MA1100 Homework 3

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T04

1st October 2017

Q1

Statement. *For any sets X, Y, Z and any maps $f : X \mapsto Y$ and $g : Y \mapsto Z$, if f is injective and g is injective, then $g \circ f$ is injective.*

Statement is **true**.

Proof. If f is injective, by definition,

$$\forall x_1, x_2 \in X.\ f(x_1) = f(x_2) \implies x_1 = x_2.$$

If g is injective, by definition,

$$\forall y_1, y_2 \in Y.\ g(y_1) = g(y_2) \implies y_1 = y_2.$$

$g \circ f$ is defined as

$$\forall x \in X.\ (g \circ f)(x) := g(f(x)).$$

Then given $a, b \in X$,

if $(g \circ f)(a) = (g \circ f)(b)$, then

by definition of the composite map $g \circ f$, $g(f(a)) = g(f(b))$.

Since g is injective and $f(a), f(b) \in Y$, this implies $f(a) = f(b)$.

Since f is injective and $a, b \in X$, this implies $a = b$.

Therefore, we can conclude that given f is injective and g is injective,

$$\forall a, b \in X.\ (g \circ f)(a) = (g \circ f)(b) \implies a = b,$$

$g \circ f$ is injective. □

Q2

Statement. *For any sets X, Y, Z and any maps $f : X \mapsto Y$ and $g : Y \mapsto Z$, if f is injective and g is surjective, then $g \circ f$ is injective.*

Statement is **false**.

Negation. *There exists sets X, Y, Z and maps $f : X \mapsto Y$ and $g : Y \mapsto Z$ such that, f is injective and g is surjective, but $g \circ f$ is not injective.*

Proof. Let

$$
\begin{aligned}
X &:= \{1, 2, 3\}, \\
Y &:= \{4, 5, 6, 7\}, \\
Z &:= \{10, 11\}, \\
\Gamma f \subseteq X \times Y &:= \{(1, 4), (2, 5), (3, 6)\}, \\
\Gamma g \subseteq Y \times Z &:= \{(4, 10), (5, 10), (6, 11), (7, 11)\}.
\end{aligned}
$$

Trivially, it can be visually verified that f and g are totally-defined and well-defined.
f is injective, because

$$\forall x_1, x_2 \in X.\ x_1 \neq x_2 \implies f(x_1) \neq f(x_2).$$

g is surjective, because

$$\forall z \in Z.\ \exists y \in Y.\ g(y) = z.$$

$g \circ f$ is defined as

$$\forall x \in X.\ (g \circ f)(x) := g(f(x)).$$

In this example,

$\Gamma(g \circ f) \subseteq X \times Z := \{(1, 10), (2, 10), (3, 11)\}.$

Take $a, b \in X$ to be 1 and 2 respectively,

$$(g \circ f)(1) = (g \circ f)(2) = 10.$$

Since there exists $a, b \in X$ such that $(g \circ f)(a) = (g \circ f)(b)$ and $a \neq b$,

$g \circ f$ is not injective.

Therefore, we can conclude that there exists sets X, Y, Z and maps $f : X \mapsto Y$ and $g : Y \mapsto Z$ such that f is injective and g is surjective, but $g \circ f$ is not injective. $\square$

Q3

Statement. *For any sets X, Y, Z and any maps $f : X \mapsto Y$ and $g : Y \mapsto Z$, if f is surjective and g is injective, then $g \circ f$ is injective.*

Statement is **false**.

Negation. *There exists sets X, Y, Z and maps $f : X \mapsto Y$ and $g : Y \mapsto Z$ such that, f is surjective and g is injective, but $g \circ f$ is not injective.*

Proof. Let

$$\begin{aligned} X &:= \{1, 2, 3\}, \\ Y &:= \{4, 5\}, \\ Z &:= \{10, 11\}, \\ \Gamma f \subseteq X \times Y &:= \{(1, 4), (2, 5), (3, 4)\}, \\ \Gamma g \subseteq Y \times Z &:= \{(4, 10), (5, 11)\}. \end{aligned}$$

Trivially, it can be visually verified that f and g are totally-defined and well-defined.
f is surjective, because

$$\forall y \in Y.\ \exists x \in X.\ f(x) = y.$$

g is injective, because

$$\forall y_1, y_2 \in Y.\ y_1 \neq y_2 \implies g(y_1) \neq g(y_2).$$

$g \circ f$ is defined as

$$\forall x \in X.\ (g \circ f)(x) := g(f(x)).$$

In this example,

$\Gamma(g \circ f) \subseteq X \times Z := \{(1, 10), (2, 11), (3, 10)\}.$

Take $a, b \in X$ to be 1 and 3 respectively,

$$(g \circ f)(1) = (g \circ f)(3) = 10.$$

Since there exists $a, b \in X$ such that $(g \circ f)(a) = (g \circ f)(b)$ and $a \neq b$,

$(g \circ f)$ is not injective.

