CP3208 Literature Review – Automaticity of Algebraic Structures

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There are numerous connections between group theory and automata theory. Algebraic structures like groups could be described using automata theory, while concepts from algebra also come in useful in automata theory.

1 Semigroups, Monoids and Groups

We will first introduce some relevant concepts regarding common algebraic structures.

Let G be a set and let $\circ : G \times G \to G$.

Definition 1. The structure $(G, \circ)$ is called a **semigroup** iff $\circ$ is associative, that is for all $x, y, z, \in G$, $x \circ (y \circ z) = (x \circ y) \circ z$.

Definition 2. The structure $(G, \circ, e)$ is called a **monoid** iff $(G, \circ)$ is a semigroup and for all $x \in G$, $x \circ e = e \circ x = x$.

Definition 3. The structure $(G, \circ, e)$ is called a **group** iff $(G, \circ, e)$ is a monoid and for all $x \in G$, there exists $y \in G$ satisfying $x \circ y = e$.

Example 4. The structure $(\{1, 2, ...\}, +)$ is a semigroup. Adding 0 to it gives us the monoid of natural numbers $(\mathbb{N}, +, 0)$. Considering its closure gives us the additive group of integers $(\mathbb{Z}, +, 0)$.

Definition 5. A semigroup $(G, \circ)$ is called **finitely-generated** iff there exists finite subset $F \subseteq G$ where for each $x \in G$, there exists $y_1, y_2, \dots, y_n \in A$ satisfying $x = y_1 \circ y_2 \circ ... \circ y_n$. The set F is set of generators for G.

Remark. Definition 5 generalises to monoids and groups in the obvious way. Also note that the structures in Example 4 are finitely-generated, with the generator 1.

Groups have been studied by mathematicians for centuries in their own right, but only recently have the connection between group theory and automata theory been made.[Hod76; Hod83; KN94; ECHLPT92]

2 Notions of Automaticity for Groups and Monoids

The low complexity of automata motivates the search for automatic presentations of various algebraic structures. For example, the word problem for groups in general is well-known to be undecidable. However, most reasonable formalisations of automatic groups will have their word problem solvable with a quadratic time complexity.[ECHLPT92; KKM11]

We first introduce the framework proposed by Hodgson[Hod76; Hod83] and Khoussainov and Nerode[KN94].

Definition 6. A semigroup $(G, \circ)$ is called **fully automatic** iff

- G is regular over Σ^* with Σ being a finite alphabet,
- $\circ : G \times G \to G$ is an automatic function.

Fully automatic monoids and groups are defined analogously.

Remark. In the original literature the authors referred to their definition as just 'automatic". We follow the convention in [Ste18] and use the term "fully automatic" in the sense that the *full* semigroup operation is automatic. At the same time, we also disambiguate it from the following more popular definition, attributed to Epstein, Cannon, Holt, Levy, Paterson and Thurston[ECHLPT92].

Definition 7. Let $(G, \circ)$ be a semigroup generated by a finite subset $F \subseteq G$. $(G, \circ)$ is **automatic** iff

- G is a regular subset of F^*,
- each $x \in G$ has exactly a unique representative in F^*, and
- for each $y \in G$, the multiplication map $(\circ y) : G \to G$ defined as $x \mapsto x \circ y$ is automatic.

The notion of having exactly 1 representative is equivalent to allowing multiple representatives but with equality being automatic. To see why, consider how we can unambiguously choose the lexiographically smallest element as the unique representative.

We can make some observations to highlight the differences between Definition 6 and Definition 7.

Theorem 8. *There exists a semigroup which is automatic but not fully automatic.*

Proof. Theorem 12.22 in [Ste18] provides an explicit construction. □

Kharlampovich, Khoussainov and Miasnikov were the first to formally consider the related concept of a Cayley automatic group [KKM11]. It is sometimes also called graph automatic because it considers the Cayley graph of a group as the automatic structure.

Definition 9. A finitely generated group G generated by F is **Cayley automatic** iff the following conditions hold for some finite alphabet Σ,

- representatives of G is regular in Σ^*,
- each $x \in G$ has a unique representative in Σ^*,
- for each $y \in F$, the right multiplication by y map is automatic.

Remark. We note that Definition 7 is a special case of this, with the extra condition that natural representatives are chosen, that is $\Sigma = F$ the generating set. As a result Definition 9 defines a strictly bigger class of automatic groups.

