# Courcelle's Theorem and Computational Social Choice

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## Abstract

Courcelle's theorem is a meta-theorem that can be used to proof complexity results of all sort of computational problems. Although Courcelle's theorem has found applications for graph theoretical problems its application to computational social choice (ComSoC) problems is still limited. In this paper we argue that this is for good reasons. Although it seems natural to apply Courcelle's theorem to some problems of ComSoC, such as allocation of indivisible goods. We argue that for most other ComSoC problems one will find some difficulties, that one maybe could overcome, but which make the application of Courcelle's theorem a lot less attractive than one at first glance might think. In this paper we apply Courcelle's theorem to the problem of allocation of indivisible goods and to the finite manipulation games. In the latter the game has to be constrained heavily to make Courcelle's theorem applicable. Lastly we argue about why for other ComSoC problems applying Courcelle's theorem is probably not the best approach to get complexity results.

## 1 Introduction

Courcelle's theorem [Courcelle, 1986] can be used to show that certain computational problems are solvable in *fixed parameter tractable (fpt)* time. The beauty of Courcelle's theorem is that one does not have to find an algorithm, but rather just a translation from the logic of a given problem to *Monadic Second Order* (MSO) logic to show that a problem is fpt. Although the bound given by Courcelle's theorem is not very useful in practice, it is still helpful theoretically as one does not have to think about all the details that come into play when designing an algorithm.

But Courcelle's theorem also has its drawbacks, for example the size of the universe of the $\sigma-$structure should be bounded. When one has to introduce, for example, the natural numbers, Courcelle's theorem becomes useless. Because of this it seems that there is only limited applicability of Courcelle's theorem on problems in Computational Social Choice (ComSoC).

In this paper we show how one could apply Courcelle's theorem to *the allocation of indivisible goods* and *finite manipulation games*. To be precise the contribution of this paper are as follows:

- Courcelle's theorem is used to show that the problem of $\phi$-AIG is finite parameter tractable with respect to the length of the formula $\phi$, the treewidth and the degree of its graph.
- Courcelle's theorem is used to show that Problem of $\phi$-FMG is finite parameter tractable with respect to the length of the formula $\phi$, the number of time steps, the number of agents in the coalition, the treewidth and the degree of its graph.
- Lastly we argue aboput the limitations of Courcelle's theorem and use $\phi$-FMG as an example why Courcelle's theorem is probably not nicely applicable to ComSoC problems.

## 2 Related work

[Beviá, 1998] discusses the problem of fair allocation of indivisible goods. In this problem there are a set of agents and a set of resources in which an allocation of goods of one or more goods to the each agents has to be found. This allocation could have certain properties, such as *envy-freeness*, *Pareto optimal* and *neutrality*. Finding solutions to such a problem is often hard. For example finding a solution that is both envy-free and parato optimal lies above NP [Bouveret and Lang, 2008]

[Grandi *et al.*, 2021] looks at games of influences, sometimes also called opinion diffusion ( e.g. in [Bredereck and Elkind, 2017]). In such a game, agents are placed on a graph, which represents a network of influence. Furthermore each agent has an (often binary) opinion about a set of issues. An agent i can influence agent j when there is a directed edge between agent i and agent j. In these games we can ask questions like: Is there a nash-equilibrium for this game? Is there a perfect-recall strategy such that agent i can achieve its goal? It turns out that finding a perfect-recall strategy is in EXPTIME.

In both the allocation of indivisible goods and in games of influences it seems that restricting parameters seems sensible. For example, in real-life not everyone cares about anything. Thus restricting the number of goods an agent is interested in seems sensible. Furthermore often we do not only care if there exists a winning strategy, but also if this strategy can be executed in a certain amount of time, thus limiting the time horizon of the game seems sensible. In these cases, restricting some parameter of the problem, such as treewidth of the influence graph, can result in better hardness results such as the problem being in fpt.

As a good example for restricting parameters and applying Courcelle's theorem, [Peters, 2016] discusses *graphical hedonic games of bounded treewidth*. Finding solutions to Hedonic Games with certain properties (e.g. core-stable) is NP-HARD. But when restricting the game by the treewidth, finding solutions to Hedonic games with certain properties in which the treewidth is bounded becomes FPT.

Although Courcelle's theorem has not found many applications in CoMSoc it has found application on graph-theoretic problems such as finding a *minimum vertex cover*, *minimum dominating set* and *maximum independent set* [Kneis and Langer, 2009]

## 3 Preleminairies

In the preleminairies we will introduce what Monadic Second Order logic (MSO) is and discuss relevant definitions as well as Courcelle's theorem, this follows closely the work of [Peters, 2016]. Lastly we discuss both the game of *allocation of indivisible goods* and *finite manipulation games*.

