

# MA2101S Homework 1

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**Problem 1.** Let  $\alpha \in \mathbb{Q}$  be a rational number such that the polynomial  $T^2 - \alpha = 0$  has no solutions in  $\mathbb{Q}$ .

Show that  $\mathbb{Q}(\sqrt{\alpha}) := \{a + b\sqrt{\alpha} \in \mathbb{C} : a, b \in \mathbb{Q}\}$ , where  $\sqrt{\alpha} \in \mathbb{C}$  is a square root of  $\alpha$ , is a field under the usual arithmetic operations in  $\mathbb{C}$  (is a subfield of  $\mathbb{C}$ ).

*Proof.* To prove  $\mathbb{Q}(\sqrt{\alpha})$  is a subfield of  $\mathbb{C}$ , it suffices to just check for closure under addition, multiplication, negation, reciprocation, and the existence of 0 and 1.

**Presence of 0 and 1.**  $0, 1 \in \mathbb{Q}$ , and since  $0 = 0 + 0\sqrt{\alpha}$  and  $1 = 1 + 0\sqrt{\alpha}$ ,  $0, 1 \in \mathbb{Q}(\sqrt{\alpha})$ .

**Closure under  $+$ .** For any pair  $p, q \in \mathbb{Q}(\sqrt{\alpha})$ ,  $\exists x, y, w, z \in \mathbb{Q}$  such that

$$\begin{aligned}p &= x + y\sqrt{\alpha} \\ q &= w + z\sqrt{\alpha}\end{aligned}$$

Then by associativity of  $+$  and distributivity of  $\cdot$  over  $+$  in  $\mathbb{C}$ ,

$$\begin{aligned}p + q &= x + y\sqrt{\alpha} + w + z\sqrt{\alpha} \\ &= (x + w) + (y + z)\sqrt{\alpha}\end{aligned}$$

Because  $\mathbb{Q}$  is a field,  $x + w$  and  $y + z$  are in  $\mathbb{Q}$ , thus  $p + q \in \mathbb{Q}(\sqrt{\alpha})$ .

**Closure under  $\cdot$ .** For any pair  $p, q \in \mathbb{Q}(\sqrt{\alpha})$ ,  $\exists x, y, w, z \in \mathbb{Q}$  such that

$$\begin{aligned}p &= x + y\sqrt{\alpha} \\ q &= w + z\sqrt{\alpha}\end{aligned}$$

Then by associativity of  $\cdot$  and distributivity,

$$\begin{aligned}p \cdot q &= (x + y\sqrt{\alpha}) \cdot (w + z\sqrt{\alpha}) \\ &= x(w + z\sqrt{\alpha}) + y\sqrt{\alpha}(w + z\sqrt{\alpha}) \\ &= xw + xz\sqrt{\alpha} + yw\sqrt{\alpha} + yz\sqrt{\alpha}\sqrt{\alpha} \\ &= (xw + yz\alpha) + (xz + yw)\sqrt{\alpha}\end{aligned}$$

Again because  $\mathbb{Q}$  is a field,  $xw + yz\alpha$  and  $xz + yw$  are in  $\mathbb{Q}$ , thus  $p \cdot q \in \mathbb{Q}(\sqrt{\alpha})$ .

**Closure under  $-$ .** For any  $p \in \mathbb{Q}(\sqrt{\alpha})$ ,  $\exists x, y \in \mathbb{Q}$  such that  $p = x + y\sqrt{\alpha}$ . Then

$$-p = -(x + y\sqrt{\alpha}) = -x + (-y)\sqrt{\alpha}.$$

$-p \in \mathbb{Q}(\sqrt{\alpha})$  because  $-x, -y$  in  $\mathbb{Q}$  due to  $\mathbb{Q}$  being a field.

