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## **MA2101S Homework 5**

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12th March 2018

## Question 1

For any  $n \in \mathbb{N}$ ,  $p_n(X) := nX^{n+1} - (n+1)X^n + 1 \in \mathbb{Q}[X]$ . Show that there exists  $q_n \in \mathbb{Q}[X]$  such that  $p_n(X) = (X-1)^2 q_n(X)$ .

*Proof.* Consider  $q_n(X) := \sum_{i=0}^{n-1} (i+1)X^i \in \mathbb{Q}[X]$ . Now compute  $(X-1)^2 q_n(X)$ ,

$$\begin{aligned}
 (X-1)^2 q_n(X) &= (X^2 - 2X + 1)q_n(X) \\
 &= X^2 q_n(X) - 2X q_n(X) + q_n(X) \\
 &= X^2 \sum_{i=0}^{n-1} (i+1)X^i - 2X \sum_{i=0}^{n-1} (i+1)X^i + \sum_{i=0}^{n-1} (i+1)X^i \\
 &= \sum_{i=0}^{n-1} (i+1)X^{i+2} - \sum_{i=0}^{n-1} 2(i+1)X^{i+1} + \sum_{i=0}^{n-1} (i+1)X^i \\
 &= \sum_{i=2}^{n+1} (i-1)X^i - \sum_{i=1}^n 2iX^i + \sum_{i=0}^{n-1} (i+1)X^i \\
 &= \sum_{i=1}^{n+1} (i-1)X^i - \sum_{i=1}^n 2iX^i + \sum_{i=0}^{n-1} (i+1)X^i \\
 &= \sum_{i=1}^{n-1} (i-1)X^i - \sum_{i=1}^n 2iX^i + \sum_{i=0}^{n-1} (i+1)X^i + nX^{n+1} + (n-1)X^n \\
 &= \sum_{i=1}^{n-1} (i-1)X^i - \sum_{i=1}^{n-1} 2iX^i + \sum_{i=0}^{n-1} (i+1)X^i + nX^{n+1} - (n+1)X^n \\
 &= \sum_{i=1}^{n-1} [(i-1)X^i - 2iX^i + (i+1)X^i] + nX^{n+1} - (n+1)X^n + 1 \\
 &= \sum_{i=1}^{n-1} 0 + nX^{n+1} - (n+1)X^n + 1 \\
 &= nX^{n+1} - (n+1)X^n + 1 = p_n
 \end{aligned}$$

Therefore  $p_n(X)$  is divisible by  $(X-1)^2$ . □

## Question 2

Let  $K$  be a field, and let  $a, b \in K$  with  $a \neq 0$ . Show that  $(aX + b)^0, (aX + b)^1, (aX + b)^2, \dots$  form a basis for  $K[X]$ .

**Linear independence.** *Proof.* Consider any finite subset of naturals  $S \subseteq \mathbb{N}$ . The claim is that  $\{(aX + b)^s\}_{s \in S}$  – an arbitrary finite subset of  $\{(aX + b)^i\}_{i \in \mathbb{N}}$ , is linearly independent. To prove linear independence, proceed by induction on  $|S|$ .

**Base cases.** If  $|S| = 0$  or  $|S| = 1$ , linear independence is trivial.

**Induction hypothesis.** Suppose for any  $T \subseteq \mathbb{N}$  with  $|T| = n - 1$ ,  $\{(aX + b)^t\}_{t \in T}$  is linearly independent.

Now consider  $S \subseteq \mathbb{N}$  with  $|S| = n$ . Let  $\omega \in S$  be the largest element in  $S$ , that is for any  $s \in S$ ,  $s \leq \omega$ . Because  $S$  is finite and non-empty,  $\omega$  actually exists. Consider this equation,

$$\sum_{s \in S} c_s (aX + b)^s = 0 \quad \text{in } \mathbb{Q}[X]$$

where  $(c_s)_{s \in S} \in K$  are coefficients indexed by  $S$ . Comparing the coefficient of  $X^\omega$ ,  $c_\omega a^\omega = 0$ , then because  $a^\omega \neq 0$ ,  $c_\omega = 0$ . Then the equation reduces to,

$$\sum_{s \in S \setminus \{\omega\}} c_s (aX + b)^s = 0 \quad \text{in } \mathbb{Q}[X]$$

then from induction hypothesis, because  $|S \setminus \{\omega\}| = n - 1$ , using linear independence, all the coefficients  $(c_s)_{s \in S \setminus \{\omega\}}$  are zero, together with our earlier conclusion that  $c_\omega = 0$ , completes the proof that  $\{(aX + b)^s\}_{s \in S}$  is linearly independent.

