13 Axiom of Infinity and Natural Numbers

Axiom 13.1 (Zermelo's Axiom of Infinity)**.** There exists a set X such that $\emptyset \in X$ and $\forall x \in X.\ \{ x \} \in X$.

Definition 13.2. A set X is inductive iff $\emptyset \in X$ and $\forall x \in X.\ \{ x \} \in X$. *(Axiom of Infinity specifies the existence of a inductive set.)*

Definition 13.3. A sequential system consists of:

- a set X
- an element $x_0 \in X$
- a map $T : X \to X$. In set X: $x_0 \to T(x_0) \to T(T(x_0)) \to \ldots$

Axiom 13.4 (Peano Axioms)**.** A system of natural numbers is a sequential system:

1. a set $\mathbb{N}$
2. an element $0 \in \mathbb{N}$
3. a map $S : \mathbb{N} \to \mathbb{N}$ *(succ function)*
   satisfying:
   (i) $\forall n \in \mathbb{N}.\ 0 \neq S(n)$ *(0 is not a succ to any $\mathbb{N}$)*
   (ii) S is injective, $\forall n, m \in \mathbb{N}.\ n \neq m \implies S(n) \neq S(m)$.
   (iii) for any subset $M \subseteq \mathbb{N}$ *(induction property)*
   if M has the properties
   - $0 \in M$
   - $\forall n \in M.\ S(n) \in M$

   then $M = \mathbb{N}$.

Lemma 13.5. *If $\mathcal{C}$ is any non-empty collection of inductive sets, then $\bigcap \mathcal{C}$ is also a inductive set.*

Reminder: Inductive set X means $\emptyset \in X$ and $\forall x \in X.\ \{ x \} \in X$. *(Definition 13.2)*

Proof.

1. $\forall X \in \mathcal{C}.\ X$ is inductive.
2. $\forall X \in \mathcal{C}.\ \emptyset \in X$.
3. Take an arbitary $x \in \bigcap \mathcal{C}$, then $\forall X \in \mathcal{C}.\ x \in X$
4. X is inductive, so $x \in X \implies \{ x \} \in X$.
5. then $\forall X \in \mathcal{C}.\ \{ x \} \in X$, so $\{ x \} \in \bigcap \mathcal{C}$.
6. $\bigcap \mathcal{C}$ is inductive is shown. $\square$

Definition 13.6. $\mathbb{N}$ is the intersection of all subsets from A which are inductive.

1. Take the inductive set A given by Axiom of Infinity
2. Let $\mathcal{C} := \{ X \in \mathcal{P}(A) : X \text{ is inductive} \}$. $\mathcal{C}$ consists of all subsets of A which are inductive.
3. Since A itself is inductive, and $A \in \mathcal{P}(A)$, so $A \in \mathcal{C}$.
4. Hence $\mathcal{C}$ is non-empty.
5. By Lemma 13.5, define $\mathbb{N} := \bigcap \mathcal{C}$ and $\mathbb{N}$ is an inductive set. (satisfies Axiom 13.4.1)

Lemma 13.7. *For any inductive set X, one has $\mathbb{N} \subseteq X$.*

Proof.

1. X and A are inductive sets, by Lemma 13.5, $\bigcap \{ X, A \}$ is an inductive set

$$\begin{aligned} &X \cap A \subseteq A \\ &X \cap A \in \mathcal{P}(A) \\ &X \cap A \in \mathcal{C} \end{aligned}$$

2. So $\mathbb{N}$, being $\cap\mathcal{C}$, is the subset of any element in $\mathcal{C}$, so $\mathbb{N} \subseteq X \cap A$.
   (by $\forall F \in \mathcal{F}.\ a \in \cap\mathcal{F} \implies a \in F$) □

Lemma 13.8. $\mathbb{N}$ *is the* <u>*unique*</u> *inductive set such that* $\forall$ *inductive set X, one has $\mathbb{N} \subseteq X$.*

Proof.

