

## Matrices

$\mathbf{A} = (a_{ij})_{m \times p}$  and  $\mathbf{B} = (b_{ij})_{p \times n}$ , then

$$\mathbf{AB} = (ab_{ij})_{m \times n} = \sum_{k=1}^p a_{ik}b_{kj}.$$

Matrix Multiplication is associative and distributive (left and right) over addition.

For square matrix  $\mathbf{A}$ ,  $\mathbf{A}^m \mathbf{A}^n = \mathbf{A}^{(m+n)}$

(if  $\mathbf{A}$  invertible, also works for negative  $m, n$ )

$$\mathbf{A}^{\top\top} = \mathbf{A}$$

$$(\mathbf{A} + \mathbf{B})^{\top} = \mathbf{A}^{\top} + \mathbf{B}^{\top}$$

$$(\mathbf{AB})^{\top} = \mathbf{B}^{\top} \mathbf{A}^{\top}$$

for invertible  $\mathbf{A}, \mathbf{B}$ :

$$\mathbf{AA}^{-1} = \mathbf{I} = \mathbf{A}^{-1} \mathbf{A}$$

$$(c\mathbf{A})^{-1} = \frac{1}{c} \mathbf{A}^{-1}$$

$$(\mathbf{AB})^{-1} = \mathbf{B}^{-1} \mathbf{A}^{-1}$$

## Elementary Row Operations

$$\mathbf{E}_k \dots \mathbf{E}_2 \mathbf{E}_1 \mathbf{A} = \mathbf{B}$$

$$\mathbf{A} = (\mathbf{E}_k \dots \mathbf{E}_2 \mathbf{E}_1)^{-1} \mathbf{B}$$

$$\mathbf{A} = \mathbf{E}_1^{-1} \mathbf{E}_2^{-1} \dots \mathbf{E}_k^{-1} \mathbf{B}$$

Using row-reduction to find inverse:

$$(\mathbf{A} \mid \mathbf{I}) \xrightarrow{\text{Gauss-Jordan}} (\mathbf{I} \mid \mathbf{A}^{-1})$$

$$\mathbf{A}^{-1} = \mathbf{E}_k \dots \mathbf{E}_2 \mathbf{E}_1 \Rightarrow \mathbf{A} = \mathbf{E}_1^{-1} \mathbf{E}_2^{-1} \dots \mathbf{E}_k^{-1}$$

## Determinants

$$A_{ij} = (-1)^{i+j} \det(\langle \text{cover row } i \text{ col } j \text{ in } \mathbf{A} \rangle)$$

$$\det(\mathbf{E}_{add}) = 1$$

$$\det(\mathbf{E}_{swap}) = -1$$

$$\det(\mathbf{E}_{mult}) = c$$

$$\det(\mathbf{A}_{\Delta}) = a_{11}a_{22} \dots a_{nn}$$

$$\det(\mathbf{AB}) = \det(\mathbf{A}) \det(\mathbf{B})$$

$$\det(c\mathbf{A}) = c^n \det(\mathbf{A})$$

$$\mathbf{A} \operatorname{adj}(\mathbf{A}) = \det(\mathbf{A}) \mathbf{I} \Rightarrow \mathbf{A}^{-1} = \frac{1}{\det(\mathbf{A})} \operatorname{adj}(\mathbf{A})$$

$$\operatorname{adj}(\mathbf{A}) = \begin{bmatrix} A_{11} & A_{21} & \dots & A_{n1} \\ A_{12} & A_{22} & \dots & A_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ A_{1n} & A_{2n} & \dots & A_{nn} \end{bmatrix}$$

Cramer's rule: for  $\mathbf{Ax} = \mathbf{b}$ ,

$$x_n = \frac{\det(\langle \mathbf{A} : \text{replace } n\text{-th col with } \mathbf{b} \rangle)}{\det(\mathbf{A})}$$

## Spaces

Space notation:

- $\{(a, a - b, 2b + c) \mid a, b, c \in \mathbb{R}\}$  (explicit)
- $\{(x, y, z) \mid x + y + z = 0\}$  (implicit)

## Spans & Containment

Take  $U := \{\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_k\}$

$\operatorname{span}(U)$  = set of all linear combinations of  $U$

$\begin{bmatrix} \mathbf{u}_1 & \mathbf{u}_2 & \dots & \mathbf{u}_k & \mathbf{v} \end{bmatrix}$  consistent  $\Rightarrow \mathbf{v} \in \operatorname{span}(U)$ .

ref  $\begin{bmatrix} \mathbf{u}_1 & \dots & \mathbf{u}_k \end{bmatrix}$  no zero-row  $\Rightarrow \operatorname{span}(U) = \mathbb{R}^k$ .