### 3.1 Monadic second order logic

Here we introduce mondacis second-order logic (MSO). For this we need some definitions.

- A *signature* $\sigma$ is a finite collection of relation symbols $(R_1, \dots, R_k)$, each with an *arity* of at least 1. The function $ar(R)$ maps each relation symbol to their respective arity.
- A $\sigma$-structure $\mathcal{A} := \langle A, ((R_1^{\mathcal{A}}, \dots, R_k^{\mathcal{A}})) \rangle$ consists of a finite set $A$ called the *universe* of $\mathcal{A}$ and a realisation $R_i^{\mathcal{A}} \subseteq A^{ar(R_i)}$ for each relation symbol $R_i$. Lastly the size of $\mathcal{A}$ is given by

$$||\mathcal{A}|| = |\sigma| + |A| + |\sum_{R_i \in \sigma} |R_i^{\mathcal{A}}| \cdot ar(R_i)$$

- Given a signature $\sigma$, the language of $MSO(\sigma)$ of monadic second order logice is given by the grammar:

$$\phi ::= x = y|R_i x_1, \dots, x_{ar(R_i)}|Xx|$$
$$|\neg\phi|(\phi \vee \psi)|\forall X\phi|\exists X\phi|\forall x\phi|\exists x\phi|$$

  for which $x, y, x_1, x_2, ...$ are first-order variables and X denotes monadic relations (subsets of A). The semantics $A \models \phi$ of $MSO[\sigma]$ are defined in the obvious way.

### 3.2 Tree decompositions and Courcelle's theorem

Before we can introduce Courcelle's theorem we need to define what the treewidth $TW(\mathcal{A})$ of a $\sigma$-structure $\mathcal{A}$ is.

**Definition 1.** *A* ***tree decomposition*** *of a $\sigma$-structure $\mathcal{A}$ is given by a tree $T = (V, E)$ in which vertex $v \in V$ is associated with a bag $\beta(v) \subseteq A$. In which $\beta(v) : V \mapsto 2^A$. The following two conditions should be satisfied:*

*(TD 1) Each item is associated with at least one bag and its bags are connected: for each $a \in A$ there exists vertex $v_a \in V$ such that $a \in \beta(v_a)$ and the set $\beta^{-1}(a)$ form a connected subtree of $T$*

*(TD 2) All the relations are represented in at least one bag: for each $R_i \in \sigma$ and all $a_1, \dots, a_{ar(R_i)} \in A$ if $(a_1, \dots a_{ar(R_i)}) \in R_i^{\mathcal{A}}$ then there exists a vertex $v \in V$ such that $\{a_1, \dots a_{ar(R_i)}\} \subseteq \beta(v)$.*

The width of a tree decomposition T is the size of the largest bag minus 1.

**Definition 2.** *The* ***treewidth*** *of a $\sigma$-structure $\mathcal{A}$ denoted by $TW(\mathcal{A})$ is the minimun size of the width of any of the tree decompositions of $\mathcal{A}$.*

We are now ready to introduce Courcelle's theorem:

**Theorem 1.** *Courcelle's theorem Given a formula $\phi$ of $MSO[\sigma]$ and a $\sigma$-structure $\mathcal{A}$, we can in time $g(|\phi|, tw(\mathcal{A})) \cdot |A| + O(||\mathcal{A}||)$ decide whether $\mathcal{A} \models \phi$, where g is a computable function.*

Courcelle's theorem can be used to prove fpt for certain problems with respect to parameters set $P$, for example P contains the treewidth of the graph related to the problem. This can be done as follows: One has to translate the given problem and its logic to some $\sigma$-structure and its corresponding $MSO[\sigma]$-logic. This can be done by first defining $\sigma$ and the related $\sigma$-structure. One also has to show to express the properties that one cares about in $MSO[\sigma]$-logic. We also need to show that the treewidth of the $\sigma$-structure and the length of the formula $tr(\phi)$ are bounded by some computable function f and g respectively, both which depends only on the parameters in P. Lastly the size of the universe A has to be bounded by some polynomial depending on parameters not in P, for example the number of agents.

