MA2101S Homework 5

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Question 1

> For any $n \in \mathbb{N}$, $p_n(X) := nX^{n+1} - (n+1)X^n + 1 \in \mathbb{Q}[X]$. Show that there exists $q_n \in \mathbb{Q}[X]$ such that $p_n(X) = (X-1)^2 q_n(X)$.

Proof. Consider $q_n(X) := \sum_{i=0}^{n-1}(i+1)X^i \in \mathbb{Q}[X]$. Now compute $(X-1)^2 q_n(X)$,

$$\begin{aligned}
(X-1)^2 q_n(X) &= (X^2 - 2X + 1)q_n(X) \\
&= X^2 q_n(X) - 2Xq_n(X) + q_n(X) \\
&= X^2 \sum_{i=0}^{n-1}(i+1)X^i - 2X\sum_{i=0}^{n-1}(i+1)X^i + \sum_{i=0}^{n-1}(i+1)X^i \\
&= \sum_{i=0}^{n-1}(i+1)X^{i+2} - \sum_{i=0}^{n-1}2(i+1)X^{i+1} + \sum_{i=0}^{n-1}(i+1)X^i \\
&= \sum_{i=2}^{n+1}(i-1)X^i - \sum_{i=1}^{n}2iX^i + \sum_{i=0}^{n-1}(i+1)X^i \\
&= \sum_{i=1}^{n+1}(i-1)X^i - \sum_{i=1}^{n}2iX^i + \sum_{i=0}^{n-1}(i+1)X^i \\
&= \sum_{i=1}^{n-1}(i-1)X^i - \sum_{i=1}^{n}2iX^i + \sum_{i=0}^{n-1}(i+1)X^i + nX^{n+1} + (n-1)X^n \\
&= \sum_{i=1}^{n-1}(i-1)X^i - \sum_{i=1}^{n-1}2iX^i + \sum_{i=0}^{n-1}(i+1)X^i + nX^{n+1} - (n+1)X^n \\
&= \sum_{i=1}^{n-1}\left[(i-1)X^i - 2iX^i + (i+1)X^i\right] + nX^{n+1} - (n+1)X^n + 1 \\
&= \sum_{i=1}^{n-1} 0 + nX^{n+1} - (n+1)X^n + 1 \\
&= nX^{n+1} - (n+1)X^n + 1 = p_n
\end{aligned}$$

Therefore $p_n(X)$ is divisible by $(X-1)^2$. $\square$

Question 2

> Let K be a field, and let $a, b \in K$ with $a \neq 0$. Show that $(aX+b)^0, (aX+b)^1, (aX+b)^2, \ldots$ form a basis for $K[X]$.

Linear independence. *Proof.* Consider any finite subset of naturals $S \subseteq \mathbb{N}$. The claim is that $\{ (aX+b)^s \}_{s \in S}$ – an arbitrary finite subset of $\{ (aX+b)^i \}_{i \in \mathbb{N}}$, is linearly independent. To prove linear independence, proceed by induction on $|S|$.

Base cases. If $|S| = 0$ or $|S| = 1$, linear independence is trivial.

Induction hypothesis. Suppose for any $T \subseteq \mathbb{N}$ with $|T| = n-1$, $\{ (aX+b)^t \}_{t \in T}$ is linearly independent.

Now consider $S \subseteq \mathbb{N}$ with $|S| = n$. Let $\omega \in S$ be the largest element in S, that is for any $s \in S$, $s \leqslant \omega$. Because S is finite and non-empty, ω actually exists. Consider this equation,

$$\sum_{s \in S} c_s (aX+b)^s = 0 \qquad \text{in } \mathbb{Q}[X]$$

where $(c_s)_{s \in S} \in K$ are coefficients indexed by S. Comparing the coefficient of X^ω, $c_\omega a^\omega = 0$, then because $a^\omega \neq 0$, $c_\omega = 0$. Then the equation reduces to,

$$\sum_{s \in S \setminus \{ \omega \}} c_s (aX+b)^s = 0 \qquad \text{in } \mathbb{Q}[X]$$

then from induction hypothesis, because $|S \setminus \{ \omega \}| = n-1$, using linear independence, all the coefficients $(c_s)_{s \in S \setminus \{ \omega \}}$ are zero, together with our earlier conclusion that $c_\omega = 0$, completes the proof that $\{ (aX+b)^s \}_{s \in S}$ is linearly independent.

