MA2101S Homework 1

Qi Ji
A0167793L

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Problem 1. Let $\alpha \in \mathbb{Q}$ be a rational number such that the polynomial $T^2 - \alpha = 0$ has no solutions in $\mathbb{Q}$.
Show that $\mathbb{Q}(\sqrt{\alpha}) := \{ a + b\sqrt{\alpha} \in \mathbb{C} : a, b \in \mathbb{Q} \}$, where $\sqrt{\alpha} \in \mathbb{C}$ is a square root of α, is a field under the usual arithmetic operations in $\mathbb{C}$ (is a subfield of $\mathbb{C}$).

Proof. To prove $\mathbb{Q}(\sqrt{\alpha})$ is a subfield of $\mathbb{C}$, it suffices to just check for closure under addition, multiplication, negation, reciprociation, and the existence of 0 and 1.

Presence of 0 and 1. $0, 1 \in \mathbb{Q}$, and since $0 = 0 + 0\sqrt{\alpha}$ and $1 = 1 + 0\sqrt{\alpha}$, $0, 1 \in \mathbb{Q}(\sqrt{\alpha})$.

Closure under $+$. For any pair $p, q \in \mathbb{Q}(\sqrt{\alpha})$, $\exists x, y, w, z \in \mathbb{Q}$ such that

$$\begin{aligned} p &= x + y\sqrt{\alpha} \\ q &= w + z\sqrt{\alpha} \end{aligned}$$

Then by associativity of $+$ and distributivity of $\cdot$ over $+$ in $\mathbb{C}$,

$$\begin{aligned} p + q &= x + y\sqrt{\alpha} + w + z\sqrt{\alpha} \\ &= (x + w) + (y + z)\sqrt{\alpha} \end{aligned}$$

Because $\mathbb{Q}$ is a field, $x + w$ and $y + z$ are in $\mathbb{Q}$, thus $p + q \in \mathbb{Q}(\sqrt{\alpha})$.

Closure under $\cdot$. For any pair $p, q \in \mathbb{Q}(\sqrt{\alpha})$, $\exists x, y, w, z \in \mathbb{Q}$ such that

$$\begin{aligned} p &= x + y\sqrt{\alpha} \\ q &= w + z\sqrt{\alpha} \end{aligned}$$

Then by associativity of $\cdot$ and distributivity,

$$\begin{aligned} p \cdot q &= (x + y\sqrt{\alpha}) \cdot (w + z\sqrt{\alpha}) \\ &= x(w + z\sqrt{\alpha}) + y\sqrt{\alpha}(w + z\sqrt{\alpha}) \\ &= xw + xz\sqrt{\alpha} + yw\sqrt{\alpha} + yz\sqrt{\alpha}\sqrt{\alpha} \\ &= (xw + yz\alpha) + (xz + yw)\sqrt{\alpha} \end{aligned}$$

Again because $\mathbb{Q}$ is a field, $xw + yz\alpha$ and $xz + yw$ are in $\mathbb{Q}$, thus $p \cdot q \in \mathbb{Q}(\sqrt{\alpha})$.

Closure under $-$. For any $p \in \mathbb{Q}(\sqrt{\alpha})$, $\exists x, y \in \mathbb{Q}$ such that $p = x + y\sqrt{\alpha}$. Then

$$-p = -(x + y\sqrt{\alpha}) = -x + (-y)\sqrt{\alpha}.$$

$-p \in \mathbb{Q}(\sqrt{\alpha})$ because $-x, -y$ in $\mathbb{Q}$ due to $\mathbb{Q}$ being a field.

Closure under $(-)^{-1}$. For any $p \in \mathbb{Q}(\sqrt{\alpha}) \setminus \{0\}$, $\exists x, y \in \mathbb{Q}$ such that $p = x + y\sqrt{\alpha}$. Then compute p^{-1} in $\mathbb{C}$ as follows,

$$\begin{aligned} p^{-1} &= \frac{1}{p} = \frac{1}{x + y\sqrt{\alpha}} \\ &= \frac{x - y\sqrt{\alpha}}{x^2 - y^2\alpha} \end{aligned}$$

