

13 Axiom of Infinity and Natural Numbers

Axiom 13.1 (Zermelo's Axiom of Infinity). There exists a set X such that $\emptyset \in X$ and $\forall x \in X. \{x\} \in X$.

Definition 13.2. A set X is inductive iff $\emptyset \in X$ and $\forall x \in X. \{x\} \in X$. (*Axiom of Infinity specifies the existence of a inductive set.*)

Definition 13.3. A sequential system consists of:

- a set X
- an element $x_0 \in X$
- a map $T : X \rightarrow X$. In set X : $x_0 \rightarrow T(x_0) \rightarrow T(T(x_0)) \rightarrow \dots$

Axiom 13.4 (Peano Axioms). A system of natural numbers is a sequential system:

1. a set \mathbb{N}
2. an element $0 \in \mathbb{N}$
3. a map $S : \mathbb{N} \rightarrow \mathbb{N}$ (*succ function*)
satisfying:
 - (i) $\forall n \in \mathbb{N}. 0 \neq S(n)$ (*0 is not a succ to any \mathbb{N}*)
 - (ii) S is injective, $\forall n, m \in \mathbb{N}. n \neq m \implies S(n) \neq S(m)$.
 - (iii) for any subset $M \subseteq \mathbb{N}$ (*induction property*)
if M has the properties
 - $0 \in M$
 - $\forall n \in M. S(n) \in M$
 then $M = \mathbb{N}$.

Lemma 13.5. If \mathcal{C} is any non-empty collection of inductive sets, then $\bigcap \mathcal{C}$ is also a inductive set.

Reminder: Inductive set X means $\emptyset \in X$ and $\forall x \in X. \{x\} \in X$. (*Definition 13.2*)

Proof.

1. $\forall X \in \mathcal{C}. X$ is inductive.
2. $\forall X \in \mathcal{C}. \emptyset \in X$.
3. Take an arbitrary $x \in \bigcap \mathcal{C}$, then $\forall X \in \mathcal{C}. x \in X$
4. X is inductive, so $x \in X \implies \{x\} \in X$.
5. then $\forall X \in \mathcal{C}. \{x\} \in X$, so $\{x\} \in \bigcap \mathcal{C}$.
6. $\bigcap \mathcal{C}$ is inductive is shown. □

Definition 13.6. \mathbb{N} is the intersection of all subsets from A which are inductive.

1. Take the inductive set A given by Axiom of Infinity
2. Let $\mathcal{C} := \{ X \in \mathcal{P}(A) : X \text{ is inductive} \}$. \mathcal{C} consists of all subsets of A which are inductive.
3. Since A itself is inductive, and $A \in \mathcal{P}(A)$, so $A \in \mathcal{C}$.
4. Hence \mathcal{C} is non-empty.
5. By Lemma 13.5, define $\mathbb{N} := \bigcap \mathcal{C}$ and \mathbb{N} is an inductive set. (satisfies Axiom 13.4.1)

Lemma 13.7. For any inductive set X , one has $\mathbb{N} \subseteq X$.

Proof.

1. X and A are inductive sets, by Lemma 13.5, $\bigcap \{ X, A \}$ is an inductive set

$$\begin{aligned} X \cap A &\subseteq A \\ X \cap A &\in \mathcal{P}(A) \\ X \cap A &\in \mathcal{C} \end{aligned}$$

2. So \mathbb{N} , being $\bigcap \mathcal{C}$, is the subset of any element in \mathcal{C} , so $\mathbb{N} \subseteq X \cap A$.
(by $\forall F \in \mathcal{F}. a \in \bigcap \mathcal{F} \implies a \in F$) □

Lemma 13.8. \mathbb{N} is the unique inductive set such that \forall inductive set X , one has $\mathbb{N} \subseteq X$.

Proof.

1. \mathbb{N} is inductive. (by Definition 13.6)
2. for all inductive set X , $\mathbb{N} \subseteq X$. (by Lemma 13.7)
3. Take a competitor set \mathbb{N}' is also inductive(1') and for all inductive set X , $\mathbb{N}' \subseteq X$ (2').
4. Apply (1) to (2'), for inductive set $\mathbb{N}, \mathbb{N}' \subseteq \mathbb{N}$.
5. Apply (1') to (2), for inductive set $\mathbb{N}', \mathbb{N} \subseteq \mathbb{N}'$.
6. Any set with properties (1) and (2) $\mathbb{N}' = \mathbb{N}$, uniqueness proven. □

Definition 13.9. 0 and succ function for \mathbb{N}

1. $0 := \emptyset \in \mathbb{N}$ ($\because \mathbb{N}$ is inductive, by Definition 13.2)
(satisfies Axiom 13.4.2)
2. $S : \mathbb{N} \rightarrow \mathbb{N}$ is defined as $\forall x \in \mathbb{N}. S(x) := \{ x \}$.