Therefore, we can conclude that there exists sets X, Y, Z and maps $f : X \mapsto Y$ and $g : Y \mapsto Z$ such that f is surjective and g is injective, but $g \circ f$ is not injective. $\square$

Q4

Statement. *For any sets X, Y, Z and any maps $f : X \mapsto Y$ and $g : Y \mapsto Z$, if f is injective and g is surjective, then $g \circ f$ is surjective.*

Statement is **false**.

Negation. *There exists sets X, Y, Z and maps $f : X \mapsto Y$ and $g : Y \mapsto Z$ such that, f is injective and g is surjective, but $g \circ f$ is not surjective.*

Proof. Let

$$\begin{aligned} X &:= \{1, 2\}, \\ Y &:= \{4, 5, 6\}, \\ Z &:= \{10, 11, 12\}, \\ \Gamma f \subseteq X \times Y &:= \{(1, 4), (2, 5)\}, \\ \Gamma g \subseteq Y \times Z &:= \{(4, 10), (5, 11), (6, 12)\}. \end{aligned}$$

Trivially, it can be visually verified that f and g are totally-defined and well-defined.
f is injective, because

$$\forall x_1, x_2 \in X.\ f(x_1) = f(x_2) \implies x_1 = x_2.$$

g is surjective, because

$$\forall z \in Z.\ \exists y \in Y.\ g(y) = z.$$

$g \circ f$ is defined as

$$\forall x \in X.\ (g \circ f)(x) := g(f(x)).$$

In this example,

$\Gamma(g \circ f) \subseteq X \times Z := \{(1, 10), (2, 11)\}$.

Take $12 \in Z$,

$$\forall x \in X.\ (g \circ f)(x) \neq 12.$$

Since $\exists z \in Z.\ \forall x \in X.\ (g \circ f)(x) \neq z$,

$g \circ f$ is not surjective.

Therefore, we can conclude that there exists sets X, Y, Z and maps $f : X \mapsto Y$ and $g : Y \mapsto Z$ such that f is injective and g is surjective, but $g \circ f$ is not surjective. □

Q5

Statement. *For any sets X, Y, Z and any maps $f : X \mapsto Y$ and $g : Y \mapsto Z$, if f is surjective and g is injective, then $g \circ f$ is surjective.*

Statement is **false**.

Negation. *There exists sets X, Y, Z and maps $f : X \mapsto Y$ and $g : Y \mapsto Z$ such that, f is surjective and g is injective, but $g \circ f$ is not surjective.*

Proof. Let

$$\begin{aligned}
X &:= \{1, 2, 3\}, \\
Y &:= \{4, 5\}, \\
Z &:= \{10, 11, 12\}, \\
\Gamma f \subseteq X \times Y &:= \{(1, 4), (2, 5), (3, 4)\}, \\
\Gamma g \subseteq Y \times Z &:= \{(4, 10), (5, 11)\}.
\end{aligned}$$

Trivially, it can be visually verified that f and g are totally-defined and well-defined.
f is surjective, because

$$\forall y \in Y.\ \exists x \in X.\ f(x) = y.$$

g is injective, because

$$\forall y_1, y_2 \in Y.\ y_1 \neq y_2 \implies g(y_1) \neq g(y_2).$$

$g \circ f$ is defined as

$$\forall x \in X.\ (g \circ f)(x) := g(f(x)).$$

In this example,

$$\Gamma(g \circ f) \subseteq X \times Z := \{(1, 10), (2, 11), (3, 10)\}.$$

Take $12 \in Z$,

$$\forall x \in X.\ (g \circ f)(x) \neq 12.$$

Since $\exists z \in Z.\ \forall x \in X.\ (g \circ f)(x) \neq z$, $g \circ f$ is not surjective.