**Closure under  $(-)^{-1}$ .** For any  $p \in \mathbb{Q}(\sqrt{\alpha}) \setminus \{0\}$ ,  $\exists x, y \in \mathbb{Q}$  such that  $p = x + y\sqrt{\alpha}$ . Then compute  $p^{-1}$  in  $\mathbb{C}$  as follows,

$$\begin{aligned} p^{-1} &= \frac{1}{p} = \frac{1}{x + y\sqrt{\alpha}} \\ &= \frac{x - y\sqrt{\alpha}}{x^2 - y^2\alpha} \end{aligned}$$

**Claim.**  $x^2 - y^2\alpha \neq 0$ .

Case  $y = 0$ , then because  $p \neq 0, x \neq 0$ , so  $x^2 - y^2\alpha \neq 0$ . Case  $y \neq 0$ , then  $\exists y^{-1} \in \mathbb{Q}$ . Suppose for a contradiction  $x^2 - y^2\alpha = 0$ , then we have

$$\begin{aligned} (y^{-1})^2(x^2 - y^2\alpha) &= 0 \\ \left(\frac{x}{y}\right)^2 - \alpha &= 0 \end{aligned}$$

but since  $\mathbb{Q}$  is a field and  $y \neq 0$  by assumption,  $(\frac{x}{y})^2 \in \mathbb{Q}$ , contradicting with fact that  $T^2 - \alpha = 0$  has no solution in  $\mathbb{Q}$ . Hence  $x^2 - y^2\alpha \neq 0$  and its reciprocal exists in  $\mathbb{Q}$ .

Therefore  $p^{-1} = \frac{x}{x^2 - y^2\alpha} - \frac{y}{x^2 - y^2\alpha}\sqrt{\alpha}$ , and because  $x, y, \alpha \in \mathbb{Q}$ ,  $p^{-1} \in \mathbb{Q}(\sqrt{\alpha})$ .  $\square$

**Problem 2.** Define  $V := \mathbb{C}^{\mathbb{C}}$ , consider the following subsets of  $V$ . Which are  $\mathbb{R}$ -vector spaces? Which are  $\mathbb{C}$ -vector spaces? Justify.

**Notation.** Let  $\theta_V : \mathbb{C} \rightarrow \mathbb{C}, z \mapsto 0$  denote the zero vector which is the constant function of  $0_{\mathbb{C}}$ . For each part, let the subset be called  $W$ .

Due to the sets below being subsets of  $V$  under the same operations, it suffices to check if  $W$  in each part is a subspace having  $\theta_V$ , closure under addition and scalar multiplication.

- (i) all  $f \in V$  such that  $f(0) = 1$ ;

Because  $\theta_V(0) = 0 \neq 1$ ,  $W$  is not a vector space due to absense of  $\theta_V$ .  $\blacksquare$

- (ii) all  $f \in V$  such that  $f(0) = f(1)$ ;

$\theta_V(0) = \theta_V(1) = 0$ , so the zero vector is in  $W$ .

Take any pair  $f, g \in W$ , then

$$\begin{aligned} (f + g)(0) &= f(0) + g(0) \\ &= f(1) + g(1) \\ &= (f + g)(1) \end{aligned}$$

closure under vector addition holds.

Take any  $f \in W$ , then for any  $k \in \mathbb{C}$  and  $k \in \mathbb{R}$ ,

$$\begin{aligned}(kf)(0) &= k \cdot f(0) \\ &= k \cdot f(1) \\ &= (kf)(1)\end{aligned}$$

$W$  is both a  $\mathbb{R}$  and  $\mathbb{C}$ -vector space. ■

- (iii) all  $f \in V$  such that for every  $z \in \mathbb{C}$ , one has  $\overline{f(z)} = f(z)$ ;

$\forall z \in \mathbb{C}. \overline{\theta_V(z)} = \overline{0} = 0$ , so  $\theta_V$  is in  $W$ .

For any  $z \in \mathbb{C}$ ,  $\bar{z} = z \iff z \in \mathbb{R}$ .

