Universität Regensburg
FAKULTÄT FÜR MATHEMATIK

Master Thesis

Immersions and Stiefel-Whitney Classes of Manifolds

Gesina Schwalbe

October 4, 2018

Supervisor: Dr. Georgios Raptis

Abstract

In 1944, Whitney's immersion theorem [Whi44] showed that every n-manifold can be immersed into real space of dimension $2n - 1$. Thereby the following question arose:

> What is the minimum codimension k such that for every n-dimensional compact manifold there exists an immersion $M \hookrightarrow \mathbb{R}^{n+k}$ into real space?

The immersion conjecture asserts that $k = n - \alpha(n)$ where $\alpha(n)$ is the number of ones in the binary notation of n. The goal of this thesis is to give a proof of a famous theorem by Massey [Mas60] that motivates this initial guess for k, and to prove the conjecture up to cobordism, which is a theorem by R. L. Brown [Bro71]. In both cases characteristic classes of manifolds will provide the key tooling.

More precisely, details on the following aspects will be given: In Chapter 2, a famous result by Hirsch and Smale [Hir59] is used to turn the problem of finding an immersion of certain codimension into a problem of finding a certain vector bundle monomorphism. This then admits an obstruction by characteristic classes, which is shown to vanish exactly for codimensions $k \geq n - \alpha(n)$ due to a theorem of Massey [Mas60] in Chapter 3, thus motivating the value of k in the immersion conjecture. Finally, a criterion due to Thom [Tho07] for indecomposability in the cobordism ring given by characteristic classes is the main ingredient for a theorem of R. L. Brown [Bro71], which states that the immersion conjecture is true up to cobordism. This is shown in detail in Chapter 4.

Both of the main theorems discussed in this thesis—the one by Massey and the one by R. L. Brown—were important steps towards the proof of the immersion conjecture, which was finalized as recent as 1985 by R. L. Cohen [Coh85] building on previous efforts by E. H. Brown and F. P. Peterson. An outline of the proof can be found in Chapter 5.

Contents

1 Introduction

Immersions are roughly speaking mappings between smooth manifolds that locally look like embeddings, globally however allow self-intersection, or “worse” in case the domain is not compact. First cornerstones of modern immersion theory are Whitney’s papers on this topic, remarkably his embedding and immersion theorems [Whi44] published in 1944, making immersions of certain codimension into real space always available for the investigation of manifolds [Spr05; Hir94].

Attempts to reduce the above codimension of $n-1$ for n manifolds were heated up, when a theorem of M. W. Hirsch [Hir59] building on important work of Smale allowed to answer the question about the existence of an immersion of certain codimension by the existence of a vector bundle monomorphism into Euclidean space of certain dimension. Using the theory of characteristic classes the latter can again be reformulated to the existence of a vector bundle of this certain dimension which is dual to the tangent bundle, i.e. looks like the pullback of a normal bundle. Amongst other valuable tooling, this brought in obstructions by Stiefel-Whitney classes.

In 1960, Massey [Mas60] showed with the help of Wu’s theorem [Wu50] from 1950 that these obstructions for compact n-manifolds vanish for exactly all codimensions greater than $n-\alpha(n)$, where $\alpha(n)$ is the number of ones in the binary notation of n. This motivated the conjecture that the answer to the question

> What is the minimum codimension k, such that for every n-dimensional compact manifold there exists an immersion $M \hookrightarrow \mathbb{R}^{n+k}$ into Euclidean space?

is $k = n - \alpha(n)$, later known as the immersion conjecture [Coh85].

The guess was partly reassured in 1971 by a result of R. L. Brown’s investigations of the cobordism ring [Bro71], proving the conjecture up to cobordism. It heavily relies on Thom’s results from 1954 on the structure of the cobordism ring [Tho54], and its idea is to provide generators of the cobordism ring that fulfill the immersion property, partly inspired by constructions introduced in Dold’s work on a complete generating set of the cobordism ring published in 1956 [Dol56]. However, the key point prior to the construction of generators is to find a criterion for indecomposability of cobordism classes, which is given by an indicator characteristic class.

After a long series of steps, notably by E. H. Brown Jr. and F. P. Peterson beginning in 1963 (see [Coh85]), the proof of the conjecture was eventually finalized in 1985 by R. L. Cohen, by showing that all stable normal bundles of n-manifolds factor over

$\mathrm{B}O(n-\alpha(n))$. Its main idea is to refine Steenrod's classification theorem for vector bundles by finding for each n a classifying space specifically for stable normal bundles of n-manifolds, over which all classifying maps factor. It is then shown that this factorization admits a lift to $\mathrm{B}O(n-\alpha(n))$. Important intermediate steps are to prove both, the existence and the lift, up to cobordism, and then *de-Thom-ify*, as they called it, the results. In [BP79] E. H. Brown Jr. and F. P. Peterson conducted all these steps except for de-Thom-ifying the lift, which was done by R. L. Cohen in [Coh85].

The focus of this thesis is on the results by Massey and R. L. Brown, especially the role of characteristic classes in tackling these problems. The needed reformulation using the Hirsch-Smale theorem, as well as the obstruction by characteristic classes, is given in Chapter 2. A detailed proof of Massey's theorem follows in Chapter 3, and a proof of Brown's theorem finalizes the efforts in Chapter 4. A final, more detailed outlook on the proof of the immersion conjecture can be found in Chapter 5.

The reader is assumed to be familiar with the concepts of Steenrod squares, characteristic classes, especially Stiefel-Whitney classes, and the unoriented cobordism ring. Those theories will merely be recapitulated replacing a couple of proofs by references.

Notation

If not stated otherwise, the following notation is used.

