

01. FUNCTIONS & LIMITS

Rules of Limits

- $\lim_{x \rightarrow a} (f \pm g)(x) = L \pm L'$
- $\lim_{x \rightarrow a} (fg)(x) = LL'$
- $\lim_{x \rightarrow a} \frac{f}{g}(x) = \frac{L}{L'}$, provided $L' \neq 0$
- $\lim_{x \rightarrow a} kf(x) = kL$ for any real number k .

02. DIFFERENTIATION

extreme values:

- $f'(x) = 0$
- $f'(x)$ does not exist
- end points of the domain of f

parametric differentiation: $\frac{d^2y}{dx^2} = \frac{d}{dx} \left(\frac{dy}{dx} \right) = \frac{\frac{d}{dt} \left(\frac{dy}{dx} \right)}{\frac{dx}{dt}}$

Differentiation Techniques

$f(x)$	$f'(x)$
$\tan x$	$\sec^2 x$
$\csc x$	$-\csc x \cot x$
$\sec x$	$\sec x \tan x$
$\cot x$	$-\csc^2 x$
$a^{f(x)}$	$\ln a \cdot f'(x) a^{f(x)}$
$\log_a f(x)$	$\log_a e \cdot \frac{f'(x)}{f(x)}$
$\sin^{-1} f(x)$	$\frac{f'(x)}{\sqrt{1-[f(x)]^2}}, f(x) < 1$
$\cos^{-1} f(x)$	$-\frac{f'(x)}{\sqrt{1-[f(x)]^2}}, f(x) < 1$
$\tan^{-1} f(x)$	$\frac{f'(x)}{1+[f(x)]^2}$
$\cot^{-1} f(x)$	$-\frac{f'(x)}{1+[f(x)]^2}$
$\sec^{-1} f(x)$	$\frac{f'(x)}{ f(x) \sqrt{[f(x)]^2-1}}$
$\csc^{-1} f(x)$	$-\frac{f'(x)}{ f(x) \sqrt{[f(x)]^2-1}}$

L'Hopital's Rule

$$\lim_{x \rightarrow x_0} \frac{f(x)}{g(x)} = \lim_{x \rightarrow x_0} \frac{f'(x)}{g'(x)}$$

- for indeterminate forms ($\frac{0}{0}$ or $\frac{\infty}{\infty}$), cannot directly substitute $x = a$.
- for other forms: convert to ($\frac{0}{0}$ or $\frac{\infty}{\infty}$) then apply L'Hopital's rule
- for exponents: use \ln , then sub into $e^{f(x)}$

03. INTEGRATION

Integration Techniques

$f(x)$	$\int f(x)$
$\tan x$	$\ln(\sec x), x < \frac{\pi}{2}$
$\cot x$	$\ln(\sin x), 0 < x < \pi$
$\csc x$	$-\ln(\csc x + \cot x), 0 < x < \pi$
$\sec x$	$\ln(\sec x + \tan x), x < \frac{\pi}{2}$
$\frac{1}{x^2+a^2}$	$\frac{1}{a} \tan^{-1} \left(\frac{x}{a} \right)$
$\frac{1}{\sqrt{a^2-x^2}}$	$\sin^{-1} \left(\frac{x}{a} \right), x < a$
$\frac{1}{x^2-a^2}$	$\frac{1}{2a} \ln \left(\frac{x-a}{x+a} \right), x > a$
$\frac{1}{a^2-x^2}$	$\frac{1}{2a} \ln \left(\frac{x+a}{x-a} \right), x < a$
a^x	$\frac{a^x}{\ln a}$

$$\frac{d}{dx} \int_a^x f(t) dt = f(x)$$

- indefinite integral** — the integral of the function without any limits
- antiderivative** — any function whose derivative will be the same as the original function

substitution: $\int_a^b f(g(x))g'(x)dx = \int_{g(a)}^{g(b)} f(u)du$

by parts: $\int uv' dx = uv - \int u'v dx$

Volume of Revolution

about x-axis:

- (with hollow area) $V = \pi \int_a^b [f(x)]^2 - [g(x)]^2 dx$
- (about line $y = k$) $V = \pi \int_a^b [f(x) - k]^2 dx$

04. SERIES

Geometric Series

sum (divergent)	sum (convergent)
$\frac{a(1-r^n)}{1-r}$	$\frac{a}{1-r}$

Power Series

power series about $x = 0$

$$\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + \dots$$

power series about $x = a$ (a is the centre of the power series) $\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1(x-a) + c_2(x-a)^2 + \dots$

Taylor series

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!} (x-a)^k$$

MacLaurin series:

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$$

Taylor polynomial of f at a :

$$P_n(x) = \sum_{k=0}^n \frac{f^{(k)}(a)}{k!} (x-a)^k$$

Radius of Convergence

power series converges where $\lim_{n \rightarrow \infty} \left| \frac{u_{n+1}}{u_n} \right| < 1$

converge at	R	$\lim_{n \rightarrow \infty} \left \frac{u_{n+1}}{u_n} \right $
$x = a$	0	∞
$(x-h, x+h)$	$h, \frac{1}{N}$	$N \cdot x-a $
all x	∞	0

MacLaurin Series

For $-\infty < x < \infty$

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

For $-1 < x < 1$

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$$

$$\frac{1}{1+x} = \sum_{n=0}^{\infty} (-1)^n x^n$$

$$\frac{1}{1+x^2} = \sum_{n=0}^{\infty} (-1)^n x^{2n}$$

$$\ln(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^n}{n}$$

$$\tan^{-1} x = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} x^{2n+1}$$

$$\frac{1}{(1+x)^2} = \sum_{n=1}^{\infty} (-1)^{n-1} n x^{n-1}$$

$$\frac{1}{(1-x)^2} = \sum_{n=1}^{\infty} n x^{n-1}$$

$$\frac{1}{(1-x)^3} = \frac{1}{2} \sum_{n=2}^{\infty} n(n-1) x^{n-2}$$

$$(1+x)^k = \sum_{n=0}^{\infty} \binom{k}{n} x^n = 1 + kx + \frac{k(k-1)}{2!} x^2 + \dots$$

Differentiation/Integration

For $f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n$ and $a-h < x < a+h$,

differentiation of power series:

$$f'(x) = \sum_{n=0}^{\infty} n c_n (x-a)^{n-1}$$

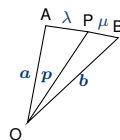
integration of power series:

$$\int f(x) dx = \sum_{n=0}^{\infty} c_n \frac{(x-a)^{n+1}}{n+1} + c$$

if $R = \infty$, $f(x)$ can be integrated to $\int_0^1 f(x) dx$

05. VECTORS

unit vector, $\hat{p} = \frac{p}{|p|}$



ratio theorem
 $p = \frac{\mu a + \lambda b}{\lambda + \mu}$

midpoint theorem
 $p = \frac{a+b}{2}$

Dot product

$$a \cdot b = |a||b| \cos \theta$$

$$\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \cdot \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = a_1 b_1 + a_2 b_2 + a_3 b_3$$

$a \perp b \Rightarrow a \cdot b = 0$	$a \cdot b > 0 : a$ is acute
$a \parallel b \Rightarrow a \cdot b = a b $	$a \cdot b < 0 : a$ is obtuse

Cross product

$$a \times b = |a||b| \sin \theta \hat{n}$$

$$\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \times \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = \begin{pmatrix} a_2 b_3 - a_3 b_2 \\ -(a_1 b_3 - a_3 b_1) \\ a_1 b_2 - a_2 b_1 \end{pmatrix}$$

$a \perp b \Rightarrow a \times b = a b $	$a \times b = -(b \times a)$
$a \parallel b \Rightarrow a \times b = 0$	$\lambda a \times \mu b = \lambda \mu (a \times b)$

Projection

$$\triangle ANO = \frac{1}{2} |\vec{OA} \times \vec{ON}|$$

$$|\vec{ON}| = |a \cdot \hat{b}| = \frac{|a \cdot b|}{|b|}$$

$$\vec{ON} = (a \cdot \hat{b}) \hat{b} = \frac{|a \cdot b|}{|b|^2} b$$

Planes

Equation of a Plane

\hat{n} is a perpendicular to the plane; A is a point on the plane.

