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## **MA2101S Homework 6**

Qi Ji (A0167793L)

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## Question 1

Let  $K$  be a field with  $\text{char}(K) \neq 2$  (i.e.  $1 + 1 \neq 0$  in  $K$ ), let  $n \in \mathbb{N}$  be an **odd** natural number, and let  $X, Y \in \mathbb{M}_n(K)$  be two  $n \times n$  square matrices over  $K$ .

- (a) Show that if  $X^t = -X$ , then  $X$  is not invertible.
- (b) Show that if  $XY = -YX$ , then  $X$  or  $Y$  is not invertible.

- (a) *Proof.* Suppose  $X^t = -X$ , using the facts that  $-X = (-1_n)X$ , determinant is multiplicative, and  $(-1)^n = -1$  as  $n$  is odd,

$$\begin{aligned}\det(X) &= \det(X^t) = \det(-X) = \det((-1_n)X) \\ \det(X) &= \det(-1_n) \det(X) \\ \det(X) &= (-1)^n \det(X) \\ \det(X) &= -\det(X) \\ \det(X) + \det(X) &= 0 \\ \det(X)(1 + 1) &= 0\end{aligned}$$

as  $\text{char}(K) \neq 2$ ,  $\det(X) = 0$ , so  $X$  is not invertible. □

- (b) *Proof.* Suppose  $XY = -YX$ , then similarly,

$$\begin{aligned}\det(XY) &= \det((-1_n)YX) \\ \det(X) \det(Y) &= -\det(Y) \det(X) \\ \det(X) \det(Y)(1 + 1) &= 0\end{aligned}$$

again as  $\text{char}(K) \neq 2$ ,  $\det(X) \det(Y) = 0$ , so  $X$  or  $Y$  is not invertible. □

## Question 2

Let  $K$  be a field, and let  $a, b, c, d, e, f \in K$  be elements of  $K$ . Consider the  $4 \times 4$  skew-symmetric matrix

$$X := \begin{pmatrix} 0 & a & b & c \\ -a & 0 & d & e \\ -b & -d & 0 & f \\ -c & -e & -f & 0 \end{pmatrix} \text{ in } \mathbb{M}_4(K).$$

Show that  $\det(X) = (af - be + cd)^2$ .

*Proof.* As  $X$  is only  $4 \times 4$ , expand  $\det(X)$ ,

$$\begin{aligned} \det(X) &= 0 - a \begin{vmatrix} -a & d & e \\ -b & 0 & f \\ -c & -f & 0 \end{vmatrix} + b \begin{vmatrix} -a & 0 & e \\ -b & -d & f \\ -c & -e & 0 \end{vmatrix} - c \begin{vmatrix} -a & 0 & d \\ -b & -d & 0 \\ -c & -e & -f \end{vmatrix} \\ &= -a(-cdf + bef - af^2) + b(be^2 - aef - cde) - c(-adf + bde - cd^2) \\ &= acdf - abef + a^2f^2 + b^2e^2 - abef - bcde + acdf - bcde + c^2d^2 \\ &= (af)^2 + (cd)^2 + (be)^2 + 2acdf - 2abef - 2bcde \end{aligned}$$

On the other hand,

$$\begin{aligned} (af - be + cd)^2 &= af(af - be + cd) - be(af - be + cd) + cd(af - be + cd) \\ &= (af)^2 - abef + acdf - abef + (be)^2 - bcde + acdf - bcde + (cd)^2 \\ &= (af)^2 + (cd)^2 + (be)^2 + 2acdf - 2abef - 2bcde \end{aligned}$$

Therefore  $\det(X) = (af - be + cd)^2$ . □

### Question 3

Let  $K$  be a field, and let  $n \in \mathbb{N}$  be any natural number with  $n > 1$ . Consider an  $n \times n$  square matrix  $A \in \mathbb{M}_n(K)$ .

- Show that  $\det(\operatorname{adj}(A)) = \det(A)^{n-1}$ .
- Show that if  $A$  is an invertible upper-triangular matrix, then the same is true for  $\operatorname{adj}(A)$ .

