

## 13 Axiom of Infinity and Natural Numbers

**Axiom 13.1** (Zermelo's Axiom of Infinity). There exists a set  $X$  such that  $\emptyset \in X$  and  $\forall x \in X. \{x\} \in X$ .

**Definition 13.2.** A set  $X$  is inductive iff  $\emptyset \in X$  and  $\forall x \in X. \{x\} \in X$ . (*Axiom of Infinity specifies the existence of a inductive set.*)

**Definition 13.3.** A sequential system consists of:

- a set  $X$
- an element  $x_0 \in X$
- a map  $T : X \rightarrow X$ . In set  $X$ :  $x_0 \rightarrow T(x_0) \rightarrow T(T(x_0)) \rightarrow \dots$

**Axiom 13.4** (Peano Axioms). A system of natural numbers is a sequential system:

1. a set  $\mathbb{N}$
2. an element  $0 \in \mathbb{N}$
3. a map  $S : \mathbb{N} \rightarrow \mathbb{N}$  (*succ function*)  
satisfying:
  - (i)  $\forall n \in \mathbb{N}. 0 \neq S(n)$  (*0 is not a succ to any  $\mathbb{N}$* )
  - (ii)  $S$  is injective,  $\forall n, m \in \mathbb{N}. n \neq m \implies S(n) \neq S(m)$ .
  - (iii) for any subset  $M \subseteq \mathbb{N}$  (*induction property*)  
if  $M$  has the properties
    - $0 \in M$
    - $\forall n \in M. S(n) \in M$
 then  $M = \mathbb{N}$ .

**Lemma 13.5.** If  $\mathcal{C}$  is any non-empty collection of inductive sets, then  $\bigcap \mathcal{C}$  is also a inductive set.

*Reminder:* Inductive set  $X$  means  $\emptyset \in X$  and  $\forall x \in X. \{x\} \in X$ . (*Definition 13.2*)

*Proof.*

1.  $\forall X \in \mathcal{C}. X$  is inductive.
2.  $\forall X \in \mathcal{C}. \emptyset \in X$ .
3. Take an arbitrary  $x \in \bigcap \mathcal{C}$ , then  $\forall X \in \mathcal{C}. x \in X$
4.  $X$  is inductive, so  $x \in X \implies \{x\} \in X$ .
5. then  $\forall X \in \mathcal{C}. \{x\} \in X$ , so  $\{x\} \in \bigcap \mathcal{C}$ .
6.  $\bigcap \mathcal{C}$  is inductive is shown. □

**Definition 13.6.**  $\mathbb{N}$  is the intersection of all subsets from  $A$  which are inductive.

1. Take the inductive set  $A$  given by Axiom of Infinity
2. Let  $\mathcal{C} := \{ X \in \mathcal{P}(A) : X \text{ is inductive} \}$ .  $\mathcal{C}$  consists of all subsets of  $A$  which are inductive.
3. Since  $A$  itself is inductive, and  $A \in \mathcal{P}(A)$ , so  $A \in \mathcal{C}$ .
4. Hence  $\mathcal{C}$  is non-empty.
5. By Lemma 13.5, define  $\mathbb{N} := \bigcap \mathcal{C}$  and  $\mathbb{N}$  is an inductive set. (satisfies Axiom 13.4.1)

**Lemma 13.7.** For any inductive set  $X$ , one has  $\mathbb{N} \subseteq X$ .

*Proof.*

1.  $X$  and  $A$  are inductive sets, by Lemma 13.5,  $\bigcap \{ X, A \}$  is an inductive set

$$\begin{aligned} X \cap A &\subseteq A \\ X \cap A &\in \mathcal{P}(A) \\ X \cap A &\in \mathcal{C} \end{aligned}$$

2. So  $\mathbb{N}$ , being  $\bigcap \mathcal{C}$ , is the subset of any element in  $\mathcal{C}$ , so  $\mathbb{N} \subseteq X \cap A$ .  
(by  $\forall F \in \mathcal{F}. a \in \bigcap \mathcal{F} \implies a \in F$ ) □

**Lemma 13.8.**  $\mathbb{N}$  is the unique inductive set such that  $\forall$  inductive set  $X$ , one has  $\mathbb{N} \subseteq X$ .

