# MA1521

AY22/23 Sem 2

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## 01. FUNCTIONS & LIMITS

### Rules of Limits

1. $\lim_{x\to a}(f \pm g)(x) = L \pm L'$
2. $\lim_{x\to a}(fg)(x) = LL'$
3. $\lim_{x\to a}\frac{f}{g}(x) = \frac{L}{L'}$, provided $L' \neq 0$
4. $\lim_{x\to a} kf(x) = kL$ for any real number $k$.

## 02. DIFFERENTIATION

extreme values:

- $f'(x) = 0$
- $f'(x)$ does not exist
- end points of the domain of $f$

parametric differentiaton: $\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}}$

### Differentiation Techniques

| $f(x)$ | $f'(x)$ |
|---|---|
| $\tan x$ | $\sec^2 x$ |
| $\csc x$ | $-\csc x \cot x$ |
| $\sec x$ | $\sec x \tan x$ |
| $\cot x$ | $-\csc^2 x$ |
| $a^{f(x)}$ | $\ln a \cdot f'(x)a^{f(x)}$ |
| $\log_a f(x)$ | $\log_a e \cdot \frac{f'(x)}{f(x)}$ |
| $\sin^{-1} f(x)$ | $\frac{f'(x)}{\sqrt{1-[f(x)]^2}}$, $\lvert f(x)\rvert<1$ |
| $\cos^{-1} f(x)$ | $-\frac{f'(x)}{\sqrt{1-[f(x)]^2}}$, $\lvert f(x)\rvert<1$ |
| $\tan^{-1} f(x)$ | $\frac{f'(x)}{1+[f(x)]^2}$ |
| $\cot^{-1} f(x)$ | $-\frac{f'(x)}{1+[f(x)]^2}$ |
| $\sec^{-1} f(x)$ | $\frac{f'(x)}{\lvert f(x)\rvert\sqrt{[f(x)]^2-1}}$ |
| $\csc^{-1} f(x)$ | $-\frac{f'(x)}{\lvert f(x)\rvert\sqrt{[f(x)]^2-1}}$ |

### L'Hopital's Rule

$$\lim_{x\to x_0}\frac{f(x)}{g(x)} = \lim_{x\to x_0}\frac{f'(x)}{g'(x)}$$

- for indeterminate forms ($\frac{0}{0}$ or $\frac{\infty}{\infty}$), cannot directly substitute $x = a$.
- for other forms: convert to ($\frac{0}{0}$ or $\frac{\infty}{\infty}$) then apply L'Hopital's rule
- for exponents: use $\ln$, then sub into $e^{f(x)}$

## 03. INTEGRATION

### Integration Techniques

| $f(x)$ | $\int f(x)$ |
|---|---|
| $\tan x$ | $\ln(\sec x)$, $\lvert x\rvert < \frac{\pi}{2}$ |
| $\cot x$ | $\ln(\sin x)$, $0 < x < \pi$ |
| $\csc x$ | $-\ln(\csc x + \cot x)$, $0 < x < \pi$ |
| $\sec x$ | $\ln(\sec x + \tan x)$, $\lvert x\rvert < \frac{\pi}{2}$ |
| $\frac{1}{x^2+a^2}$ | $\frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)$ |
| $\frac{1}{\sqrt{a^2-x^2}}$ | $\sin^{-1}\left(\frac{x}{a}\right)$, $\lvert x\rvert < a$ |
| $\frac{1}{x^2-a^2}$ | $\frac{1}{2a}\ln\left(\frac{x-a}{x+a}\right)$, $x > a$ |
| $\frac{1}{a^2-x^2}$ | $\frac{1}{2a}\ln\left(\frac{x+a}{x-a}\right)$, $x < a$ |
| $a^x$ | $\frac{a^x}{\ln a}$ |

19. $\int \frac{1}{\sqrt{(x+b)^2+a^2}}\,dx = \ln\left|(x+b)+\sqrt{(x+b)^2+a^2}\right| + C$

