Matrices

$\mathbf{A} = (a_{ij})_{m\times p}$ and $\mathbf{B} = (b_{ij})_{p\times n}$, then
$\mathbf{AB} = (ab_{ij})_{m\times n} = \sum_{k=1}^{p} a_{ik}b_{kj}$.
Matrix Multiplication is associative and distributive (left and right) over addition.
For square matrix $\mathbf{A}$, $\mathbf{A}^m\mathbf{A}^n = \mathbf{A}^{(m+n)}$
(if $\mathbf{A}$ invertible, also works for negative m, n)

$$\mathbf{A}^{\intercal\intercal} = \mathbf{A}$$
$$(\mathbf{A}+\mathbf{B})^\intercal = \mathbf{A}^\intercal + \mathbf{B}^\intercal$$
$$(\mathbf{AB})^\intercal = \mathbf{B}^\intercal\mathbf{A}^\intercal$$

for invertible $\mathbf{A}, \mathbf{B}$:

$$\mathbf{AA}^{-1} = \mathbf{I} = \mathbf{A}^{-1}\mathbf{A}$$
$$(c\mathbf{A})^{-1} = \frac{1}{c}\mathbf{A}^{-1}$$
$$(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}$$

Elementary Row Operations

$$\mathbf{E}_k \dots \mathbf{E}_2\mathbf{E}_1\mathbf{A} = \mathbf{B}$$
$$\mathbf{A} = (\mathbf{E}_k \dots \mathbf{E}_2\mathbf{E}_1)^{-1}\mathbf{B}$$
$$\mathbf{A} = \mathbf{E}_1^{-1}\mathbf{E}_2^{-1} \dots \mathbf{E}_k^{-1}\mathbf{B}$$

Using row-reduction to find inverse:

$$(\mathbf{A} \mid \mathbf{I}) \xrightarrow{\text{Gauss-Jordan}} (\mathbf{I} \mid \mathbf{A}^{-1})$$

$\mathbf{A}^{-1} = \mathbf{E}_k \dots \mathbf{E}_2\mathbf{E}_1 \Rightarrow \mathbf{A} = \mathbf{E}_1^{-1}\mathbf{E}_2^{-1} \dots \mathbf{E}_k^{-1}$

Determinants

$A_{ij} = (-1)^{i+j} \det(\langle\text{cover row } i \text{ col } j \text{ in } \mathbf{A}\rangle)$

$$\det(\mathbf{E}_{add}) = 1$$
$$\det(\mathbf{E}_{swap}) = -1$$
$$\det(\mathbf{E}_{mult}) = c$$
$$\det(\mathbf{A}_\Delta) = a_{11}a_{22}\cdots a_{nn}$$
$$\det(\mathbf{AB}) = \det(\mathbf{A})\det(\mathbf{B})$$
$$\det(c\mathbf{A}) = c^n\mathbf{A}$$

$\mathbf{A}\ \mathrm{adj}(\mathbf{A}) = \det(\mathbf{A})\ \mathbf{I} \Rightarrow \mathbf{A}^{-1} = \frac{1}{\det(\mathbf{A})}\mathrm{adj}(\mathbf{A})$

$$\mathrm{adj}(\mathbf{A}) = \begin{bmatrix} A_{11} & A_{21} & \dots & A_{n1} \\ A_{12} & A_{22} & \dots & A_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ A_{1n} & A_{2n} & \dots & A_{nn} \end{bmatrix}$$

Cramer's rule: for $\mathbf{A}\boldsymbol{x} = \boldsymbol{b}$,

$$x_n = \frac{\det(\langle\mathbf{A} : \text{replace } n\text{-th col with } \boldsymbol{b}\rangle)}{\det(\mathbf{A})}$$

Spaces

Space notation:

- $\{(a, a-b, 2b+c) \mid a, b, c \in \mathbb{R}\}$ (explicit)
- $\{(x, y, z) \mid x+y+z = 0\}$ (implicit)

Spans & Containment

Take $U := \{\boldsymbol{u}_1, \boldsymbol{u}_2, \dots, \boldsymbol{u}_k\}$
$\mathrm{span}(U)$ = set of all linear combinations of U
$\begin{bmatrix}\boldsymbol{u}_1 & \boldsymbol{u}_2 & \dots & \boldsymbol{u}_k & \boldsymbol{v}\end{bmatrix}$ consistent $\Rightarrow v \in \mathrm{span}(U)$.
ref $\begin{bmatrix}\boldsymbol{u}_1 & \dots & \boldsymbol{u}_k\end{bmatrix}$ no zero-row $\Rightarrow \mathrm{span}(U) = \mathbb{R}^k$.
Take $V := \{\boldsymbol{v}_1, \boldsymbol{v}_2, \dots, \boldsymbol{v}_k\}$
each $\boldsymbol{u}_i \in \mathrm{span}(V) \iff \mathrm{span}(U) \subseteq \mathrm{span}(V)$

