Chapter 7: Techniques of Integration

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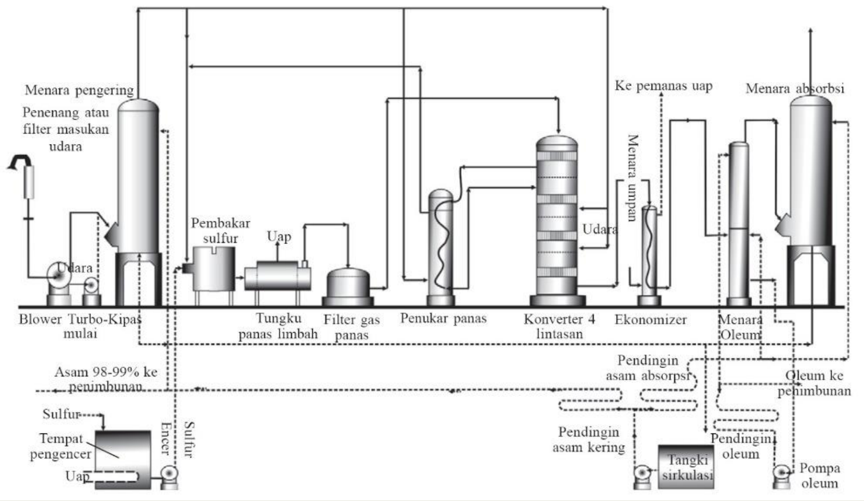
Reference: Varberg, Purcell, and Rigdon. Calculus. 9 edition.

After This Lecture You **<u>Should</u>**:

- Understand basic integration rules
- Understand integration by parts
- Understand some trigonometric integrals

Why do we care?



7.1 Basic Integration Rules

$$f(x) = \frac{e^x + e^{-x}}{2} = \cosh x$$

$$g(x) = (1 + \cos^4 x)^{1/2}$$

$$h(x) = \frac{3^{x^2 - 2x}}{\ln(x^2 + 1)} - \sin[\cos(\cosh x)]$$

Elementary Function!

<u>Differentiation</u>

-Straightforward

-Easy

-Result is always elementary function

<u>Integration</u>

-Does not always elementary function

-Two principle techniques: *substitution* and *integration by parts*

7.1 Standard Integral Form

Note:
This table will be provided on the exam

Constants, Powers	1. $\int k\,du = ku + C$	2. $\int u^r\,du = \begin{cases} \dfrac{u^{r+1}}{r+1} + C & r \neq -1 \\ \ln\lvert u\rvert + C & r = -1 \end{cases}$
Exponentials	3. $\int e^u\,du = e^u + C$	4. $\int a^u\,du = \dfrac{a^u}{\ln a} + C, a \neq 1, a > 0$
Trigonometric Functions	5. $\int \sin u\,du = -\cos u + C$	6. $\int \cos u\,du = \sin u + C$
	7. $\int \sec^2 u\,du = \tan u + C$	8. $\int \csc^2 u\,du = -\cot u + C$
	9. $\int \sec u \tan u\,du = \sec u + C$	10. $\int \csc u \cot u\,du = -\csc u + C$
	11. $\int \tan u\,du = -\ln\lvert\cos u\rvert + C$	12. $\int \cot u\,du = \ln\lvert\sin u\rvert + C$
Algebraic Functions	13. $\int \dfrac{du}{\sqrt{a^2 - u^2}} = \sin^{-1}\left(\dfrac{u}{a}\right) + C$	14. $\int \dfrac{du}{a^2 + u^2} = \dfrac{1}{a}\tan^{-1}\left(\dfrac{u}{a}\right) + C$
	15. $\int \dfrac{du}{u\sqrt{u^2 - a^2}} = \dfrac{1}{a}\sec^{-1}\left(\dfrac{\lvert u\rvert}{a}\right) + C = \dfrac{1}{a}\cos^{-1}\left(\dfrac{a}{\lvert u\rvert}\right) + C$	
Hyperbolic Functions	16. $\int \sinh u\,du = \cosh u + C$	17. $\int \cosh u\,du = \sinh u + C$