20. $\int \frac{1}{\sqrt{(x+b)^2-a^2}}\,dx = \ln\left|(x+b)+\sqrt{(x+b)^2-a^2}\right| + C$

21. $\int \sqrt{a^2-x^2}\,dx = \frac{x}{2}\sqrt{a^2-x^2}+\frac{a^2}{2}\sin^{-1}\frac{x}{a}+C$

22. $\int \sqrt{x^2-a^2}\,dx = \frac{x}{2}\sqrt{x^2-a^2}-\frac{a^2}{2}\ln\left|x+\sqrt{x^2-a^2}\right|+C$

$$\frac{d}{dx}\int_a^x f(t)dt = f(x)$$

substitution: $\int_a^b f(g(x))g'(x)dx = \int_{g(a)}^{g(b)} f(u)du$

by parts: $\int uv'\,dx = uv - \int u'v\,dx$

### Volume of Revolution

about x-axis:

- (with hollow area) $V = \pi\int_a^b [f(x)]^2 - [g(x)]^2 dx$
- (about line $y = k$) $V = \pi\int_a^b [f(x)-k]^2 dx$

## 04. SERIES

### Geometric Series

| sum (**divergent**) | sum (**convergent**) |
|---|---|
| $\frac{a(1-r^n)}{1-r}$ | $\frac{a}{1-r}$ |

### Power Series

power series about $x = 0$

$$\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1x + c_2x^2 + \ldots$$

power series about $x = a$ ($a$ is the centre of the power series) $\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1(x-a) + c_2(x-a)^2 + \ldots$

### Taylor series

$$\sum_{k=0}^{\infty}\frac{f^k(a)}{k!}(x-a)^k$$

MacLaurin series:

$$f(x) = \sum_{n=0}^{\infty}\frac{f^n(0)}{n!}x^n$$

Taylor polynomial of $f$ at $a$:

$$P_n(x) = \sum_{k=0}^{n}\frac{f^k(a)}{k!}(x-a)^k$$

### Radius of Convergence

power series converges where $\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| < 1$

| converge at | $R$ | $\lim_{n\to\infty}\left\lvert\frac{u_{n+1}}{u_n}\right\rvert$ |
|---|---|---|
| $x = a$ | $0$ | $\infty$ |
| $(x-h, x+h)$ | $h$, $\frac{1}{N}$ | $N \cdot \lvert x-a\rvert$ |
| all $x$ | $\infty$ | $0$ |

### MacLaurin Series

For $-\infty < x < \infty$

$$\sin x = \sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

$$\cos x = \sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{(2n)!}$$

$$e^x = \sum_{n=0}^{\infty}\frac{x^n}{n!}$$

For $-1 < x < 1$

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$$

$$\frac{1}{1+x} = \sum_{n=0}^{\infty}(-1)^n x^n$$

$$\frac{1}{1+x^2} = \sum_{n=0}^{\infty}(-1)^n x^{2n}$$

$$\ln(1+x) = \sum_{n=1}^{\infty}\frac{(-1)^{n-1}x^n}{n}$$

$$\tan^{-1} x = \sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}x^{2n+1}$$

$$\frac{1}{(1+x)^2} = \sum_{n=1}^{\infty}(-1)^{n-1}nx^{n-1}$$

$$\frac{1}{(1-x)^2} = \sum_{n=1}^{\infty}nx^{n-1}$$

$$\frac{1}{(1-x)^3} = \frac{1}{2}\sum_{n=2}^{\infty}n(n-1)x^{n-2}$$

$$(1+x)^k = \sum_{n=0}^{\infty}\binom{k}{n}x^n = 1 + kx + \frac{k(k-1)}{2!}x^2 + \ldots$$

### Differentiation/Integration

For $f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n$ and $a-h < x < a+h$,

