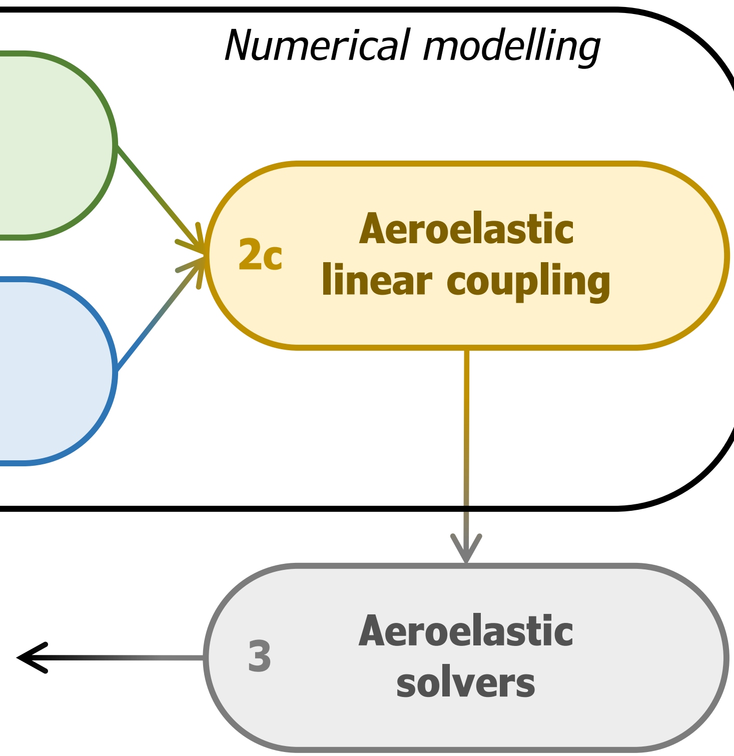
UNIVERSITAT POLITÈCNICA DE CATALUNYA
BARCELONATECH
Escola Superior d'Enginyeries Industrial, Aeroespacial i Audiovisual de Terrassa

Project

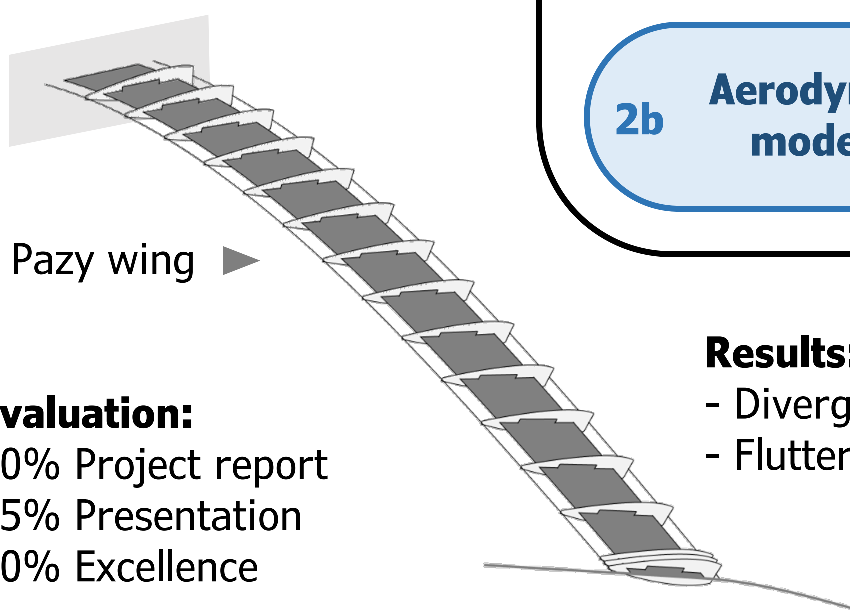
Setup of a virtual laboratory for studying aeroelastic problems

Goal: Implement a set of MATLAB functions to perform different kinds of aeroelastic analysis (e.g. assess divergence conditions, flutter study, unsteady aerodynamics, etc.).



Results:
- Divergence
- Flutter study

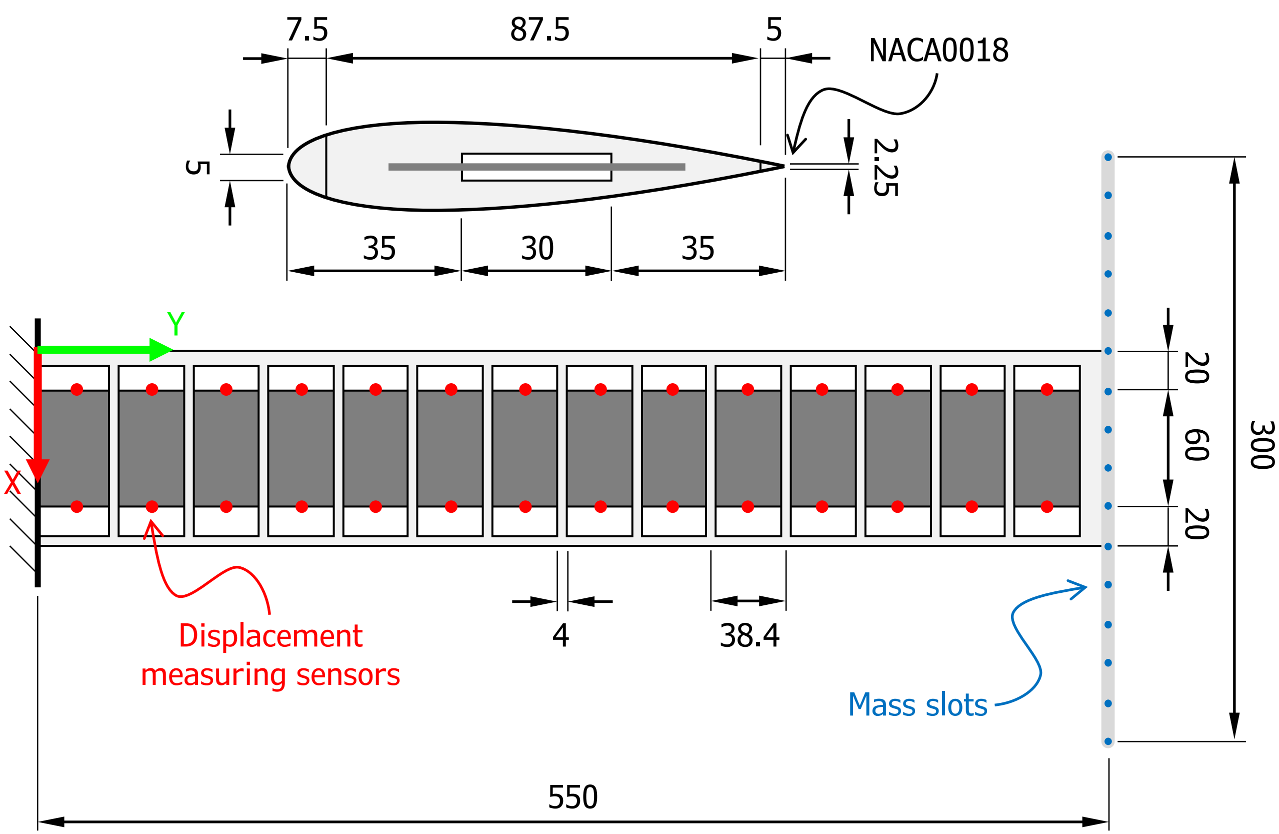
Pazy wing ▶



Evaluation:
40% Project report
15% Presentation
10% Excellence

Pazy wing

Highly flexible wing model designed to study aeroelastic phenomena associated with geometrically nonlinear deflections with the aim of becoming a benchmark for nonlinear aeroelastic simulation models.



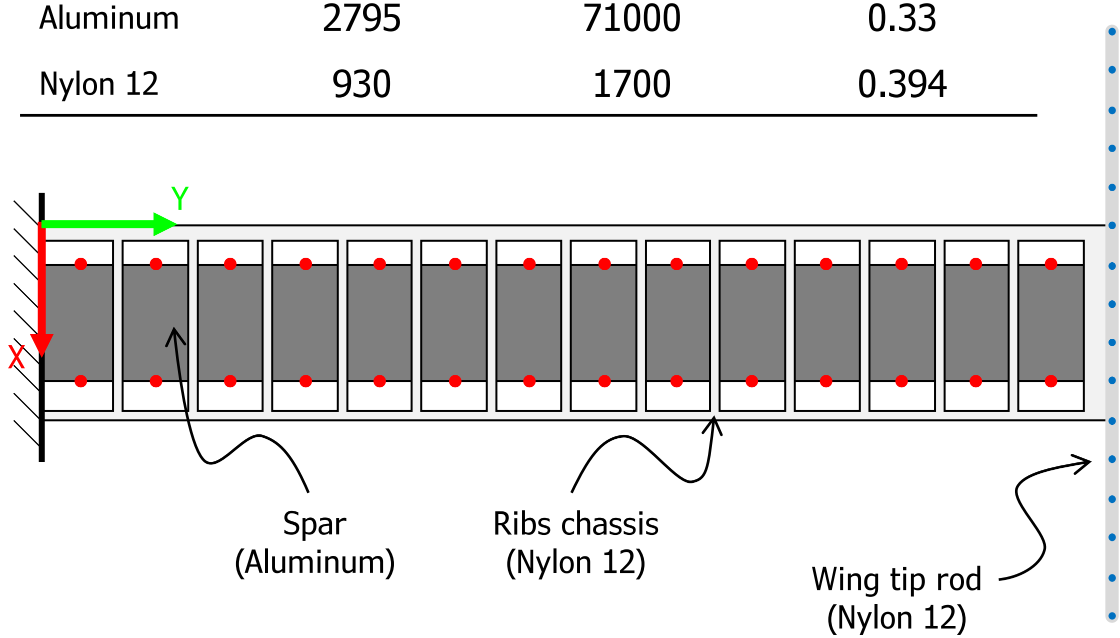
Avin, *et al*. Experimental Aeroelastic Benchmark of a Very Flexible Wing. *AIAA* (2022)

(*) Magnitudes in mm

Pazy wing

Highly flexible wing model designed to study aeroelastic phenomena associated with geometrically nonlinear deflections with the aim of becoming a benchmark for nonlinear aeroelastic simulation models.