Take any pair  $f, g \in W$ , then for any  $z \in \mathbb{C}$ ,  $(f+g)(z) = f(z) + g(z)$ , since  $f(z), g(z) \in \mathbb{R}$  and  $\mathbb{R}$  is a subfield of  $\mathbb{C}$ ,  $(f+g)(z) \in \mathbb{R}$  and thus closure under vector addition holds.

Take any non-zero  $f$  from  $W$ , take any  $k \in \mathbb{C}$  where  $k = a + bi, a, b \in \mathbb{R}, b \neq 0$ , then for any  $z \in \mathbb{C}$  where  $f(z) \neq 0$ ,

$$\begin{aligned}(kf)(z) &= k \cdot f(z) \\ &= (a + bi) \cdot f(z) \\ &= a \cdot f(z) + (b \cdot f(z))i\end{aligned}$$

Since  $\exists k \in \mathbb{C}, f \in W$  where  $\text{Im}((kf)(z)) \neq 0 \iff (kf)(z) \notin \mathbb{R}$ , closure under scalar multiplication is broken and this set is not a  $\mathbb{C}$ -vector space.

However, for any  $f \in V$  where for every  $z \in \mathbb{C}$ ,  $f(z) \in \mathbb{R}$ . For any  $k \in \mathbb{R}, z \in \mathbb{C}$ , because  $\mathbb{R}$  is a subfield,  $(kf)(z) = k \cdot f(z) \in \mathbb{R}$ . Hence  $W$  is a  $\mathbb{R}$ -vector space. ■

- (iv) all  $f \in V$  such that for every  $z \in \mathbb{C}$ , one has  $f(\bar{z}) = f(z)$ ;

$\theta_V$  is a constant function and ignores its parameter, satisfying the condition, thus  $\theta_V \in W$ .

For any pair  $f, g \in W$ , then for any  $z \in \mathbb{C}$ ,

$$\begin{aligned}(f+g)(\bar{z}) &= f(\bar{z}) + g(\bar{z}) \\ &= f(z) + g(z) \\ &= (f+g)(z)\end{aligned}$$

Thus closure under vector addition holds.

For any  $f \in W, k \in \mathbb{C}$ , for all  $z \in \mathbb{C}$ ,

$$\begin{aligned}(kf)(\bar{z}) &= k \cdot f(\bar{z}) \\ &= k \cdot f(z) \\ &= (kf)(z)\end{aligned}$$

Closure under scalar multiplication holds (also holds for  $k \in \mathbb{R}$ ).  $W$  is both a  $\mathbb{R}$  and  $\mathbb{C}$ -vector space. ■

- (v) all  $f \in V$  such that for every  $z \in \mathbb{C}$ , one has  $f(z^2) = f(z)^2$ ;

Take  $f = g = \text{id}_{\mathbb{C}}$ , clear that property above holds. Take  $z = 2 \in \mathbb{C}$ ,

$$\begin{aligned}(f+g)(2^2) &= f(4) + g(4) \\ &= 8 \\ (f+g)(2)^2 &= (f(2) + g(2))^2 \\ &= 4^2 = 16\end{aligned}$$

We can see that  $(f + g)(2^2) \neq (f + g)(2)^2$ , thus  $f + g \notin W$ , closure under vector addition is broken and  $W$  is not a  $\mathbb{R}$  or  $\mathbb{C}$ -vector space.  $\blacksquare$

**Definition.** Let  $K$  be a field,  $V$  a  $K$ -vector space. For any  $K$ -subspaces  $U, W \subseteq V$ , define

$$U + W := \{v \in V : \exists u \in U, w \in W. v = u + w\}.$$

**Problem 3.** Let  $V := \mathbb{R}^{\mathbb{R}}$ . Consider  $V_{\text{even}}$  (resp.  $V_{\text{odd}}$ ) as subsets of all even (resp. odd) functions.

