Problem 5

PROBLEM 5. One of the most extensively used aerodynamic models that is capable to explain flutter induced instabilities was developed by Theodorsen back in 1935. According to Theodorsen's aerodynamic model, the expressions for the lift and moment are given by:

$$\ell = \pi\rho_\infty b^2\left(U_\infty\dot{\theta} - ba\ddot{\theta} - \ddot{w}\right) + 2\pi\rho_\infty U_\infty bC(k)\left(U_\infty\theta + b\left(\frac{1}{2} - a\right)\dot{\theta} - \dot{w}\right)$$

$$m_{sc} = -\pi\rho_\infty b^3\left(U_\infty\left(\frac{1}{2} - a\right)\dot{\theta} + b\left(\frac{1}{8} + a^2\right)\ddot{\theta} + a\ddot{w}\right) + 2\pi\rho_\infty U_\infty b^2 C(k)\left(a + \frac{1}{2}\right)\left(U_\infty\theta + b\left(\frac{1}{2} - a\right)\dot{\theta} - \dot{w}\right)$$

where $b = c/2$, $a = x_{sc}/b - 1$ and

$$C(k) = \frac{H_1^{(2)}(k)}{H_1^{(2)}(k) + iH_0^{(2)}(k)}, \qquad k = \frac{\omega b}{U_\infty} \text{ : reduced frequency}$$

is the Theodorsen's function, a transfer function to account for attenuation by the wake vorticity. A typical approximation for this function is given by:

$$C(k) = 1 - \frac{0.165}{1 - i\dfrac{0.0455}{k}} - \frac{0.335}{1 - i\dfrac{0.3}{k}}$$

Theodorsen's aerodynamic model works with harmonic motions. Then, we can assume:

$$\theta = \bar{\theta}e^{i\omega t} \quad w = \bar{w}e^{i\omega t}$$

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Problem 5. Using Theodorsen's aerodynamic model:

(a) Obtain the dynamic equations of motion in terms of the following non-dimensional parameters:

- Stiffness ratio: $\sigma = \frac{\omega_w}{\omega_\theta}$, with $\omega_\theta^2 = \frac{k_\theta}{I_{sc}}$, $\omega_w^2 = \frac{k_w}{m}$
- Shear center location: a
- Static unbalance: $x_\theta = \frac{x_{cm} - x_{sc}}{b}$
- Non-dimensional squared radius of gyration: $r_\theta^2 = \frac{I_{sc}}{mb^2}$
- Mass/density ratio: $\mu = \frac{m}{\pi \rho_\infty b^2}$

Hint: $k = \omega b / U_\infty$ and $\lambda = (\omega_\theta/\omega)^2$ should appear as the two unknowns.

(b) Implement an algorithm to compute the flutter velocity U_F for a given set of non-dimensional parameters. **Hint**: The algorithm should compute λ_F and κ_F for a given set of $\sigma, a, x_\theta, r_\theta^2$ and μ. You can check the results of the following article for benchmarking: F. Behestinia et al. Journal of Fluids and Structures 73 (2017) 1-15

(c) Plot $\frac{U_F}{b\omega_\theta}$ against different values of σ, a, x_θ, r_θ^2 and μ.

Note: Assume $c_w = c_\theta = 0$.

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Theodorsen's function:

$$C(\kappa) = \frac{H_1^{(2)}(k)}{H_1^{(2)}(k) + iH_0^{(2)}(k)}, \qquad k = \frac{\omega b}{U_\infty} : \text{reduced frequency}$$

In general, one can express $C(k) = F(k) + iG(k)$, where

$$F(k) = \frac{\sum_{n=0}^{N} p_n k^n}{\sum_{m=0}^{M} r_m k^m} \quad \rightarrow \quad F'(k) = \frac{dF}{dk} = \frac{\sum_{n=1}^{N} n p_n k^{n-1}}{\sum_{m=0}^{M} r_m k^m} - \frac{\sum_{m=1}^{M} m r_m k^{m-1}}{\sum_{m=0}^{M} r_m k^m} F(k)$$

$$G(k) = \frac{\sum_{\ell=0}^{L} q_\ell k^\ell}{\sum_{m=0}^{M} r_m k^m} \quad \rightarrow \quad G'(k) = \frac{dG}{dk} = \frac{\sum_{\ell=1}^{L} \ell q_\ell k^{\ell-1}}{\sum_{m=0}^{M} r_m k^m} - \frac{\sum_{m=1}^{M} m r_m k^{m-1}}{\sum_{m=0}^{M} r_m k^m} G(k)$$

In our case:

$$C(k) = 1 - \frac{0.165}{1 - i\frac{0.0455}{k}} - \frac{0.335}{1 - i\frac{0.3}{k}}$$

$$C(k) = \frac{0.5k^4 + 0.0765k^2 + 1.8632 \times 10^{-4}}{k^4 + 0.0921k^2 + 1.8632 \times 10^{-4}} + i\frac{-0.1080k^3 - 8.8374 \times 10^{-4}k}{k^4 + 0.0921k^2 + 1.8632 \times 10^{-4}}$$

Third order approximation:

$$C(k) = \frac{0.5(ik)^3 + 1.0761(ik)^2 + 0.524855(ik) + 0.0451331}{(ik)^3 + b_2(ik)^2 + b_1(ik) + b_0}$$

$$C(k) = \frac{0.5k^6 + 1.172549k^4 + 0.232122k^2 + 0.0020537}{k^6 + 2.220145k^4 + 0.315667k^2 + 0.0020706} + i\frac{-0.124995k^5 - 0.223670k^3 - 0.0076711k}{k^6 + 2.220145k^4 + 0.315667k^2 + 0.0020706}$$