- "Space" always means topological space, and all maps between spaces are continuous.
- For X, Y spaces, $[X, Y]$ denotes the set of homotopy classes of base point preserving maps $X \to Y$.
- All manifolds are smooth and closed.
- Cobordism always means unoriented cobordism.
- All vector bundles are real.
- The bundle ε^r over a space X is the rank-r trivial bundle $X \times \mathbb{R}^r \to X$.
- $\oplus$ denotes the usual Whitney sum of vector bundles.
- For a fiber bundle $\xi\colon E \to B$ denote by: $\mathrm{E}\xi := E$ the total space, $\mathrm{B}\xi := B$ the base space, $p_\xi := p$ the underlying surjection, $\mathrm{E}_b := p^{-1}(b)$ the fiber over a point $b \in \mathrm{B}$, 0_ξ the zero section if it is exists, and by $E^0 := E \setminus \mathrm{im} 0_\xi$ the total space without the zero section.
- $\mathbb{Z}_2 := \mathbb{Z}/2\mathbb{Z}$, $O(n)$ is the n-dimensional real orthogonal group, and O their limit, i.e. the group of all orthogonal matrices over $\mathbb{R}$.
- All homology groups have $\mathbb{Z}_2$ coefficients. Singular cohomology in degree r of a pair (X, A) with coefficients in a ring R is denoted by $H^r(X, A; R)$, with the short form

$H^r(X;R) \coloneqq H^r(X,\emptyset;R)$. Relative cohomology of a pointed space $(X,*)$ is denoted by $\widetilde{H}^r(X;R) \cong H^r(X,*;R)$. Analogously for singular homology.

- For pairs of spaces (X,A), (X,B), R a unital commutative ring, and $i,j \in \mathbb{N}$ denote
  - the (relative) cup-product by
  $$\begin{aligned} -\cup-\colon H^i(X,A;R)\times H^j(X,B;R) &\longrightarrow H^{i+j}(X,A\cup B;R)\\ (x,y) &\longmapsto x\cup y\ , \end{aligned}$$
  - and the (relative) cap-product by
  $$\begin{aligned} -\cap-\colon H^i(X,A;R)\times H_{i+j}(X,A\cup B;R) &\longrightarrow H_j(X,B;R)\\ (x,\alpha) &\longmapsto x\cap\alpha \eqqcolon \langle x,\,\alpha\rangle\ \ . \end{aligned}$$
  - For a manifold M^n, $[M]\in H_n(M)$ is the unique fundamental class and generator.

2 Formulation of the Immersion Conjecture

This chapter is dedicated to reviewing the concepts and results, which are needed to formulate the immersion conjecture and its connection to the theory of characteristic classes.

The (re)formulation in Section 2.3 uses as main ingredient a theorem by Hirsch and Smale on the relation between immersions and vector bundle monomorphisms presented in Subsection 2.1.2. This, together with other required definitions and properties of immersions, is contained in Section 2.1. Thereafter, in Section 2.2, characteristic classes of vector bundles are reviewed. Most importantly, the Stiefel-Whitney characteristic classes are recalled in Subsection 2.2.2, together with an outline of the way these will be particularly useful throughout this thesis. The last section then contains an explanation on how they contribute to the immersion problem via obstructions.

This chapter is meant as revision and outline, therefore a couple of preliminary results are merely referenced without proof.

2.1 Immersions

This section recapitulates the definition some properties of immersions.

2.1.1 Definition

As already mentioned, immersions are technically local embeddings of manifolds. From a different point of view, immersions are merely a special case of monomorphisms of vector bundles. So, recall that a morphism $(\xi_1 \colon \mathrm{E}_1 \to X_1) \to (\xi_2 \colon E_2 \to X_2)$ of vector bundles over different spaces is a map $F \colon E_1 \to E_2$ which is linear on fibers, and that covers its restriction to the zero section, i.e. it makes the following diagram commute:

$$\begin{tikzcd} E_1 \arrow[r, "F"] \arrow[d, "\xi_1"] & E_2 \arrow[d, "\xi_2"] \\ X_1 \arrow[r, "F|_{0_{\xi_1}}"] & X_2 \end{tikzcd}$$

Further, remember the fact that such a morphism is a monomorphism in the category of vector bundles if and only if its restriction to each fiber is injective.

Definition 2.1. A smooth map $f\colon M \to N$ of smooth manifolds is called an *immersion*, written $M \hookrightarrow N$, if its differential $\mathrm{D}f\colon \mathrm{T}M \to \mathrm{T}N$ is a monomorphism of vector bundles. A homotopy $H\colon M \times I \to N$ which is an immersion in each stage is called *regular*.

Remark 2.2. Let M and N be manifolds.

i) Immersions are local embeddings, i.e. for an immersion $f\colon M \hookrightarrow N$, around every point in M there is an open neighborhood on which f is a diffeomorphism onto its image. More descriptive, immersions are mappings that do not allow creases, respectively sharp bends, or puncturing (see e.g. [95] for nicely illustrated examples) However, globally, immersions need not be injective since e.g. self-intersections of the image are allowed.

   Proof. This is a conclusion from the implicit function theorem. For details see e.g. [Hir94, Chap. 1, Theorem 3.1]. □

ii) Embeddings of manifolds are exactly those injective immersions that are topological embeddings. If M is compact, any injective immersion $f\colon M \to N$ is an embedding.

   Proof. Use the fact that for manifolds compact is equivalent to sequentially compact, in order to get

$$\left\{ y = \lim_n f(x_n) \;\middle|\; (x_n)_{n\in\mathbb{N}} \subset M \text{ without limit} \right\} \cap M = \emptyset$$

   directly from injectivity of f, and compactness of M and $f(M)$. Then apply [Ada93, Chap. II, Lemma 2.6]. □

2.1.2 The Hirsch-Smale Theorem

It is easy to see that in general not all vector bundle monomorphisms between tangent bundles of smooth manifolds need to be the differential of an immersion. However, taking the differential gives a canonical inclusion of the set of immersions into the set of vector bundle monomorphisms. A theorem of Hirsch and Smale states that this inclusion actually is a homotopy equivalence, translating questions on the existence of immersions into the context of characteristic classes of vector bundles.

For the formulation, one has to equip the respective sets with a topology as follows.