*Proof.*

1.  $\mathbb{N}$  is inductive. (by Definition 13.6)
2. for all inductive set  $X$ ,  $\mathbb{N} \subseteq X$ . (by Lemma 13.7)
3. Take a competitor set  $\mathbb{N}'$  is also inductive(1') and for all inductive set  $X$ ,  $\mathbb{N}' \subseteq X$  (2').
4. Apply (1) to (2'), for inductive set  $\mathbb{N}, \mathbb{N}' \subseteq \mathbb{N}$ .
5. Apply (1') to (2), for inductive set  $\mathbb{N}', \mathbb{N} \subseteq \mathbb{N}'$ .
6. Any set with properties (1) and (2)  $\mathbb{N}' = \mathbb{N}$ , uniqueness proven. □

**Definition 13.9.** 0 and succ function for  $\mathbb{N}$

1.  $0 := \emptyset \in \mathbb{N}$  ( $\because \mathbb{N}$  is inductive, by Definition 13.2)  
(satisfies Axiom 13.4.2)
2.  $S : \mathbb{N} \rightarrow \mathbb{N}$  is defined as  $\forall x \in \mathbb{N}. S(x) := \{ x \}$ .

- $S$  is defined for all  $x \in \mathbb{N}$ ,  $S$  is totally-defined.
- $S(x) = \{x\} \neq x$ ,  $S$  is well-defined.
- Given  $x \in \mathbb{N}$ ,  $\{x\} \in \mathbb{N}$ . ( $\because \mathbb{N}$  is inductive, by Definition 13.2)

**Theorem 13.10.** *The sequential system  $\mathbb{N}$  and  $S$  we defined satisfies property (i), (ii), (iii) in Axiom 13.4.3.*

Property (i):  $\forall n \in \mathbb{N}. 0 \neq S(n)$

*Proof.*

1. take an arbitrary  $n \in \mathbb{N}$ , then  $S(n) = \{n\}$
2.  $0 = \emptyset$  by definition, and for all  $n$ ,  $n \notin \emptyset$ ,
3. so  $\emptyset \neq \{n\}$  □

Property (ii):  $S$  is injective,  $\forall m, n \in \mathbb{N}. S(m) = S(n) \implies m = n$ .

*Proof.*

1. take  $m, n \in \mathbb{N}$ , if  $S(m) = S(n)$ , then
2.  $\{m\} = \{n\}$
3. by Axiom of Extensionality:  $m \in \{m\} \implies m \in \{n\}$ , so  $m = n$
4.  $S$  is injective. □

Property (iii): For any subset  $M \subseteq \mathbb{N}$ , if

- $0 \in M$
- $\forall n \in M. S(n) \in M$

then  $M = \mathbb{N}$ .

*Proof.*

1. Let  $M \subseteq \mathbb{N}$ ,
2. Then  $M$  is an inductive set (*by properties above*).
3. Then by Lemma 13.8,  $\mathbb{N} \subseteq M$ .
4. Assumed  $M \subseteq \mathbb{N}$ , therefore  $M = \mathbb{N}$ . □

**Conclusion.** *The sequential system  $\mathbb{N}$  and successor function  $S$  we defined above satisfies Axiom 13.4.*

## 14 Axiom of Infinity

**Principle of Induction.** Suppose  $P(-)$  is a statement about natural numbers.  $\forall n \in \mathbb{N}$ ,  $P(n)$  is a proposition with truth value.