Material	Density (kg m^{-3})	Young's Modulus (MPa)	Poisson's ratio
Aluminum	2795	71000	0.33
Nylon 12	930	1700	0.394



Avin, *et al*. Experimental Aeroelastic Benchmark of a Very Flexible Wing. *AIAA* (2022)

Virtual experiments

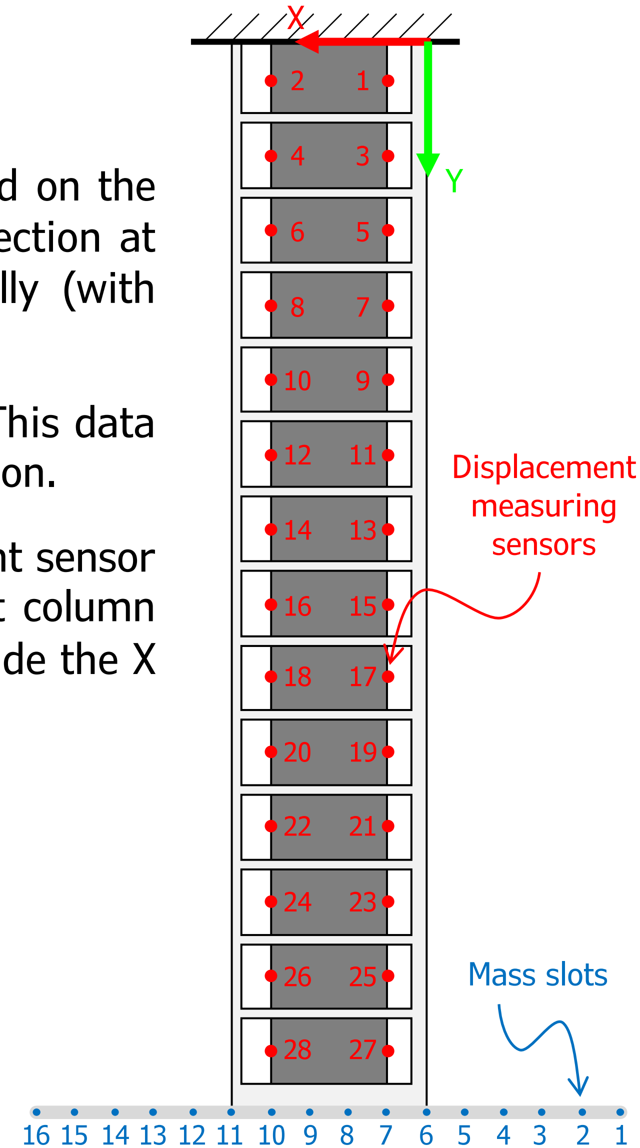
We can put masses on the wing tip rod to apply a certain load on the wing structure and obtain a measurement of the vertical deflection at the locations with displacement sensors. This is done virtually (with Matlab) using the **`PazyWingLoad`** function.

1) Determine the mass that will be added to each mass slot. This data is provided in an array as input to the **`PasyWingLoad`** function.

2) Obtain the vertical deflection measured at each displacement sensor in a matrix as output of the **`PazyWingLoad`** function. First column provides the measurement. Second and third columns provide the X and Y coordinates of each sensor, respectively.

Goal: Obtain the effective structural properties of the wing:

- Shear center position: x_{sc}
- Torsional stiffness: $\overline{GJ}$
- Bending stiffness: $\overline{EI}$

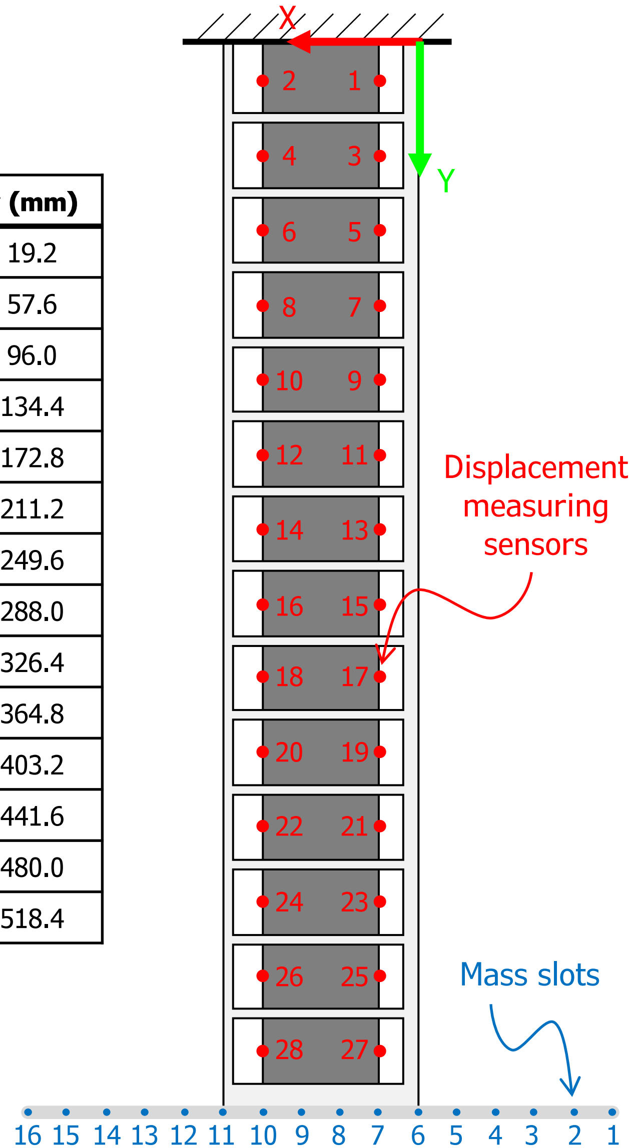


Virtual experiments

Mass slot	x (mm)
1	-100
2	-80
3	-60
4	-40
5	-20
6	0
7	20
8	40
9	60
10	80
11	100
12	120
13	140
14	160
15	180
16	200

Sensor	x (mm)	y (mm)
1	20	19.2
3	20	57.6
5	20	96.0
7	20	134.4
9	20	172.8
11	20	211.2
13	20	249.6
15	20	288.0
17	20	326.4
19	20	364.8
21	20	403.2
23	20	441.6
25	20	480.0
27	20	518.4

Sensor	x (mm)	y (mm)
2	80	19.2
4	80	57.6
6	80	96.0
8	80	134.4
10	80	172.8
12	80	211.2
14	80	249.6
16	80	288.0
18	80	326.4
20	80	364.8
22	80	403.2
24	80	441.6
26	80	480.0
28	80	518.4