Definition 2.3. Let M, N be closed smooth manifolds of dimensions $\dim M < \dim N$.

i) Equip the set of all vector bundle monomorphisms from ξ_1 to ξ_2 with the compact-open topology (see e.g. [Hat02]), and denote that space by $\mathrm{Mono}(\xi_1, \xi_2)$. Note that a path between monomorphisms F_1 and F_2 in the space $\mathrm{Mono}(\xi, \eta)$ is an homotopy from F_1 to F_2 which is a vector bundle monomorphism in each stage.

ii) Taking the differential yields an injection

$$\mathrm{Imm}(M,N) \longrightarrow \mathrm{Mono}(\mathrm{T}M,\mathrm{T}N)\ , \quad f \longmapsto \mathrm{D}f$$

of the set $\mathrm{Imm}(M,N)$ of all immersions from M to N. Equip $\mathrm{Imm}(M,N)$ in the following with the subspace topology. This results in the weak topology described in [Hir94, Section 2.1], which equals the Whitney C^1-topology since M was chosen to be compact. By the way, $\mathrm{Imm}(M,N)$ is open in $C^1(M,N)$ equipped with the Whitney C^1-topology (see [Hir94, Section 2.1, Theorem 1.1]).

Now one can state the major result in immersion theory by Hirsch using preliminary work of Smale [Hir59, Sections 5 and 6]. The following formulation is according to [CT89, Theorem 1.2].

Theorem 2.4 (Hirsch-Smale)**.** *Let M, N be closed manifolds with* $\dim M < \dim N$. *Then the differential map* $\mathrm{D}\colon \mathrm{Imm}(M,N) \to \mathrm{Mono}(\mathrm{T}M,\mathrm{T}N)$ *induces isomorphisms on the homotopy groups. Especially,*

$$\mathrm{D}_*\colon \pi_0(\mathrm{Imm}(M,N)) \xrightarrow{\sim} \pi_0(\mathrm{Mono}(\mathrm{T}M,\mathrm{T}N))$$

describes an isomorphism of path-connected components. Therefore, every vector bundle monomorphism $F\colon \mathrm{T}M \to \mathrm{T}N$ is homotopic (through vector bundle monomorphisms) to a monomorphism which is the differential $\mathrm{D}f$ *of a smooth map $f\colon M \to N$, i.e. of an immersion.*

Proof. See [EM02, Theorem 8.2.1] or the original paper [Hir59]. □

Thus, any monomorphism of vector bundles over smooth, closed manifolds M and N implies the existence of an immersion from M to N. This conclusion will be needed to reformulate the immersion problem.

2.1.3 Normal Bundles

Another nice property of immersions is that every immersion gives rise to a normal bundle. This will finally make it possible to translate the existence of an immersion of certain codimension into the existence of a vector bundle of certain rank fulfilling a homotopy invariant lifting property.

Definition 2.5. Let $\iota\colon M^n \hookrightarrow N^{n+r}$ be an immersion of smooth manifolds. The *normal bundle ν_ι of ι* is the well-defined, quotient bundle $\iota^*\mathrm{T}N/\mathrm{T}M$ of rank r, respectively the one fulfilling $\nu_\iota \oplus \mathrm{T}M \cong \iota^*\mathrm{T}N$.

Recall that manifolds have the very handy property that they admit a unique tangent bundle. Similarly, a normal bundle of an immersion into Euclidean space is unique up to a notion of stable equivalence, under which characteristic classes will turn out to be invariant.

Definition 2.6. Call two vector bundles ξ_1, ξ_2 over the same space *stably equivalent* in case there are $s_1, s_2 \in \mathbb{N}$ such that $\xi_1 \oplus \varepsilon^{s_1} \cong \xi_2 \oplus \varepsilon^{s_2}$.

Now the promised notion of the stable normal bundle can be clarified.

Lemma/Definition 2.7. Let M^n be a closed, smooth manifold. Then all normal bundles of immersions of M into Euclidean spaces are stably equivalent. The resulting equivalence class is called the *stable normal bundle of* M, written ν_M. When working with vector bundles in a context that is stable in the above sense, like e.g. characteristic classes, the stable normal bundle of M may be identified with an arbitrary representative of its class.

Proof (sketch). First show that every normal bundle of an immersion is stably equivalent to the normal bundle of *some* embedding (i.e. some injective immersion). Then ensure that all normal bundles of embeddings are stably equivalent.

Immersions Any immersion $\iota\colon M \hookrightarrow \mathbb{R}^{n+r}$ can be raised to higher codimension by concatenation with the linear embedding $l\colon \mathbb{R}^{n+r} \hookrightarrow \mathbb{R}^{n+r+s}$ into the first components. As the normal bundles ν_ι and $\nu_{l\circ\iota} \cong \nu_\iota \oplus \varepsilon^s$ are stably equivalent, raising the codimension does not change the stable equivalence class. Furthermore, since a regular homotopy yields an isomorphism on the normal bundles, it suffices to show:
claim. For $r > n$ every immersion $M \hookrightarrow \mathbb{R}^{n+r}$ is regularly homotopic to an embedding.

For the claim use bumping techniques to show that for $r > n$ every immersion $\iota\colon M \hookrightarrow \mathbb{R}^{n+r}$ is regularly homotopic to an injective immersion. This is e.g. [Ada93, Chap. II, Lemma 2.5]. However, as M is compact, injective immersions are embeddings.

Embeddings By Whitney's embedding theorem (see e.g. [Ada93, Chap. II.2]), it is known that every manifold admits an embedding into some real space. Further, by e.g. the General Position theorem (compare [Sko08, Chap. 2]) or Haefliger's theorem (see e.g. [Ada93, Chap. II.1]), it is known that for sufficiently large $k \in \mathbb{N}$ all embeddings $M \hookrightarrow \mathbb{R}^{n+k}$ are isotopic, i.e. homotopic through embeddings, and hence their normal bundles are isomorphic. Therefore, all normal bundles of embeddings of a manifold are stably equivalent. $\square$

2.2 Characteristic Classes of Vector Bundles

The theory of characteristic classes provides the key tools for the rest of this thesis. Therefore, this section revises basic results, and recalls in detail several properties of Stiefel-Whitney classes. The latter are generators of all characteristic classes of vector bundles, and will be essential in proving the theorems of the subsequent chapters.

2.2.1 General Definition and Properties

Before starting off with the definition of characteristic classes, we recall the definition of universal bundles and Steenrod's classification theorem. For more details see [Die08, Chapter 14.4].