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System of equations:

$$\left(-\omega^2[\mathbf{M}] + [\mathbf{K}] - ([\mathbf{A}_\mathrm{R}] + i[\mathbf{A}_\mathrm{I}])\right)\{\bar{\mathbf{x}}\}e^{i\omega t} = \{\mathbf{0}\}$$

where

$$\{\bar{\mathbf{x}}\} = \begin{Bmatrix} \bar{\theta} \\ \bar{w} \end{Bmatrix}$$

$$[\mathbf{M}] = \begin{bmatrix} I_\mathrm{sc} & mdb \\ mdb & mb^2 \end{bmatrix}$$

$$[\mathbf{K}] = \begin{bmatrix} k_\theta & 0 \\ 0 & k_w b^2 \end{bmatrix}$$

$$[\mathbf{A}_\mathrm{R}] = \pi\rho_\infty b^2 U_\infty^2 \left(k^2 \begin{bmatrix} 1/8 + a^2 & a \\ a & 1 \end{bmatrix} + kG(k) \begin{bmatrix} 2a^2 - 1/2 & 2a + 1 \\ 2a - 1 & 2 \end{bmatrix} + F(k) \begin{bmatrix} 1 + 2a & 0 \\ 2 & 0 \end{bmatrix}\right) = \pi\rho_\infty b^2 U_\infty^2 [\widehat{\mathbf{A}}_\mathrm{R}(k)]$$

$$[\mathbf{A}_\mathrm{I}] = \pi\rho_\infty b^2 U_\infty^2 \left(k \begin{bmatrix} a - 1/2 & 0 \\ 1 & 0 \end{bmatrix} - kF(k) \begin{bmatrix} 2a^2 - 1/2 & 2a + 1 \\ 2a - 1 & 2 \end{bmatrix} + G(k) \begin{bmatrix} 1 + 2a & 0 \\ 2 & 0 \end{bmatrix}\right) = \pi\rho_\infty b^2 U_\infty^2 [\widehat{\mathbf{A}}_\mathrm{I}(k)]$$

Note that $\bar{w}$ is considered the non-dimensional amplitude (by the half-chord b) of w. Also, to make the matrices units consistent, the second equation (whole second row of each matrix) has been multiplied by b.

The system can then be expressed:

$$\left(-\omega^2[\mathbf{M}] + [\mathbf{K}] - \pi\rho_\infty b^2 U_\infty^2\left([\widehat{\mathbf{A}}_\mathrm{R}(k)] + i[\widehat{\mathbf{A}}_\mathrm{I}(k)]\right)\right)\{\bar{\mathbf{x}}\} = \{\mathbf{0}\}$$

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By dividing everything by $\pi\rho_\infty b^2 U_\infty^2$:

$$\left(-\frac{\omega^2}{\pi\rho_\infty b^2 U_\infty^2}[\mathbf{M}] + \frac{1}{\pi\rho_\infty b^2 U_\infty^2}[\mathbf{K}] - \left([\widehat{\mathbf{A}}_\mathrm{R}(k)] + i[\widehat{\mathbf{A}}_\mathrm{I}(k)]\right)\right)\{\bar{\mathbf{x}}\} = \{\mathbf{0}\}$$

we can identify the terms:

$$\frac{\omega^2}{\pi\rho_\infty b^2 U_\infty^2}\begin{bmatrix} I_\mathrm{sc} & mdb \\ mdb & mb^2 \end{bmatrix} = \frac{mb^2\omega^2}{\pi\rho_\infty b^2 U_\infty^2}\begin{bmatrix} I_\mathrm{sc}/mb^2 & d/b \\ d/b & 1 \end{bmatrix} = k^2\mu \underbrace{\begin{bmatrix} r_\theta^2 & -x_\theta \\ -x_\theta & 1 \end{bmatrix}}_{[\widehat{\mathbf{M}}]}$$

$$\frac{1}{\pi\rho_\infty b^2 U_\infty^2}\begin{bmatrix} k_\theta & 0 \\ 0 & k_w b^2 \end{bmatrix} = \frac{mb^2\omega^2\omega_\theta^2}{\pi\rho_\infty b^2 U_\infty^2 \omega^2}\begin{bmatrix} I_\mathrm{sc}/mb^2 & 0 \\ 0 & \omega_w^2/\omega_\theta^2 \end{bmatrix} = k^2\lambda\mu \underbrace{\begin{bmatrix} r_\theta^2 & 0 \\ 0 & \sigma^2 \end{bmatrix}}_{[\widehat{\mathbf{K}}]}$$

$$[\widehat{\mathbf{A}}_\mathrm{R}(k)] = k^2 \underbrace{\begin{bmatrix} 1/8 + a^2 & a \\ a & 1 \end{bmatrix}}_{[\widehat{\mathbf{M}}']} + kG(k) \underbrace{\begin{bmatrix} 2a^2 - 1/2 & 2a+1 \\ 2a-1 & 2 \end{bmatrix}}_{[\widehat{\mathbf{C}}']} + F(k) \underbrace{\begin{bmatrix} 1+2a & 0 \\ 2 & 0 \end{bmatrix}}_{[\widehat{\mathbf{K}}']}$$

$$[\widehat{\mathbf{A}}_\mathrm{I}(k)] = k \underbrace{\begin{bmatrix} a - 1/2 & 0 \\ 1 & 0 \end{bmatrix}}_{[\widehat{\mathbf{C}}'']} - kF(k) \underbrace{\begin{bmatrix} 2a^2 - 1/2 & 2a+1 \\ 2a-1 & 2 \end{bmatrix}}_{[\widehat{\mathbf{C}}']} + G(k) \underbrace{\begin{bmatrix} 1+2a & 0 \\ 2 & 0 \end{bmatrix}}_{[\widehat{\mathbf{K}}']}$$

where

$$\mu = \frac{m}{\pi\rho_\infty b^2}, \qquad \sigma = \frac{\omega_w}{\omega_\theta}, \qquad r_\theta^2 = \frac{I_\mathrm{sc}}{mb^2}, \qquad x_\theta = -\frac{d}{b}, \qquad a = \frac{x_\mathrm{sc}}{b} - 1, \qquad k = \frac{b\omega}{U_\infty}, \qquad \lambda = \left(\frac{\omega_\theta}{\omega}\right)^2$$