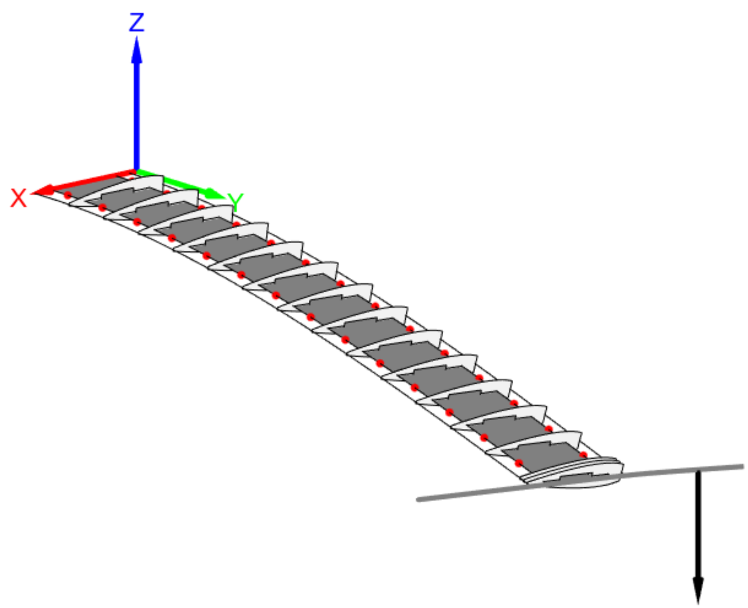


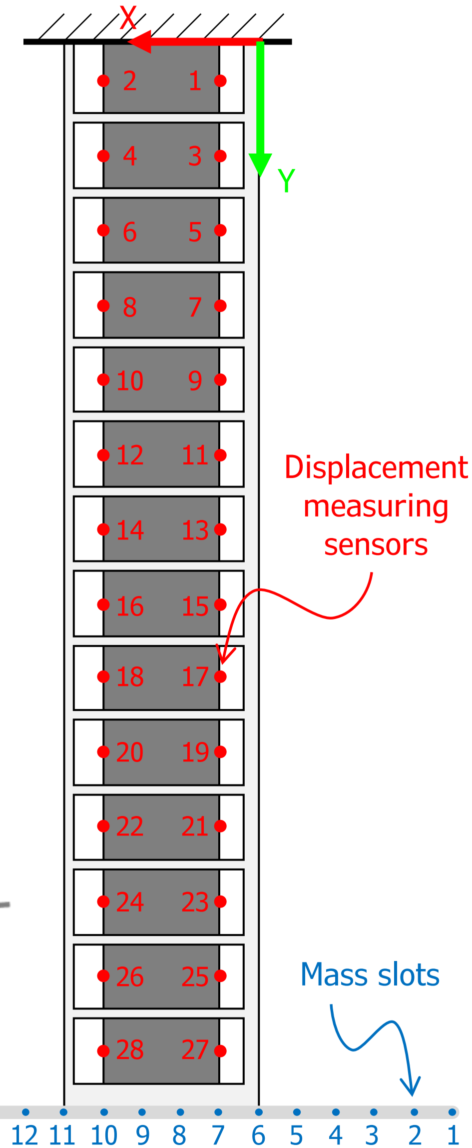
Virtual experiments

Example: 1 kg mass applied on slot #3

```
fload = [0; 0; 1; 0; 0; 0; 0; 0; 0; 0; 0; 0; 0; 0; 0; 0];
displ = PazyWingLoad(fload);
w1 = displ(1:2:end,1)*1000;
w2 = displ(2:2:end,1)*1000;
```

Sensor	w_1 (mm)	Sensor	w_2 (mm)
1	-0.1118	2	-0.0715
3	-1.6801	4	-1.328
5	-4.6564	6	-3.9671
7	-9.0056	8	-7.928
9	-14.5062	10	-13.0452
11	-21.0775	12	-19.2978
13	-28.5962	14	-26.5707
15	-37.043	16	-34.553
17	-46.1637	18	-43.282
19	-55.932	20	-52.7517
21	-66.2797	22	-62.7152
23	-77.0194	24	-73.1441
25	-88.057	26	-83.792
27	-99.3177	28	-94.7388





Virtual experiments

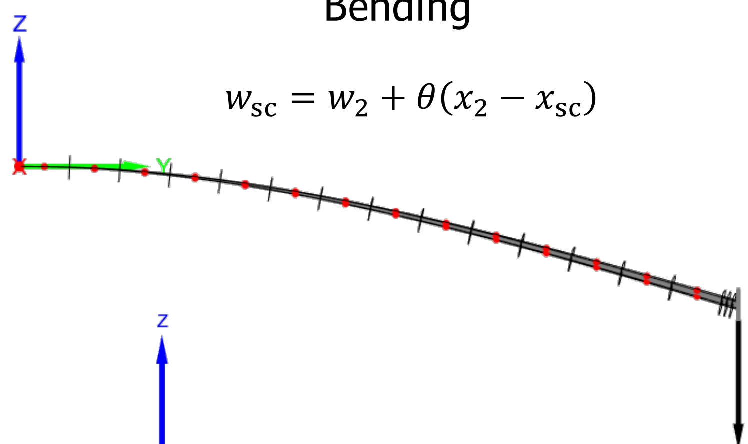
Example: 1 kg mass applied on slot #3

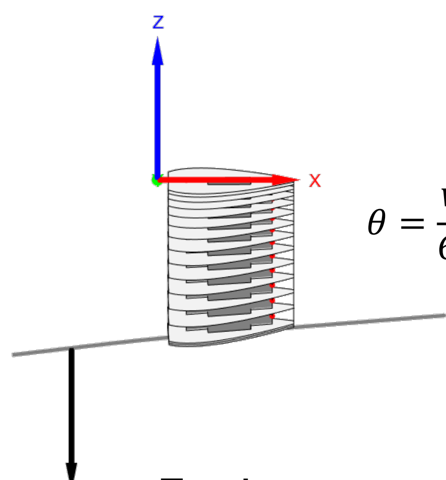
```
fload = [0; 0; 1; 0; 0; 0; 0; 0; 0; 0; 0; 0; 0; 0; 0; 0];
displ = PazyWingLoad(fload);
w1 = displ(1:2:end,1)*1000;
w2 = displ(2:2:end,1)*1000;
```

Sensor	w_1 (mm)	Sensor	w_2 (mm)
1	-0.1118	2	-0.0715
3	-1.6801	4	-1.328
5	-4.6564	6	-3.9671
7	-9.0056	8	-7.928
9	-14.5062	10	-13.0452
11	-21.0775	12	-19.2978
13	-28.5962	14	-26.5707
15	-37.043	16	-34.553
17	-46.1637	18	-43.282
19	-55.932	20	-52.7517
21	-66.2797	22	-62.7152
23	-77.0194	24	-73.1441
25	-88.057	26	-83.792
27	-99.3177	28	-94.7388

Bending

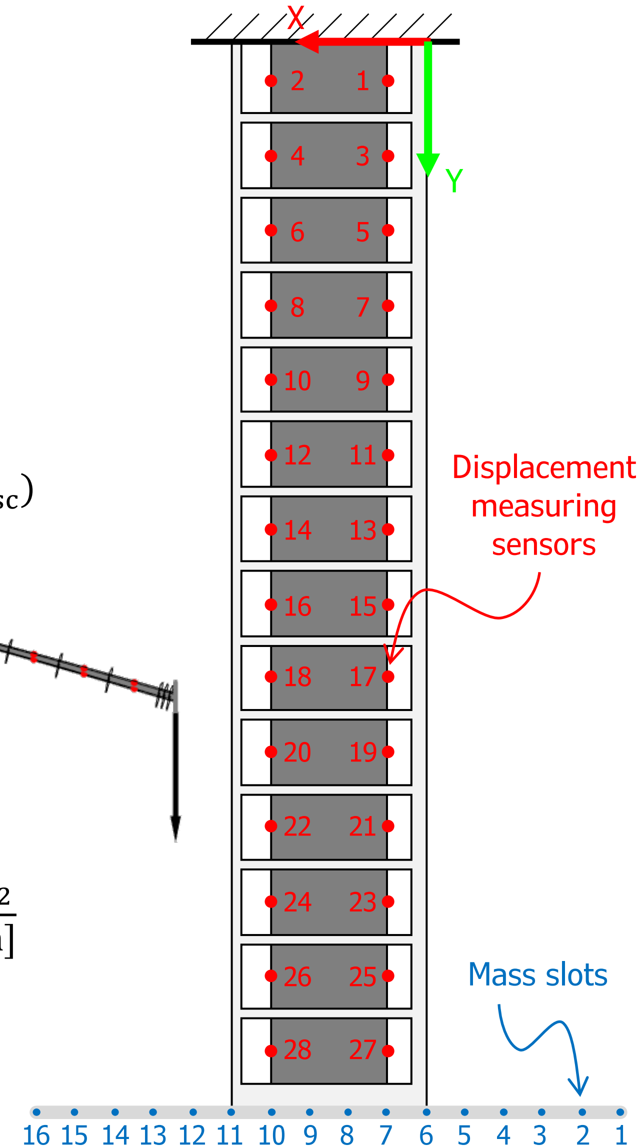
$$w_{sc} = w_2 + \theta(x_2 - x_{sc})$$





$$\theta = \frac{w_1 - w_2}{60\ [\text{mm}]}$$

Torsion



Virtual experiments

Beams theory (Euler-Bernoulli bending + St. Venant torsion):

