault condition.
 - b) The e-mail address of every user can be retrieved whenever the archive contains at least one message sent by every user on the system.
 - c) For every security breach there is at least one mechanism that can detect that breach if and only if there is a process that has not been compromised.
 - d) There are at least two paths connecting every two distinct endpoints on the network.
 - e) No one knows the password of every user on the system except for the system administrator, who knows all passwords.[
- 19.** Express each of these statements using mathematical and logical operators, predicates, and quantifiers, where the domain consists of all integers.
- a) The sum of two negative integers is negative.
 - b) The difference of two positive integers is not necessarily positive.

- c) The sum of the squares of two integers is greater than or equal to the square of their sum.
d) The absolute value of the product of two integers is the product of their absolute values.
20. Express each of these statements using predicates, quantifiers, logical connectives, and mathematical operators where the domain consists of all integers.
- The product of two negative integers is positive.
 - The average of two positive integers is positive.
 - The difference of two negative integers is not necessarily negative.
 - The absolute value of the sum of two integers does not exceed the sum of the absolute values of these integers.
21. Use predicates, quantifiers, logical connectives, and mathematical operators to express the statement that every positive integer is the sum of the squares of four integers.
22. Use predicates, quantifiers, logical connectives, and mathematical operators to express the statement that there is a positive integer that is not the sum of three squares.
23. Express each of these mathematical statements using predicates, quantifiers, logical connectives, and mathematical operators.
- The product of two negative real numbers is positive.
 - The difference of a real number and itself is zero.
 - Every positive real number has exactly two square roots.
 - A negative real number does not have a square root that is a real number.
24. Translate each of these nested quantifications into an English statement that expresses a mathematical fact. The domain in each case consists of all real numbers.
- $\exists x \forall y (x + y = y)$
 - $\forall x \forall y (((x \geq 0) \wedge (y < 0)) \rightarrow (x - y > 0))$
 - $\exists x \exists y (((x \leq 0) \wedge (y \leq 0)) \wedge (x - y > 0))$
 - $\forall x \forall y ((x \neq 0) \wedge (y \neq 0) \leftrightarrow (xy \neq 0))$
25. Translate each of these nested quantifications into an English statement that expresses a mathematical fact. The domain in each case consists of all real numbers.
- $\exists x \forall y (xy = y)$
 - $\forall x \forall y (((x < 0) \wedge (y < 0)) \rightarrow (xy > 0))$
 - $\exists x \exists y ((x^2 > y) \wedge (x < y))$
 - $\forall x \forall y \exists z (x + y = z)$
26. Let $Q(x, y)$ be the statement " $x + y = x - y$." If the domain for both variables consists of all integers, what are the truth values?
- $Q(1, 1)$
 - $Q(2, 0)$
 - $\forall y Q(1, y)$
 - $\exists x Q(x, 2)$
 - $\exists x \exists y Q(x, y)$
 - $\forall x \forall y Q(x, y)$
 - $\forall x \forall y Q(x, y)$
27. Determine the truth value of each of these statements if the domain for all variables consists of all integers.
- $\forall n \exists m (n^2 < m)$
 - $\exists n \forall m (n < m^2)$
 - $\forall n \exists m (n + m = 0)$
 - $\exists n \forall m (nm = m)$
- e) $\exists n \exists m (n^2 + m^2 = 5)$ f) $\exists n \exists m (n^2 + m^2 = 6)$
g) $\exists n \exists m (n + m = 4 \wedge n - m = 1)$
h) $\exists n \exists m (n + m = 4 \wedge n - m = 2)$
i) $\forall n \forall m \exists p (p = (m + n)/2)$
28. Determine the truth value of each of these statements if the domain of each variable consists of all real numbers.
- $\forall x \exists y (x^2 = y)$
 - $\forall x \exists y (x = y^2)$
 - $\exists x \forall y (xy = 0)$
 - $\exists x \exists y (x + y \neq y + x)$
 - $\forall x (x \neq 0 \rightarrow \exists y (xy = 1))$
 - $\exists x \forall y (y \neq 0 \rightarrow xy = 1)$
 - $\forall x \exists y (x + y = 1)$
 - $\exists x \exists y (x + 2y = 2 \wedge 2x + 4y = 5)$
 - $\forall x \exists y (x + y = 2 \wedge 2x - y = 1)$
 - $\forall x \forall y \exists z (z = (x + y)/2)$
29. Suppose the domain of the propositional function $P(x, y)$ consists of pairs x and y , where x is 1, 2, or 3 and y is 1, 2, or 3. Write out these propositions using disjunctions and conjunctions.
- $\forall x \forall y P(x, y)$
 - $\exists x \exists y P(x, y)$
 - $\exists x \forall y P(x, y)$
 - $\forall y \exists x P(x, y)$
30. Rewrite each of these statements so that negations appear only within predicates (that is, so that no negation is outside a quantifier or an expression involving logical connectives).
- $\neg \exists y \exists x P(x, y)$
 - $\neg \forall x \exists y P(x, y)$
 - $\neg \exists y (Q(y) \wedge \forall x \neg R(x, y))$
 - $\neg \exists y (\exists x R(x, y) \vee \forall x S(x, y))$
 - $\neg \exists y (\forall x \exists z T(x, y, z) \vee \exists x \forall z U(x, y, z))$
31. Express the negations of each of these statements so that all negation symbols immediately precede predicates.
- $\forall x \exists y \forall z T(x, y, z)$
 - $\forall x \exists y P(x, y) \vee \forall x \exists y Q(x, y)$
 - $\forall x \exists y (P(x, y) \wedge \exists z R(x, y, z))$
 - $\forall x \exists y (P(x, y) \rightarrow Q(x, y))$
32. Express the negations of each of these statements so that all negation symbols immediately precede predicates.
- $\exists z \forall y \forall x T(x, y, z)$
 - $\exists x \exists y P(x, y) \wedge \forall x \forall y Q(x, y)$
 - $\exists x \exists y (Q(x, y) \leftrightarrow Q(y, x))$
 - $\forall y \exists x \exists z (T(x, y, z) \vee Q(x, y))$
33. Rewrite each of these statements so that negations appear only within predicates (that is, so that no negation is outside a quantifier or an expression involving logical connectives).
- $\neg \forall x \forall y P(x, y)$
 - $\neg \forall y \exists x P(x, y)$
 - $\neg \forall y \forall x (P(x, y) \vee Q(x, y))$
 - $\neg (\exists x \exists y \neg P(x, y) \wedge \forall x \forall y Q(x, y))$
 - $\neg \forall x (\exists y \forall z P(x, y, z) \wedge \exists z \forall y P(x, y, z))$
34. Find a common domain for the variables x , y , and z for which the statement $\forall x \forall y ((x \neq y) \rightarrow \forall z ((z = x) \vee (z = y)))$ is true and another domain for which it is false.
35. Find a common domain for the variables x , y , z , and w for which the statement $\forall x \forall y \forall z \exists w ((w \neq x) \wedge (w \neq y) \wedge (w \neq z))$ is true and another common domain for these variables for which it is false.