Example 10. The Heisenberg group $\mathcal{H}_3(\mathbb{Z})$ defined as

$$\left\{ \begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix} : a, b, c \in \mathbb{Z} \right\}$$

is Cayley automatic (6.6 in [KKM11]), but not automatic (8.1.1 in [ECHLPT92]).

Definition 11. G is **Cayley biautomatic** if the third condition in Definition 9 also holds for left multiplication.

It is interesting to note that there exists a Cayley automatic group that is not Cayley biautomatic. [MŠ12]

3 General Automatic Structures

For general structures we consider the more general framework of Hodgson[Hod76; Hod83] and Khoussainov and Nerode[KN94].

Definition 12. A structure $(A, R_1, R_2, \dots, R_k, f_1, f_2, \dots, f_h)$ is automatic iff

- the set A is regular in Σ^* where Σ is some finite alphabet,
- all the relations $R_1, R_2, \dots, R_k$ are automatic, and
- all the functions $f_1, f_2, \dots, f_h$ are automatic.

We do not distinguish between structures that are automatic and structures that are merely isomorphic to an automatic structure.

Example 13. A fully automatic group $(G, \circ)$ satisfying Definition 6 is an automatic structure.

Definition 14 (Ordinals)**.** A set α is an ordinal iff every $\beta \in \alpha$ is a subset of α and $(\alpha, \in)$ is a well-order.

Ordinals can be thought of as equivalence classes of well-ordered sets in set theory. They naturally describe how many times a process is iterated, possibly transfinitely many times, and are commonly encountered in model theory and recursion Theory. It is hence expected for there to be a suitable characterisation of ordinals in automata theory.[Del04]

Theorem 15 (Delhommé)**.** *Let α be an ordinal, $(\alpha, +, \in)$ is automatic iff $\alpha < \omega^\omega$.*

Proof. Theorem 13.10 in [Ste18] provides a proof in English. □

4 Semiautomatic Structures and Groups

Seeking more general ways to utilise finite automata for representing non-automatic structures, Jain, Khoussainov, Stephan, Teng and Zou proposed semiautomatic structures as a generalisation of Definition 12[JKSTZ17].

Definition 16. Let $f : R^n \to R$ be a function. f is **semiautomatic** iff fixing $n-1$ inputs, the resultant $R \to R$ function is automatic. The definition of a semiautomatic relation is analogous since we can view it as a $\{0, 1\}$-valued function.

Definition 17. A structure $(A, f_1, f_2, \dots, f_k; g_1, g_2, g_h)$ is **semiautomatic** iff

- the set A is regular in Σ^* where Σ is some finite alphabet,
- all the functions $f_1, f_2, \dots, f_k$ are automatic, and
- all the functions $g_1, g_2, \dots, g_h$ are semiautomatic

where without loss of generality we treat relations as functions.

Jain, Khoussainov and Stephan used semiautomaticity to describe group-like structures.[JKS18] They cited a related result by Tsankov.[Tsa09]

Theorem 18. *The structure $(\mathbb{Q}, +, =)$ has no automatic presentation.*

Theorem 19 (21 in [JKSTZ17])**.** *The ordered group $(\mathbb{Q}, <, =; +)$ of rationals is semiautomatic.*

We note that $\mathbb{Q}$ is not finitely generated, which leads on to this still open question.

Question 20. *Are all finitely generated semiautomatic groups Cayley automatic?*

Several important results presented in [JKSTZ17] concern the automatic and semiautomatic presentations of naturally-occuring algebraic structures.

Theorem 21. *The ordered rings*

- $(\mathbb{Z}[\phi], +, <, =; \cdot)$ *where* $\phi = \frac{1+\sqrt{5}}{2}$ *is the golden ratio, and*
- $(\mathbb{Z}(\sqrt{n}), \mathbb{Z}, +, <, =; \cdot)$ *for every natural number* n

admit semiautomatic presentations.

5 Future Directions

We can possibly gain more insights in the automaticity of various algebraic structures. A good place to start would be to consider a natural generalisation of Theorem 21

Question 22. *Let n be a natural number. Consider the ordered ring $(\mathbb{Z}(\sqrt[3]{n}), +, \cdot, <)$. What can be made semiautomatic? What can be made automatic?*

This could be generalised further.

Question 23. *Let n, p be a natural numbers. What about $(\mathbb{Z}(\sqrt[p]{n}), +, \cdot, <)$?*

We could also explore the structures of ordinals and consider some generalisations to Theorem 15.

Question 24. *Let $\alpha \geq \omega^\omega$ be an ordinal. Consider the structure $(\alpha, \in, +, \cdot)$. What relations admit semiautomatic representations?*

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