### 3.3 Allocation of indivisible goods

**Definition 3.** *A* ***graphical game of allocation of indivisible goods*** *is a game $\mathcal{G}$ together with a graph G in which we have a set of $n$ agents $N$ and a set of $m$ resources $R$. Each agent $a_i \in N$ as a complete and transitive preference relation over the possible subsets of the resources $\mathcal{R} = 2^R$ denoted by $\succeq_{a_i}$. The game has an outcome $\pi = (\pi_{a_1}, \dots \pi_{a_n})$ of pairwise non-overlapping allocations, such that $\bigcup_{a_i \in N} \pi_{a_i} = R$ in which $\pi_{a_i}$ denotes the allocation for agent $a_i$*

*The graph $G = (N \cup R, E)$ consists of nodes that represents the agents and the resources, and the edges represent which agents actually care about which resources. We add the following constraint to the preference relations: for each agent $a_i$ and all allocations $S, T \in \mathcal{R}$ we have that $S \succeq_{a_i} T$ if and only if $S \cap \Gamma(a_i) \succeq_{a_i} T \cap \Gamma(a_i)$. In which $\Gamma_G(a_i)$ is the neighborhood of $a_i$ in G. This constraint captures the idea that each agent only cares about a subset of all the available resources.*

For the rest of the paper we assume that $n \leq m$. We also define *game of indivisible goods logic* (GIG-logic) with the following grammar:

$$\phi ::= a_i = a_j|a_i \in Y|r_k \in L|S \succeq_i T|S = \pi_i$$

$$|\neg\phi|(\phi \vee \psi)|\forall a_i\phi|\exists a_i\phi|\forall r_k\phi|\exists r_k\phi|$$

$$\forall S\phi|\exists S\phi|\forall\pi\phi|\exists\pi\phi|$$

In which $Y \subseteq N$, $L \subseteq R$, and $S$ and $T$ are allocations. The semantics are again defined in the obvious way. With this grammar we can express certain properties for example:

- Envy-freeness: $\forall a_i \forall a_j(\pi_{a_i} \succeq_{a_i} \pi_{a_j})$
- Perfect: $\forall S \forall a_j(\pi_{a_i} \succeq_{a_i} S)$

Next we define the problem of $\phi$-AIG and a corresponding theorem.
*$\phi$-AIG*
**Instance**: a graphical game of allocation of indivisible goods $\mathcal{G} = (\langle N, R, (\succeq_{a_i \in N})\rangle, G)$ and a formula $\phi$ of AIG logic.
**Question** does $\mathcal{G} \models \phi$?

**Theorem 2.** *The problem of $\phi$-AIG can be solved in $O(g(|\phi|, d, k) \cdot m^2)$ for some computable function g in which d is the degree of the graph and $k = TW(G)$.*

### 3.4 Finite Manipulation Games

The second type of game we look at is the finite manipulation game. Variants of this game are discussed in [Grandi *et al.*, 2021], such as variants with perfect and imperfect information. For out purposes we restrict the game to a finite version. Furthermore we restrict the treewidth and the degree of the influence graph. Lastly we also only allow for strategies that are expressible in 3-cnf formulas of a certain length. We do this because otherwise it seems impossible to apply Courcelle's theorem.

**Definition 4.** *A **finite manipulation game** consists of a set of n agents N with $T \in \mathbf{N}^+$ time steps. The agents are all placed on a directed graph $G = (G, E)$ called the influence graph. Each agent has an initial opinion $o_{a_i} \in O = \{pos, neg\}$ about the issue. Each agent can choose to announce its opinion or to keep quiet. Hence each agent as action $Act = \{quiet, announce\}$. Each agent also has an update rule for its opinion $r_{ai} : Act^n \times O^n \mapsto O$ that maps the opinion of a given time step t to the new opinion $t+1$. The graph induces a constraint on the update function, that is: $r_{a_i}(act_{a_0}, \ldots, act_{a_n}) = o$ with $o \in O$ if and only if $r_{a_i}(\{act'_{a_0}, \ldots, act'_{a_n}\}) = o$ for all action profiles such that $act'_{a_j} = act_{a_j}$ if $a_j \in influence_G(a_i)$ and else $act'_{a_j} \in Act$. In which $influence_G(a_i)$ is the set of all agents that can influence agent $a_i$. This constraint basically says that the update function of agent $a_i$ only depends on the agents that can actually influence agent $a_i$. Each time step the agents choose there action and based on that all the new opinions in the next time step are calculated. This updating creates a history of opinions and actions denoted by $H \subseteq O^n \times [T] \cup Act^n \times [T]$. In which $H_k$ denotes the kth timestep of the history.*

Lastly a set of agents can have a goal. This goal can be formulated with the help of $\mathcal{L}_{LTL-1}$ with the following grammar:

$$\phi ::= o_{a_i}|\neg\phi|\phi_1 \wedge \phi_2| \bigcirc \phi|\phi_1 \mathcal{U} \phi_2$$