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Dividing by $-k^2$, the resulting system yields:

$$\left([\widehat{\mathbf{M}}] - \lambda[\widehat{\mathbf{K}}] + \left([\widehat{\mathbf{M}}'] + ik^{-1}\left([\widehat{\mathbf{C}}''] - \left(F(k) + iG(k)\right)[\widehat{\mathbf{C}}']\right) + k^{-2}\left(F(k) + iG(k)\right)[\widehat{\mathbf{K}}']\right)\right)\{\bar{\mathbf{x}}\} = \{\mathbf{0}\}$$

Or

$$\Bigg(\underbrace{[\widehat{\mathbf{K}}]^{-1}[\widehat{\mathbf{M}}] + [\widehat{\mathbf{K}}]^{-1}\left([\widehat{\mathbf{M}}'] + ik^{-1}\left([\widehat{\mathbf{C}}''] - \left(F(k) + iG(k)\right)[\widehat{\mathbf{C}}']\right) + k^{-2}\left(F(k) + iG(k)\right)[\widehat{\mathbf{K}}']\right)}_{[\widehat{\mathbf{D}}(k)]} - \lambda[\mathbf{1}]\Bigg)\{\bar{\mathbf{x}}\} = \{\mathbf{0}\}$$

Flutter condition:

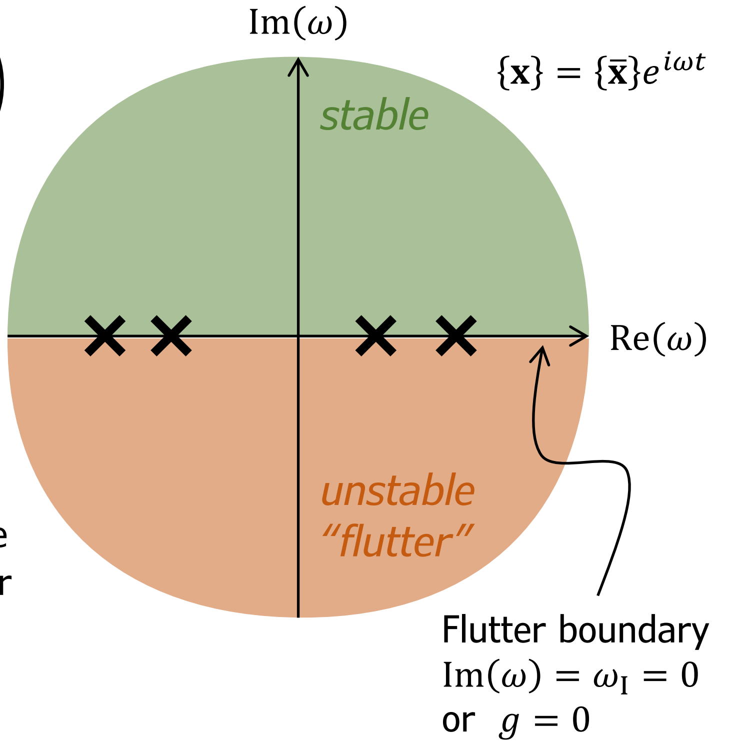
$$\lambda_{\mathrm{F}} = \frac{\omega_\theta^2}{{\omega_{\mathrm{F}}}^2}(1 + ig) \equiv \frac{\omega_\theta^2}{(\omega_{\mathrm{R}} + i\omega_{\mathrm{I}})^2} = \omega_\theta^2\left(\frac{\omega_{\mathrm{R}}^2 - \omega_{\mathrm{I}}^2}{(\omega_{\mathrm{R}}^2 + \omega_{\mathrm{I}}^2)^2} + i\,\frac{-2\omega_{\mathrm{I}}\omega_{\mathrm{R}}}{(\omega_{\mathrm{R}}^2 + \omega_{\mathrm{I}}^2)^2}\right)$$

$${\omega_{\mathrm{F}}}^2 = \frac{\omega_\theta^2}{\mathrm{Re}(\lambda_{\mathrm{F}})} \equiv \frac{\left(\omega_{\mathrm{R}}^2 + \omega_{\mathrm{I}}^2\right)^2}{\omega_{\mathrm{R}}^2 - \omega_{\mathrm{I}}^2}; \quad g \equiv \frac{\mathrm{Im}(\lambda_F)}{\mathrm{Re}(\lambda_F)} = \frac{-2\omega_{\mathrm{I}}\omega_{\mathrm{R}}}{\omega_{\mathrm{R}}^2 - \omega_{\mathrm{I}}^2}$$

$$k_{\mathrm{F}} = \frac{\omega_{\mathrm{F}} b}{U_{\mathrm{F}}}$$

Notice that the only physically admissible solutions are those for which $\mathrm{Re}(\lambda_{\mathrm{F}}) > 0$ (since this guarantees that both ω_{F} and U_{F} are positive and real-valued). Additionally, for $\mathrm{Im}(\lambda_{\mathrm{F}}) > 0$ (or $g > 0$), the system becomes unstable (i.e., $\omega_{\mathrm{I}} < 0$). In the limit, $g = 0$ the flutter condition is satisfied:

$$\det\left([\widehat{\mathbf{D}}(k_{\mathrm{F}})] - \lambda_{\mathrm{F}}[\mathbf{1}]\right) = 0$$