**Claim.**  $A \operatorname{adj}(A) = \det(A) 1_n$ .

*Proof (of Claim).* Expanding the  $(i, j)$  entries of  $A \operatorname{adj}(A)$ , we have

$$\begin{aligned} (A \operatorname{adj}(A))_{ij} &= \sum_{k=1}^n A_{ik} \operatorname{adj}(A)_{kj} \\ &= \sum_{k=1}^n (-1)^{j+k} A_{ik} \det(\tilde{A}_{jk}) \end{aligned}$$

- Case  $i = j$ , we get the co-factor expansion along the  $i$ -th row, which evaluates to  $\det(A)$ .
- Case  $i \neq j$ , consider the matrix  $B$  obtained by copying  $A$ , then replacing its  $j$ -th with the  $i$ -th row of  $A$ . Then for any  $k \in \{1, \dots, n\}$ ,  $A_{ik} = B_{ik} = B_{jk}$  and  $\tilde{A}_{jk} = \tilde{B}_{jk}$ , then

$$\begin{aligned} (A \operatorname{adj}(A))_{ij} &= \sum_{k=1}^n (-1)^{j+k} B_{jk} \det(\tilde{B}_{jk}) \\ &= \det(B) \end{aligned}$$

as  $B$  by construction has two equal rows, it has determinant 0.

Therefore

$$(A \operatorname{adj}(A))_{ij} = \begin{cases} \det(A) & \text{if } i = j \\ 0 & \text{otherwise} \end{cases}$$

$$A \operatorname{adj}(A) = \det(A) 1_n. \quad \square$$

- Proof.* Consider the equality proven, taking determinants,

$$\begin{aligned} A \operatorname{adj}(A) &= \det(A) 1_n \\ \det(A) \det(\operatorname{adj}(A)) &= \det(A)^n \end{aligned}$$

If  $A$  is invertible ( $\det(A) \neq 0$ ), we obtain the conclusion.

As  $n > 1$ ,  $0^{n-1} = 0$ . It remains to show that when  $\det(A) = 0$ ,  $\det(\operatorname{adj}(A)) = 0$ . Suppose  $A$  is

singular, from claim,

$$A \operatorname{adj}(A) = 0.$$

Reading this equality in terms of left-multiplication means that  $\operatorname{Im}(\operatorname{adj}(A)) \subseteq \ker(A)$ , which means  $\operatorname{rank}(\operatorname{adj}(A)) \leq \operatorname{nullity}(A)$ .

- Suppose  $\operatorname{nullity}(A) = n$ , then  $A$  is the zero matrix which trivially implies that  $\operatorname{adj}(A)$  is also the zero matrix, in this case  $\operatorname{adj}(A)$  will be singular.
- Now suppose  $\operatorname{nullity}(A) < n$ , then  $\operatorname{rank}(\operatorname{adj}(A)) \leq \operatorname{nullity}(A) < n$ , which implies  $\operatorname{adj}(A)$  is not full rank, and thus singular too.

Therefore, the equation  $\det(\operatorname{adj}(A)) = \det(A)^{n-1}$  holds too when  $A$  is singular.  $\square$

(b) *Proof.* Suppose  $A$  is an invertible upper-triangular matrix, then by claim,  $\operatorname{adj}(A)$  is invertible too and has inverse  $\frac{1}{\det(A)}A$ . Since  $A$  is upper-triangular, whenever  $i > j$ ,  $A_{ij} = 0$ . The  $(i, j)$ -entries for  $\operatorname{adj}(A)$  is given by

$$\operatorname{adj}(A)_{ij} = (-1)^{i+j} \det(\tilde{A}_{ji}).$$

Then to show that  $\operatorname{adj}(A)$  is upper-triangular, it suffices to show that for any  $k, l \in \{1, \dots, n\}$ ,  $l > k \implies \det(\tilde{A}_{kl}) = 0$ .

Take any  $k, l \in \{1, \dots, n\}$  with  $k < l$ . Let  $d_1, \dots, d_n \in K$  be diagonal entries of  $A$ , then  $\tilde{A}_{kl}$  can be expressed as

$$\tilde{A}_{kl} = \begin{pmatrix} d_1 & & & & & & & & \\ & \ddots & & & & & & & \\ & & d_{k-1} & & & & & & \\ & & & 0 & d_{k+1} & & & & \star \\ & & & & \ddots & \ddots & & & \\ & & & & & \ddots & d_{l-1} & & \\ & & & & & & 0 & & \\ & 0 & & & & & & d_{l+1} & \\ & & & & & & & & \ddots \\ & & & & & & & & & d_n \end{pmatrix}.$$