- 36.** Express each of these statements using quantifiers. Then form the negation of the statement so that no negation is to the left of a quantifier. Next, express the negation in simple English. (Do not simply use the phrase “It is not the case that.”)
- No one has lost more than one thousand dollars playing the lottery.
 - There is a student in this class who has chatted with exactly one other student.
 - No student in this class has sent e-mail to exactly two other students in this class.
 - Some student has solved every exercise in this book.
 - No student has solved at least one exercise in every section of this book.
- 37.** Express each of these statements using quantifiers. Then form the negation of the statement so that no negation is to the left of a quantifier. Next, express the negation in simple English. (Do not simply use the phrase “It is not the case that.”)
- Every student in this class has taken exactly two mathematics classes at this school.
 - Someone has visited every country in the world except Libya.
 - No one has climbed every mountain in the Himalayas.
 - Every movie actor has either been in a movie with Kevin Bacon or has been in a movie with someone who has been in a movie with Kevin Bacon.
- 38.** Express the negations of these propositions using quantifiers, and in English.
- Every student in this class likes mathematics.
 - There is a student in this class who has never seen a computer.
 - There is a student in this class who has taken every mathematics course offered at this school.
 - There is a student in this class who has been in at least one room of every building on campus.
- 39.** Find a counterexample, if possible, to these universally quantified statements, where the domain for all variables consists of all integers.
- $\forall x \forall y (x^2 = y^2 \rightarrow x = y)$
 - $\forall x \exists y (y^2 = x)$
 - $\forall x \forall y (xy \geq x)$
- 40.** Find a counterexample, if possible, to these universally quantified statements, where the domain for all variables consists of all integers.
- $\forall x \exists y (x = 1/y)$
 - $\forall x \exists y (y^2 - x < 100)$
 - $\forall x \forall y (x^2 \neq y^3)$
- 41.** Use quantifiers to express the associative law for multiplication of real numbers.
- 42.** Use quantifiers to express the distributive laws of multiplication over addition for real numbers.
- 43.** Use quantifiers and logical connectives to express the fact that every linear polynomial (that is, polynomial of degree 1) with real coefficients and where the coefficient of x is nonzero, has exactly one real root.
- 44.** Use quantifiers and logical connectives to express the fact that a quadratic polynomial with real number coefficients has at most two real roots.
- 45.** Determine the truth value of the statement $\forall x \exists y (xy = 1)$ if the domain for the variables consists of
- the nonzero real numbers.
 - the nonzero integers.
 - the positive real numbers.
- 46.** Determine the truth value of the statement $\exists x \forall y (x \leq y^2)$ if the domain for the variables consists of
- the positive real numbers.
 - the integers.
 - the nonzero real numbers.
- 47.** Show that the two statements $\neg \exists x \forall y P(x, y)$ and $\forall x \exists y \neg P(x, y)$, where both quantifiers over the first variable in $P(x, y)$ have the same domain, and both quantifiers over the second variable in $P(x, y)$ have the same domain, are logically equivalent.
- *48.** Show that $\forall x P(x) \vee \forall x Q(x)$ and $\forall x \forall y (P(x) \vee Q(y))$, where all quantifiers have the same nonempty domain, are logically equivalent. (The new variable y is used to combine the quantifications correctly.)
- *49.**
 - Show that $\forall x P(x) \wedge \exists x Q(x)$ is logically equivalent to $\forall x \exists y (P(x) \wedge Q(y))$, where all quantifiers have the same nonempty domain.
 - Show that $\forall x P(x) \vee \exists x Q(x)$ is equivalent to $\forall x \exists y (P(x) \vee Q(y))$, where all quantifiers have the same nonempty domain.
- A statement is in **prenex normal form (PNF)** if and only if it is of the form
- $$Q_1 x_1 Q_2 x_2 \cdots Q_k x_k P(x_1, x_2, \dots, x_k),$$
- where each Q_i , $i = 1, 2, \dots, k$, is either the existential quantifier or the universal quantifier, and $P(x_1, \dots, x_k)$ is a predicate involving no quantifiers. For example, $\exists x \forall y (P(x, y) \wedge Q(y))$ is in prenex normal form, whereas $\exists x P(x) \vee \forall x Q(x)$ is not (because the quantifiers do not all occur first).
- Every statement formed from propositional variables, predicates, **T**, and **F** using logical connectives and quantifiers is equivalent to a statement in prenex normal form. Exercise 51 asks for a proof of this fact.
- *50.** Put these statements in prenex normal form. [Hint: Use logical equivalence from Tables 6 and 7 in Section 1.3, Table 2 in Section 1.4, Example 19 in Section 1.4, Exercises 45 and 46 in Section 1.4, and Exercises 48 and 49.]
- $\exists x P(x) \vee \exists x Q(x) \vee A$, where A is a proposition not involving any quantifiers.
 - $\neg(\forall x P(x) \vee \forall x Q(x))$
 - $\exists x P(x) \rightarrow \exists x Q(x)$
- **51.** Show how to transform an arbitrary statement to a statement in prenex normal form that is equivalent to the given statement. (Note: A formal solution of this exercise requires use of structural induction, covered in Section 5.3.)
- *52.** Express the quantification $\exists! x P(x)$, introduced in Section 1.4, using universal quantifications, existential quantifications, and logical operators.