We assume the section moves as a rigid body, so the displacement field is assumed as:

$$u_x(x,y,z) = z\theta(y)$$

$$u_y(x,y,z) = -z\frac{\partial w_{sc}(y)}{\partial y} = -z\gamma(y)$$

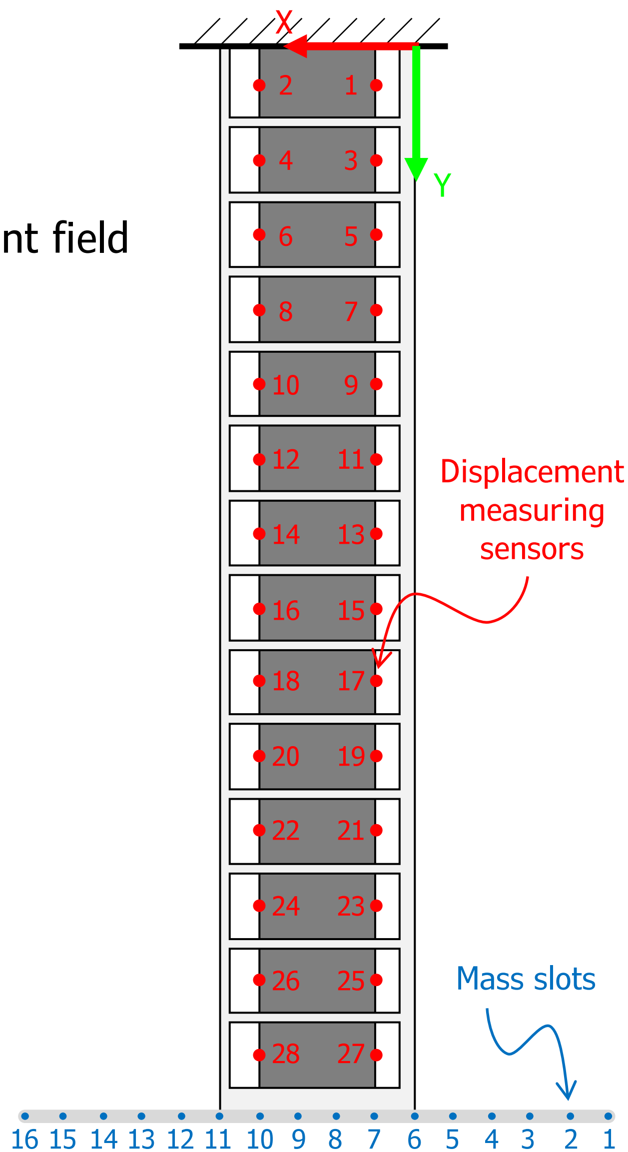
$$u_z(x,y,z) = w_{sc}(y) - (x - x_{sc})\theta(y)$$

Then, the expected strain field is:

$$\varepsilon(x,y,z) = -z\frac{\partial^2 w_{sc}(y)}{\partial y^2} = -z\frac{\partial \gamma(y)}{\partial y}$$

$$\gamma_{xy}(x,y,z) = z\frac{\partial \theta(y)}{\partial y}$$

$$\gamma_{yz}(x,y,z) = -(x - x_{sc})\frac{\partial \theta(y)}{\partial y}$$



Virtual experiments

Beams theory (Euler-Bernoulli bending + St. Venant torsion):

The stress strain constitutive relationship gives:

$$\sigma(x,y,z) = E\varepsilon(x,y,z)$$

$$\tau_{xy}(x,y,z) = G\gamma_{xy}(x,y,z)$$

$$\tau_{yz}(x,y,z) = G\gamma_{yz}(x,y,z)$$

Integrating over the section, we get:

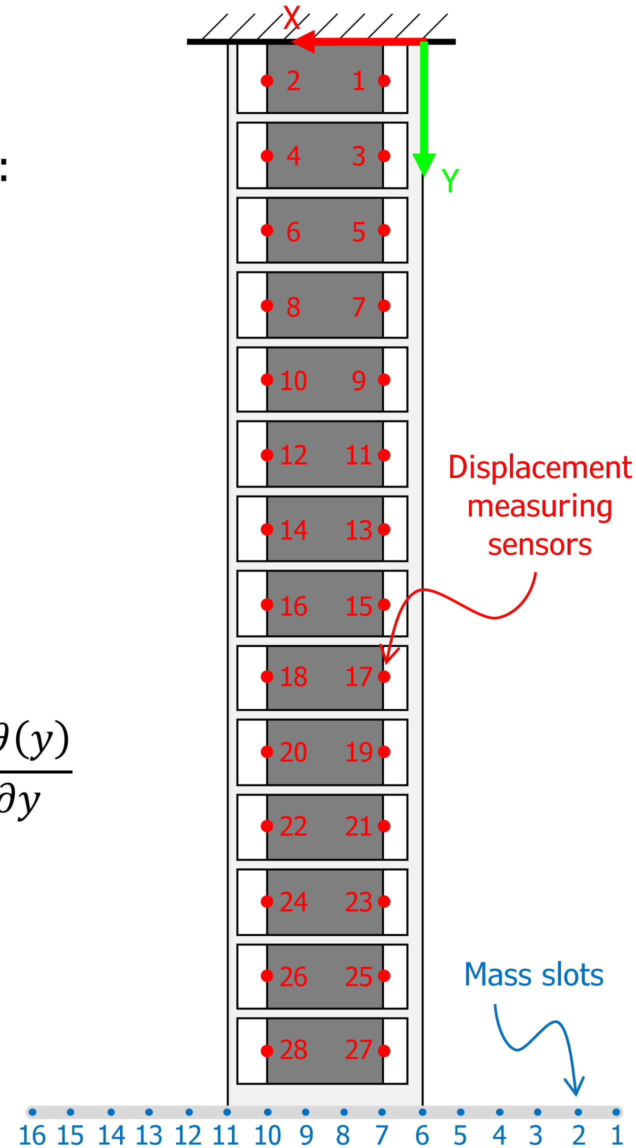
- Internal torsion moment:

$$T_{\text{int}}(y) = \int_A \Big(z\tau_{xy}(x,y,z) - (x - x_{\text{sc}})\tau_{yz}(x,y,z)\Big)\,dA = \overline{GJ}\,\frac{\partial\theta(y)}{\partial y}$$

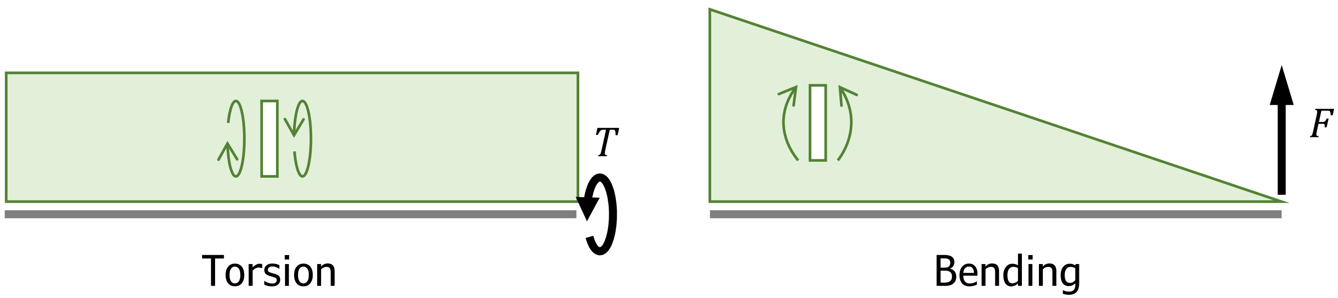
- Internal bending moment and shear force:

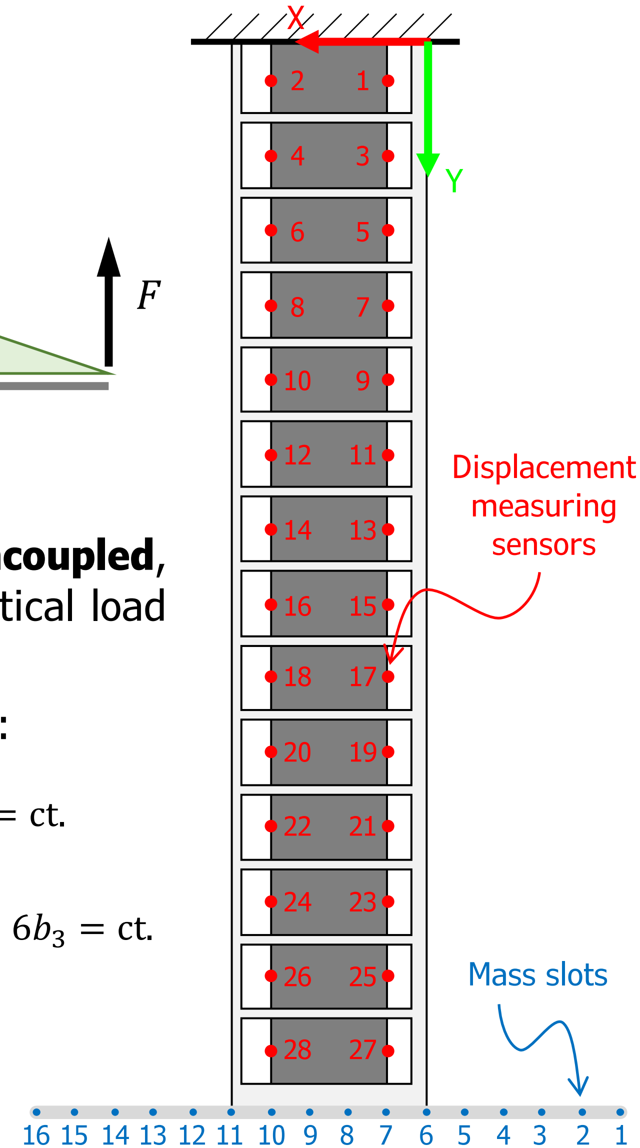
$$M_{\text{int}}(y) = -\int_A z\sigma(x,y,z)dA = \overline{EI}\,\frac{\partial^2 w_{\text{sc}}(y)}{\partial y^2}$$

$$Q_{\text{int}}(y) = -\frac{\partial M}{\partial y} = -\overline{EI}\,\frac{\partial^3 w_{\text{sc}}(y)}{\partial y^3}$$



Virtual experiments





We work under the hypothesis that torsion and bending are **uncoupled**, i.e., we can define a shear center, x_{sc}, in which applying a vertical load does not generate an apparent twist of the wing.

