MA2101S Homework 6

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Question 1

> Let K be a field with $\mathrm{char}(K) \neq 2$ (i.e. $1 + 1 \neq 0$ in K), let $n \in \mathbb{N}$ be an **odd** natural number, and let $X, Y \in \mathbb{M}_n(K)$ be two $n \times n$ square matrices over K.
>
> (a) Show that if $X^t = -X$, then X is not invertible.
>
> (b) Show that if $XY = -YX$, then X or Y is not invertible.

(a) *Proof.* Suppose $X^t = -X$, using the facts that $-X = (-1_n)X$, determinant is multiplicative, and $(-1)^n = -1$ as n is odd,

$$\begin{aligned}
\det(X) = \det(X^t) &= \det(-X) = \det((-1_n)X) \\
\det(X) &= \det(-1_n)\, \det(X) \\
\det(X) &= (-1)^n\, \det(X) \\
\det(X) &= -\det(X) \\
\det(X) + \det(X) &= 0 \\
\det(X)\,(1+1) &= 0
\end{aligned}$$

as $\mathrm{char}(K) \neq 2$, $\det(X) = 0$, so X is not invertible. □

(b) *Proof.* Suppose $XY = -YX$, then similarly,

$$\begin{aligned}
\det(XY) &= \det((-1_n)YX) \\
\det(X)\det(Y) &= -\det(Y)\det(X) \\
\det(X)\det(Y)(1+1) &= 0
\end{aligned}$$

again as $\mathrm{char}(K) \neq 2$, $\det(X)\det(Y) = 0$, so X or Y is not invertible. □

Question 2

> Let K be a field, and let $a, b, c, d, e, f \in K$ be elements of K. Consider the 4×4 skew-symmetric matrix
> $$X := \begin{pmatrix} 0 & a & b & c \\ -a & 0 & d & e \\ -b & -d & 0 & f \\ -c & -e & -f & 0 \end{pmatrix} \quad \text{in } \mathbb{M}_4(K).$$
> Show that $\det(X) = (af - be + cd)^2$.

Proof. As X is only 4×4, expand $\det(X)$,

$$\begin{aligned}
\det(X) = 0 - a \begin{vmatrix} -a & d & e \\ -b & 0 & f \\ -c & -f & 0 \end{vmatrix} + b \begin{vmatrix} -a & 0 & e \\ -b & -d & f \\ -c & -e & 0 \end{vmatrix} - c \begin{vmatrix} -a & 0 & d \\ -b & -d & 0 \\ -c & -e & -f \end{vmatrix} \\
&= -a\left(-cdf + bef - af^2\right) + b\left(be^2 - aef - cde\right) - c\left(-adf + bde - cd^2\right) \\
&= acdf - abef + a^2f^2 + b^2e^2 - abef - bcde + acdf - bcde + c^2d^2 \\
&= (af)^2 + (cd)^2 + (be)^2 + 2acdf - 2abef - 2bcde
\end{aligned}$$

On the other hand,

$$\begin{aligned}
(af - be + cd)^2 &= af\left(af - be + cd\right) - be\left(af - be + cd\right) + cd\left(af - be + cd\right) \\
&= (af)^2 - abef + acdf - abef + (be)^2 - bcde + acdf - bcde + (cd)^2 \\
&= (af)^2 + (cd)^2 + (be)^2 + 2acdf - 2abef - 2bcde
\end{aligned}$$

Therefore $\det(X) = (af - be + cd)^2$. □

Question 3

> Let K be a field, and let $n \in \mathbb{N}$ be any natural number with $n > 1$. Consider an $n \times n$ square matrix $A \in \mathbb{M}_n(K)$.
>
> (a) Show that $\det(\operatorname{adj}(A)) = \det(A)^{n-1}$.
> (b) Show that if A is an invertible upper-triangular matrix, then the same is true for $\operatorname{adj}(A)$.

Claim. $A \operatorname{adj}(A) = \det(A)\, 1_n$.