Claim. $x^2 - y^2\alpha \neq 0$.
Case $y = 0$, then because $p \neq 0, x \neq 0$, so $x^2 - y^2\alpha \neq 0$. Case $y \neq 0$, then $\exists y^{-1} \in \mathbb{Q}$. Suppose for a contradiction $x^2 - y^2\alpha = 0$, then we have

$$\begin{aligned} (y^{-1})^2(x^2 - y^2\alpha) &= 0 \\ \left(\frac{x}{y}\right)^2 - \alpha &= 0 \end{aligned}$$

but since $\mathbb{Q}$ is a field and $y \neq 0$ by assumption, $(\frac{x}{y})^2 \in \mathbb{Q}$, contradicting with fact that $T^2 - \alpha = 0$ has no solution in $\mathbb{Q}$. Hence $x^2 - y^2\alpha \neq 0$ and its reciprocal exists in $\mathbb{Q}$.
Therefore $p^{-1} = \dfrac{x}{x^2 - y^2\alpha} - \dfrac{y}{x^2 - y^2\alpha}\sqrt{\alpha}$, and because $x, y, \alpha \in \mathbb{Q}$, $p^{-1} \in \mathbb{Q}(\sqrt{\alpha})$. □

Problem 2. Define $V := \mathbb{C}^{\mathbb{C}}$, consider the following subsets of V. Which are $\mathbb{R}$-vector spaces? Which are $\mathbb{C}$-vector spaces? Justify.

Notation. Let $\mathit{0}_V : \mathbb{C} \to \mathbb{C}, z \mapsto 0$ denote the zero vector which is the constant function of $0_{\mathbb{C}}$. For each part, let the subset be called W.

Due to the sets below being subsets of V under the same operations, it suffices to check if W in each part is a subspace having $\mathit{0}_V$, closure under addition and scalar multiplication.

(i) all $f \in V$ such that $f(0) = 1$;
Because $\mathit{0}_V(0) = 0 \neq 1$, W is not a vector space due to absense of $\mathit{0}_V$. ■

(ii) all $f \in V$ such that $f(0) = f(1)$;
$\mathit{0}_V(0) = \mathit{0}_V(1) = 0$, so the zero vector is in W.
Take any pair $f, g \in W$, then

$$\begin{aligned} (f + g)(0) &= f(0) + g(0) \\ &= f(1) + g(1) \\ &= (f + g)(1) \end{aligned}$$

closure under vector addition holds.

Take any $f \in W$, then for any $k \in \mathbb{C}$ and $k \in \mathbb{R}$,

$$\begin{aligned}(kf)(0) &= k \cdot f(0) \\ &= k \cdot f(1) \\ &= (kf)(1)\end{aligned}$$

W is both a $\mathbb{R}$ and $\mathbb{C}$-vector space. ■

(iii) all $f \in V$ such that for every $z \in \mathbb{C}$, one has $\overline{f(z)} = f(z)$;
$\forall z \in \mathbb{C}$. $\overline{\mathit{0}_V(z)} = \overline{0} = 0$, so $\mathit{0}_V$ is in W.

For any $z \in \mathbb{C}$, $\overline{z} = z \iff z \in \mathbb{R}$.
Take any pair $f, g \in W$, then for any $z \in \mathbb{C}, (f+g)(z) = f(z) + g(z)$, since $f(z), g(z) \in \mathbb{R}$ and $\mathbb{R}$ is a subfield of $\mathbb{C}$, $(f+g)(z) \in \mathbb{R}$ and thus closure under vector addition holds.

Take any non-zero f from W, take any $k \in \mathbb{C}$ where $k = a + bi, a, b \in \mathbb{R}, b \neq 0$, then for any $z \in \mathbb{C}$ where $f(z) \neq 0$,

$$\begin{aligned}(kf)(z) &= k \cdot f(z) \\ &= (a+bi) \cdot f(z) \\ &= a \cdot f(z) + (b \cdot f(z))i\end{aligned}$$

Since $\exists k \in \mathbb{C}, f \in W$ where $\mathrm{Im}((kf)(z)) \neq 0 \iff (kf)(z) \notin \mathbb{R}$, closure under scalar multiplication is broken and this set is not a $\mathbb{C}$-vector space.