In this case, applying a point load F at any point in the wing tip:

$$T_{\text{int}}(y) = F(x_{sc} - x) = \text{ct.} \;\rightarrow\; \theta(y) = a_0 + a_1 y \;\rightarrow\; \frac{\partial \theta(y)}{\partial y} = a_1 = \text{ct.}$$

$$Q_{\text{int}}(y) = F = \text{ct.} \;\rightarrow\; w_{sc}(y) = b_0 + b_1 y + b_2 y^2 + b_3 y^3 \;\rightarrow\; \frac{\partial^3 w_{sc}(y)}{\partial y^3} = 6 b_3 = \text{ct.}$$

Therefore:

$$\overline{GJ} = \frac{F(x_{sc} - x)}{a_1}; \quad \overline{EI} = -\frac{F}{6 b_3}$$

Virtual experiments

Step 1: Perform test/s to find the average shear center position x_{sc}.

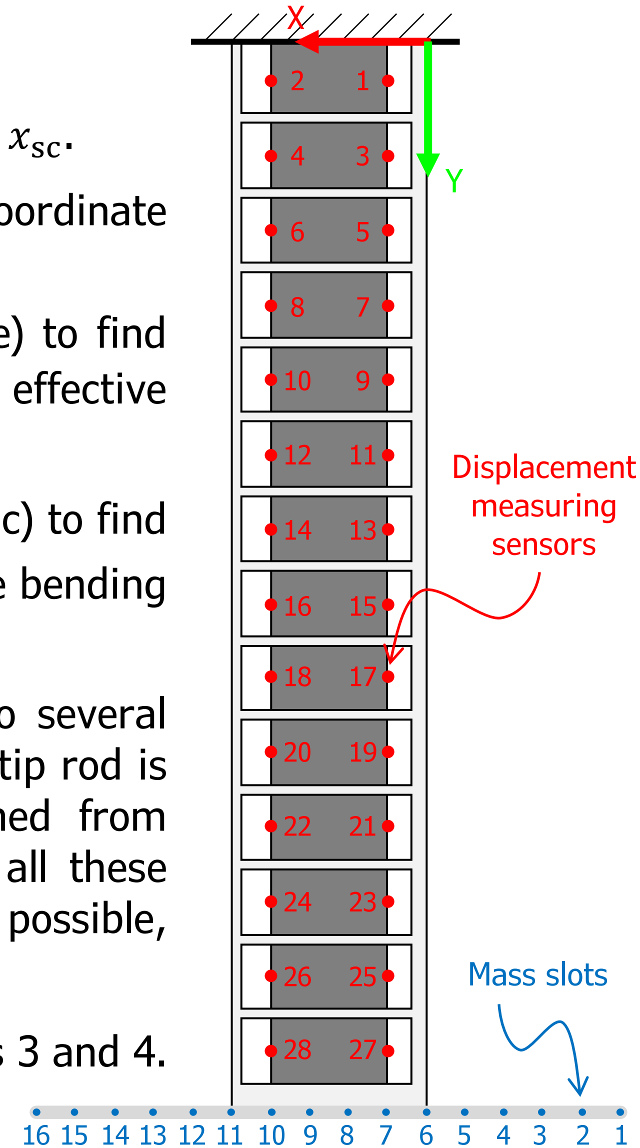
Step 2: Perform test/s and evaluate $\theta^{(i)}$ and $w_{sc}^{(i)}$ at each $y^{(i)}$ coordinate of the displacement measurement sensors.

Step 3: Fit $\left(y^{(i)}, \theta^{(i)}\right)$ points into a first order polynomial (line) to find the corresponding a_1 coefficient, and then compute the effective torsional stiffness $\overline{GJ}$.

Step 4: Fit $\left(y^{(i)}, w_{sc}^{(i)}\right)$ points into a third order polynomial (cubic) to find the corresponding b_3 coefficient, and then compute the effective bending stiffness $\overline{EI}$.

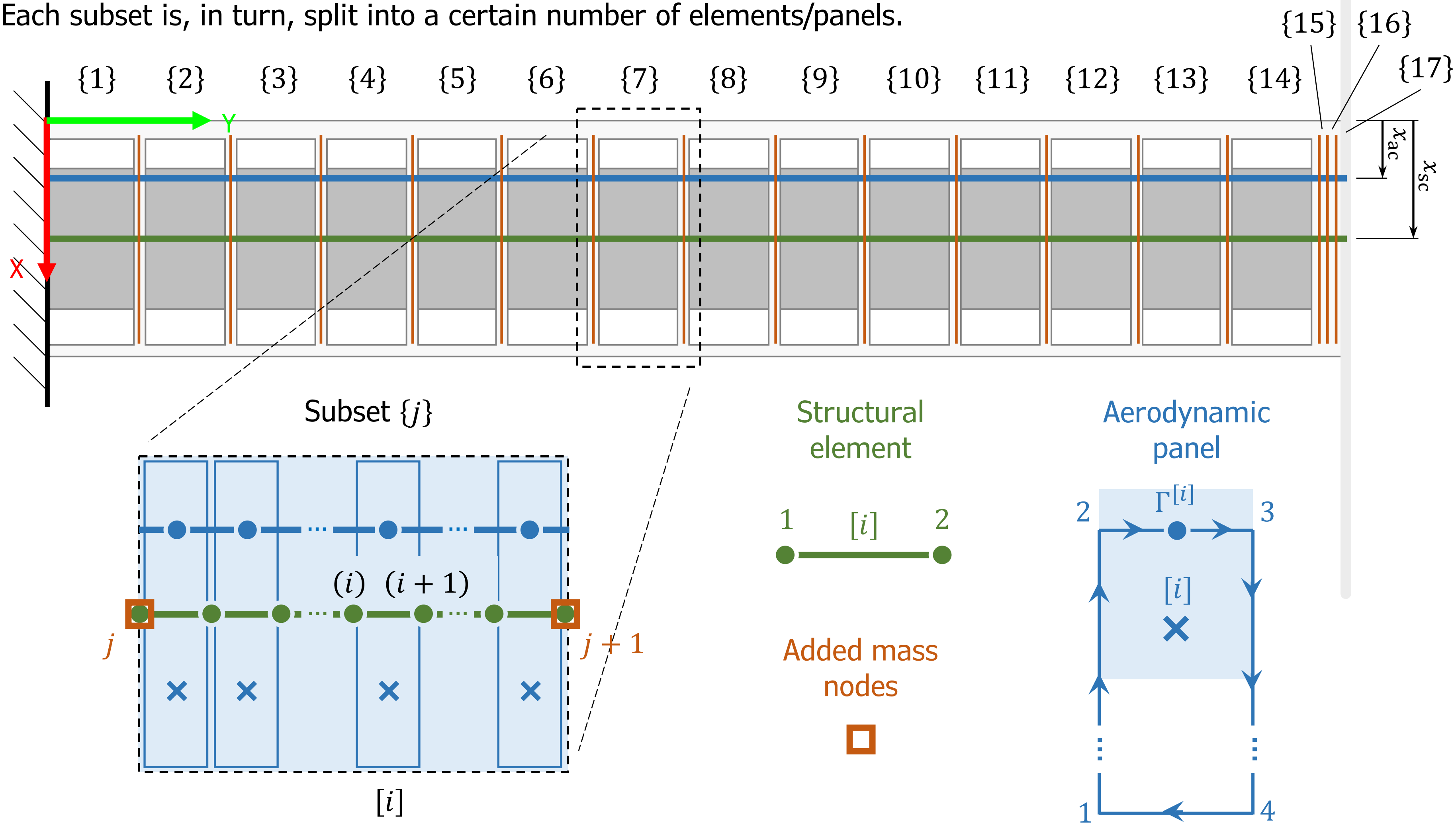
Note! There might be certain variability in the results due to several factors (actual setting violates beam's theory hypothesis, wing tip rod is not infinitely stiff, applied forces and measurements obtained from sensors are not exact, etc.). It is important to be aware of all these potential factors when justifying the results obtained and, when possible, try to quantify their influence.

Hint! You can use **`polyfit`** function of Matlab to carry out steps 3 and 4.



Numerical modelling

The wing is split into 17 subsets, already predetermined by regions between consecutive ribs.
Each subset is, in turn, split into a certain number of elements/panels.



Numerical modelling

PRECPROCESS

- Nodal coordinates array: $\{\mathbf{y}\} = \{\cdots \quad y^{(i)} \quad \cdots\}^{\mathrm{T}}$

  Note: At least, one must guarantee that a global node exists at the location of each rib (see Table).

