ST2334 Summary Notes

AY23/24 Sem 1, github.com/gerteck

1. Basic Probability Concepts

- **Sample Space:** S All possible outcomes of stat. expt.
- **Null Event**: Event that contains no element, empty set, $\varnothing$
- **Axioms of Probability**:
  For any event X, $0 \leq P(X) \leq 1$. $P(S) = 1$.
  If $A \cap B = \emptyset$ (Mut Excl), $P(A \cup B) = P(A) + P(B)$.
- Finite sample space with equally likely outcomes: $P(A)$ = ($\frac{\#samplepointsA}{\#totalsamplepointsS}$). (e.g. birthday problem)

Event Operation & Relationships

- **Event Operations:** Union, Intersection, Complement.
- **Event Relationships:** Contained: $A \subset B$
  Equivalence: $A \subset B$ with $A \supset B \rightarrow A = B$
  Mutually Exclusive: $A \cap B = \emptyset$.
- **De Morgan's Law:** $(A \cup B)' = A' \cap B'$ and $(A \cap B)' = A' \cup B'$

Counting Methods

- Multiplication Principle: (Sequential Events)
- Addition Principle: (Pairwise Disjoin sets)
- **Permutation**: ${}_nP_r = \frac{n!}{(n-r)!}$
- **Combination**: $\binom{n}{r} = \frac{n!}{(n-r)!r!}$

Conditional Probability

- Understand conditional as reduced sample space.

$$P(B|A) = \frac{P(B \cap A)}{P(A)} = \frac{P(A|B)P(B)}{P(A)}$$

Independence

$A \perp B \leftrightarrow P(A \cap B) = P(A)P(B)$
$A \perp B \leftrightarrow P(A|B) = P(A)$

Law of Total Probability

- **Partition:** If $A_1, \cdots, A_n$ mutually exclusive, $\bigcup_{i=1}^n A_i = S$, then $A_1, \cdots, A_n$ are partitions.
- If $A_1, \cdots, A_n$ are partitions of S, then for any event B:

$$P(B) = \sum_{i=1}^{n} P(B \cap A_i) = \sum_{i=1}^{n} P(B|A_i)P(A_i)$$

Bayes' Theorem

Let $A_1, \cdots, A_n$ be partitions of S. For any event B:

$$P(A_k|B) = \frac{P(B|A_k)P(A_k)}{\sum_{i=1}^n P(B|A_k)P(A_i)}$$

For when $n = 2$, $\{A, A'\}$ becomes a partition of S.

$$P(A|B) = \frac{P(A)P(B|A))}{P(A)P(B|A) + P(A')P(B|A')}$$

2. Random Variables

A function X, which assigns a real number to every s $\in$ S is called a random variable.
- **Range space**: $Rx = \{x|x = X(s), s \in S\}$
- Likewise, the set X $\in$ A, for A being a subset of R, is also a subset of $S : s \in S : X(s) \in A$.

Probability Distribution

Two main types of RV used in practice: discrete and continuous.
- Probability assigned to each possible X
- Given RV X with range of R_x:
  Discrete: Numbers in R_x are finite or countable
  Continuous: R_x is interval

(Discrete) Probability Mass Function $f(x)$:

$$f(x) \begin{cases} P(X = x), & \text{for } x \in R_X \\ 0, & \text{for } x \notin R_X \end{cases}$$

1. $f(x_i) = P(X = x_i) \geq 0$ for $x_i \in R_x$
2. $f(x_i) = 0$ for $x_i \notin R_x$
3. $\sum_{i=1}^{\infty} f(x_i) = 1$ (PSum = 1)
4. $\forall B \subseteq \mathbb{R}, P(X \in B) = \sum_{x_i \in B \cap R_x} f(x_i)$

(Continuous) Probability Density Function $f(x)$:

- Given R_x is interval. Quantifies probability that X is in some range.
- $p.f.$ must satisfy:
  1. $f(x) \geq 0$, $f(x) = 0$ for $x \notin R_x$
  2. No need $f(x) \leq 1$ (Concerned with area)
  3. $\int_{R_x} f(x)dx = 1$ (Integration over R_X = 1)
  4. $\forall a, b$ s.t. $a \leq b, P(a \leq X \leq b) = \int_a^b f(x)dx$
- **Note:** $P(X = x_0) = \int_{x_0}^{x_0} f(x)dx = 0$
- Hence, to check if a function is a pdf,
  1. $f(x) \geq 0$ for $x \in R_x$, $f(x) = 0$ for $x \notin R_x$
  2. $\int_{R_x} f(x)dx = 1$.

Cumulative Distribution Function

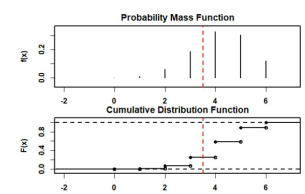
Describes distribution of a RV X: cumulative distribution function (cdf), applicable for discrete or continuous RV.

$$F(x) = P(X \leq x)$$

$F(x)$ is non-decreasing and $0 \leq F(x) \leq 1$
- Probability fn & cumulative distribution fn have one-to-one correspondence. For any probability fn given, the cdf is uniquely determined, vice versa.

CDF Discrete RV: Step Function $F(x)$

$$F(x) = \sum_{t \in R_x; t \leq x} f(t)$$

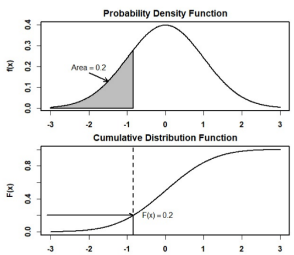


- $P(a \leq X \leq b) = P(X \leq b) - P(X < a)$
- $P(a \leq X \leq b) = F(b) - F(a-)$
- $P(a \leq X \leq b) = F(b) - \lim_{x \to a^-} F(x)$
- $0 \leq f(x) \leq 1$
- c.d.f has to be **right continuous** (• —)

CDF Continuous RV: $F(x)$

$$F(x) = \int_{-\infty}^{x} f(t)dt$$

$$impt : f(x) = \frac{d(F(x))}{dx}$$



- $P(a \le X \le b) = P(a < X < b) = F(b) - F(a)$
- $0 \le f(x)$.
  e.g. $f(x) = 3x^2$ is a valid $p.f.$ since $\int_{R_x} f(x)dx = 1$

Expectation μ & Variance σ

Expectation of Random Variable: μ

- **Mean of discrete RV**:

$$\mu = E(X) = \sum_{x \in R_x} x_i f(x_i)$$

- **E.g.**: X discrete RV with p.m.f. $f(x)$ and range R_X
  $\mu = E(g(x)) = \sum_{x \in R_x} g(x)f(x)$

- **Mean of continuous RV**:

$$\mu = E(X) = \int_{x \in R_x} xf(x)dx$$

- **E.g.**: X continuous RV with p.d.f. $f(x)$ and range R_X
  $\mu = E(g(x)) = \int_{x \in R_x} g(x)f(x)dx$
- **Properties of Expectation:**
- $E(aX + b) = aE(X) + b$
- Linearity of expectation: $E(X + Y) = E(X) + E(Y)$

Variance of Random Variable: σ

$$\sigma_X^2 = V(X) = E[(X - \mu_X)^2]$$

- **Variance of discrete RV**:

$$V(X) = \sum_{x \in R_x} (x - \mu_X)^2 f(x)$$

- **Variance of continuous RV**:

$$V(X) = \int_{x \in R_x} (x - \mu_X)^2 f(x)dx$$

- $V(X) \ge 0$ and $V(X) = 0$ when X is a constant
- $V(aX + b) = a^2 V(X)$
- **alt. form:** $V(X) = E(X^2) - (E(X))^2$
- **Standard Deviation:** $\sigma_X = \sqrt{V(X)}$

3. Joint Distributions

- Consider more than 1 RV simultaneously,
- Given sample space S. Let X and Y be functions mapping $s \in S \to \mathbb{R}$: (X, Y) is 2D random vector.

