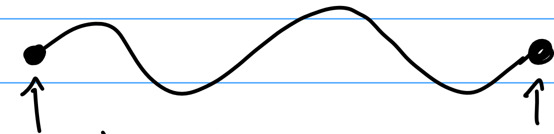
Problemas de condição de contorno

- CORDA:



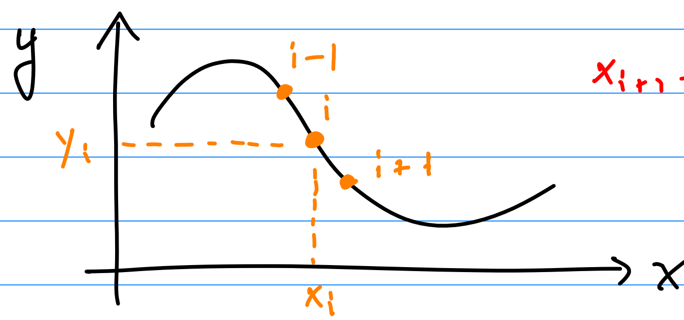
pontos fixos no contorno

- eq. schrödinger: $\psi(\pm L) = 0$

- MODOS NORMAIS: $\frac{\partial^2}{\partial x^2} y(x) = \lambda\, y(x)$

Abordagem numérica: diferenças finitas

Derivada discreta



$x_{i+1} - x_i = \Delta x$

Derivada p/ frente: $y'_i = \frac{y_{i+1} - y_i}{\Delta x}$

Derivada p/ trás: $y'_i = \frac{y_i - y_{i-1}}{\Delta x}$

Derivada simétrica: $y'_i = \frac{y_{i+1} - y_{i-1}}{2\Delta x}$

Série de Taylor

$f(z)$, série p/ $z \approx 0$

$$f(z) \approx f(0) + \frac{(z-0)}{1!} f'(0) + \frac{(z-0)^2}{2!} f''(0) + \mathcal{O}(z^3)$$

Agora: $y(x+\Delta x)$ p/ $\Delta x \approx 0$

Faça: $f(\Delta x) = y(x+\Delta x)$

$$f(\Delta x) \approx f(0) + \Delta x f'(0) + \frac{\Delta x^2}{2} f''(0) + \mathcal{O}(\Delta x^3)$$

$$y(x+\Delta x) \approx y(x) + \Delta x \left[\frac{\partial}{\partial \Delta x} y(x+\Delta x) \Big|_{\Delta x = 0} \right] + \\ + \frac{\Delta x^2}{2} \left[\frac{\partial^2}{\partial \Delta x^2} y(x+\Delta x) \Big|_{\Delta x = 0} \right] + \mathcal{O}(\Delta x^3)$$

* Regra da cadeia:

$$\frac{\partial}{\partial \Delta x} y(\underbrace{x+\Delta x}_{=z}) \Big|_{\Delta x = 0} = \overbrace{\frac{\partial z}{\partial \Delta x}}^{=1} \cdot \frac{\partial}{\partial z} y(z) \Big|_{\substack{\Delta x = 0 \\ z = x}} = y'(x)$$

Assim:

$$y(x+\Delta x) \approx y(x) + \Delta x\, y'(x) + \frac{\Delta x^2}{2} y''(x) + \mathcal{O}(\Delta x^3)$$

$$y(x-\Delta x) \approx y(x) - \Delta x\, y'(x) + \frac{\Delta x^2}{2} y''(x) + \mathcal{O}(\Delta x^3)$$