By axiom of specification, define  $M := \{n \in \mathbb{N} : P(n) \text{ is true}\}$  Suppose we show

- (1) Base case:  $P(0)$  is true
- (2) Induction step:  $\forall k \in \mathbb{N}, P(k) \implies P(k+1)$

Then we know  $0 \in M$  and  $\forall k \in \mathbb{N}, k \in M \implies S(k) \in M$ . Then by property (iii) of Peano Axiom 13.4.3,  $M = \mathbb{N}$ . *(induction property)*

**Definition 14.1.** If  $f : A \rightarrow B$  is a map, then the *f-image of A* (or the range of  $f$ ) is

$$f(A) := \{b \in B : \exists a \in A. b = f(a)\}$$

Example:  $S(\mathbb{N}) = \{n \in \mathbb{N} : \exists k \in \mathbb{N}. n = S(k)\}$

**Lemma 14.2.**  $S(\mathbb{N}) = \mathbb{N} \setminus \{0\}$

*Proof.*

1. Let  $P(n) := (n = 0) \vee (\exists k \in \mathbb{N}. n = S(k))$ .
2.  $P(0)$  is trivially true.
3. Suppose  $n \in \mathbb{N}$  such that  $P(n)$  is true, either  $n = 0$  or  $\exists k \in \mathbb{N}. n = S(k)$ 
  - case  $n = 0$ , then  $S(n) = S(0)$  which is  $\in S(\mathbb{N})$
  - case  $\exists k \in \mathbb{N}. n = S(k)$ , then  $S(n) = S(S(k))$  which is  $\in S(\mathbb{N})$
4. so  $P(S(n))$  is true.
5. By Principle of Induction,  $\forall n \in \mathbb{N}. P(n)$  is true.

$n$  is either 0 or a successor of some  $k \in \mathbb{N}$ .

□

**Theorem 14.3** (Recursion Theorem (universal property of  $\mathbb{N}$ )). *Let  $(X, x_0, T)$  be any sequential system where*

- $X$  is a set.
- $x_0 \in X$  is a given element.
- $T : X \rightarrow X$  is a map.

*Then there exists a unique map*

$$\varphi : \mathbb{N} \rightarrow X$$

*such that*

1.  $\varphi(0) = x_0 \in X$
2. *The diagram commutes*

$$\begin{array}{ccc} \mathbb{N} & \xrightarrow{\varphi} & X \\ S \downarrow & & \downarrow T \\ \mathbb{N} & \xrightarrow{\varphi} & X \end{array}$$

$$\begin{aligned} \text{ie. } T \circ \varphi &= \varphi \circ S : \mathbb{N} \rightarrow X, \\ \forall n \in \mathbb{N}. T(\varphi(n)) &= \varphi(S(n)) \end{aligned}$$

Intuitively:

$$\begin{array}{ccccccc} \mathbb{N} : & 0 & \xrightarrow{S} & 1 & \xrightarrow{S} & 2 & \xrightarrow{S} & 3 & \xrightarrow{S} & \dots \\ \varphi \downarrow & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ T : & x_0 & \xrightarrow{T} & T(x_0) & \xrightarrow{T} & T^2(x_0) & \xrightarrow{T} & T^3(x_0) & \xrightarrow{T} & \dots \end{array}$$

*Proof.* later

## Consequence of Recursion Theorem

**Theorem 14.4** (Uniqueness of Natural Number System). *Let  $(\mathbb{N}, 0, S)$  be our natural number system. Suppose  $(\mathbb{N}', 0', S')$  is another natural number system satisfying Peano Axioms 13.4. Then there exists maps*

$$\varphi : \mathbb{N} \rightarrow \mathbb{N}' \text{ and } \varphi' : \mathbb{N}' \rightarrow \mathbb{N}$$

*such that*

- (i)  $\varphi(0) = 0'$  and  $\varphi'(0') = 0$ .
- (ii) *this diagram commutes,*

$$\begin{array}{ccc} \mathbb{N} & \xrightarrow{\varphi} & \mathbb{N}' \\ S \downarrow & & \downarrow S' \\ \mathbb{N} & \xrightarrow{\varphi} & \mathbb{N}' \end{array} \quad \begin{array}{ccc} \mathbb{N}' & \xrightarrow{\varphi'} & \mathbb{N} \\ S' \downarrow & & \downarrow S \\ \mathbb{N}' & \xrightarrow{\varphi'} & \mathbb{N} \end{array}$$

- (iii)  $\varphi' \circ \varphi = id_{\mathbb{N}}$  and  $\varphi \circ \varphi' = id_{\mathbb{N}'}$ .