Subspaces, Linear Independence

Definition of subspace V: $\mathbf{0} \in V$, and
$\forall \boldsymbol{u}, \boldsymbol{v} \in V.\ \forall c, d \in \mathbb{R}.\ c\boldsymbol{u} + d\boldsymbol{v} \in V$
U is LI means only trivial solution for
$c_1\boldsymbol{u}_1 + c_2\boldsymbol{u}_2 + \dots + c_k\boldsymbol{u}_k = \mathbf{0}$.

Basis and Coordinate systems

A set of vectors S is a basis for vector space V iff

- S is linearly independent
- S spans V

Given basis S, and $\boldsymbol{v} \in V$

$$\boldsymbol{v} = c_1\boldsymbol{s}_1 + c_2\boldsymbol{s}_2 + \dots + c_k\boldsymbol{s}_k$$
$$(\boldsymbol{v})_S = (c_1, c_2, \dots, c_k) \in \mathbb{R}^k$$

Coordinate Systems

$\forall \boldsymbol{u}, \boldsymbol{v} \in V.\ \boldsymbol{u} = \boldsymbol{v} \iff (\boldsymbol{u})_S = (\boldsymbol{v})_S$ (uniq.)
$\forall \boldsymbol{u}, \boldsymbol{v} \in V. c, d \in \mathbb{R}.\ (c\boldsymbol{u} + d\boldsymbol{v})_S = c(\boldsymbol{u})_S + d(\boldsymbol{v})_S$
Let $k = |S|$.
$\boldsymbol{v}_1, \boldsymbol{v}_2, \dots \in V$, are linear independent $\iff$ $(\boldsymbol{v}_1)_S, (\boldsymbol{v}_2)_S, \dots \in \mathbb{R}^k$ are linear independent.
$\mathrm{span}\{\boldsymbol{v}_1, \boldsymbol{v}_2, \dots\} = V \iff$
$\mathrm{span}\{(\boldsymbol{v}_1)_S, (\boldsymbol{v}_2)_S, \dots\} = \mathbb{R}^k$.

Dimensions

$\dim(V) := |S|$ where S is a basis for V, is unique.

- S is linearly independent
- $\mathrm{span}(S) = V$
- $|S| = \dim(V)$

2 of above true $\Rightarrow$ all true.

Transition Matrices

S and T are bases for V, for $v \in V$.

$$\mathbf{P}_{S,T} = \begin{bmatrix}[\boldsymbol{s}_1]_T & [\boldsymbol{s}_2]_T & \dots & [\boldsymbol{s}_k]_T\end{bmatrix}$$

is transition matrix from basis S to T. ie

$$[\boldsymbol{v}]_T = P_{S,T}[\boldsymbol{v}]_S$$
$$[\boldsymbol{v}]_S = P_{S,T}^{-1}[\boldsymbol{v}]_T$$

Rowsp and Range(Colsp)

Take $\mathbf{R} := \mathrm{ref}(\mathbf{A})$ is $m \times n$.

- Row operations preserve rowsp.
- Rows in $\mathbf{R}$ form basis for rowsp.
- Pivot columns in $\mathbf{R}$ correspond to linearly independent columns in $\mathbf{A}$
  Note: row operations preserve linear (in)-dependence of columns but could destroy other information like colsp.

Column space of $\mathbf{A} = \{\mathbf{A}\boldsymbol{u} \mid \boldsymbol{u} \in \mathbb{R}^n\}$.
So $\mathbf{A}\boldsymbol{x} = \boldsymbol{b}$ consistent $\iff \boldsymbol{b} \in \mathrm{colsp}(\mathbf{A})$.

