1 Counting

Useful formulas.

$$a + ar + ar^2 + \cdots + ar^n = \frac{a(r^{n+1} - 1)}{r - 1}$$

$$a + (a + d) + \cdots + (a + nd) = \frac{(n + 1)(a + a + nd)}{2}$$

$${}^nP_k = \frac{n!}{(n - k)!}$$

$$\binom{n}{k} = {}^nC_k = \frac{n!}{k!(n - k)!}$$

$$(a + b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k$$

2 Graphing

Definitions.

Tree. graph with no cycle, **Leaf.** vertex with degree 1.
Weight of graph. sum of weights of all its edges.
Spanning tree. subgraph that is a tree and contains *all* vertices of original graph.

Theorems.

Degree Theorem. In any graph, sum of degrees $= 2\times$ no. edges.
Leaf Lemma. Every tree with ≥ 2 vertices has ≥ 2 leaves (deg1 vertices).
Tree Theorem. Every tree with n vertices has exactly $n - 1$ edges.
Euler Walk Theorem I. A connected graph contains a closed Euler walk iff every vertex has an even degree.
directed version. iff for every vertex the no. arrows in = no. arrows out.
Euler Walk Theorem II. A connected graph contains an open Euler walk iff
(i) start/end vertices have odd degree; and
(ii) all other vertices have even degree.
directed version. iff
(i) for start vertex, no. arrows out exactly one more than no. arrows in;
(ii) for end vertex, no. arrows in exactly one more than no. arrows out;
(iii) for all other vertices no. arrows in = no. arrows out.

Prim's Algorithm

1. Choose any vertex to initalise tree.
2. Grow the tree by one edge: of all edges that connect to vertices not yet in tree, find one with minimum-weight and add it in tree.
3. Repeat step 2 until tree spans.

Finding Euler circuit.

1. Check all vertices are even.
2. Construct any cycle.
3. If there's still remaining unused edges, construct cycle with those and combine the cycles.
4. Repeat step 3 until all edges used.

Chinese Postman Problem (Choosing edges to repeat)

1. Enumerate all pairings of odd vertices.
2. For each pair, find paths that join the vertices with minimum weight.
3. Use the set of pairings which sum of weights is minimised.
4. Walk the edges found above twice.

Vertex colouring If graph G contains a complete graph on n vertices, then $\chi(G) \geq n$.
Upper bound. Arrange degrees in decreasing order, place integers $1, 2, 3, \ldots$ below the degrees until you reach integer k such that $k + 1 > d_{k+1}$, then $\chi(G) \leq k + 1$.

3 Clocking

Congruence properties If $a \equiv b \pmod{m}$, then I can

- add multiples of modulo m to any side
- multiply by integers
- exponentiate both sides with positive powers only
- two congruences with the same modulo can be added/multiplied with each other

Calendrical Knowledge

YYYY is a leap year iff

- it is not a century year and divisible by 4 (1996); or
- it is a century year and divisible by 400 (2000).

Calendar (mod 7)

Jan	Feb	Mar	Apr	May	Jun	Jul	Aug	Sep	Oct	Nov	Dec
31	28/29	31	30	31	30	31	31	30	31	30	31
3	0/1	3	2	3	2	3	3	2	3	2	3

4 Coding

Binary representation. Divide by 2 and write it **right-to-left**.

bin	oct	hex	bin	hex
0	0	0	1000	8
1	1	1	1001	9
10	2	2	1010	A
11	3	3	1011	B
100	4	4	1100	C
101	5	5	1101	D
110	6	6	1110	E
111	7	7	1111	F

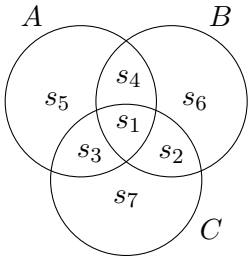
Modulo 37 encoding and ECC. 0-9 becomes 0-9 and _(space) is 36, and

sym	A	B	C	D	E	F	G	H	I	J	K	L	M
num	10	11	12	13	14	15	16	17	18	19	20	21	22
sym	N	O	P	Q	R	S	T	U	V	W	X	Y	Z
num	23	24	25	26	27	28	29	30	31	32	33	34	35

Weighted sum $w = 1(\text{last char}) + 2(\text{second last char}) + \cdots + n(\text{first char})$, and for modulo 37 encoding, append a checksum char such that $w \equiv 0 \pmod{37}$.

ISBN. 10 digits, X in last digit means 10, and a valid ISBN will have $w \equiv 0 \pmod{11}$.

Hamming $(7,4)$ **and** $(8,4)$ **Codes.** Every circle has even parity. In $(8,4)$, s_8 keeps total parity even.



5 Cryptography

Affine Cryptosystems with modulo n, (a,b) is the encryption key
encrypt: $y \equiv ax + b \pmod{n}$
decrypt: $y \equiv a'x - a'b \pmod{n}$ where a' is a inverse modulo n.

Finding modulo Inverse. Use Euclid $a = qd + r$, $\gcd(a,d) = \gcd(d,r)$.

Modular exponentiation. Finding $a^n \pmod{m}$. Express n as multiples(usually binary), then split a^n and find remainder for each term.

RSA. p, q are 2 primes, $n = pq$, k is an integer that has an inverse $\pmod{(p-1)(q-1)}$.

- Let P be a chunk of plaintext such that $0 \le P < n$.
- j is inverse of $k \bmod (p-1)(q-1)$, $kj \equiv 1 \pmod{(p-1)(q-1)}$
- Public key is (n,k). Private key is (j).
- Procedure encrypt: $C \equiv P^k \pmod{n}$.
- Procedure decrypt: $P \equiv C^j \pmod{n}$.

6 Chancing

Probability. With event E and sample space S, $\mathbb{P}(E) = \dfrac{|E|}{|S|}$.

Independence. characterising property: $\mathbb{P}(A \cap B) = \mathbb{P}(A)\mathbb{P}(B)$ where A, B are events.

Conditional Probability. $\mathbb{P}(A \mid B) = \dfrac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)}$.

Binomial distribution. For n independent events each with probability p,

$$X \sim B(n,p) \iff \mathbb{P}(X = k) = \binom{n}{k} p^k (1-p)^{n-k} \text{ and } \mathbb{E}[X] = np$$

Expectation. $\mathbb{E}[X] = \sum_x x \cdot \mathbb{P}(X = x)$ iterated through all possible values for x.

Expectation is linear so suppose total winnings $X = X_1 + \cdots + X_k$, then

$$\mathbb{E}[X] = \mathbb{E}[X_1] + \cdots + \mathbb{E}[X_k].$$

Poisson Distribution For Binomial distribution with large N and moderate $\mathbb{E}[X] = c = Np$, then

$$\mathbb{P}(X = k) \approx \frac{e^{-c} c^k}{k!}$$