Concretely:

$$\begin{array}{ccccccc} \mathbb{N} : & 0 & \xrightarrow{S} & 1 & \xrightarrow{S} & 2 & \xrightarrow{S} & 3 & \xrightarrow{S} & \dots \\ \varphi \updownarrow \varphi' & \updownarrow & & \updownarrow & & \updownarrow & & \updownarrow & & \\ \mathbb{N}' : & 0' & \xrightarrow{S'} & 1' & \xrightarrow{S'} & 2' & \xrightarrow{S'} & 3' & \xrightarrow{S'} & \dots \end{array}$$

*Proof.*

1. We have our natural number system  $(\mathbb{N}, 0, S)$ .
2. Given sequential system  $(\mathbb{N}', 0', S')$ , by recursion theorem, there exists a map

$$\varphi : \mathbb{N} \rightarrow \mathbb{N}'$$

such that

- (i)  $\varphi(0) = 0'$ , and
- (ii) this diagram commutes

$$\begin{array}{ccc} \mathbb{N} & \xrightarrow{\varphi} & \mathbb{N}' \\ S \downarrow & & \downarrow S' \\ \mathbb{N} & \xrightarrow{\varphi} & \mathbb{N}' \end{array}$$

3. Now we have natural number system  $(\mathbb{N}', 0', S')$ ,
4. Given sequential system  $(\mathbb{N}, 0, S)$ , by recursion theorem, there exists a map

$$\varphi' : \mathbb{N}' \rightarrow \mathbb{N}$$

such that

- (i)  $\varphi'(0') = 0$ , and
- (ii) this diagram commutes

$$\begin{array}{ccc} \mathbb{N}' & \xrightarrow{\varphi'} & \mathbb{N} \\ S' \downarrow & & \downarrow S \\ \mathbb{N}' & \xrightarrow{\varphi'} & \mathbb{N} \end{array}$$

5. for  $\varphi' \circ \varphi : \mathbb{N} \rightarrow \mathbb{N}$ ,

- note  $(\varphi' \circ \varphi)(0) = \varphi'(\varphi(0)) = \varphi'(0') = 0$
- and this commutes

$$\begin{array}{ccccc} \mathbb{N} & \xrightarrow{\varphi} & \mathbb{N}' & \xrightarrow{\varphi'} & \mathbb{N} \\ S \downarrow & & \downarrow S' & & \downarrow S \\ \mathbb{N} & \xrightarrow{\varphi} & \mathbb{N}' & \xrightarrow{\varphi'} & \mathbb{N} \end{array}$$

$$\text{ie. } S \circ (\varphi' \circ \varphi) = (\varphi' \circ \varphi) \circ S$$

6. But  $\text{id}_{\mathbb{N}} : \mathbb{N} \rightarrow \mathbb{N}$  also enjoys properties

- $\text{id}_{\mathbb{N}}(0) = 0$
- $S \circ \text{id}_{\mathbb{N}} = \text{id}_{\mathbb{N}} \circ S$

7. By applying Recursion Theorem of natural number system  $(\mathbb{N}, 0, S)$  to the sequential system  $(\mathbb{N}, 0, S)$  (itself), there exists a unique map

$$f : \mathbb{N} \rightarrow \mathbb{N}$$

such that

- $f(0) = 0$ , and
- this diagram commutes

$$\begin{array}{ccc} \mathbb{N} & \xrightarrow{f} & \mathbb{N} \\ S \downarrow & & \downarrow S \\ \mathbb{N} & \xrightarrow{f} & \mathbb{N} \end{array}$$

8. We just showed that  $\text{id}_{\mathbb{N}}$  is unique and has the same properties as  $\varphi' \circ \varphi$ , so  $\varphi' \circ \varphi = \text{id}_{\mathbb{N}}$ .
9. Repeating from (5.), symmetrically,  $\varphi \circ \varphi' = \text{id}_{\mathbb{N}'}$  □