Rank Nullity

$$\begin{aligned}
\operatorname{rank}(\mathbf{0}) &= 0 \\
\operatorname{rank}(\mathbf{I}_n) &= n \\
\operatorname{rank}(\mathbf{A}) &\leqslant \min\{m, n\} \qquad (\ddagger) \\
\operatorname{rank}(\mathbf{AB}) &\leqslant \min\{\operatorname{rank}(\mathbf{A}), \operatorname{rank}(\mathbf{B})\}
\end{aligned}$$

If equality holds in ($\ddagger$), $\mathbf{A}$ has **full rank**.

$$\begin{aligned}
\operatorname{rank}(\mathbf{A}) &= \operatorname{rank}(\mathbf{A}^\intercal) \\
\operatorname{rank}(\mathbf{A}) + \operatorname{nullity}(\mathbf{A}) &= \text{no. columns}
\end{aligned}$$

Kernel/Nullspace

$$\operatorname{null}(\mathbf{A}) := \{\boldsymbol{u} \in \mathbb{R}^n \mid \mathbf{A}\boldsymbol{u} = \mathbf{0}\}$$

Suppose $\mathbf{A}\boldsymbol{v} = \boldsymbol{b}$, then general solution of $\mathbf{A}\boldsymbol{x} = \boldsymbol{b}$

$$\boldsymbol{x} \in \{\boldsymbol{u} + \boldsymbol{v} \mid \boldsymbol{u} \in \operatorname{null}(\mathbf{A})\}$$

Vectors

Inner product: $\boldsymbol{u} \cdot \boldsymbol{v} := \mathbf{u}^\intercal \mathbf{v}$

$$\begin{aligned}
\|\boldsymbol{u}\| &= \sqrt{\boldsymbol{u} \cdot \boldsymbol{u}} \\
\|c\boldsymbol{u}\| &= |c|\,\|\boldsymbol{u}\| \\
d(\boldsymbol{u}, \boldsymbol{v}) &= \|\boldsymbol{u} - \boldsymbol{v}\| \\
\cos\angle(\boldsymbol{u}, \boldsymbol{v}) &= \frac{\boldsymbol{u} \cdot \boldsymbol{v}}{\|\boldsymbol{u}\|\,\|\boldsymbol{v}\|} \\
\|\boldsymbol{u} + \boldsymbol{v}\| &\leqslant \|\boldsymbol{u}\| + \|\boldsymbol{v}\| \qquad (\triangle \text{ ineq.})
\end{aligned}$$

Orthogonal

$\boldsymbol{u} \cdot \boldsymbol{v} = 0 \iff \boldsymbol{u} \perp \boldsymbol{v}$
orthogonal $\Rightarrow$ linear independence.
orthonormal := orthogonal $\wedge$ norm 1.
To check if S is orthogonal basis for V:

(i) S is orthogonal
(ii) $|S| = \dim(V)$ or $\operatorname{span}(S) = V$ (ref: dim)

Projections

Let S be an orthogonal basis for V, then $\forall \boldsymbol{w} \in \mathbb{R}^n$,

$$\boldsymbol{p} = \frac{\boldsymbol{w} \cdot \boldsymbol{s}_1}{\boldsymbol{s}_1 \cdot \boldsymbol{s}_1}\boldsymbol{s}_1 + \frac{\boldsymbol{w} \cdot \boldsymbol{s}_2}{\boldsymbol{s}_2 \cdot \boldsymbol{s}_2}\boldsymbol{s}_2 + \cdots + \frac{\boldsymbol{w} \cdot \boldsymbol{s}_k}{\boldsymbol{s}_k \cdot \boldsymbol{s}_k}\boldsymbol{s}_k$$

is projection of $\boldsymbol{w}$ on V. (existence of projections)

Case $\boldsymbol{w} \in V$, then $\boldsymbol{p} = \boldsymbol{w}$. For orthonormal basis, simplify expr as denominator becomes 1.

Gram-Schmidt Algorithm

Basis $U := \{\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_k\}$ for V.

$$\boldsymbol{v}_1 = \boldsymbol{u}_1$$

for $i \in \{2, 3, \ldots, k\}$

$$\boldsymbol{v}_i = \boldsymbol{u}_i - \sum_{j=1}^{i-1} \frac{\boldsymbol{u}_i \cdot \boldsymbol{v}_j}{\boldsymbol{v}_j \cdot \boldsymbol{v}_j}\,\boldsymbol{v}_j$$

$\{\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_k\}$ form orthogonal basis for V.