Let $k, k', k'' \in [T]$ the semantics are as follows:

$$H_{\leq k} \models o_{a_i} \iff o_{a_i} \in H_k$$
$$H_{\leq k} \models \neg\phi \iff H_{\leq k} \not\models \phi$$
$$H_{\leq k} \models \phi_1 \wedge \phi_2 \iff H_{\leq k} \models \phi_1 \text{ and } H_{\leq k} \models \phi_2$$
$$H_{\leq k} \models \bigcirc\phi \iff H_{\leq k+1} \models \phi$$
$$H_{\leq k} \models \phi_1 \mathcal{U} \phi_2 \iff \exists k' : (k \leq k' \wedge H_{\leq k'} \models \phi_2) \text{ and}$$
$$\forall k'' : (k \leq k'' < k' \to H_{\leq k''} \models \phi_2)$$

We allow agents to have a strategy $S_{a_i}$ which is defined as a 3-cnf formula $\psi_{a_i}$ with at most $c \leq \lfloor \frac{T}{3} \rfloor$ clauses containing the propositions $o_{a_0}, \cdots, o_{a_n}$. Such that the agent announces its opinion if $\psi_{a_i}$ is true and keeps quiet otherwise. Such a strategy limits the possible histories of a game. Namely $H$ is a history of a finite manipulation game with a set of agents A of size m playing strategy $(\psi_{a_0}, \ldots \psi_{a_m})$ if and only if the strategy was generated by following the strategy defined by the agents in A.

Note that, although the strategies are memoryless, with $c$ large enough we can express any propositional formula. Hence in theory we do not lose much expressibility compared to allowing for any formula.

Lastly we can now question if a strategy makes some goal $\phi$ true in a finite manipulation game $\mathcal{G}$:

$\mathcal{G}, (\psi_0, \cdots \psi_{n'}) \models \phi$ if and only if for all history $H$ in which we followed the strategy $(\psi_0, \cdots \psi_{n'})$ we have that $H_{\leq T} \models \phi$. From this we can define the following problem and theorem:
**$\phi$-FMG**
**Input** A finite game of influence $\mathcal{G}$ and a goal $\phi$ and a set of agents $Col \subseteq N$ of size m
**Question** Does there exists a collective strategy $(\psi_0, \cdots \psi_m)$ such that $G, (\psi_0, \cdots \psi_m) \models \phi$?

**Theorem 3.** *The problem of $\phi$-FMG can be solved in $O(f(k, d, T, m, |\phi|) \cdot n))$ for some computable function g in which d is the degree of the graph, $k = TW(G)$. T is the number of timesteps, m is the size of the coalition and n is the number of agents.*

## 4 Applying Courcelle's theorem

### 4.1 Proof of Theorem 2

Proving theorem 2 consists of a number of steps. Given a graphical game of indivisible goods $\mathcal{G}$ we need to 1) Define a corresponding $\sigma$-structure 2) Bound the treewidth 3) Show how to translate formulas from GIG-logic to $MSO[\sigma]$-logic and argue about the bound on the length of the resulting formula's 4) Apply Courcelle's theorem to get our result. This proof is heavinly inspired by the proof of theorem 2 from [Peters, 2016], but it had to be changed to fit the problem of $\phi-$AIG. Furthermore we work out some steps in more detail.

**Step 1: Defining the $\sigma$-structure.** First define the $\sigma$-structure for a given game $\mathcal{G}$.

We get a $\sigma-$signature consists of unary relation AGENT, RESOURCE, EDGE, binary relation INCI and $(2m+1)-$ary relation PREF. The universe $A = N \cup R \cup E \cup \{*\}$ Then the relations will be defined as follows: $AGENT = N$, $RESOURCE = R$, $INCI = \{(a_i, e)|a_i \in N, e \in E :$

$a_i \in E\}$ and $(a_i, r_1 \dots r_m, r_{m+1} \dots r_{2m}) \in PREF$ if and only if $a_i \in N$, $r_s \in R \cup \{*\}$ for $1 \leq s \leq 2m$ and $\{r_1, \dots r_m\} \backslash \{*\} \succeq_{a_i} \{r_{m+1}, \dots r_{2m}\} \backslash \{*\}$

**Step 2: Bounding the treewidth** We construct a tree decomposition of $\sigma$-structure and argue about its treewidth. First we know that by assumption the graph $G$ has treewidth at most k and degree d. Let this treewidth be given by some tree-decomposition $T$. We will alter it to some tree-decomposition $T'$ with at most treewidth $k + (k+1) \cdot d + 1$.

