Introduction to Mathematical Structures - Steven Galovich

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1.1 Logic

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2 Chapter 2: Sets

2.1 Sets

2.2 Operations on Sets

2.2.1

Let $U = \{1, 2, 3, 4, 5, 6\}$, $A = \{1, 2, 3\}$, $B = \{2, 3, 4\}$, and $C = \{4, 5, 6\}$. Find each of the following sets:

(a) $A \cup B$.

Solution. $A \cup B = \{1, 2, 3, 4\}$ □

(b) $A \cap B$.

Solution. $A \cap B = \{2, 3\}$ □

(c) A^c.

Solution. $A^c = \{4, 5, 6\}$ □

(d) B^c.

Solution. $B^c = \{1, 5, 6\}$ □

(e) $B \cap C$.

Solution. $B \cap C = \{4\}$ □

(f) $A \cap (B \cup C)$.

Solution. $A \cap (B \cup C) = \{2, 3\}$ $\square$

(g) $(A \cap B) \cup (A \cap C)$.

Solution. $(A \cap B) \cup (A \cap C) = \{2, 3\}$ $\square$

(h) $A \cap C^c$.

Solution. $A \cap C^c = \{1, 2, 3\}$ $\square$

(i) $A - C$.

Solution. $A - C = \{1, 2, 3\}$ $\square$

(j) $P(A) \cap P(B)$.

Solution. $P(A) \cap P(B) = \{\emptyset, \{2\}, \{3\}, \{2, 3\}\}$ $\square$

2.2.2

Let A and B be arbitrary sets.

(a) Prove: $A \cup A = A$ and $A \cap A = A$.

Proof. Let $x \in A \cup A$, then $x \in A$ or $x \in A$. Since both expressions are the same, $x \in A$. Hence $A \cup A \subseteq A$. Let $x \in A$, then $x \in A \cup A$. Hence $A \subseteq A \cup A$. We conclude that $A \cup A = A$.

Let $x \in A \cap A$, then $x \in A$. Hence $A \cap A \subseteq A$. Let $x \in A$, then $x \in A$. Hence $x \in A \cap A$, therefore $A \subseteq A \cap A$. We conclude that $A \cap A = A$. $\square$

(b) Prove: $A \cup B = B \cup A$ and $A \cap B = B \cap A$.

Proof. Let $x \in A \cup B$, then $x \in A$ or $x \in B$; if $x \in A$ then $x \in B \cup A$. If $x \in B$ then $x \in B \cup A$, therefore $A \cup B \subseteq B \cup A$. In the same way $B \cup A \subseteq A \cup B$. We conclude that $A \cup B = B \cup A$.

Let $x \in A \cap B$, then $x \in A$ and $x \in B$. Hence $x \in B \cap A$, therefore $A \cap B \subseteq B \cap A$. In the same way $B \cap A \subseteq A \cap B$. We conclude that $A \cap B = B \cap A$. $\square$

(c) Prove: $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ in two ways.

Proof. Let $x \in A \cup (B \cap C)$. Then $x \in A$ or $x \in B \cap C$. If $x \in A$, then $x \in A \cup B$ and $x \in A \cup C$. Hence, $x \in (A \cup B) \cap (A \cup C)$. If $x \in B \cap C$, then $x \in B$ and $x \in C$. Since $x \in B$, $x \in A \cup B$. Since $x \in C$, $x \in A \cup C$. Since $x \in A \cup B$ and $x \in A \cup C$, $x \in (A \cup B) \cap (A \cup C)$. It follows that $A \cup (B \cap C) \subseteq (A \cup B) \cap (A \cup C)$.