  Range spc: $R_{X,Y} = \{(x, y) | x = X(s), y = Y(s), s \in S\}$
- **Discrete 2D RV:**
  \# of possible values of $(X(s), Y(s))$ finite / countable
- **Continuous 2D RV**:
  \# of possible values of $(X(s), Y(s))$ assume any value in some region of the Euclidean space $\mathbb{R}^2$
- If both X and Y are discrete/continuous, then (X, Y) is discrete/continuous respectively.

Joint Probability Function

- **Joint Probability (mass) function, 2D discrete RV**:

$$f_{X,Y}(x, y) = P(X = x, Y = y)$$

  - $f_{X,Y}(x, y) \ge 0$ for any $(x, y) \in R_{X,Y}$
  - $f_{X,Y}(x, y) = 0$ for any $(x, y) \notin R_{X,Y}$
  - $\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} P(X = x_i, Y = y_i) = 1$
  - Let $A \subseteq R_{X,Y}$.
    $P((X, Y) \in A) = \sum\sum_{(x,y) \in A} f_{X,Y}(x, y)$

- **Joint Probability (density) function, 2D cont. RV**:

$$P(a \le X \le b, c \le Y \le d) = \int_a^b \int_c^d f_{X,Y}(x, y)dydx$$

  - $f_{X,Y}(x, y) \ge 0$ for any $(x, y) \in R_{X,Y}$
  - $f_{X,Y}(x, y) = 0$ for any $(x, y) \notin R_{X,Y}$
  - $\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_{X,Y}(x, y)dxdy = 1$
    or equivalently:
  - $\int\int_{(x,y) \in R_{X,Y}} f_{X,Y}(x, y)dxdy = 1$

Marginal Probability Function

Marginal distribution of X is individual distribution of X, ignoring the value of Y. "Projection" of 2D function $f_{X,Y}(x, y)$ to 1D function.

Let (X, Y) be 2D RV with joint probability function $f_{X,Y}(x, y)$:

If Y is **discrete**, $f_X(x) = \sum_y f_{X,Y}(x, y)$

If Y is **continuous**, $f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x, y)dy$

- $f_Y(y)$ defined similarly
- $f_X(x)$ is a $p.f.$, satisfies all properties of prob. fn.

Conditional Distribution

Let (X, Y) be 2D RV with joint probability function $f_{X,Y}(x, y)$. Then $\forall x$ s.t. $f_X(x) > 0$: (X marg prob fn.)
Conditional probability function of Y given X = x:

$$f_{Y|X}(y|x) = \frac{f_{X,Y}(x, y)}{f_X(x)}$$

- Intuition: Distribution of Y given $X = x$
- Only defined for x s.t. $f_X(x) > 0$
- $f_{Y|X}(y|x)$ is a $p.f.$ if we fix x, satisfies prop. of prob.fn.
- But, $f_{Y|X}(y|x)$ is not a $p.f.$ for x: No need for sum / integral over x = 1. Hence,
  If $f_X(x) > 0$: $f_{X,Y}(x, y) = f_X(x)f_{Y|X}(y|x)$
  If $f_Y(y) > 0$: $f_{X,Y}(x, y) = f_Y(y)f_{X|Y}(x|y)$
- **Probability Y $\le y$, Average Y given $X = x$**
- $P(Y \le y | X = x) = \int_{-\infty}^{y} f_{Y|X}(y|x)dy$
- $E(Y | X = x) = \int_{-\infty}^{\infty} y f_{Y|X}(y|x)dy$

Independent Random Variables

$$X \perp Y : \forall x, y, f_{X,Y}(x, y) = f_X(x)f_Y(y)$$

- Necessary condition: $R_{X,Y}$ must be a product space. i.e. $R_{X,Y} = \{(x, y)|x \in R_X; y \in R_y\} = R_X \times R_Y$ Else, dependent.