Step (1) for each $e = \{a_i, r_k\}$ of G find a bag $\beta(w)$ that contains both $a_i$ and $r_k$ and introduce a new bag $\{e, a_i, r_k\}$ that we attach as a leaf to w. This does not increase the width. Step (2) for each agent $a_i$ and every bag $\beta(w)$ of T such that $a_i \in \beta(w)$ add the set $\Gamma_G(a_i)$ to $\beta(w)$. Step (3) we add $*$ to every bag.

Now we show that the treewidth of $T'$ is at most $k + (k+1) \cdot d + 1$. First we know that each bag in $T$ contained at most $k+1$ agents as each bag was at most size $k+1$. Then for each bag we added at most $(k+1) \cdot d$ resources to the bag. Lastly we added $*$ to the bag which increases the sizes with 1. Hence the treewidth is at most $k + (k+1) \cdot d + 1$.

Lastly we need to show that this tree decomposition is a valid tree decomposition, for this we need to check the two conditions (TD 1) and (TD 2):

(TD 1) By definition of G we know that each agent and resource must be present in at least one bag and those bags are connected. In step 2 we add the edges together with the agents and resources in such a way that for each element $a \in N \cup R \cup E$ its bags form a connected subtree of $T'$. Lastly $*$ is added to each bag and thus $*$ is in every bag and its bags are arbitrarily connected.

(TD 2) First by definition of T we know that AGENT and RESOURCE are represented in at least 1 bag. We need to check if that is also the case for EDGE, INCI and PREF. By the first step we get that both EDGE and INCI are represented in at least 1 bag. In the second step we add for each agent all the resources that the agent $a_i$ is connected to. In the last step we add $*$. Then by construction of PREF we know that for each $(a_i, r_1, ...r_l) \in PREF$ we have that there exists a bag $\beta(w)$ with in which $\{a_i, r_1, ...r_l\} \in \beta(w)$. Therefore we know that every relation is represented in at least 1 bag. We can conclude that $T'$ is a valid tree decomposition and its width is at most $k + (k+1) \cdot d + 1$.

**Step 3: Translation of the formula's** We need to define a translation $tr$. This translation can be defined recursively. We note that we trivial can translate all the atoms (except for the $S = \pi_{a_i}$), boolean operators, quantification over agents and resources and subsets of resources $S$. We thus need to encode the quantification over allocations and $S = \pi_{a_i}$

For a subset $X \subseteq E$ we have that we can say that $X$ is an allocation denoted by $ALLOC(X)$ when :

$$\begin{aligned} &\forall (a_k, r_k) \in X (\forall r_k \exists x \in X : r_k \in x) \\ &\qquad \wedge (\forall a_j \neq a_k : a_j \notin X) \end{aligned}$$

We thus express that X is an allocation if every resource is assigned and no resource is assigned to two distinct agents. We define $\pi_{a_i} := \{r_k \in R | (a, r_k) \in \pi : a_i = a\}$ Then:

$$\begin{aligned} tr(\forall \pi \phi) &:= \forall S(alloc(S) \wedge tr(\phi)) \\ tr(\exists \pi \phi) &:= \exists S(alloc(S) \wedge tr(\phi)) \\ tr(S = \pi_{a_i}) &:= (\forall r_k \in \pi_{a_i} : r_k \in S) \\ &\quad \wedge (\forall s \in S : s \in \pi_{a_i}) \end{aligned}$$

Note that the first to translation quantify over every subset of the universe that is an allocation. The third rule is just the definition of a set being equal.

Lastly we need to translate the preferences.

$$\begin{aligned} &tr(S \succeq_{a_i} T) := \\ &\exists s_1, \dots_d, t_1 \dots, t_d \in R \cup \{*\} \\ &\forall s \in S(INCI(a_i, s) \rightarrow s = s_1 \vee \dots \vee s = s_d) \\ &\wedge \forall t \in T(INCI(a_i, t) \rightarrow t = t_1 \vee \dots \vee t = t_d) \\ &\wedge PREF(a_i, s_1, \dots_d, t_1 \dots, t_d) \end{aligned}$$

This translation says that $S$ is preferred over $T$ when, reducing the resources to the neighborhood of $a_i$ we have that the PREF relation hold over those resources.

Note that this translation is of size O(d) as all the translations are of constant size and only the translation of $S \succeq_{a_i} T$ grows as the maximum degree d of the graph grows.

