Impedance Spectroscopy

Prelab

2.1) Use $v = \hat{V}e^{j\omega t}$ and $i_{(t)} = \hat{I}e^{j\omega t}$ to show $Z = \frac{\hat{V}}{\hat{I}} = \frac{1}{j\omega C}$ for a capacitor

$Z \equiv \frac{\hat{V}}{\hat{I}} = R$ $\qquad i_{(t)} = C\frac{dv}{dt} = Cj\omega\hat{V}e^{j\omega t}$

$\frac{v_{(t)}}{i_{(t)}} = \frac{\hat{V}\cancel{e^{j\omega t}}}{\hat{I}\cancel{e^{j\omega t}}} = R$ so $Z = R = \frac{\cancel{\hat{V}e^{j\omega t}}}{Cj\omega\cancel{\hat{V}e^{j\omega t}}}$ so $Z = \frac{1}{Cj\omega}$

2.2) Verify that $v_{(t)} = \frac{I}{\omega C}\sin(\omega t) = \frac{I}{\omega C}\cos(\omega t - 90°)$ satisfies $i = C\frac{dv}{dt}$ when $i = I\cos(\omega t)$

$\frac{dv}{dt} = \frac{I}{\cancel{\omega}C}\cancel{\omega}\cos(\omega t)$, so $i = \cancel{C}\frac{I}{\cancel{C}}\cos(\omega t)$, as desired.

2.3) Show that, for an inductor, $Z = j\omega L$

$v = L\frac{di}{dt} = L\hat{I}e^{j\omega t}j\omega$ $\qquad \frac{v_{(t)}}{i_{(t)}} = Z = Lj\omega\frac{\cancel{\hat{I}e^{j\omega t}}}{\cancel{\hat{I}e^{j\omega t}}}$ so $Z = j\omega L$

2.4) Taking the real part of $\hat{I}Ze^{i\omega t}$ gives $v_{(t)} = -\omega L\sin(\omega t) = \omega L\cos(\omega t + 90°)$. Verify this, and that it satisfies $v = L\frac{di}{dt}$ when $i = I\cos(\omega t)$.

$Z = Lj\omega$ so $\hat{I}Ze^{i\omega t} = Lj\omega\hat{I}e^{i\omega t} = Lj\omega\hat{I}(\cos(\omega t) + j\sin(\omega t))$

$v_{(t)} = L\omega\hat{I}(j\cos(\omega t) - \sin(\omega t))$, the real part of which is

$v_{real(t)} = -L\omega I\sin(\omega t)$

$v_{(t)} = L\frac{di}{dt} = -LI\omega\sin(\omega t)$ ✓

2.5) Use trig identities to show that $p_{(t)} = \frac{I^2}{2}|Z|[\cos\phi + \cos(2\omega t + \phi)]$

We wish to show that $\cos\omega t\cos(\omega t + \phi) = \frac{1}{2}(\cos\phi + \cos(2\omega t + \phi))$

$\cos(a+b) = \cos a\cos b - \sin a\sin b$

$\cos(\omega t + \phi) = \cos(\omega t)\cos\phi - \sin(\omega t)\sin\phi$

$\cos^2(\omega t)\cos\phi - \cos(\omega t)\sin(\omega t)\sin\phi$

$\sin a\sin b = \frac{1}{2}(\cos(a-b) - \cos(a+b))$

$\sin(\omega t)\sin\phi = \frac{1}{2}(\cos(\omega t - \phi) - \cos(\omega t + \phi))$

$\cos^2(\omega t)\cos\phi - \cos(\omega t)\frac{1}{2}(\cos(\omega t - \phi) - \cos(\omega t + \phi))$

$\cos a\cos b = \frac{1}{2}(\cos(a-b) + \cos(a+b))$

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(2.5) (a) $\cos(\omega t)\cos(\omega t-\phi)=\frac{1}{2}\left(\cos(\cancel{\omega t-\omega t}+\phi)+\cos(2\omega t-\phi)\right)$

$\cos(\omega t)\cos(\omega t+\phi)=\frac{1}{2}\left(\cos(\cancel{\omega t-\omega t}-\phi)+\cos(2\omega t+\phi)\right)$

$\cos^2(\omega t)\cos\phi-\frac{1}{4}\left(\cos(2\omega t-\phi)-\cos(2\omega t+\phi)\right)=\cos^2(\omega t)\cos\phi+\frac{1}{4}\left(\cos(2\omega t+\phi)-\cos(2\omega t-\phi)\right)$

$\cos a+\cos b=2\cos\frac{a+b}{2}\cos\frac{a-b}{2}$

$\cancel{\cos(2\omega t+\phi)}$ $\cos a-\cos b=-2\sin\frac{a+b}{2}\sin\frac{a-b}{2}$