Proof (of Claim). Expanding the (i, j) entries of $A \operatorname{adj}(A)$, we have

$$\begin{aligned}(A \operatorname{adj}(A))_{ij} &= \sum_{k=1}^{n} A_{ik} \operatorname{adj}(A)_{kj} \\ &= \sum_{k=1}^{n} (-1)^{j+k} A_{ik} \det(\widetilde{A}_{jk})\end{aligned}$$

1. Case $i = j$, we get the co-factor expansion along the i-th row, which evaluates to $\det(A)$.
2. Case $i \neq j$, consider the matrix B obtained by copying A, then replacing its j-th with the i-th row of A. Then for any $k \in \{1, \dots, n\}$, $A_{ik} = B_{ik} = B_{jk}$ and $\widetilde{A}_{jk} = \widetilde{B}_{jk}$, then

$$\begin{aligned}(A \operatorname{adj}(A))_{ij} &= \sum_{k=1}^{n} (-1)^{j+k} B_{jk} \det(\widetilde{B}_{jk}) \\ &= \det(B)\end{aligned}$$

   as B by construction has two equal rows, it has determinant 0.

Therefore

$$\begin{aligned}(A \operatorname{adj}(A))_{ij} &= \begin{cases}\det(A) & \text{if } i = j \\ 0 & \text{otherwise}\end{cases} \\ A \operatorname{adj}(A) &= \det(A)\, 1_n.\end{aligned}$$

□

(a) *Proof.* Consider the equality proven, taking determinants,

$$\begin{aligned}A \operatorname{adj}(A) &= \det(A)\, 1_n \\ \det(A) \det(\operatorname{adj}(A)) &= \det(A)^n\end{aligned}$$

If A is invertible ($\det(A) \neq 0$), we obtain the conclusion.

As $n > 1$, $0^{n-1} = 0$. It remains to show that when $\det(A) = 0$, $\det(\operatorname{adj}(A)) = 0$. Suppose A is singular, from claim,

$$A \operatorname{adj}(A) = 0.$$

Reading this equality in terms of left-multiplication means that $\operatorname{Im}(\operatorname{adj}(A)) \subseteq \ker(A)$, which means $\operatorname{rank}(\operatorname{adj}(A)) \leqslant \operatorname{nullity}(A)$.

- Suppose $\operatorname{nullity}(A) = n$, then A is the zero matrix which trivially implies that $\operatorname{adj}(A)$ is also the zero matrix, in this case $\operatorname{adj}(A)$ will be singular.
- Now suppose $\operatorname{nullity}(A) < n$, then $\operatorname{rank}(\operatorname{adj}(A)) \leqslant \operatorname{nullity}(A) < n$, which implies $\operatorname{adj}(A)$ is not full rank, and thus singular too.

Therefore, the equation $\det(\operatorname{adj}(A)) = \det(A)^{n-1}$ holds too when A is singular. □

(b) *Proof.* Suppose A is an invertible upper-triangular matrix, then by claim, $\operatorname{adj}(A)$ is invertible too and has inverse $\frac{1}{\det(A)}A$. Since A is upper-triangular, whenever $i > j$, $A_{ij} = 0$. The (i, j)-entries for $\operatorname{adj}(A)$ is given by

$$\operatorname{adj}(A)_{ij} = (-1)^{i+j} \det(\widetilde{A}_{ji}).$$

Then to show that $\operatorname{adj}(A)$ is upper-triangular, it suffices to show that for any $k, l \in \{1, \dots, n\}$, $l > k \implies \det(\widetilde{A}_{kl}) = 0$.

Take any $k, l \in \{1, \dots, n\}$ with $k < l$. Let $d_1, \dots, d_n \in K$ be diagonal entries of A, then $\widetilde{A}_{kl}$ can be expressed as

$$\widetilde{A}_{kl} = \begin{pmatrix} d_1 & & & & & & & & & \\ & \ddots & & & & & & & & \\ & & d_{k-1} & & & & & \star & & \\ & & & 0 & d_{k+1} & & & & & \\ & & & & \ddots & \ddots & & & & \\ & & & & & \ddots & d_{l-1} & & & \\ & & & & & & 0 & & & \\ & \mathbf{0} & & & & & & d_{l+1} & & \\ & & & & & & & & \ddots & \\ & & & & & & & & & d_n \end{pmatrix}.$$