Hence any finite subset of  $\{(aX + b)^0, (aX + b)^1, (aX + b)^2, \dots\}$  is linearly independent.  $\square$

**Spanning.** *Proof.* To show that  $\{(aX + b)^i\}_{i \in \mathbb{N}}$  spans  $K[X]$ , proceed by induction on the degree of the polynomial that lies in  $K[X]$ .

**Base cases.** Trivial to see that zero polynomial is spanned. Since  $(aX + b)^0 = 1$ , all degree 0 polynomials are spanned too.

**Induction hypothesis.** Suppose any polynomial of degree strictly less than  $n$  is spanned by  $\{(aX + b)^i\}_{i \in \mathbb{N}}$ .

Let  $f \in K[X]$  with  $\deg(f) = n$ , so  $f = \sum_{i=0}^n f_i X^i$ , where  $f_0, \dots, f_n \in K$  are coefficients with  $f_n \neq 0$ . From binomial theorem,

$$\begin{aligned} (aX + b)^n &= \sum_{r=0}^n \binom{n}{r} (aX)^r b^{n-r} \\ &= a^n X^n + \sum_{r=0}^{n-1} \binom{n}{r} (aX)^r b^{n-r} \end{aligned}$$

as  $a^n \neq 0$ , proceed to compute  $f - \frac{f_n}{a^n} (aX + b)^n$ ,

$$\begin{aligned} f - \frac{f_n}{a^n} (aX + b)^n &= f_n X^n + \sum_{i=0}^{n-1} f_i X^i - \frac{f_n}{a^n} \left( a^n X^n + \sum_{r=0}^{n-1} \binom{n}{r} (aX)^r b^{n-r} \right) \\ &= f_n X^n + \sum_{i=0}^{n-1} f_i X^i - f_n X^n - \frac{f_n}{a^n} \sum_{r=0}^{n-1} \binom{n}{r} a^r X^r b^{n-r} \\ &= \sum_{r=0}^{n-1} \left( f_r X^r - \frac{f_n}{a^n} \binom{n}{r} a^r b^{n-r} X^r \right) \\ &= \sum_{r=0}^{n-1} \left( f_r - \frac{f_n}{a^n} \binom{n}{r} a^r b^{n-r} \right) X^r \end{aligned}$$

This means  $f - \frac{f_n}{a^n} (aX + b)^n$  is a polynomial with degree at most  $n - 1$ , so by induction hypothesis, it is spanned by  $\{(aX + b)^i\}_{i \in \mathbb{N}}$ . So there exists a finite subset  $S \subseteq \mathbb{N}$ , and coefficients  $(c_s)_{s \in S} \in K$  indexed by  $S$  such that

$$f - \frac{f_n}{a^n} (aX + b)^n = \sum_{s \in S} c_s (aX + b)^s,$$

which gives

$$f = \sum_{s \in S} c_s (aX + b)^s + \frac{f_n}{a^n} (aX + b)^n.$$

By strong induction, any polynomial is spanned by  $\{(aX + b)^i\}_{i \in \mathbb{N}}$ . Therefore  $\{(aX + b)^i\}_{i \in \mathbb{N}}$  forms a basis for  $K[X]$ .  $\square$

### Question 3

Let  $K$  be a field, and let  $h \in K[X]$  be a polynomial with  $\deg(h) \geq 1$ . Consider the linear endomorphism  $\Phi$  of  $K[X]$  given by

$$\Phi : K[X] \rightarrow K[X], \quad f \mapsto f(h).$$

- (a) Show that  $\Phi$  is injective.  
 (b) Show that  $\Phi$  is an isomorphism if and only if  $\deg(h) = 1$ .

**Proposition.** For any nonzero polynomials  $f, g \in K[X]$ ,  $\deg(f(g)) = \deg(f) \deg(g)$ .

*Proof.* Let  $f_0, \dots, f_m \in K$  such that  $f = \sum_{i=0}^m f_i X^i$  and  $g_0, \dots, g_n \in K$  such that  $g = \sum_{j=0}^n g_j X^j$ , with  $f_m \neq 0$  and  $g_n \neq 0$ , where  $m = \deg(f)$ ,  $n = \deg(g)$ ,  $m, n \geq 0$ , then

$$\begin{aligned} f(g) &= \sum_{i=0}^m f_i g^i \\ &= \sum_{i=0}^m f_i \left( \sum_{j=0}^n g_j X^j \right)^i \end{aligned}$$

As  $\deg(g^i) = i \cdot \deg(g)$  for any  $i \in \mathbb{N}$ ,  $\deg(f(g)) \leq m \cdot \deg(g)$ . Also note that in  $f(g)$ , the coefficient of  $X^{mn}$  is  $f_m g_n^m$ , which is nonzero, therefore  $\deg(f(g)) = mn = \deg(f) \deg(g)$ .  $\square$