However, for any $f \in V$ where for every $z \in \mathbb{C}$, $f(z) \in \mathbb{R}$. For any $k \in \mathbb{R}, z \in \mathbb{C}$, because $\mathbb{R}$ is a subfield, $(kf)(z) = k \cdot f(z) \in \mathbb{R}$. Hence W is a $\mathbb{R}$-vector space. ■

(iv) all $f \in V$ such that for every $z \in \mathbb{C}$, one has $f(\overline{z}) = f(z)$;
$\mathit{0}_V$ is a constant function and ignores its parameter, satisfying the condition, thus $\mathit{0}_V \in W$.

For any pair $f, g \in W$, then for any $z \in \mathbb{C}$,

$$\begin{aligned}(f+g)(\overline{z}) &= f(\overline{z}) + g(\overline{z}) \\ &= f(z) + g(z) \\ &= (f+g)(z)\end{aligned}$$

Thus closure under vector addition holds.

For any $f \in W, k \in \mathbb{C}$, for all $z \in \mathbb{C}$,

$$\begin{aligned}(kf)(\overline{z}) &= k \cdot f(\overline{z}) \\ &= k \cdot f(z) \\ &= (kf)(z)\end{aligned}$$

Closure under scalar multiplication holds (also holds for $k \in \mathbb{R}$). W is both a $\mathbb{R}$ and $\mathbb{C}$-vector space. ■

(v) all $f \in V$ such that for every $z \in \mathbb{C}$, one has $f(z^2) = f(z)^2$;
Take $f = g = \mathrm{id}_{\mathbb{C}}$, clear that property above holds. Take $z = 2 \in \mathbb{C}$,

$$\begin{aligned}(f+g)(2^2) &= f(4) + g(4) \\ &= 8 \\ (f+g)(2)^2 &= (f(2) + g(2))^2 \\ &= 4^2 = 16\end{aligned}$$

We can see that $(f+g)(2^2) \neq (f+g)(2)^2$, thus $f+g \notin W$, closure under vector addition is broken and W is not a $\mathbb{R}$ or $\mathbb{C}$-vector space. ■

Definition. Let K be a field, V a K-vector space. For any K-subspaces $U, W \subseteq V$, define

$$U + W := \{ v \in V : \exists u \in U, w \in W.\ v = u + w \}.$$

Problem 3. Let $V := \mathbb{R}^{\mathbb{R}}$. Consider V_{even} (resp. V_{odd}) as subsets of all even (resp. odd) functions.

Notation. Let $0_V : \mathbb{R} \to \mathbb{R}, x \mapsto 0$ denote the zero vector which is the constant function of $0_{\mathbb{R}}$.

(a) Show that V_{even} and V_{odd} are $\mathbb{R}$-subspaces of V.

Proof. Firstly, $\forall x \in \mathbb{R}.\ 0_V(x) = 0$, trivially $0_V \in V_{\text{odd}}$ and $0_V \in V_{\text{even}}$.
Consider $f, g \in V_{\text{even}}$, then $\forall x \in \mathbb{R}$,

$$\begin{aligned}(f+g)(-x) &= f(-x) + g(-x)\\ &= f(x) + g(x)\\ &= (f+g)(x)\end{aligned}$$

Thus $f + g \in V_{\text{even}}$. Consider any $f \in V_{\text{even}}, k \in \mathbb{R}$, then $\forall x \in \mathbb{R}$,

$$\begin{aligned}(kf)(-x) &= k \cdot f(-x)\\ &= k \cdot f(x)\\ &= (kf)(x)\end{aligned}$$

Thus $kf \in V_{\text{even}}$. Therefore V_{even} is a subspace of V.
Now consider $f, g \in V_{\text{odd}}$, then $\forall x \in \mathbb{R}$,

$$\begin{aligned}(f+g)(-x) &= f(-x) + g(-x)\\ &= -f(x) - g(x)\\ &= -(f(x) + g(x))\\ &= -(f+g)(x)\end{aligned}$$

Thus $f + g \in V_{\text{odd}}$. Consider any $f \in V_{\text{odd}}, k \in \mathbb{R}$, then $\forall x \in \mathbb{R}$,

$$\begin{aligned}(kf)(-x) &= k \cdot f(-x)\\ &= k \cdot (-f(x))\\ &= -k \cdot f(x)\\ &= -(k \cdot f(x)) = -(kf)(x)\end{aligned}$$

Thus $kf \in V_{\text{odd}}$. Therefore V_{odd} is a subspace of V. □

(b) Show that $V_{\text{even}} \cap V_{\text{odd}} = \{ 0_V \}$ and $V_{\text{even}} + V_{\text{odd}} = V$.

