
MA2104 Assignment 3

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Question 1

Find the point on the paraboloid $z = \frac{x^2}{4} + \frac{y^2}{25}$ that is closest to the point $(3, 0, 0)$.

Solution. For any point (x, y, z) , the distance between (x, y, z) and $(3, 0, 0)$ is given by

$$D(x, y, z) := \sqrt{(x-3)^2 + y^2 + z^2}.$$

By hint, D is minimum if and only if D^2 is minimum, so it suffice to find minimum of D^2 given the constraint that

$$g(x, y, z) := \frac{x^2}{4} + \frac{y^2}{25} - z = 0.$$

Proceed to use Lagrange Multipliers to find maximum of D^2 given $g(x, y, z) = 0$, since $\nabla g \neq \mathbf{0}$,

$$\nabla(D^2)(x, y, z) = \langle 2(x-3), 2y, 2z \rangle$$

$$\nabla g(x, y, z) = \left\langle \frac{x}{2}, \frac{2y}{25}, -1 \right\rangle$$

Now proceed to solve $\nabla(D^2)(x, y, z) = \lambda \nabla g(x, y, z)$, which gives the following system of equations

$$\left. \begin{array}{l} 2(x-3) = \lambda \frac{x}{2} \\ 2y = \lambda \frac{2y}{25} \\ 2z = -\lambda \\ \frac{x^2}{4} + \frac{y^2}{25} = z \end{array} \right\} \Rightarrow \left. \begin{array}{l} 4(x-3) = \lambda x \\ 25y = \lambda y \\ z = -\frac{\lambda}{2} \\ \frac{x^2}{4} + \frac{y^2}{25} = z \end{array} \right\}$$

From second equation, $y = 0$ or $\lambda = 25$.

- Case $\lambda = 25$, then $z = -\frac{25}{2}$, and

$$4x - 12 = 25x$$

$$x = -\frac{4}{7}$$

This has no solution in \mathbb{R} as $\frac{y^2}{25} \geq 0 > z - \frac{x^2}{4} = -\frac{25}{2} - \frac{4}{49}$.

- Case $y = 0$, then fourth equation reduces to $x^2 = 4z$,

$$x^2 = 4z$$

$$x^2 = -2\lambda$$

$$\lambda = -\frac{x^2}{2}$$

substituting that into our first equation we get

$$4x - 12 = -\frac{x^3}{2}$$

$$x^3 + 8x - 24 = 0$$

$$x = 2$$

Then we get $z = 1$. The only critical point is $(2, 0, 1)$. ■

Question 2

Suppose that the temperature of a metal plate is given by $T(x, y) = x^2 + 2x + y^2$ for points (x, y) on the elliptical plate defined by $x^2 + 4y^2 \leq 24$.

Find the maximum and minimum temperatures on the plate.

Solution. The gradient vector for T is given by

$$\nabla T(x, y) = \langle 2x + 2, 2y \rangle.$$

The critical points are when $\nabla T = \mathbf{0}$, so

$$\left. \begin{array}{l} 2x + 2 = 0 \\ 2y = 0 \end{array} \right\}$$

The only critical point obtained is $(-1, 0)$, which is eyeballed to be inside the elliptical plate.

Next, proceed to use Lagrange multipliers to find critical points on the boundary, let $g(x, y) := x^2 + 4y^2 = 24$ be our constraint. Then

$$\nabla g(x, y) = \langle 2x, 8y \rangle.$$

Solving $\nabla T(x, y) = \lambda \nabla g(x, y)$, we obtain the system of equations

$$\left. \begin{aligned} x + 1 &= \lambda x \\ y &= 4\lambda y \\ x^2 + 4y^2 &= 24 \end{aligned} \right\}$$

From second equation, $\lambda = \frac{1}{4}$ or $y = 0$,

- Case $\lambda = \frac{1}{4}$, then

$$\begin{aligned} x + 1 &= \frac{x}{4} \\ x &= -\frac{4}{3} \end{aligned}$$

Substituting that into our constraint,

$$\begin{aligned} \frac{16}{9} + 4y^2 &= 24 \\ y^2 &= \frac{50}{9} \\ y &= \pm \frac{5\sqrt{2}}{3} \end{aligned}$$

- Case $y = 0$, then $x^2 = 24$, so $x = \pm 2\sqrt{6}$.

Tabulating the critical points,

(x, y)	$(-1, 0)$	$(-\frac{4}{3}, \pm \frac{5\sqrt{2}}{3})$	$(2\sqrt{6}, 0)$	$(-2\sqrt{6}, 0)$
$T(x, y)$	-1	$\frac{14}{3}$	$24 + 4\sqrt{6}$	$24 - 4\sqrt{6}$
	min		max	

■

Question 3

Evaluate the following integral

$$\int_0^2 \int_{\sqrt{y}}^2 \sqrt{x^2 + y} \, dx \, dy.$$

Solution. Let the region of integration be called D , so

$$D = \{ (x, y) : 0 \leq y \leq 4, \sqrt{y} \leq x \leq 2 \}.$$

But D can also be expressed as

$$D = \{ (x, y) : 0 \leq x \leq 2, 0 \leq y \leq x^2 \}.$$

This allows us to rewrite the integral as

$$\begin{aligned} \iint_D \sqrt{x^2 + y} \, dA &= \int_0^2 \int_0^{x^2} \sqrt{x^2 + y} \, dy \, dx \\ &= \int_0^2 \left[\frac{2}{3} (x^2 + y)^{3/2} \right]_0^{x^2} dx \\ &= \frac{2}{3} \int_0^2 ((2x^2)^{3/2} - (x^2)^{3/2}) \, dx \\ &= \frac{2}{3} \int_0^2 (2^{3/2} - 1) x^3 \, dx \\ &= \frac{2}{3} (2^{3/2} - 1) \left[\frac{x^4}{4} \right]_0^2 \\ &= \frac{8}{3} (2^{3/2} - 1) \end{aligned}$$

■

Question 4

Rewrite the following iterated integral in the order $dy \, dx \, dz$:

$$\int_{-1}^1 \int_0^{\sqrt{1-x^2}} \int_0^{y/2} f(x, y, z) \, dz \, dy \, dx.$$

Solution. Let D denote the region of integration, then it can be given by

$$D = \left\{ (x, y, z) : -1 \leq x \leq 1, 0 \leq y \leq \sqrt{1-x^2}, 0 \leq z \leq \frac{y}{2} \right\}.$$

To integrate in the order $dy \, dx \, dz$, first find absolute bounds for z .

$$z \leq \frac{y}{2} \leq \frac{\sqrt{1-x^2}}{2} \leq \frac{1}{2},$$

so $0 \leq z \leq 1/2$.

Next up, find bounds for x , note that because $x^2 + y^2 \leq 1$, and $y \geq 2z$,

$$\begin{aligned}|x| &\leq \sqrt{1 - y^2} \\ |x| &\leq \sqrt{1 - 4z^2}\end{aligned}$$

so $-\sqrt{1 - 4z^2} \leq x \leq \sqrt{1 - 4z^2}$.

Lastly, note that because $z \leq y/2$, y is bounded below as $2z \leq y$. So $2z \leq y \leq \sqrt{1 - x^2}$. Then rewriting the integral, we have

$$\iiint_D f(x, y, z) \, dV = \int_0^{1/2} \int_{-\sqrt{1-4z^2}}^{\sqrt{1-4z^2}} \int_{2z}^{\sqrt{1-x^2}} f(x, y, z) \, dy \, dx \, dz. \quad \blacksquare$$