**differentiation** of power series:

$$f'(x) = \sum_{n=0}^{\infty} nc_n(x-a)^{n-1}$$

**integration** of power series:

$$\int f(x)dx = \sum_{0}^{\infty} c_n\frac{(x-1)^{n+1}}{n+1} + c$$

if $R = \infty$, $f(x)$ can be integrated to $\int_0^1 f(x)dx$

## 05. VECTORS

### Dot (Scalar) product

$$\boldsymbol{a}\cdot\boldsymbol{b} = |\boldsymbol{a}||\boldsymbol{b}|\cos\theta$$

$$\begin{pmatrix}a_1\\a_2\\a_3\end{pmatrix}\cdot\begin{pmatrix}b_1\\b_2\\b_3\end{pmatrix} = a_1b_1 + a_2b_2 + a_3b_3$$

$\boldsymbol{a}\perp\boldsymbol{b}\Rightarrow\boldsymbol{a}\cdot\boldsymbol{b} = 0$

$\boldsymbol{a}\parallel\boldsymbol{b}\Rightarrow\boldsymbol{a}\cdot\boldsymbol{b} = |\boldsymbol{a}||\boldsymbol{b}|$

$\boldsymbol{a}\cdot\boldsymbol{b} > 0$ : $\boldsymbol{a}$ is acute

$\boldsymbol{a}\cdot\boldsymbol{b} < 0$ : $\boldsymbol{a}$ is obtuse

### Cross (Vector) product

$$\boldsymbol{a}\times\boldsymbol{b} = |\boldsymbol{a}||\boldsymbol{b}|\sin\theta\hat{\boldsymbol{n}}$$

$$\begin{pmatrix}a_1\\a_2\\a_3\end{pmatrix}\times\begin{pmatrix}b_1\\b_2\\b_3\end{pmatrix} = \begin{pmatrix}a_2b_3-a_3b_2\\-(a_1b_3-a_3b_1)\\a_1b_2-a_2b_1\end{pmatrix}$$

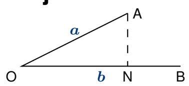
$\boldsymbol{a}\perp\boldsymbol{b}\Rightarrow\boldsymbol{a}\times\boldsymbol{b} = |\boldsymbol{a}||\boldsymbol{b}|$

$\boldsymbol{a}\parallel\boldsymbol{b}\Rightarrow\boldsymbol{a}\times\boldsymbol{b} = 0$

$\boldsymbol{a}\times\boldsymbol{b} = -(\boldsymbol{b}\times\boldsymbol{a})$

$\lambda\boldsymbol{a}\times\mu\boldsymbol{b} = \lambda\mu(\boldsymbol{a}\times\boldsymbol{b})$

### Projection




$\triangle ANO = \frac{1}{2}|\overrightarrow{OA}\times\overrightarrow{ON}|$

$|\overrightarrow{ON}| = |\boldsymbol{a}\cdot\hat{\boldsymbol{b}}| = \frac{|\boldsymbol{a}\cdot\boldsymbol{b}|}{|\boldsymbol{b}|}$

$\overrightarrow{ON} = (\boldsymbol{a}\cdot\hat{\boldsymbol{b}})\hat{\boldsymbol{b}} = \frac{|\boldsymbol{a}\cdot\boldsymbol{b}|}{|\boldsymbol{b}|^2}\boldsymbol{b}$

**Sphere**: Normal Vector: Passes through centre of sphere

### Planes

### Equation of a Plane

$\boldsymbol{n}$ is a perpendicular to the plane; $A$ is a point on the plane.