Let $x \in (A \cup B) \cap (A \cup C)$. Hence $x \in A \cup B$ and $x \in A \cup C$. Since $x \in A \cup B$, $x \in A$ or $x \in B$. If $x \in A$, then $x \in A \cup (B \cap C)$. Since $x \in A \cup C$, $x \in A$ or $x \in C$. If $x \in A$, then $x \in A \cup (B \cap C)$. Since $x \in B$ and $x \in C$, then $x \in B \cap C$, then $x \in A \cup (B \cap C)$. Since $x \in A \cup (B \cap C)$, it follows that $(A \cup B) \cap (A \cup C) \subseteq A \cup (B \cap C)$.
Therefore $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$. □

2.2.3

(a) Prove: $A \subseteq B$ if and only if $A \cup B = B$.

Proof. Suppose $A \subseteq B$, if $x \in A$ then $x \in B$. Let $y \in A \cup B$, then $y \in A$ or $y \in B$. If $y \in A$ then $y \in B$, then $y \in B$. $A \cup B \subseteq B$. If $y \in B$ then $y \in A \cup B$. $B \subseteq A \cup B$. Therefore $A \subseteq B \Rightarrow A \cup B = B$.
Suppose $A \cup B = B$. If $x \in A$ then $x \in B$. $A \subseteq B$. Therefore $A \cup B = B \Rightarrow A \subseteq B$.
We conclude that $A \subseteq B \Leftrightarrow A \cup B = B$. □

(b) Prove: $A \subseteq B$ if and only if $A \cap B = A$.

Proof. Suppose $A \subseteq B$, if $x \in A$ then $x \in B$. Let $y \in A \cap B$, then $y \in A$, $A \cap B \subseteq A$. Let $y \in A$, then $y \in B$, so $A \subseteq A \cap B$. Therefore $A \subseteq B \Rightarrow A \cap B = A$.
Suppose $A \cap B = A$. Then if $x \in A$, $x \in A \cap B$. Then $x \in B$. Therefore $A \cap B = A \Rightarrow A \subseteq B$. We conclude that $A \subseteq B \Leftrightarrow A \cap B = A$. □

2.2.4

Let U be a set and A, B, C be subsets of U.

(a) Prove: $A - B = A \cap B^c$ and $A - B = B^c - A^c$.

Proof. Let $x \in A - B$ then $x \in A$ and $x \notin B$. Therefore $A - B \subseteq A \cap B^c$.
Let $x \in A \cap B^c$ then $x \in A$ and $x \notin B$. Therefore $A \cap B^c \subseteq A - B$.
Thus we show that $A - B = A \cap B^c$.

Let $x \in A - B$, then $x \in A$ and $x \notin B$, then $A - B \subseteq B^c - A^c$.
Let $x \in B^c - A^c$ then $x \in B^c$ and $x \notin A^c$. Since $x \notin A^c$ then $x \in A$ and since $x \in B^c$, $x \notin B$. Therefore $B^c - A^c \subseteq A - B$. We conclude that $A - B = B^c - A^c$ □

(b) Prove: $A - (A - B) = A \cap B$.

Proof. Let $x \in A-(A-B)$, then $x \in A$ and $x \notin (A-B)$. Then $x \in A$ and $x \notin (A-B)$, then $x \in A$ and $x \in B$. Therefore $A - (A - B) \subseteq A \cap B$.
Let $x \in A \cap B$, then $x \in A$ and $x \in B$, since $x \in B$, $x \notin A - B$. Therefore $x \in A-(A-B)$. Hence $A \cap B \subseteq A-(A-B)$. We conclude that $A-(A-B) = A \cap B$. □

2.3 Ordered Pairs and Cartesian Products

2.3.1

(a) Find $A \times B$ if $A = \{1, 2, 3\}$ and $B = \{3, 4, 5\}$.

Solution. $A \times B = \{(1, 3), (1, 4), (1, 5), (2, 3), (2, 4), (2, 5), (3, 3), (3, 4), (3, 5)\}$ □

(b) Find $\{a, b\} \times \{a, b\}$.