Therefore, we can conclude that there exists sets X, Y, Z and maps $f : X \mapsto Y$ and $g : Y \mapsto Z$ such that f is surjective and g is injective, but $g \circ f$ is not surjective. $\square$

Q6

Statement. *For any sets X, Y, Z and any maps $f : X \mapsto Y$ and $g : Y \mapsto Z$, if f is surjective and g is surjective, then $g \circ f$ is surjective.*

Statement is **true**.

Proof. If f is surjective, by definition,

$$\forall y \in Y.\ \exists x \in X.\ f(x) = y.$$

If g is surjective, by definition,

$$\forall z \in Z.\ \exists y \in Y.\ g(y) = z.$$

$g \circ f$ is defined as

$$\forall x \in X.\ (g \circ f)(x) := g(f(x)).$$

Then given $c \in Z$,

Since g is surjective, $\exists b \in Y.\ g(b) = c$.

f is also surjective, so given $b \in Y$, $\exists a \in X.\ f(a) = b$.

Therefore, $\exists a \in X.\ g(f(a)) = c$.

Therefore, we can conclude that given f is surjective and g is surjective,

$$\forall c \in Z.\ \exists a \in X.\ (g \circ f)(a) = c,$$

$g \circ f$ is surjective. $\square$

Q7

(a)

Claim. *Given sets A, B, $A \subseteq B$ iff $A \cup B = B$.*

Proof. Assume $A \subseteq B$, then $\forall x.\ x \in A \implies x \in B$. $(\implies)$
Let $x \in A \cup B$ be arbitary, but fixed, then,

$$(x \in A) \lor (x \in B).$$

Case $x \in A$, since $A \subseteq B, x \in B$.

Case $x \in B$, trivially, $x \in B$.

Because for any arbitary x, $x \in A \cup B \implies x \in B$, we have $A \cup B \subseteq B$.
Conversely let $x \in B$ be arbitary, but fixed, then trivially,

$$\begin{gathered} x \in B \\ (x \in A) \lor (x \in B) \\ x \in A \cup B \end{gathered}$$

Since for any arbitary x, $x \in B \implies x \in A \cup B$, we have $B \subseteq A \cup B$. Now because $A \cup B \subseteq B$ and $B \subseteq A \cup B$, we conclude that if $A \subseteq B$, then $A \cup B = B$.
Assume $A \cup B = B$, then by axiom of extentionality, $(\impliedby)$

$$\begin{gathered} \forall x.\ x \in A \cup B \iff x \in B \\ \forall x.\ (x \in A) \lor (x \in B) \iff x \in B \end{gathered}$$

Let $x \in A$ be arbitary, but fixed, then by above statement, $x \in B$. Because for any arbitary x, $x \in A \implies x \in B$, we conclude that if $A \cup B = B$, then $A \subseteq B$.
We have $A \subseteq B \implies A \cup B = B$ and $A \cup B = B \implies A \subseteq B$, so $A \subseteq B$ iff $A \cup B = B$. $\square$

(b)

Claim. *Given sets A, B, $A \cap B = A$ iff $A \cup B = B$.*

Proof. Assume $A \cap B = A$, then by axiom of extentionality, $(\implies)$

$$\begin{aligned} &\forall x.\ x \in A \cap B \iff x \in A \\ &\forall x.\ (x \in A) \wedge (x \in B) \iff x \in A \end{aligned} \tag{1}$$

Let $x \in A \cup B$ be arbitary, but fixed, then,

$$(x \in A) \vee (x \in B).$$

Case $x \in A$, by (1), $(x \in A) \wedge (x \in B)$, so $x \in B$.

Case $x \in B$, trivially, $x \in B$.

Because for any arbitary x, $x \in A \cup B \implies x \in B$, we have $A \cup B \subseteq B$.
Conversely let $x \in B$ be arbitary, but fixed, then trivially,

$$\begin{aligned} &x \in B \\ &(x \in A) \vee (x \in B) \\ &x \in A \cup B \end{aligned}$$

Since for any arbitary x, $x \in B \implies x \in A \cup B$, we have $B \subseteq A \cup B$. Because $A \cup B \subseteq B$ and $B \subseteq A \cup B$, we conclude that if $A \cap B = A$, then $A \cup B = B$.
Now assume $A \cup B = B$, then by axiom of extentionality, $(\impliedby)$

$$\begin{aligned} &\forall x.\ x \in A \cup B \iff x \in B \\ &\forall x.\ (x \in A) \vee (x \in B) \iff x \in B \end{aligned} \tag{2}$$