1.6 Rules of Inference

Introduction

Later in this chapter we will study proofs. Proofs in mathematics are valid arguments that establish the truth of mathematical statements. By an **argument**, we mean a sequence of statements that end with a conclusion. By **valid**, we mean that the conclusion, or final statement of the argument, must follow from the truth of the preceding statements, or **premises**, of the argument. That is, an argument is valid if and only if it is impossible for all the premises to be true and the conclusion to be false. To deduce new statements from statements we already have, we use rules of inference which are templates for constructing valid arguments. Rules of inference are our basic tools for establishing the truth of statements.

Before we study mathematical proofs, we will look at arguments that involve only compound propositions. We will define what it means for an argument involving compound propositions to be valid. Then we will introduce a collection of rules of inference in propositional logic. These rules of inference are among the most important ingredients in producing valid arguments. After we illustrate how rules of inference are used to produce valid arguments, we will describe some common forms of incorrect reasoning, called **fallacies**, which lead to invalid arguments.

After studying rules of inference in propositional logic, we will introduce rules of inference for quantified statements. We will describe how these rules of inference can be used to produce valid arguments. These rules of inference for statements involving existential and universal quantifiers play an important role in proofs in computer science and mathematics, although they are often used without being explicitly mentioned.

Finally, we will show how rules of inference for propositions and for quantified statements can be combined. These combinations of rule of inference are often used together in complicated arguments.

Valid Arguments in Propositional Logic

Consider the following argument involving propositions (which, by definition, is a sequence of propositions):

“If you have a current password, then you can log onto the network.”

“You have a current password.”

Therefore,

“You can log onto the network.”

We would like to determine whether this is a valid argument. That is, we would like to determine whether the conclusion “You can log onto the network” must be true when the premises “If you have a current password, then you can log onto the network” and “You have a current password” are both true.

Before we discuss the validity of this particular argument, we will look at its form. Use p to represent “You have a current password” and q to represent “You can log onto the network.” Then, the argument has the form

$$\begin{array}{c} p \rightarrow q \\ p \\ \hline \therefore q \end{array}$$

where \therefore is the symbol that denotes “therefore.”

We know that when p and q are propositional variables, the statement $((p \rightarrow q) \wedge p) \rightarrow q$ is a tautology (see Exercise 10(c) in Section 1.3). In particular, when both $p \rightarrow q$ and p are true, we know that q must also be true. We say this form of argument is **valid** because whenever all its premises (all statements in the argument other than the final one, the conclusion) are true, the conclusion must also be true. Now suppose that both “If you have a current password, then you can log onto the network” and “You have a current password” are true statements. When we replace p by “You have a current password” and q by “You can log onto the network,” it necessarily follows that the conclusion “You can log onto the network” is true. This argument is **valid** because its form is valid. Note that whenever we replace p and q by propositions where $p \rightarrow q$ and p are both true, then q must also be true.

What happens when we replace p and q in this argument form by propositions where not both p and $p \rightarrow q$ are true? For example, suppose that p represents “You have access to the network” and q represents “You can change your grade” and that p is true, but $p \rightarrow q$ is false. The argument we obtain by substituting these values of p and q into the argument form is

“If you have access to the network, then you can change your grade.”
 “You have access to the network.”

∴ “You can change your grade.”

The argument we obtained is a valid argument, but because one of the premises, namely the first premise, is false, we cannot conclude that the conclusion is true. (Most likely, this conclusion is false.)

In our discussion, to analyze an argument, we replaced propositions by propositional variables. This changed an argument to an **argument form**. We saw that the validity of an argument follows from the validity of the form of the argument. We summarize the terminology used to discuss the validity of arguments with our definition of the key notions.

DEFINITION 1

An *argument* in propositional logic is a sequence of propositions. All but the final proposition in the argument are called *premises* and the final proposition is called the *conclusion*. An argument is *valid* if the truth of all its premises implies that the conclusion is true.

An *argument form* in propositional logic is a sequence of compound propositions involving propositional variables. An argument form is *valid* no matter which particular propositions are substituted for the propositional variables in its premises, the conclusion is true if the premises are all true.

From the definition of a valid argument form we see that the argument form with premises p_1, p_2, \dots, p_n and conclusion q is valid, when $(p_1 \wedge p_2 \wedge \dots \wedge p_n) \rightarrow q$ is a tautology.

The key to showing that an argument in propositional logic is valid is to show that its argument form is valid. Consequently, we would like techniques to show that argument forms are valid. We will now develop methods for accomplishing this task.

Rules of Inference for Propositional Logic

We can always use a truth table to show that an argument form is valid. We do this by showing that whenever the premises are true, the conclusion must also be true. However, this can be a tedious approach. For example, when an argument form involves 10 different propositional variables, to use a truth table to show this argument form is valid requires $2^{10} = 1024$ different rows. Fortunately, we do not have to resort to truth tables. Instead, we can first establish the validity of some relatively simple argument forms, called **rules of inference**. These rules of inference can be used as building blocks to construct more complicated valid argument forms. We will now introduce the most important rules of inference in propositional logic.