- S is defined for all $x \in \mathbb{N}$, S is totally-defined.
- $S(x) = \{x\} \neq x$, S is well-defined.
- Given $x \in \mathbb{N}$, $\{x\} \in \mathbb{N}$. ($\because \mathbb{N}$ is inductive, by Definition 13.2)

Theorem 13.10. *The sequential system \mathbb{N} and S we defined satisfies property (i), (ii), (iii) in Axiom 13.4.3.*

Property (i): $\forall n \in \mathbb{N}. 0 \neq S(n)$

Proof.

1. take an arbitrary $n \in \mathbb{N}$, then $S(n) = \{n\}$
2. $0 = \emptyset$ by definition, and for all n , $n \notin \emptyset$,
3. so $\emptyset \neq \{n\}$ □

Property (ii): S is injective, $\forall m, n \in \mathbb{N}. S(m) = S(n) \implies m = n$.

Proof.

1. take $m, n \in \mathbb{N}$, if $S(m) = S(n)$, then
2. $\{m\} = \{n\}$
3. by Axiom of Extensionality: $m \in \{m\} \implies m \in \{n\}$, so $m = n$
4. S is injective. □

Property (iii): For any subset $M \subseteq \mathbb{N}$, if

- $0 \in M$
- $\forall n \in M. S(n) \in M$

then $M = \mathbb{N}$.

Proof.

1. Let $M \subseteq \mathbb{N}$,
2. Then M is an inductive set (*by properties above*).
3. Then by Lemma 13.8, $\mathbb{N} \subseteq M$.
4. Assumed $M \subseteq \mathbb{N}$, therefore $M = \mathbb{N}$. □

Conclusion. *The sequential system \mathbb{N} and successor function S we defined above satisfies Axiom 13.4.*

14 Axiom of Infinity

Principle of Induction. Suppose $P(-)$ is a statement about natural numbers. $\forall n \in \mathbb{N}$, $P(n)$ is a proposition with truth value.

By axiom of specification, define $M := \{n \in \mathbb{N} : P(n) \text{ is true}\}$. Suppose we show

- (1) Base case: $P(0)$ is true
- (2) Induction step: $\forall k \in \mathbb{N}, P(k) \implies P(k+1)$

Then we know $0 \in M$ and $\forall k \in \mathbb{N}, k \in M \implies S(k) \in M$. Then by property (iii) of Peano Axiom 13.4.3, $M = \mathbb{N}$. *(induction property)*

Definition 14.1. If $f : A \rightarrow B$ is a map, then the f -image of A (or the range of f) is

$$f(A) := \{b \in B : \exists a \in A. b = f(a)\}$$

Example: $S(\mathbb{N}) = \{n \in \mathbb{N} : \exists k \in \mathbb{N}. n = S(k)\}$

Lemma 14.2. $S(\mathbb{N}) = \mathbb{N} \setminus \{0\}$

Proof.

1. Let $P(n) := (n = 0) \vee (\exists k \in \mathbb{N}. n = S(k))$.
2. $P(0)$ is trivially true.
3. Suppose $n \in \mathbb{N}$ such that $P(n)$ is true, either $n = 0$ or $\exists k \in \mathbb{N}. n = S(k)$
 - case $n = 0$, then $S(n) = S(0)$ which is $\in S(\mathbb{N})$
 - case $\exists k \in \mathbb{N}. n = S(k)$, then $S(n) = S(S(k))$ which is $\in S(\mathbb{N})$
4. so $P(S(n))$ is true.
5. By Principle of Induction, $\forall n \in \mathbb{N}. P(n)$ is true.

n is either 0 or a successor of some $k \in \mathbb{N}$.

□

Theorem 14.3 (Recursion Theorem (universal property of \mathbb{N})). *Let (X, x_0, T) be any sequential system where*

- X is a set.
- $x_0 \in X$ is a given element.
- $T : X \rightarrow X$ is a map.

Then there exists a unique map

$$\varphi : \mathbb{N} \rightarrow X$$

such that

1. $\varphi(0) = x_0 \in X$
2. *The diagram commutes*

$$\begin{array}{ccc} \mathbb{N} & \xrightarrow{\varphi} & X \\ S \downarrow & & \downarrow T \\ \mathbb{N} & \xrightarrow{\varphi} & X \end{array}$$

$$\begin{aligned} \text{ie. } T \circ \varphi &= \varphi \circ S : \mathbb{N} \rightarrow X, \\ \forall n \in \mathbb{N}. T(\varphi(n)) &= \varphi(S(n)) \end{aligned}$$

Intuitively:

$$\begin{array}{ccccccc} \mathbb{N} : & 0 & \xrightarrow{S} & 1 & \xrightarrow{S} & 2 & \xrightarrow{S} & 3 & \xrightarrow{S} & \dots \\ \varphi \downarrow & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ T : & x_0 & \xrightarrow{T} & T(x_0) & \xrightarrow{T} & T^2(x_0) & \xrightarrow{T} & T^3(x_0) & \xrightarrow{T} & \dots \end{array}$$