Orthogonal matrices

iff $\mathbf{A}^\intercal = \mathbf{A}^{-1}$. rows and cols form orthonormal basis for $\mathbb{R}^n$.
Note: Transition matrix between two orthonormal bases is orthogonal. So $\mathbf{P}_{T,S} = (\mathbf{P}_{S,T})^{-1} = (\mathbf{P}_{S,T})^\intercal$

Eigenvalues

λ is an eigenvalue of A iff $\exists \boldsymbol{v} \neq \mathbf{0}$. $\mathbf{A}\boldsymbol{v} = \lambda\boldsymbol{v}$.
Characteristic polynomial is $\det(\mathbf{A} - \lambda\mathbf{I}) = 0$
Eigenspace $E_\lambda :=$ nullspace of $(\mathbf{A} - \lambda\mathbf{I})$.
To diagonalise $n \times n$ matrix $\mathbf{A}$,

1. Find all distinct $\lambda_1, \lambda_2, \ldots, \lambda_k$,
2. For each λ_i, find basis $S_{\lambda i}$ for eigenspace $E_{\lambda i}$ (If $|S_{\lambda i}| < p_i$ where p_i is power of $(\lambda - \lambda_i)$ in polynomial, then not diagonalisable, abort.)
3. Set $S = \bigcup_{i \in \{1,\ldots,k\}} S_{\lambda i} = \{\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_n\}$ (union the eigenbases).
   $\mathbf{A}$ is diagonalisable if $|S| = n$.

$\mathbf{P} = \begin{bmatrix}\boldsymbol{u}_1 & \boldsymbol{u}_2 & \ldots & \boldsymbol{u}_n\end{bmatrix}$ st. $\mathbf{A} = \mathbf{PDP}^{-1}$.

For orthogonally diagonalisable (symmetric) matrix $\mathbf{A}$, Gram-Schmidt and normalise the bases in step 2, then $\mathbf{A} = \mathbf{PDP}^\intercal$.

Linear Transformations

Suppose $T : V \to W$ is a linear transformation and $S = \{\,\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_n\,\}$ is a basis for V.
If $T(S)$ is known, and $\mathbf{P}_{E,S}$ is the transition matrix from basis E to S,
$\mathbf{B} = \begin{bmatrix}T(\boldsymbol{u}_1) & T(\boldsymbol{u}_2) & \cdots & T(\boldsymbol{u}_n)\end{bmatrix}$.

$$\begin{tikzcd}
V \arrow[r, "\mathbf{P}_{E,S}"] \arrow[d, "T"'] & {[V]_S} \arrow[dl, "\mathbf{B}"] \\
W &
\end{tikzcd}$$

(generalisable).

Corner case matrices

Standard non-diagonalisable matrix: $\begin{bmatrix}a & 1\\ 0 & a\end{bmatrix}$.

$\mathbf{A} = \begin{bmatrix}1 & 1\\ 0 & 0\end{bmatrix}; \mathbf{B} = \begin{bmatrix}1 & 1\\ -1 & -1\end{bmatrix}$ then

$$\mathbf{AB} = \mathbf{0} \text{ but } \mathbf{BA} \neq \mathbf{0}$$

$\mathbf{A} = \begin{bmatrix}1 & 2\\ 0 & -1\end{bmatrix}; \mathbf{B} = \begin{bmatrix}1 & 3\\ 0 & -1\end{bmatrix}$ then

$$\mathbf{A}^2 = \mathbf{B}^2 = \mathbf{I} \text{ but } \mathbf{A} \neq \mathbf{B}, (\mathbf{AB})^2 \neq \mathbf{I}$$

$\mathbf{A} = \begin{bmatrix}2 & 1\\ 3 & 2\end{bmatrix}; \mathbf{B} = \begin{bmatrix}1 & -1\\ 1 & 1\end{bmatrix}$ then

$(\mathbf{AB})^k \neq \mathbf{A}^k\mathbf{B}^k, (\mathbf{AB})^\intercal \neq \mathbf{A}^\intercal\mathbf{B}^\intercal, (\mathbf{AB})^{-1} \neq \mathbf{A}^{-1}\mathbf{B}^{-1}$

Theorem. Invertible Square Matrix
For any square matrix $\mathbf{A}$:

$\quad$ $\mathbf{A}$ is invertible,
$\iff$ $\mathbf{A}\boldsymbol{x} = \mathbf{0}$ only has trivial solution,
$\iff$ $\operatorname{rref}(\mathbf{A}) = \mathbf{I}$,
$\iff$ $\mathbf{A} = \mathbf{E}_k \ldots \mathbf{E}_2\mathbf{E}_1$,
$\iff$ $\det(\mathbf{A}) \neq 0$,
$\iff$ rows/cols in $\mathbf{A}$ form basis for $\mathbb{R}^n$,
$\iff$ $\mathbf{A}$ has full rank,
$\iff$ 0 is not an eigenvalue of $\mathbf{A}$.