The tautology $(p \wedge (p \rightarrow q)) \rightarrow q$ is the basis of the rule of inference called **modus ponens**, or the **law of detachment**. (Modus ponens is Latin for *mode that affirms*.) This tautology leads to the following valid argument form, which we have already seen in our initial discussion about arguments (where, as before, the symbol \therefore denotes “therefore”):

$$\begin{array}{c} p \\ p \rightarrow q \\ \hline \therefore q \end{array}$$

Using this notation, the hypotheses are written in a column, followed by a horizontal bar, followed by a line that begins with the therefore symbol and ends with the conclusion. In particular, modus ponens tells us that if a conditional statement and the hypothesis of this conditional statement are both true, then the conclusion must also be true. Example 1 illustrates the use of modus ponens.

EXAMPLE 1 Suppose that the conditional statement “If it snows today, then we will go skiing” and its hypothesis, “It is snowing today,” are true. Then, by modus ponens, it follows that the conclusion of the conditional statement, “We will go skiing,” is true. ◀

As we mentioned earlier, a valid argument can lead to an incorrect conclusion if one or more of its premises is false. We illustrate this again in Example 2.

EXAMPLE 2 Determine whether the argument given here is valid and determine whether its conclusion must be true because of the validity of the argument.

“If $\sqrt{2} > \frac{3}{2}$, then $(\sqrt{2})^2 > (\frac{3}{2})^2$. We know that $\sqrt{2} > \frac{3}{2}$. Consequently, $(\sqrt{2})^2 = 2 > (\frac{3}{2})^2 = \frac{9}{4}$.”

Solution: Let p be the proposition “ $\sqrt{2} > \frac{3}{2}$ ” and q the proposition “ $2 > (\frac{3}{2})^2$.” The premises of the argument are $p \rightarrow q$ and p , and q is its conclusion. This argument is valid because it is constructed by using modus ponens, a valid argument form. However, one of its premises, $\sqrt{2} > \frac{3}{2}$, is false. Consequently, we cannot conclude that the conclusion is true. Furthermore, note that the conclusion of this argument is false, because $2 < \frac{9}{4}$. ◀

There are many useful rules of inference for propositional logic. Perhaps the most widely used of these are listed in Table 1. Exercises 9, 10, 15, and 30 in Section 1.3 ask for the verifications that these rules of inference are valid argument forms. We now give examples of arguments that use these rules of inference. In each argument, we first use propositional variables to express the propositions in the argument. We then show that the resulting argument form is a rule of inference from Table 1.

TABLE 1 Rules of Inference.

<i>Rule of Inference</i>	<i>Tautology</i>	<i>Name</i>
$\begin{array}{c} p \\ p \rightarrow q \\ \hline \therefore q \end{array}$	$(p \wedge (p \rightarrow q)) \rightarrow q$	Modus ponens
$\begin{array}{c} \neg q \\ p \rightarrow q \\ \hline \therefore \neg p \end{array}$	$(\neg q \wedge (p \rightarrow q)) \rightarrow \neg p$	Modus tollens
$\begin{array}{c} p \rightarrow q \\ q \rightarrow r \\ \hline \therefore p \rightarrow r \end{array}$	$((p \rightarrow q) \wedge (q \rightarrow r)) \rightarrow (p \rightarrow r)$	Hypothetical syllogism
$\begin{array}{c} p \vee q \\ \neg p \\ \hline \therefore q \end{array}$	$((p \vee q) \wedge \neg p) \rightarrow q$	Disjunctive syllogism
$\begin{array}{c} p \\ \hline \therefore p \vee q \end{array}$	$p \rightarrow (p \vee q)$	Addition
$\begin{array}{c} p \wedge q \\ \hline \therefore p \end{array}$	$(p \wedge q) \rightarrow p$	Simplification
$\begin{array}{c} p \\ q \\ \hline \therefore p \wedge q \end{array}$	$((p) \wedge (q)) \rightarrow (p \wedge q)$	Conjunction
$\begin{array}{c} p \vee q \\ \neg p \vee r \\ \hline \therefore q \vee r \end{array}$	$((p \vee q) \wedge (\neg p \vee r)) \rightarrow (q \vee r)$	Resolution

EXAMPLE 3 State which rule of inference is the basis of the following argument: “It is below freezing now. Therefore, it is either below freezing or raining now.”

Solution: Let p be the proposition “It is below freezing now” and q the proposition “It is raining now.” Then this argument is of the form

$$\frac{p}{\therefore p \vee q}$$

This is an argument that uses the addition rule. ◀

EXAMPLE 4 State which rule of inference is the basis of the following argument: “It is below freezing and raining now. Therefore, it is below freezing now.”

Solution: Let p be the proposition “It is below freezing now,” and let q be the proposition “It is raining now.” This argument is of the form

$$\frac{p \wedge q}{\therefore p}$$

This argument uses the simplification rule. ◀

EXAMPLE 5 State which rule of inference is used in the argument:

If it rains today, then we will not have a barbecue today. If we do not have a barbecue today, then we will have a barbecue tomorrow. Therefore, if it rains today, then we will have a barbecue tomorrow.

Solution: Let p be the proposition “It is raining today,” let q be the proposition “We will not have a barbecue today,” and let r be the proposition “We will have a barbecue tomorrow.” Then this argument is of the form

$$\begin{array}{c} p \rightarrow q \\ q \rightarrow r \\ \therefore p \rightarrow r \end{array}$$

Hence, this argument is a hypothetical syllogism. ◀

Using Rules of Inference to Build Arguments

When there are many premises, several rules of inference are often needed to show that an argument is valid. This is illustrated by Examples 6 and 7, where the steps of arguments are displayed on separate lines, with the reason for each step explicitly stated. These examples also show how arguments in English can be analyzed using rules of inference.

