Lecture 6: Column space and Null-space

The intersection and union of subspace

Suppose two subspace $\mathbb{S}$ and $\mathbb{T}$,

- their intersection $\mathbb{S} \cap \mathbb{T}$ is also a subspace.

$$\exists \boldsymbol{v}, \boldsymbol{w} \in \mathbb{S} \cap \mathbb{T} \Rightarrow \begin{cases} \boldsymbol{v} \in \mathbb{S}, \boldsymbol{v} \in \mathbb{T} \\ \boldsymbol{w} \in \mathbb{S}, \boldsymbol{w} \in \mathbb{T} \end{cases} \Rightarrow$$
$$\boldsymbol{v} + \boldsymbol{w} \in \mathbb{S} \text{ and } \in \mathbb{T}, \boldsymbol{kv} \in \mathbb{S} \text{ and } \in \mathbb{T}, \boldsymbol{kw} \in \mathbb{S} \text{ and } \in \mathbb{T}$$

- their union $\mathbb{S} \cup \mathbb{T}$ is not a subspace.

  Proof by contradiction, there are so many counter-example.

Column space

Give a matrix $A = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 3 \\ 3 & 1 & 4 \\ 4 & 1 & 5 \end{bmatrix}$, each column vector of A is a subspace of $\mathbb{R}^4$, and all the linear combination of column vectors form a subspace called **column space**.

What we concern about is the scale of column space of a given matrix, and we can use a equation $Ax = b$ to represent this question.

Ax=b

Suppose $Ax = b$ like following:

$$Ax = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 3 \\ 3 & 1 & 4 \\ 4 & 1 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix} = b$$

We can propose two question(actually is one):

- Does $Ax = b$ always has a solution for every b ?
- What b can make $Ax = b$ have a solution ?

From forward lecture, we know that Ax is the combination of column vectors of A:

$$Ax = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 3 \\ 3 & 1 & 4 \\ 4 & 1 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} x_2 \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} x_3 \begin{bmatrix} 2 \\ 3 \\ 4 \\ 5 \end{bmatrix}$$

or in other word, Ax is represent the column space of A.

Obviously, three 4-dimension vectors can not full-fill $\mathbb{R}^4$, so Ax(which is the column space of A) subspace of $\mathbb{R}^4$, it can only expression a part of 4-dimension vector b that is $Ax = b$ doesn't have solution for every b.

And only b is in A's' column space, $Ax = b$ have solution.

Null space

Null space of A: all solution of $Ax = 0$.

Solution of Ax=0

$$Ax = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 3 \\ 3 & 1 & 4 \\ 4 & 1 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = 0$$

Solve the equation and $x = k \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}, k \in R$.

Check that solution of $Ax = 0$ always give a subspace:

$$\begin{aligned} &\exists \boldsymbol{v}, \boldsymbol{w} \Rightarrow Av = 0, Aw = 0 \\ &\Rightarrow \begin{cases} A(v+w) = 0 \\ A(kv) = 0 \\ A(kw) = 0 \end{cases} \end{aligned}$$

Solution of Ax=b: explain from a space aspect

So for a special $b(b \neq 0)$ the solutions of $Ax = b$ can't form a subspace: x won't go through $\mathbf{0}$, and when $Ax = b$ exist solution, b is in the column space of A, all the solution of $Ax = b$ construct a plain that don't go through zero-point.