MA2202S Homework 4

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i. We already have |S|=m, just need to show |gS|=m. From definition of gS, we see that it is the image of (g*)(S) the left-multiply by g map under S. Since the left-multiply map (g*) is injective we have |gS|=|S|=m.

ii. Let $S \in X$, we can verify that $\pi'(e,S) = eS = S$. Now let $g,h \in G$, $\pi'(g,\pi'(h,S)) = \pi'(g,hS) = ghS$. On the other hand $\pi'(gh,S) = ghS$.

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i. Since x_i and x_j are in the same orbit, we have a $g \in G$ such that

$$x_i = \pi(g)x_j$$
.

Suppose $z \in G_{x_i}$ such that $\pi(z)x_i = x_i$, then we see that

$$\pi(g^{-1}zg)x_j = \pi(g^{-1}z)x_i = \pi(g^{-1})x_i = x_j$$

so $g^{-1}zg\in G_{x_j}$. We see that $z\mapsto g^{-1}zg$ defines a map $G_{x_i}\to G_{x_j}$. This map of conjugation is bijective, as a symmetric argument shows that $z'\mapsto gz'g^{-1}$ defines a map $G_{x_j}\to G_{x_i}$, which is its inverse. \Box

ii. By part (i) and proposition 79,

$$\sum_{i=1}^{r}\left|G_{x_{i}}\right|=r\left|G_{x_{1}}\right|=\left|Gx_{1}\right|\left|G_{x_{1}}\right|=\left|G\right|.$$

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i. From definition of the matrix A, we have

$$\sum_{i=1}^n a_{ij} = \left|\left\{\,g_i \in G : \pi(g_i)x_j = x_j\,\right\}\right| = \left|G_{x_j}\right|. \quad \Box$$

ii. Also from definition of matrix A, we have

$$\sum_{i=1}^{m}a_{ij}=\left|\left\{\,x_{j}\in X:\pi(g_{i})x_{j}=x_{j}\,\right\}\right|=\left|F(g_{i})\right|. \label{eq:continuous_problem}$$

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iii. By parts (i) and (ii),

$$\begin{split} \sum_{j=1}^m \left| G_{x_j} \right| &= \sum_{j=1}^m \sum_{i=1}^n a_{ij} \\ &= \sum_{i=1}^n \sum_{j=1}^m a_{ij} \\ &= \sum_{i=1}^n \left| F(g_i) \right|. \end{split}$$

iv. By part (ii) of previous question,

$$\sum_{j=1}^m \left|G_{x_j}\right| = |G| \cdot \left|\left\{\,Gx : x \in X\,\right\}\right|.$$

By part (iii) we have the number of G-orbits being

$$\frac{1}{|G|} \sum_{j=1}^{m} \left| G_{x_j} \right| = \frac{1}{|G|} \sum_{i=1}^{n} |F(g_i)|.$$

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