- Element connectivities matrix:

$$[\mathbf{T}_{\mathrm{n}}] = \begin{bmatrix} & \vdots \\ i & i+1 \\ & \vdots \end{bmatrix}$$

- Subset connectivities arrays. For each subset $\{j\}$, an array $\mathbf{T}_{\mathrm{s}}\{j\}$ is defined containing the indices of the elements that are part of the subset. According to this, $\mathbf{T}_{\mathrm{n}}(\mathbf{T}_{\mathrm{s}}\{j\}(1),1)$ and $\mathbf{T}_{\mathrm{n}}\left(\mathbf{T}_{\mathrm{s}}\{j\}\left(n_j\right),2\right)$ give the indices of the global nodes corresponding to the bounds of the j-th subset (considering the number of elements in such subset is n_j):

$$y^{(\mathbf{T}_{\mathrm{n}}(\mathbf{T}_{\mathrm{s}}\{j\}(1),1))} = y_j; \quad y^{\left(\mathbf{T}_{\mathrm{n}}\left(\mathbf{T}_{\mathrm{s}}\{j\}\left(n_j\right),2\right)\right)} = y_{j+1}$$

Rib	Node j	y_j (mm)
-	1	0.0
1	2	38.4
2	3	76.8
3	4	115.2
4	5	153.6
5	6	192.0
6	7	230.4
7	8	268.8
8	9	307.2
9	10	345.6
10	11	384.0
11	12	422.4
12	13	460.8
13	14	499.2
14	15	537.6
15	16	541.6
16	17	545.6
-	18	550.0

Numerical modelling – Slender beams model

STRUCTURAL MATRICES

We have 3 degrees of freedom (DOFs) per node: the elastic twist θ, the vertical deflection w_{sc}, and its rate of change $\gamma = \partial w_{\mathrm{sc}}/\partial y$ (= bending angle). The total number of DOFs will be $N = 3n$, with n being the total number of (global) nodes.

Initialization:

$$[\bar{\mathbf{K}}] = [\mathbf{0}]_{N\times N}$$

$$[\bar{\mathbf{M}}] = [\mathbf{0}]_{N\times N}$$

For each element $[i]$:

$$l^{[i]} = \mathbf{y}\big(\mathbf{T}_{\mathrm{n}}(i,2)\big) - \mathbf{y}\big(\mathbf{T}_{\mathrm{n}}(i,2)\big)$$

For each degree of freedom $k = 1 \dots 3$

$$I(k,1) = 3 \times (\mathbf{T}_{\mathrm{n}}(i,1) - 1) + k$$

$$I(3+k,1) = 3 \times (\mathbf{T}_{\mathrm{n}}(i,2) - 1) + k$$

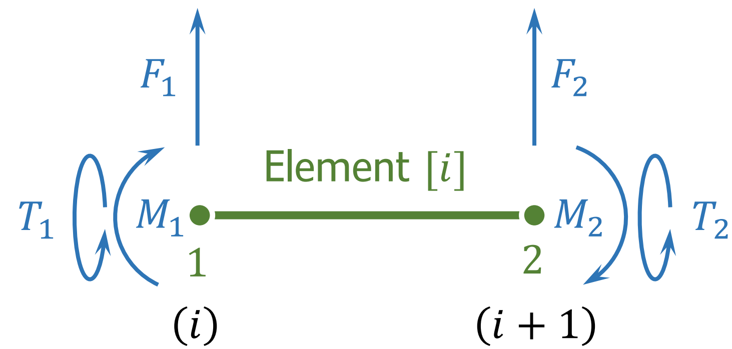
Next k

$$\bar{\mathbf{K}}(I,I) = \bar{\mathbf{K}}(I,I) + \boxed{\left[\mathbf{K}^{[i]}\right]}$$

$$\bar{\mathbf{M}}(I,I) = \bar{\mathbf{M}}(I,I) + \boxed{\left[\mathbf{M}^{[i]}\right]}$$

See next slides

Next element $[i]$



$$\left\{\mathbf{u}^{[i]}\right\} = \begin{Bmatrix} \theta^{(i)} \\ w_{\mathrm{sc}}^{(i)} \\ \gamma^{(i)} \\ \theta^{(i+1)} \\ w_{\mathrm{sc}}^{(i+1)} \\ \gamma^{(i+1)} \end{Bmatrix}; \quad \left\{\mathbf{f}^{[i]}\right\} = \begin{Bmatrix} T_1^{[i]} \\ F_1^{[i]} \\ M_1^{[i]} \\ T_2^{[i]} \\ F_2^{[i]} \\ M_2^{[i]} \end{Bmatrix}$$

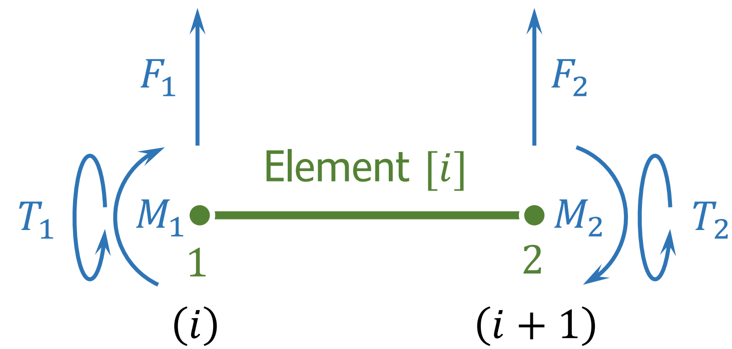
Numerical modelling – Slender beams model

Element stiffness matrix

$$[\mathbf{K}^{[i]}] = [\mathbf{K}_\mathrm{t}^{[i]}] + [\mathbf{K}_\mathrm{b}^{[i]}]$$

- Torsion:

$$[\mathbf{K}_\mathrm{t}^{[i]}] = \frac{\overline{GJ}}{l^{[i]}}\begin{bmatrix} 1 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$



- Bending:

$$[\mathbf{K}_\mathrm{b}^{[i]}] = \frac{\overline{EI}}{(l^{[i]})^3}\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 12 & 6l^{[i]} & 0 & -12 & 6l^{[i]} \\ 0 & 6l^{[i]} & 4\left(l^{[i]}\right)^2 & 0 & -6l^{[i]} & 2\left(l^{[i]}\right)^2 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -12 & -6l^{[i]} & 0 & 12 & -6l^{[i]} \\ 0 & 6l^{[i]} & 2\left(l^{[i]}\right)^2 & 0 & -6l^{[i]} & 4\left(l^{[i]}\right)^2 \end{bmatrix}$$

$$\{\mathbf{u}^{[i]}\} = \begin{Bmatrix} \theta^{(i)} \\ w_{sc}^{(i)} \\ \gamma^{(i)} \\ \theta^{(i+1)} \\ w_{sc}^{(i+1)} \\ \gamma^{(i+1)} \end{Bmatrix}; \quad \{\mathbf{f}^{[i]}\} = \begin{Bmatrix} T_1^{[i]} \\ F_1^{[i]} \\ M_1^{[i]} \\ T_2^{[i]} \\ F_2^{[i]} \\ M_2^{[i]} \end{Bmatrix}$$

Numerical modelling – Slender beams model

Element mass matrix

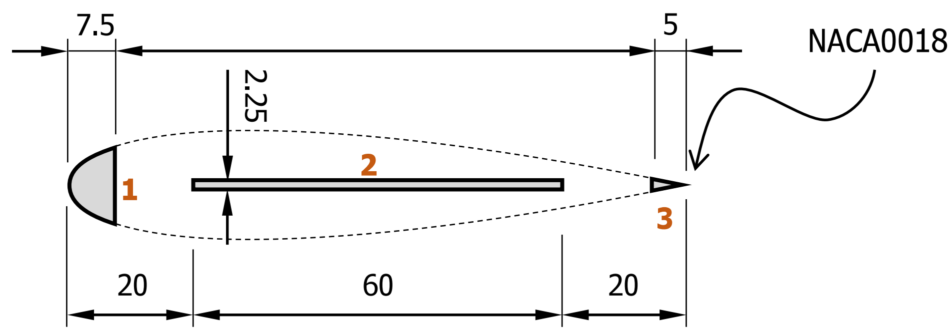
$$\left[\mathbf{M}^{[i]}\right] = [\mathbf{d}]^{\mathrm{T}}\left(\left[\mathbf{M}_{\mathrm{t}}'^{[i]}\right] + \left[\mathbf{M}_{\mathrm{b}}'^{[i]}\right]\right)[\mathbf{d}]$$

$$\overline{\rho A} = \sum_{m=1}^{3} \rho^{(m)} A^{(m)}\,; \qquad x_{\mathrm{cm}} = \frac{1}{\overline{\rho A}} \sum_{m=1}^{3} \rho^{(m)} \int_{A^{(m)}} x\,dA$$

$$I_{\mathrm{cm}} = \sum_{m=1}^{3} \rho^{(m)} \int_{A^{(m)}} (x - x_{\mathrm{cm}})^2\,dA\,; \qquad d = x_{\mathrm{cm}} - x_{\mathrm{sc}}$$