- parametric: $\boldsymbol{r} = \boldsymbol{a} + \lambda\boldsymbol{b} + \mu\boldsymbol{c}$
- scalar product: $\boldsymbol{r}\cdot\boldsymbol{n} = \boldsymbol{a}\cdot\boldsymbol{n}$
- standard form: $\boldsymbol{r}\cdot\hat{\boldsymbol{n}} = d$
- cartesian: $\boldsymbol{a}x + \boldsymbol{b}y + \boldsymbol{c}z = p$

Length of projection of $\boldsymbol{a}$ on $\boldsymbol{n} = |\boldsymbol{a}\cdot\hat{\boldsymbol{n}}| = \perp$ from $O$ to $\pi$

### Distance from a point to a plane

Shortest distance from a point $S(x_0, y_0, z_0)$ to a plane $\Pi : ax + by + c = d$ is given by: $\frac{|ax_0+by_0+cz_0-d|}{\sqrt{a^2+b^2+c^2}}$

## 06. PARTIAL DIFFERENTIATION

### Partial Derivatives

For $f(x, y)$, (some function of 2 variables)

first-order partial derivatives:

$f_x = \frac{d}{dx}f(x,y)$ | $f_y = \frac{d}{dy}f(x,y)$

second-order partial derivatives:

$f_{xx} = (f_x)_x = \frac{d}{dx}f_x$

$f_{yy} = (f_y)_y = \frac{d}{dy}f_y$

$f_{xy} = (f_x)_y = \frac{d}{dy}f_x$

$f_{yx} = (f_y)_x = \frac{d}{dx}f_y$

### Chain Rule

For $z(t) = f(x(t), y(t))$,

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$$

For $z(s,t) = f(x(s,t), y(s,t))$,

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}$$

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s}$$

## Directional Derivatives

The directional derivative of $f$ at $(a, b)$ in the direction of unit vector $\hat{\boldsymbol{u}} = u_1\boldsymbol{i} + u_2\boldsymbol{j}$ is

$D_u f(a,b) = f_x(a,b) \cdot u_1 + f_y(a,b) \cdot u_2$
$D_u f(a,b) = \nabla f(a,b) \cdot \hat{\boldsymbol{u}}$
(gradient vector . unit direction)

- **geometric meaning**: $D_u f(a,b)$ is the gradient of the tangent at $(a,b)$ to curve $C$ on a surface $z = f(x,y)$
  - rate of change of $f(x,y)$ at $(a,b)$ in the direction of $\boldsymbol{u}$

## Gradient Vector

The **gradient** at $f(x,y)$ is the vector
$\nabla f = f_x\boldsymbol{i} + f_y\boldsymbol{j} = (f_x, f_y)$

**In Direction of $\boldsymbol{u}$**:
$$D_u f(a,b) = \nabla f(a,b) \cdot \hat{\boldsymbol{u}} = |\nabla f(a,b)| \cos\theta$$

- $f$ increases most rapidly in the direction $\nabla f(a,b)$
- $f$ decreases most rapidly in the direction $-\nabla f(a,b)$
- largest possible value of $D_u f(a,b) = |\nabla f(a,b)|$
- (occurs in the same direction as $(f_x, f_y)$)

## Maximum & Minimum Values

$f(x,y)$ has a **local maximum** at $(a,b)$ if $f(x,y) \leq f(a,b)$ for all points $(x,y)$ near $(a,b)$.
$f(x,y)$ has a **local minimum** at $(a,b)$ if $f(x,y) \geq f(a,b)$ for all points $(x,y)$ near $(a,b)$.

## Critical Points

- $f_x(a,b)$ or $f_y(a,b)$ does not exist; OR
- $f_x(a,b) = 0$ and $f_y(a,b) = 0$
  - $f_x(0,b) \leq 0$ - maximum point *along the x axis*
  - $f_y(a,0) \geq 0$ - minimum point *along the y axis*

## Saddle Points

- $f_x(a,b) = 0, f_y(a,b) = 0$
- neither a local minimum nor a local maximum

## Second Derivative Test

Where $f_x(a,b) = 0$ and $f_y(a,b) = 0$.