Let $x \in A \cap B$ be arbitary, but fixed, then,

$$\begin{aligned} &(x \in A) \wedge (x \in B) \\ &x \in A \end{aligned}$$

Because for any arbitary x, $x \in A \cap B \implies x \in A$, we have $A \cap B \subseteq A$.
Conversely let $x \in A$ be arbitary, but fixed, then by (2), $x \in B$.
Since $x \in A$ to begin with, we have

$$\begin{aligned} &(x \in A) \wedge (x \in B) \\ &x \in A \cap B \end{aligned}$$

Since for any arbitary x, $x \in A \implies x \in A \cap B$, we have $A \subseteq A \cap B$. Because $A \cap B \subseteq A$ and $A \subseteq A \cap B$, we conclude that if $A \cup B = B$, then $A \cap B = A$.
We have $A \cap B = A \implies A \cup B = B$ and $A \cup B = B \implies A \cap B = A$, so $A \cap B = A$ iff $A \cup B = B$. $\square$

Q8

Claim. *Let A, B and U be sets so that $A \subseteq U$ and $B \subseteq U$. $A = \emptyset$ iff the equality $((U \setminus A) \cap B) \cup (A \cap (U \setminus B)) = B$ holds.*

Proof. Assume $A = \emptyset$, then $\forall x.\ x \notin A$. Since $B \subseteq U$, so $\forall x.\ x \in B \implies x \in U$. $(\implies)$

$$\begin{aligned}
&((U \setminus A) \cap B) \cup (A \cap (U \setminus B)) \\
&= \{\, x \in U : (x \in (U \setminus A) \cap B) \vee (x \in A \cap (U \setminus B)) \,\} \\
&= \{\, x \in U : ((x \in U \setminus A) \wedge (x \in B)) \vee ((x \in A) \wedge (x \in U \setminus B)) \,\} \\
&= \{\, x \in U : (x \in U \setminus A) \wedge (x \in B) \,\} && \text{by } x \notin A \\
&= \{\, x \in U : (x \in U) \wedge \neg(x \in A) \wedge (x \in B) \,\} \\
&= \{\, x \in U : (x \in U) \wedge (x \in B) \,\} \\
&= \{\, x \in U : x \in B \,\} && \text{by } x \in B \implies x \in U \\
&= B
\end{aligned}$$

If $A = \emptyset$, then the equality $((U \setminus A) \cap B) \cup (A \cap (U \setminus B)) = B$ holds.
Now assume $((U \setminus A) \cap B) \cup (A \cap (U \setminus B)) = B$. $(\impliedby)$
By axiom of extentionality,

$$\begin{aligned}
&\forall x.\ x \in ((U \setminus A) \cap B) \cup (A \cap (U \setminus B)) \iff x \in B \\
&\forall x.\ (x \in (U \setminus A) \cap B) \vee (x \in A \cap (U \setminus B)) \iff x \in B \\
&\forall x.\ ((x \in U \setminus A) \wedge (x \in B)) \vee ((x \in A) \wedge (x \in U \setminus B)) \iff x \in B \\
&\forall x.\ ((x \in U) \wedge \neg(x \in A) \wedge (x \in B)) \vee ((x \in A) \wedge (x \in U) \wedge \neg(x \in B)) \iff x \in B \\
&\forall x.\ ((x \in A) \wedge (x \in U) \wedge \neg(x \in B)) \implies x \in B
\end{aligned}$$

Suppose for a contradiction that $\exists x \in A$, since $A \subseteq U$, $x \in U$,
if $x \notin B$, $(x \in A) \wedge (x \in U) \wedge \neg(x \in B)$ is true, but $x \in B$ false, a contradiction.
Therefore if the equality $((U \setminus A) \cap B) \cup (A \cap (U \setminus B)) = B$ holds, there must not exist x where $x \in A$, that is, $\forall x.\ x \notin A$, which means $A = \emptyset$.
Because $A = \emptyset \implies ((U \setminus A) \cap B) \cup (A \cap (U \setminus B)) = B$ and
$((U \setminus A) \cap B) \cup (A \cap (U \setminus B)) = B \implies A = \emptyset$,
we can conclude that $A = \emptyset$ iff the equality $((U \setminus A) \cap B) \cup (A \cap (U \setminus B)) = B$ holds. $\square$

Q9

Claim. *Suppose $f : X \mapsto Y$ is injective. Then for any set T, the map Φ_T of "post-composition with f" is injective.*