Hence visually verify that whenever  $k < l$ ,  $\tilde{A}_{kl}$  is an upper-triangular matrix with at least one zero on the diagonal, then  $\det(\tilde{A}_{kl}) = 0$ . This completes the proof that  $\operatorname{adj}(A)$  is upper-triangular.  $\square$

### Question 4

Let  $K$  be a field, and let  $m, n \in \mathbb{N}_{>0}$  be positive integers, and let  $V := \mathbb{M}_{m \times n}(K)$  be the  $K$ -vector space of  $m \times n$  matrices over  $K$ . Fix a  $m \times m$  square matrix  $A \in \mathbb{M}_{m \times m}(K)$  and a  $n \times n$  square matrix  $B \in \mathbb{M}_{n \times n}(K)$ , and consider the map

$$\Phi : V \rightarrow V \quad \text{given by} \quad X \mapsto AXB.$$

**Note.** Throughout this question, let  $\mathcal{H} := (e_{11}, \dots, e_{1n}, \dots, e_{m1}, \dots, e_{mn})$  denote the standard basis for  $\mathbb{M}_{m \times n}(K)$  ordered this way. Where for any  $(r, s) \in \{1, \dots, m\} \times \{1, \dots, n\}$ ,  $e_{rs} \in \mathbb{M}_{m \times n}(K)$  is characterised by

$$(e_{rs})_{ij} = \delta_{ir}\delta_{js} = \begin{cases} 1 & \text{if } (i, j) = (r, s) \\ 0 & \text{otherwise} \end{cases}.$$

**(a) Show that  $\Phi$  is a  $K$ -linear operator on  $V$ , and compute its trace  $\text{Tr}(\Phi)$  in terms of  $A$  and  $B$ .**

*Solution.* First note that  $\Phi = (X \mapsto AX) \circ (Y \mapsto YB)$ . Then because matrix multiplication is bi-linear,  $\Phi$  is a composition of linear maps and is hence a  $K$ -linear operator on  $V$ .

In order to compute the trace, first figure out where  $\Phi$  sends the standard basis vectors to. For any  $(r, s) \in \{1, \dots, m\} \times \{1, \dots, n\}$ ,

$$\begin{aligned} \Phi(e_{rs}) &= A e_{rs} B \\ &= A \begin{pmatrix} 0 & & & \\ & \ddots & & \\ B_{s1} & \cdots & B_{sn} & \\ & \ddots & & \\ 0 & & & \end{pmatrix} \leftarrow \text{in } r\text{-th row} \\ &= \begin{pmatrix} A_{1r}B_{s1} & A_{1r}B_{s2} & \cdots & A_{1r}B_{sn} \\ A_{2r}B_{s1} & A_{2r}B_{s2} & \cdots & A_{2r}B_{sn} \\ \vdots & \vdots & \ddots & \vdots \\ A_{mr}B_{s1} & A_{mr}B_{s2} & \cdots & A_{mr}B_{sn} \end{pmatrix} \\ (\Phi(e_{rs}))_{ij} &= A_{ir}B_{sj} \end{aligned}$$

Then the trace can be computed by

$$\begin{aligned}\mathrm{Tr}(\Phi) &= \sum_{(r,s)} (\Phi(e_{rs}))_{rs} \\ &= \sum_{(r,s)} A_{rr} B_{ss} \\ &= \sum_{r=1}^m \sum_{s=1}^n A_{rr} B_{ss} \\ &= \mathrm{Tr}(A) \mathrm{Tr}(B)\end{aligned}$$

■

**(b) Compute the determinant  $\det(\Phi)$  of  $\Phi$  in terms of  $A, B, m$  and  $n$ .** *Solution.* Since we established that  $\Phi = (X \mapsto AX) \circ (Y \mapsto YB)$ , and since determinant is multiplicative, it suffices to compute the determinant for each  $L_A, R_B : V \rightarrow V$ , where  $L_A := X \mapsto AX$  and  $R_B := Y \mapsto YB$ .