Lemma/Definition 2.8. i) Any topological group G admits a contractible space EG with a free G-action, and a corresponding principal G-bundle

$$\gamma^G \colon \mathrm{EG} \longrightarrow \mathrm{B}G \coloneqq \mathrm{EG}/G\ ,$$

called the *universal G-bundle*, where γ_G, EG, and BG are all unique up to homotopy. BG is called the *classifying space* for principal G-bundles. For construction and uniqueness see [Hat02, Example 1B.7 ff.], respectively note that universal coverings are unique up to homotopy.

ii) γ^G fulfills the following universal property: For any space X admitting the homotopy type of a CW-complex there is a bijection between $[X, \mathrm{B}G]$, which denotes the homotopy classes of maps from X to BG, and the isomorphism classes of principal G-bundles over X, given by

$$(f \colon X \to \mathrm{B}G) \longmapsto f^*\gamma^G\ .$$

This correspondence is natural in X, and is a version of Steenrod's classification theorem, see [Die08, Theorem 14.4.1], or [CT89, Theorem 1.4, p. 75].

As becomes clear directly from the statement, the classifying theorem serves in translating bundle theoretic problems into homotopy theoretic ones, which will be a crucial step in reformulating the immersion problem in Section 2.3. Such homotopy theoretic questions can then be tackled using known cohomological tools, which yields the general concept of characteristic classes. Some important examples for vector bundles, namely the Stiefel-Whitney classes, will be discussed in detail in Subsection 2.2.2.

Definition 2.9. A *characteristic class*

- of degree i
- with coefficients in a ring R
- for principal G-bundles for a group G

is a natural transformation

$$\mathrm{Cl} \colon [-, \mathrm{B}G] \Longrightarrow H^i(-; R)\ .$$

of contravariant functors from the category of spaces with the homotopy type of a CW-complex to the category of sets.

Remark 2.10. By Brown's representation theorem (e.g. [Hat02, Chap. 4.E]), $H^i(-;R)$ is a representable functor represented by the Eilenberg-MacLane space $K(i,R)$. Thus, by the Yoneda lemma, a characteristic class is represented by a morphism

$$\mathrm{cl}\colon \mathrm{B}G \longrightarrow K(i,R)$$

in Top, i.e. by a cohomology class cl of $\mathrm{B}G$. Thus, for a space X, which admits the homotopy type of a CW-complex, applying Cl to a principal G-bundle over X that is represented by a morphism $\eta\colon X \to \mathrm{B}G$ as in Definition 2.8.ii), yields

$$\mathrm{Cl}(X) = \eta^*\mathrm{cl} \in H^i(X;R) \ .$$

This describes a one-to-one correspondence between characteristic classes as above and cohomology classes in $H^i(BG;R)$, and in the following any characteristic class will be identified with its corresponding cohomology class.

As this thesis is mainly concerned with vector bundles, we have a closer look at the significance of classifying spaces in that context, especially at their stability property and its implications for normal bundles.

Lemma 2.11. *Let X be any space, let M^n be a manifold, and $r,s \in \mathbb{N}$.*

i) There is a natural equivalence between the category of n-dimensional vector bundles and that of principal $O(n)$-, respectively $\mathrm{GL}_n(\mathbb{R})$-bundles.

ii) The inclusion $\mathrm{B}O(r) \to \mathrm{B}O(r+s)$ is r-connected.

iii) On vector bundles, the inclusion $\mathrm{B}O(r) \to \mathrm{B}O(r+s)$ represents the direct sum with the trivial bundle ε^s. I.e. for vector bundles ξ_1 and ξ_2 over X with classifying maps f_1 respectively f_2, there is a lift up to homotopy of the form

$$\begin{tikzcd} X \arrow[r, "f_1"] \arrow[dr, bend right, "f_2"'] & \mathrm{B}O(r) \arrow[d, "\mathrm{incl}"] \\ & \mathrm{B}O(r+s) \end{tikzcd}$$

if and only if $\xi_1 \oplus \varepsilon^s \cong \xi_2$.

iv) Taking the limit of all classifying maps of the normal bundles of embeddings of M yields a homotopy class in $[M, \mathrm{BO} = \lim_{k\to\infty} \mathrm{B}O(k)]$ that classifies the stable normal bundle of M uniquely. Any lift of it in $[M, \mathrm{B}O(r)]$ represents a vector bundle ν with the property $\nu \oplus \mathrm{T}M \cong \varepsilon^{n+r}$.

Proof. The natural equivalence is given by the known construction of associated vector bundles, and the stability property of the family $(\mathrm{B}O(n))_n$ becomes clear from $\mathrm{B}O(n) \cong \lim_{k\to\infty} G_n(\mathbb{R}^k)$ where $G_n(\mathbb{R}^k)$ is the Grassmann manifold of n-dimensional vector subspaces of $\mathbb{R}^k$.

For the r-connectivity in ii), observe that the diagram

$$\begin{tikzcd}
O(r) \arrow[r] \arrow[d, "\mathrm{incl}"] & \mathrm{E}O(r) \arrow[r] \arrow[d, "\mathrm{incl}"] & \mathrm{B}O(r) \arrow[d, "\mathrm{incl}"] \\
O(r+1) \arrow[r] \arrow[d] & \mathrm{E}O(r+1) \arrow[r] & \mathrm{B}O(r+1) \\
S^r & &
\end{tikzcd}$$

commutes, where the rows are the defining fiber bundles for the classifying spaces, and $O(r) \to O(r+1) \to S^r$ is the well-known fiber bundle of the inclusion of orthogonal groups. Since $\mathrm{E}O(s)$ is contractible for any $s \in \mathbb{N}$, the long exact sequences of homotopy for the horizontal fiber bundles yield $\pi_i(O(r)) \cong \pi_{i+1}(\mathrm{B}O(r))$ for $i \in \mathbb{N}$, analogously for $r+1$. The sequence for the vertical fiber bundle yields that $O(r) \to O(r+1)$ is r-connected, and commutativity gives the same for $\mathrm{incl}\colon \mathrm{B}O(r) \to \mathrm{B}O(r+1)$.

