# MA2104 Assignment 3

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## Question 1

> Find the point on the paraboloid $z = \dfrac{x^2}{4} + \dfrac{y^2}{25}$ that is closest to the point $(3, 0, 0)$.

Solution. For any point $(x, y, z)$, the distance between $(x, y, z)$ and $(3, 0, 0)$ is given by

$$D(x, y, z) := \sqrt{(x-3)^2 + y^2 + z^2}.$$

By hint, $D$ is minimum if and only if $D^2$ is minimum, so it suffice to find minimum of $D^2$ given the constraint that

$$g(x, y, z) := \frac{x^2}{4} + \frac{y^2}{25} - z = 0.$$

Proceed to use Lagrange Multipliers to find maximum of $D^2$ given $g(x, y, z) = 0$, since $\nabla g \neq \mathbf{0}$,

$$\begin{aligned} \nabla(D^2)(x, y, z) &= \langle 2(x-3), 2y, 2z \rangle \\ \nabla g(x, y, z) &= \left\langle \frac{x}{2}, \frac{2y}{25}, -1 \right\rangle \end{aligned}$$

Now proceed to solve $\nabla(D^2)(x, y, z) = \lambda \nabla g(x, y, z)$, which gives the following system of equations

$$\left.\begin{aligned} 2(x-3) &= \lambda \frac{x}{2} \\ 2y &= \lambda \frac{2y}{25} \\ 2z &= -\lambda \\ \frac{x^2}{4} + \frac{y^2}{25} &= z \end{aligned}\right\} \implies \left.\begin{aligned} 4(x-3) &= \lambda x \\ 25y &= \lambda y \\ z &= -\frac{\lambda}{2} \\ \frac{x^2}{4} + \frac{y^2}{25} &= z \end{aligned}\right\}$$

From second equation, $y = 0$ or $\lambda = 25$.

- Case $\lambda = 25$, then $z = -\frac{25}{2}$, and

$$\begin{aligned} 4x - 12 &= 25x \\ x &= -\frac{4}{7} \end{aligned}$$

This has no solution in $\mathbb{R}$ as $\frac{y^2}{25} \geq 0 > z - \frac{x^2}{4} = -\frac{25}{2} - \frac{4}{49}$.

- Case $y = 0$, then fourth equation reduces to $x^2 = 4z$,

$$\begin{aligned} x^2 &= 4z \\ x^2 &= -2\lambda \\ \lambda &= -\frac{x^2}{2} \end{aligned}$$

substituting that into our first equation we get

$$\begin{aligned} 4x - 12 &= -\frac{x^3}{2} \\ x^3 + 8x - 24 &= 0 \\ x &= 2 \end{aligned}$$

Then we get $z = 1$. The only critical point is $(2, 0, 1)$. ■

## Question 2

> Suppose that the temperature of a metal plate is given by $T(x, y) = x^2 + 2x + y^2$ for points $(x, y)$ on the elliptical plate defined by $x^2 + 4y^2 \leq 24$.
>
> Find the maximum and minimum temperatures on the plate.

Solution. The gradient vector for $T$ is given by

$$\nabla T(x, y) = \langle 2x + 2, 2y \rangle .$$

The critical points are when $\nabla T = \mathbf{0}$, so

$$\left.\begin{aligned} 2x + 2 &= 0 \\ 2y &= 0 \end{aligned}\right\}$$

The only critical point obtained is $(-1, 0)$, which is eyeballed to be inside the elliptical plate.