Solution. $\{a, b\} \times \{a, b\} = \{(a, a), (a, b), (b, a), (b, b)\}$. □

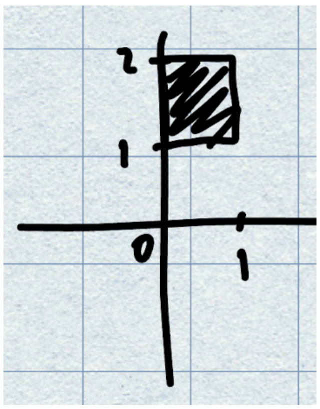
(c) Find $\{\{a, b\}\} \times \{a, b\}$.

Solution. $\{\{a, b\}\} \times \{a, b\} = \{(\{a, b\}, a), (\{a, b\}, b)\}$ □

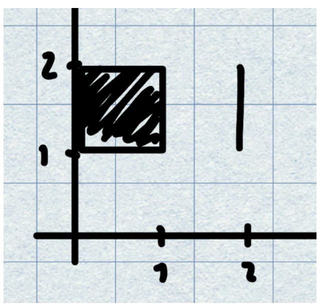
2.3.2

Draw a sketch of each of the following subsets of $\mathbb{R}^2$.

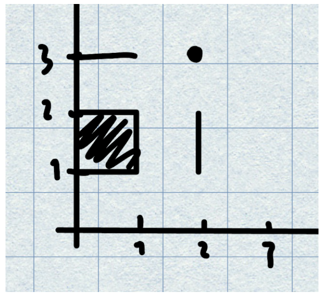
(a) $[0, 1] \times [1, 2]$ (Recall $[a, b] = \{x \in \mathbb{R} | a \leq x \leq b\}$).



(b) $([0, 1] \cup \{2\}) \times [1, 2]$.



(c) $([0, 1] \cup \{2\}) \times ([1, 2] \cup \{3\})$.



2.3.12

Suppose A and B are finite sets.

(a) Formulate a conjecture that expresses the number of elements of $A \times B$ in terms of the number of elements of A and B. Check your conjecture in some simple cases.

Solution. The cardinality of the cartesian product of 2 sets is the product between the cardinality of them. $|A \times B| = |A| \times |B|$.

$A = \{1, 2\}$ and $B = \{10, 20, 30\}$ $A \times B = \{(1, 10), (1, 20), (2, 10), (2, 20), (3, 10), (3, 20)\}$
$|A \times B| = |A| \times |B| = 3 \times 2 = 6$. □

(b) Prove your conjecture. (Hint: Use mathematical induction on the number of elements of A.)

Proof. Let A be a finite set with $|A| = n$, and let B be a set with $|B| = 5$. We want to show that $|A \times B| = 5n$.

Base Case
Let $n = 1$. This means $A = \{a_1\}$ for some element a_1. Then $A \times B = \{(a_1, b) \mid b \in B\}$. Since $|B| = 5$, there are 5 such pairs. So, $|A \times B| = 5$. According to the formula, $5n = 5 \times 1 = 5$. Thus, the base case holds: $|A \times B| = 5n$ when $n = 1$.

Inductive Hypothesis
Assume that for some non-negative integer k, if A' is a finite set with $|A'| = k$, then $|A' \times B| = 5k$.

Inductive Step
Now consider a finite set A with $|A| = k+1$. Since A has $k+1$ elements, we can write $A = A' \cup \{a_{k+1}\}$, where A' is a set with k elements ($|A'| = k$) and $a_{k+1} \notin A'$.
Now let's consider the Cartesian product: $A \times B = (A' \cup \{a_{k+1}\}) \times B$ Using the distributive property of Cartesian product over union, we have: $A \times B = (A' \times B) \cup (\{a_{k+1}\} \times B)$
We know that $A' \times B$ and $\{a_{k+1}\} \times B$ are disjoint. This is because any element in $A' \times B$ is of the form (a', b) where $a' \in A'$, and any element in $\{a_{k+1}\} \times B$ is of the form (a_{k+1}, b). Since $a_{k+1} \notin A'$, these sets cannot have any common elements.