$\cos^2 a=\frac{1+\cos(2a)}{2}$

$\cos^2(\omega t)=\frac{1+\cos(2\omega t)}{2}$

$\frac{1+\cos(2\omega t)}{2}\cos\phi+\frac{1}{4}\left(\cos(2\omega t+\phi)-\cos(2\omega t-\phi)\right)$

$\frac{1}{2}\cos\phi+\frac{1}{2}\cos(2\omega t)\cos\phi+\frac{1}{2}\left(\frac{1}{2}\right)\left(\cos(2\omega t+\phi)-\cos(2\omega t-\phi)\right)$

$\frac{1}{2}\left(\cos\phi+\frac{1}{2}\left(\cancel{\cos(2\omega t-\phi)}+\cos(2\omega t+\phi)\right)+\frac{1}{2}\left(\cos(2\omega t+\phi)-\cancel{\cos(2\omega t-\phi)}\right)\right)$

$=\frac{1}{2}\left(\cos\phi+\cos(2\omega t+\phi)\right)$ ✓

(b) Explain why $P(t)\neq R\,\frac{1}{2}\hat{I}e^{j\omega t}\hat{V}e^{j\omega t}=Riv=v^2\neq iv$

Because $P_{(t)}=i_{(t)}v_{(t)}\neq v^2$ unless $v=i$, which it cannot as they have different units.

(c) Show that $\langle p\rangle=\frac{1}{2}I^2|Z|\cos\phi$ over one period of oscillation

$\langle p\rangle=\frac{1}{2}I^2|Z|\left(\cos\phi+\underbrace{\langle\cos(2\omega t+\phi)\rangle}\right)=\frac{1}{2}I^2|Z|\cos\phi$

This ranges evenly from -1 to 1, making $\langle\,\rangle\to 0$. So, this term simply drops out.

(d) Show that $\langle p\rangle=\frac{1}{2}R\,\hat{I}\hat{V}=\frac{1}{2}R(\hat{Z})$

~~$\hat{V}=\hat{I}\,R$ ... so ... $\hat{V}=\hat{I}R$~~

○ $\langle i_{(t)}v_{(t)}\rangle=$ stuff, getting ahead of myself

$P_{(t)}=i_{(t)}v_{(t)}=I\cos(\omega t)\,V\cos(\omega t+\phi)=I^2\cos(\omega t)|Z|\cos(\omega t+\phi)=I^2|Z|\frac{1}{2}(\cos\phi+\ldots$

$\langle p\rangle=\frac{1}{2}I^2|Z|\left(\cos(-\phi)+\cancel{0}\right)$, $\quad R(Z)=|Z|\cos(-\phi)$

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2.6) Show that the average power dissipated in a capacitor filled with a material of relative complex permitivity is given by

$$\langle p \rangle = \frac{I^2}{2\omega|C|}\sin\delta,$$ where C is the magnitude of the complex-valued capacitance and δ is the loss angle defined by $\sin\delta = \frac{\varepsilon_r''}{|\varepsilon_r|}$.

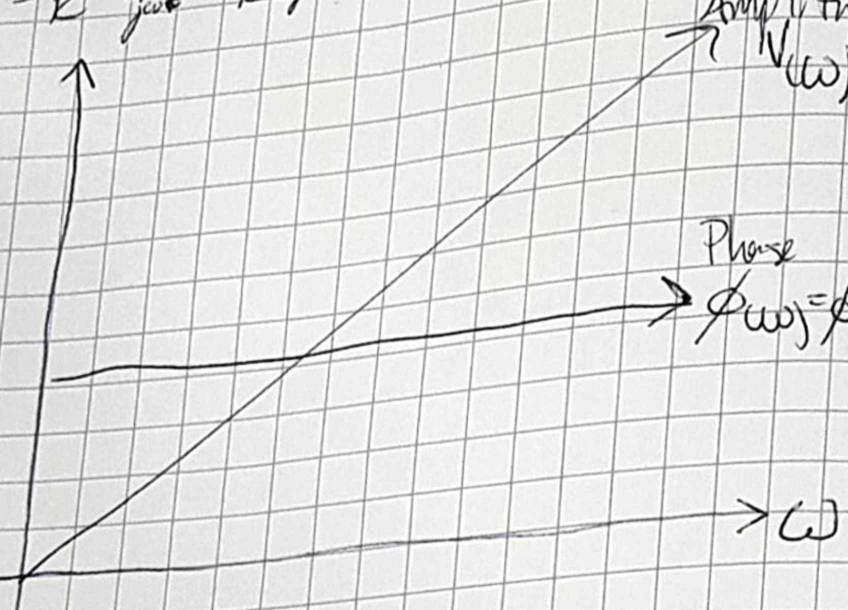
$$\langle p \rangle = \frac{1}{2}I^2|Z|\cos(-\phi) = \frac{1}{2}I^2\left|\frac{d}{j\omega\varepsilon_0(\varepsilon_r' - j\varepsilon_r'')A}\right| = \frac{I^2}{2}\frac{\cos(-\phi)}{j\omega C} = \frac{I^2}{2\omega|C|}\frac{\cos(-\phi)}{\cancel{I}}$$

$$= \frac{I^2}{2\omega|C|}\sin\delta$$

2.7) $\boxed{a}$ Show that $Z = \frac{R}{1+j\omega\tau}$, where $\tau = RC$

$$\frac{1}{Z} = \frac{1}{R} + \frac{1}{\frac{1}{j\omega C}} = \frac{1}{R} + j\omega C = \frac{j\omega CR+1}{R} = \frac{j\omega\tau+1}{R} \Rightarrow Z = \frac{R}{1+j\omega\tau}$$