For the lifting property of the stable normal bundle, consider a lift $\nu \in [M, \mathrm{B}O(r)]$, $r > 1$, by i) classifying a vector bundle over M, which we will also call ν. By iii) and the definition of the stable normal bundle, there is some $s \in \mathbb{N}$ such that

$$\mathrm{T}M \oplus \nu \oplus \varepsilon^s \cong \varepsilon^{n+r+s} \ .$$

Assume $s > 0$. In order to show that this still implies $\mathrm{T}M \oplus \nu \cong \varepsilon^{n+r}$ (i.e. that their classifying maps are homotopic), consider the corresponding homotopy commutative diagram of classifying maps

$$\begin{tikzcd}
M \arrow[r, shift left, "\mathrm{T}M\oplus\nu"] \arrow[r, shift right, "\varepsilon^{n+r}"'] \arrow[dr, bend right, "\varepsilon^{n+r+s}"'] & \mathrm{B}O(n+r) \arrow[d, "\mathrm{incl}"] \\
 & \mathrm{B}O(n+r+s)
\end{tikzcd}$$

This says that $\mathrm{incl} \circ \mathrm{T}M \oplus \nu$ and $\mathrm{incl} \circ \varepsilon^{n+r}$ must be homotopic via some homotopy

$$H\colon M \times I \to \mathrm{B}O(n+r+s) \ .$$

The trick now is to use that by Morse theory M has the homotopy type of an n-dimensional CW-complex, together with incl being $(n+r)$-connected, $r > 0$, and both $\mathrm{B}O(n+r+s)$ and $\mathrm{B}O(n+r)$ being path-connected. Because with these assumptions obstruction theory yields that H lifts to a homotopy $M \times I \to \mathrm{B}O(n+r)$ between the classifying maps of $\mathrm{T}M \oplus \nu$ and ε^{n+r}, as was needed (see e.g. [Hat02, Lemma 4.6]). $\square$

Notation 2.12. Throughout this thesis assume $R = \mathbb{Z}_2$ and $G = O(n)$ respectively $G = O$ if not stated otherwise.

2.2.2 Stiefel-Whitney Classes

Now that the general concept is known, this section reviews the defining and immediate properties of the Stiefel-Whitney classes, a generating set for the ring $H^*(\mathrm{B}O)$ of characteristic classes of vector bundles. This makes them especially interesting for investigation, as any property stable under cohomology ring operations only needs to be checked on the generating set. Furthermore, they will be invaluable for constructing new characteristic classes used as obstructions or indicators. More precisely, their duals, their inverses under certain Steenrod operations called Wu classes, and a couple of special polynomials evaluated on them will be used.

First start with the defining properties of the Stiefel-Whitney classes. Compare e.g. [MS74, compare §4, p. 37].

Definition 2.13. The *Stiefel-Whitney classes* are characteristic classes for principal O-bundles respectively vector bundles, i.e. cohomology classes $w_i \in H^i(\mathrm{B}O; \mathbb{Z}_2)$, $i \in \mathbb{N}$, fulfilling the following properties for any vector bundles ξ and η over a space B, and any map $f\colon A \to B$ of spaces:

Axiom 1 *(Naturality).* $f^*w_i(\xi) = w_i(f^*\xi)$,

Axiom 2. $w_0(\xi) = 1$,

Axiom 3. $w(\gamma_1) = 1 + x$,

Axiom 4 *(Multiplicativity).* $w(\xi \oplus \eta) = w(\xi) \cup w(\eta)$,
i.e. in degree n we have $w_n(\xi \oplus \eta) = \sum_{i+j=n} w_i(\xi) \cup w_j(\eta)$,

where the *total Stiefel-Whitney class* $w := \sum_{i\geq 0} w_i$ is the formal sum of all Stiefel-Whitney classes, γ_1 is the tautological line bundle over $\mathbb{R}\mathrm{P}^1$, $\mathbb{R}\mathrm{P}^\infty \cong \mathrm{B}O(1) \to \mathrm{B}O$, and x is the[1] generator of $H^*(\mathbb{R}\mathrm{P}^\infty; \mathbb{Z}_2) \cong \mathbb{Z}_2[x]$.

Note that naturality is already implied by the requirements for a characteristic class. However, given only the above axioms:

Theorem 2.14. *Stiefel-Whitney classes exist and are uniquely defined by the above properties. Furthermore, they generate the ring* $H^*(\mathrm{B}O; \mathbb{Z}_2)$*, which is isomorphic to* $\mathbb{Z}_2[w_i \mid i \geq 1]$.

Proof. One possible concrete construction utilizes the Euler class, another one via Steenrod squares can be found in Theorem 3.17. For uniqueness see [MS74, Theorem 7.3]. For the generating property see e.g. [MS74, Theorem 7.1 ff.], or [May99, Chap. 7.6]. □

[1] This is well-defined: A ring R of the form $\mathbb{Z}_2[x]$ with $\deg(x) = 1$ only admits two elements in degree 1, 0 and a generator. Therefore, there exists exactly one ring isomorphism, and this sends the unique generator in degree 1 to x.

As already mentioned, the above generating property means that every characteristic class of vector bundles of a fixed dimension can be represented as a certain combination of Stiefel-Whitney classes. Moreover, they—and hence all characteristic classes—behave extremely well with respect to vector bundle operations as emphasized below.

Remark 2.15. Let ξ, η be vector bundles over a space X.

i) $w_i(\eta) = 0$ for any vector bundle η with $\operatorname{rk}\eta < i$. Therefore, the total Stiefel-Whitney class $w(\xi)$ is well-defined (i.e. the sum is finite) for any vector bundle ξ of finite rank.

   Proof. See [Die08, Sec. 19.4]. □

ii) $w_i(\varepsilon) = 0$ for $i > 0$, and one immediately concludes from multiplicativity:

   a) The Stiefel-Whitney classes are stable, i.e. $w_i(\xi \oplus \varepsilon) = w_i(\xi)$, which once more proves the stability property of characteristic classes of vector bundles. Thus, for a manifold M^n, all normal bundles ν_ι of embeddings $\iota\colon M \to \mathbb{R}^{n+k+r}$ share the same Stiefel-Whitney classes, written $w(\nu_M)$ accordingly. Note that $w(\nu_M) = {\nu_M}^* w_i$, where ν_M denotes the classifying map of the stable normal bundle.

   b) If $\xi \oplus \eta = \varepsilon^{\operatorname{rk}\xi + \operatorname{rk}\eta}$, then $w(\xi) \cup w(\eta) = 1$. Especially, for any choice of embedding $\iota\colon M^n \to \mathbb{R}^{n+k}$ with normal bundle ν_ι of a smooth manifold M we have $\mathrm{T}M \oplus \nu_\iota = \varepsilon$, and therefore $1 = w(\mathrm{T}M) \cup w(\nu_\iota) = w(\mathrm{T}M) \cup w(\nu_M)$.