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Method 1: Use a non-linear system solver to find $\{\mathbf{x}\} = \{\lambda_\mathrm{F}, \kappa_\mathrm{F}\}^T$ satisfying the equations:

$$F_1(\lambda_\mathrm{F}, k_\mathrm{F}) = \mathrm{Re}\Big(\det\big([\widehat{\mathbf{D}}(k_\mathrm{F})] - \lambda_\mathrm{F}[\mathbf{1}]\big)\Big) = D_{11}^\mathrm{R} D_{22}^\mathrm{R} - D_{11}^\mathrm{I} D_{22}^\mathrm{I} - D_{12}^\mathrm{R} D_{21}^\mathrm{R} + D_{12}^\mathrm{I} D_{21}^\mathrm{I} - \big(D_{11}^\mathrm{R} + D_{22}^\mathrm{R}\big)\lambda_\mathrm{F} + \lambda_\mathrm{F}^2 = 0$$

$$F_2(\lambda_\mathrm{F}, k_F) = \mathrm{Im}\Big(\det\big([\widehat{\mathbf{D}}(k_\mathrm{F})] - \lambda_\mathrm{F}[\mathbf{1}]\big)\Big) = D_{11}^\mathrm{R} D_{22}^\mathrm{I} + D_{11}^\mathrm{I} D_{22}^\mathrm{R} - D_{12}^\mathrm{R} D_{21}^\mathrm{I} - D_{12}^\mathrm{I} D_{21}^\mathrm{R} - \big(D_{11}^\mathrm{I} + D_{22}^\mathrm{I}\big)\lambda_\mathrm{F} = 0$$

1. Start with an initial guess $\{\mathbf{x}^{(0)}\} = \left\{\lambda_\mathrm{F}^{(0)}, k_\mathrm{F}^{(0)}\right\}^\mathrm{T}$ and evaluate $\{\mathbf{F}^{(n)}\} = \left\{F_1\left(\lambda_\mathrm{F}^{(n)}, k_\mathrm{F}^{(n)}\right), F_2\left(\lambda_\mathrm{F}^{(n)}, k_\mathrm{F}^{(n)}\right)\right\}^T$ and the Jacobian $[\boldsymbol{\mathcal{J}}^{(n)}]$:

$$\mathcal{J}_{11}^{(n)} = \frac{\partial F_1}{\partial \lambda_\mathrm{F}} = -\left(D_{11}^{\mathrm{R}(n)} + D_{22}^{\mathrm{R}(n)}\right) + 2\lambda_\mathrm{F}^{(n)}$$

$$\mathcal{J}_{21}^{(n)} = \frac{\partial F_2}{\partial \lambda_\mathrm{F}} = -\left(D_{11}^{\mathrm{I}(n)} + D_{22}^{\mathrm{I}(n)}\right)$$

$$\begin{aligned}\mathcal{J}_{12}^{(n)} = \frac{\partial F_1}{\partial k_\mathrm{F}} = {} & D_{22}^{\mathrm{R}(n)} \frac{\partial D_{11}^{\mathrm{R}(n)}}{\partial k_\mathrm{F}} + D_{11}^{\mathrm{R}(n)} \frac{\partial D_{22}^{\mathrm{R}(n)}}{\partial k_\mathrm{F}} - D_{22}^{\mathrm{I}(n)} \frac{\partial D_{11}^{\mathrm{I}(n)}}{\partial k_\mathrm{F}} - D_{11}^{\mathrm{I}(n)} \frac{\partial D_{22}^{\mathrm{I}(n)}}{\partial k_\mathrm{F}} \\ & - D_{21}^{\mathrm{R}(n)} \frac{\partial D_{12}^{\mathrm{R}(n)}}{\partial k_\mathrm{F}} - D_{12}^{\mathrm{R}(n)} \frac{\partial D_{21}^{\mathrm{R}(n)}}{\partial k_\mathrm{F}} + D_{21}^{\mathrm{I}(n)} \frac{\partial D_{12}^{\mathrm{I}(n)}}{\partial k_\mathrm{F}} + D_{12}^{\mathrm{I}(n)} \frac{\partial D_{21}^{\mathrm{I}(n)}}{\partial k_\mathrm{F}} - \left(\frac{\partial D_{11}^{\mathrm{R}(n)}}{\partial k_\mathrm{F}} + \frac{\partial D_{22}^{\mathrm{R}(n)}}{\partial k_\mathrm{F}}\right)\lambda_\mathrm{F}^{(n)}\end{aligned}$$

$$\begin{aligned}\mathcal{J}_{22}^{(n)} = \frac{\partial F_2}{\partial k_\mathrm{F}} = {} & D_{22}^{\mathrm{I}(n)} \frac{\partial D_{11}^{\mathrm{R}(n)}}{\partial k_\mathrm{F}} + D_{11}^{\mathrm{R}(n)} \frac{\partial D_{22}^{\mathrm{I}(n)}}{\partial k_\mathrm{F}} + D_{22}^{\mathrm{R}(n)} \frac{\partial D_{11}^{\mathrm{I}(n)}}{\partial k_\mathrm{F}} + D_{11}^{\mathrm{I}(n)} \frac{\partial D_{22}^{\mathrm{R}(n)}}{\partial k_\mathrm{F}} \\ & - D_{21}^{\mathrm{I}(n)} \frac{\partial D_{12}^{\mathrm{R}(n)}}{\partial k_\mathrm{F}} - D_{12}^{\mathrm{R}(n)} \frac{\partial D_{21}^{\mathrm{I}(n)}}{\partial k_\mathrm{F}} - D_{21}^{\mathrm{R}(n)} \frac{\partial D_{12}^{\mathrm{I}(n)}}{\partial k_\mathrm{F}} - D_{12}^{\mathrm{I}(n)} \frac{\partial D_{21}^{\mathrm{R}(n)}}{\partial k_\mathrm{F}} - \left(\frac{\partial D_{11}^{\mathrm{I}(n)}}{\partial k_\mathrm{F}} + \frac{\partial D_{22}^{\mathrm{I}(n)}}{\partial k_\mathrm{F}}\right)\lambda_\mathrm{F}^{(n)}\end{aligned}$$

