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## **MA2202S Homework 2**

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25th September 2018

## 1

Claim that  $(\mu_n, \times) \simeq (\mathbb{Z}/n\mathbb{Z}, +)$ . Define  $\phi : \mu_n \rightarrow \mathbb{Z}/n\mathbb{Z}$ , with the knowledge that

$$\mu_n = \{ e^{2\pi i k/n} : k \in \{0, \dots, n-1\} \}$$

so we can define

$$\phi(z) = \frac{n \log z}{2\pi i}$$

such that  $\phi$  satisfies

$$\phi(e^{2\pi i k/n}) = k.$$

Then we can observe that  $\phi^{-1} : \mathbb{Z}/n\mathbb{Z} \rightarrow \mu_n$  is given by

$$\phi^{-1}(k) = e^{2\pi i k/n}.$$

Let  $H$  be a subgroup of  $(\mu_n, \times)$ . If  $H$  is a trivial subgroup, we are done, so suppose it's not trivial.

Consider  $H' = \phi(H)$  the subgroup of  $\mathbb{Z}/n\mathbb{Z}$ . Let  $d$  denote the smallest number in  $H' \setminus \{0\}$ .

**Claim.**  $d \mid n$  and  $H' = \{0, d, 2d, \dots, n-d\}$  **exactly.** Suppose on the contrary that  $d \nmid n$ , then there exists  $q \in \mathbb{Z}_0^+, r \in \{1, \dots, d-1\}$  such that

$$\begin{aligned} n &= qd + r \\ n - \underbrace{d - d - \dots - d}_{q \text{ times}} &= r \end{aligned}$$

which implies that  $r \in H'$ , contradicting minimality of  $d$ . So  $d \mid n$  which shows that  $\{0, d, 2d, \dots, n-d\} \subseteq H'$ .

For second part of claim, suppose on the contrary we have  $H' \supsetneq D = \{0, d, 2d, \dots, n-d\}$ . We take  $k \in H' \setminus D$ , then divide  $k$  by  $d$ , because  $k \notin D$ , we have  $q \in \mathbb{Z}_0^+, r \in \{1, \dots, d-1\}$  such that

$$k = qd + r$$

then by a similar argument as just now,  $r \in H'$  which contradicts minimality of  $d$ .

Letting  $r \in \mathbb{Z}^+$  such that  $dr = n$ , we have  $H' = \{0, d, 2d, \dots, (r-1)d\}$ , unravel  $\phi$  to get

$$H = \{1, e^{2\pi i d/n}, e^{2\pi i 2d/n}, \dots, e^{2\pi i (r-1)d/n}\}$$

as  $r = n/d$ ,

$$H = \{1, e^{2\pi i/r}, e^{2\pi i 2/r}, \dots, e^{2\pi i (e-1)/r}\}$$

then it can be observed that  $H = \mu_r$  with  $r \mid n$ .

Conversely suppose  $H = \mu_r$  where  $r \mid n$ , let  $d \in \mathbb{N}$ ,  $rd = n$ . Elements of  $\mu_r$  can be enumerated as

$$\mu_r = \{1, e^{2\pi i/r}, e^{2\pi i 2/r}, \dots, e^{2\pi i(r-1)/r}\}$$

as  $r = n/d$ ,

$$\mu_r = \{1, e^{2\pi i d/n}, e^{2\pi i 2d/n}, \dots, e^{2\pi i(r-1)d/n}\} \subseteq \mu_n$$

take  $e^{2\pi i a d/n}, e^{2\pi i b d/n} \in \mu_r$  where  $a, b \in \{0, \dots, r-1\}$ , then

$$\begin{aligned} e^{2\pi i a d/n} e^{2\pi i b d/n} &= e^{2\pi i(a+b)d/n} \\ &= e^{2\pi i(a+b-r)d/n} \end{aligned}$$

as  $e^{2\pi i r d/n} = e^{2\pi i} = 1$ , so in both cases  $a+b \geq r$  and  $a+b < r$ , we have  $e^{2\pi i a d/n} e^{2\pi i b d/n} \in \mu_r$ , so  $\mu_r$  is a subgroup.