Derivada p/ frente:

$$y'(x) = \frac{y(x+\Delta x) - y(x)}{\Delta x} - \frac{\sigma(\Delta x^2)}{\Delta x}$$

$$\boxed{y'(x) = \frac{y(x+\Delta x) - y(x)}{\Delta x}} + \sigma(\Delta x)$$

Derivada simétrica

$$y'(x) = \frac{y(x+\Delta x) - y(x-\Delta x)}{2\Delta x} + \sigma(\Delta x^2)$$

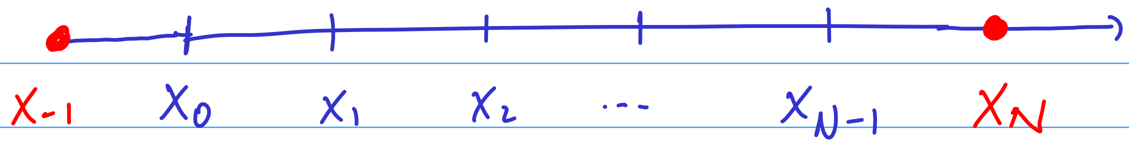
Segunda derivada simétrica

$$y''(x) = \frac{y(x+\Delta x) - 2y(x) + y(x-\Delta x)}{\Delta x^2} + \sigma(\Delta x^2)$$

na forma discreta: $\boxed{x+\Delta x \longrightarrow x_{i+1}}$

$$\boxed{y''_i = \frac{y_{i+1} - 2y_i + y_{i-1}}{\Delta x^2}} + \sigma(\Delta x^2)$$

Forma matricial e cond. contorno



Cond. contorno: caixa fechada

$$y(x_N) = y_N = 0$$

$$y(x_{-1}) = y_{-1} = 0$$

Forma matricial

$i=0$: $$y_0'' = \frac{y_1 - 2y_0 + \cancel{y_{-1}}}{\Delta x^2} = \frac{y_1 - 2y_0}{\Delta x^2}$$

$i=1$: $$y_1'' = \frac{y_2 - 2y_1 + y_0}{\Delta x^2}$$

i genérico: ver acima

$i = N-2$: $$y_{N-2}'' = \frac{y_{N-1} - 2y_{N-2} + y_{N-3}}{\Delta x^2}$$

$i = N-1$: $$y_{N-1}'' = \frac{\cancel{y_N} - 2y_{N-1} + y_{N-2}}{\Delta x^2}$$

$$\begin{pmatrix} y''_0 \\ y''_1 \\ \vdots \\ y''_i \\ \vdots \\ y''_{N-2} \\ y''_{N-1} \end{pmatrix} = \frac{1}{\Delta x^2} \begin{pmatrix} -2 & 1 & & & & & \\ 1 & -2 & 1 & & & & \\ & \ddots & \ddots & \ddots & & & \\ & & 1 & -2 & 1 & & \\ & & & \ddots & \ddots & \ddots & \\ & & & & 1 & -2 & 1 \\ & & & & & 1 & -2 \end{pmatrix} \begin{pmatrix} y_0 \\ y_1 \\ \vdots \\ y_i \\ \vdots \\ y_{N-2} \\ y_{N-1} \end{pmatrix}$$

$$\vec{y}'' = \vec{D}_2 \cdot \vec{y}$$

$$\underbrace{\text{matriz } \vec{D}_2}_{\text{Tridiagonal}} : \begin{cases} \text{Diagonal} : -2/\Delta x^2 \\ \underset{(\pm 1)}{\text{Subdiagonais}} : 1/\Delta x^2 \end{cases}$$

Eq. CORDA:

$$\frac{d^2}{dx^2} y(x,t) = \frac{1}{v^2} \frac{d^2}{dt^2} y(x,t)$$

$$y(x,t) = y(x)\, e^{i\omega t}$$

$$\frac{d^2}{dx^2} y(x) = -\frac{\omega^2}{v^2} y(x)$$

$$\boxed{-\frac{d^2}{dx^2} y(x) = \lambda\, y(x), \qquad \lambda = \frac{\omega^2}{v^2}}$$

modos normais: ω_n, $y_n(x)$

solução geral:

$$y(x,t) = \sum_n c_n\, e^{i\omega_n t}\, y_n(x)$$

$$y(x,0) = \sum_n c_n\, y_n(x) \; : \; \underline{\text{cond. inicial}}$$