- (a) *Proof.* To show injectivity, proceed to show that  $\Phi$  has a trivial kernel. Suppose for a contradiction  $\Phi$  has a non-trivial kernel, that is there exists  $f \in K[X]$ , with  $\deg(f) \geq 0$ , and  $\Phi(f) = 0$ . This means  $\deg(\Phi(f)) = \deg(0) = -\infty$ , but because both  $f, h$  are nonzero polynomials, by proposition above,  $\deg(f(h)) = \deg(f) \deg(h) \geq 0$  which is a contradiction.  $\square$
- (b) *Proof.* If  $\deg(h) = 1$ , from Question 2, since  $h = aX + b$  where  $a, b \in K$  with  $a \neq 0$ , the set  $\{h^i\}_{i \in \mathbb{N}}$  forms a basis of  $K[X]$ . Evaluating  $\Phi$  on the standard basis  $\{X^i\}_{i \in \mathbb{N}}$  for  $K[X]$  gives that for any  $i \in \mathbb{N}$ ,  $\Phi(X^i) = h^i$ . Since  $\Phi$  sends basis to basis, it is an isomorphism.

Conversely suppose  $\deg(h) \geq 2$ , the claim is that  $X \notin \text{Im}(\Phi)$ . Consider the degree of the polynomial (point) we evaluate  $\Phi$  at, for any  $f \in K[X]$ ,

- Case  $\deg(f) = -\infty$ ,  $\Phi(f) = 0$ , and  $\deg(\Phi(f)) = -\infty$ ,
- Case  $\deg(f) = 0$ ,  $\Phi(f) = f$  is degree 0,
- Case  $\deg(f) \geq 1$ ,  $\Phi(f) = f(h)$  has degree  $\deg(f) \deg(h) \geq 2$ .

This means that no degree 1 polynomial lies in  $\text{Im}(\Phi)$ , therefore  $\Phi$  is not an isomorphism.  $\square$

### Question 4

Let  $K$  be a field of characteristic 0. Consider the linear endomorphism  $S$  of  $K[X]$  given by

$$S : K[X] \rightarrow K[X], \quad \sum_{n=0}^d a_n X^n \mapsto \sum_{n=0}^d \frac{a_n}{n+1} X^{n+1}.$$

Let  $V \subseteq K[X]$  be a non-zero subspace which is stable under  $S$ . Show that  $V$  is not finite-dimensional.

*Proof.* Suppose for a contradiction that  $V \subseteq K[X]$  is non-zero, stable under  $S$  and is finite-dimensional, then  $V$  has a finite basis  $\mathcal{B}$ . Note that since  $V$  is not the zero subspace,  $\mathcal{B}$  is non-empty. Consider  $\deg(\mathcal{B}) \subseteq \mathbb{N}$ , a finite and non-empty subset of natural numbers. Let  $\omega \in \deg(\mathcal{B})$  be the largest element, that is, for any  $d \in \deg(\mathcal{B})$ ,  $d \leq \omega$ . This means that there exists  $z \in \mathcal{B}$  such that  $\deg(z) = \omega$ , and for any  $b \in \mathcal{B}$ ,  $\deg(b) \leq \deg(z)$ .

As linear combination of polynomials do not increase the degree, for any  $v \in \text{span}(\mathcal{B}) = V$ ,  $\deg(v) \leq \omega$ . But now, consider  $S(z)$ . Let  $z_0, \dots, z_\omega \in K$  with  $z_\omega \neq 0$  such that  $z = \sum_{i=0}^{\omega} z_i X^i$ , then

$$\begin{aligned} S(z) &= S\left(\sum_{i=0}^{\omega} z_i X^i\right) \\ &= \sum_{i=0}^{\omega} \frac{z_i}{i+1} X^{i+1} \end{aligned}$$

which has degree  $\omega + 1$ , as  $\frac{z_\omega}{\omega+1} \neq 0$ . Then from our earlier conclusion that any  $v \in V$  has degree less than or equal to  $\omega$ , we have  $z \in V$ , but  $S(z) \notin V$ , which contradicts fact that  $V$  is stable under  $S$ .  $\square$

## Question 5

Let  $K$  be a field of characteristic 0. Consider the linear endomorphism  $D$  of  $K[X]$  given by

$$D : K[X] \rightarrow K[X], \quad \sum_{n=0}^d a_n X^n \mapsto \sum_{n=1}^d n a_n X^{n-1}.$$

Let  $V \subseteq K[X]$  be a finite dimensional subspace. Show that  $D$  is nilpotent on  $V$ , i.e. there exists  $m \in \mathbb{N}$  such that for any  $f \in V$ , one has  $D^m(f) = 0$ .