## 2

Factors of 15 are 1, 3, 5, 15. Using question 1, we have trivial subgroups  $\{0\}$  and  $\langle 1 \rangle = \mathbb{Z}/15\mathbb{Z}$ , we also have the non-trivial subgroups  $\langle 3 \rangle$  and  $\langle 5 \rangle$ .

## 3

**i.  $H = \text{Stab}_G(s_0)$  is a subgroup of  $G$ .**

Take  $h_1, h_2 \in H$ , then

$$\begin{aligned} \pi(h_1 h_2, s_0) &= \pi(h_1, \pi(h_2, s_0)) \\ &= \pi(h_1, s_0) \\ &= s_0 \end{aligned}$$

so  $h_1 h_2 \in H$ .

Also let  $h \in H$ ,

$$\begin{aligned}
 s_0 &= \pi(e, s_0) \\
 &= \pi(h^{-1}h, s_0) \\
 &= \pi(h^{-1}, \pi(h, s_0)) \\
 &= \pi(h^{-1}, s_0)
 \end{aligned}$$

then  $h^{-1} \in H$ . Therefore  $H$  is a subgroup.

**ii.**

$\pi(g_1, s_0) = \pi(g_2, s_0)$  if and only if  $g_1 \in g_2H$ .

Suppose  $\pi(g_1, s_0) = \pi(g_2, s_0)$ , then

$$\begin{aligned}
 \pi(g_2^{-1}, \pi(g_1, s_0)) &= \pi(g_2^{-1}, \pi(g_2, s_0)) \\
 \pi(g_2^{-1}g_1, s_0) &= \pi(g_2^{-1}g_2, s_0) \\
 &= \pi(e, s_0) \\
 &= s_0
 \end{aligned}$$

so  $g_2^{-1}g_1 \in H$  which implies  $g_1 \in g_2H$ .

Conversely suppose  $g_1 \in g_2H$ , then  $g_2^{-1}g_1 \in H$ ,

$$\begin{aligned}
 \pi(g_1, s_0) &= \pi(g_1, \pi(g_2^{-1}g_2, s_0)) \\
 &= \pi(g_1, \pi(g_2^{-1}, \pi(g_2, s_0))) \\
 &= \pi(g_1g_2^{-1}, \pi(g_2, s_0)) \\
 &= \pi(e, \pi(g_2, s_0)) \\
 &= \pi(g_2, s_0)
 \end{aligned}$$

**iii. Show  $f$  is well-defined and injective**

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where  $f$  is defined as

$$\begin{aligned} f : G/H &\rightarrow S \\ gH &\mapsto \pi(g, s_0). \end{aligned}$$

Let  $g, g' \in G$ ,

$$\begin{aligned} gH &= g'H \\ \Leftrightarrow g &\in g'H && \text{by tutorial 3A Q1} \\ \Leftrightarrow \pi(g, s_0) &= \pi(g', s_0) && \text{by part ii} \\ \Leftrightarrow f(gH) &= f(g'H) && \text{by definition of } f \end{aligned}$$

the  $\Rightarrow$  argument gives well-definedness and the  $\Leftarrow$  argument gives injectivity.

**iv.**  $|G| = |O| |H|$ .

Since  $G$  is finite, by theorem 38 we have

$$|G/H| = \frac{|G|}{|H|}.$$

Consider  $f' : G/H \rightarrow O$  defined by  $f'(gH) = f(gH)$ , which is just  $f$  contracted to its image. As  $f$  is already an injection, restricting it to its image will make  $f'$  a bijection, then we have

$$\begin{aligned} \frac{|G|}{|H|} &= |G/H| = |O| \\ |G| &= |H| |O|. \end{aligned}$$