Lecture 7: Solving the Ax=0

Elimination

Suppose matrix $A = \begin{bmatrix} 1 & 2 & 2 & 2 \\ 2 & 4 & 6 & 8 \\ 3 & 6 & 8 & 10 \end{bmatrix}$, solve the null space of x from $Ax = 0$.

$$\begin{bmatrix} 1 & 2 & 2 & 2 \\ 2 & 4 & 6 & 8 \\ 3 & 6 & 8 & 10 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = 0$$

Elimination:

$$\begin{bmatrix} 1 & 2 & 2 & 2 \\ 2 & 4 & 6 & 8 \\ 3 & 6 & 8 & 10 \end{bmatrix} \to \begin{bmatrix} \color{red}{1} & 2 & 2 & 2 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 2 & 4 \end{bmatrix} \to \begin{bmatrix} \color{red}{1} & 2 & 2 & 2 \\ 0 & 0 & \color{red}{2} & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

After elimination we get two pivots: 1 and 2, the number of pivot is equal to the rank of the matrix.

For free column we could free choice, and figure out the pivot columns by back substitution:

- two pivot columns and two free columns:
  - choose value for free variable, $\begin{bmatrix} x_2 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ or $\begin{bmatrix} x_2 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, other value will be the linear combination of $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$
  - solve the $\begin{bmatrix} x_1 \\ x_3 \end{bmatrix} = \begin{bmatrix} -2 \\ 0 \end{bmatrix}$ or $\begin{bmatrix} x_1 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ -0 \end{bmatrix}$
  - two special solution: $\begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ 0 \\ -2 \\ 1 \end{bmatrix}$
- all the linear combinations of special solutions form all the solution of A called null space:
  $k_1 \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + k_2 \begin{bmatrix} 2 \\ 0 \\ -2 \\ 1 \end{bmatrix}$

Reduced echelon matrix R

Reduce the eliminated matrix and we will got:

$$\begin{bmatrix} 1 & 2 & 2 & 2 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 0 & -2 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 0 & -2 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 2 & -2 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} = R$$

and concerned of the pivot columns and free columns dividedly:

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, F = \begin{bmatrix} 2 & -2 \\ 0 & 2 \end{bmatrix}$$
$$\Rightarrow R = \begin{bmatrix} I & F \\ 0 & 0 \end{bmatrix}$$

where I is a $r * r$ size matrix represent the pivot columns and F is $r * (n-r)$ size matrix represent the free columns.

Solving the $Ax = 0$ becomes to solve $Rx = 0$, suppose null-space matrix N--all the special solution of $Ax = 0$, we can infer that:

$$Ax = Rx = \begin{bmatrix} I & F \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_{pivot} \\ x_{free} \end{bmatrix} = RN = 0$$

$$\begin{bmatrix} I & F \end{bmatrix} \begin{bmatrix} x_{pivot} \\ x_{free} \end{bmatrix} = 0 \Longrightarrow I \cdot x_{pivot} + F \cdot x_{free} = 0 \Longrightarrow x_{pivot} = -F \cdot x_{free}$$

Usually we choose the column of I as the value of x_{free}, so $x_{pivot} = -F$, thus $N = \begin{bmatrix} -F \\ I_{(n-r)*(n-r)} \end{bmatrix}$.

So back to the example $R = \begin{bmatrix} 1 & 0 & 2 & -2 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$, we can solve the $N = \begin{bmatrix} -2 & 2 \\ 0 & -2 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$.

Notice that we did a column exchange when reduce the matrix, this will change the null space, so we need to exchange the row 2 and row 3 to clear the influence and get the special solutions : $\begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ 0 \\ -2 \\ 1 \end{bmatrix}$.