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## **MA2104 Assignment 4**

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## Question 1

Find

$$\iiint_E z(x^2 + y^2 + z^2)^{-3/2} dV$$

where

$$E := \{ (x, y, z) : x^2 + y^2 + z^2 \leq 16, z \geq 2 \}.$$

Upper bound for  $\phi$  happens at  $z = 2$ ,  $x^2 + y^2 = 12$ , then  $\rho = 4$  and  $\phi = \arccos(2/4) = \pi/3$ .

Since  $z = \rho \cos \phi$  and  $2 \leq z$ , we have  $2 \sec \phi \leq \rho$ .

Hence, converting to spherical coordinates, we have

$$E = \left\{ (\rho \sin \phi \cos \theta, \rho \sin \phi \sin \theta, \rho \cos \theta) : 0 \leq \theta \leq 2\pi, 0 \leq \phi \leq \frac{\pi}{3}, 2 \sec \phi \leq \rho \leq 4 \right\}$$

then computing the integral, we have

$$\begin{aligned} \iiint_E z(x^2 + y^2 + z^2)^{-3/2} dV &= \int_0^{2\pi} \int_0^{\pi/3} \int_{2 \sec \phi}^4 \rho \cos \phi (\rho^2)^{-3/2} (\rho^2 \sin \phi) d\rho d\phi d\theta \\ &= 2\pi \int_0^{\pi/3} \int_{2 \sec \phi}^4 \cos \phi \sin \phi d\rho d\phi \\ &= 2\pi \int_0^{\pi/3} (4 - 2 \sec \phi) \cos \phi \sin \phi d\phi \\ &= 4\pi \int_0^{\pi/3} 2 \sin \phi \cos \phi - \sin \phi d\phi \\ &= 4\pi \int_0^{\pi/3} \sin(2\phi) - \sin \phi d\phi \\ &= 4\pi \left[ -\frac{\cos(2\phi)}{2} + \cos \phi \right]_0^{\pi/3} \\ &= 4\pi \left( \frac{1}{4} + \frac{1}{2} + \frac{1}{2} - 1 \right) = \pi \end{aligned}$$

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## Question 2

Find

$$\iint_D x dA$$

where

$$D := \{ (x, y) : x^2 + (y - 1)^2 \leq 1, x^2 + y^2 \geq 1 \}.$$

To convert polar coordinates, we need the ranges of  $(r, \theta)$ . To find range of  $\theta$ , we first need to find points where the two circles intersect. By inspection, we have  $y = \frac{1}{2}$ . Solving  $x^2 + \frac{1}{4} = 1$  gives the solution that  $x = \pm \frac{\sqrt{3}}{2}$ .

Then

$$\arctan\left(\frac{1}{\sqrt{3}}\right) \leq \theta \leq \arctan\left(-\frac{1}{\sqrt{3}}\right)$$

$$\frac{\pi}{6} \leq \theta \leq \frac{5\pi}{6}$$

Next find bounds for  $r$ , it is obvious that  $r \geq 1$ , it remains to find upper bound.

$$x^2 + (y-1)^2 \leq 1$$

$$x^2 + y^2 - 2y \leq 0$$

$$r^2 \leq 2r \sin \theta$$

$$r \leq 2 \sin \theta$$

because  $r \geq 1$ . So

$$D = \left\{ (r \cos \theta, r \sin \theta) : \frac{\pi}{6} \leq \theta \leq \frac{5\pi}{6}, 1 \leq r \leq 2 \sin \theta \right\}$$

Therefore,

$$\begin{aligned} \iint_D x \, dA &= \int_{\pi/6}^{5\pi/6} \int_1^{2 \sin \theta} r \cos \theta \, r \, dr \, d\theta \\ &= \int_{\pi/6}^{5\pi/6} \cos \theta \left[ \frac{r^3}{3} \right]_1^{2 \sin \theta} d\theta \\ &= \int_{\pi/6}^{5\pi/6} \frac{8}{3} \sin^3 \theta \cos \theta - \frac{1}{3} \cos \theta \, d\theta \\ &= \left[ \frac{2}{3} \sin^4 \theta - \frac{1}{3} \sin \theta \right]_{\pi/6}^{5\pi/6} \\ &= 0 \quad \because \sin\left(\frac{\pi}{6}\right) = \sin\left(\frac{5\pi}{6}\right) \end{aligned}$$

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### Question 3

Find

$$\iint_D \frac{2x^2 + y^2}{xy} dA$$

where

$$D := \{(x, y) : 1 \leq \frac{y}{\sqrt{x}} \leq 2, 1 \leq x^2 + y^2 \leq 4\}.$$

Let  $u = x^2 + y^2$  and  $v = \frac{y}{\sqrt{x}}$ . Clearly we have  $1 \leq u \leq 4$  and  $1 \leq v \leq 2$ . We have

$$\begin{aligned}\frac{\partial(u, v)}{\partial(x, y)} &= \begin{vmatrix} 2x & 2y \\ -\frac{1}{2}yx^{-3/2} & x^{-1/2} \end{vmatrix} \\ &= 2x^{1/2} + x^{-3/2}y^2\end{aligned}$$

Using the identity that

$$\frac{\partial(u, v)}{\partial(x, y)} = \frac{1}{\frac{\partial(x, y)}{\partial(u, v)}}$$

we can compute the change of variable

$$\begin{aligned}\iint_D \frac{2x^2 + y^2}{xy} dA &= \int_1^2 \int_1^4 \frac{2x^2 + y^2}{xy} \cdot \frac{1}{2x^{1/2} + x^{-3/2}y^2} du dv \\ &= \int_1^2 \int_1^4 \frac{2x^2 + y^2}{y(2x^{3/2} + x^{-1/2}y^2)} du dv \\ &= \int_1^2 \int_1^4 \frac{2x^2 + y^2}{\frac{y}{\sqrt{x}}(2x^2 + y^2)} du dv \\ &= \int_1^2 \int_1^4 \frac{1}{v} du dv \\ &= 3 [\ln v]_1^2 = 3 \ln 2\end{aligned}$$

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