**Notation.** Let  $\theta_V : \mathbb{R} \rightarrow \mathbb{R}, x \mapsto 0$  denote the zero vector which is the constant function of  $0_{\mathbb{R}}$ .

(a) Show that  $V_{\text{even}}$  and  $V_{\text{odd}}$  are  $\mathbb{R}$ -subspaces of  $V$ .

*Proof.* Firstly,  $\forall x \in \mathbb{R}. \theta_V(x) = 0$ , trivially  $\theta_V \in V_{\text{odd}}$  and  $\theta_V \in V_{\text{even}}$ .

Consider  $f, g \in V_{\text{even}}$ , then  $\forall x \in \mathbb{R}$ ,

$$\begin{aligned} (f + g)(-x) &= f(-x) + g(-x) \\ &= f(x) + g(x) \\ &= (f + g)(x) \end{aligned}$$

Thus  $f + g \in V_{\text{even}}$ . Consider any  $f \in V_{\text{even}}, k \in \mathbb{R}$ , then  $\forall x \in \mathbb{R}$ ,

$$\begin{aligned} (kf)(-x) &= k \cdot f(-x) \\ &= k \cdot f(x) \\ &= (kf)(x) \end{aligned}$$

Thus  $kf \in V_{\text{even}}$ . Therefore  $V_{\text{even}}$  is a subspace of  $V$ .

Now consider  $f, g \in V_{\text{odd}}$ , then  $\forall x \in \mathbb{R}$ ,

$$\begin{aligned} (f + g)(-x) &= f(-x) + g(-x) \\ &= -f(x) - g(x) \\ &= -(f(x) + g(x)) \\ &= -(f + g)(x) \end{aligned}$$

Thus  $f + g \in V_{\text{odd}}$ . Consider any  $f \in V_{\text{odd}}, k \in \mathbb{R}$ , then  $\forall x \in \mathbb{R}$ ,

$$\begin{aligned} (kf)(-x) &= k \cdot f(-x) \\ &= k \cdot (-f(x)) \\ &= -k \cdot f(x) \\ &= -(k \cdot f(x)) = -(kf)(x) \end{aligned}$$

Thus  $kf \in V_{\text{odd}}$ . Therefore  $V_{\text{odd}}$  is a subspace of  $V$ .  $\square$

(b) Show that  $V_{\text{even}} \cap V_{\text{odd}} = \{\theta_V\}$  and  $V_{\text{even}} + V_{\text{odd}} = V$ .

*Proof.* Take  $f \in V_{\text{even}} \cap V_{\text{odd}}$ ,  $f$  is both even and odd, so  $\forall x \in \mathbb{R}$ ,

$$f(-x) = f(x) \text{ and } f(-x) = -f(x).$$

So  $f(x) = -f(x)$  for all  $x \in \mathbb{R}$ , which is the case if and only if for all  $x \in \mathbb{R}$ ,  $f(x) = 0$ . This means  $f$  is the constant function of 0, that is  $0_V$ , this implies  $V_{\text{even}} \cap V_{\text{odd}} = \{0_V\}$ .

Consider any  $h \in V_{\text{even}} + V_{\text{odd}}$ , by definition,  $\exists f \in V_{\text{even}}, g \in V_{\text{odd}}. h = f + g$ . Since  $V_{\text{even}}$  and  $V_{\text{odd}}$  are subspaces of  $V$ ,  $h = f + g \in V$ , so  $V_{\text{even}} + V_{\text{odd}} \subseteq V$ .