where

$$\left[\frac{\partial \widehat{\mathbf{D}}^\mathrm{R}}{\partial k_\mathrm{F}}\right] = [\widehat{\mathbf{K}}]^{-1}\left(\left(\frac{F'(k_\mathrm{F})}{k_\mathrm{F}^2} - \frac{2F(k_\mathrm{F})}{k_\mathrm{F}^3}\right)[\widehat{\mathbf{K}}'] + \left(\frac{G'(k_\mathrm{F})}{k_\mathrm{F}} - \frac{G(k_\mathrm{F})}{k_\mathrm{F}^2}\right)[\widehat{\mathbf{C}}']\right)$$

$$\left[\frac{\partial \widehat{\mathbf{D}}^\mathrm{I}}{\partial k_\mathrm{F}}\right] = [\widehat{\mathbf{K}}]^{-1}\left(\left(\frac{G'(k_\mathrm{F})}{k_\mathrm{F}^2} - \frac{2G(k_\mathrm{F})}{k_\mathrm{F}^3}\right)[\widehat{\mathbf{K}}'] + \left(\frac{F'(k_\mathrm{F})}{k_\mathrm{F}} - \frac{F(k_\mathrm{F})}{k_\mathrm{F}^2}\right)[\widehat{\mathbf{C}}'] - \frac{1}{k_\mathrm{F}^2}[\widehat{\mathbf{C}}'']\right)$$

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2. Update solution:

$$\{\Delta\mathbf{x}^{(n)}\} = -\left[\boldsymbol{\mathcal{J}}^{(n)}\right]^{-1}\{\mathbf{F}^{(n)}\},$$

$$\{\mathbf{x}^{(n+1)}\} = \{\mathbf{x}^{(n)}\} + \beta\{\Delta\mathbf{x}^{(n)}\}$$

4. Repeat steps 1 and 2 until the solution is converged.

<u>Method 2</u>: Solve the quadratic equation for λ_{F} with complex-valued coefficients (function of k_{F}) resulting from:

$$\det\left(\left[\widehat{\mathbf{D}}(k_{\mathrm{F}})\right] - \lambda_{\mathrm{F}}[\mathbf{1}]\right) = (D_{11} - \lambda_{\mathrm{F}})(D_{22} - \lambda_{\mathrm{F}}) - D_{12}D_{21} = 0$$

$$D_{11}D_{22} - D_{12}D_{21} - (D_{11} + D_{22})\lambda_{\mathrm{F}} + \lambda_{\mathrm{F}}^2 = 0$$

1. Specify a set of trial values for k_{F}.
2. Solve the quadratic equation for λ_{F} that correspond to each of the selected values of k_{F}.
3. For each root, $\lambda_{\mathrm{F}}^{(i)} = \lambda_{\mathrm{F}}^{(i)\mathrm{R}} + i\lambda_{\mathrm{F}}^{(i)\mathrm{I}}$, compute:

$$\omega_{\mathrm{F}}^{(i)}(k_{\mathrm{F}}) = \frac{\omega_\theta}{\sqrt{\lambda_{\mathrm{F}}^{(i)\mathrm{R}}}}; \qquad g^{(i)}(k_{\mathrm{F}}) = \frac{\lambda_{\mathrm{F}}^{(i)\mathrm{I}}}{\lambda_{\mathrm{F}}^{(i)\mathrm{R}}}; \qquad U_{\mathrm{F}}^{(i)}(k_{\mathrm{F}}) = \frac{b\omega_{\mathrm{F}}^{(i)}}{k_{\mathrm{F}}}$$

4. Interpolate to find the value of k_{F}^* at which $g^{(i)}(k_{\mathrm{F}}^*) = 0$. Then, the flutter speed will correspond to $U_{\mathrm{F}}^{(i)}(k_{\mathrm{F}}^*)$.

