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## **MA2202S Homework 4**

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**1**

- i. We already have  $|S| = m$ , just need to show  $|gS| = m$ . From definition of  $gS$ , we see that it is the image of  $(g*)(S)$  the left-multiply by  $g$  map under  $S$ . Since the left-multiply map  $(g*)$  is injective we have  $|gS| = |S| = m$ .  $\square$
- ii. Let  $S \in X$ , we can verify that  $\pi'(e, S) = eS = S$ . Now let  $g, h \in G$ ,  $\pi'(g, \pi'(h, S)) = \pi'(g, hS) = ghS$ . On the other hand  $\pi'(gh, S) = ghS$ .  $\square$

**2**

- i. Since  $x_i$  and  $x_j$  are in the same orbit, we have a  $g \in G$  such that

$$x_i = \pi(g)x_j.$$

Suppose  $z \in G_{x_i}$  such that  $\pi(z)x_i = x_i$ , then we see that

$$\pi(g^{-1}zg)x_j = \pi(g^{-1}z)x_i = \pi(g^{-1})x_i = x_j$$

so  $g^{-1}zg \in G_{x_j}$ . We see that  $z \mapsto g^{-1}zg$  defines a map  $G_{x_i} \rightarrow G_{x_j}$ . This map of conjugation is bijective, as a symmetric argument shows that  $z' \mapsto gz'g^{-1}$  defines a map  $G_{x_j} \rightarrow G_{x_i}$ , which is its inverse.  $\square$

- ii. By part (i) and proposition 79,

$$\sum_{i=1}^r |G_{x_i}| = r |G_{x_1}| = |Gx_1| |G_{x_1}| = |G|. \quad \square$$

**3**

- i. From definition of the matrix  $A$ , we have

$$\sum_{i=1}^n a_{ij} = |\{g_i \in G : \pi(g_i)x_j = x_j\}| = |G_{x_j}|. \quad \square$$

- ii. Also from definition of matrix  $A$ , we have

$$\sum_{j=1}^m a_{ij} = |\{x_j \in X : \pi(g_i)x_j = x_j\}| = |F(g_i)|. \quad \square$$

iii. By parts (i) and (ii),

$$\begin{aligned}
 \sum_{j=1}^m |G_{x_j}| &= \sum_{j=1}^m \sum_{i=1}^n a_{ij} \\
 &= \sum_{i=1}^n \sum_{j=1}^m a_{ij} \\
 &= \sum_{i=1}^n |F(g_i)|.
 \end{aligned}
 \quad \square$$

iv. By part (ii) of previous question,

$$\sum_{j=1}^m |G_{x_j}| = |G| \cdot |\{Gx : x \in X\}|.$$

By part (iii) we have the number of  $G$ -orbits being

$$\frac{1}{|G|} \sum_{j=1}^m |G_{x_j}| = \frac{1}{|G|} \sum_{i=1}^n |F(g_i)|. \quad \square$$