MA2202S Homework 4

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i. We already have $|S| = m$, just need to show $|gS| = m$. From definition of gS, we see that it is the image of $(g*)(S)$ the left-multiply by g map under S. Since the left-multiply map $(g*)$ is injective we have $|gS| = |S| = m$. $\square$

ii. Let $S \in X$, we can verify that $\pi'(e, S) = eS = S$. Now let $g, h \in G$, $\pi'(g, \pi'(h, S)) = \pi'(g, hS) = ghS$. On the other hand $\pi'(gh, S) = ghS$. $\square$

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i. Since x_i and x_j are in the same orbit, we have a $g \in G$ such that

$$x_i = \pi(g)x_j.$$

Suppose $z \in G_{x_i}$ such that $\pi(z)x_i = x_i$, then we see that

$$\pi(g^{-1}zg)x_j = \pi(g^{-1}z)x_i = \pi(g^{-1})x_i = x_j$$

so $g^{-1}zg \in G_{x_j}$. We see that $z \mapsto g^{-1}zg$ defines a map $G_{x_i} \to G_{x_j}$. This map of conjugation is bijective, as a symmetric argument shows that $z' \mapsto gz'g^{-1}$ defines a map $G_{x_j} \to G_{x_i}$, which is its inverse. $\square$

ii. By part (i) and proposition 79,

$$\sum_{i=1}^{r} \left|G_{x_i}\right| = r\left|G_{x_1}\right| = |Gx_1|\left|G_{x_1}\right| = |G|. \qquad \square$$

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i. From definition of the matrix A, we have

$$\sum_{i=1}^{n} a_{ij} = \left|\left\{ g_i \in G : \pi(g_i)x_j = x_j \right\}\right| = \left|G_{x_j}\right|. \qquad \square$$

ii. Also from definition of matrix A, we have

$$\sum_{j=1}^{m} a_{ij} = \left|\left\{ x_j \in X : \pi(g_i)x_j = x_j \right\}\right| = |F(g_i)|. \qquad \square$$

iii. By parts (i) and (ii),

$$\begin{aligned}\sum_{j=1}^{m}\left|G_{x_j}\right| &= \sum_{j=1}^{m}\sum_{i=1}^{n} a_{ij} \\ &= \sum_{i=1}^{n}\sum_{j=1}^{m} a_{ij} \\ &= \sum_{i=1}^{n}\left|F(g_i)\right|. \end{aligned}\qquad\square$$

iv. By part (ii) of previous question,

$$\sum_{j=1}^{m}\left|G_{x_j}\right| = |G| \cdot |\{\, Gx : x \in X \,\}|.$$

By part (iii) we have the number of G-orbits being

$$\frac{1}{|G|}\sum_{j=1}^{m}\left|G_{x_j}\right| = \frac{1}{|G|}\sum_{i=1}^{n}\left|F(g_i)\right|. \qquad\square$$