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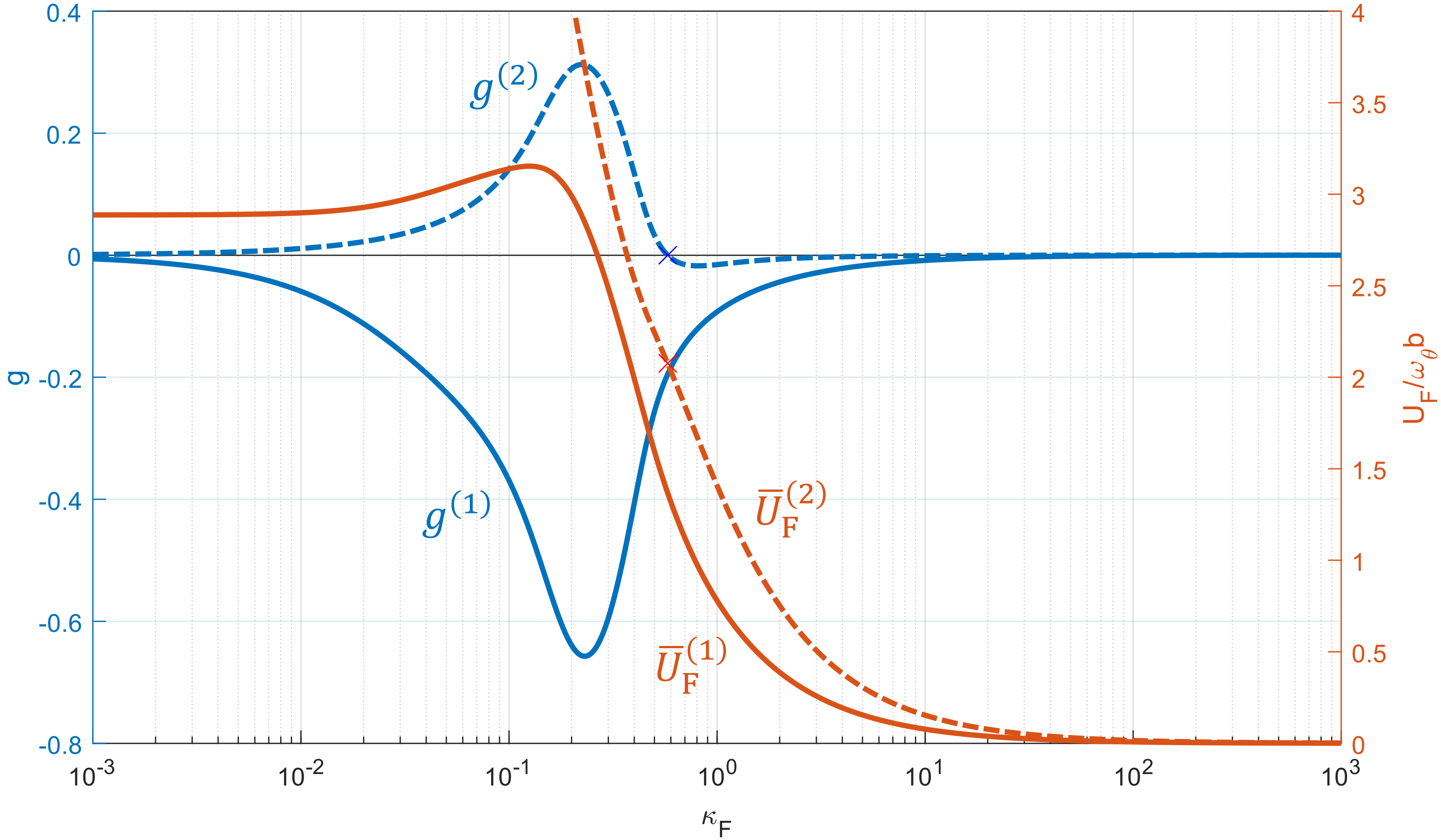

$$\sigma = 1, \mu = 20, a = -0.2, x_\theta = 0.3, r_\theta = 0.5$$

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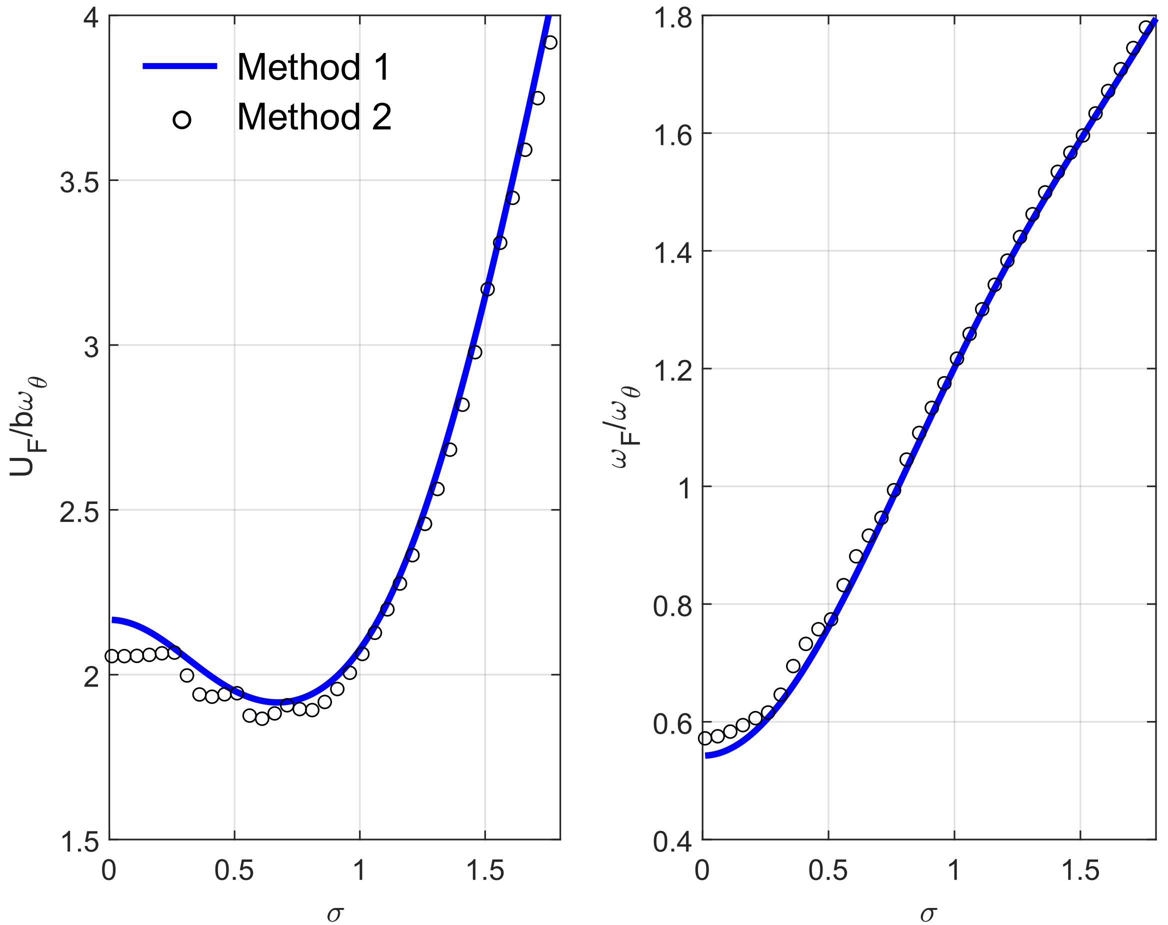