EXAMPLE 6 Show that the premises “It is not sunny this afternoon and it is colder than yesterday,” “We will go swimming only if it is sunny,” “If we do not go swimming, then we will take a canoe trip,” and “If we take a canoe trip, then we will be home by sunset” lead to the conclusion “We will be home by sunset.”



Solution: Let p be the proposition “It is sunny this afternoon,” q the proposition “It is colder than yesterday,” r the proposition “We will go swimming,” s the proposition “We will take a canoe trip,” and t the proposition “We will be home by sunset.” Then the premises become $\neg p \wedge q$, $r \rightarrow p$, $\neg r \rightarrow s$, and $s \rightarrow t$. The conclusion is simply t . We need to give a valid argument with premises $\neg p \wedge q$, $r \rightarrow p$, $\neg r \rightarrow s$, and $s \rightarrow t$ and conclusion t .

We construct an argument to show that our premises lead to the desired conclusion as follows.

Step	Reason
1. $\neg p \wedge q$	Premise
2. $\neg p$	Simplification using (1)
3. $r \rightarrow p$	Premise
4. $\neg r$	Modus tollens using (2) and (3)
5. $\neg r \rightarrow s$	Premise
6. s	Modus ponens using (4) and (5)
7. $s \rightarrow t$	Premise
8. t	Modus ponens using (6) and (7)

Note that we could have used a truth table to show that whenever each of the four hypotheses is true, the conclusion is also true. However, because we are working with five propositional variables, p , q , r , s , and t , such a truth table would have 32 rows. ◀

EXAMPLE 7 Show that the premises “If you send me an e-mail message, then I will finish writing the program,” “If you do not send me an e-mail message, then I will go to sleep early,” and “If I go to sleep early, then I will wake up feeling refreshed” lead to the conclusion “If I do not finish writing the program, then I will wake up feeling refreshed.”

Solution: Let p be the proposition “You send me an e-mail message,” q the proposition “I will finish writing the program,” r the proposition “I will go to sleep early,” and s the proposition “I will wake up feeling refreshed.” Then the premises are $p \rightarrow q$, $\neg p \rightarrow r$, and $r \rightarrow s$. The desired conclusion is $\neg q \rightarrow s$. We need to give a valid argument with premises $p \rightarrow q$, $\neg p \rightarrow r$, and $r \rightarrow s$ and conclusion $\neg q \rightarrow s$.

This argument form shows that the premises lead to the desired conclusion.

Step	Reason
1. $p \rightarrow q$	Premise
2. $\neg q \rightarrow \neg p$	Contrapositive of (1)
3. $\neg p \rightarrow r$	Premise
4. $\neg q \rightarrow r$	Hypothetical syllogism using (2) and (3)
5. $r \rightarrow s$	Premise
6. $\neg q \rightarrow s$	Hypothetical syllogism using (4) and (5)



Resolution

Computer programs have been developed to automate the task of reasoning and proving theorems. Many of these programs make use of a rule of inference known as **resolution**. This rule of inference is based on the tautology



$$((p \vee q) \wedge (\neg p \vee r)) \rightarrow (q \vee r).$$

(Exercise 30 in Section 1.3 asks for the verification that this is a tautology.) The final disjunction in the resolution rule, $q \vee r$, is called the **resolvent**. When we let $q = r$ in this tautology, we obtain $(p \vee q) \wedge (\neg p \vee q) \rightarrow q$. Furthermore, when we let $r = F$, we obtain $(p \vee q) \wedge (\neg p) \rightarrow q$ (because $q \vee F \equiv q$), which is the tautology on which the rule of disjunctive syllogism is based.

EXAMPLE 8 Use resolution to show that the hypotheses “Jasmine is skiing or it is not snowing” and “It is snowing or Bart is playing hockey” imply that “Jasmine is skiing or Bart is playing hockey.”



Solution: Let p be the proposition “It is snowing,” q the proposition “Jasmine is skiing,” and r the proposition “Bart is playing hockey.” We can represent the hypotheses as $\neg p \vee q$ and $p \vee r$, respectively. Using resolution, the proposition $q \vee r$, “Jasmine is skiing or Bart is playing hockey,” follows.



Resolution plays an important role in programming languages based on the rules of logic, such as Prolog (where resolution rules for quantified statements are applied). Furthermore, it can be used to build automatic theorem proving systems. To construct proofs in propositional logic using resolution as the only rule of inference, the hypotheses and the conclusion must be expressed as **clauses**, where a clause is a disjunction of variables or negations of these variables. We can replace a statement in propositional logic that is not a clause by one or more equivalent statements that are clauses. For example, suppose we have a statement of the form $p \vee (q \wedge r)$. Because $p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$, we can replace the single statement $p \vee (q \wedge r)$ by two statements $p \vee q$ and $p \vee r$, each of which is a clause. We can replace a statement of the form $\neg(p \vee q)$ by the two statements $\neg p$ and $\neg q$ because De Morgan’s law tells us that $\neg(p \vee q) \equiv \neg p \wedge \neg q$. We can also replace a conditional statement $p \rightarrow q$ with the equivalent disjunction $\neg p \vee q$.

EXAMPLE 9 Show that the premises $(p \wedge q) \vee r$ and $r \rightarrow s$ imply the conclusion $p \vee s$.

Solution: We can rewrite the premises $(p \wedge q) \vee r$ as two clauses, $p \vee r$ and $q \vee r$. We can also replace $r \rightarrow s$ by the equivalent clause $\neg r \vee s$. Using the two clauses $p \vee r$ and $\neg r \vee s$, we can use resolution to conclude $p \vee s$. ◀

Fallacies

Several common fallacies arise in incorrect arguments. These fallacies resemble rules of inference, but are based on contingencies rather than tautologies. These are discussed here to show the distinction between correct and incorrect reasoning.