**Step 4: applying courcelle's theorem:** We note that with the above steps we can translate any game into $MSO[\sigma]$ with size of universe $O(m^2)$ and tree width at most $k+(k+1) \cdot d+1$ and any AIG formula $\phi$ into a formula of $\phi'$ $MSO[\sigma]$ with size $O(d \cdot |\phi|)$ . Then when we apply Courcelle's theorem to conclude that $O(g(|\phi|, d, k) \cdot m^2)$ for some computable function g. $\square$

## 4.2 Proof of Theorem 3

First we note that we had to alter the definition of Manipulation games in [Grandi *et al.*, 2021] quite a bit to make sure that it is possible to apply Courcelle's theorem. First of all we need to bound the length of the game because otherwise we had to include the natural numbers into the universe of the $\sigma$-structure. Furthermore we couldn't allow for any strategy as then the size of the universe would grow with O($|Act|^n$) which could not give us an fpt result. Even with these restrictions there are still a lot of technicial details that make applying Courcelle's theorem not that straightforward. One has to check, for example, when something is a valid history that follows from some strategy. With these things in mind we show how one can apply Courcelle's theorem to $\phi$-FMG.

**step 1: $\sigma$-structure** $\sigma$-signature consists of unary relation AGENT, EDGE, ACTION, TIME, OPINION, binary relation INCI and finally a $2n+3$-ary relation UPDATE. The idea behind UPDATE is that it encodes if a given opinion is the updated opinion of a certain agent after the agents played their action.

The universe should contain the agents, the opinion of each agent in each timestep t: $Ot = N \times (O \cup \{*\}) \times [T]$ and the actions an agent can perform in each timestep $Actt = N \times (Act \cup \{*\}) \times [T]$ as well as a place holder $*$. Note that we use $[T] = \{t \in \mathbb{N} | t \leq T\}$ We get:

$$A = N \cup E \cup Act \cup O \cup Ot \cup Actt \cup [T] \cup \{*\}$$

We define the relations over this universe as:

$$AGENT = N, EDGE = E,$$

$$ACTION = Act, TIME = [T]$$

$$INCI = \{(a_i, a_j)|(a_i, a_j) \in E\}, OPINION = O$$

Finally for UPDATE we have that for:

$act_0, \ldots act_n \in Act \cup \{*\}$ and $o_0, \ldots, o_n \in O \cup \{*\}$ :
$(a_i, o_{t+1}, (a_0, o_0, t) \ldots (a_n, o_n, t),$
$(a_0, act_0, t), \ldots (a_n, act_n, t)) \in UPDATE$
if and only if $r_{a_i}(\vec{Act}_{a_i}) = o_{t+1}$
and $\forall a_j \notin influence_G(a_i) : act_j = o_j = *$

In which $\vec{Act}_{a_i}$ are the actions restricted to actions of the agents that influence agent $a_i$. We add this constraint such that we keep the size of the UPDATE relation of size $O(n \cdot d \cdot T)$ while still making sure that UPDATE encodes the idea of updating the opinion of an agent.

**Bounding the treewidth** We construct a tree decomposition of $\sigma$-structure and argue about its treewidth. We know by assumption that G has treewidth at most k and degree d. We alter it to some tree-decomposition with a treewidth of size $O(k \cdot d \cdot T)$
Step (1) for each $e = \{a_i, a_j\}$ of G find a bag $\beta(w)$ that contains both $a_i$ and $a_j$ and introduce a new bag $\{e, a_i, a_j\}$ that we attach as a leaf to w. For each $o_t = (a_i, o, t) \in O_t$ in which $a_i \in \beta(w)$ we add a leaf that contains $\{o_t, a_i, o, t\}$ and for each $act_t = (a_i, act, t) \in Actt$ in which $a_i \in \beta(w)$ we add a leaf that contains $\{act_t, a_i, act, t\}$
Step (2) For each agent $a_i$ and every bag $\beta(w)$ of T such that $a_i \in \beta(w)$ add the set $\Gamma_G(a_i) \times O \times [T]$ and $\Gamma_G(a_i) \times Act \times [T]$ to $\beta(w)$.
Step (3) finally we add $*, O, Act$ and $[T]$ to every bag. This increases the treewidth $1 + 2 + 2 + T$.

Next we need to show this tree decomposition is a valid tree decomposition of $\mathcal{A}$ by showing that for this tree decomposition (TD 1) and (TD 2) hold.
(TD 1) first we note that we know by construction of the graph that each agent is represented. In step 1 we add the edges, in step 2 we add $Ot$ and $Actt$. We note that by the fact that the bags for each agent form a connected subtree we made sure by construction that the bags for each of the elements in $Ot$ and $Actt$ also form a connected subtree. Lastly we note that $Act$, $Ot$ , $[T]$ and $\{*\}$ are all represented in a bag and represent a connected subtree. that because we add $Act$, $O$ and $[T]$ to the bags we have that for each element in $Ot$ and $Actt$ it bags form a connected subtree. Therefore for this tree decomposition (TD 1) holds.