**Finding determinant of  $L_A$ .** For any  $(r, s) \in \{1, \dots, m\} \times \{1, \dots, n\}$ , compute  $L_A(e_{rs})$ ,

$$\begin{aligned} L_A(e_{rs}) &= A e_{rs} \\ &= \begin{pmatrix} & A_{1r} & & \\ 0 & \cdots & \vdots & \cdots & 0 \\ & & A_{mr} & \end{pmatrix} \\ &\quad \text{in column } s \uparrow \\ &= A_{1r}e_{1s} + \cdots + A_{mr}e_{ms} \end{aligned}$$

Then by substituting in different values of  $r$  and  $s$ , we derive the matrix representation of  $L_A$  (with respect to ordered basis  $\mathcal{H}$ ) in block form as

$$[L_A]_{\mathcal{H}} = \begin{pmatrix} A_{11}1_n & A_{12}1_n & \cdots & A_{1m}1_n \\ A_{21}1_n & A_{22}1_n & \cdots & A_{2m}1_n \\ \vdots & \vdots & \ddots & \vdots \\ A_{m1}1_n & A_{m2}1_n & \cdots & A_{mm}1_n \end{pmatrix} \quad (1)$$

If  $A$  is singular, it is clear that the left-multiplication by  $A$  operator has no inverse, which implies  $\det(L_A) = 0 = \det(A)$ . If  $A$  is an invertible matrix, then  $A$  is a product of elementary matrices, so there exists elementary matrices  $E_1, \dots, E_k \in \mathbb{M}_{m \times m}(K)$  such that  $A = E_k \cdots E_1$ . Then  $L_A = L_{E_k} \circ \cdots \circ L_{E_1}$ . Then we are reduced to finding out the determinant of the left-multiply by elementary matrix operator.

**Claim.** For any elementary matrix  $E \in \mathbb{M}_{m \times m}(K)$ ,  $\det(L_E) = \det(E)^n$ .

1. Case  $E$  is a “row swap” elementary matrix, then by substituting  $A = E$  in (1),  $[L_E]_{\mathcal{H}}$  consists of  $n$  row swaps from  $1_{mn}$ . Then  $\det(L_E) = (-1)^n = \det(E)^n$ .
2. Case  $E$  is of a “multiply a row by  $c \in K$ ” matrix, then examine (1) again,  $[L_E]_{\mathcal{H}}$  is a diagonal matrix with all ones except  $n$  occurrences of  $c$ . Then  $\det(L_E) = c^n = \det(E)^n$ .
3. Case  $E$  is “add multiple of row to another row” matrix, then from (1),  $[L_E]_{\mathcal{H}}$  will be triangular with 1’s on the diagonal, so  $\det(L_E) = 1 = \det(E)^n$ .



Then from multiplicativity of determinant, recall that  $\det(A) = \det(E_k) \cdots \det(E_1)$ , then

$$\begin{aligned}\det(L_A) &= \det(L_{E_k}) \cdots \det(L_{E_1}) \\ &= \det(E_k)^n \cdots \det(E_1)^n \\ &= (\det(E_k) \cdots \det(E_1))^n \\ &= \det(A)^n\end{aligned}$$

**Finding determinant of  $R_B$ .** For any  $(r, s) \in \{1, \dots, m\} \times \{1, \dots, n\}$ , compute  $R_B(e_{rs})$ ,

$$\begin{aligned}R_B(e_{rs}) &= e_{rs} B \\ &= \begin{pmatrix} 0 \\ \vdots \\ B_{s1} & \cdots & B_{sn} \\ \vdots \\ 0 \end{pmatrix} \leftarrow \text{in } r\text{-th row} \\ &= B_{s1}e_{r1} + \cdots + B_{sn}e_{rn}\end{aligned}$$

This time, obtain the matrix representation of  $R_B$  (with respect to ordered basis  $\mathcal{H}$ ) in block form as

$$[R_B]_{\mathcal{H}} = \begin{pmatrix} B^t & & \\ & B^t & \\ & & \ddots \\ & & & B^t \end{pmatrix} \leftarrow \text{repeats } m \text{ times on diagonal} \quad (2)$$

If  $B$  is singular, it is again clear that  $R_B$  has no inverse, and  $\det(R_B) = 0$ . If  $B$  is invertible, exists elementary matrices  $E_1, \dots, E_k \in \mathbb{M}_{n \times n}(K)$  such that  $B = E_1 \cdots E_k$ , then  $R_B = R_{E_k} \circ \cdots \circ R_{E_1}$ . Now using a similar argument, we can find the determinant of  $R_B$ .