   Proof. The trivial rank n bundle over X is defined as the pullback $\pi^*\varepsilon^n$ of the rank n bundle $\varepsilon^n\colon \mathbb{R}^n \to *$ over the point by the trivial map $\pi\colon X \to *$. The naturality of the Stiefel-Whitney classes gives

$$w(\varepsilon^n) = \pi^* w(\varepsilon^n) \in \pi^*\left(H^*(*;\mathbb{Z}_2)\right) ,$$

   and the result follows from $H^i(*;\mathbb{Z}_2) = 0$ for $i > 0$. □

In order to algebraically work with the Stiefel-Whitney classes, the formal inverse is often handy. Especially, since it is well-known for manifolds as explained below.

Definition 2.16. Define the *dual Stiefel-Whitney (characteristic) classes* $\overline{w}_i$ in degree i inductively by

$$\begin{aligned} 1 &= \overline{w}_0 \cup w_0 = \overline{w}_0 && \text{in degree 0} \\ 0 &= \sum_{i+j=n} \overline{w}_i \cup w_j && \text{in degree n>0} \end{aligned}$$

Denoting the formal sum by $\overline{w} := \sum_{i\geq 0} \overline{w}_i$ as above this can be reformulated as

$$1 = w \cup \overline{w}$$

in the completion of the polynomial ring $H^*(\mathrm{B}O) \cong \mathbb{Z}_2[w_i \mid i \in \mathbb{N}]$.

By Remark 2.15.ii)b), a first example of dual Stiefel-Whitney classes is given by the canonical tangent and normal bundle of a manifold, which makes them especially handy in the context relevant for the immersion conjecture.

Definition 2.17. For a manifold M use the following abbreviation

$$w(M) \coloneqq w(\mathrm{T}M) \;, \qquad \text{and thus} \qquad \overline{w}(M) \coloneqq \overline{w}(\mathrm{T}M) = w(\nu_M) \;.$$

2.3 Reformulation of the Immersion Conjecture

The immersion problem can finally be clearly stated with the definitions from Section 2.1. The goal of this section is to reformulate the immersion conjecture to a statement that can be analyzed with means of homotopy theory of vector bundles, and show how characteristic classes relate to this by finding a powerful obstruction. The latter will be followed up in the subsequent chapter.

Before reformulating, recall the actual immersion conjecture.

Definition 2.18. For $n \in \mathbb{N}$ consider the unique minimal binary expansion

$$n = 2^{i_1} + \cdots + 2^{i_{l_n}}, \quad \text{with} \quad i_1 < \cdots < i_{l_n} \;.$$

Define $\alpha(n) \coloneqq l_n$, i.e. $\alpha(n)$ is the number of ones in the binary notation of n.

Theorem 2.19. *For $n \in \mathbb{N}$, every closed, smooth, n-dimensional manifold immerses into $\mathbb{R}^{2n-\alpha(n)}$.*

In the style of this conjecture, an n-manifold that immerses into some $\mathbb{R}^{2n-\alpha(n)}$ will be said to have the *immersion property.* And the question, whether a particular manifold does have the immersion property, will be referred to as the *immersion problem* for this manifold.

Recall, that by the Theorem 2.4 of Hirsch and Smale, any vector bundle monomorphism between tangent bundles of smooth manifolds implies the existence of an immersion. This is the main ingredient for the subsequent reformulation of the immersion problem.

Theorem 2.20. *Let $n, k \in \mathbb{N}$ and M^n be a closed, smooth, n-dimensional manifold. The following statements are equivalent.*

i) M immerses into $\mathbb{R}^{n+k}$.

ii) There is a vector bundle monomorphism $F\colon \mathrm{T}M \to \mathrm{T}\mathbb{R}^{n+k}$.

iii) There is a k-dimensional vector bundle $\nu\colon \mathrm{E}\nu \to M$ over M with

$$\nu \oplus \mathrm{T}M \cong \varepsilon^{n+k} \;.$$

iv) For the map $\nu_M\colon M \to \mathrm{B}O$ classifying the stable normal bundle over M there is a lift $\nu\colon M \to \mathrm{B}O(k)$ making the following diagram commute up to homotopy

$$\begin{tikzcd} M \arrow[r, "\nu"] \arrow[dr, bend right, "\nu_M"'] & \mathrm{B}O(k) \arrow[d, hook, "\mathrm{incl}"] \\ & \mathrm{B}O \end{tikzcd}$$

Proof of Theorem 2.20. The strategy is to show i)⇒iv)⇒iii)⇒ii)⇔i).

i)⇒iv): The classifying map of an immersion's normal bundle lifts ν_M as required, using Steenrod's classification theorem 2.8.ii) and the properties of the stable normal bundle from Lemma 2.11.iv).

iv)⇒iii): Also, by Lemma 2.11.iv), any rank k vector bundle that is represented by a lift of ν_M to $[M, \mathrm{B}O(k)]$ as in iv), has the property needed for iii).

iii)⇒ii): When given some rank k vector bundle ν such that $\nu \oplus \mathrm{T}M \cong \varepsilon^{n+k}$, there is a vector bundle monomorphism $\mathrm{T}M \to \varepsilon^{n+k}$ over M. Then the following chain of vector bundle morphisms

$$\begin{tikzcd} \mathrm{T}M \arrow[r, hook] \arrow[d] & M \times \mathbb{R}^{n+k} \arrow[r] \arrow[d, "\varepsilon^{n+k}"] & * \times \mathbb{R}^{n+k} \arrow[r, hook] \arrow[d, "\varepsilon^{n+k}"] & \mathbb{R}^{n+k} \times \mathbb{R}^{n+k} \arrow[d, "\varepsilon^{n+k}"] \\ M \arrow[r, equal] & M \arrow[r] & * \arrow[r, hook] & \mathbb{R}^{n+k} \end{tikzcd}$$

is fiber-wise injective in each stage, and hence a monomorphism as was needed.

i)⇔ii): The tricky part is to relate i) and ii), even though it is easily seen that i) implies ii) by simply taking F to be the differential $\mathrm{D}f$ of the immersion from i).