Take  $V := \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k\}$

each  $\mathbf{u}_i \in \operatorname{span}(V) \iff \operatorname{span}(U) \subseteq \operatorname{span}(V)$

## Subspaces, Linear Independence

Definition of subspace  $V$ :  $\mathbf{0} \in V$ , and

$\forall \mathbf{u}, \mathbf{v} \in V. \forall c, d \in \mathbb{R}. c\mathbf{u} + d\mathbf{v} \in V$

$U$  is LI means only trivial solution for

$$c_1 \mathbf{u}_1 + c_2 \mathbf{u}_2 + \dots + c_k \mathbf{u}_k = \mathbf{0}.$$

## Basis and Coordinate systems

A set of vectors  $S$  is a basis for vector space  $V$  iff

- $S$  is linearly independent
- $S$  spans  $V$

Given basis  $S$ , and  $\mathbf{v} \in V$

$$\mathbf{v} = c_1 \mathbf{s}_1 + c_2 \mathbf{s}_2 + \dots + c_k \mathbf{s}_k$$

$$(\mathbf{v})_S = (c_1, c_2, \dots, c_k) \in \mathbb{R}^k$$

## Coordinate Systems

$\forall \mathbf{u}, \mathbf{v} \in V. \mathbf{u} = \mathbf{v} \iff (\mathbf{u})_S = (\mathbf{v})_S$  (uniq.)

$\forall \mathbf{u}, \mathbf{v} \in V. c, d \in \mathbb{R}. (c\mathbf{u} + d\mathbf{v})_S = c(\mathbf{u})_S + d(\mathbf{v})_S$

Let  $k = |S|$ .

$\mathbf{v}_1, \mathbf{v}_2, \dots \in V$ , are linear independent  $\iff$

$(\mathbf{v}_1)_S, (\mathbf{v}_2)_S, \dots \in \mathbb{R}^k$  are linear independent.

$\operatorname{span}\{\mathbf{v}_1, \mathbf{v}_2, \dots\} = V \iff$

$\operatorname{span}\{(\mathbf{v}_1)_S, (\mathbf{v}_2)_S, \dots\} = \mathbb{R}^k$ .

## Dimensions

$\dim(V) := |S|$  where  $S$  is a basis for  $V$ , is unique.

- $S$  is linearly independent
- $\operatorname{span}(S) = V$
- $|S| = \dim(V)$

2 of above true  $\Rightarrow$  all true.

## Transition Matrices

$S$  and  $T$  are bases for  $V$ , for  $\mathbf{v} \in V$ .

$$\mathbf{P}_{S,T} = \begin{bmatrix} [\mathbf{s}_1]_T & [\mathbf{s}_2]_T & \dots & [\mathbf{s}_k]_T \end{bmatrix}$$

is transition matrix from basis  $S$  to  $T$ . ie

$$[\mathbf{v}]_T = \mathbf{P}_{S,T} [\mathbf{v}]_S$$

$$[\mathbf{v}]_S = \mathbf{P}_{S,T}^{-1} [\mathbf{v}]_T$$

## Rowsp and Range(Colsp)

Take  $\mathbf{R} := \operatorname{ref}(\mathbf{A})$  is  $m \times n$ .

- Row operations preserve rowsp.
- Rows in  $\mathbf{R}$  form basis for rowsp.
- Pivot columns in  $\mathbf{R}$  correspond to linearly independent columns in  $\mathbf{A}$

**Note:** row operations preserve linear (in)-dependence of columns but could destroy other information like colsp.

Column space of  $\mathbf{A} = \{\mathbf{A}\mathbf{u} \mid \mathbf{u} \in \mathbb{R}^n\}$ .

So  $\mathbf{Ax} = \mathbf{b}$  consistent  $\iff \mathbf{b} \in \operatorname{colsp}(\mathbf{A})$ .