Hence visually verify that whenever $k < l$, $\widetilde{A}_{kl}$ is an upper-triangular matrix with at least one zero on the diagonal, then $\det(\widetilde{A}_{kl}) = 0$. This completes the proof that $\operatorname{adj}(A)$ is upper-triangular. □

Question 4

> Let K be a field, and let $m, n \in \mathbb{N}_{>0}$ be positive integers, and let $V := \mathbb{M}_{m\times n}(K)$ be the K-vector space of $m \times n$ matrices over K. Fix a $m \times m$ square matrix $A \in \mathbb{M}_{m\times m}(K)$ and a $n \times n$ square matrix $B \in \mathbb{M}_{n\times n}(K)$, and consider the map
>
> $$\Phi : V \to V \quad \text{given by} \quad X \mapsto AXB.$$

Note. Throughout this question, let $\mathcal{H} := (e_{11}, \dots, e_{1n}, \dots, e_{m1}, \dots, e_{mn})$ denote the standard basis for $\mathbb{M}_{m\times n}(K)$ ordered this way. Where for any $(r, s) \in \{1, \dots, m\} \times \{1, \dots, n\}$, $e_{rs} \in \mathbb{M}_{m\times n}(K)$ is characterised by

$$(e_{rs})_{ij} = \delta_{ir}\delta_{js} = \begin{cases} 1 & \text{if } (i,j) = (r,s) \\ 0 & \text{otherwise} \end{cases}.$$

(a) Show that Φ is a K-linear operator on V, and compute its trace $\mathrm{Tr}(\Phi)$ in terms of A and B.

Solution. First note that $\Phi = (X \mapsto AX) \circ (Y \mapsto YB)$. Then because matrix multiplication is bi-linear, Φ is a composition of linear maps and is hence a K-linear operator on V.

In order to compute the trace, first figure out where Φ sends the standard basis vectors to. For any $(r, s) \in \{1, \dots, m\} \times \{1, \dots, n\}$,

$$\begin{aligned}
\Phi(e_{rs}) &= A\, e_{rs} B \\
&= A \begin{pmatrix} & 0 & \\ & \vdots & \\ B_{s1} & \cdots & B_{sn} \\ & \vdots & \\ & 0 & \end{pmatrix} \leftarrow \text{in } r\text{-th row} \\
&= \begin{pmatrix} A_{1r}B_{s1} & A_{1r}B_{s2} & \cdots & A_{1r}B_{sn} \\ A_{2r}B_{s1} & A_{2r}B_{s2} & \cdots & A_{2r}B_{sn} \\ \vdots & \vdots & \ddots & \vdots \\ A_{mr}B_{s1} & A_{mr}B_{s2} & \cdots & A_{mr}B_{sn} \end{pmatrix} \\
(\Phi(e_{rs}))_{ij} &= A_{ir}B_{sj}
\end{aligned}$$

Then the trace can be computed by

$$\begin{aligned}
\operatorname{Tr}(\Phi) &= \sum_{(r,s)} \left(\Phi(e_{rs})\right)_{rs} \\
&= \sum_{(r,s)} A_{rr} B_{ss} \\
&= \sum_{r=1}^{m} \sum_{s=1}^{n} A_{rr} B_{ss} \\
&= \operatorname{Tr}(A) \operatorname{Tr}(B)
\end{aligned}$$

■

(b) Compute the determinant $\det(\Phi)$ of Φ in terms of A, B, m and n. *Solution.* Since we established that $\Phi = (X \mapsto AX) \circ (Y \mapsto YB)$, and since determinant is multiplicative, it suffices to compute the determinant for each $L_A, R_B : V \to V$, where $L_A := X \mapsto AX$ and $R_B := Y \mapsto YB$.