The converse direction is an application of the Hirsch-Smale theorem 2.4. However, first substitute the non-compact manifold $\mathbb{R}^{n+k}$ with the compact sphere $N = S^{n+k}$, to make M and N comply with the assumptions of the theorem: As $\dim M < n+k$ by assumption, the image of M under any immersion $M \to S^{n+k}$ is a zero-set by Sard's theorem (see e.g. [Hir94, Chap. 3, Theorem 1.3]), and hence every such immersion misses a point in S^{n+k}, thus factoring over an immersion $M \to \mathbb{R}^{n+k}$. This then shows that also ii) implies i) which makes them equivalent. □

This now gives rise to involve the powerful obstruction theory of characteristic classes of vector bundles as follows.

Corollary 2.21. *Let $n, k \in \mathbb{N}$, and M^n be a smooth, closed manifold. If M immerses into $\mathbb{R}^{n+k}$, then $\overline{w}_i(M) = 0$ for all $i > k$.*

Proof. By Theorem 2.20 M immerses into $\mathbb{R}^{n+k}$ if and only if there is a rank-k normal bundle ν of M. Since the Stiefel-Whitney classes are stable, $w(\nu) = w(\nu_M) = \overline{w}(M)$. However, as explained in Remark 2.15.i), all Stiefel-Whitney classes $w_i(\nu)$ of degree i exceeding the rank k of ν are zero. $\square$

As a result, the immersion conjecture requires that all n-manifolds have vanishing dual Stiefel-Whitney classes in degrees $i > n-\alpha(n)$. That this is true, is a theorem of Massey which will be proven in Chapter 3. It was an inspiration to state the conjecture with the value $k = n - \alpha(n)$ in the first place.

3 Results on the Stiefel-Whitney Classes of Manifolds

Recall from Corollary 2.21 that a manifold can only immerse in codimension k if its dual Stiefel-Whitney classes in degrees higher than k are zero. This chapter is dedicated to the proof of a theorem by Massey [Mas60], which says that this obstruction vanishes in general for $k = n - \alpha(n)$ (Section 3.3), and that this is the best possible general result (Section 3.4), i.e. for $k < n - \alpha(n)$ there are examples of manifolds not immersing into $\mathbb{R}^{n+k}$.

The preliminary work for Massey's theorem encompasses a review on Steenrod squares in Section 3.1, and the investigation of Wu characteristic classes in Section 3.2. In the course of this a couple of other important results on Stiefel-Whitney classes of manifolds are discussed. Most notably, Wu's formula on their relation to the Wu characteristic classes is proven in Subsection 3.2.4

3.1 Steenrod Squares

Similar to the cup and cap product Steenrod squares add further structure to the cohomology ring, thus adding more tools to differentiate spaces by means of cohomology. As applications, they will serve in constructing two kinds of characteristic classes, the Stiefel-Whitney and the Wu classes, and are a key ingredient for the proof of Massey's theorem on obstructions for normal bundles of manifolds.

The following definition is due to Steenrod and Epstein [Ste62, Chap. I.1, p. 1].

Definition 3.1. The *Steenrod squares* Sq^i for $i \in \mathbb{N}$ are each a family of cohomology operations, i.e. families of homomorphisms, of the form

$$\left(\mathrm{Sq}^i\colon H^n(X, A; \mathbb{Z}_2) \to H^{n+i}(X, A; \mathbb{Z}_2) \;\middle|\; n \in \mathbb{N}\right)$$

that satisfy the following relations for any pair of spaces (X, A), and any map of pairs of spaces $f\colon (X, A) \to (Y, B)$:

(Naturality) $f^* \circ \mathrm{Sq}^i = \mathrm{Sq}^i \circ f^*$.

(Stability) $\Sigma \circ \mathrm{Sq}^i = \mathrm{Sq}^i \circ \Sigma$ where Σ is the suspension isomorphism on cohomology.

(Cartan formula) For any $n \in \mathbb{N}$, and $x, y \in H^n(X)$ holds

$$\mathrm{Sq}^i(x \cup y) = \sum_{r+s=i} \mathrm{Sq}^r(x) \cup \mathrm{Sq}^s(y) \tag{Cartan's formula}$$

(Fixed values) The following values are fixed for $x \in H^n(X, A)$:

$$\mathrm{Sq}^i(x) = 0 \qquad \text{for } n < i \tag{3.1}$$
$$\mathrm{Sq}^i(x) = x^2 \qquad \text{for } n = i \tag{3.2}$$
$$\mathrm{Sq}^0 = \mathrm{id} \tag{3.3}$$

(Adem relations) For $\alpha < 2\beta$ holds

$$\mathrm{Sq}^\alpha \circ \mathrm{Sq}^\beta = \sum_{j=0}^{\lfloor \frac{\alpha}{2} \rfloor} \binom{\beta - j - 1}{\alpha - 2j} \mathrm{Sq}^{\alpha+\beta+j}\mathrm{Sq}^j \tag{Adem's formula}$$

The formal sum of all Steenrod squares $\mathrm{Sq} := \sum_{j \in \mathbb{N}} \mathrm{Sq}^j$ is called the *total Steenrod square*. Note that for any degree $n \in \mathbb{N}$ the total Steenrod square $\mathrm{Sq}\colon H^n(X) \to H^*(X)$ is well-defined since the sum is finite by (3.1). Also Cartan's formula can be reformulated to

$$\mathrm{Sq}(x \cup y) = \mathrm{Sq}(x) \cup \mathrm{Sq}(y) \ ,$$

i.e. Sq is a group homomorphism with respect to the cup-product.

Theorem 3.2. *The Steenrod squares exist and are uniquely determined by naturality, Cartan's formula, and the fixed values from Definition 3.1.*

Proof. For existence see [MT68, Chapter 2], for uniqueness see [Ste62, VIII §3]. □

The fact that Steenrod squares, or in general cohomology operations, can be added and concatenated, already gives a hint that they might form a ring, which was proven by Steenrod. The following notation and facts are according to [MT68, Chap. 6].

