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## **MA2101S Homework 7**

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## Question 1

It is trivial that (i)  $\implies$  (ii).  $\square$

For (ii)  $\implies$  (i), suppose (ii), let  $\mathcal{B} = \{v_1, \dots, v_n\}$  be a basis for  $V$ . Clearly these vectors are non-zero, so they are eigenvectors of  $\varphi$ . Then there exists eigenvalues  $\lambda_1, \dots, \lambda_n \in K$  such that for any  $i \in \{1, \dots, n\}$ ,  $\varphi(v_i) = \lambda_i v_i$ . As vectors in  $\mathcal{B}$  form a basis,  $v_1 + \dots + v_n \neq 0$ , then there exist an eigenvalue  $c \in K$  such that

$$\begin{aligned}\varphi(v_1 + \dots + v_n) &= c(v_1 + \dots + v_n) \\ &= c v_1 + \dots + c v_n\end{aligned}$$

but by linearity,

$$\varphi(v_1 + \dots + v_n) = \lambda_1 v_1 + \dots + \lambda_n v_n$$

Then by uniqueness of basis coefficients,  $c = \lambda_1 = \dots = \lambda_n$ , then it becomes clear that  $\varphi = c \cdot \text{id}_V$ .

$\square$

## Question 2

(a) **False.**

*Counter-example.* Set  $K = \mathbb{R}$ ,  $V = \mathbb{R}^2$ , taking all matrices with respect to standard basis, set

$$B = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \quad A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad v = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

We can now compute that  $ABv = A0 = 0$ , which means  $v$  is an eigenvector of  $AB$  with eigenvalue 0, but  $BAv = B \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$  which is not a scalar multiple of  $v$ .  $\square$

(b) **True.**

*Proof.* Let  $\lambda \in K$  be an eigenvalue of  $AB$ , then  $\exists v \in V \setminus \{0_V\}$  such that

$$ABv = \lambda v.$$

- Case  $\lambda = 0$ , then  $AB$  is singular  $\implies BA$  is not invertible too, which lets us conclude that  $\lambda = 0$  is also an eigenvalue of  $BA$ .

- Case  $\lambda \neq 0$ , then  $Bv \neq 0_V$ , then pre-multiplying by  $B$  gives

$$BA(Bv) = \lambda(Bv)$$

which shows that  $\lambda$  is also an eigenvalue of  $BA$  with eigenvector  $Bv$ .  $\square$

### Question 3

(a) **True.**

**Claim.** Let  $v$  be an eigenvector of  $\varphi$  corresponding to eigenvalue  $\lambda$ . For any  $n \in \mathbb{N}$ ,  $\varphi^n v = \lambda^n v$ .

*Proof(of Claim).* It is given that  $\varphi v = \lambda v$ . Suppose  $\varphi^{n-1} v = \lambda^{n-1} v$ , then  $\varphi^n v = \varphi^{n-1}(\varphi v) = \varphi^{n-1}(\lambda v) = \lambda \varphi^{n-1} v$  by linearity. Applying induction hypothesis gives us the conclusion that  $\varphi^n v = \lambda^n v$ .  $\square$

*Proof(of 3a).* Let  $\lambda \in \mathbb{C}$  be an eigenvalue of  $\varphi$ , so  $\exists v \in V \setminus \{0\} \cdot \varphi v = \lambda v$ . Let  $f(T) \in \mathbb{C}[T]$  be given by

$$f(T) = \sum_{i=0}^{\deg(f)} f_i T^i.$$

Evaluating  $f$  at  $\varphi$  gives us an endomorphism,

$$\begin{aligned} f(\varphi) &= \sum_{i=0}^{\deg(f)} f_i \varphi^i \quad \text{in } \text{End}(V) \\ f(\varphi)v &= \sum_{i=0}^{\deg(f)} f_i \varphi^i v \quad \text{in } V \\ &= \sum_{i=0}^{\deg(f)} f_i \lambda^i v \quad \text{by Claim} \\ &= f(\lambda)v \end{aligned}$$

this shows that  $f(\lambda) \in \mathbb{C}$  is an eigenvalue of  $f(\varphi)$ .  $\square$