1. $\mathbb{N}$ is inductive. *(by Definition 13.6)*
2. for all inductive set X, $\mathbb{N} \subseteq X$. *(by Lemma 13.7)*
3. Take a competitor set $\mathbb{N}'$ is also inductive$(1')$ and for all inductive set X, $\mathbb{N}' \subseteq X(2')$.
4. Apply (1) to $(2')$, for inductive set $\mathbb{N}, \mathbb{N}' \subseteq \mathbb{N}$.
5. Apply $(1')$ to (2), for inductive set $\mathbb{N}', \mathbb{N} \subseteq \mathbb{N}'$.
6. Any set with properties (1) and (2) $\mathbb{N}' = \mathbb{N}$, uniqueness proven. □

Definition 13.9. 0 and succ function for $\mathbb{N}$

1. $0 := \emptyset \in \mathbb{N}$ *($\because$ $\mathbb{N}$ is inductive, by Definition 13.2)*

   (satisfies Axiom 13.4.2)
2. $S : \mathbb{N} \to \mathbb{N}$ is defined as $\forall x \in \mathbb{N}.\ S(x) := \{ x \}$.

- S is defined for all $x \in \mathbb{N}$, S is totally-defined.
- $S(x) = \{ x \} \neq x$, S is well-defined.
- Given $x \in \mathbb{N}$, $\{ x \} \in \mathbb{N}$. *($\because$ $\mathbb{N}$ is inductive, by Definition 13.2)*

Theorem 13.10. *The sequential system $\mathbb{N}$ and S we defined satisfies property (i), (ii), (iii) in* Axiom 13.4.3.

Property (i): $\forall n \in \mathbb{N}.\ 0 \neq S(n)$

Proof.

1. take an arbitary $n \in \mathbb{N}$, then $S(n) = \{ n \}$
2. $0 = \emptyset$ by definition, and for all n, $n \notin \emptyset$,
3. so $\emptyset \neq \{ n \}$ □

Property (ii): S is injective, $\forall m, n \in \mathbb{N}.\ S(m) = S(n) \implies m = n$.

Proof.

1. take $m, n \in \mathbb{N}$, if $S(m) = S(n)$, then
2. $\{ m \} = \{ n \}$
3. by Axiom of Extentionality: $m \in \{ m \} \implies m \in \{ n \}$, so $m = n$
4. S is injective. □

Property (iii): For any subset $M \subseteq \mathbb{N}$, if

- $0 \in M$
- $\forall n \in M.\ S(n) \in M$

then $M = \mathbb{N}$.

Proof.

1. Let $M \subseteq \mathbb{N}$,
2. Then M is an inductive set *(by properties above)*.
3. Then by Lemma 13.8, $\mathbb{N} \subseteq M$.
4. Assumed $M \subseteq \mathbb{N}$, therefore $M = \mathbb{N}$. □

Conclusion. *The sequential system $\mathbb{N}$ and successor function S we defined above satisfies* Axiom 13.4.

14 Axiom of Infinity

Principle of Induction. Suppose $P(-)$ is a statement about natural numbers. $\forall n \in \mathbb{N}$, $P(n)$ is a proposition with truth value.
By axiom of specification, define $M := \{ n \in \mathbb{N} : P(n) \text{ is true} \}$ Suppose we show

(1) Base case: $P(0)$ is true

(2) Induction step: $\forall k \in \mathbb{N}, P(k) \implies P(k+1)$

Then we know $0 \in M$ and $\forall k \in \mathbb{N}, k \in M \implies S(k) \in M$. Then by property (iii) of Peano Axiom 13.4.3, $M = \mathbb{N}$. *(induction property)*

Definition 14.1. If $f : A \to B$ is a map, then the *f-image of A* (or the range of f) is

$$f(A) := \{ b \in B : \exists a \in A.\ b = f(a) \}$$

Example: $S(\mathbb{N}) = \{ n \in \mathbb{N} : \exists k \in N.\ n = S(k) \}$

Lemma 14.2. $S(\mathbb{N}) = \mathbb{N} \setminus \{ 0 \}$

Proof.