Discriminant D:

$$D = f_{xx}(a,b)f_{yy}(a,b) - f_{xy}(a,b)^2$$

| $D$ | $f_{xx}(a,b)$ | local |
|---|---|---|
| + | + | min |
| + | - | max |
| - | any | saddle point |
| 0 | any | no conclusion |

# 07. DOUBLE INTEGRALS

Let $\Delta A_i$ be the area of $R_i$ and $(x_i, y_i)$ be a point on $R_i$. Let $f(x,y)$ be a function of two variables. The **double integral** of $f$ over $R$ is

$$\iint_R f(x,y)dA = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i, y_i)\Delta A_i$$

## Properties of Double Integrals

1. $\iint_R (f(x,y) + g(x,y))dA$
   $= \iint_R f(x,y)dA + \iint_R g(x,y)dA$
2. $\iint_R cf(x,y)dA = c\iint_R f(x,y)dA$, where $c$ is a constant
3. If $f(x,y) \geq g(x,y)$ for all $(x,y) \in \mathbb{R}$, then
   $\iint_R f(x,y)dA \geq \iint_R g(x,y)dA$
4. If $R$ = $R1 \cup R2$, $R1$ and $R2$ do not overlap, then
   $\iint_R f(x,y)dA = \iint_{R1} f(x,y)dA + \iint_{R2} f(x,y)dA$
5. The area of $R$,
   $A(R) = \iint_R dA = \iint_R 1dA$
6. If $m \leq f(x,y) \leq M$ for all $(x,y) \in R$, then
   $mA(R) \leq \iint_R f(x,y)dA \leq MA(R)$

## Rectangular Regions

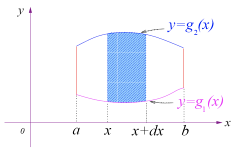
For a rectangular region $R$ in the $xy$-plane,
$a \leq x \leq b, \quad c \leq y \leq d$

$$\iint_R f(x,y)dA = \int_c^d \left[\int_a^b f(x,y)dx\right] dy = \int_a^b \left[\int_c^d f(x,y)dy\right] dx$$

If $f(x,y) = g(x)h(y)$, then
$$\iint_R g(x)h(y)dA = \left(\int_a^b g(x)dx\right)\left(\int_c^d h(y)dy\right)$$

## General Regions

### Type I: Integrate against complicated y first

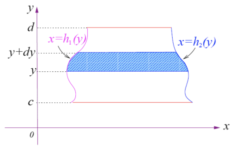



lower/upper bounds:
$g_1(x) \leq y \leq g_2(x)$

left/right bounds:
$a \leq x \leq b$

The region $R$ is given by
$$\iint_R f(x,y)dA = \int_a^b \left[\int_{g_1(x)}^{g_2(x)} f(x,y)dy\right] dx$$

### Type II: Integrate against complicated x first

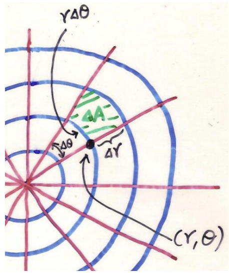



lower/upper bounds:
$c \leq y \leq d$

left/right bounds:
$h_1(y) \leq x \leq h_2(y)$

The region $R$ is given by
$$\iint_R f(x,y)dA = \int_c^d \left[\int_{h_1(y)}^{h_2(y)} f(x,y)dx\right] dy$$

## Polar Coordinates




$x = r\cos\theta$
$y = r\sin\theta$
Hence, $x^2 + y^2 = r^2$
$dxdy \Rightarrow rdrd\theta$

$\Delta A \approx (r\Delta\theta)(\Delta r)$
$= r\Delta r\Delta\theta$

$\boxed{dA = rdrd\theta}$

The region $R$ is given by
$R: a \leq r \leq b, \ \alpha \leq \theta \leq \beta$
$$\iint_R f(x,y)dA = \int_\alpha^\beta \int_a^b f(r\cos\theta, r\sin\theta)r \, drd\theta$$