(b) **True.**

*Proof.* Let  $a$  be an eigenvalue of  $f(\varphi)$ , so

$$\exists v \in V \setminus \{0\} \cdot (f(\varphi) - aI)v = 0.$$

Consider the polynomial  $f(T) - a \in \mathbb{C}[T]$ , by Fundamental Theorem of Algebra, there exists

$\lambda_1, \dots, \lambda_k, c \in \mathbb{C}$  such that

$$f(T) - a = c \cdot \prod_{i=1}^k (T - \lambda_i) \quad \text{in } \mathbb{C}[T] \quad (\dagger)$$

evaluating at  $\varphi$  gives

$$f(\varphi) - aI = c \cdot \prod_{i=1}^k (\varphi - \lambda_i I) \quad \text{in } \text{End}_{\mathbb{C}}(V)$$

As it is known that LHS is singular, by multiplicativity of determinant, RHS is necessarily singular, so (at least) one of  $\varphi - \lambda_i I$  is singular, so

$$\exists \lambda \in \{\lambda_1, \dots, \lambda_k\} \cdot \det(\varphi - \lambda I) = 0$$

which implies that  $\lambda$  is an eigenvalue of  $\varphi$ , then evaluating  $(\dagger)$  at  $\lambda$  gives

$$f(\lambda) - a = c \cdot 0 \implies a = f(\lambda). \quad \square$$

## Question 4

(a) **True.**

*Proof.* Let  $f(T) \in \mathbb{C}[T]$  be given by

$$f(T) := T^k - 1.$$

then  $f(A) = A^k - 1_n = 0$  in  $\mathbb{M}_n(\mathbb{C})$ , so  $f$  annihilates  $A$ . We see that  $f$  has the following factorisation in  $\mathbb{C}[T]$

$$T^k - 1 = \prod_{j=0}^{k-1} (T - e^{j \cdot 2\pi i / k})$$

Let  $m(T) \in \mathbb{C}[T]$  be the minimal polynomial of  $A$ , then it is necessary that  $m(T) \mid f(T)$ . As  $f(T)$  splits completely into distinct linear factors,  $m(T)$  also has this property. This means that  $A$  is diagonalisable.  $\square$

(b) **False.**

*Counter-example.* Consider  $A := \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \in \mathbb{M}_2(\mathbb{C})$ .  $A^2 = 0$  is certainly diagonalisable, but  $A$  is already in Jordan canonical form and is not diagonalisable.  $\square$

## Question 5

**Definition.** For any  $z \in \mathbb{C}$  that is algebraic over  $\mathbb{Q}$ , let the **minimal polynomial** of  $z$  refer to the (necessarily unique) monic generator of the ideal of polynomials over  $\mathbb{Q}$  which annihilates  $z$ . (ie. The lowest degree monic polynomial with rational coefficients which has  $z$  as a root.)

**Lemma.** Let  $f \in \mathbb{Q}[T]$ , let  $z \in \mathbb{C}$  be algebraic over  $\mathbb{Q}$  with  $f(z) = 0$ , then  $m_z$  divides  $f$  in  $\mathbb{Q}[T]$ .

*Proof (Lemma).* First apply division algorithm in  $\mathbb{Q}[T]$ , so  $\exists q, r \in \mathbb{Q}[T]$  such that

$$f(T) = q(T) \cdot m_z(T) + r(T) \quad \text{in } \mathbb{Q}[T],$$

with  $\deg(r) < \deg(m_z)$ . Then evaluation at  $z$  gives

$$0 = q(z) \cdot 0 + r(z) \quad \text{in } \mathbb{C}.$$

By minimality of  $m_z$ ,  $r$  is necessarily the zero polynomial. Hence  $m_z \mid f$ . □

*Proof (Q5).* Let  $n = \dim_{\mathbb{Q}} V$ . When  $n = 0$ , the conclusion trivially holds, so suppose  $n \neq 0$ . Fix any ordered basis, and let  $A \in M_n(\mathbb{Q})$  be the matrix representation of  $\varphi$ . Let  $f(T) \in \mathbb{Q}[T]$  be the characteristic polynomial of  $A$ . We know that  $\deg(f) = n$ .