Finding determinant of L_A. For any $(r, s) \in \{1, \dots, m\} \times \{1, \dots, n\}$, compute $L_A(e_{rs})$,

$$\begin{aligned} L_A(e_{rs}) &= A\, e_{rs} \\ &= \begin{pmatrix} & & A_{1r} & & \\ 0 & \cdots & \vdots & \cdots & 0 \\ & & A_{mr} & & \end{pmatrix} \\ &\qquad\quad \text{in column } s \uparrow \\ &= A_{1r}e_{1s} + \cdots + A_{mr}e_{ms} \end{aligned}$$

Then by substituting in different values of r and s, we derive the matrix representation of L_A (with respect to ordered basis $\mathcal{H}$) in block form as

$$[L_A]_{\mathcal{H}} = \begin{pmatrix} A_{11}\,1_n & A_{12}\,1_n & \cdots & A_{1m}\,1_n \\ A_{21}\,1_n & A_{22}\,1_n & \cdots & A_{2m}\,1_n \\ \vdots & \vdots & \ddots & \vdots \\ A_{m1}\,1_n & A_{m2}\,1_n & \cdots & A_{mm}\,1_n \end{pmatrix} \tag{1}$$

If A is singular, it is clear that the left-multiplication by A operator has no inverse, which implies $\det(L_A) = 0 = \det(A)$. If A is an invertible matrix, then A is a product of elementary matrices, so there exists elementary matrices $E_1, \dots, E_k \in \mathbb{M}_{m\times m}(K)$ such that $A = E_k \cdots E_1$. Then $L_A = L_{E_k} \circ \cdots \circ L_{E_1}$. Then we are reduced to finding out the determinant of the left-multiply by elementary matrix operator.

Claim. For any elementary matrix $E \in \mathbb{M}_{m\times m}(K)$, $\det(L_E) = \det(E)^n$.

1. Case E is a "row swap" elementary matrix, then by substituting $A = E$ in (1), $[L_E]_{\mathcal{H}}$ consists of n row swaps from 1_{mn}. Then $\det(L_E) = (-1)^n = \det(E)^n$.
2. Case E is of a "multiply a row by $c \in K$" matrix, then examine (1) again, $[L_E]_{\mathcal{H}}$ is a diagonal matrix with all ones except n occurrences of c. Then $\det(L_E) = c^n = \det(E)^n$.
3. Case E is "add multiple of row to another row" matrix, then from (1), $[L_E]_{\mathcal{H}}$ will be triangular with 1's on the diagonal, so $\det(L_E) = 1 = \det(E)^n$.

Then from multiplicativity of determinant, recall that $\det(A) = \det(E_k) \cdots \det(E_1)$, then

$$\begin{aligned}\det(L_A) &= \det(L_{E_k}) \cdots \det(L_{E_1}) \\ &= \det(E_k)^n \cdots \det(E_1)^n \\ &= (\det(E_k) \cdots \det(E_1))^n \\ &= \det(A)^n\end{aligned}$$

Finding determinant of R_B. For any $(r, s) \in \{1, \ldots, m\} \times \{1, \ldots, n\}$, compute $R_B(e_{rs})$,

$$\begin{aligned}R_B(e_{rs}) &= e_{rs} B \\ &= \begin{pmatrix} & 0 & \\ & \vdots & \\ B_{s1} & \cdots & B_{sn} \\ & \vdots & \\ & 0 & \end{pmatrix} \leftarrow \text{in } r\text{-th row} \\ &= B_{s1} e_{r1} + \cdots + B_{sn} e_{rn}\end{aligned}$$

This time, obtain the matrix representation of R_B (with respect to ordered basis $\mathcal{H}$) in block form as

$$[R_B]_{\mathcal{H}} = \begin{pmatrix} B^t & & & \\ & B^t & & \\ & & \ddots & \\ & & & B^t \end{pmatrix} \leftarrow \text{repeats } m \text{ times on diagonal} \tag{2}$$

If B is singular, it is again clear that R_B has no inverse, and $\det(R_B) = 0$. If B is invertible, exists elementary matrices $E_1, \ldots, E_k \in \mathbb{M}_{n \times n}(K)$ such that $B = E_1 \cdots E_k$, then $R_B = R_{E_k} \circ \ldots \circ R_{E_1}$. Now using a similar argument, we can find the determinant of R_B.

Claim. For any elementary matrix $E \in \mathbb{M}_{n \times n}(K)$, $\det(R_E) = \det(E)^m$.

