MA2104 Assignment 1

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Problem 1

Vector parallel to L_1,$\overrightarrow{AB} = \langle 1, 0, -1 \rangle$, vector parallel to L_2,$\overrightarrow{CD} = \langle -2, 5, -1 \rangle$.

$$\begin{aligned} L_1 &: \langle 1, 1, 1 \rangle + s\langle 1, 0, -1 \rangle, \\ L_2 &: \langle 3, 0, -1 \rangle + t\langle -2, 5, -1 \rangle. \end{aligned}$$

Find a vector orthorgonal to both $\overrightarrow{AB}$ and $\overrightarrow{CD}$,

$$\mathbf{n} := \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & -1 \\ -2 & 5 & -1 \end{vmatrix} = \langle 5, 3, 5 \rangle.$$

The shortest distance is the absolute value of scalar projection of $\overrightarrow{AC} = \langle 2, -1, -2 \rangle$ on $\mathbf{n}$.

$$\begin{aligned} \left| \operatorname{comp}_{\mathbf{n}} \overrightarrow{AC} \right| &= \left| \frac{\overrightarrow{AC} \cdot \mathbf{n}}{\|\mathbf{n}\|} \right| \\ &= \left| \frac{-3}{\sqrt{59}} \right| = \frac{3}{\sqrt{59}}. \end{aligned} \qquad \square$$

Problem 2

A vector orthogonal to both $\mathbf{j} - \mathbf{k}$ and $\mathbf{i} + \mathbf{j}$ will be their cross product,

$$\mathbf{n} := \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 1 & -1 \\ 1 & 1 & 0 \end{vmatrix} = \langle 1, -1, -1 \rangle.$$

Two unit vectors orthogonal to $\mathbf{j} - \mathbf{k}$ and $\mathbf{i} + \mathbf{j}$ are

$$\begin{aligned} \frac{\mathbf{n}}{\|\mathbf{n}\|} &= \frac{1}{\sqrt{3}} \langle 1, -1, -1 \rangle \\ &= \left\langle \frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}} \right\rangle, \text{ and} \\ -\frac{\mathbf{n}}{\|\mathbf{n}\|} &= \left\langle -\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}} \right\rangle. \end{aligned} \qquad \square$$

Problem 3

$\overrightarrow{PQ} = \langle 2,3,1\rangle, \overrightarrow{PS} = \langle 4,2,5\rangle, \overrightarrow{QR} = \langle 4,2,5\rangle, \overrightarrow{SR} = \langle 2,3,1\rangle$. Parallelogram is spanned by vectors $\overrightarrow{PQ}$ and $\overrightarrow{PS}$.

$$\begin{aligned} \text{area} = \left\|\overrightarrow{PQ} \times \overrightarrow{PS}\right\| &= \left\|\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 3 & 1 \\ 4 & 2 & 5 \end{vmatrix}\right\| \\ &= \|\langle 13, -6, -8\rangle\| = \sqrt{269}. \end{aligned}$$

□

Problem 4

$\overrightarrow{PQ} = \langle 1,2,1\rangle$, $\overrightarrow{PR} = \langle 5,0,-2\rangle$, a vector normal to the plane will be their cross product.

$$\begin{aligned} \mathbf{n} := \overrightarrow{PQ} \times \overrightarrow{PR} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & 1 \\ 5 & 0 & -2 \end{vmatrix} \\ &= \langle -4, 7, -10\rangle \end{aligned}$$

Define point S as $\overrightarrow{OS} := \overrightarrow{OP} + \overrightarrow{PQ} + \overrightarrow{PR}$. Area of triangle PQR is half the area of parallelogram $PQRS$, which can be computed as $\|\mathbf{n}\|$. Therefore

$$\begin{aligned} \text{area of } PQR &= \frac{1}{2}\|\mathbf{n}\| \\ &= \frac{\sqrt{165}}{2} \end{aligned}$$

□

Problem 5

$$\begin{aligned} \text{signed volume} &= \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) \\ &= \begin{vmatrix} 1 & 1 & -1 \\ 1 & -1 & 1 \\ -1 & 1 & 1 \end{vmatrix} \\ &= -1 - 1 - 1 + 1 - 1 - 1 \\ &= -4 \end{aligned}$$