Since the sets are disjoint, the cardinality of their union is the sum of their cardinalities: $|A \times B| = |A' \times B| + |\{a_{k+1}\} \times B|$
By the Inductive Hypothesis, since $|A'| = k$, we have $|A' \times B| = 5k$.
Now let's find the cardinality of $\{a_{k+1}\} \times B$. The set $\{a_{k+1}\} \times B$ consists of pairs where the first element is a_{k+1} and the second element is from B. Since $|B| = 5$, there are exactly 5 such pairs: $(a_{k+1}, b_1), (a_{k+1}, b_2), (a_{k+1}, b_3), (a_{k+1}, b_4), (a_{k+1}, b_5)$. Therefore, $|\{a_{k+1}\} \times B| = 5$.
Substituting these values into the equation for $|A \times B|$: $|A \times B| = 5k + 5$ $|A \times B| = 5(k+1)$
Since we assumed $|A| = k + 1$, this shows that if the formula holds for k, it also holds for $k + 1$. We conclude that $|A \times B| = |A| \times |B|$. □

3 Chapter 3

3.1 Relations

3.1.2

Give several examples of relations on $\{a, b, c\}$.

Solution. $R_1 = \{(a, b), (b, a), (c, b)\}$, $R_2 = \{(b, c)\}$, $R_3 = \{(a, a), (b, b), (c, c)\}$ and $R_4 = \{(a, b), (b, c)\}$ □

3.1.3

(a) How many relations on $\{1, 2\}$ are reflexive?

Solution. $R_1 = \{(1, 1), (2, 2)\}$ $R_2 = \{(1, 1), (2, 2), (1, 2)\}$ $R_3 = \{(1, 1), (2, 2), (2, 1)\}$ $R_4 = \{(1, 1), (2, 2), (1, 2), (2, 1)\}$. There are 4. □

(b) How many are symmetric?

Solution. $R_1 = \{(1, 2), (2, 1)\}$ $R_2 = \{(1, 2), (2, 1), (1, 1)\}$ $R_3 = \{(1, 2), (2, 1), (2, 2)\}$ $R_4 = \{(1, 2), (2, 1), (1, 1), (2, 2)\}$ $R_5 = \{(1, 1)\}$ $R_6 = \{(1, 1), (2, 2)\}$ $R_7 = \{(2, 2)\}$ $R_8 = \emptyset$. There are 8. □

(c) How many are both reflexive and symmetric?

Solution. $R_1 = \{(1, 1), (2, 2)\}$ $R_2 = \{(1, 1), (2, 2), (1, 2), (2, 1)\}$. There are 2. □

(d) How many are neither reflexive nor symmetric?

Solution. $R_1 = \{(1, 1), (1, 2)\}$ $R_2 = \{(1, 1), (2, 1)\}$ $R_3 = \{(2, 2), (2, 1)\}$ $R_4 = \{(2, 2), (1, 2)\}$ $R_5 = \{(1, 2)\}$ $R_6 = \{(2, 1)\}$. There are 6. □

3.1.5

On a single set of your choice, give examples of relations that possess exactly one and exactly two of the following three properties—reflexivity, symmetry, and transitivity. You should give six examples. Let $A = \{a, b, c\}$.

- Only reflexive:

  Solution. $R_1 = \{(a, a), (b, b), (c, c), (b, c), (c, a)\}$ □

- Only symmetric:

  Solution. $R_2 = \{(a, c), (c, a)\}$ □

- Only transitive:

  Solution. $R_3 = \{(a, b), (b, c), (a, c)\}$ □

- Reflexive and symmetric but not transitive:

  Proof. $R_4 = \{(a, a), (b, b), (c, c), (b, c), (c, b), (a, b), (b, a)\}$ □

- Reflexive and transitive but not symmetric:

  Solution. $R_5 = \{(a, a), (b, b), (c, c), (a, b), (b, c), (a, c)\}$ □

- Symmetric and transitive but not reflexive:

  Solution. $R_6 = \{(a, a)\}$ □

3.1.10

Let A be a set and let R and S be relations on A.