Hence any finite subset of $\{ (aX+b)^0, (aX+b)^1, (aX+b)^2, \ldots \}$ is linearly independent. $\square$

Spanning. *Proof.* To show that $\{ (aX+b)^i \}_{i\in\mathbb{N}}$ spans $K[X]$, proceed by induction on the degree of the polynomial that lies in $K[X]$.

Base cases. Trivial to see that zero polynomial is spanned. Since $(aX+b)^0 = 1$, all degree 0 polynomials are spanned too.

Induction hypothesis. Suppose any polynomial of degree strictly less than n is spanned by $\{ (aX+b)^i \}_{i\in\mathbb{N}}$.

Let $f \in K[X]$ with $\deg(f) = n$, so $f = \sum_{i=0}^{n} f_i X^i$, where $f_0, \dots, f_n \in K$ are coefficients with $f_n \neq 0$. From binomial theorem,

$$\begin{aligned}(aX+b)^n &= \sum_{r=0}^{n} \binom{n}{r} (aX)^r b^{n-r} \\ &= a^n X^n + \sum_{r=0}^{n-1} \binom{n}{r} (aX)^r b^{n-r}\end{aligned}$$

as $a^n \neq 0$, proceed to compute $f - \frac{f_n}{a^n}(aX+b)^n$,

$$\begin{aligned}f - \frac{f_n}{a^n}(aX+b)^n &= f_n X^n + \sum_{i=0}^{n-1} f_i X^i - \frac{f_n}{a^n}\left(a^n X^n + \sum_{r=0}^{n-1} \binom{n}{r} (aX)^r b^{n-r} \right) \\ &= f_n X^n + \sum_{i=0}^{n-1} f_i X^i - f_n X^n - \frac{f_n}{a^n} \sum_{r=0}^{n-1} \binom{n}{r} a^r X^r b^{n-r} \\ &= \sum_{r=0}^{n-1} \left(f_r X^r - \frac{f_n}{a^n} \binom{n}{r} a^r b^{n-r} X^r \right) \\ &= \sum_{r=0}^{n-1} \left(f_r - \frac{f_n}{a^n} \binom{n}{r} a^r b^{n-r} \right) X^r\end{aligned}$$

This means $f - \frac{f_n}{a^n}(aX+b)^n$ is a polynomial with degree at most $n-1$, so by induction hypothesis, it is spanned by $\{ (aX+b)^i \}_{i\in\mathbb{N}}$. So there exists a finite subset $S \subseteq \mathbb{N}$, and coefficients $(c_s)_{s\in S} \in K$ indexed by S such that

$$f - \frac{f_n}{a^n}(aX+b)^n = \sum_{s\in S} c_s (aX+b)^s,$$

which gives

$$f = \sum_{s\in S} c_s (aX+b)^s + \frac{f_n}{a^n}(aX+b)^n.$$

By strong induction, any polynomial is spanned by $\{ (aX+b)^i \}_{i\in\mathbb{N}}$. Therefore $\{ (aX+b)^i \}_{i\in\mathbb{N}}$ forms a basis for $K[X]$. $\square$

Question 3

> Let K be a field, and let $h \in K[X]$ be a polynomial with $\deg(h) \geqslant 1$. Consider the linear endormorphism Φ of $K[X]$ given by
>
> $$\Phi : K[X] \to K[X], \qquad f \mapsto f(h).$$
>
> (a) Show that Φ is injective.
> (b) Show that Φ is an isomorphism if and only if $\deg(h) = 1$.

Proposition. For any nonzero polynomials $f, g \in K[X]$, $\deg(f(g)) = \deg(f)\deg(g)$.

Proof. Let $f_0, \dots, f_m \in K$ such that $f = \sum_{i=0}^m f_i X^i$ and $g_0, \dots, g_n \in K$ such that $g = \sum_{j=0}^n g_j X^j$, with $f_m \neq 0$ and $g_n \neq 0$, where $m = \deg(f), n = \deg(g), m, m \geqslant 0$, then

$$\begin{aligned} f(g) &= \sum_{i=0}^m f_i g^i \\ &= \sum_{i=0}^m f_i \left(\sum_{j=0}^n g_j X^j \right)^i \end{aligned}$$

As $\deg(g^i) = i \cdot \deg(g)$ for any $i \in \mathbb{N}$, $\deg(f(g)) \leqslant m \cdot \deg(g)$. Also note that in $f(g)$, the coefficient of X^{mn} is $f_m g_n^m$, which is nonzero, therefore $\deg(f(g)) = mn = \deg(f)\deg(g)$. ☐