1. Case E is a row swap matrix, then from (2), $[R_E]_{\mathcal{H}}$ contains m row swaps from 1_{mn}, so $\det(R_E) = (-1)^m = \det(E)^m$.
2. Case E is of "multiply a row by $c \in K$" type, then in (2), $[R_E]_{\mathcal{H}}$ is a diagonal matrix with all ones except for m occurrences of c. Then $\det(L_E) = c^m = \det(E)^m$.
3. Case E is "add multiple of row to another row" matrix, then from (2), $[R_E]_{\mathcal{H}}$ will be triangular with 1's on diagonal, so $\det(R_E) = 1 = \det(E)^m$.

Then from multiplicativity of determinant, we get $\det(R_B) = \det(B)^m$.

Finally, as $\Phi = L_A \circ R_B$, $\det(\Phi) = \det(L_A) \det(R_B) = \det(A)^n \det(B)^m$. ■

Question 5

> Let K be a field, and let $x_1, \dots, x_n \in K$ be n elements of K. The $n \times n$ *van der Monde determinant* of $x_1, \dots, x_n$ is defined as
>
> $$V(x_1, x_2, \dots, x_n) := \det \begin{pmatrix} 1 & 1 & \cdots & 1 \\ x_1 & x_2 & \cdots & x_n \\ \vdots & \vdots & \ddots & \vdots \\ x_1^{n-1} & x_2^{n-1} & \cdots & x_n^{n-1} \end{pmatrix}.$$
>
> Show that
>
> $$V(x_1, x_2, \dots, x_n) = \prod_{1 \leqslant i < j \leqslant n} (x_j - x_i) \quad \text{in } K.$$

Proof. Proceed by induction on n.

Base case. For $n = 2$, $x_1, x_2 \in K$,

$$\begin{aligned} V(x_1, x_2) &= \det \begin{pmatrix} 1 & 1 \\ x_1 & x_2 \end{pmatrix} \\ &= x_2 - x_1 = \prod_{1 \leqslant i < j \leqslant n} (x_j - x_i) \end{aligned}$$

Induction hypothesis. Suppose for any $n-1$ elements $x_2, \dots, x_n \in K$, we have $V(x_2, \dots, x_n) = \prod_{2 \leqslant i < j \leqslant n} (x_j - x_i)$.

Then for n elements $x_1, \dots, x_n \in K$,

$$V(x_1, x_2, \dots, x_n) = \begin{vmatrix} 1 & 1 & \cdots & 1 \\ x_1 & x_2 & \cdots & x_n \\ \vdots & \vdots & \ddots & \vdots \\ x_1^{n-1} & x_2^{n-1} & \cdots & x_n^{n-1} \end{vmatrix}$$

subtract x_1 times of $n-1$-th row from n-th row

$$= \begin{vmatrix} 1 & 1 & \cdots & 1 \\ x_1 & x_2 & \cdots & x_n \\ \vdots & \vdots & \ddots & \vdots \\ 0 & x_2^{n-2}(x_2 - x_1) & \cdots & x_n^{n-2}(x_n - x_1) \end{vmatrix}$$

successively subtract $k-1$-th row from k-th row as k iterates from $n-1$ to 2, and get

$$= \begin{vmatrix} 1 & 1 & \cdots & 1 \\ 0 & x_2 - x_1 & \cdots & x_n - x_1 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & x_2^{n-2}(x_2 - x_1) & \cdots & x_n^{n-2}(x_n - x_1) \end{vmatrix}$$

co-factor expansion along first column

$$= \begin{vmatrix} x_2 - x_1 & x_3 - x_1 & \cdots & x_n - x_1 \\ x_2(x_2 - x_1) & x_3(x_3 - x_1) & \cdots & x_n(x_n - x_1) \\ \vdots & \vdots & \ddots & \vdots \\ x_2^{n-2}(x_2 - x_1) & x_3^{n-2}(x_3 - x_1) & \cdots & x_n^{n-2}(x_n - x_1) \end{vmatrix}$$

since every column has a scalar I can factor out, take determinant of the transpose then use multi-linearity