Proof. f is injective, by definition,

$$\forall x_1, x_2 \in X.\ f(x_1) = f(x_2) \implies x_1 = x_2.$$

For any set T, the map Φ_T of "post-composition with f" is defined as

$$\forall \phi \in \mathrm{Maps}(T, X).\ \Phi_T(\phi) := (f \circ \phi).$$

Given any set T and $\phi_1, \phi_2 \in \mathrm{Maps}(T, X)$,
if $f \circ \phi_1 = f \circ \phi_2$, then

$$\begin{aligned}
\forall t \in T.\ \forall y \in Y.\ (t, y) \in \Gamma(f \circ \phi_1) &\iff (t, y) \in \Gamma(f \circ \phi_2) \\
\forall t \in T.\ \forall y \in Y.\ (f \circ \phi_1)(t) = y &\iff (f \circ \phi_2)(t) = y \\
\forall t \in T.\ (f \circ \phi_1)(t) &= (f \circ \phi_2)(t) \\
\forall t \in T.\ f(\phi_1(t)) &= f(\phi_2(t))
\end{aligned}$$

Since $\phi_1(t), \phi_2(t) \in X$, by injectivity of f,

$$\begin{aligned}
\forall t \in T.\ \phi_1(t) &= \phi_2(t) \\
\forall t \in T.\ \forall x \in X.\ \phi_1(t) = x &\iff \phi_2(t) = x \\
\forall t \in T.\ \forall x \in X.\ (t, x) \in \Gamma\phi_1 &\iff (t, x) \in \Gamma\phi_2
\end{aligned}$$

Therefore $\phi_1 = \phi_2$.
For any set T, for all $\phi_1, \phi_2 \in \mathrm{Maps}(T, X)$, we have $(f \circ \phi_1) = (f \circ \phi_2)$, implies $\phi_1 = \phi_2$.
This means that if f is injective, the map Φ_T of "post-composition with f" is injective for any set T. $\square$

Q10

Claim. *Suppose for any set T, the map Φ_T of "post-composition with f" is injective. Then $f : X \mapsto Y$ is injective.*

Proof. For any set T, the map Φ_T of "post-composition with f" is defined as

$$\forall \phi \in \text{Maps}(T, X).\ \Phi_T(\phi) := (f \circ \phi).$$

Φ_T of "post-composition with f" is injective, by definition, for any set T,

$$\forall \phi_1, \phi_2 \in \text{Maps}(T, X).\ (f \circ \phi_1) = (f \circ \phi_2) \implies \phi_1 = \phi_2 \tag{1}$$

By definition, $\text{Maps}(T, X)$ contains *all* maps from T to X, this means that given $T \neq \emptyset$,

$$\begin{gathered}\forall x \in X.\ \forall t \in T.\ \exists \phi \in \text{Maps}(T, X).\ (t, x) \in \Gamma\phi \\ \forall x \in X.\ \forall t \in T.\ \exists \phi \in \text{Maps}(T, X).\ \phi(t) = x\end{gathered}$$

Given $x_1, x_2 \in X$, if $f(x_1) = f(x_2)$, then

> Take $x_1 = \phi_1(t_0)$ and $x_2 = \phi_2(t_0)$, where $\phi_1, \phi_2 \in \text{Maps}(T, X)$ and $t_0 \in T$ is arbitary, but fixed, then
>
> $$f(\phi_1(t_0)) = f(\phi_2(t_0)).$$
>
> Since t_0 is arbitary,
>
> $$\begin{gathered}\forall t \in T.\ f(\phi_1(t)) = f(\phi_2(t)) \\ \forall t \in T.\ (f \circ \phi_1)(t) = (f \circ \phi_2)(t) \\ \forall t \in T.\ \forall y \in Y.\ (f \circ \phi_1)(t) = y \iff (f \circ \phi_2)(t) = y \\ \forall t \in T.\ \forall y \in Y.\ (t, y) \in \Gamma(f \circ \phi_1) \iff (t, y) \in \Gamma(f \circ \phi_2) \\ (f \circ \phi_1) = (f \circ \phi_2)\end{gathered}$$
>
> Because Φ_T of "post-composition with f" is injective, by (1),
>
> $$\begin{gathered}\phi_1 = \phi_2 \\ \phi_1(t_0) = \phi_2(t_0) \\ x_1 = x_2\end{gathered}$$