Next, proceed to use Lagrange multipliers to find critical points on the boundary, let $g(x, y) := x^2 + 4y^2 = 24$ be our constraint. Then

$$\nabla g(x, y) = \langle 2x, 8y \rangle .$$

Solving $\nabla T(x,y) = \lambda \nabla g(x,y)$, we obtain the system of equations

$$\left.\begin{aligned} x+1 &= \lambda x \\ y &= 4\lambda y \\ x^2+4y^2 &= 24 \end{aligned}\right\}$$

From second equation, $\lambda = \frac{1}{4}$ or $y=0$,

- Case $\lambda = \frac{1}{4}$, then

$$\begin{aligned} x+1 &= \frac{x}{4} \\ x &= -\frac{4}{3} \end{aligned}$$

  Substituting that into our constraint,

$$\begin{aligned} \frac{16}{9}+4y^2 &= 24 \\ y^2 &= \frac{50}{9} \\ y &= \pm\frac{5\sqrt{2}}{3} \end{aligned}$$

- Case $y=0$, then $x^2 = 24$, so $x = \pm 2\sqrt{6}$.

Tabulating the critical points,

| $(x,y)$ | $(-1,0)$ | $(-\frac{4}{3}, \pm\frac{5\sqrt{2}}{3})$ | $(2\sqrt{6},0)$ | $(-2\sqrt{6},0)$ |
|---|---|---|---|---|
| $T(x,y)$ | $-1$ | $\frac{14}{3}$ | $24+4\sqrt{6}$ | $24-4\sqrt{6}$ |
| | min | | max | |

■

## Question 3

> Evaluate the following integral
> $$\int_0^2 \int_{\sqrt{y}}^2 \sqrt{x^2+y}\, dx\, dy.$$

Solution. Let the region of integration be called $D$, so

$$D = \{\,(x,y) : 0 \le y \le 4, \sqrt{y} \le x \le 2\,\}\,.$$

But $D$ can also be expressed as

$$D = \{\,(x,y) : 0 \le x \le 2, 0 \le y \le x^2\,\}\,.$$

This allows us to rewrite the integral as

$$\begin{aligned}
\iint_D \sqrt{x^2+y}\,dA &= \int_0^2 \int_0^{x^2} \sqrt{x^2+y}\,dy\,dx \\
&= \int_0^2 \left[\frac{2}{3}\left(x^2+y\right)^{3/2}\right]_0^{x^2} dx \\
&= \frac{2}{3}\int_0^2 \left((2x^2)^{3/2} - (x^2)^{3/2}\right) dx \\
&= \frac{2}{3}\int_0^2 \left(2^{3/2}-1\right) x^3\,dx \\
&= \frac{2}{3}\left(2^{3/2}-1\right)\left[\frac{x^4}{4}\right]_0^2 dx \\
&= \frac{8}{3}\left(2^{3/2}-1\right)
\end{aligned}$$

■

## Question 4

> Rewrite the following iterated integral in the order $dy\,dx\,dz$:
>
> $$\int_{-1}^{1}\int_0^{\sqrt{1-x^2}}\int_0^{y/2} f(x,y,z)\,dz\,dy\,dx.$$

Solution. Let $D$ denote the region of integration, then it can be given by

$$D = \left\{\,(x,y,z) : -1 \le x \le 1, 0 \le y \le \sqrt{1-x^2}, 0 \le z \le \frac{y}{2}\,\right\}.$$

To integrate in the order $dy\,dx\,dz$, first find absolute bounds for $z$.

$$z \le \frac{y}{2} \le \frac{\sqrt{1-x^2}}{2} \le \frac{1}{2},$$

so $0 \le z \le 1/2$.

Next up, find bounds for $x$, note that because $x^2 + y^2 \leq 1$, and $y \geq 2z$,

$$
\begin{aligned}
|x| &\leq \sqrt{1 - y^2} \\
|x| &\leq \sqrt{1 - 4z^2}
\end{aligned}
$$

so $-\sqrt{1 - 4z^2} \leq x \leq \sqrt{1 - 4z^2}$.

Lastly, note that because $z \leq y/2$, $y$ is bounded below as $2z \leq y$. So $2z \leq y \leq \sqrt{1 - x^2}$. Then rewriting the integral, we have

$$
\iiint_D f(x, y, z)\ dV = \int_0^{1/2} \int_{-\sqrt{1-4z^2}}^{\sqrt{1-4z^2}} \int_{2z}^{\sqrt{1-x^2}} f(x, y, z)\ dy\ dx\ dz. \qquad \blacksquare
$$