The polynomial  $T^p - 1$  can also be factorised in  $\mathbb{Q}[T]$  like so

$$T^p - 1 = (T - 1)(T^{p-1} + \dots + T + 1) \quad \text{in } \mathbb{Q}[T]$$

Let  $m(T) \in \mathbb{Q}[T]$  be given by  $m(T) := T^{p-1} + \dots + T + 1$ . Evaluation at  $\varphi$  gives

$$\varphi^p - \text{id}_V = 0 = (\varphi - \text{id}_V)m(\varphi) \quad \text{in } \text{End}(V)$$

as  $(\varphi - \text{id}_V)v = 0$  implies  $v = 0_V$ , we have  $m(\varphi) = 0$  in  $\text{End}(V)$ . So  $m$  annihilates  $\varphi$ , and similarly also annihilates its matrix representation  $A$ .

Now consider the field of complex numbers, and the corresponding linear map  $\mathbb{C}^n \rightarrow \mathbb{C}^n$  that  $A$  characterises. Because  $m$  annihilates  $A$ , for any of its eigenvalues  $\lambda \in \mathbb{C}$ ,  $m(\lambda) = 0$  (corollary of 3a). As it is given that  $m$  is irreducible over  $\mathbb{Q}$ ,  $m$  will be the minimal polynomial of  $\lambda$ .

Lastly, proceed to repeatedly apply lemma to obtain the result that

$$f(T) = m(T)^k \cdot l \quad \text{for some } k \in \mathbb{N} \setminus \{0\}, l \in \mathbb{Q} \setminus \{0\}.$$

Choose any root  $\lambda \in \mathbb{C}$  of  $f(T)$ , then because  $\lambda$  is an eigenvalue of  $A$ , by Lemma we have  $m \mid f$  in  $\mathbb{Q}[T]$ , so  $\exists q \in \mathbb{Q}[T]. f = m \cdot q$ .

1. Case  $q$  has no roots in  $\mathbb{C}$ , by our earlier assumption that  $f$  nonzero,  $q$  is a constant polynomial.
2. Case  $q$  has a root, say  $z \in \mathbb{C}$ , then  $f(z) = 0$ , which means  $z$  is an eigenvalue of  $A$ . Using the same argument, we obtain that  $z$  is also a root of  $m(T)$  and by Lemma,  $m \mid q$ . Then  $\exists q'(T) \in \mathbb{Q}[T]$  such that  $f = m^2 \cdot q'$ . Repeat this process until  $q_k(T)$  is degree 0, and we obtain the result stated earlier.

Then taking degrees,

$$\deg(f) = k \deg(m) \implies n = k(p-1) \text{ for some } k \in \mathbb{N} \setminus \{0\},$$

which shows  $p-1 \mid \dim_{\mathbb{Q}} V$ . □

## Question 6

*Proof.* Rewriting the recurrence equation in matrix form gives us that for any  $n \geq 1$ ,

$$\begin{pmatrix} P_{n+1} \\ P_n \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} P_n \\ P_{n-1} \end{pmatrix}$$

recursive expansion gives that for any  $n \geq 0$ ,

$$\begin{aligned} \begin{pmatrix} P_{n+1} \\ P_n \end{pmatrix} &= \begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix}^n \begin{pmatrix} P_1 \\ P_0 \end{pmatrix} \\ &= \begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix}^n \begin{pmatrix} 1 \\ 0 \end{pmatrix} \end{aligned}$$

Finding a closed form formula for Pell numbers reduces to diagonalising the matrix  $A := \begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix}$ .