(TD 2) By construction we have that AGENT is represented in at least 1 bag. Then by step (1) we have that EDGE, INCI is represented in at least 1 bag. Furthermore in step 2 for each agent $a_i$ we added the actions and opinions of the neighborhood of $a_i$ to its bags hence we know that UPDATE is also represented. Lastly step 3 ensures that ACTION, TIME and OPINION is represented in at least 1 bag. Therefore TD (2) holds.

Lastly we look at the size of the tree decomposition. We note that step 1 does not increase the size and step 2 does also not increase the size. Step 3 increases the size with at most $(k + 1) \cdot d \cdot (2 + 2) \cdot T$ as each bag contains at most $k + 1$ agents. Finally step 4 increases the size with $5 + T$. Hence we get a treewidth of $O(k \cdot d \cdot T)$.

**Step 3: Translation of the formula's** First we want to express in $MSO[\sigma]$ the following: there exists a strategy for a group of agents A of size m that makes some formula $\phi$ true:

$$\phi' = \exists S_0 \ldots S_m : str(S_0), \ldots str(S_a) :$$
$$\forall H \in A : hist(S_0, a_0 \ldots, S_m, a_m, H) \rightarrow holds(H, \phi)$$

In which $S_0, \ldots S_m \subseteq Ot$ captures the idea that $S_i$ is a valid strategy for $a_i$, $hist(S_0, a_0 \ldots, S_m, a_m, H)$ checks if the given strategies lead to history H and finally $holds(H, \phi)$ checks if $\phi$ holds according to history $H$. We will continue with defining those functions.

Note that when defining $holds(H, \phi)$ we need to make sure that the $LTL_l$ $\phi$ is translated to an $MSO[\sigma]$-formula.

The definition of $holds(H, \phi)$ will be as follows:

$$\begin{aligned} holds(H_{\leq k}, o_{a_i}) &:= (a_i, o, k) \in H_{\leq k} \\ holds(H_{\leq k}, \neg\phi) &:= \neg holds(H_{\leq k}, \phi) \\ holds(H_{\leq k}, \phi_1 \wedge \phi_2) &:= holds(H_{\leq k}, \phi_1) \\ &\quad \wedge holds(H_{\leq k}, \phi_2) \\ holds(H_{\leq k}, \bigcirc\phi) &:= holds(H_{\leq k+1}, \phi) \\ holds(H_{\leq k}, \phi_1 \mathcal{U} \phi_2) &:= \exists k \geq k' : \\ &\quad H_{\leq k'} holds(H_{\leq k+1}, \phi_1) \\ &\quad \wedge \forall k'' : (k \leq k'' < k' : \\ &\quad \rightarrow holds(H_{\leq k''}, \phi_2) \end{aligned}$$

The function $str(S_i)$ encodes that $S_i$ represents a strategy, that is, it represents a valid 3-cnf formula with at most $T$ variables:

$$\begin{aligned} &(\forall(a, o, i) \in S : (OPINION(o) \vee o = *) \\ &\wedge \forall(a', o', i') \in S : i = i' \rightarrow o = o' \wedge a = a') \\ &\wedge \forall i \in [T] : \exists(a, o, i') \in S : i = i' \end{aligned}$$

Thus we see that each proposition that encodes a opinion is mapped to a location in the CNF formula and we can only have one opinion on each location.

The last function that we need to define is $hist(S_0, a_0, \ldots, S_m, a_m, H)$, in which we want to encode that H is a history that adheres to the strategies of agents $a_0, \ldots a_m$.

The idea is as follows:

1. Encode what it means that one set of opinions $O_{t+1}$ follows from the next given actions $A_t$ and the current set of opinion $O_t$ with a function $follows(O_{t+1}, A_t, O_t)$. We can do this with help of the UPDATE relation.

2. Then encode if $A_t$ is a valid set of possible actions given a list of strategies $S_0, \ldots, S_m$ for agents $a_0, \ldots, a_m$ and a set of opinions $O_t$ with the function $PosActions(S_0, \ldots, S_m, a_0, \ldots, a_m, O_t, A_t)$

3. With the help of two functions above we can check for a history if for each time step (1) the given actions adhere to the given strategies and (2) for each set of actions and opinions the next set of opinions actually follow from the current set of opinions. If both (1) and (2) holds then we know that we have a history that adheres to the strategies.

Note that we (probably) can encode $hist(S_0, a_0, \ldots, S_m, a_m, H)$ in O(m) as we can quantify over all the actions, opinions and timesteps.

Finally we can see that the formula $\phi'$ then is of size $O(m + |\phi|)$.