$$\begin{aligned}
&= \begin{vmatrix} x_2 - x_1 & x_2(x_2 - x_1) & \cdots & x_2^{n-2}(x_2 - x_1) \\ x_3 - x_1 & x_3(x_3 - x_1) & \cdots & x_3^{n-2}(x_3 - x_1) \\ \vdots & \vdots & \ddots & \vdots \\ x_n - x_1 & x_n(x_n - x_1) & \cdots & x_n^{n-2}(x_n - x_1) \end{vmatrix} \\
&= \prod_{j=2}^{n}(x_j - x_1) \begin{vmatrix} 1 & x_2 & \cdots & x_2^{n-2} \\ 1 & x_3 & \cdots & x_3^{n-2} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & x_n & \cdots & x_n^{n-2} \end{vmatrix} \\
&= \prod_{j=2}^{n}(x_j - x_1) \begin{vmatrix} 1 & 1 & \cdots & 1 \\ x_2 & x_3 & \cdots & x_n \\ \vdots & \vdots & \ddots & \vdots \\ x_2^{n-2} & x_3^{n-2} & \cdots & x_n^{n-2} \end{vmatrix} \\
&= \prod_{j=2}^{n}(x_j - x_1)\, V(x_2, \dots, x_n)
\end{aligned}$$

now applying induction hypothesis,

$$\begin{aligned}
&= \prod_{j=2}^{n}(x_j - x_1) \prod_{2 \leqslant i < j \leqslant n}(x_j - x_i) \\
&= \prod_{1 \leqslant i < j \leqslant n}(x_j - x_i)
\end{aligned}$$

□

Question 6

Proof. Proceed by induction on n.

Base case. For $n = 2$, let $a_1, a_2 \in K$,

$$\begin{aligned}\frac{(a_1, a_2)}{(a_2)} &= \frac{\det\begin{pmatrix} a_1 & 1 \\ -1 & a_2 \end{pmatrix}}{a_2} \\ &= \frac{a_1 a_2 + 1}{a_2} \\ &= a_1 + \frac{1}{a_2}\end{aligned}$$

Induction hypothesis. Suppose for any $n-1$ elements $a_2, \dots, a_n \in K$,

$$a_2 + \cfrac{1}{\cfrac{\ddots}{a_3 + \cfrac{1}{\ddots + \cfrac{1}{a_{n-1} + \cfrac{1}{a_n}}}}} = \frac{(a_2, a_3, \dots, a_n)}{(a_3, \dots, a_n)}.$$

Then for any n elements $a_1, \dots, a_n \in K$, compute $(a_1, \dots, a_n)$ by expanding along first row,

$$(a_1, \dots, a_n) = a_1 \begin{vmatrix} a_2 & 1 & & & \\ -1 & a_3 & \ddots & \mathbf{0} & \\ & \ddots & \ddots & \ddots & \\ & \mathbf{0} & \ddots & a_{n-1} & 1 \\ & & & -1 & a_n \end{vmatrix} - \begin{vmatrix} -1 & 1 & & & \\ 0 & a_3 & \ddots & \mathbf{0} & \\ & -1 & \ddots & \ddots & \\ & \mathbf{0} & \ddots & a_{n-1} & 1 \\ & & & -1 & a_n \end{vmatrix}$$

expand second term along its first column

$$\begin{aligned} &= a_1(a_2, a_3, \dots, a_n) + \begin{vmatrix} a_3 & 1 & & \mathbf{0} \\ -1 & \ddots & \ddots & \\ & \ddots & \ddots & 1 \\ \mathbf{0} & & -1 & a_n \end{vmatrix} \\ &= a_1(a_2, a_3, \dots, a_n) + (a_3, \dots, a_n) \end{aligned}$$

then division throughout by $(a_2, \dots, a_n)$ (assuming it makes sense) will allow us to apply the induction hypothesis

$$\begin{aligned}
\frac{(a_1, a_2, \dots, a_n)}{(a_2, \dots, a_n)} &= a_1 + \frac{(a_3, \dots, a_n)}{(a_2, a_3, \dots, a_n)} \\
&= a_1 + \cfrac{1}{\cfrac{(a_2, a_3, \dots, a_n)}{(a_3, \dots, a_n)}} \\
&= a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cfrac{\ddots}{\ddots + \cfrac{1}{a_{n-1} + \cfrac{1}{a_n}}}}}
\end{aligned}$$

□