- parametric: $r = a + \lambda b + \mu c$
- scalar product: $r \cdot n = a \cdot n$
- standard form: $r \cdot \hat{n} = d$
- cartesian: $ax + by + cz = p$

Length of projection of a on $n = |a \cdot \hat{n}| = \perp$ from O to π

Distance from a point to a plane

Shortest distance from a point $S(x_0, y_0, z_0)$ to a plane

$\Pi : ax + by + c = d$ is given by:

$$\frac{|ax_0 + by_0 + cz_0 - d|}{\sqrt{a^2 + b^2 + c^2}}$$

06. PARTIAL DIFFERENTIATION

Partial Derivatives

For $f(x, y)$,

first-order partial derivatives:

$$f_x = \frac{d}{dx} f(x, y) \quad \left| \quad f_y = \frac{d}{dy} f(x, y) \right.$$

second-order partial derivatives:

$$f_{xx} = (f_x)_x = \frac{d}{dx} f_x \quad f_{xy} = (f_x)_y = \frac{d}{dy} f_x$$

$$f_{yy} = (f_y)_y = \frac{d}{dy} f_y \quad f_{yx} = (f_y)_x = \frac{d}{dx} f_y$$

Chain Rule

For $z(t) = f(x(t), y(t))$,

$$\frac{dz}{dt} = \frac{\partial z}{\partial x} \frac{dx}{dt} + \frac{\partial z}{\partial y} \frac{dy}{dt}$$

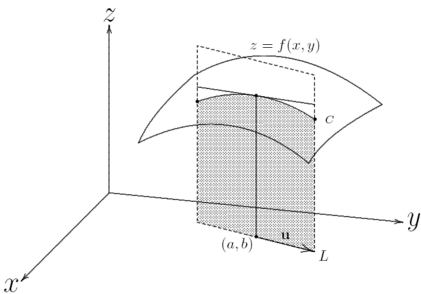
For $z(s, t) = f(x(s, t), y(s, t))$,

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x} \frac{\partial x}{\partial t} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial t}$$

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x} \frac{\partial x}{\partial s} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial s}$$

Directional Derivatives

The directional derivative of f at (a, b) in the direction of unit vector $\hat{u} = u_1\hat{i} + u_2\hat{j}$ is $D_u f(a, b) = f_x(a, b) \cdot u_1 + f_y(a, b) \cdot u_2$



- geometric meaning:** $D_u f(a, b)$ is the gradient of the tangent at (a, b) to curve C on a surface $z = f(x, y)$
 - rate of change of $f(x, y)$ at (a, b) in the direction of u

Gradient Vector

The **gradient** at $f(x, y)$ is the vector $\nabla f = f_x\hat{i} + f_y\hat{j}$

$D_u f(a, b) = \nabla f(a, b) \cdot \hat{u} = |\nabla f(a, b)| \cos \theta$

- f increases most rapidly in the direction $\nabla f(a, b)$
- f decreases most rapidly in the direction $-\nabla f(a, b)$
- largest possible value of $D_u f(a, b) = |\nabla f(a, b)|$
 - occurs in the same direction as $f_x(a, b)\hat{i} + f_y(a, b)\hat{j}$

Physical Meaning

Suppose a point p moves a small distance Δt along a unit vector \hat{u} to a new point q .

increment in f , $\Delta f \approx D_u f(p)(\Delta t)$

Maximum & Minimum Values

$f(x, y)$ has a **local maximum** at (a, b) if $f(x, y) \leq f(a, b)$ for all points (x, y) near (a, b) .

$f(x, y)$ has a **local minimum** at (a, b) if $f(x, y) \geq f(a, b)$ for all points (x, y) near (a, b) .