Since

$$\forall x_1, x_2 \in X.\ f(x_1) = f(x_2) \implies x_1 = x_2$$

We can conlude that if the map Φ_T of "post-composition with f" is injective for any set T, f is injective. $\square$

Q11

Claim. *Suppose $f : X \mapsto Y$ is surjective. Then for any set T, the map Ψ_T of "pre-composition with f" is injective.*

Proof. f is surjective, by definition,

$$\forall y \in Y.\ \exists x \in X.\ f(x) = y. \tag{1}$$

The map Ψ_T of "pre-composition with f" is defined as

$$\forall \psi \in \text{Maps}(Y, T).\ \Psi_T(\psi) := (\psi \circ f).$$

Given any set T and $\psi_1, \psi_2 \in \text{Maps}(Y, T)$,
if $\Psi_T(\psi_1) = \Psi_T(\psi_2)$, then

$$\begin{aligned}
(\psi_1 \circ f) &= (\psi_2 \circ f) \\
\forall x \in X.\ (\psi_1 \circ f)(x) &= (\psi_2 \circ f)(x) \\
\forall x \in X.\ \psi_1(f(x)) &= \psi_2(f(x)) \\
\forall y \in Y.\ \psi_1(y) &= \psi_2(y) && \text{by (1)} \\
\forall y \in Y.\ \forall t \in T.\ \psi_1(y) = t &\iff \psi_2(y) = t \\
\forall y \in Y.\ \forall t \in T.\ (y, t) \in \Gamma\psi_1 &\iff (y, t) \in \Gamma\psi_2
\end{aligned}$$

Therefore $\psi_1 = \psi_2$.
For any set T, for all $\psi_1, \psi_2 \in \text{Maps}(Y, T)$, we have $(\psi_1 \circ f) = (\psi_2 \circ f) \implies \psi_1 = \psi_2$.
This means that if f is surjective, the map Ψ_T of "pre-composition with f" is injective for any set T. $\square$

Q12

Claim. *Suppose for any set T, the map Ψ_T of "pre-composition with f" is injective. Then $f : X \mapsto Y$ is surjective.*

Proof. For any set T, the map Ψ_T of "pre-composition with f" is defined as

$$\forall \psi \in \text{Maps}(Y, T).\ \Psi_T(\psi) := (\psi \circ f).$$

The map Ψ_T of "pre-composition with f" is injective, by definition, for any set T,

$$\forall \psi_1, \psi_2 \in \text{Maps}(Y, T).\ \psi_1 \neq \psi_2 \implies (\psi_1 \circ f) \neq (\psi_2 \circ f) \tag{*}$$

Suppose for a contradiction that f is not surjective, meaning

$$\exists y \in Y.\ \forall x \in X.\ f(x) \neq y$$

Take $Y_0 \subseteq Y$ to be when the above condition holds,

$$Y_0 := \{\ y \in Y : \forall x \in X.\ f(x) \neq y\ \}$$

$$\forall y \in Y \setminus Y_0.\ \exists x \in X.\ f(x) = y.$$

Take $\psi_1, \psi_2 \in \text{Maps}(Y, T)$ where $\psi_1 \neq \psi_2$, specifically

$$\begin{aligned} \forall y \in Y \setminus Y_0.\ \psi_1(y) &= \psi_2(y) \\ \forall y \in Y_0.\ \psi_1(y) &\neq \psi_2(y) \end{aligned} \tag{1}$$

Then for all $x \in X$, $f(x) \in Y \setminus Y_0$, then by (1)

$$\begin{gathered} \forall x \in X.\ \psi_1(f(x)) = \psi_2(f(x)) \\ \forall x \in X.\ (\psi_1 \circ f)(x) = (\psi_2 \circ f)(x) \\ \forall x \in X.\ \forall t \in T.\ (\psi_1 \circ f)(x) = t \iff (\psi_2 \circ f)(x) = t \\ \forall x \in X.\ \forall t \in T.\ (x, t) \in \Gamma(\psi_1 \circ f) \iff (x, t) \in \Gamma(\psi_2 \circ f) \\ (\psi_1 \circ f) = (\psi_2 \circ f) \end{gathered}$$

There exists maps $\psi_1, \psi_2 \in \text{Maps}(Y, T)$ where $\psi_1 \neq \psi_2$ and $(\psi_1 \circ f) = (\psi_2 \circ f)$, a contradiction with (*).

Therefore, if the map Ψ_T of "pre-composition with f" is injective for any set T, f is surjective. □