(a) *Proof.* To show injectivity, proceed to show that Φ has a trivial kernel. Suppose for a contradiction Φ has a non-trivial kernel, that is there exists $f \in K[X]$, with $\deg(f) \geqslant 0$, and $\Phi(f) = 0$. This means $\deg(\Phi(f)) = \deg(0) = -\infty$, but because both f, h are nonzero polynomials, by proposition above, $\deg(f(h)) = \deg(f)\deg(h) \geqslant 0$ which is a contradiction. ☐

(b) *Proof.* If $\deg(h) = 1$, from Question 2, since $h = aX + b$ where $a, b \in K$ with $a \neq 0$, the set $\{ h^i \}_{i \in \mathbb{N}}$ forms a basis of $K[X]$. Evaluating Φ on the standard basis $\{ X^i \}_{i \in \mathbb{N}}$ for $K[X]$ gives that for any $i \in \mathbb{N}$, $\Phi(X^i) = h^i$. Since Φ sends basis to basis, it is an isomorphism.

Conversely suppose $\deg(h) \geqslant 2$, the claim is that $X \notin \mathrm{Im}(\Phi)$. Consider the degree of the polynomial (point) we evaluate Φ at, for any $f \in K[X]$,

- Case $\deg(f) = -\infty$, $\Phi(f) = 0$, and $\deg(\Phi(f)) = -\infty$,
- Case $\deg(f) = 0$, $\Phi(f) = f$ is degree 0,
- Case $\deg(f) \geqslant 1$, $\Phi(f) = f(h)$ has degree $\deg(f)\deg(h) \geqslant 2$.

This means that no degree 1 polynomial lies in $\mathrm{Im}(\Phi)$, therefore Φ is not an isomorphism. ☐

Question 4

> Let K be a field of characteristic 0. Consider the linear endormorphism S of $K[X]$ given by
>
> $$S : K[X] \to K[X], \qquad \sum_{n=0}^{d} a_n X^n \mapsto \sum_{n=0}^{d} \frac{a_n}{n+1} X^{n+1}.$$
>
> Let $V \subseteq K[X]$ be a non-zero subspace which is stable under S. Show that V is not finite-dimensional.

Proof. Suppose for a contradiction that $V \subseteq K[X]$ is non-zero, stable under S and is finite-dimensional, then V has a finite basis $\mathcal{B}$. Note that since V is not the zero subspace, $\mathcal{B}$ is non-empty. Consider $\deg(\mathcal{B}) \subseteq \mathbb{N}$, a finite and non-empty subset of natural numbers. Let $\omega \in \deg(\mathcal{B})$ be the largest element, that is, for any $d \in \deg(\mathcal{B})$, $d \leqslant \omega$. This means that there exists $z \in \mathcal{B}$ such that $\deg(z) = \omega$, and for any $b \in \mathcal{B}$, $\deg(b) \leqslant \deg(z)$.

As linear combination of polynomials do not increase the degree, for any $v \in \operatorname{span}(\mathcal{B}) = V$, $\deg(v) \leqslant \omega$. But now, consider $S(z)$. Let $z_0, \ldots, z_\omega \in K$ with $z_\omega \neq 0$ such that $z = \sum_{i=0}^{\omega} z_i X^i$, then

$$\begin{aligned} S(z) &= S\left(\sum_{i=0}^{\omega} z_i X^i\right) \\ &= \sum_{i=0}^{\omega} \frac{z_i}{i+1} X^{i+1} \end{aligned}$$

which has degree $\omega + 1$, as $\frac{z_\omega}{\omega+1} \neq 0$. Then from our earlier conclusion that any $v \in V$ has degree less than or equal to ω, we have $z \in V$, but $S(z) \notin V$, which contradicts fact that V is stable under S. □

Question 5

> Let K be a field of characteristic 0. Consider the linear endomorphism D of $K[X]$ given by
>
> $$D : K[X] \to K[X], \qquad \sum_{n=0}^{d} a_n X^n \mapsto \sum_{n=1}^{d} n a_n X^{n-1}.$$
>
> Let $V \subseteq K[X]$ be a finite dimensional subspace. Show that D is nilpotent on V, i.e. there exists $m \in \mathbb{N}$ such that for any $f \in V$, one has $D^m(f) = 0$.

Claim. For any nonzero $f \in K[X]$, $D^{\deg(f)+1}(f) = 0$.