$$\mu = 20, a = -0.2, x_\theta = 0.3, r_\theta = 0.5$$

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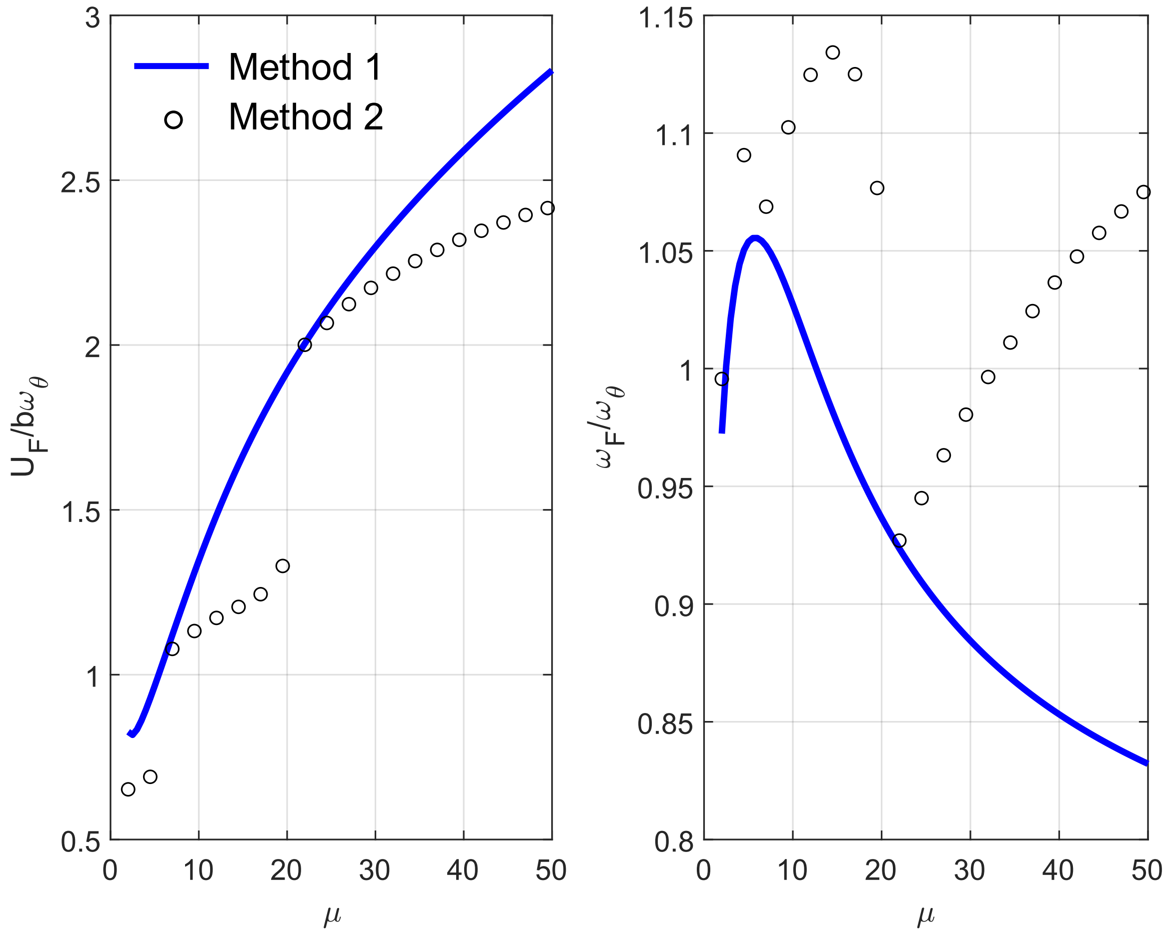

