Lecture 8: Complete solution of Ax=b

Solution of Ax=b

Solvability of $Ax = b$

$$\begin{bmatrix} 1 & 2 & 2 & 2 \\ 2 & 4 & 6 & 8 \\ 3 & 6 & 8 & 10 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

Use elimination with augment matrix:

$$\left[\begin{array}{cccc|c} 1 & 2 & 2 & 2 & b_1 \\ 2 & 4 & 6 & 8 & b_2 \\ 3 & 6 & 8 & 10 & b_3 \end{array}\right] \Longrightarrow \left[\begin{array}{cccc|c} 1 & 2 & 2 & 2 & b_1 \\ 0 & 0 & 2 & 4 & b_2 - 2b_1 \\ 0 & 0 & 0 & 0 & b_3 - b_2 - b_1 \end{array}\right]$$

Back substitution row 3 into $Ax = b$ we can get $0 = b_3 - b_2 - b_1$, thus $b_3 = b_1 + b_2$, which means the third element of b is equal to the plus of first and second element. Reflect on the matrix A is that row 3 is equal to row 1 plus row 2. Combine with the conclusion of lecture 7, we can know the condition of the solution:

- in column consider:
  - b is in the column space of A or,
  - b is the combination of the columns of A.
- in row consider: if the combination of rows of A gives zero-row, the same combination of b should also be real number 0.

Solution: $x_{general}$

Suppose $b = \begin{bmatrix} 1 \\ 5 \\ 6 \end{bmatrix}$, then eliminated augment matrix:

$$\left[\begin{array}{cccc|c} 1 & 2 & 0 & -2 & -2 \\ 0 & 0 & 1 & 2 & \frac{3}{2} \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

General solution is all the solutions satisfy the equation, usually express as a infinite solution form. We already know that $Ax_{nullspace} \equiv 0$ and we can solve a $x_{practice}$ for a specific $b (b \in \mathbb{R}^{columnspace})$, so the definition is as following:

$x_{general}$: all the solutions satisfy the equation, general include two part $x_{particular}$ and $x_{nullspace}$.

From the eliminated augment matrix, we can solve the $x_{particular} = \begin{bmatrix} -2 \\ 0 \\ \frac{3}{2} \\ 0 \end{bmatrix}$.

And set the augment column zero we can solve the $x_{nullspace} = k_1 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + k_2 \begin{bmatrix} 2 \\ 0 \\ -2 \\ 1 \end{bmatrix} (k_1, k_2 \in R)$.

So the $x_{general} = x_{nullspace} + x_{particular} = k_1 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + k_2 \begin{bmatrix} 2 \\ 0 \\ -2 \\ 1 \end{bmatrix} + \begin{bmatrix} -2 \\ 0 \\ \frac{3}{2} \\ 0 \end{bmatrix} (k_1, k_2 \in R)$

Solution cases for a A_{m*n}

Suppose a matrix A_{m*n} and rank $r(A) = r$, debate its solution cases:

- full rank of column: r=n<m

  after elimination we will get reduced matrix $R = \begin{bmatrix} I \\ 0 \end{bmatrix}$, r=n means there aren't free variables, the solution of $Ax = b$ if exist(when $b \in \mathbb{R}^{columnspace}$), then is the only $X_{particular}$.
- full rank of row: r=m<n

  after elimination we will get reduced matrix $R = \begin{bmatrix} I & F \end{bmatrix}$, there are n-r free variables, so there always have solution for $Ax = b$.
- full rank of matrix: square matrix, r=n=m

  after elimination we will get reduced matrix $R = \begin{bmatrix} I \end{bmatrix}$, null space only 0, exist only solution for every b.
- r<m and r<n, $R = \begin{bmatrix} I & F \\ 0 & 0 \end{bmatrix}$, it depends on b, zero solution or infinite solutions.