The proposition $((p \rightarrow q) \wedge q) \rightarrow p$ is not a tautology, because it is false when p is false and q is true. However, there are many incorrect arguments that treat this as a tautology. In other words, they treat the argument with premises $p \rightarrow q$ and q and conclusion p as a valid argument form, which it is not. This type of incorrect reasoning is called the **fallacy of affirming the conclusion**.

EXAMPLE 10 Is the following argument valid?

If you do every problem in this book, then you will learn discrete mathematics. You learned discrete mathematics.

Therefore, you did every problem in this book.

Solution: Let p be the proposition “You did every problem in this book.” Let q be the proposition “You learned discrete mathematics.” Then this argument is of the form: if $p \rightarrow q$ and q , then p . This is an example of an incorrect argument using the fallacy of affirming the conclusion. Indeed, it is possible for you to learn discrete mathematics in some way other than by doing every problem in this book. (You may learn discrete mathematics by reading, listening to lectures, doing some, but not all, the problems in this book, and so on.) ◀

The proposition $((p \rightarrow q) \wedge \neg p) \rightarrow \neg q$ is not a tautology, because it is false when p is false and q is true. Many incorrect arguments use this incorrectly as a rule of inference. This type of incorrect reasoning is called the **fallacy of denying the hypothesis**.

EXAMPLE 11 Let p and q be as in Example 10. If the conditional statement $p \rightarrow q$ is true, and $\neg p$ is true, is it correct to conclude that $\neg q$ is true? In other words, is it correct to assume that you did not learn discrete mathematics if you did not do every problem in the book, assuming that if you do every problem in this book, then you will learn discrete mathematics?

Solution: It is possible that you learned discrete mathematics even if you did not do every problem in this book. This incorrect argument is of the form $p \rightarrow q$ and $\neg p$ imply $\neg q$, which is an example of the fallacy of denying the hypothesis. ◀

Rules of Inference for Quantified Statements

We have discussed rules of inference for propositions. We will now describe some important rules of inference for statements involving quantifiers. These rules of inference are used extensively in mathematical arguments, often without being explicitly mentioned.

Universal instantiation is the rule of inference used to conclude that $P(c)$ is true, where c is a particular member of the domain, given the premise $\forall x P(x)$. Universal instantiation is used when we conclude from the statement “All women are wise” that “Lisa is wise,” where Lisa is a member of the domain of all women.

TABLE 2 Rules of Inference for Quantified Statements.

<i>Rule of Inference</i>	<i>Name</i>
$\frac{\forall x P(x)}{\therefore P(c)}$	Universal instantiation
$\frac{P(c) \text{ for an arbitrary } c}{\therefore \forall x P(x)}$	Universal generalization
$\frac{\exists x P(x)}{\therefore P(c) \text{ for some element } c}$	Existential instantiation
$\frac{P(c) \text{ for some element } c}{\therefore \exists x P(x)}$	Existential generalization

Universal generalization is the rule of inference that states that $\forall x P(x)$ is true, given the premise that $P(c)$ is true for all elements c in the domain. Universal generalization is used when we show that $\forall x P(x)$ is true by taking an arbitrary element c from the domain and showing that $P(c)$ is true. The element c that we select must be an arbitrary, and not a specific, element of the domain. That is, when we assert from $\forall x P(x)$ the existence of an element c in the domain, we have no control over c and cannot make any other assumptions about c other than it comes from the domain. Universal generalization is used implicitly in many proofs in mathematics and is seldom mentioned explicitly. However, the error of adding unwarranted assumptions about the arbitrary element c when universal generalization is used is all too common in incorrect reasoning.

Existential instantiation is the rule that allows us to conclude that there is an element c in the domain for which $P(c)$ is true if we know that $\exists x P(x)$ is true. We cannot select an arbitrary value of c here, but rather it must be a c for which $P(c)$ is true. Usually we have no knowledge of what c is, only that it exists. Because it exists, we may give it a name (c) and continue our argument.

Existential generalization is the rule of inference that is used to conclude that $\exists x P(x)$ is true when a particular element c with $P(c)$ true is known. That is, if we know one element c in the domain for which $P(c)$ is true, then we know that $\exists x P(x)$ is true.

We summarize these rules of inference in Table 2. We will illustrate how some of these rules of inference for quantified statements are used in Examples 12 and 13.

EXAMPLE 12

Show that the premises “Everyone in this discrete mathematics class has taken a course in computer science” and “Marla is a student in this class” imply the conclusion “Marla has taken a course in computer science.”

Solution: Let $D(x)$ denote “ x is in this discrete mathematics class,” and let $C(x)$ denote “ x has taken a course in computer science.” Then the premises are $\forall x(D(x) \rightarrow C(x))$ and $D(\text{Marla})$. The conclusion is $C(\text{Marla})$.



The following steps can be used to establish the conclusion from the premises.

Step	Reason
1. $\forall x(D(x) \rightarrow C(x))$	Premise
2. $D(\text{Marla}) \rightarrow C(\text{Marla})$	Universal instantiation from (1)
3. $D(\text{Marla})$	Premise
4. $C(\text{Marla})$	Modus ponens from (2) and (3)



EXAMPLE 13 Show that the premises “A student in this class has not read the book,” and “Everyone in this class passed the first exam” imply the conclusion “Someone who passed the first exam has not read the book.”

Solution: Let $C(x)$ be “ x is in this class,” $B(x)$ be “ x has read the book,” and $P(x)$ be “ x passed the first exam.” The premises are $\exists x(C(x) \wedge \neg B(x))$ and $\forall x(C(x) \rightarrow P(x))$. The conclusion is $\exists x(P(x) \wedge \neg B(x))$. These steps can be used to establish the conclusion from the premises.