Properties of Independent RV

For X, Y independent RV:

- If $A, B \subseteq \mathbb{R}$, then events $X \in A$ and $Y \in B$ are independent:

$$P(X \in A; Y \in B) = P(X \in A)P(Y \in B)$$

$$P(X \leq x; Y \leq y) = P(X \leq x)P(Y \leq y)$$

- Then, $g_1(X)$ and $g_2(Y)$ are **independent**, for arbitrary g.
- **Conditional distribution** given Independence:

$$f_X(x) > 0 \rightarrow f_{Y|X}(y|x) = f_Y(y)$$

$$f_Y(y) > 0 \rightarrow f_{X|Y}(x|y) = f_X(x)$$

To check independence

1. $R_{X,Y}$ is a product space. i.e. R_X does not depend on Y, vice versa. (e.g. $0 < y < x$ is NOT a product space)
2. Additionally, $f_{X,Y}(x, y)$ **= some** $C * g_1(x)g_2(y)$ **where** g_1 **depends on** x **only and** g_2 **depends on** y **only.**

Marginal Distribution under Independence

- Since, $f_{X,Y}(x, y) = f_X(x)f_Y(y)$ for independent RV, we derive marginal distribution by standardising $g_1(x)$ and $g_2(y)$.
- For discrete: $f_X(x) = \frac{g_1(x)}{\sum_{t \in R_X} g_1(t)}$
- For continuous: $f_X(x) = \frac{g_1(x)}{\int_{t \in R_X} g_1(t)dt}$

Expectation of a Random Vector

Given **2 variable function** $g(x, y)$**:**

- If (X, Y) is discrete:

$$E(g(X, Y)) = \sum_x \sum_y g(x, y)f_{X,Y}(x, y)$$

- If (X, Y) is continuous:

$$E(g(X, Y)) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(x, y)f_{X,Y}(x, y)dydx$$

- If $X \perp Y$:

$$E(XY) = E(X)E(Y)$$

- (If $X \perp Y$, follows that $cov(X, Y) = 0$). However, converse not always true.

Covariance

- For random variables X, Y:

$$cov(X, Y) = E((X - E(X))(Y - E(Y)))$$

- If (X, Y) both **discrete**:

$$cov(X, Y) = \sum_x \sum_y (x - \mu_X)(y - \mu_Y)f_{X,Y}(x, y)$$

- If (X, Y) both **continuous**:

$$cov(X, Y) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (x-\mu_X)(y-\mu_Y)f_{X,Y}(x, y)dxdy$$

- **Alt:** $cov(X, Y) = E(XY) - E(X)E(Y)$
- **Hence, for**$X \perp Y \rightarrow cov(X, Y) = 0$. (However, converse not always true).
- **Properties of covariance:**
- $cov(aX + b, cY + d) = (ac)cov(X, Y)$
- $V(aX + bY) = a^2V(X) + b^2V(Y) + 2ab * cov(X, Y)$
- $X \perp Y \rightarrow V(X \pm Y) = V(X) + V(Y)$

4.1 Special Probability Distributions

- **Discrete Distributions**: Study whole classes of discrete RVs that arise frequently in applications.

