Lecture 2: Elimination

This lecture mainly introduced how to solve equations with elimination.

Solve the Ax=b with elimination

In short, elimination is doing row operation to a given matrix until the matrix become an upper triangular matrix.

elimination

Suppose we have following equations:

$$\left\{\begin{array}{rcl} x+2y+z & = & 2 \\ 3x+8y+z & = & 12 \\ 4y+z & = & 2 \end{array}\right.$$

transform it to a $Ax=b$ format:

$$\begin{bmatrix} 1 & 2 & 1 \\ 3 & 8 & 1 \\ 0 & 4 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 12 \\ 2 \end{bmatrix}$$

following is the detail steps of elimination:

$$\begin{bmatrix} 1 & 2 & 1 \\ 3 & 8 & 1 \\ 0 & 4 & 1 \end{bmatrix} \xrightarrow{E_{21}} \begin{bmatrix} 1 & 2 & 1 \\ 0 & 2 & -2 \\ 0 & 4 & 1 \end{bmatrix} \xrightarrow{E_{32}} \begin{bmatrix} 1 & 2 & 1 \\ 0 & 2 & -2 \\ 0 & 0 & 5 \end{bmatrix}$$

where the E_{21} and E_{32} are permutation matrix.

solution

To solve the equations, we need add the vector b in the matrix A, they form a augment matrix like this:

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 2 \\ 3 & 8 & 1 & 12 \\ 0 & 4 & 1 & 2 \end{array}\right]$$

Now, we do elimination:

$$\left[\begin{array}{ccc|c}1&2&1&2\\3&8&1&12\\0&4&1&2\end{array}\right]\xRightarrow{E_{21}}\left[\begin{array}{ccc|c}1&2&1&2\\0&2&-2&6\\0&4&1&2\end{array}\right]\xRightarrow{E_{21}}\left[\begin{array}{ccc|c}1&2&1&2\\0&2&-2&6\\0&0&5&-10\end{array}\right]$$

then back substitute the coefficients to the equations:

$$\begin{cases}x+2y+z &= & 2\\ \quad 2y-z &= & 6\\ \quad\quad 5z &= & -10\end{cases}$$

finally we got the solution is $\begin{bmatrix}2\\1\\-2\end{bmatrix}$.

Elimination matrix

row vector multiple a matrix

Above is an simple raw exchange example for elimination so that you can have a intuitional understand of elimination, here we will show a more systematic way about exchange matrix.

We know that matrix multiple a vector(specific a column vector) is about columns' combination of the matrix in lecture 1, but we do raw operation in elimination, so let's consider how about vector(specific a row vector) multiple a matrix:

$$\begin{bmatrix}1&2&7\end{bmatrix}\begin{bmatrix}?&?&?\\?&?&?\\?&?&?\end{bmatrix}=\text{the combinations of the row of the matrix}$$

elimination matrix

elimination matrix: format the row operations of one matrix to a matrix multiplication form.

Look at the elimination process in section 1:

$$\begin{bmatrix}1&2&1\\3&8&1\\0&4&1\end{bmatrix}\xRightarrow{E_{21}}\begin{bmatrix}1&2&1\\0&2&-2\\0&4&1\end{bmatrix}\xRightarrow{E_{32}}\begin{bmatrix}1&2&1\\0&2&-2\\0&0&5\end{bmatrix}$$

It takes two steps to finish the elimination work:

- row2 minus three times row1, so the $E_{21}=\begin{bmatrix}1&0&0\\-3&1&0\\0&0&1\end{bmatrix}$
- then, row3 minus two times row2, so the $E_{32}=\begin{bmatrix}1&0&0\\0&1&0\\0&-2&1\end{bmatrix}$

which is:

$$\begin{bmatrix}1&0&0\\0&1&0\\0&-2&1\end{bmatrix}\cdot\begin{bmatrix}1&0&0\\-3&1&0\\0&0&1\end{bmatrix}\cdot\begin{bmatrix}1&2&1\\3&8&1\\0&4&1\end{bmatrix}=\begin{bmatrix}1&2&1\\0&2&-2\\0&0&5\end{bmatrix}$$

Summary,

$$E_{32}\cdot E_{21}\cdot A=U(\text{an upper triangular matrix})$$

inverse matrix

Now we know how to use matrix multiplication to do matrix exchange, we consider a inverse process: how to get the original matrix A from the eliminated matrix U?

Answer: use inverse matrix.

Define: if exist a matrix satisfy that,

$$A \cdot A^{-1} = I$$

we call matrix A^{-1} is the inversion of matrix A and matrix A is invertible.