Conversely take  $f \in V$ . For all  $x \in \mathbb{R}$ ,

$$\begin{aligned} f(x) &= f(x) + 0_V(x) \\ &= \frac{1}{2}f(x) + \frac{1}{2}f(x) + \frac{1}{2}f(-x) - \frac{1}{2}f(-x) \\ &= \frac{1}{2}(f(x) + f(-x)) + \frac{1}{2}(f(x) - f(-x)) \end{aligned}$$

Define  $g, h : \mathbb{R} \rightarrow \mathbb{R}$  as

$$\begin{aligned} g : x &\mapsto \frac{1}{2}(f(x) + f(-x)) \\ h : x &\mapsto \frac{1}{2}(f(x) - f(-x)) \end{aligned}$$

Verify that  $g \in V_{\text{even}}$  and  $h \in V_{\text{odd}}$ .

$$\begin{aligned} g(-x) &= \frac{1}{2}(f(-x) + f(-(-x))) \\ &= \frac{1}{2}(f(x) + f(-x)) \\ h(-x) &= \frac{1}{2}(f(-x) - f(-(-x))) \\ &= -\frac{1}{2}(f(x) - f(-x)) \end{aligned}$$

Because  $f = g + h$ ,  $f \in V_{\text{even}} + V_{\text{odd}}$ . So  $V \subseteq V_{\text{even}} + V_{\text{odd}}$  and this completes the proof that  $V_{\text{even}} + V_{\text{odd}} = V$ .  $\square$

**Problem 4.** Let  $K$  be any field, let  $V$  be a  $K$ -vector space, and let  $V_1, V_2 \subseteq V$  be  $K$ -subspaces of  $V$ . Suppose  $V_1 \cap V_2 = \{0_V\}$  and  $V_1 + V_2 = V$ . Show that for any  $v \in V$ , there exist unique vectors  $v_1 \in V_1$  and  $v_2 \in V_2$  such that  $v = v_1 + v_2$  in  $V$ .

*Proof.* Take any arbitrary  $v \in V$ , since  $V_1 + V_2 = V$ , by definition of  $V_1 + V_2$ , there exists  $v_1 \in V_1, v_2 \in V_2$  such that  $v = v_1 + v_2$ .

Suppose  $\exists v'_1 \in V_1, v'_2 \in V_2$  where  $v = v'_1 + v'_2$ .

$$\begin{aligned} v &= v_1 + v_2 = v'_1 + v'_2 \\ v_1 - v'_1 &= v'_2 - v_2 \end{aligned}$$

Clearly  $LHS \in V_1$  and  $RHS \in V_2$  due to closure under vector addition. This implies  $LHS = RHS = 0_V$  since  $V_1 \cap V_2 = \{0_V\}$ , thus we have  $v_1 = v'_1$  and  $v_2 = v'_2$ , completing the uniqueness proof.  $\square$

**Problem 5.** Let  $K$  be any field, let  $V$  be a  $K$ -vector space, and let  $V_1, V_2 \subseteq V$  be  $K$ -subspaces of  $V$ . Suppose the set-theoretic union  $V_1 \cup V_2$  is also a  $K$ -subspace of  $V$ . Show that one of the subspaces  $V_1$  or  $V_2$  is contained in the other.

*Proof.*  $V_1 \cup V_2$  is also a  $K$ -subspace of  $V$ , Suppose for a contradiction neither  $V_1$  nor  $V_2$  is contained in the other. Means that  $V_1 \setminus (V_1 \cap V_2)$  and  $V_2 \setminus (V_1 \cap V_2)$  are both non-empty, namely there exists  $v_1 \in V_1, v_2 \in V_2$  where  $v_1, v_2 \notin V_1 \cap V_2$ . Clearly  $v_1, v_2 \in V_1 \cup V_2$ . By closure property of vector addition in a subspace,  $v_1 + v_2 \in V_1 \cup V_2$ . Case  $v_1 + v_2 \in V_1$ , then  $v_1 + v_2 - v_1 = v_2 \in V_1$ , contradicting fact that  $v_2 \notin V_1 \cap V_2$ . A symmetric argument shows that  $v_1 + v_2$  cannot be in  $V_2$ , thus a contradiction.  $\square$

**Problem 6.** Let  $K$  be an infinite field, let  $V$  be a vector space over  $K$ , and let  $V_1, \dots, V_n \subset V$  be a finite list of proper  $K$ -subspaces over  $V$ . Show that  $V \neq \bigcup_{j=1}^n V_j$ .