Discrete Uniform Distribution

- If X has values $x_1, x_2, \cdots, x_k$ with **equal probability**

$$f(x) \begin{cases} \frac{1}{k}, & \text{for } x = x_1, x_2, ..., x_k \\ 0, & \text{otherwise} \end{cases}$$

- **Expectation**:
$\mu_X = E(X) = \sum_{i=1}^{k} x_i f_X(x_i) = \frac{1}{k}\sum x_i$
- **Variance**:
$\sigma_X^2 = V(X) = E(X^2) - (E(X))^2 = \frac{1}{k}\sum x_i^2 - \mu_X^2$

Bernoulli, $Ber(p)$

- **Bernoulli Trial**: Random experiment has 2 possible outcomes (success and failure).
- **Bernoulli Random Variable**: X represents number of success in a single Bernoulli Trial. X has only two possible values: 1, or 0.
- **Probability mass function:** Let $0 \le p \le 1$ be the probability of success in Bernoulli trial

$$f_X(x) = P(X = x) = \begin{cases} p & x = 1 \\ 1 - p & x = 0 \\ 0 & otherwise \end{cases}$$

- $f_X(x) = p^x(1-p)^{1-x}$ for $x = 0$ or 1
- Bernoulli distr. is case of binomial distr. where $n = 1$.
- **Notation:** $X \sim Ber(p)$ and $q = 1 - p$

$$f_x(1) = p, f_x(0) = q$$

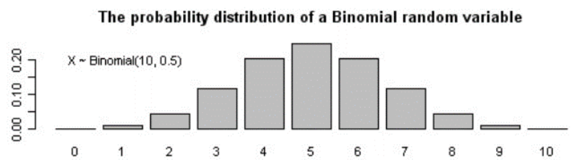
- **Expectation:** $\mu_X = E(X) = p$
- **Variance:** $\sigma_X^2 = V(X) = p(1-p)$
- **Bernoulli Process**: Sequence of repeatedly performed independent and identical Ber. trials.
- Generates sequence of **independent and identically distributed (i.i.d.)** Ber. RVs: $X_1, X_2, \cdots$

Binomial Distribution, $B(n,p)$

- **Binomial RV:** counts **number of successes** in n trials in a Ber. process.
- Given n independent trials with each trial having same probability p of success:

$$P(X = x) = \binom{n}{x} p^x (1-p)^{n-x}$$

- **Notation**: $X \sim B(n,p)$
- $E(X) = np$, $V(X) = np(1-p)$



Negative Binomial Distribution, $NB(k,p)$ (k^{th} success)

- Let X = no. of independent identical distributed Bernoulli(p) trials until k^{th} **success** occurs.
- **Probability mass function of X**:

$$P(X = x) = \binom{x-1}{k-1} p^k (1-p)^{x-k}$$

- **Notation**: $X \sim NB(k,p)$
- $E(X) = \frac{k}{p}$ and $V(X) = \frac{(1-p)k}{p^2}$

Geometric Distribution, $G(p)$ (till 1^{st} success)

- Let X = no. of i.i.d. Bernoulli(p) trials until 1st success occurs.

$$P(X = x) = p(1-p)^{x-1}$$

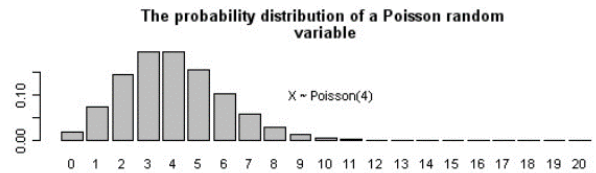
- **Notation**: $X \sim G(p)$
- $E(X) = \frac{1}{p}$ and $V(X) = \frac{1-p}{p^2}$

Poisson Distribution

- **Poisson RV**: Denotes number of events occuring in **fixed period of time or fixed region**, k = no. of occurences.

$$P(X = k) = \frac{e^{-\lambda}\lambda^k}{k!}$$

- **Notation**: $X \sim Poisson(\lambda)$ where $\lambda > 0$ is expected number of occurrences during given period/region
- $E(X) = \lambda$ and $V(X) = \lambda$



The number of infections X in a hospital each week has been shown to follow a Poisson distribution with a mean of 3 infections per week. What is the probability that

(a) there is *no* infection for a week?
(b) there are *less than* 4 infections for a week?