$\boxed{b}$



let $i = I\cos(\omega t)$

~~$V_{(t)} = I\omega - \sin(\omega t)$~~

$V = iR = iZ(1+j\omega\tau)$

$V_{(\omega)} \propto \omega$

phase shouldn't change with ω

2.8) $\boxed{a}$ Without analysis, give zero- and high-frequency limits of impedance

$0: Z \to R$

$\infty: Z \to 0$

$\boxed{b}$ Show that the impedance of this circuit can be written as

$$\frac{Z}{R_1} = \frac{1 + j\frac{\omega}{\omega_c}\frac{R_2}{R_1+R_2}}{1 + j\frac{\omega}{\omega_c}},$$ where $\omega_c = \frac{1}{(R_1+R_2)C_2}$

$$Z = \frac{R_2+R_1}{1+j\omega(R_2+R_1)C_2} = R_1\frac{1+\frac{R_2}{R_1}}{1+j\omega\frac{1}{\omega_c}} = \frac{1+\frac{R_2}{R_1}}{1+j\frac{\omega}{\omega_c}}R_1 \Rightarrow \frac{1+\frac{R_2}{R_1}}{1+j\frac{\omega}{\omega_c}} = \frac{Z}{R_1}$$

$$\frac{R_2}{R_1} = \frac{R_2}{R_1}\frac{(R_1+R_2)C}{(R_1+R_2)C} = \frac{R_2}{\omega_c}\frac{1}{R_1C(R_1+R_2)} = \frac{R_2}{\omega_c}\frac{\omega_c}{R_1} - \frac{R_2}{\omega_c}j\omega$$

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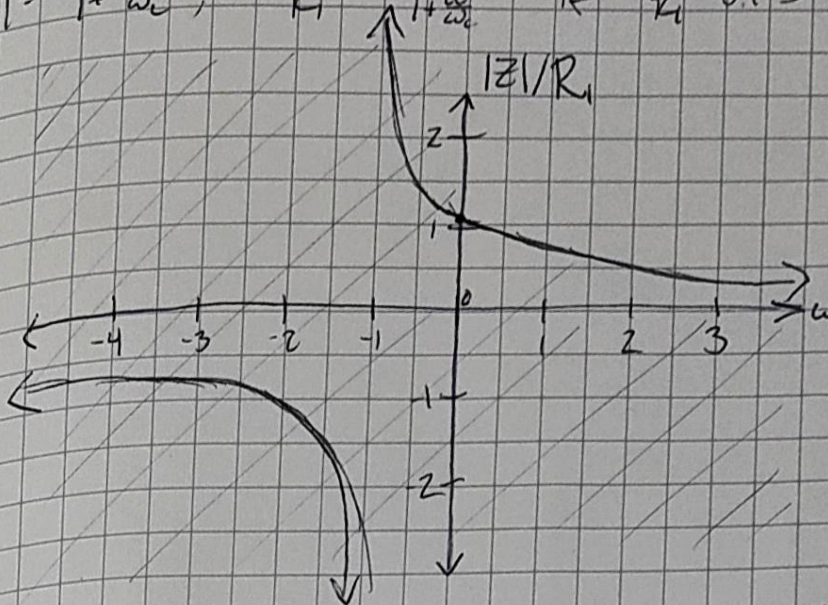
(2.8) c) Find an expression for $\frac{|Z|}{R_1}$. Now suppose $\frac{R_2}{R_1}=0.1$. Plot $\frac{|Z|}{R_1}$ vs $\frac{\omega}{\omega_c}$.

$|Z| = \frac{R_2+R_1}{1+\frac{\omega}{\omega_c}}$, $\frac{|Z|}{R_1} = \frac{1+\frac{R_2}{R_1}}{1+\frac{\omega}{\omega_c}}$ if $\frac{R_2}{R_1}=0.1 \Rightarrow \frac{|Z|}{R_1} = \frac{1.1}{1+\frac{\omega}{\omega_c}}$



d) Find an expression for the phase ϕ of the impedance ($\tan\phi = \frac{Im Z}{Re Z}$)

$\phi = \tan^{-1}\left(\frac{Im\left(\frac{1+\frac{R_2}{R_1}}{1+\omega/\omega_c}\right)}{Re\left(\frac{1+R_2/R_1}{1+\omega/\omega_c}\right)}\right) = \tan^{-1}$