Lecture 2: Elimination

This lecture mainly introduced how to solve equations with elimination.

Solve the Ax=b with elimination

In short, elimination is doing row operation to a given matrix until the matrix become an upper triangular matrix.

elimination

Suppose we have following equations:

$$\begin{cases} x+2y+z &= 2 \\ 3x+8y+z &= 12 \\ 4y+z &= 2 \end{cases}$$

transform it to a $Ax=b$ format:

$$\begin{bmatrix} 1 & 2 & 1 \\ 3 & 8 & 1 \\ 0 & 4 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 12 \\ 2 \end{bmatrix}$$

following is the detail steps of elimination:

$$\begin{bmatrix} 1 & 2 & 1 \\ 3 & 8 & 1 \\ 0 & 4 & 1 \end{bmatrix} \xRightarrow{E_{21}} \begin{bmatrix} 1 & 2 & 1 \\ 0 & 2 & -2 \\ 0 & 4 & 1 \end{bmatrix} \xRightarrow{E_{32}} \begin{bmatrix} 1 & 2 & 1 \\ 0 & 2 & -2 \\ 0 & 0 & 5 \end{bmatrix}$$

where the E_{21} and E_{32} are permutation matrix.

solution

To solve the equations, we need add the vector b in the matrix A, they form a augment matrix like this:

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 2 \\ 3 & 8 & 1 & 12 \\ 0 & 4 & 1 & 2 \end{array}\right]$$

Now, we do elimination:

$$
\left[\begin{array}{ccc|c} 1 & 2 & 1 & 2 \\ 3 & 8 & 1 & 12 \\ 0 & 4 & 1 & 2 \end{array}\right] \xrightarrow{E_{21}} \left[\begin{array}{ccc|c} 1 & 2 & 1 & 2 \\ 0 & 2 & -2 & 6 \\ 0 & 4 & 1 & 2 \end{array}\right] \xrightarrow{E_{21}} \left[\begin{array}{ccc|c} 1 & 2 & 1 & 2 \\ 0 & 2 & -2 & 6 \\ 0 & 0 & 5 & -10 \end{array}\right]
$$

then back substitute the coefficients to the equations:

$$
\begin{cases} x + 2y + z & = & 2 \\ 2y - z & = & 6 \\ 5z & = & -10 \end{cases}
$$

finally we got the solution is $\begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}$.

Elimination matrix

row vector multiple a matrix

Above is an simple raw exchange example for elimination so that you can have a intuitional understand of elimination, here we will show a more systematic way about exchange matrix.

We know that matrix multiple a vector(specific a column vector) is about columns' combination of the matrix in lecture 1, but we do raw operation in elimination, so let's consider how about vector(specific a row vector) multiple a matrix:

$$
\begin{bmatrix} 1 & 2 & 7 \end{bmatrix} \begin{bmatrix} ? & ? & ? \\ ? & ? & ? \\ ? & ? & ? \end{bmatrix} = \text{the combinations of the row of the matrix}
$$

elimination matrix

elimination matrix: format the row operations of one matrix to a matrix multiplication form.

Look at the elimination process in section 1:

$$
\begin{bmatrix} 1 & 2 & 1 \\ 3 & 8 & 1 \\ 0 & 4 & 1 \end{bmatrix} \xrightarrow{E_{21}} \begin{bmatrix} 1 & 2 & 1 \\ 0 & 2 & -2 \\ 0 & 4 & 1 \end{bmatrix} \xrightarrow{E_{32}} \begin{bmatrix} 1 & 2 & 1 \\ 0 & 2 & -2 \\ 0 & 0 & 5 \end{bmatrix}
$$

It takes two steps to finish the elimination work:

- row2 minus three times row1, so the $E_{21} = \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$
- then, row3 minus two times row2, so the $E_{32} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix}$

which is:

$$
\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 2 & 1 \\ 3 & 8 & 1 \\ 0 & 4 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 2 & -2 \\ 0 & 0 & 5 \end{bmatrix}
$$

Summary,

$$
E_{32} \cdot E_{21} \cdot A = U(\text{an upper triangular matrix})
$$

inverse matrix

Now we know how to use matrix multiplication to do matrix exchange, we consider a inverse process: how to get the original matrix A from the eliminated matrix U?

Answer: use inverse matrix.

Define: if exist a matrix satisfy that,

$$A \cdot A^{-1} = I = A^{-1} \cdot A$$

we call matrix A^{-1} is the inversion of matrix A and matrix A is invertible.