## Rank Nullity

$$\text{rank}(\mathbf{0}) = 0$$

$$\text{rank}(\mathbf{I}_n) = n$$

$$\text{rank}(\mathbf{A}) \leq \min\{m, n\} \quad (\ddagger)$$

$$\text{rank}(\mathbf{AB}) \leq \min\{\text{rank}(\mathbf{A}), \text{rank}(\mathbf{B})\}$$

If equality holds in  $(\ddagger)$ ,  $\mathbf{A}$  has **full rank**.

$$\text{rank}(\mathbf{A}) = \text{rank}(\mathbf{A}^\top)$$

$$\text{rank}(\mathbf{A}) + \text{nullity}(\mathbf{A}) = \text{no. columns}$$

## Kernel/Nullspace

$$\text{null}(\mathbf{A}) := \{\mathbf{u} \in \mathbb{R}^n \mid \mathbf{A}\mathbf{u} = \mathbf{0}\}$$

Suppose  $\mathbf{A}\mathbf{v} = \mathbf{b}$ , then general solution of  $\mathbf{A}\mathbf{x} = \mathbf{b}$

$$\mathbf{x} \in \{\mathbf{u} + \mathbf{v} \mid \mathbf{u} \in \text{null}(\mathbf{A})\}$$

## Vectors

Inner product:  $\mathbf{u} \cdot \mathbf{v} := \mathbf{u}^\top \mathbf{v}$

$$\|\mathbf{u}\| = \sqrt{\mathbf{u} \cdot \mathbf{u}}$$

$$\|c\mathbf{u}\| = |c| \|\mathbf{u}\|$$

$$d(\mathbf{u}, \mathbf{v}) = \|\mathbf{u} - \mathbf{v}\|$$

$$\cos \angle(\mathbf{u}, \mathbf{v}) = \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\| \|\mathbf{v}\|}$$

$$\|\mathbf{u} + \mathbf{v}\| \leq \|\mathbf{u}\| + \|\mathbf{v}\| \quad (\triangle \text{ ineq.})$$

## Orthogonal

$$\mathbf{u} \cdot \mathbf{v} = 0 \iff \mathbf{u} \perp \mathbf{v}$$

orthogonal  $\Rightarrow$  linear independence.

orthonormal := orthogonal  $\wedge$  norm 1.

To check if  $S$  is orthogonal basis for  $V$ :

(i)  $S$  is orthogonal

(ii)  $|S| = \dim(V)$  or  $\text{span}(S) = V$  (ref: dim)

## Projections

Let  $S$  be an orthogonal basis for  $V$ , then  $\forall \mathbf{w} \in \mathbb{R}^n$ ,

$$\mathbf{p} = \frac{\mathbf{w} \cdot \mathbf{s}_1}{\mathbf{s}_1 \cdot \mathbf{s}_1} \mathbf{s}_1 + \frac{\mathbf{w} \cdot \mathbf{s}_2}{\mathbf{s}_2 \cdot \mathbf{s}_2} \mathbf{s}_2 + \cdots + \frac{\mathbf{w} \cdot \mathbf{s}_k}{\mathbf{s}_k \cdot \mathbf{s}_k} \mathbf{s}_k$$

is projection of  $\mathbf{w}$  on  $V$ . (existence of projections)

Case  $\mathbf{w} \in V$ , then  $\mathbf{p} = \mathbf{w}$ . For orthonormal basis, simplify expr as denominator becomes 1.

## Gram-Schmidt Algorithm

Basis  $U := \{\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_k\}$  for  $V$ .

$$\mathbf{v}_1 = \mathbf{u}_1$$

for  $i \in \{2, 3, \dots, k\}$

$$\mathbf{v}_i = \mathbf{u}_i - \sum_{j=1}^{i-1} \frac{\mathbf{u}_i \cdot \mathbf{v}_j}{\mathbf{v}_j \cdot \mathbf{v}_j} \mathbf{v}_j$$

$\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k\}$  form orthogonal basis for  $V$ .

## Orthogonal matrices

iff  $\mathbf{A}^\top = \mathbf{A}^{-1}$ . rows and cols form orthonormal basis for  $\mathbb{R}^n$ .