Proof (of claim). Proceed by induction on $\deg(f)$, case $\deg(f) = 0$, it is clear that $D^1(0) = 0$. (There are no terms in a sum from 1 to 0.) Suppose for any $g \in K[X]$ with $\deg(g) = n-1$, $D^n(g) = 0$.

Consider $f \in K[X]$ with $\deg(f) = n$, so $f_0, \ldots, f_n \in K$ with $f_n \neq 0$ such that $f = \sum_{i=0}^{n} f_i X_i$, then by induction hypothesis,

$$\begin{aligned} D^{n+1}(f) &= D^n\left(D(f)\right) \\ &= D^n\left(D\left(\sum_{i=0}^{n} f_i X^i\right)\right) \\ &= D^n\left(\sum_{i=1}^{n} i f_i X^{i-1}\right) \\ &= 0 \end{aligned}$$

Therefore by induction, for any nonzero $f \in K[X]$, $D^{\deg(f)+1}(f) = 0$. □
An immediate corollary is that for any $f \in K[X]$, for any $m \in \mathbb{N}$, where $m > \deg(f)$, $D^m(f) = 0$.

Proof (of Q5). V is finite dimensional, so V has a finite basis $\mathcal{B}$. In the case that V is the zero subspace, $D(0) = 0$ so D is nilpotent. For cases where V is a non-zero subspace of $K[X]$, $\mathcal{B}$ is non-empty. Consider $\deg(\mathcal{B}) \subseteq \mathbb{N}$, which is a finite and non-empty subset of natural numbers. It has the largest element ω, where for any $d \in \deg(\mathcal{B})$, $d \leqslant \omega$. This means that there exists $z \in \mathcal{B}$ such that $\deg(z) = \omega$, and for any $b \in \mathcal{B}$, $\deg(b) \leqslant \deg(z)$.

As linear combination of polynomials do not increase the degree, for any $v \in \operatorname{span}(\mathcal{B}) = V$, $\deg(v) \leqslant \omega$. For $0 \in V$, $D^{\omega+1}(0) = 0$ is trivial. For any nonzero $v \in V$, as $\omega + 1 > \deg(v)$, by claim, $D^{\omega+1}(v) = 0$. Therefore D is nilpotent. □

Question 6

> Let K be a field. For each $t \in K$, "evaluation at t" gives a linear functional $\mathrm{eval}_t \in K[X]^\vee$ on the K-vector space $K[X]$:
> $$\mathrm{eval}_t : K[X] \to K, \qquad f \mapsto f(t),$$
> which has the property that for any $f, g \in K[X]$, one has
> $$\mathrm{eval}(fg) = \mathrm{eval}(f)\,\mathrm{eval}(g) \qquad \text{in } K.$$
> Show that for any linear functional $\varphi \in K[X]^\vee$ with property that for any $f, g \in K[X]$, one has
> $$\varphi(fg) = \varphi(f)\varphi(g) \qquad \text{in } K,$$
> then either $\varphi = 0$ in $K[X]^\vee$ or there exists $t \in K$ such that $\varphi = \mathrm{eval}_t$ in $K[X]^\vee$.

Proof. Let $\varphi \in K[X]^\vee$ be any multiplicative linear functional. By multiplicative property, $\varphi(1) = \varphi(1) \cdot \varphi(1)$, then $\varphi(1) = 0$ or $\varphi(1) = 1$.

Case $\varphi(1) = 0$, then for any $f \in K[X]$, $\varphi(f) = \varphi(1 \cdot f) = \varphi(1) \cdot \varphi(f) = 0$, so φ is the zero functional.

Case φ nonzero and $\varphi(1) = 1$, for any $f \in K[X]$, let $f_0, \ldots, f_d \in K$ such that $f = \sum_{i=0}^d f_i X^i$, where $d = \deg(f)$, then by linearity and multiplicative property,

$$\begin{aligned}
\varphi(f) &= \varphi\left(\sum_{i=0}^d f_i X^i\right) \\
&= f_0 \varphi(1) + \sum_{i=1}^d f_i \varphi(X^i) \\
&= f_0 + \sum_{i=1}^d f_i \varphi(X)^i
\end{aligned}$$

define $t := \varphi(X) \in K$, then

$$\begin{aligned}
\mathrm{eval}_t(f) &= f(t) \\
&= \sum_{i=0}^d f_i t^i \\
&= f_0 + \sum_{i=1}^d f_i t^i
\end{aligned}$$

Since f was arbitrary, we see that by setting $t = \varphi(X) \in K$, $\varphi = \mathrm{eval}_t$. □