Let  $f(t) \in \mathbb{R}[t]$  be characteristic polynomial of  $A$ ,

$$\begin{aligned} f(t) &= (2-t)(-t) - 1 \\ &= t^2 - 2t - 1 \end{aligned}$$

Roots of  $f$  are  $\frac{2 \pm \sqrt{8}}{2} = 1 \pm \sqrt{2}$ . Let  $\alpha := 1 + \sqrt{2}$  and  $\beta := 1 - \sqrt{2}$ , note that they can be characterised as solutions of the equation  $t^2 = 2t + 1$ . Using this property, it becomes clear that  $\begin{pmatrix} \alpha \\ 1 \end{pmatrix}$  is an eigenvector

(with eigenvalue  $\alpha$ ), because

$$\begin{aligned} \begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \alpha \\ 1 \end{pmatrix} &= \begin{pmatrix} 2\alpha + 1 \\ \alpha \end{pmatrix} \\ &= \begin{pmatrix} \alpha^2 \\ \alpha \end{pmatrix} \\ &= \alpha \begin{pmatrix} \alpha \\ 1 \end{pmatrix} \end{aligned}$$

As  $\beta$  has the same characterising property, the same computation will also show that  $\begin{pmatrix} \beta \\ 1 \end{pmatrix}$  is an eigenvector for eigenvalue  $\beta$ . Since the eigenspace has enough dimensions,  $A$  is diagonalisable, in fact

$$\begin{aligned} A &= \begin{pmatrix} \alpha & \beta \\ 1 & 1 \end{pmatrix} \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix} \begin{pmatrix} \alpha & \beta \\ 1 & 1 \end{pmatrix}^{-1} \\ A^n &= \begin{pmatrix} \alpha & \beta \\ 1 & 1 \end{pmatrix} \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}^n \begin{pmatrix} \alpha & \beta \\ 1 & 1 \end{pmatrix}^{-1} \\ &= \frac{1}{\alpha - \beta} \begin{pmatrix} \alpha & \beta \\ 1 & 1 \end{pmatrix} \begin{pmatrix} \alpha^n & 0 \\ 0 & \beta^n \end{pmatrix} \begin{pmatrix} 1 & -\beta \\ -1 & \alpha \end{pmatrix} \end{aligned}$$

substituting that into our original expression for  $\begin{pmatrix} P_{n+1} \\ P_n \end{pmatrix}$ , we can derive the closed form,

$$\begin{aligned} \begin{pmatrix} P_{n+1} \\ P_n \end{pmatrix} &= A^n \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ &= \frac{1}{\alpha - \beta} \begin{pmatrix} \alpha & \beta \\ 1 & 1 \end{pmatrix} \begin{pmatrix} \alpha^n & 0 \\ 0 & \beta^n \end{pmatrix} \begin{pmatrix} 1 & -\beta \\ -1 & \alpha \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ &= \frac{1}{\alpha - \beta} \begin{pmatrix} \alpha^{n+1} & \beta^{n+1} \\ \alpha^n & \beta^n \end{pmatrix} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \\ &= \frac{1}{\alpha - \beta} \begin{pmatrix} \alpha^{n+1} - \beta^{n+1} \\ \alpha^n - \beta^n \end{pmatrix} \end{aligned}$$

then we have

$$\begin{aligned} P_n &= \frac{1}{\alpha - \beta} (\alpha^n - \beta^n) \\ &= \frac{1}{2\sqrt{2}} \cdot \alpha^n - \frac{1}{2\sqrt{2}} \cdot \beta^n \end{aligned}$$

and  $\alpha = 1 + \sqrt{2}$ ,  $\beta = 1 - \sqrt{2}$ ,  $c = \frac{1}{2\sqrt{2}}$ ,  $d = -\frac{1}{2\sqrt{2}}$  can all be verified to be real numbers.  $\square$