- Torsion:

$$\left[\mathbf{M}_{\mathrm{t}}'^{[i]}\right] = \frac{I_{\mathrm{cm}}\, l^{[i]}}{6}\begin{bmatrix} 2 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$



- Bending:

$$\left[\mathbf{M}_{\mathrm{b}}'^{[i]}\right] = \frac{\overline{\rho A}\, l^{[i]}}{420}\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 156 & 22l^{[i]} & 0 & 54 & -13l^{[i]} \\ 0 & 22l^{[i]} & 4\left(l^{[i]}\right)^2 & 0 & 13l^{[i]} & -3\left(l^{[i]}\right)^2 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 54 & 13l^{[i]} & 0 & 156 & -22l^{[i]} \\ 0 & -13l^{[i]} & -3\left(l^{[i]}\right)^2 & 0 & -22l^{[i]} & 4\left(l^{[i]}\right)^2 \end{bmatrix}$$

- Offset matrix:

$$[\mathbf{d}] = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ -d & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -d & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Numerical modelling – Added ribs masses

Once the mass matrix is assembled and $[\bar{\mathbf{M}}]$ is obtained, the inertial effect of the ribs can be accounted for by adding the following terms at the degrees of freedom associated to the corresponding y_j positions. In particular:

For each $j = 1 \dots 16$

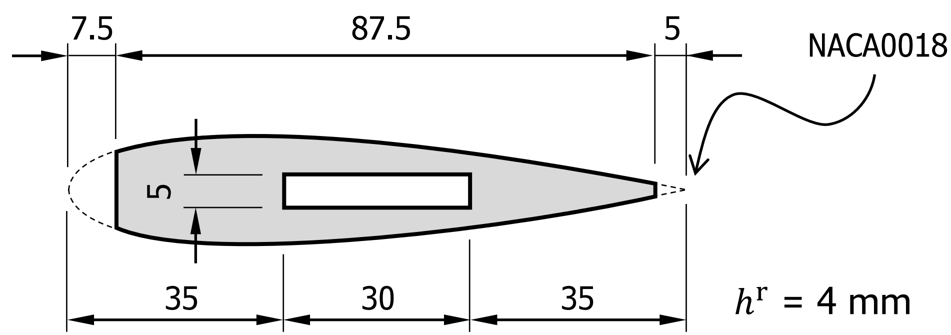
For each degree of freedom k

$$I(k) = 3 \times \left(\mathbf{T}_\mathrm{n}\left(\mathbf{T}_\mathrm{s}\{j\}(n_j), 2\right) - 1\right) + k$$

Next k

$$\bar{\mathbf{M}}(I,I) = \bar{\mathbf{M}}(I,I) + [\mathbf{M}_\mathrm{r}]$$

Next j

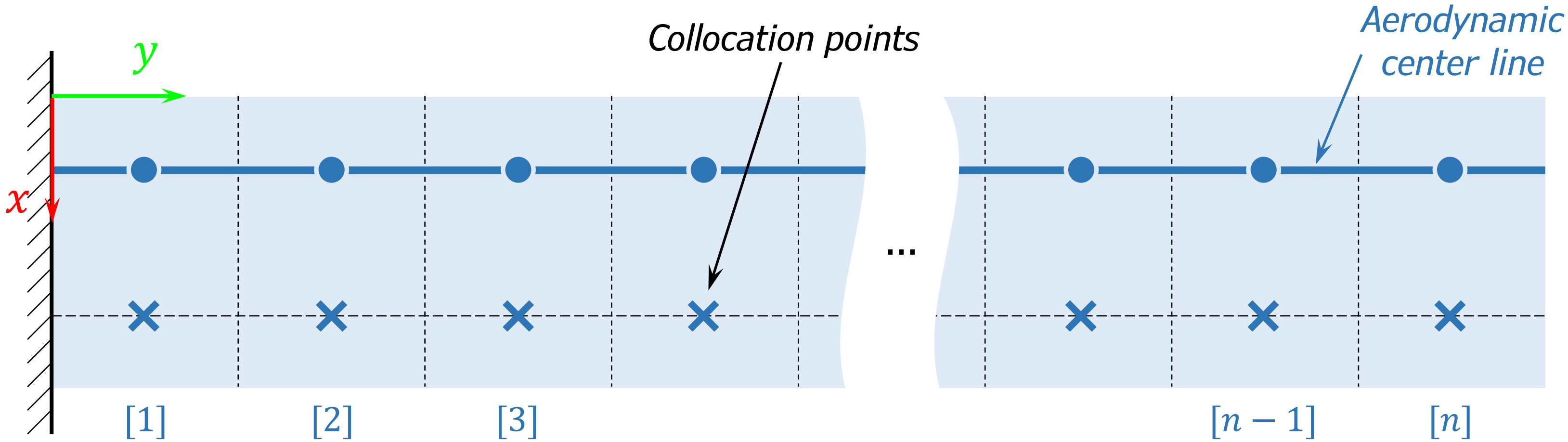


with:

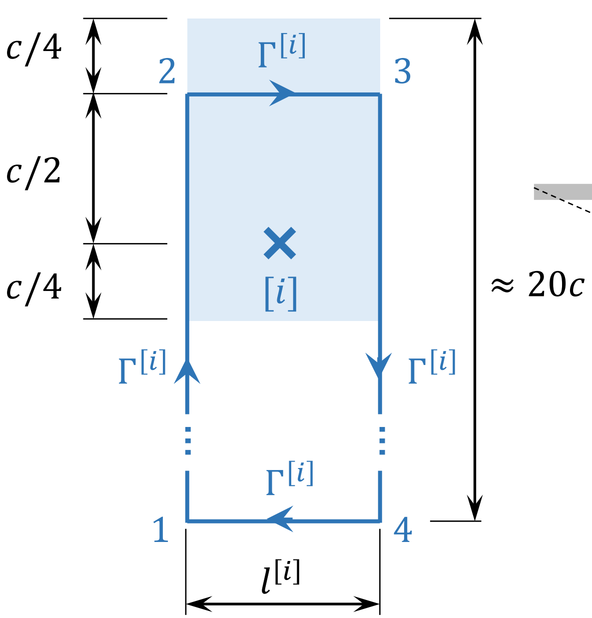
$$[\mathbf{M}_\mathrm{r}] = h^\mathrm{r} \begin{bmatrix} I^\mathrm{r}_\mathrm{cm} + \rho^\mathrm{r} A^\mathrm{r} (x^\mathrm{r}_\mathrm{cm} - x_\mathrm{sc})^2 & -\rho^\mathrm{r} A^\mathrm{r} (x^\mathrm{r}_\mathrm{cm} - x_\mathrm{sc}) & 0 \\ -\rho^\mathrm{r} A^\mathrm{r} (x^\mathrm{r}_\mathrm{cm} - x_\mathrm{sc}) & \rho^\mathrm{r} A^\mathrm{r} & 0 \\ 0 & 0 & I^\mathrm{r}_x \end{bmatrix}$$

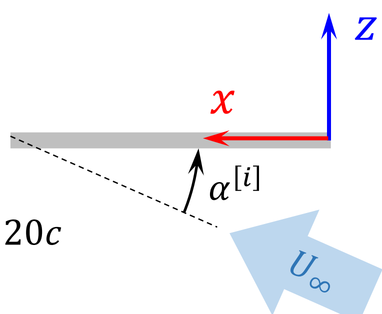
$$x^\mathrm{r}_\mathrm{cm} = \frac{1}{A^\mathrm{r}} \int_{A^\mathrm{r}} x dA\,; \quad I^\mathrm{r}_\mathrm{cm} = \int_{A^\mathrm{r}} \rho^\mathrm{r} (x - x_\mathrm{cm})^2 dA\,; \quad I^\mathrm{r}_x = \int_{A^\mathrm{r}} \rho^\mathrm{r} z^2 dA$$

Numerical modelling – Lifting-line surface analysis



Horseshoe element $[i]$**:**





Discretization:

- Surface area of the element: $S^{[i]}$
- Normal vector of the element: $\mathbf{n}^{[i]}$
- Coordinates of collocation point: $\mathbf{x}^{[i]}$

Kutta condition for each element:

$$\left(\sum_{j=1}^{n} \left(\boldsymbol{v}_{12}^{[j]} + \boldsymbol{v}_{23}^{[j]} + \boldsymbol{v}_{34}^{[j]} \right) \Big|_{\mathbf{x}=\mathbf{x}^{[i]}} + \mathbf{U}_{\infty} \right) \cdot \mathbf{n}^{[i]} = 0$$

Numerical modelling – Lifting-line surface analysis

Induced velocity at point $\mathbf{x}$ due to a vortex segment from $\mathbf{x}_j$ to $\mathbf{x}_k$ with vorticity $\Gamma^{[i]}$:

$$\boldsymbol{v}_{jk}^{[i]}(\mathbf{x}) = \frac{\Gamma^{[i]}}{4\pi}\frac{\mathbf{r}_j \times \mathbf{r}_k}{\left\|\mathbf{r}_j \times \mathbf{r}_k\right\|^2}\left(\frac{\mathbf{l}^{[i]}\cdot\mathbf{r}_j}{\left\|\mathbf{r}_j\right\|} - \frac{\mathbf{l}^{[i]}\cdot\mathbf{r}_k}{\left\|\mathbf{r}_k\right\|}\right); \quad \mathbf{r}_j = \mathbf{x} - \mathbf{x}_j; \quad \mathbf{l}^{[i]} = \mathbf{x}_k - \mathbf{x}_j$$