Proof. Take $f \in V_{\text{even}} \cap V_{\text{odd}}$, f is both even and odd, so $\forall x \in \mathbb{R}$,

$$f(-x) = f(x) \text{ and } f(-x) = -f(x).$$

So $f(x) = -f(x)$ for all $x \in \mathbb{R}$, which is the case if and only if for all $x \in \mathbb{R}$, $f(x) = 0$. This means f is the constant function of 0, that is 0_V, this implies $V_{\text{even}} \cap V_{\text{odd}} = \{ 0_V \}$.

Consider any $h \in V_{\text{even}} + V_{\text{odd}}$, by definition, $\exists f \in V_{\text{even}}, g \in V_{\text{odd}}$. $h = f + g$. Since V_{even} and V_{odd} are subspaces of V, $h = f + g \in V$, so $V_{\text{even}} + V_{\text{odd}} \subseteq V$.

Conversely take $f \in V$. For all $x \in \mathbb{R}$,

$$\begin{aligned} f(x) &= f(x) + 0_V(x) \\ &= \frac{1}{2}f(x) + \frac{1}{2}f(x) + \frac{1}{2}f(-x) - \frac{1}{2}f(-x) \\ &= \frac{1}{2}(f(x) + f(-x)) + \frac{1}{2}(f(x) - f(-x)) \end{aligned}$$

Define $g, h : \mathbb{R} \to \mathbb{R}$ as

$$\begin{aligned} g &: x \mapsto \frac{1}{2}(f(x) + f(-x)) \\ h &: x \mapsto \frac{1}{2}(f(x) - f(-x)) \end{aligned}$$

Verify that $g \in V_{\text{even}}$ and $h \in V_{\text{odd}}$.

$$\begin{aligned} g(-x) &= \frac{1}{2}(f(-x) + f(-(-x))) \\ &= \frac{1}{2}(f(x) + f(-x)) \\ h(-x) &= \frac{1}{2}(f(-x) - f(-(-x))) \\ &= -\frac{1}{2}(f(x) - f(-x)) \end{aligned}$$

Because $f = g + h$, $f \in V_{\text{even}} + V_{\text{odd}}$. So $V \subseteq V_{\text{even}} + V_{\text{odd}}$ and this completes the proof that $V_{\text{even}} + V_{\text{odd}} = V$. □

Problem 4. Let K be any field, let V be a K-vector space, and let $V_1, V_2 \subseteq V$ be K-subspaces of V. Suppose $V_1 \cap V_2 = \{ 0_V \}$ and $V_1 + V_2 = V$. Show that for any $v \in V$, there exist unique vectors $v_1 \in V_1$ and $v_2 \in V_2$ such that $v = v_1 + v_2$ in V.

Proof. Take any arbitary $v \in V$, since $V_1 + V_2 = V$, by definition of $V_1 + V_2$, there exists $v_1 \in V_1, v_2 \in V_2$ such that $v = v_1 + v_2$.
Suppose $\exists v_1' \in V_1, v_2' \in V_2$ where $v = v_1' + v_2'$.

$$\begin{aligned} v = v_1 + v_2 &= v_1' + v_2' \\ v_1 - v_1' &= v_2' - v_2 \end{aligned}$$

Clearly $LHS \in V_1$ and $RHS \in V_2$ due to closure under vector addition. This implies $LHS = RHS = 0_V$ since $V_1 \cap V_2 = \{ 0_V \}$, thus we have $v_1 = v_1'$ and $v_2 = v_2'$, completing the uniqueness proof. □

Problem 5. Let K be any field, let V be a K-vector space, and let $V_1, V_2 \subseteq V$ be K-subspaces of V. Suppose the set-theoretic union $V_1 \cup V_2$ is also a K-subspace of V. Show that one of the subspaces V_1 or V_2 is contained in the other.