(a) Prove: $(R^{-1})^{-1} = R$.

Proof. $(R^{-1})^{-1} = \{(x, y) \in A \times A \mid (y, x) \in R^{-1}\}$
$(R^{-1})^{-1} = \{(x, y) \in A \times A \mid (x, y) \in R\}$
$(R^{-1})^{-1} = R$ □

(b) Prove: $(R \cap S)^{-1} = R^{-1} \cap S^{-1}$.

Proof. $(x, y) \in (R \cap S)^{-1} \iff (y, x) \in (R \cap S) \iff (y, x) \in R$ and$(y, x) \in S$, $\iff$ $(x, y) \in R^{-1}$ and $(x, y) \in S^{-1}$. Therefore $(x, y) \in R^{-1} \cap S^{-1}$. □

3.1.15

Let $A = \{a, b, c, d\}$, $R = \{(a, a), (a, b), (a, c), (b, c), (c, a), (c, c), (c, d), (d, c)\}$, and $S = \{(a, a), (b, a), (b, c), (b, d), (d, b)\}$. Find:

(a) $R \circ R$.

Solution. $R \circ R = \{(a, a), (a, b), (a, c), (a, d), (b, a), (b, c), (b, d), (c, a), (c, b), (c, c), (c, d), (d, a), (d, c), (d, d)\}$ □

(b) $R \circ S$.

Solution. $R \circ S = \{(a, a), (a, b), (a, c), (b, a), (b, b), (b, c), (b, d), (d, c)\}$ □

(c) $S \circ R$.

Solution. $S \circ R = \{(a, a), (a, c), (a, d), (c, a), (c, b)\}$ □

(d) $S \circ S$.

Solution. $S \circ S = \{(a, a), (b, a), (b, b), (d, a), (d, c), (d, d)\}$ □

3.1.17

In each case give the composition $R \circ S$.

(a) $A = \mathbb{R}$; xRy if $y = \sin(x)$; xSy if $y = x^2$.

Solution. $R \circ S = \{(x, z) \in \mathbb{R} \times \mathbb{R} \mid z = \sin(x^2)\}$ □

(b) $A = \mathbb{R}$; xRy if $y = \sin(x)$; xSy if $y = x$.

Solution. $R \circ S = \{(x, z) \in \mathbb{R} \times \mathbb{R} \mid z = \sin(x)\}$ □

3.2 Order Relations

3.3 Equivalence Relations

3.3.1

(a) How many equivalence relations are there on $\{a, b, c\}$?

Solution. $E_1 = \{(a, a), (b, b), (c, c), (a, b), (b, c), (a, c), (b, a), (c, b), (c, a)\}$
$E_2 = \{(a, a), (b, b), (c, c)\}$ $E_3 = \{(a, a), (b, b), (c, c), (a, b), (b, a)\}$
$E_4 = \{(a, a), (b, b), (c, c), (a, c), (c, a)\}$ $E_5 = \{(a, a), (b, b), (c, c), (b, c), (c, b)\}$.
There are 5. □

(b) Give four examples of equivalence relations on $\{a, b, c, d\}$.

Solution. $E_1 = \{(a,a),(b,b),(c,c),(d,d)\}$ $E_2 = \{(a,a),(b,b),(c,c),(d,d),(a,b),(b,a)\}$ $E_3 = \{(a,a),(b,b),(c,c),(d,d),(a,d),(d,a)\}$ $E_4 = \{(a,a),(b,b),(c,c),(d,d),(c,b),(b,c)\}$ □

3.3.2

Let $\mathcal{L}$ be the set of lines in the Euclidean plane. For $\ell_1, \ell_2 \in \mathcal{L}$ define $\ell_1 \perp \ell_2$ to mean ℓ_1 and ℓ_2 are perpendicular. Is $\perp$ an equivalence relation?

Solution. If $\perp$ was an equivalence relation, then for all $\ell \in \mathcal{L}$ $\ell_1 \perp \ell_1$ (reflexivity). However, this is not true, because a line is not perpendicular with itself since the angle that it form with itself is $0°$. □

3.3.5

Which of the following relations R are equivalence relations on $\mathbb{Z} - \{0\}$?