1. Let $P(n) := (n = 0) \vee (\exists k \in \mathbb{N}.\ n = S(k))$.
2. $P(0)$ is trivially true.
3. Suppose $n \in \mathbb{N}$ such that $P(n)$ is true, either $n = 0$ or $\exists k \in \mathbb{N}.\ n = S(k)$
   - case $n = 0$, then $S(n) = S(0)$ which is $\in S(\mathbb{N})$
   - case $\exists k \in \mathbb{N}.\ n = S(k)$, then $S(n) = S(S(k))$ which is $\in S(\mathbb{N})$
4. so $P(S(n))$ is true.
5. By Principle of Induction, $\forall n \in \mathbb{N}.\ P(n)$ is true.

   n is either 0 or a successor of some $k \in \mathbb{N}$. $\square$

Theorem 14.3 (Recursion Theorem (universal property of $\mathbb{N}$)). *Let (X, x_0, T) be any sequential system where*

- *X is a set.*
- *$x_0 \in X$ is a given element.*
- *$T : X \to X$ is a map.*

Then there exists a unique map

$$\varphi : \mathbb{N} \to X$$

such that

1. *$\varphi(0) = x_0 \in X$*
2. *The diagram commutes*

$$\begin{tikzcd} \mathbb{N} \arrow[r, "\varphi"] \arrow[d, "S"'] & X \arrow[d, "T"] \\ \mathbb{N} \arrow[r, "\varphi"] & X \end{tikzcd}$$

ie. $T \circ \varphi = \varphi \circ S : \mathbb{N} \to X$,
$\forall n \in \mathbb{N}.\ T(\varphi(n)) = \varphi(S(n))$

Intuitively:

$$\begin{tikzcd} \mathbb{N}: \arrow[d, "\varphi"'] & 0 \arrow[r, "S"] \arrow[d] & 1 \arrow[r, "S"] \arrow[d] & 2 \arrow[r, "S"] \arrow[d] & 3 \arrow[r, "S"] \arrow[d] & \dots \\ T: & x_0 \arrow[r, "T"] & T(x_0) \arrow[r, "T"] & T^2(x_0) \arrow[r, "T"] & T^3(x_0) \arrow[r, "T"] & \dots \end{tikzcd}$$

Proof. later

Consequence of Recursion Theorem

Theorem 14.4 (Uniqueness of Natural Number System). *Let $(\mathbb{N}, 0, S)$ be our natural number system. Suppose $(\mathbb{N}', 0', S')$ is another natural number system satisfying Peano Axioms 13.4. Then there exists maps*

$$\varphi : \mathbb{N} \to \mathbb{N}' \text{ and } \varphi' : \mathbb{N}' \to \mathbb{N}$$

such that

(i) $\varphi(0) = 0'$ *and* $\varphi'(0') = 0$.

(ii) this diagram commutes,

$$\begin{tikzcd} \mathbb{N} \arrow[r, "\varphi"] \arrow[d, "S"'] & \mathbb{N}' \arrow[d, "S'"] \\ \mathbb{N} \arrow[r, "\varphi"] & \mathbb{N}' \end{tikzcd} \qquad \begin{tikzcd} \mathbb{N}' \arrow[r, "\varphi'"] \arrow[d, "S'"'] & \mathbb{N} \arrow[d, "S"] \\ \mathbb{N}' \arrow[r, "\varphi'"] & \mathbb{N} \end{tikzcd}$$

(iii) $\varphi' \circ \varphi = id_{\mathbb{N}}$ *and* $\varphi \circ \varphi' = id_{\mathbb{N}'}$.