We are given than $X \sim$ Poisson(3). Then required probabilities are

(a) $P(X = 0) = e^{-3}$.
(b) $P(X \le 3) = e^{-3}\left(1 + 3 + \frac{3^2}{2!} + \frac{3^3}{3!}\right)$.

Poisson Process

- Continuous time process, count number of occurrences within some interval of time. (given **rate** α)
- Properties of **Poisson process with rate parameter** α:
  - Expected no. of occurrences in interval length T: αT
  - No simultaneous occurrences, and no. of occurrences in disjoint intervals independent.
- **Number of occurrences in any interval** T of Poisson process follows $Poisson(\alpha T)$ distribution.
(**Apply** $X\ Poisson(\alpha T)$ **directly)**

Poisson Approximation of Binomial Distribution

- Let $X \sim B(n,p)$. Suppose $n \to \infty$ and $p \to 0$ s.t. $\lambda = np$ remains constant.
- Then, approximately, $X \sim Poisson(\lambda)$.

$$\lim_{p \to 0; n \to \infty} P(X = x) = \frac{e^{-np}(np)^x}{x!}$$

- Approximation is good when ($n \ge 20$ and $p \le 0.05$), or ($n \ge 100$ and $np \le 10$)

4.2 Special Probability Distributions

- **Continuous Distributions**: Many "natural" RVs whose set of possible values **uncountable**. Model with classes of continuous random variables.

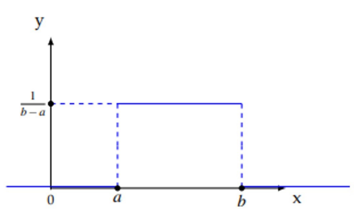
Continuous Uniform Distribution, $U(a, b)$

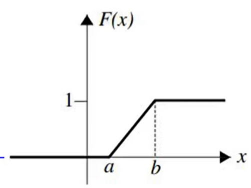
RV X follows uniform distribution over interval (a, b) if $p.d.f.$ given by:

$$f_X(x) = \begin{cases} \frac{1}{b-a} & a \leq x \leq b \\ 0 & otherwise \end{cases}$$

- **Notation**: $X \sim U(a, b)$
- $E(X) = \frac{a+b}{2}$ and $V(X) = \frac{(b-a)^2}{12}$ (derive by integration).
- **Cumulative distr. func.** $c.d.f.$ is given by:

$$F_X(x) = \begin{cases} 0 & x < a \\ \frac{x-a}{b-a} & a \leq x \leq b \\ 1 & x > b \end{cases}$$





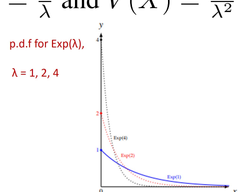
Exponential Distribution, $Exp(\lambda)$

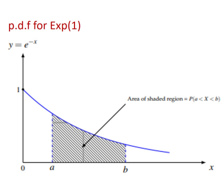
- Continuous counterpart to **geometric distribution**.

X follows exponential distribution, with parameter $\lambda > 0$ if $p.f.$ is given by:

$$f_x(x) = \begin{cases} \lambda e^{-\lambda x} & x \geq 0 \\ 0 & x < 0 \end{cases}$$

- **Notation**: $X \sim Exp(\lambda)$
- $E(X) = \frac{1}{\lambda}$ and $V(X) = \frac{1}{\lambda^2}$





- We can **derive λ from mean / expectation of X**, since $E(X) = \frac{1}{\lambda}$.
- $c.d.f.$ is given by:

$$F_X(x) = P(X \leq x) = \begin{cases} 1 - e^{-\lambda x} & x > 0 \\ 0 & x \leq 0 \end{cases}$$

- Additionally, $P(X > x) = e^{-\lambda x}$, for $x > 0$.
- **Exponential distribution "Memoryless"**: Suppose X has exponential distribution with $\lambda > 0$. Then for any positive numbers s and t, we have:

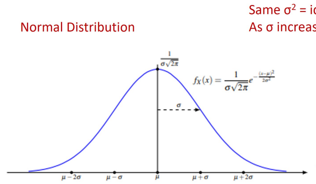
$$P(X > s + t | X > s) = P(X > t)$$

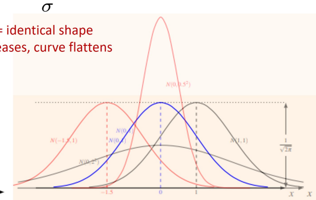
Normal Distribution, $N(\mu, \sigma^2)$

X said to follow normal distribution with mean μ and variance σ^2 if $p.f.$ given by:

$$f_X(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-(x-\mu)^2/(2\sigma^2)}$$

- **Notation**: $X \sim N(\mu, \sigma^2)$
- $E(X) = \mu$ and $V(X) = \sigma^2$
- $p.f.$ is **bell-shaped curve and symmetric** about $x = \mu$
- Total area under curve is 1
- 2 normal curves are identical in shape if they have same σ^2. They differ in location by $\mu_1 - \mu_2$.
- As σ increases, curve becomes more spread out
- If $X \sigma N(\mu, \sigma^2)$ and let $Z = \frac{X-\mu}{\sigma}$





Standardized Normal Distribution, $Z = N(0, 1)$

If $X \sim N(\mu, \sigma^2)$, then $Z \sim N(0, 1)$:

$$Z = \frac{X - \mu}{\sigma}$$

- $E(Z) = 0$ and $V(Z) = 1$
- $p.f$ **of** Z **is given by**:

$$\phi(z) = f_Z(z) = \frac{1}{\sqrt{2\pi}} e^{-z^2/2}$$

- **Standardizing normal distribution** allows us to use tables to find probabilities:
- For $X \sim N(\mu, \sigma^2)$, compute $P(x_1 < X < x_2)$ by standardization:

$$x_1 < X < x_2 \leftrightarrow \frac{x_1 - \mu}{\sigma} < \frac{X - \mu}{\sigma} < \frac{x_2 - \mu}{\sigma}$$

- Then, $P(z_1 < Z < z_2)$, **use** $f_Z(z)$ **table to calculate**.
- **Cumulative d.f. of standard Normal**:

$$\Phi(z) = F_Z(z) = \int_{-\infty}^{z} f_Z(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-t^2/2} dt$$

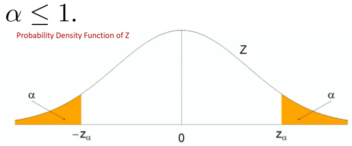
- $P(Z \geq 0) = P(Z \leq 0) = \phi(0) = 0.5$
- For any z, $\Phi(z) = P(Z \leq z) = P(Z \geq -z) = 1 - \phi(-z)$
- $-Z \sim N(0, 1)$
- If $Z \sim N(0, 1)$, then $\sigma Z + \mu \sim N(\mu, \sigma^2)$

Quantile

- **Upper Quantile:** x_α that satisfies:

$$P(X \geq x_\alpha) = \alpha$$

- where $0 \leq \alpha \leq 1$.



e.g. The 0.05th (upper) quantile of $Z \sim N(0, 1)$ is 1.645, i.e. $z_{0.05} = 1.645$.

- $P(Z \geq z_\alpha) = P(Z \leq -z_\alpha) = \alpha$
- Upper z_α = Lower $z_{1-\alpha}$

Normal Approximation to Binomial Distribution

Let $X \sim B(n, p)$, then as $n \rightarrow \infty$:

$$Z = \frac{X - E(X)}{\sqrt{V(X)}} = \frac{X - np}{\sqrt{np(1-p)}} \sim N(0, 1)$$

- Approximation is good when $np > 5$ and $n(1 - p) > 5$

5. Sampling, Sampling Distributions