(a) aRb if $2|(a+b)$.

Solution. Let us analyze if this relation is reflexive, a and b are equal, therefore the sum is $2(a)$ or $2(b)$, then that is a number divisible by 2. Therefore the relations in reflexive. Let us check if it is symmetric, $a+b=b+a$, therefore this relation is symmetric. Finally, let us analyze if it is transitive, either a and b are both even or both are odd. If they are even, then c is even and therefore aRc; if they are odd, then c must be odd and therefore aRc. We conclude that this is an equivalence relation. □

(b) aRb if $3|(a+b)$.

Solution. Let us give a counter example, we know that $1R2$ since $1+2=3$ and $3 \equiv 0 \pmod 3$, also $2R7$ since $2+7=9$ and $9 \equiv 0 \pmod 3$. If this relation is an equivalence relation, then it is transitive. However $1 \not R 7$. We conclude that this is not an equivalence relation. □

(c) aRb if $a|b$ and $b|a$.

Solution. Let us analyze if this relations is reflexive, every number is divisible by itself, therefore it is reflexive. It is symmetric by the condition of the relation. Let us check if it is transitive, since $a|b$ and $b|a$... □

(d) aRb if $a|b$ or $b|a$.

Solution. If it is an equivalence relation, then it is symmetric. However, we can use a counter example, $3R27$ since $3|27$ but $27 \not R 3$ since $27 \nmid 3$. Therefore is not an equivalence relation. □

(e) aRb if $a|b$.

Solution. If it is an equivalence relation, then it is symmetric. However, we can use a counter example, $4R20$ since $4|20$ but $20 \not R 4$ since $20 \nmid 4$. Therefore is not an equivalence relation. $\square$

3.3.8

Let $A = \mathbb{N} \times \mathbb{N}$. For $x, y \in A$ with $x = (a, b)$ and $y = (c, d)$, define $x \sim y$ if $a + d = b + c$.

(a) Prove: $\sim$ is an equivalence relation on A.

Proof. Let us analyze if it is reflexive, let $x = (a, b)$, $a + b = a + b$ is always true, therefore $\sim$ is reflexive. It is also symmetric since $a + d = b + c$ implies that $c + b = d + a$. Finally, let us check if it is transitive, let $x = (a, b)$, $y = (c, d)$ and $z = (e, f)$, we need to show that $x \sim y$ and $y \sim z$ imply $x \sim z$. We have that

$$\begin{aligned} a + d &= b + c \\ c + f &= d + e \\ a + d + c + f &= b + c + d + e \\ a + \cancel{d} + \cancel{c} + f &= b + \cancel{c} + \cancel{d} + e \end{aligned}$$

Since $a + f = b + e$, we conclude that $x \sim z$, therefore $\sim$ is transitive, thus $\sim$ is an equivalence relation on A. $\square$

(b) Show: $\{(n, 0)|n \in \mathbb{N}\} \cup \{(0, m)|m \in \mathbb{N} \text{ and } m \neq 0\}$ is a complete set of representatives for $A/\sim$.

Proof. I have yet to solved this problem; I will do it ASAP. $\square$

3.3.16

Define the relation $\equiv_1$ on $\mathbb{R}$ by the rule: If $x, y \in \mathbb{R}$, then $x \equiv_1 y$ if $x - y$ is an integer.

(a) Show that $\equiv_1$ is an equivalence relation on $\mathbb{R}$.