**Claim.** For any nonzero  $f \in K[X]$ ,  $D^{\deg(f)+1}(f) = 0$ .

*Proof (of claim).* Proceed by induction on  $\deg(f)$ , case  $\deg(f) = 0$ , it is clear that  $D^1(0) = 0$ . (There are no terms in a sum from 1 to 0.) Suppose for any  $g \in K[X]$  with  $\deg(g) = n - 1$ ,  $D^n(g) = 0$ .

Consider  $f \in K[X]$  with  $\deg(f) = n$ , so  $f_0, \dots, f_n \in K$  with  $f_n \neq 0$  such that  $f = \sum_{i=0}^n f_i X^i$ , then by induction hypothesis,

$$\begin{aligned} D^{n+1}(f) &= D^n(D(f)) \\ &= D^n\left(D\left(\sum_{i=0}^n f_i X^i\right)\right) \\ &= D^n\left(\sum_{i=1}^n i f_i X^{i-1}\right) \\ &= 0 \end{aligned}$$

Therefore by induction, for any nonzero  $f \in K[X]$ ,  $D^{\deg(f)+1}(f) = 0$ . □

An immediate corollary is that for any  $f \in K[X]$ , for any  $m \in \mathbb{N}$ , where  $m > \deg(f)$ ,  $D^m(f) = 0$ .

*Proof (of Q5).*  $V$  is finite dimensional, so  $V$  has a finite basis  $\mathcal{B}$ . In the case that  $V$  is the zero subspace,  $D(0) = 0$  so  $D$  is nilpotent. For cases where  $V$  is a non-zero subspace of  $K[X]$ ,  $\mathcal{B}$  is non-empty. Consider  $\deg(\mathcal{B}) \subseteq \mathbb{N}$ , which is a finite and non-empty subset of natural numbers. It has the largest element  $\omega$ , where for any  $d \in \deg(\mathcal{B})$ ,  $d \leq \omega$ . This means that there exists  $z \in \mathcal{B}$  such that  $\deg(z) = \omega$ , and for any  $b \in \mathcal{B}$ ,  $\deg(b) \leq \deg(z)$ .

As linear combination of polynomials do not increase the degree, for any  $v \in \text{span}(\mathcal{B}) = V$ ,  $\deg(v) \leq \omega$ . For  $0 \in V$ ,  $D^{\omega+1}(0) = 0$  is trivial. For any nonzero  $v \in V$ , as  $\omega + 1 > \deg(v)$ , by claim,  $D^{\omega+1}(v) = 0$ . Therefore  $D$  is nilpotent. □

## Question 6

Let  $K$  be a field. For each  $t \in K$ , “evaluation at  $t$ ” gives a linear functional  $\text{eval}_t \in K[X]^\vee$  on the  $K$ -vector space  $K[X]$ :

$$\text{eval}_t : K[X] \rightarrow K, \quad f \mapsto f(t),$$

which has the property that for any  $f, g \in K[X]$ , one has

$$\text{eval}(fg) = \text{eval}(f) \text{eval}(g) \quad \text{in } K.$$

Show that for any linear functional  $\varphi \in K[X]^\vee$  with property that for any  $f, g \in K[X]$ , one has

$$\varphi(fg) = \varphi(f)\varphi(g) \quad \text{in } K,$$

then either  $\varphi = 0$  in  $K[X]^\vee$  or there exists  $t \in K$  such that  $\varphi = \text{eval}_t$  in  $K[X]^\vee$ .

*Proof.* Let  $\varphi \in K[X]^\vee$  be any multiplicative linear functional. By multiplicative property,  $\varphi(1) = \varphi(1) \cdot \varphi(1)$ , then  $\varphi(1) = 0$  or  $\varphi(1) = 1$ .

Case  $\varphi(1) = 0$ , then for any  $f \in K[X]$ ,  $\varphi(f) = \varphi(1 \cdot f) = \varphi(1) \cdot \varphi(f) = 0$ , so  $\varphi$  is the zero functional.

Case  $\varphi$  nonzero and  $\varphi(1) = 1$ , for any  $f \in K[X]$ , let  $f_0, \dots, f_d \in K$  such that  $f = \sum_{i=0}^d f_i X^i$ , where  $d = \deg(f)$ , then by linearity and multiplicative property,

$$\begin{aligned} \varphi(f) &= \varphi\left(\sum_{i=0}^d f_i X^i\right) \\ &= f_0 \varphi(1) + \sum_{i=1}^d f_i \varphi(X^i) \\ &= f_0 + \sum_{i=1}^d f_i \varphi(X)^i \end{aligned}$$

define  $t := \varphi(X) \in K$ , then

$$\begin{aligned} \text{eval}_t(f) &= f(t) \\ &= \sum_{i=0}^d f_i t^i \\ &= f_0 + \sum_{i=1}^d f_i t^i \end{aligned}$$

Since  $f$  was arbitrary, we see that by setting  $t = \varphi(X) \in K$ ,  $\varphi = \text{eval}_t$ . □