**Step 4: Applying Courcelle's theorem** We showed how to translate an $\phi$-FMG to a $\sigma$-structure $\mathcal{A}$ with universe of size $O(n \cdot d \cdot T)$ together with a $MSO[\sigma]$ formula $\phi'$. We now know that we can solve if $\mathcal{A} \models \phi'$ in time $O(g(|\phi'|, tw(\mathcal{A}))|A| + ||\mathcal{A}||) = O(f(k, d, T, m, |\phi|) \cdot n)$ for some computable functions $f$ and $g$. Hence we have showed that $\phi$-FMG problem is solvable in fpt with respect to the degree and treewidht of the graph, the number of timesteps T, the size of the coalition and the size of the goal formula $\phi$.

## 5 Limitations of Courcelle's Theorem

The beauty of Courcelle's theorem is that one does not have to find an algorithm, but a translation from the logic of a given problem to Monadic Second Order Logic to show that a problem is fpt. But Courcelle's theorem also has its drawbacks, for example, when looking at allocation of indivisible goods, we need to bound the number of resources an agent is interested in, otherwise applying Courcelle's theorem will not result in a fpt bound.

But there are some problems that need even more restrictions, as shown with the finite manipulation games. To apply Courcelle's theorem we not only had to bound the degree, and the treewidth of the graph of the game but we also had to make the game finite and only allowed for strategies that are representable in 3cnf form not containing more variables than the number of timesteps.

Clearly the restrictions to the degree, treewidth and the number of timesteps were needed. Furthermore we had to restrict the kind of strategies that are allowed, because we had to encode the strategies somehow. Allowing for any strategy would again lead to a to fast growing universe of our $\sigma$-structure as we would need to include aggregate functions with domain of size $O(2^n)$. For this reason it seems very unlikely that it is easy (or possible at all) to apply Courcelle's theorem to problems that are all about aggregate functions such as *preference aggregation* problems such as *opinion aggregation*, *voting rules* and *welfare aggregation* [List, 2013]

Lastly, sometimes there are no good ways to bound parameters for a problem. For example with the problem of budget allocation, a budget has to be allocated to 1 or more projects. If we would allow for partial budgets one has to introduce the rational numbers the the universe of the $\sigma$-structure. But even using each rational between 0 and 1 already leads to an universe of infinite size. It seems to use Courcelle's theorem one has to restrict the problem in ways that not seem very "natural". As in that maybe it would be possible to apply Courcelle's theorem but only on a strong distillation of the original problem one likes to solve.

## 6 Conclusion and Future Work

In this paper we looked at Courcelle's theorem and applied it to two social choice problems: *Allocation of Indivisible goods* and *Finite Manipulation games*. We constrained the problems in such a way that we could apply Courcelle's theorem and get fpt results. Furthermore we briefly looked at other ComSoC problems and argued why Courcelle's theorem seems not a good fit to (try to) show finite parameter tractability of these problems. The main problem is that the universe of the $\sigma$-universe would grow to fast to make use of Courcelle's theorem. At the moment constraining these problems in such a way that we could make use of Courcelle's theorem does not seem very "natural". Future work could include finding ways in constraining these problems in a sensible way and applying Courcelle's theorem to new ComSoC problems that, we are sure, will be discovered.


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## References


[Beviá, 1998] Carmen Beviá. Fair allocation in a general model with indivisible goods. *Review of Economic Design*, 3(3):195–213, 1998.

[Bouveret and Lang, 2008] Sylvain Bouveret and Jérôme Lang. Efficiency and envy-freeness in fair division of indivisible goods: Logical representation and complexity. *Journal of Artificial Intelligence Research*, 32:525–564, 2008.

[Bredereck and Elkind, 2017] Robert Bredereck and Edith Elkind. Manipulating opinion diffusion in social networks. In *IJCAI*, pages 894–900, 2017.

[Courcelle, 1986] Bruno Courcelle. On context-free sets of graphs and their monadic second-order theory. In *International Workshop on Graph Grammars and Their Application to Computer Science*, pages 133–146. Springer, 1986.

[Grandi *et al.*, 2021] Umberto Grandi, Emiliano Lorini, Arianna Novaro, and Laurent Perrussel. Games of influence. *Journal of Logic and Computation*, 2021.

[Kneis and Langer, 2009] Joachim Kneis and Alexander Langer. A practical approach to courcelle's theorem. *Electronic Notes in Theoretical Computer Science*, 251:65–81, 2009.

[List, 2013] Christian List. Social choice theory. 2013.

[Peters, 2016] Dominik Peters. Graphical hedonic games of bounded treewidth. In *Proceedings of the AAAI Conference on Artificial Intelligence*, volume 30, 2016.