MA2104 Assignment 2

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Question 1.

(a) $\lim\limits_{(x,y)\to(0,0)} \dfrac{2xy^3}{x^2+8y^6}$.

Solution. First take limit along the line $y = 0$,

$$\begin{aligned}\lim_{(x,0)\to(0,0)} \frac{2x(0)^3}{x^2+8(0)^6} &= \lim_{x\to 0} \frac{0}{x^2} \\ &= 0\end{aligned}$$

Next, take the limit along the curve $x = y^3$,

$$\begin{aligned}\lim_{(y^3,y)\to(0,0)} \frac{2(y^3)y^3}{(y^3)^2+8y^6} &= \lim_{y\to 0} \frac{2y^6}{9y^6} \\ &= \frac{2}{9}\end{aligned}$$

Hence the limit does not exist. ∎

(b) $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^3+4x^2+2y^2}{2x^2+y^2}$.

Solution.

$$\lim_{(x,y)\to(0,0)} \frac{x^3+4x^2+2y^2}{2x^2+y^2} = \lim_{(x,y)\to(0,0)} \frac{x^3}{2x^2+y^2} + 2$$

Now it remains to compute $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^3}{2x^2+y^2}$, note that

$$\begin{aligned} y^2 &\geq 0 \\ 2x^2+y^2 &\geq 2x^2 \geq 0 \\ \frac{x^3}{2x^2+y^2} \leq \left|\frac{x^3}{2x^2+y^2}\right| &\leq \frac{\left|x^3\right|}{2x^2} = \frac{|x|}{2}\end{aligned}$$

and similarly

$$-\frac{|x|}{2} \leq -\left|\frac{x^3}{2x^2+y^2}\right| \leq \frac{x^3}{2x^2+y^2}$$

Since $\lim_{(x,y)\to(0,0)} -\frac{|x|}{2} = \lim_{(x,y)\to(0,0)} \frac{|x|}{2} = 0$, by Squeeze theorem,

$$\lim_{(x,y)\to(0,0)} \frac{x^3}{2x^2+y^2} = 0,$$

then we have

$$\lim_{(x,y)\to(0,0)} \frac{x^3+4x^2+2y^2}{2x^2+y^2} = 2. \qquad \blacksquare$$

Question 2.

(i)

$$\begin{aligned}\frac{\partial z}{\partial x} &= \frac{\partial z}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial x} \\ &= f_u(u,v)(2xy) + f_v(u,v)(0) \\ &= 2xyf_u(u,v)\end{aligned} \qquad \blacksquare$$

(ii)

$$\begin{aligned}\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) &= \frac{\partial}{\partial y}\left(2xyf_u(u,v)\right) \\ &= 2x\left(y\frac{\partial}{\partial y}(f_u(u,v)) + f_u(u,v)\right)\end{aligned}$$

Note that

$$\begin{aligned}\frac{\partial}{\partial y}(f_u(u,v)) &= f_{uu}(u,v)\frac{\partial u}{\partial y} + f_{uv}(u,v)\frac{\partial v}{\partial y} \\ &= x^2 f_{uu}(u,v) + 2yf_{uv}(u,v)\end{aligned}$$

So

$$\begin{aligned}\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) &= 2x\left(x^2yf_{uu}(u,v) + 2y^2f_{uv}(u,v) + f_u(u,v)\right) \\ &= 2x^3yf_{uu}(u,v) + 4xy^2f_{uv}(u,v) + 2xf_u(u,v)\end{aligned} \qquad \blacksquare$$

Question 3.

Solution. Let $f(x, y) = 2x^2 + y^2$ be the ellipse the insect moves on, then

$$\nabla f(x, y) = \langle 4x, 2y \rangle$$
$$\nabla f(1, 1) = \langle 4, 2 \rangle$$

By inspection, a unit tangent vector to f at $(1, 1)$ will be

$$\mathbf{u} := \frac{1}{2\sqrt{5}} \langle 2, -4 \rangle = \left\langle \frac{1}{\sqrt{5}}, \frac{-2}{\sqrt{5}} \right\rangle.$$

Now compute $\nabla T(1, 1)$,

$$\nabla T(x, y) = \langle 6x - 2y, -2x \rangle$$
$$\nabla T(1, 1) = \langle 4, -2 \rangle$$

Taking the directional derivative

$$\begin{aligned} D_{\mathbf{u}} T(1, 1) &= \nabla T(1, 1) \cdot \mathbf{u} \\ &= \langle 4, -2 \rangle \cdot \left\langle \frac{1}{\sqrt{5}}, \frac{-2}{\sqrt{5}} \right\rangle \\ &= \frac{8}{\sqrt{5}} \ {}^\circ\mathrm{C\ cm}^{-1} \end{aligned}$$

Then by chain rule, rate of change of temperature will be

$$\frac{24}{\sqrt{5}} \ {}^\circ\mathrm{C\ s}^{-1}. \qquad \blacksquare$$

Question 4.

If $\mathbf{r}(t) \neq \mathbf{0}$, show that $\frac{d}{dt} |\mathbf{r}(t)| = \frac{1}{|\mathbf{r}(t)|} \mathbf{r}(t) \cdot \mathbf{r}'(t)$.

Proof. Suppose $\mathbf{r}(t) \neq \mathbf{0}$,

$$\frac{d}{dt} |\mathbf{r}(t)| = \frac{d}{dt} \left(\sqrt{\mathbf{r}(t) \cdot \mathbf{r}(t)} \right)$$

by chain rule outside and product rule inside,

$$\begin{aligned} &= \frac{1}{2\sqrt{\mathbf{r}(t) \cdot \mathbf{r}(t)}} \cdot (\mathbf{r}(t)\mathbf{r}'(t) + \mathbf{r}'(t)\mathbf{r}(t)) \\ &= \frac{2\mathbf{r}(t)\mathbf{r}'(t)}{2\sqrt{|\mathbf{r}(t)|^2}} \\ &= \frac{1}{|\mathbf{r}(t)|} \mathbf{r}(t) \cdot \mathbf{r}'(t) \end{aligned} \qquad \square$$