**Claim.** For any elementary matrix  $E \in \mathbb{M}_{n \times n}(K)$ ,  $\det(R_E) = \det(E)^m$ .

1. Case  $E$  is a row swap matrix, then from (2),  $[R_E]_{\mathcal{H}}$  contains  $m$  row swaps from  $1_{mn}$ , so  $\det(R_E) = (-1)^m = \det(E)^m$ .
2. Case  $E$  is of “multiply a row by  $c \in K$ ” type, then in (2),  $[R_E]_{\mathcal{H}}$  is a diagonal matrix with all ones except for  $m$  occurrences of  $c$ . Then  $\det(R_E) = c^m = \det(E)^m$ .
3. Case  $E$  is “add multiple of row to another row” matrix, then from (2),  $[R_E]_{\mathcal{H}}$  will be triangular with 1’s on diagonal, so  $\det(R_E) = 1 = \det(E)^m$ .

Then from multiplicativity of determinant, we get  $\det(R_B) = \det(B)^m$ .

Finally, as  $\Phi = L_A \circ R_B$ ,  $\det(\Phi) = \det(L_A) \det(R_B) = \det(A)^n \det(B)^m$ . ■

## Question 5

Let  $K$  be a field, and let  $x_1, \dots, x_n \in K$  be  $n$  elements of  $K$ . The  $n \times n$  *van der Monde determinant* of  $x_1, \dots, x_n$  is defined as

$$V(x_1, x_2, \dots, x_n) := \det \begin{pmatrix} 1 & 1 & \cdots & 1 \\ x_1 & x_2 & \cdots & x_n \\ \vdots & \vdots & \ddots & \vdots \\ x_1^{n-1} & x_2^{n-1} & \cdots & x_n^{n-1} \end{pmatrix}.$$

Show that

$$V(x_1, x_2, \dots, x_n) = \prod_{1 \leq i < j \leq n} (x_j - x_i) \quad \text{in } K.$$

*Proof.* Proceed by induction on  $n$ .

**Base case.** For  $n = 2$ ,  $x_1, x_2 \in K$ ,

$$\begin{aligned} V(x_1, x_2) &= \det \begin{pmatrix} 1 & 1 \\ x_1 & x_2 \end{pmatrix} \\ &= x_2 - x_1 = \prod_{1 \leq i < j \leq n} (x_j - x_i) \end{aligned}$$

**Induction hypothesis.** Suppose for any  $n - 1$  elements  $x_2, \dots, x_n \in K$ , we have  $V(x_2, \dots, x_n) = \prod_{2 \leq i < j \leq n} (x_j - x_i)$ .

Then for  $n$  elements  $x_1, \dots, x_n \in K$ ,

$$V(x_1, x_2, \dots, x_n) = \begin{vmatrix} 1 & 1 & \cdots & 1 \\ x_1 & x_2 & \cdots & x_n \\ \vdots & \vdots & \ddots & \vdots \\ x_1^{n-1} & x_2^{n-1} & \cdots & x_n^{n-1} \end{vmatrix}$$

subtract  $x_1$  times of  $n - 1$ -th row from  $n$ -th row

$$= \begin{vmatrix} 1 & 1 & \cdots & 1 \\ x_1 & x_2 & \cdots & x_n \\ \vdots & \vdots & \ddots & \vdots \\ 0 & x_2^{n-2}(x_2 - x_1) & \cdots & x_n^{n-2}(x_n - x_1) \end{vmatrix}$$

successively subtract  $k - 1$ -th row from  $k$ -th row as  $k$  iterates from  $n - 1$  to 2, and get

$$= \begin{vmatrix} 1 & 1 & \cdots & 1 \\ 0 & x_2 - x_1 & \cdots & x_n - x_1 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & x_2^{n-2}(x_2 - x_1) & \cdots & x_n^{n-2}(x_n - x_1) \end{vmatrix}$$

co-factor expansion along first column

$$= \begin{vmatrix} x_2 - x_1 & x_3 - x_1 & \cdots & x_n - x_1 \\ x_2(x_2 - x_1) & x_3(x_3 - x_1) & \cdots & x_n(x_n - x_1) \\ \vdots & \vdots & \ddots & \vdots \\ x_2^{n-2}(x_2 - x_1) & x_3^{n-2}(x_3 - x_1) & \cdots & x_n^{n-2}(x_n - x_1) \end{vmatrix}$$

since every column has a scalar I can factor out, take determinant of the transpose then use multilinearity