7.1 Substitution in Indefinite Integrals

Theorem A **Substitution in Indefinite Integrals**

Let g be a differentiable function and suppose that F is an antiderivative of f. Then, if $u = g(x)$,

$$\int f(g(x))g'(x)\,dx = \int f(u)\,du = F(u) + C = F(g(x)) + C$$

$$\int \frac{x}{\cos^2(x^2)}dx.$$

Let: $u = x^2$ ⟶ $du = 2x\,dx$ ⟶ $x\,dx = \frac{1}{2}\,du$

Substitute then we get:

$$\int \frac{x}{cos^2(x^2)}dx = \frac{1}{2}\int \frac{du}{cos^2(u)} = \frac{1}{2}\int sec^2\,(u)$$

Look at the table, we will get:

$$\frac{1}{2}\int sec^2\,(u) = \frac{1}{2}\tan(u) + C = \frac{1}{2}\tan(x^2) + C$$

7.1 More Examples

$$\int \frac{3}{\sqrt{5-9x^2}}\,dx.$$

$$= \sin^{-1}\left(\frac{3x}{\sqrt{5}}\right) + C$$

Hint: let u= 3x

$$\int \frac{6e^{1/x}}{x^2}\,dx.$$

$$= -6e^u + C = -6e^{1/x} + C$$

Hint: let u= 1/x

$$\int \frac{e^x}{4+9e^{2x}}\,dx.$$

$$= \frac{1}{3}\cdot\frac{1}{2}\tan^{-1}\left(\frac{u}{2}\right) + C = \frac{1}{6}\tan^{-1}\left(\frac{3e^x}{2}\right) + C$$

Hint: let u= $3e^x$

Note: The derivation is presented manually at whiteboard

Task 1 [Deadline Submission: 12 Feb 2018]

Find:

1. $\int (x-2)^2 \, dx$
2. $\int \frac{dx}{x^2+4}$
3. $\int 6z\sqrt{4+z^2} \, dz$
4. $\int \frac{\sin(\ln 4x^2)}{x} \, dx$
5. $\int \frac{(6t-1)sin\sqrt{3t^2-t-1}}{\sqrt{3t^2-t-1}} \, dt$

7.2 Integration by Parts

If integration by substitution fails, it maybe possible to use double substitution (integration by parts)

$$\int u\,dv = uv - \int v\;du$$

$$\int x \cos x\, dx.$$

Let $u = x$ and $dv = \cos x\; dx$

$du = dx$ $\qquad$ $v = \sin x$

$$\int x \cos x\; dx = x\; sinx - \int \sin x\; dx$$

$$\int x \cos x\; dx = x\; sinx + \cos x + C$$

7.2 More Examples

$$\int_1^2 \ln x\, dx. \quad \Rightarrow \quad = 2\ln 2 - 1 \approx 0.386$$

$$\int \arcsin x\, dx. \quad \Rightarrow \quad = x \arcsin x + \sqrt{1 - x^2} + C$$

$$\int_1^2 t^6 \ln t\, dt. \quad \Rightarrow \quad = \frac{128}{7}\ln 2 - \frac{127}{49} \approx 10.083$$

Note: The derivation is presented manually at whiteboard

Task 2 [Deadline Submission: 12 Feb 2018]

Find:

1. $\int xe^{3x}\,dx$

2. $\int \frac{\ln x}{x^2}\,dx$

3. $\int x^5\sqrt{x^3+4}\,dx$

4. $\int \arctan 5x\,dx$

7.3 Trigonometric Integrals

1. $\int \sin^n x\, dx$ and $\int \cos^n x\, dx$
2. $\int \sin^m x \cos^n x\, dx$
3. $\int \sin mx \cos nx\, dx$, $\int \sin mx \sin nx\, dx$, $\int \cos mx \cos nx\, dx$
4. $\int \tan^n x\, dx$, $\int \cot^n x\, dx$
5. $\int \tan^m x \sec^n x\, dx$, $\int \cot^m x \csc^n x\, dx$

Useful Identities

Some trigonometric identities needed in this section are the following.