Where, $D = \{(r,\theta) \mid 0 \leq \theta \leq \_, 0 \leq r \leq \_\}$

## Applications

### Volume

Suppose $D$ is a solid under the surface of $z = f(x,y)$ over a plane region $R$

Volume of $D = \iint_R f(x,y)dA$

### Surface Area

For area $S$ of that portion of the surface $z = f(x,y)$ that projects onto $R$,
$$S = \iint_R \sqrt{(\frac{\partial z}{\partial x})^2 + (\frac{\partial z}{\partial y})^2 + 1}dA$$

# 08. ORDINARY DIFFERENTIAL EQUATIONS

- **general solution**: solution containing arbitrary constants
- **particular solution**: gives specific values to arbitrary constants
- the general solution of the $n$-th order DE will have $n$ arbitrary constants

## Separable Equations (dx/dy separable)

A first-order DE is **separable** if it can be written in the form
$M(x) - N(y)y' = 0$ or $M(x)dx = N(y)dy$

## Reductions to Separable Form

| form | rec change of variable |
|---|---|
| $y' = g(\frac{y}{x})$ | set $v = \frac{y}{x}$ <br> $\Rightarrow y' = v + xv'$ |
| $y' = f(ax+by+c)$ <br> $\Rightarrow y' = \frac{ax+by+c}{\alpha x+\beta y+\gamma}$ | set $v = ax + by$ <br> get eqn in terms of $v$ & $v'$ |
| **Standard Form:** <br> $y' + P(x)y = Q(x)$ | $R = e^{\int Pdx}$ <br> $\Rightarrow y = \frac{1}{R}\int RQdx$ |
| **Bernoulli Equation:** <br> $y' + P(x)y = Q(x)y^n$ | set $z = y^{1-n}$ <br> sub $z$ and $z'$ obtain eqn <br> $\Rightarrow y' = \frac{y^n}{1-n}z'$ <br> $R = e^{\int Pdx}$ <br> $\Rightarrow z = \frac{1}{R}\int RQdx$ |

## Population Models

$N$ - number; $B$ - birth rate; $t$ - time; $D$ - death rate

**Logistic Model**
$$N = \frac{N_{t=\infty}}{1+(\frac{N_{t=\infty}}{N_{t=0}}-1)e^{-Bt}}$$

**Malthus Model**
$N(t) = N_0e^{kt}$
where $k = B - D$

**Common Scenarios: Uranium decays into Thorium**

amount of uranium,
$U(t) = U_0e^{-k_Ut}$
$\frac{dU}{dt} = -k_UU$
amount of thorium,
$T(t) = \frac{k_UU_0}{k_T-k_U}(e^{-k_Ut} - e^{-k_Tt})$
$\frac{dT}{dt} = k_UU - k_TT$
decay rate constant, $k = \frac{\ln 2}{t_{1/2}}$
ratio of thorium to uranium,
$\frac{T}{U} = \frac{k_U}{k_T-k_U}(1 - e^{-(k_T-k_U)t})$

**Radioactive decay**
$Q(t) = Q_0e^{-kt}$
$k = \frac{\ln 2}{t_{1/2}}$

**Cooling/Heating**
$\frac{dT}{dt} = k(T - T_{env})$
$\frac{1}{T-T_{env}}dT = kdt$

**Falling objects (N2L)**
Resistance = $bv^2$
$m\frac{dv}{dt} = mg - bv^2$
Let $k = \sqrt{\frac{mg}{b}}$
$\Rightarrow \frac{1}{v^2-k^2}dv = -\frac{b}{m}dt$

**Resistive medium**
Resistance = $kv$
$m\frac{dv}{dt} = mg - kv$
$v' + \frac{k}{m}v = g$ (linear)

**Concentration of salt in liquid**
Let $R$ = rate of flow (in and out), $Q$ = total amount of salt, $V$ = total volume, $C_{in}$ = concentration of inflow
Rate of flow, $\frac{dQ}{dt} = RC_{in} - \frac{R}{V}Q$
$\Rightarrow Q' + \frac{R}{V}Q = RC_{in}$