System of equations:

$$[\mathbf{A}]\{\boldsymbol{\Gamma}\} = -U_\infty\{\boldsymbol{\alpha}\}$$

$$\begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1n} \\ A_{21} & A_{22} & \cdots & A_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ A_{n1} & A_{n2} & \cdots & A_{nn} \end{bmatrix}\begin{Bmatrix} \Gamma^{[1]} \\ \Gamma^{[2]} \\ \vdots \\ \Gamma^{[n]} \end{Bmatrix} = -U_\infty\begin{Bmatrix} \alpha^{[1]} \\ \alpha^{[2]} \\ \vdots \\ \alpha^{[n]} \end{Bmatrix}$$

where $[\mathbf{A}]$ is the aerodynamic influence coefficients matrix:

$$A_{ij} = \left(\boldsymbol{v}_{12}^{[j]}\left(\mathbf{x}^{[i]}\right) + \boldsymbol{v}_{23}^{[j]}\left(\mathbf{x}^{[i]}\right) + \boldsymbol{v}_{34}^{[j]}\left(\mathbf{x}^{[i]}\right)\right)\Big|_{\Gamma^{[j]}=1} \cdot \mathbf{n}^{[i]}$$

Total lift on the element:

$$L^{[i]} = \rho_\infty U_\infty S^{[i]}\Gamma^{[i]} \quad \rightarrow \quad \{\mathbf{L}\} = -\rho_\infty U_\infty^2[\mathbf{S}][\mathbf{A}]^{-1}\{\boldsymbol{\alpha}\}; \quad [\mathbf{S}] = \begin{bmatrix} S^{[1]} & & 0 \\ & \ddots & \\ 0 & & S^{[n]} \end{bmatrix}$$

Numerical modelling – Aeroelastic coupling

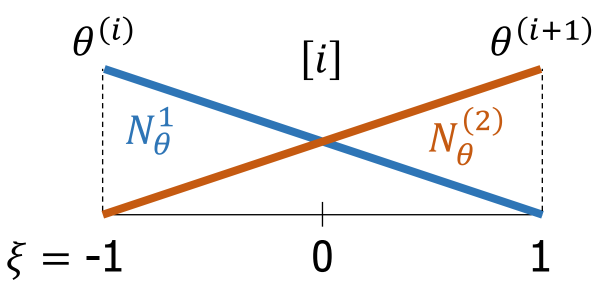
We want to find the coupling matrices $\mathbf{I}_{\alpha u}$ and $\mathbf{I}_{fL}$. Some useful information to obtain them:

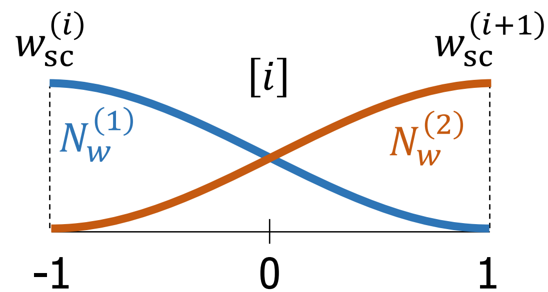
- Interpolation of the degrees of freedom inside the structural element:

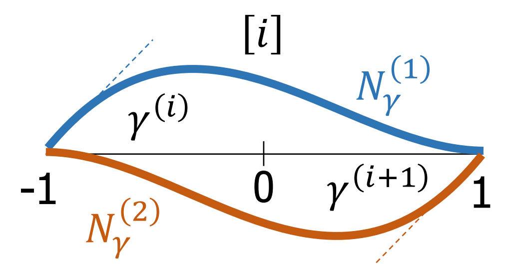
$$\theta^{[i]}(\xi) = N_\theta^{(1)}(\xi)\theta^{(i)} + N_\theta^{(2)}(\xi)\theta^{(i+1)}$$

$$w_{\mathrm{sc}}^{[i]}(\xi) = N_w^{(1)}(\xi)w_{\mathrm{sc}}^{(i)} + N_\gamma^{(1)}(\xi)\gamma^{(i)} + N_w^{(2)}(\xi)w_{\mathrm{sc}}^{(i+1)} + N_\gamma^{(2)}(\xi)\gamma^{(i+1)}$$

$$\gamma^{[i]}(\xi) = N_{w,\xi}^{(1)}(\xi)w_{\mathrm{sc}}^{(i)} + N_{\gamma,\xi}^{(1)}(\xi)\gamma^{(i)} + N_{w,\xi}^{(2)}(\xi)w_{\mathrm{sc}}^{(i+1)} + N_{\gamma,\xi}^{(2)}(\xi)\gamma^{(i+1)}$$







$$N_\theta^{(1)}(\xi) = \frac{1}{2}(1-\xi)$$

$$N_\theta^{(2)}(\xi) = \frac{1}{2}(1+\xi)$$

$$N_w^{(1)}(\xi) = \frac{1}{4}(2-3\xi+\xi^3)$$

$$N_w^{(2)}(\xi) = \frac{1}{4}(2+3\xi-\xi^3)$$

$$N_\gamma^{(1)}(\xi) = \frac{1}{4}(1-\xi-\xi^2+\xi^3)$$

$$N_\gamma^{(2)}(\xi) = \frac{1}{4}(-1-\xi+\xi^2+\xi^3)$$

$$\xi = \frac{2y - \left(y^{(i)} + y^{(i+1)}\right)}{y^{(i+1)} - y^{(i)}}$$

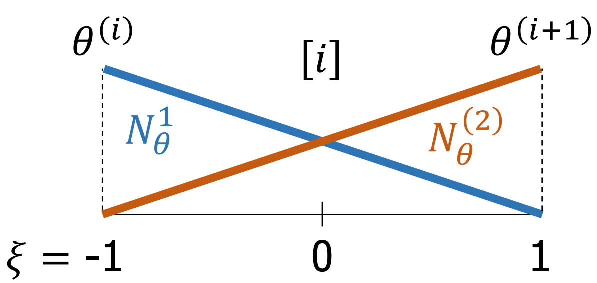
Numerical modelling – Aeroelastic coupling

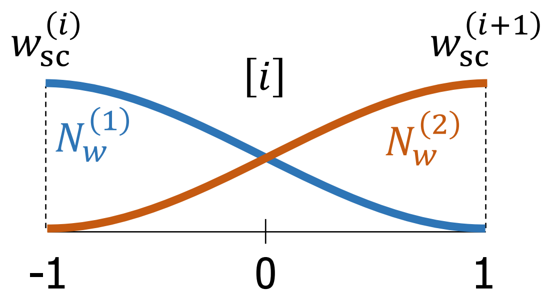
We want to find the coupling matrices $\mathbf{I}_{\alpha u}$ and $\mathbf{I}_{fL}$. Some useful information to obtain them:

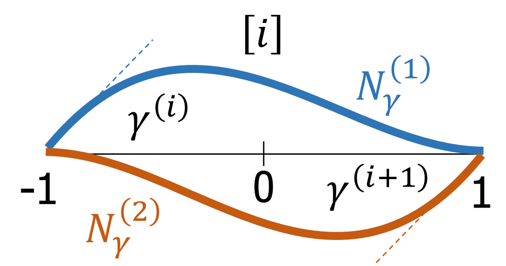
- Virtual work of an external distributed force p.u. length:

$$\delta W^{\text{ext}} = \sum_{i=1}^{n} \int_{y^{(i)}}^{y^{(i+1)}} \delta u_z f_z \, dx \underset{\uparrow}{=} \sum_{i=1}^{n} \delta u_z\left(y^{[i]}\right) F^{[i]} \,; \quad \delta u_z(y) = \delta w_{sc}(y) - (x - x_{sc})\delta\theta(y)$$

$$f_z = F^{[i]} \delta\left(y - y^{[i]}\right) \qquad \delta(y) \text{ : Dirac delta function}$$







$$N_\theta^{(1)}(\xi) = \frac{1}{2}(1 - \xi) \qquad N_w^{(1)}(\xi) = \frac{1}{4}(2 - 3\xi + \xi^3) \qquad N_\gamma^{(1)}(\xi) = \frac{1}{4}(1 - \xi - \xi^2 + \xi^3)$$

$$N_\theta^{(2)}(\xi) = \frac{1}{2}(1 + \xi) \qquad N_w^{(2)}(\xi) = \frac{1}{4}(2 + 3\xi - \xi^3) \qquad N_\gamma^{(2)}(\xi) = \frac{1}{4}(-1 - \xi + \xi^2 + \xi^3)$$

$$\xi = \frac{2y - \left(y^{(i)} + y^{(i+1)}\right)}{y^{(i+1)} - y^{(i)}}$$