Pythagorean Identities

$$\sin^2 x + \cos^2 x = 1$$
$$1 + \tan^2 x = \sec^2 x$$
$$1 + \cot^2 x = \csc^2 x$$

Half-Angle Identities

$$\sin^2 x = \frac{1 - \cos 2x}{2}$$
$$\cos^2 x = \frac{1 + \cos 2x}{2}$$

7.3 Type 1: $(\int sin^n x\, dx, \int cos^n x\, dx)$

(*n* Odd) Find $\int \sin^5 x\, dx$.

$$\begin{aligned}\int \sin^5 x\, dx &= \int \sin^4 x \sin x\, dx\\ &= \int (1-\cos^2 x)^2 \sin x\, dx\\ &= \int (1 - 2\cos^2 x + \cos^4 x)\sin x\, dx\\ &= -\int (1 - 2\cos^2 x + \cos^4 x)(-\sin x\, dx)\\ &= -\cos x + \tfrac{2}{3}\cos^3 x - \tfrac{1}{5}\cos^5 x + C\end{aligned}$$

(*n* Even) Find $\int \sin^2 x\, dx$

$$\begin{aligned}\int \sin^2 x\, dx &= \int \frac{1-\cos 2x}{2}\, dx\\ &= \frac{1}{2}\int dx - \frac{1}{4}\int (\cos 2x)(2\, dx)\\ &= \frac{1}{2}x - \frac{1}{4}\sin 2x + C\end{aligned}$$

7.3 Type 2: ($\int \sin^m x \cos^n x\, dx$)

(*m* or *n* Odd) Find $\int \sin^3 x \cos^{-4} x\, dx$.

$$\begin{aligned}\int \sin^3 x \cos^{-4} x\, dx &= \int (1-\cos^2 x)(\cos^{-4} x)(\sin x)\, dx \\ &= -\int (\cos^{-4} x - \cos^{-2} x)(-\sin x\, dx) \\ &= -\left[\frac{(\cos x)^{-3}}{-3} - \frac{(\cos x)^{-1}}{-1}\right] + C \\ &= \frac{1}{3}\sec^3 x - \sec x + C\end{aligned}$$

(Both *m* and *n* Even) Find $\int \sin^2 x \cos^4 x\, dx$.

$$\begin{aligned}&\int \sin^2 x \cos^4 x\, dx \\ &\quad = \int \left(\frac{1-\cos 2x}{2}\right)\left(\frac{1+\cos 2x}{2}\right)^2 dx \\ &\quad = \frac{1}{8}\int (1+\cos 2x - \cos^2 2x - \cos^3 2x)\, dx \\ &\quad = \frac{1}{8}\int \left[1+\cos 2x - \frac{1}{2}(1+\cos 4x) - (1-\sin^2 2x)\cos 2x\right] dx \\ &\quad = \frac{1}{8}\int \left[\frac{1}{2} - \frac{1}{2}\cos 4x + \sin^2 2x \cos 2x\right] dx \\ &\quad = \frac{1}{8}\left[\int \frac{1}{2}dx - \frac{1}{8}\int \cos 4x(4\, dx) + \frac{1}{2}\int \sin^2 2x(2\cos 2x\, dx)\right] \\ &\quad = \frac{1}{8}\left[\frac{1}{2}x - \frac{1}{8}\sin 4x + \frac{1}{6}\sin^3 2x\right] + C\end{aligned}$$

7.3 Type 3: ($\int \sin mx \cos nx \, dx, \int \sin mx \sin nx \, dx, \int \cos mx \cos nx \, dx$)

Trigonometric Identity

1. $\sin mx \cos nx = \frac{1}{2}[\sin(m+n)x + \sin(m-n)x]$
2. $\sin mx \sin nx = -\frac{1}{2}[\cos(m+n)x - \cos(m-n)x]$
3. $\cos mx \cos nx = \frac{1}{2}[\cos(m+n)x + \cos(m-n)x]$

$$\int \sin 2x \cos 3x \, dx.$$

$$= \frac{1}{2}\int [\sin 5x + \sin(-x)] \, dx$$

$$= \frac{1}{10}\int \sin 5x(5 \, dx) - \frac{1}{2}\int \sin x \, dx$$

$$= -\frac{1}{10}\cos 5x + \frac{1}{2}\cos x + C$$

Task 3 [Deadline Submission: 12 Feb 2018]

Find:

1. $$\int \sin^4\left(\frac{w}{2}\right)\cos^2\left(\frac{w}{2}\right)\,dw$$

2. $$\int \cot^3 2t\,dt$$

3. $$\int \tan^3 x \sec^2 x\,dx$$