$$\begin{aligned} &= \begin{vmatrix} x_2 - x_1 & x_2(x_2 - x_1) & \cdots & x_2^{n-2}(x_2 - x_1) \\ x_3 - x_1 & x_3(x_3 - x_1) & \cdots & x_3^{n-2}(x_3 - x_1) \\ \vdots & \vdots & \ddots & \vdots \\ x_n - x_1 & x_n(x_n - x_1) & \cdots & x_n^{n-2}(x_n - x_1) \end{vmatrix} \\ &= \prod_{j=2}^n (x_j - x_1) \begin{vmatrix} 1 & x_2 & \cdots & x_2^{n-2} \\ 1 & x_3 & \cdots & x_3^{n-2} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & x_n & \cdots & x_n^{n-2} \end{vmatrix} \\ &= \prod_{j=2}^n (x_j - x_1) \begin{vmatrix} 1 & 1 & \cdots & 1 \\ x_2 & x_3 & \cdots & x_n \\ \vdots & \vdots & \ddots & \vdots \\ x_2^{n-2} & x_3^{n-2} & \cdots & x_n^{n-2} \end{vmatrix} \\ &= \prod_{j=2}^n (x_j - x_1) V(x_2, \dots, x_n) \end{aligned}$$

now applying induction hypothesis,

$$\begin{aligned} &= \prod_{j=2}^n (x_j - x_1) \prod_{2 \leq i < j \leq n} (x_j - x_i) \\ &= \prod_{1 \leq i < j \leq n} (x_j - x_i) \end{aligned}$$

□

## Question 6

*Proof.* Proceed by induction on  $n$ .

**Base case.** For  $n = 2$ , let  $a_1, a_2 \in K$ ,

$$\begin{aligned} \frac{(a_1, a_2)}{(a_2)} &= \frac{\det \begin{pmatrix} a_1 & 1 \\ -1 & a_2 \end{pmatrix}}{a_2} \\ &= \frac{a_1 a_2 + 1}{a_2} \\ &= a_1 + \frac{1}{a_2} \end{aligned}$$

**Induction hypothesis.** Suppose for any  $n - 1$  elements  $a_2, \dots, a_n \in K$ ,

$$a_2 + \frac{1}{\frac{a_3 + \frac{1}{\ddots + \frac{1}{a_{n-1} + \frac{1}{a_n}}}}}{\ddots} = \frac{(a_2, a_3, \dots, a_n)}{(a_3, \dots, a_n)}.$$

Then for any  $n$  elements  $a_1, \dots, a_n \in K$ , compute  $(a_1, \dots, a_n)$  by expanding along first row,

$$(a_1, \dots, a_n) = a_1 \begin{vmatrix} a_2 & 1 & & 0 \\ -1 & a_3 & \ddots & \\ & \ddots & \ddots & \\ 0 & \ddots & a_{n-1} & 1 \\ & & -1 & a_n \end{vmatrix} - \begin{vmatrix} -1 & 1 & & 0 \\ 0 & a_3 & \ddots & \\ -1 & \ddots & \ddots & \\ 0 & \ddots & a_{n-1} & 1 \\ & & -1 & a_n \end{vmatrix}$$

expand second term along its first column

$$\begin{aligned} &= a_1 (a_2, a_3, \dots, a_n) + \begin{vmatrix} a_3 & 1 & & 0 \\ -1 & \ddots & \ddots & \\ & \ddots & \ddots & 1 \\ 0 & & -1 & a_n \end{vmatrix} \\ &= a_1 (a_2, a_3, \dots, a_n) + (a_3, \dots, a_n) \end{aligned}$$

then division throughout by  $(a_2, \dots, a_n)$  (assuming it makes sense) will allow us to apply the induction

hypothesis

$$\begin{aligned}
 \frac{(a_1, a_2, \dots, a_n)}{(a_2, \dots, a_n)} &= a_1 + \frac{(a_3, \dots, a_n)}{(a_2, a_3, \dots, a_n)} \\
 &= a_1 + \frac{1}{\frac{(a_2, a_3, \dots, a_n)}{(a_3, \dots, a_n)}} \\
 &= a_1 + \frac{1}{a_2 + \frac{1}{a_3 + \frac{\ddots}{\ddots + \frac{1}{a_{n-1} + \frac{1}{a_n}}}}}
 \end{aligned}
 \quad \square$$