Proof. Let us analyze if it is reflexive, let $x \in \mathbb{R}$. Then $x - x = 0 = 1 \cdot 0$; hence $x \equiv_1 x$. Let us check symmetry, let $x \equiv_1 y$. then there exists $k \in \mathbb{Z}$ such that $x - y = 1 \cdot k$. Thus $y - x = -1 \cdot k = 1 \cdot (-k)$, therefore $y \equiv_1 x$. Let us analyze transitivity, let $x \equiv_1 y$ and $y \equiv_1 z$, then $x - y = 1 \cdot k$ and $y - z = 1 \cdot m$. Hence $x - z = x - y + y - z = 1 \cdot k + 1 \cdot m = 1 \cdot (k + m)$. Therefore $x \equiv_1 z$. Thus we show that $\equiv_1$ is an equivalence relation in $\mathbb{R}$. $\square$

(b) What is $[1/2]_{\equiv_1}$?

Solution. $\{k + 1/2 \mid k \in \mathbb{Z}\}$ $\square$

(c) Describe the set of equivalence classes.

Solution. $\mathbb{R}/\equiv_1 = \{[x] \equiv_1 |x \in \mathbb{R}\} = \{\{r + k|k \in \mathbb{Z}|r \in [0, 1)\}\}$ $\square$

3.4 Functions

3.4.1

Find all functions $f : A \to B$ when:

(a) $A = \{1, 2\}$ and $B = \{1\}$.

Solution. $f(x) = 1,\ \forall x \in A.$ □

(b) $A = \{1, 2, 3\}$ and $B = \{1\}$.

Solution. $f(x) = 1,\ \forall x \in A.$ □

(c) $A = \{a_1, \ldots, a_n\}$ and $B = \{b\}$.

Solution. $f(x) = b,\ \forall x \in A.$ □

(d) $A = \{1\}$ and $B = \{1, 2\}$.

Solution. $f_1(1) = 1,\ f_2(1) = 2$ □

(e) $A = \{1\}$ and $B = \{1, 2, 3\}$.

Solution. $f_1(1) = 1,\ f_2(1) = 2,\ f_3(1) = 3$ □

(f) $A = \{a\}$ and $B = \{b_1, \ldots, b_n\}$.

Solution. $f(a) = b_1,\ f(a) = b_2,\ f(a) = b_3, ..., f(a) = b_n$ □

3.4.2

In each case state whether the given function is injective, surjective, and/or bijective.

(a) $f : \mathbb{R} \to \mathbb{R}, f(x) = 2x$ for $x \in \mathbb{R}$.

Solution. This is bijective. □

(b) $f : \mathbb{R} \to \mathbb{R}, f(x) = 3 - x$ for $x \in \mathbb{R}$.

Solution. This is bijective. □

(c) $f : \mathbb{R} \to \mathbb{R}, f(x) = x^2 + 2x + 3$ for $x \in \mathbb{R}$.

Solution. This is neither injective nor surjective. □

(d) $f : [0, \pi] \to [0, 1], f(x) = \sin(x)$ for $x \in [0, \pi]$.

Solution. This is surjective. ☐

(e) $f : \mathbb{R} \to \mathbb{R}^+, f(x) = e^{(x^2)}$ for $x \in \mathbb{R}$.

Solution. This is neither injective nor surjective. ☐

3.4.3

Define $g : \mathbb{Z} \to \mathbb{N}$ as follows:

$$g(x) = \begin{cases} 2x & \text{if } x \geq 0. \\ -2x - 1 & \text{if } x < 0. \end{cases}$$

(a) Evaluate $g(x)$ for $-5 \leq x \leq 5$.

Solution. $g(-5) = 9,\ g(-4) = 7,\ g(-3) = 5,\ g(-2) = 3,\ g(-1) = 1,\ g(0) = 0,\ g(1) = 2,\ g(2) = 4,\ g(3) = 6,\ g(4) = 8,\ g(5) = 10.$ ☐

(b) Describe the definition of g in a sentence.

Solution. The function g maps each integer x to $2x$ if $x \geq 0$ and to $-2x+1$ if $x < 0$. ☐

(c) Prove that if $x \in \mathbb{Z}$, then $g(x) \in \mathbb{N}$.