*Proof.* Let  $n \in \mathbb{N}$ ,  $V_1, \dots, V_n \subset V$  be a finite list of proper  $K$ -subspaces over  $V$ . Suppose for a contradiction that  $V = \bigcup_{i=1}^n V_i$ . Trivially,  $n$  cannot be 0 or 1, result of Q5 implies  $n \neq 2$ , so  $n \geq 3$ .

Using an algorithm, we can remove subspaces in the list as such,

1. For each  $x \in \{1, 2, \dots, n\}$ ,
2. If  $\bigcup_{i \neq x} V_i = V$ , remove  $V_x$  from the list.

Since list is finite, algorithm halts. Therefore, without loss of generality, we can assume that for any  $x \in \{1, \dots, n\}$ ,

$$V = \bigcup_{i=1}^n V_i \neq \bigcup_{i \neq x} V_i, \text{ and}$$

$$V_x \setminus \bigcup_{i \neq x} V_i \neq \emptyset.$$

Now take vectors

$$u \in V_1 \setminus \bigcup_{i \neq 1} V_i \text{ and } w \in V_2 \setminus \bigcup_{i \neq 2} V_i$$

For any  $a \in K \setminus \{0_K\}$ , define

$$v_a := u + aw.$$

It is clear that  $v_a \in V = \bigcup_{i=1}^n V_i$ . Suppose  $v_a \in V_1$ , then  $a^{-1}(v_a - u) = w \in V_1$ , contradicting  $w \notin V_i$  for any  $i \neq 2$ . So  $v_a \notin V_1$ . Suppose  $v_a \in V_2$ , then  $v_a - aw = u \in V_2$ , contradicting  $u \notin V_i$  for any  $i \neq 1$ . So  $v_a \in \bigcup_{i=3}^n V_i$ .

$|K \setminus \{0_K\}| = \infty$  while  $|\{V_3, \dots, V_n\}| = n - 2$ . Since  $|K \setminus \{0_K\}| > |\{V_3, \dots, V_n\}|$ , there does not exist an injective map  $K \setminus \{0_K\} \rightarrow \{V_3, \dots, V_n\}$ . Now consider the following maps,

$$f : K \setminus \{0_K\} \rightarrow \bigcup_{i=3}^n V_i$$

$$a \mapsto v_a = u + aw;$$

$$g : \bigcup_{i=3}^n V_i \rightarrow \{V_3, \dots, V_n\}$$

$$v \mapsto V_j$$

where  $V_j$  is the lowest-indexed subspace fulfilling  $v \in V_j$ .

We have  $g \circ f : K \setminus \{0_K\} \rightarrow \{V_3, \dots, V_n\}$ , which as shown, cannot be injective. This means that  $\exists a, b \in K \setminus \{0_K\}, V_j \in \{V_3, \dots, V_n\} . a \neq b \wedge (g \circ f)(a) = (g \circ f)(b)$ . So we can conclude that

$$\exists V_j \in \{V_3, \dots, V_n\}, a, b \in K \setminus \{0_K\} . a \neq b \text{ and } v_a, v_b \in V_j.$$

Then  $v_a - v_b \in V_j$  due to closure property of subspace,

$$\begin{aligned} v_a - v_b &= (u + aw) - (u + bw) \\ &= (a - b)w \end{aligned}$$

Since  $a \neq b, a - b \neq 0_K$ , by closure property this implies  $(a - b)^{-1} \cdot (a - b)w \in V_j \implies w \in V_j$ , contradicting fact that  $\forall i \neq 2. w \notin V_i$ .

Therefore  $V$  cannot be a union of a finite list of proper  $K$ -subspaces.  $\square$