Therefore volume of parallelpiped is 4. □

Problem 6

To show $\mathbf{v} := (\mathbf{a} \times \mathbf{b}) + (\mathbf{b} \times \mathbf{c}) + (\mathbf{c} \times \mathbf{a})$ is perpendicular to the plane containing P, Q, R, it suffices to show that the vector is orthogonal to both $\overrightarrow{PQ}$ and $\overrightarrow{PR}$.
$\overrightarrow{PQ} = \mathbf{b} - \mathbf{a}$. To check for orthogonality, compute $\mathbf{v} \cdot \overrightarrow{PQ}$ as follows,

$$\begin{aligned}
&((\mathbf{a} \times \mathbf{b}) + (\mathbf{b} \times \mathbf{c}) + (\mathbf{c} \times \mathbf{a})) \cdot (\mathbf{b} - \mathbf{a}) \\
&= (\mathbf{a} \times \mathbf{b}) \cdot (\mathbf{b} - \mathbf{a}) + (\mathbf{b} \times \mathbf{c}) \cdot (\mathbf{b} - \mathbf{a}) + (\mathbf{c} \times \mathbf{a}) \cdot (\mathbf{b} - \mathbf{a}) \\
&= \underbrace{(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{b}}_{0} - \underbrace{(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{a}}_{0} + \underbrace{(\mathbf{b} \times \mathbf{c}) \cdot \mathbf{b}}_{0} - (\mathbf{b} \times \mathbf{c}) \cdot \mathbf{a} + (\mathbf{c} \times \mathbf{a}) \cdot \mathbf{b} - \underbrace{(\mathbf{c} \times \mathbf{a}) \cdot \mathbf{a}}_{0} \\
&= -(\mathbf{b} \times \mathbf{c}) \cdot \mathbf{a} + (\mathbf{c} \times \mathbf{a}) \cdot \mathbf{b} && \text{(matrix with duplicate rows has det 0)} \\
&= (\mathbf{c} \times \mathbf{b}) \cdot \mathbf{a} + (\mathbf{c} \times \mathbf{a}) \cdot \mathbf{b} && \text{(by anticommutativity of } \times\text{)}
\end{aligned}$$

By computing $\mathbf{a} \cdot (\mathbf{c} \times \mathbf{b})$ and $\mathbf{b} \cdot (\mathbf{c} \times \mathbf{a})$, it is clear that by swapping the 1st row with the 3rd row of the determinant in $\mathbf{a} \cdot (\mathbf{c} \times \mathbf{b})$, $\mathbf{b} \cdot (\mathbf{c} \times \mathbf{a})$ is obtained. Thus by property of determinant, $\mathbf{a} \cdot (\mathbf{c} \times \mathbf{b}) = -\mathbf{b} \cdot (\mathbf{c} \times \mathbf{a})$ and we have $\mathbf{v} \cdot \overrightarrow{PQ} = 0$, so $\mathbf{v} \perp \overrightarrow{PQ}$.
Similarly, we can compute $\mathbf{v} \cdot \overrightarrow{PR}$, where $\overrightarrow{PR} = (\mathbf{c} - \mathbf{a})$,

$$\begin{aligned}
&((\mathbf{a} \times \mathbf{b}) + (\mathbf{b} \times \mathbf{c}) + (\mathbf{c} \times \mathbf{a})) \cdot (\mathbf{c} - \mathbf{a}) \\
&= (\mathbf{a} \times \mathbf{b}) \cdot (\mathbf{c} - \mathbf{a}) + (\mathbf{b} \times \mathbf{c}) \cdot (\mathbf{c} - \mathbf{a}) + (\mathbf{c} \times \mathbf{a}) \cdot (\mathbf{c} - \mathbf{a}) \\
&= (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c} - \underbrace{(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{a}}_{0} + \underbrace{(\mathbf{b} \times \mathbf{c}) \cdot \mathbf{c}}_{0} - (\mathbf{b} \times \mathbf{c}) \cdot \mathbf{a} + \underbrace{(\mathbf{c} \times \mathbf{a}) \cdot \mathbf{c}}_{0} - \underbrace{(\mathbf{c} \times \mathbf{a}) \cdot \mathbf{a}}_{0} \\
&= (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c} - (\mathbf{b} \times \mathbf{c}) \cdot \mathbf{a} \\
&= (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c} + (\mathbf{c} \times \mathbf{b}) \cdot \mathbf{a}
\end{aligned}$$

By a similar computation, we can determine that $(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c} = -(\mathbf{c} \times \mathbf{b}) \cdot \mathbf{a}$, implying $\mathbf{v} \cdot \overrightarrow{PR} = 0$, so we have $\mathbf{v} \perp \overrightarrow{PR}$. This completes the proof. $\square$

Problem 7

Compute signed volume of parallelpiped spanned by **a**,**b** and **c**.

$$\begin{aligned}
\text{signed volume} &= \begin{vmatrix} 2 & 4 & -8 \\ 3 & -1 & 3 \\ -5 & 11 & -25 \end{vmatrix} \\
&= 50 - 60 - 264 + 40 - 66 + 300 \\
&= 0
\end{aligned}$$

This implies that the parallelpiped spanned by **a**,**b** and **c** is in fact a plane, meaning that the vectors are coplanar. $\square$