Critical Points

- $f_x(a, b)$ or $f_y(a, b)$ does not exist; OR
- $f_x(a, b) = 0$ and $f_y(a, b) = 0$
 - $f_x(0, b) \leq 0$ - maximum point *along the x axis*
 - $f_y(a, 0) \geq 0$ - minimum point *along the y axis*

Saddle Points

- $f_x(a, b) = 0, f_y(a, b) = 0$
- neither a local minimum nor a local maximum

Second Derivative Test

Let $f_x(a, b) = 0$ and $f_y(a, b) = 0$.
 $D = f_{xx}(a, b)f_{yy}(a, b) - f_{xy}(a, b)^2$

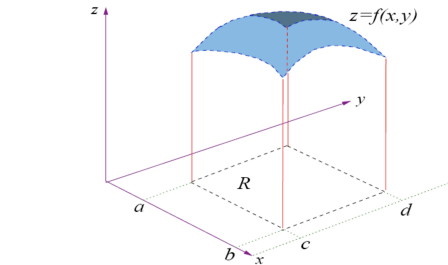
D	$f_{xx}(a, b)$	local
+	+	min
+	-	max
-	any	saddle point
0	any	no conclusion

07. DOUBLE INTEGRALS

Let ΔA_i be the area of R_i and (x_i, y_i) be a point on R_i . Let $f(x, y)$ be a function of two variables. The **double integral** of f over R is

$$\iint_R f(x, y) dA = \lim_{n \rightarrow \infty} \sum_{i=1}^n f(x_i, y_i) \Delta A_i$$

$\iint_R f(x, y) dA$ is the volume under the surface $z = f(x, y)$ and above the xy -plane over the region R .



Properties of Double Integrals

- $\iint_R (f(x, y) + g(x, y)) dA = \iint_R f(x, y) dA + \iint_R g(x, y) dA$
- $\iint_R c f(x, y) dA = c \iint_R f(x, y) dA$, where c is a constant
- If $f(x, y) \geq g(x, y)$ for all $(x, y) \in \mathbb{R}$, then $\iint_R f(x, y) dA \geq \iint_R g(x, y) dA$
- If $R = R_1 \cup R_2$, R_1 and R_2 do not overlap, then $\iint_R f(x, y) dA = \iint_{R_1} f(x, y) dA + \iint_{R_2} f(x, y) dA$
- The area of R , $A(R) = \iint_R dA = \iint_R 1 dA$
- If $m \leq f(x, y) \leq M$ for all $(x, y) \in R$, then $m A(R) \leq \iint_R f(x, y) dA \leq M A(R)$

Rectangular Regions

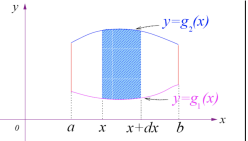
For a rectangular region R in the xy -plane, $a \leq x \leq b, c \leq y \leq d$

$$\iint_R f(x, y) dA = \int_c^d \left[\int_a^b f(x, y) dx \right] dy = \int_a^b \left[\int_c^d f(x, y) dy \right] dx$$

If $f(x, y) = g(x)h(y)$, then
$$\iint_R g(x)h(y) dA = \left(\int_a^b g(x) dx \right) \left(\int_c^d h(y) dy \right)$$

General Regions

Type A

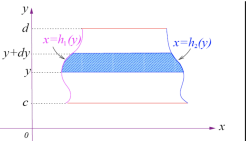


lower/upper bounds: $g_1(x) \leq y \leq g_2(x)$

left/right bounds: $a \leq x \leq b$

The region R is given by
$$\iint_R f(x, y) dA = \int_a^b \left[\int_{g_1(x)}^{g_2(x)} f(x, y) dy \right] dx$$

Type B

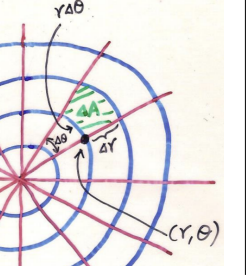


lower/upper bounds: $c \leq y \leq d$

left/right bounds: $h_1(y) \leq x \leq h_2(y)$

The region R is given by
$$\iint_R f(x, y) dA = \int_c^d \left[\int_{h_1(y)}^{h_2(y)} f(x, y) dx \right] dy$$