$$\sigma = 0.707, a = -0.2, x_\theta = 0.3, r_\theta = 0.5$$

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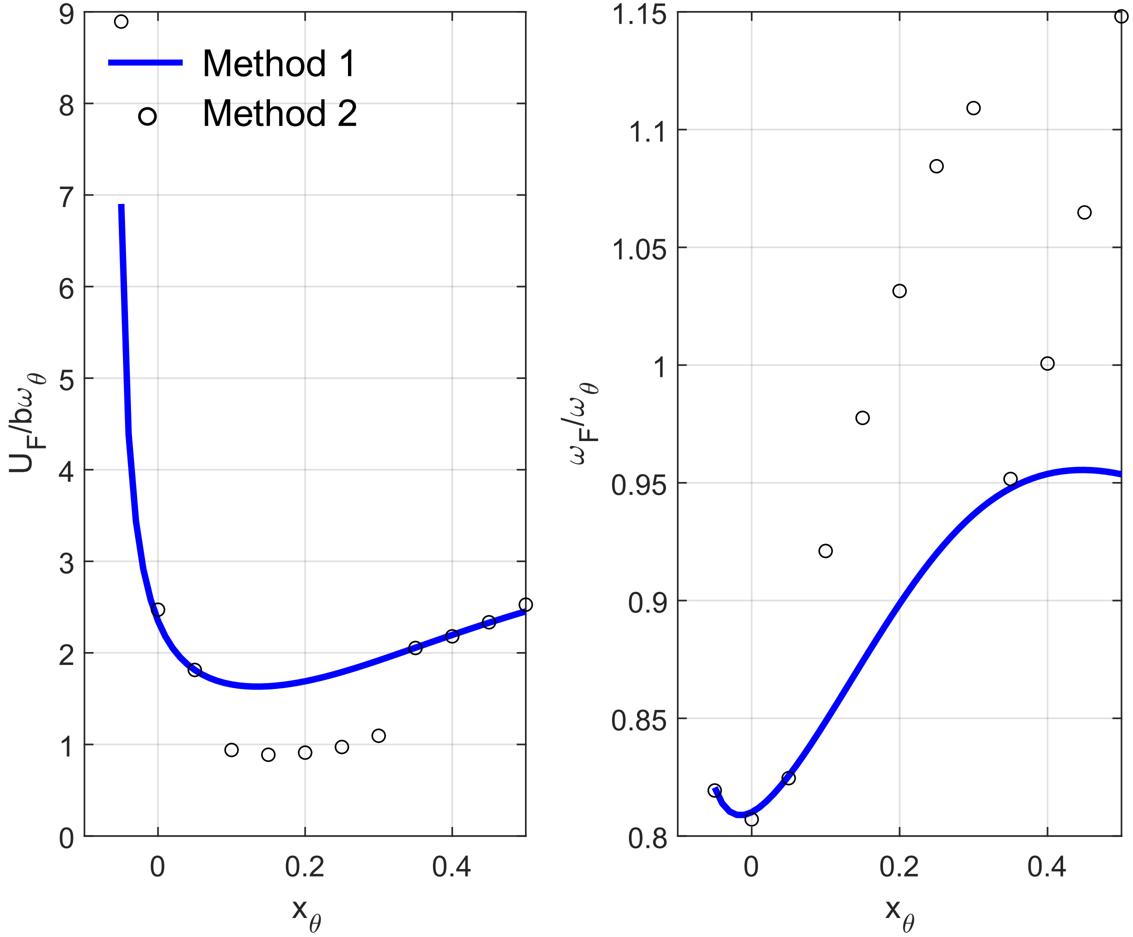

$\sigma = 0.707, \mu = 20, a = -0.2, r_\theta = 0.5$

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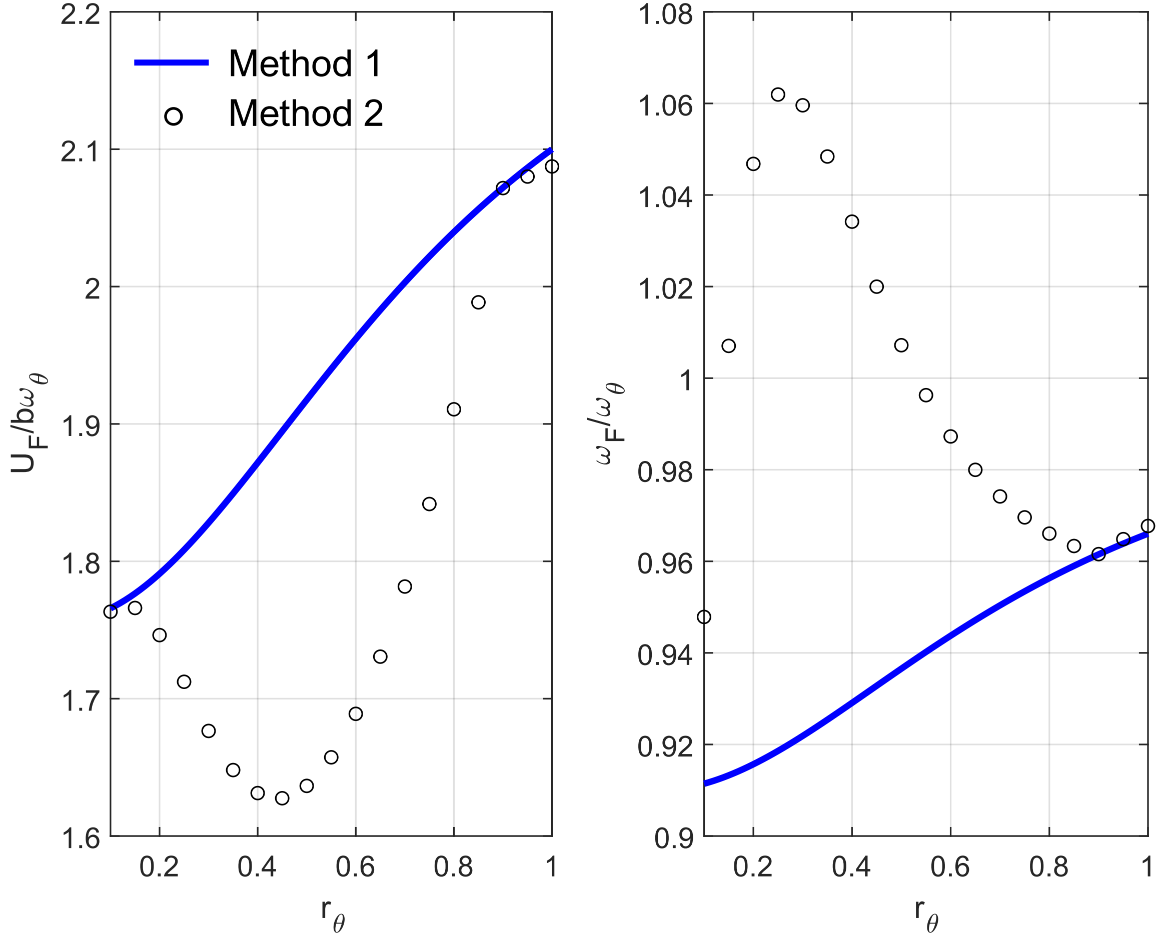

$$\sigma = 0.707, \mu = 20, a = -0.2, x_\theta = 0.3$$