Proof. If $x \geq 0$, then $g(x) = 2x \geq 0$ is a non-negative integer, therefore $g(x) \in \mathbb{N}$. If $x < 0$, then $g(x) = -2x - 1 > 0$ is a positive integer, therefore $g(x) \in \mathbb{N}$. ☐

(d) Prove that g is injective.

Proof. We need to show that if $f(p) = f(q)$ then $p = q$. Since this is a piece-wise function with 2 options, we have 3 cases

Case 1: $p \geq 0$ and $q \geq 0$
$2p = 2q$, therefore $p = q$.

Case 2: $p < 0$ and $q < 0$
$-2p - 1 = -2q - 1$, therefore $p = q$.

Case 3: $p \geq 0$ and $q < 0$
$2p = -2q - 1$, an even number cannot be equal an odd one, so this case does not count. p and q must lie in the same option. Since $p = q$ in all cases, the function g is injective. ☐

(e) Prove that g is surjective.

Proof. We need to show that for every element $y \in \mathbb{N}$, there exists an $x \in \mathbb{Z}$ such that $g(x) = y$. Let $y \in \mathbb{N}$. We consider two cases for y:

Case 1: y is an even non-negative integer.
If y is an even non-negative integer, then $y = 2k$ for some integer $k \geq 0$. Let $x = k$. Since $k \geq 0$, we use the first rule for $g(x)$: $g(x) = g(k) = 2k = y$. Here, $x = k$ is an integer, and $g(k) = y$ is in $\mathbb{N}$.

Case 2: y is an odd positive integer.
If y is an odd positive integer, then $y = 2k - 1$ for some integer $k \geq 1$. We want to find x such that $g(x) = y$. Since y is odd, x must be negative according to the definition of g. So, we set $-2x - 1 = y$. $-2x - 1 = 2k - 1$ $-2x = 2k$ $x = -k$. Since $k \geq 1$, $x = -k$ is a negative integer ($x < 0$). Let's check $g(x)$: $g(x) = g(-k) = -2(-k) - 1 = 2k - 1 = y$. Here, $x = -k$ is an integer, and $g(-k) = y$ is in $\mathbb{N}$. Since every natural number y can be mapped to by some integer x, the function g is surjective. ☐

(f) Find g^{-1}.

Solution. $y = \frac{x}{2}$ if $x \geq 0$ and $y = \frac{-x-1}{2}$ if $x < 0$ ☐

3.4.4

Let $a, b \in \mathbb{R}$ with $a \neq 0$. Define $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = a \cdot x + b$.

(a) Show f is injective.

Proof. We need to show that if $f(p) = f(q)$ then $p = q$. $f(p) = ap + b$, $f(q) = aq + b$, equalizing we get $p = q$, therefore this function is injective (one-to-one). ☐

(b) What is Ran(f)?

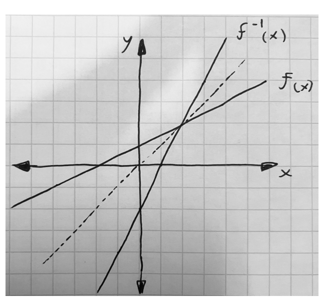
Solution. $\text{Ran}(f) = \mathbb{R} - \{b\}$ ☐

(c) Find $f^{-1}(x)$ for $x \in \text{Ran}(f)$.

Solution. $f^{-1}(x) = \frac{x-b}{a}$ ☐

(d) Sketch the graphs of f and f^{-1}.

Solution. They are symmetric with respect to $y = x$



$\square$

(e) As a check of (c) show that if $f(x) = -x$, then $f = f^{-1}$.

Proof. $f^{-1}(x) = \frac{(-ax+b)-b}{a} = \frac{-ax}{a} = -x$ $\square$

3.4.10

Let $f : A \to B$ with $A_1 \subseteq A$ and $B_1 \subseteq B$.

(a) Prove, or disprove and salvage: $f^{-1}(f(B_1)) = B_1$.

Proof. $\square$

(b) Prove, or disprove and salvage: $f(f^{-1}(A_1)) = A_1$.

Proof. $\square$