Step	Reason
1. $\exists x(C(x) \wedge \neg B(x))$	Premise
2. $C(a) \wedge \neg B(a)$	Existential instantiation from (1)
3. $C(a)$	Simplification from (2)
4. $\forall x(C(x) \rightarrow P(x))$	Premise
5. $C(a) \rightarrow P(a)$	Universal instantiation from (4)
6. $P(a)$	Modus ponens from (3) and (5)
7. $\neg B(a)$	Simplification from (2)
8. $P(a) \wedge \neg B(a)$	Conjunction from (6) and (7)
9. $\exists x(P(x) \wedge \neg B(x))$	Existential generalization from (8)



Combining Rules of Inference for Propositions and Quantified Statements

We have developed rules of inference both for propositions and for quantified statements. Note that in our arguments in Examples 12 and 13 we used both universal instantiation, a rule of inference for quantified statements, and modus ponens, a rule of inference for propositional logic. We will often need to use this combination of rules of inference. Because universal instantiation and modus ponens are used so often together, this combination of rules is sometimes called **universal modus ponens**. This rule tells us that if $\forall x(P(x) \rightarrow Q(x))$ is true, and if $P(a)$ is true for a particular element a in the domain of the universal quantifier, then $Q(a)$ must also be true. To see this, note that by universal instantiation, $P(a) \rightarrow Q(a)$ is true. Then, by modus ponens, $Q(a)$ must also be true. We can describe universal modus ponens as follows:

$$\frac{\begin{array}{l} \forall x(P(x) \rightarrow Q(x)) \\ P(a), \text{ where } a \text{ is a particular element in the domain} \end{array}}{\therefore Q(a)}$$

Universal modus ponens is commonly used in mathematical arguments. This is illustrated in Example 14.

EXAMPLE 14 Assume that “For all positive integers n , if n is greater than 4, then n^2 is less than 2^n ” is true. Use universal modus ponens to show that $100^2 < 2^{100}$.

Solution: Let $P(n)$ denote “ $n > 4$ ” and $Q(n)$ denote “ $n^2 < 2^n$.” The statement “For all positive integers n , if n is greater than 4, then n^2 is less than 2^n ” can be represented by $\forall n(P(n) \rightarrow Q(n))$, where the domain consists of all positive integers. We are assuming that $\forall n(P(n) \rightarrow Q(n))$ is true. Note that $P(100)$ is true because $100 > 4$. It follows by universal modus ponens that $Q(100)$ is true, namely that $100^2 < 2^{100}$. ◀

Another useful combination of a rule of inference from propositional logic and a rule of inference for quantified statements is **universal modus tollens**. Universal modus tollens

combines universal instantiation and modus tollens and can be expressed in the following way:

$$\begin{array}{c} \forall x(P(x) \rightarrow Q(x)) \\ \neg Q(a), \text{ where } a \text{ is a particular element in the domain} \\ \hline \therefore \neg P(a) \end{array}$$

The verification of universal modus tollens is left as Exercise 25. Exercises 26–29 develop additional combinations of rules of inference in propositional logic and quantified statements.

Exercises

1. Find the argument form for the following argument and determine whether it is valid. Can we conclude that the conclusion is true if the premises are true?

If Socrates is human, then Socrates is mortal.
Socrates is human.

∴ Socrates is mortal.

2. Find the argument form for the following argument and determine whether it is valid. Can we conclude that the conclusion is true if the premises are true?

If George does not have eight legs, then he is not a spider.
George is a spider.

∴ George has eight legs.

3. What rule of inference is used in each of these arguments?

- a) Alice is a mathematics major. Therefore, Alice is either a mathematics major or a computer science major.
- b) Jerry is a mathematics major and a computer science major. Therefore, Jerry is a mathematics major.
- c) If it is rainy, then the pool will be closed. It is rainy. Therefore, the pool is closed.
- d) If it snows today, the university will close. The university is not closed today. Therefore, it did not snow today.
- e) If I go swimming, then I will stay in the sun too long. If I stay in the sun too long, then I will sunburn. Therefore, if I go swimming, then I will sunburn.

4. What rule of inference is used in each of these arguments?

- a) Kangaroos live in Australia and are marsupials. Therefore, kangaroos are marsupials.
- b) It is either hotter than 100 degrees today or the pollution is dangerous. It is less than 100 degrees outside today. Therefore, the pollution is dangerous.
- c) Linda is an excellent swimmer. If Linda is an excellent swimmer, then she can work as a lifeguard. Therefore, Linda can work as a lifeguard.
- d) Steve will work at a computer company this summer. Therefore, this summer Steve will work at a computer company or he will be a beach bum.

- e) If I work all night on this homework, then I can answer all the exercises. If I answer all the exercises, I will understand the material. Therefore, if I work all night on this homework, then I will understand the material.

5. Use rules of inference to show that the hypotheses “Randy works hard,” “If Randy works hard, then he is a dull boy,” and “If Randy is a dull boy, then he will not get the job” imply the conclusion “Randy will not get the job.”

6. Use rules of inference to show that the hypotheses “If it does not rain or if it is not foggy, then the sailing race will be held and the lifesaving demonstration will go on,” “If the sailing race is held, then the trophy will be awarded,” and “The trophy was not awarded” imply the conclusion “It rained.”

7. What rules of inference are used in this famous argument? “All men are mortal. Socrates is a man. Therefore, Socrates is mortal.”

8. What rules of inference are used in this argument? “No man is an island. Manhattan is an island. Therefore, Manhattan is not a man.”

9. For each of these collections of premises, what relevant conclusion or conclusions can be drawn? Explain the rules of inference used to obtain each conclusion from the premises.

- a) “If I take the day off, it either rains or snows.” “I took Tuesday off or I took Thursday off.” “It was sunny on Tuesday.” “It did not snow on Thursday.”

