Lecture 4: A=LU

A=LU

Before we talked about elimination and row exchange of matrix, now we know that matrix multiplication can achieve the same process. So suppose a good matrix A_{3x3}, we can eliminate matrix A with exchange matrix use following form and finally get a upper triangular matrix U:

$$E_{32}E_{31}E_{21}A = U \tag{1}$$

Easy to know exchange matrix must be invertible, so formula (1) can transform into:

$$A = E_{21}^{-1}E_{31}^{-1}E_{32}^{-1}U \tag{2}$$

and $L = E_{21}^{-1}E_{31}^{-1}E_{32}^{-1}$, thus $A = LU$.

Why A=LU

Right now we have:

$$\begin{cases} EA = U \\ AL = U \end{cases} \tag{3}$$

if seems E is enough for elimination, why we still L, now let's look at a example.

compare L with E

Still a good matrix A_{3x3}, and $E_{32} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -5 & 1 \end{bmatrix}$, $E_{21} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $E_{31} = I$, solve the L.

Solution:

$$A = E_{21}^{-1}E_{31}^{-1}E_{32}^{-1}U$$
$$E_{21}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, E_{31}^{-1} = I \, , E_{32}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 5 & 1 \end{bmatrix}$$
$$\text{thus, } L = E_{21}^{-1}E_{31}^{-1}E_{32}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 5 & 1 \end{bmatrix}$$

Notice that elements of matrix L are all from matrix E_{21} and E_{32}, which means different with matrix $E = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 10 & -5 & 1 \end{bmatrix}$, we can solve the matrix L without doing matrix multiplication.

- e_{31} in matrix E is appear due to the multiplier of elimination.

Formulation:

Define l_{ij}: multiplier of eliminate row_i use row_j, thus:

$$row_i = row_i - l_{ij} \times row_j \tag{4}$$

and in matrix multiplication we use exchange matrix E_{ij} represent this task.

Now we can write down the matrix $L = E^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix}$.

Postscript

In book 《Linear Algebra》chapter2.3, it said:

> The special property of L is that all the multipliers l_{ij} fall into place. Those numbers are mixed up in E(forward elimination from A to U). They are perfect in L (undoing elimination, returning from U to A). Inverting puts the steps and their matrices E_{ij}^{-1} in the opposite order and that prevents the mixup.

In short is that using inverse of E avoid the mixture of l_{ij} while forward elimination do. An intuition explain is we minus eliminated row when we doing forward elimination(for example row_3 minus eliminated row_2 while eliminate element a_{32}).

Application of A=LU--solution of Ax=b

This section is an advanced contents to show what $A = LU$ can do, if you find it's difficult to understand this section, please read after you finished the lecture 8.

Now we know $A = LU$, if we want to solve $Ax = b$, we can substitute $A = LU$ into it and we get $LUx = b$.

- suppose $Ux = c$, we get $Lc = b$
- solve the c and back substitute

Example, solve the equation $\begin{aligned} u + 2v &= 5 \\ 4u + 9v &= 21 \end{aligned}$.

1. Extract the coefficient to matrix $A = \begin{bmatrix} 1 & 2 \\ 4 & 9 \end{bmatrix}$, $b = \begin{bmatrix} 5 \\ 21 \end{bmatrix}$.
2. $l_{ij} = 4$, $L = \begin{bmatrix} 1 & 0 \\ 4 & 1 \end{bmatrix}$

$$Lc = b\,, \quad \begin{bmatrix} 1 & 0 \\ 4 & 1 \end{bmatrix} \cdot [\,c\,] = \begin{bmatrix} 5 \\ 21 \end{bmatrix} \Longrightarrow c = \begin{bmatrix} 5 \\ 1 \end{bmatrix}$$

3. back substitute to $Ux = c$:

$$\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \cdot [\,x\,] = \begin{bmatrix} 5 \\ 1 \end{bmatrix} \Longrightarrow x = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

In a word, solve L is quicker than E.

The complexity of elimination

Suppose multiple and substract for one element is one operation, and matrix A_{n*n}.

1. elimination

$$a_{ij} - k \cdot a_{1j} \quad \text{while } 2 \le i \le n, 1 \le j \le n, \text{k is in} A_{i1}$$

2. if $i = 2$, there are $n-1$ rows need to substract the row_1 and every row has n numbers need to multiple and substract, so there are $(n-1)*n$ times operation, nearly n^2.
3. if $i = n$, only a_{nn-1} need to change
4. so the complexity of elimination should be:

$$n^2 + (n-1)^2 + \cdots + 2^2 + 1^1 \approx \frac{1}{3}n^2$$

5. if add one constant column b, it will take

$$(n-1) + (n-2) + \cdots + 2 + 1 = \frac{n*(n-1)}{2}$$

   times operation, and if take back substitute into consideration, it will takes nearly n^2 times operation.