Proof. $V_1 \cup V_2$ is also a K-subspace of V, Suppose for a contradiction neither V_1 nor V_2 is contained in the other. Means that $V_1 \setminus (V_1 \cap V_2)$ and $V_2 \setminus (V_1 \cap V_2)$ are both non-empty, namely there exists $v_1 \in V_1, v_2 \in V_2$ where $v_1, v_2 \notin V_1 \cap V_2$. Clearly $v_1, v_2 \in V_1 \cup V_2$. By closure property of vector addition in a subspace, $v_1 + v_2 \in V_1 \cup V_2$. Case $v_1 + v_2 \in V_1$, then $v_1 + v_2 - v_1 = v_2 \in V_1$, contradicting fact that $v_2 \notin V_1 \cap V_2$. A symmetric argument shows that $v_1 + v_2$ cannot be in V_2, thus a contradiction. □

Problem 6. Let K be an infinite field, let V be a vector space over K, and let $V_1, \dots V_n \subset V$ be a finite list of proper K-subspaces over V. Show that $V \neq \bigcup_{j=1}^{n} V_j$.

Proof. Let $n \in \mathbb{N}$, $V_1, \dots, V_n \subset V$ be a finite list of proper K-subspaces over V. Suppose for a contradiction that $V = \bigcup_{i=1}^{n} V_i$. Trivially, n cannot be 0 or 1, result of Q5 implies $n \neq 2$, so $n \geq 3$. Using an algorithm, we can remove subspaces in the list as such,

1. For each $x \in \{ 1, 2, \dots, n \}$,
2. If $\bigcup_{i \neq x} V_i = V$, remove V_x from the list.

Since list is finite, algorithm halts. Therefore, without loss of generality, we can assume that for any $x \in \{ 1, \dots, n \}$,

$$V = \bigcup_{i=1}^{n} V_i \neq \bigcup_{i \neq x} V_i, \text{ and}$$

$$V_x \setminus \bigcup_{i \neq x} V_i \neq \emptyset.$$

Now take vectors

$$u \in V_1 \setminus \bigcup_{i \neq 1} V_i \text{ and } w \in V_2 \setminus \bigcup_{i \neq 2} V_i$$

For any $a \in K \setminus \{ 0_K \}$, define

$$v_a := u + aw.$$

It is clear that $v_a \in V = \bigcup_{i=1}^{n} V_i$. Suppose $v_a \in V_1$, then $a^{-1}(v_a - u) = w \in V_1$, contradicting $w \notin V_i$ for any $i \neq 2$. So $v_a \notin V_1$. Suppose $v_a \in V_2$, then $v_a - aw = u \in V_2$, contradicting $u \notin V_i$ for any $i \neq 1$. So $v_a \in \bigcup_{i=3}^{n} V_i$.

$|K \setminus \{ 0_K \}| = \infty$ while $|\{ V_3, \dots, V_n \}| = n - 2$. Since $|K \setminus \{ 0_K \}| > |\{ V_3, \dots, V_n \}|$, there does not exist an injective map $K \setminus \{ 0_K \} \to \{ V_3, \dots, V_n \}$. Now consider the following maps,

$$\begin{aligned} f : K \setminus \{ 0_K \} &\to \bigcup_{i=3}^{n} V_i \\ a &\mapsto v_a = u + aw; \\ g : \bigcup_{i=3}^{n} V_i &\to \{ V_3, \dots, V_n \} \\ v &\mapsto V_j \end{aligned}$$

where V_j is the lowest-indexed subspace fulfilling $v \in V_j$.
We have $g \circ f : K \setminus \{ 0_K \} \to \{ V_3, \ldots, V_n \}$, which as shown, cannot be injective. This means that $\exists a, b \in K \setminus \{ 0_K \}, V_j \in \{ V_3, \ldots, V_n \} .\ a \neq b \wedge (g \circ f)(a) = (g \circ f)(b)$. So we can conclude that

$$\exists V_j \in \{ V_3, \ldots, V_n \},\ a, b \in K \setminus \{ 0_K \} .\ a \neq b \text{ and } v_a, v_b \in V_j.$$

Then $v_a - v_b \in V_j$ due to closure property of subspace,

$$\begin{aligned} v_a - v_b &= (u + aw) - (u + bw) \\ &= (a - b)w \end{aligned}$$

Since $a \neq b$, $a - b \neq 0_K$, by closure property this implies $(a - b)^{-1} \cdot (a - b)w \in V_j \implies w \in V_j$, contradicting fact that $\forall i \neq 2.\ w \notin V_i$.
Therefore V cannot be a union of a finite list of proper K-subspaces. $\square$