Concretely:

$$\begin{tikzcd} \mathbb{N}: \arrow[d, shift right, "\varphi"'] & 0 \arrow[r, "S"] \arrow[d, shift right] & 1 \arrow[r, "S"] \arrow[d, shift right] & 2 \arrow[r, "S"] \arrow[d, shift right] & 3 \arrow[r, "S"] \arrow[d, shift right] & \dots \\ \mathbb{N}': \arrow[u, shift right, "\varphi'"'] & 0' \arrow[r, "S'"] \arrow[u, shift right] & 1' \arrow[r, "S'"] \arrow[u, shift right] & 2' \arrow[r, "S'"] \arrow[u, shift right] & 3' \arrow[r, "S'"] \arrow[u, shift right] & \dots \end{tikzcd}$$

Proof.

1. We have our natural number system $(\mathbb{N}, 0, S)$.
2. Given sequential system $(\mathbb{N}', 0', S')$, by recursion theorem, there exists a map

   $$\varphi : \mathbb{N} \to \mathbb{N}'$$

   such that

   (i) $\varphi(0) = 0'$, and

   (ii) this diagram commutes

   $$\begin{array}{ccc} \mathbb{N} & \xrightarrow{\varphi} & \mathbb{N}' \\ S\downarrow & & \downarrow S' \\ \mathbb{N} & \xrightarrow{\varphi} & \mathbb{N}' \end{array}$$

3. Now we have natural number system $(\mathbb{N}', 0', S')$,
4. Given sequential system $(\mathbb{N}, 0, S)$, by recursion theorem, there exists a map

   $$\varphi' : \mathbb{N}' \to \mathbb{N}$$

   such that

   (i) $\varphi'(0') = 0$, and

   (ii) this diagram commutes

   $$\begin{array}{ccc} \mathbb{N}' & \xrightarrow{\varphi'} & \mathbb{N} \\ S'\downarrow & & \downarrow S \\ \mathbb{N}' & \xrightarrow{\varphi'} & \mathbb{N} \end{array}$$

5. for $\varphi' \circ \varphi : \mathbb{N} \to \mathbb{N}$,
   - note $(\varphi' \circ \varphi)(0) = \varphi'(\varphi(0)) = \varphi'(0') = 0$
   - and this commutes

   $$\begin{array}{ccccc} \mathbb{N} & \xrightarrow{\varphi} & \mathbb{N}' & \xrightarrow{\varphi'} & \mathbb{N} \\ S\downarrow & & \downarrow S' & & \downarrow S \\ \mathbb{N} & \xrightarrow{\varphi} & \mathbb{N}' & \xrightarrow{\varphi'} & \mathbb{N} \end{array}$$

   ie. $S \circ (\varphi' \circ \varphi) = (\varphi' \circ \varphi) \circ S$

6. But $\mathrm{id}_{\mathbb{N}} : \mathbb{N} \to \mathbb{N}$ also enjoys properties
   - $\mathrm{id}_{\mathbb{N}}(0) = 0$
   - $S \circ \mathrm{id}_{\mathbb{N}} = \mathrm{id}_{\mathbb{N}} \circ S$
7. By applying Recursion Theorem of natural number system $(\mathbb{N}, 0, S)$ to the sequential system $(\mathbb{N}, 0, S)$ (itself), there exists a <u>unique map</u>

   $$f : \mathbb{N} \to \mathbb{N}$$

   such that

- $f(0) = 0$, and
- this diagram commutes

$$\begin{array}{ccc} \mathbb{N} & \xrightarrow{f} & \mathbb{N} \\ {\scriptstyle S}\downarrow & & \downarrow{\scriptstyle S} \\ \mathbb{N} & \xrightarrow{f} & \mathbb{N} \end{array}$$

8. We just showed that $\mathrm{id}_{\mathbb{N}}$ is unique and has the same properties as $\varphi' \circ \varphi$, so $\varphi' \circ \varphi = \mathrm{id}_{\mathbb{N}}$.
9. Repeating from (5.), symmetrically, $\varphi \circ \varphi' = \mathrm{id}_{\mathbb{N}'}$ $\square$