Proof. later

Consequence of Recursion Theorem

Theorem 14.4 (Uniqueness of Natural Number System). *Let $(\mathbb{N}, 0, S)$ be our natural number system. Suppose $(\mathbb{N}', 0', S')$ is another natural number system satisfying Peano Axioms 13.4. Then there exists maps*

$$\varphi : \mathbb{N} \rightarrow \mathbb{N}' \text{ and } \varphi' : \mathbb{N}' \rightarrow \mathbb{N}$$

such that

- (i) $\varphi(0) = 0'$ and $\varphi'(0') = 0$.
- (ii) *this diagram commutes,*

$$\begin{array}{ccc} \mathbb{N} & \xrightarrow{\varphi} & \mathbb{N}' \\ S \downarrow & & \downarrow S' \\ \mathbb{N} & \xrightarrow{\varphi} & \mathbb{N}' \end{array} \quad \begin{array}{ccc} \mathbb{N}' & \xrightarrow{\varphi'} & \mathbb{N} \\ S' \downarrow & & \downarrow S \\ \mathbb{N}' & \xrightarrow{\varphi'} & \mathbb{N} \end{array}$$

- (iii) $\varphi' \circ \varphi = id_{\mathbb{N}}$ and $\varphi \circ \varphi' = id_{\mathbb{N}'}$.

Concretely:

$$\begin{array}{ccccccc} \mathbb{N} : & 0 & \xrightarrow{S} & 1 & \xrightarrow{S} & 2 & \xrightarrow{S} & 3 & \xrightarrow{S} & \dots \\ \varphi \updownarrow \varphi' & \updownarrow & & \updownarrow & & \updownarrow & & \updownarrow & & \\ \mathbb{N}' : & 0' & \xrightarrow{S'} & 1' & \xrightarrow{S'} & 2' & \xrightarrow{S'} & 3' & \xrightarrow{S'} & \dots \end{array}$$

Proof.

1. We have our natural number system $(\mathbb{N}, 0, S)$.
2. Given sequential system $(\mathbb{N}', 0', S')$, by recursion theorem, there exists a map

$$\varphi : \mathbb{N} \rightarrow \mathbb{N}'$$

such that

- (i) $\varphi(0) = 0'$, and
- (ii) this diagram commutes

$$\begin{array}{ccc} \mathbb{N} & \xrightarrow{\varphi} & \mathbb{N}' \\ S \downarrow & & \downarrow S' \\ \mathbb{N} & \xrightarrow{\varphi} & \mathbb{N}' \end{array}$$

3. Now we have natural number system $(\mathbb{N}', 0', S')$,
4. Given sequential system $(\mathbb{N}, 0, S)$, by recursion theorem, there exists a map

$$\varphi' : \mathbb{N}' \rightarrow \mathbb{N}$$

such that

- (i) $\varphi'(0') = 0$, and
- (ii) this diagram commutes

$$\begin{array}{ccc} \mathbb{N}' & \xrightarrow{\varphi'} & \mathbb{N} \\ S' \downarrow & & \downarrow S \\ \mathbb{N}' & \xrightarrow{\varphi'} & \mathbb{N} \end{array}$$

5. for $\varphi' \circ \varphi : \mathbb{N} \rightarrow \mathbb{N}$,
 - note $(\varphi' \circ \varphi)(0) = \varphi'(\varphi(0)) = \varphi'(0') = 0$
 - and this commutes

$$\begin{array}{ccccc} \mathbb{N} & \xrightarrow{\varphi} & \mathbb{N}' & \xrightarrow{\varphi'} & \mathbb{N} \\ S \downarrow & & \downarrow S' & & \downarrow S \\ \mathbb{N} & \xrightarrow{\varphi} & \mathbb{N}' & \xrightarrow{\varphi'} & \mathbb{N} \end{array}$$

$$\text{ie. } S \circ (\varphi' \circ \varphi) = (\varphi' \circ \varphi) \circ S$$

6. But $\text{id}_{\mathbb{N}} : \mathbb{N} \rightarrow \mathbb{N}$ also enjoys properties
 - $\text{id}_{\mathbb{N}}(0) = 0$
 - $S \circ \text{id}_{\mathbb{N}} = \text{id}_{\mathbb{N}} \circ S$
7. By applying Recursion Theorem of natural number system $(\mathbb{N}, 0, S)$ to the sequential system $(\mathbb{N}, 0, S)$ (itself), there exists a unique map

$$f : \mathbb{N} \rightarrow \mathbb{N}$$

such that

- $f(0) = 0$, and
- this diagram commutes

$$\begin{array}{ccc} \mathbb{N} & \xrightarrow{f} & \mathbb{N} \\ S \downarrow & & \downarrow S \\ \mathbb{N} & \xrightarrow{f} & \mathbb{N} \end{array}$$

8. We just showed that $\text{id}_{\mathbb{N}}$ is unique and has the same properties as $\varphi' \circ \varphi$, so $\varphi' \circ \varphi = \text{id}_{\mathbb{N}}$.
9. Repeating from (5.), symmetrically, $\varphi \circ \varphi' = \text{id}_{\mathbb{N}'}$ □