

# CS3230 Homework 1

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## 1 K-sorted Array

### 1.1

**Statement 1.1.1.** Fix  $j$ , suppose for any  $i \in \{1, \dots, j - k - 1\}$ ,  $B[i]$  contains the  $i$ -th smallest element of  $A$ . Then the value extracted from the heap will be the  $(j - k)$ -th smallest element of  $A$ .

### 1.2

For any  $i \in \{1, \dots, n\}$ , we let  $M(i)$  denote the  $i$ -th smallest element of  $A$ .

*Proof.* We first observe that the elements

$$X := \{A[1], \dots, A[k], \dots, A[\max(j, n)]\}$$

have been added to the heap  $S$  in previous (if any) and current iterations of the **for** loop. Since  $A$  is  $k$ -sorted,  $M(j - k) \in \{A[1], \dots, A[\max(j - k + k, n)]\} = X$ . By our assumption that  $B[i]$  contains the  $i$ -th smallest element of  $A$  for each  $i \in \{1, \dots, j - k - 1\}$ . We see that the elements

$$Y := \{M(1), \dots, M(j - k - 1)\}$$

have already been extracted from  $S$  in previous iterations. As  $A$  contains distinct integers, we see that  $M(j - k) \notin Y$ . Now we see that the heap  $S$  contains precisely  $X \setminus Y$ . All elements less than  $M(j - k)$  are not in  $S$ , so  $M(j - k)$  is minimal in  $S$ , and it will be the extracted value.  $\square$

### 1.3

*Proof.* Proceed by induction on  $j - k$ . Applying Statement 1.1.1 with  $j = k + 1$  proves the base case that  $B[1]$  will contain the smallest element of  $A$ . Similarly, Statement 1.1.1 proves the inductive case. This means for every  $i \in \{1, \dots, n\}$ ,  $B[i] = M(i)$  so in particular,  $B$  contains the elements of  $A$  in sorted order.  $\square$

## 2 Inversions

### 2.1

*Solution.* Given the array  $\langle 2, 3, 8, 6, 1 \rangle$ . The inversions are  $(1, 5), (2, 5), (3, 4), (3, 5), (4, 5)$ . ■

### 2.2

*Solution.* The array given by  $\langle n, n-1, \dots, 1 \rangle$  has inversion count  $\binom{n}{2}$ . ■

### 2.3

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**Algorithm 1:** Counting inversions with modified mergesort.

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**Data:** an array  $A[1, \dots, n]$  containing a permutation of the  $n$  elements  
**Result:** the number of inversions in  $A$

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1  inversions  $\leftarrow 0$ 
2  subroutine modified-merge (left, mid, right) is
    Data: indices of start of left subarray, start of right subarray and end
        of right subarray, where both subarrays sorted
    Result: two subarrays merged, inversions incremented
3  Initialise array  $B[\text{left}, \dots, \text{right}]$ 
4   $i \leftarrow \text{left}; j \leftarrow \text{mid}$ 
5   $k \leftarrow \text{left}$ 
6  while  $k \leq \text{right}$  do
7      if  $i < \text{mid} \wedge (j > \text{right} \vee A[i] < A[j])$  then
8           $B[k] \leftarrow A[i]$ 
9           $i \leftarrow i + 1$ 
10     else
11          $B[k] \leftarrow A[j]$ 
12          $j \leftarrow j + 1$ 
13         inversions  $\leftarrow \text{inversions} + (\text{mid} - i)$ 
14      $k \leftarrow k + 1$ 
15  copy  $B[\text{left}, \dots, \text{right}]$  into  $A[\text{left}, \dots, \text{right}]$ 
16 function mergesort ( $L, R$ ) is
    Data: start  $L$  and end  $R$  indices of subarray to mergesort
17  if  $L = R$  then
18      return
19   $M \leftarrow \lfloor \frac{L+R}{2} \rfloor + 1$ 
20  mergesort( $L, M-1$ )
21  mergesort( $M, R$ )
22  modified-merge( $L, M, R$ )
23 mergesort (1, n)
24 output inversions

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