- b) “If I eat spicy foods, then I have strange dreams.” “I have strange dreams if there is thunder while I sleep.” “I did not have strange dreams.”

- c) “I am either clever or lucky.” “I am not lucky.” “If I am lucky, then I will win the lottery.”

- d) “Every computer science major has a personal computer.” “Ralph does not have a personal computer.” “Ann has a personal computer.”

- e) “What is good for corporations is good for the United States.” “What is good for the United States is good for you.” “What is good for corporations is for you to buy lots of stuff.”

- f) “All rodents gnaw their food.” “Mice are rodents.” “Rabbits do not gnaw their food.” “Bats are not rodents.”

- 10.** For each of these sets of premises, what relevant conclusion or conclusions can be drawn? Explain the rules of inference used to obtain each conclusion from the premises.
- "If I play hockey, then I am sore the next day." "I use the whirlpool if I am sore." "I did not use the whirlpool."
 - "If I work, it is either sunny or partly sunny." "I worked last Monday or I worked last Friday." "It was not sunny on Tuesday." "It was not partly sunny on Friday."
 - "All insects have six legs." "Dragonflies are insects." "Spiders do not have six legs." "Spiders eat dragonflies."
 - "Every student has an Internet account." "Homer does not have an Internet account." "Maggie has an Internet account."
 - "All foods that are healthy to eat do not taste good." "Tofu is healthy to eat." "You only eat what tastes good." "You do not eat tofu." "Cheeseburgers are not healthy to eat."
 - "I am either dreaming or hallucinating." "I am not dreaming." "If I am hallucinating, I see elephants running down the road."
- 11.** Show that the argument form with premises p_1, p_2, \dots, p_n and conclusion $q \rightarrow r$ is valid if the argument form with premises p_1, p_2, \dots, p_n, q , and conclusion r is valid.
- 12.** Show that the argument form with premises $(p \wedge t) \rightarrow (r \vee s)$, $q \rightarrow (u \wedge t)$, $u \rightarrow p$, and $\neg s$ and conclusion $q \rightarrow r$ is valid by first using Exercise 11 and then using rules of inference from Table 1.
- 13.** For each of these arguments, explain which rules of inference are used for each step.
- "Doug, a student in this class, knows how to write programs in JAVA. Everyone who knows how to write programs in JAVA can get a high-paying job. Therefore, someone in this class can get a high-paying job."
 - "Somebody in this class enjoys whale watching. Every person who enjoys whale watching cares about ocean pollution. Therefore, there is a person in this class who cares about ocean pollution."
 - "Each of the 93 students in this class owns a personal computer. Everyone who owns a personal computer can use a word processing program. Therefore, Zeke, a student in this class, can use a word processing program."
 - "Everyone in New Jersey lives within 50 miles of the ocean. Someone in New Jersey has never seen the ocean. Therefore, someone who lives within 50 miles of the ocean has never seen the ocean."
- 14.** For each of these arguments, explain which rules of inference are used for each step.
- "Linda, a student in this class, owns a red convertible. Everyone who owns a red convertible has gotten at least one speeding ticket. Therefore, someone in this class has gotten a speeding ticket."
- b)** "Each of five roommates, Melissa, Aaron, Ralph, Venesha, and Keeshawn, has taken a course in discrete mathematics. Every student who has taken a course in discrete mathematics can take a course in algorithms. Therefore, all five roommates can take a course in algorithms next year."
- c)** "All movies produced by John Sayles are wonderful. John Sayles produced a movie about coal miners. Therefore, there is a wonderful movie about coal miners."
- d)** "There is someone in this class who has been to France. Everyone who goes to France visits the Louvre. Therefore, someone in this class has visited the Louvre."
- 15.** For each of these arguments determine whether the argument is correct or incorrect and explain why.
- All students in this class understand logic. Xavier is a student in this class. Therefore, Xavier understands logic.
 - Every computer science major takes discrete mathematics. Natasha is taking discrete mathematics. Therefore, Natasha is a computer science major.
 - All parrots like fruit. My pet bird is not a parrot. Therefore, my pet bird does not like fruit.
 - Everyone who eats granola every day is healthy. Linda is not healthy. Therefore, Linda does not eat granola every day.
- 16.** For each of these arguments determine whether the argument is correct or incorrect and explain why.
- Everyone enrolled in the university has lived in a dormitory. Mia has never lived in a dormitory. Therefore, Mia is not enrolled in the university.
 - A convertible car is fun to drive. Isaac's car is not a convertible. Therefore, Isaac's car is not fun to drive.
 - Quincy likes all action movies. Quincy likes the movie *Eight Men Out*. Therefore, *Eight Men Out* is an action movie.
 - All lobstermen set at least a dozen traps. Hamilton is a lobsterman. Therefore, Hamilton sets at least a dozen traps.
- 17.** What is wrong with this argument? Let $H(x)$ be " x is happy." Given the premise $\exists x H(x)$, we conclude that $H(\text{Lola})$. Therefore, Lola is happy.
- 18.** What is wrong with this argument? Let $S(x, y)$ be " x is shorter than y ." Given the premise $\exists s S(s, \text{Max})$, it follows that $S(\text{Max}, \text{Max})$. Then by existential generalization it follows that $\exists x S(x, x)$, so that someone is shorter than himself.
- 19.** Determine whether each of these arguments is valid. If an argument is correct, what rule of inference is being used? If it is not, what logical error occurs?
- If n is a real number such that $n > 1$, then $n^2 > 1$. Suppose that $n^2 > 1$. Then $n > 1$.
 - If n is a real number with $n > 3$, then $n^2 > 9$. Suppose that $n^2 \leq 9$. Then $n \leq 3$.
 - If n is a real number with $n > 2$, then $n^2 > 4$. Suppose that $n \leq 2$. Then $n^2 \leq 4$.