Polar Coordinates



$x = r \cos \theta$
 $y = r \sin \theta$
 $dx dy \Rightarrow r dr d\theta$

$\Delta A \approx (r \Delta \theta)(\Delta r) = r \Delta r \Delta \theta$

$\boxed{dA = r dr d\theta}$

The region R is given by $R: a \leq r \leq b, \alpha \leq \theta \leq \beta$

$$\iint_R f(x, y) dA = \int_\alpha^\beta \int_a^b f(r \cos \theta, r \sin \theta) r dr d\theta$$

Applications

Volume

Suppose D is a solid under the surface of $z = f(x, y)$ over a plane region R

Volume of $D = \iint_R f(x, y) dA$

Surface Area

For area S of that portion of the surface $z = f(x, y)$ that projects onto R ,

$$S = \iint_R \sqrt{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1} dA$$

08. ORDINARY DIFFERENTIAL EQUATIONS

- general solution:** solution containing arbitrary constants
- particular solution:** gives specific values to arbitrary constants
- the general solution of the n -th order DE will have n arbitrary constants

Separable Equations

A first-order DE is **separable** if it can be written in the form $M(x) - N(y)y' = 0$ or $M(x)dx = N(y)dy$

Reductions to Separable Form

form	change of variable
$y' = g\left(\frac{y}{x}\right)$	set $v = \frac{y}{x}$ $\Rightarrow y' = v + xv'$
$y' = f(ax + by + c)$ $\Rightarrow y' = \frac{ax + by + c}{\alpha x + \beta y + \gamma}$	set $v = ax + by$
$y' + P(x)y = Q(x)$	$R = e^{\int P dx}$ $\Rightarrow y = \frac{1}{R} \int RQ dx$
$y' + P(x)y = Q(x)y^n$	set $z = y^{1-n}$ $\Rightarrow y' = \frac{y^n}{1-n} z'$ $R = e^{\int P dx}$ $\Rightarrow y = \frac{1}{R} \int RQ dx$

Population Models

N - number; B - birth rate; t - time; D - death rate

Logistic Model $N = \frac{N_t = \infty}{1 + \left(\frac{N_t = \infty}{N_{t=0}} - 1\right)e^{-Bt}}$

Malthus Model $N(t) = N_0 e^{kt}$ where $k = B - D$

Common Scenarios: Uranium decays into Thorium

amount of uranium, $U(t) = U_0 e^{-k_U t}$
 $\frac{dU}{dt} = -k_U U$

amount of thorium, $T(t) = \frac{k_U U_0}{k_T - k_U} (e^{-k_U t} - e^{-k_T t})$
 $\frac{dT}{dt} = k_U U - k_T T$

decay rate constant, $k = \frac{\ln 2}{t_{1/2}}$

ratio of thorium to uranium, $\frac{T}{U} = \frac{k_U}{k_T - k_U} (1 - e^{-(k_T - k_U)t})$

Radioactive decay $Q(t) = Q_0 e^{-kt}$
 $k = \frac{\ln 2}{t_{1/2}}$

Cooling/Heating $\frac{dT}{dt} = k(T - T_{env})$
 $\frac{1}{T - T_{env}} dT = k dt$

Falling objects (N2L) Resistance = bv^2
 $m \frac{dv}{dt} = mg - bv^2$
Let $k = \sqrt{\frac{mg}{b}}$
 $\Rightarrow \frac{1}{v^2 - k^2} dv = -\frac{b}{m} dt$

Resistive medium Resistance = kv
 $m \frac{dv}{dt} = mg - kv$
 $v' + \frac{k}{m} v = g$ (linear)

Concentration of salt in liquid
Let R = rate of flow (in and out), Q = total amount of salt, V = total volume, C_{in} = concentration of inflow
Rate of flow, $\frac{dQ}{dt} = RC_{in} - \frac{R}{V} Q$
 $\Rightarrow Q' + \frac{R}{V} Q = RC_{in}$