Note: Transition matrix between two orthonormal bases is orthogonal. So  $\mathbf{P}_{T,S} = (\mathbf{P}_{S,T})^{-1} = (\mathbf{P}_{S,T})^\top$

## Eigenvalues

$\lambda$  is an eigenvalue of  $A$  iff  $\exists \mathbf{v} \neq \mathbf{0}$ .  $\mathbf{A}\mathbf{v} = \lambda\mathbf{v}$ .

Characteristic polynomial is  $\det(\mathbf{A} - \lambda\mathbf{I}) = 0$

Eigenspace  $E_\lambda := \text{nullspace of } (\mathbf{A} - \lambda\mathbf{I})$ .

To diagonalise  $n \times n$  matrix  $\mathbf{A}$ ,

1. Find all distinct  $\lambda_1, \lambda_2, \dots, \lambda_k$ ,
2. For each  $\lambda_i$ , find basis  $S_{\lambda_i}$  for eigenspace  $E_{\lambda_i}$  (If  $|S_{\lambda_i}| < p_i$  where  $p_i$  is power of  $(\lambda - \lambda_i)$  in polynomial, then not diagonalisable, abort.)
3. Set  $S = \bigcup_{i \in \{1, \dots, k\}} S_{\lambda_i} = \{\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_n\}$  (union the eigenbases).  
 $\mathbf{A}$  is diagonalisable if  $|S| = n$ .

$$\mathbf{P} = [\mathbf{u}_1 \quad \mathbf{u}_2 \quad \dots \quad \mathbf{u}_n] \text{ st. } \mathbf{A} = \mathbf{P}\mathbf{D}\mathbf{P}^{-1}.$$

For orthogonally diagonalisable (symmetric) matrix  $\mathbf{A}$ , Gram-Schmidt and normalise the bases in step 2, then  $\mathbf{A} = \mathbf{P}\mathbf{D}\mathbf{P}^\top$ .

## Linear Transformations

Suppose  $T: V \rightarrow W$  is a linear transformation and  $S = \{\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_n\}$  is a basis for  $V$ .

If  $T(S)$  is known, and  $\mathbf{P}_{E,S}$  is the transition matrix from basis  $E$  to  $S$ ,

$$\mathbf{B} = [T(\mathbf{u}_1) \quad T(\mathbf{u}_2) \quad \dots \quad T(\mathbf{u}_n)].$$

$$\begin{array}{ccc} V & \xrightarrow{\mathbf{P}_{E,S}} & [V]_S \\ \downarrow T & \swarrow \mathbf{B} & \\ W & & \end{array}$$

(generalisable).

## Corner case matrices

Standard non-diagonalisable matrix:  $\begin{bmatrix} a & 1 \\ 0 & a \end{bmatrix}$ .

$$\mathbf{A} = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}; \mathbf{B} = \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix} \text{ then}$$

$$\mathbf{AB} = \mathbf{0} \text{ but } \mathbf{BA} \neq \mathbf{0}$$

$$\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix}; \mathbf{B} = \begin{bmatrix} 1 & 3 \\ 0 & -1 \end{bmatrix} \text{ then}$$

$$\mathbf{A}^2 = \mathbf{B}^2 = \mathbf{I} \text{ but } \mathbf{A} \neq \mathbf{B}, (\mathbf{AB})^2 \neq \mathbf{I}$$

$$\mathbf{A} = \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix}; \mathbf{B} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \text{ then}$$

$$(\mathbf{AB})^k \neq \mathbf{A}^k \mathbf{B}^k, (\mathbf{AB})^\top \neq \mathbf{A}^\top \mathbf{B}^\top, (\mathbf{AB})^{-1} \neq \mathbf{A}^{-1} \mathbf{B}^{-1}$$

## Theorem. Invertible Square Matrix

For any square matrix  $\mathbf{A}$ :

$\mathbf{A}$  is invertible,

$\iff \mathbf{A}\mathbf{x} = \mathbf{0}$  only has trivial solution,

$\iff \text{rref}(\mathbf{A}) = \mathbf{I}$ ,

$\iff \mathbf{A} = \mathbf{E}_k \dots \mathbf{E}_2 \mathbf{E}_1$ ,

$\iff \det(\mathbf{A}) \neq 0$ ,

$\iff$  rows/cols in  $\mathbf{A}$  form basis for  $\mathbb{R}^n$ ,

$\iff \mathbf{A}$  has full rank,

$\iff 0$  is not an eigenvalue of  $\mathbf{A}$ .