Definition 3.3. The *Steenrod algebra* $\mathcal{A}$ is the quotient of the graded $\mathbb{Z}_2$-polynomial algebra $\mathbb{Z}_2[\mathrm{Sq}^i \mid i \in \mathbb{N}]$ with grading $\deg \mathrm{Sq}^i := i$ by the two-sided relations of both Adem's formula and $\mathrm{Sq}^0 = 1$. With the induced grading, multiplication, and the diagonal defined by

$$\mathcal{A} \longrightarrow \mathcal{A} \otimes \mathcal{A} \ , \qquad \mathrm{Sq}^i \longmapsto \sum_{r+s=i} \mathrm{Sq}^r \otimes \mathrm{Sq}^s \ ,$$

it is an associative, connected, non-commutative graded Hopf algebra over $\mathbb{Z}_2$.

Notation 3.4. In the following, iterated Steenrod squares $\mathrm{Sq}^{i_1} \cdot \mathrm{Sq}^{i_2} \cdots \mathrm{Sq}^{i_l}$ will have the short form $\mathrm{Sq}^{(i_1,i_2,\dots,i_l)}$, and will be evaluated on an element x of a cohomology ring as $\mathrm{Sq}^{i_1} \circ \cdots \circ \mathrm{Sq}^{i_l}(x)$ respecting the properties from Definition 3.1. Furthermore, for a sequence $I = (i_1, \dots, i_l)$, respectively Sq^I, denote by

$l(I) := l$ the *length* of I,

$d(I) := \sum_{j=1}^{l} i_j$ the *degree* of I, and by

$e(I) := 2i_1 - d(I) = \sum_{j=1}^{l-1}(i_j - 2i_{j+1})$ the *excess* of I.

The sequence I, respectively Sq^I, is called *admissible*, if $l(I) = 1$ or $i_j \geq 2i_{j+1}$ for $0 \leq j < l(I)$.

Remark 3.5. The following properties will be needed for Massey's Theorem:

i) The set of iterated Steenrod squares Sq^I of admissible sequences I forms a basis for $\mathcal{A}$ as $\mathbb{Z}_2$-vector space [MT68, Chap. 6, Theorem 1].

Let $I = (i_1, \dots, i_l)$ be a sequence in $\mathbb{N}$.

ii) $\deg(\mathrm{Sq}^I(x)) = \deg(x) + d(\mathrm{Sq}^I)$.

iii) $\mathrm{Sq}^I(x) = 0$ for $\deg(x) < e(I)$ if I is admissible.

Proof. The non-obvious part iii) follows by induction over $l(I)$ using (3.1): The case $l(I) = 1$ is (3.1). For $l(I) > 1$ and $J := (i_2, \dots, i_l)$, the condition $\deg(x) < e(I) = i_1 - d(J)$ implies

$$\deg(\mathrm{Sq}^J(x)) = \deg(x) + d(J) < e(J) + d(J) = 2i_2 \overset{\text{adm.}}{\leq} i_1$$

So, $\mathrm{Sq}^I(x) = \mathrm{Sq}^{i_1}(\mathrm{Sq}^J(x)) = 0$ by (3.1). □

The subsequent concept of formally inverting a formal sum of elements in a graded $\mathbb{Z}_2$-algebra will recur for several characteristic classes. This particular definition is needed to define the Wu classes in Subsection 3.2.3.

Definition 3.6. The *antipode* $\chi\colon \mathcal{A} \to \mathcal{A}$ of the Steenrod algebra is a graded homomorphism inductively defined by the relation

$$1 = \mathrm{Sq}^0 = \mathrm{Sq} \circ \chi(\mathrm{Sq}) = \sum_{k\geq 0} \sum_{r+s=k} \mathrm{Sq}^r \circ \chi(\mathrm{Sq}^s)$$

3.2 Wu Classes and the Wu Formula

A crucial part in the proof of Massey's theorem is to cleverly involve a new type of characteristic classes with nice properties that are directly connected to Stiefel-Whitney classes through Steenrod squares. The Wu classes were developed by Wu Wen-Tsün, and will be introduced in Subsection 3.2.3. The proof of Wu's theorem on their relation to Stiefel-Whitney classes in Subsection 3.2.4 utilizes some properties of Thom classes and the Thom isomorphism as reviewed in Subsection 3.2.1, as well as a certain construction of Stiefel-Whitney classes as introduced in Subsection 3.2.2.

3.2.1 Thom Classes and the Thom Isomorphisms

In this subsection, Thom classes and the Thom isomorphisms will be reviewed. The tools to be recalled in this section are an important ingredient for the proof of Wu's theorem, both directly as well as for constructing the Stiefel-Whitney characteristic classes from Steenrod squares.

Let B be a paracompact space, e.g. a manifold, $\xi\colon E \xrightarrow{p} B$ a vector bundle over B of rank $k > 0$, and let R be a principal ideal domain. The following definition can e.g. also be found in [Hat02, p. 441].

Definition 3.7. A *Thom class* of ξ in R-coefficients is a cohomology class $u(\xi)$ in $H^k(E, E \setminus \{0\}; R)$, such that for all points $b \in B$ and fiber inclusions

$$i_b\colon (E_b, E_b \setminus \{0\}) \to (E, E^0)$$

the restriction $u(\xi)|_{E_b} = i_b^*(u(\xi))$ is a free generator of the R-module $H^k(E_b, E_b \setminus \{0\}; R)$, i.e. a unit of the ring $H^k(E_b, E_b \setminus \{0\}; R) \cong H^k(\mathbb{R}, \mathbb{R} \setminus \{0\}; R) \cong R$.

The following corollaries will deduce notions of naturality, multiplicativity, and uniqueness for Thom classes quite directly from their above definition.

Corollary 3.8. *The Thom class construction is natural with respect to the pullback of vector bundles over paracompact spaces. I.e. given any map of paracompact spaces $f\colon A \to B$, and a vector bundle $\xi\colon E \to B$ with pullback map $F\colon f^*\xi \to \xi$, then F^*u of a Thom class u of ξ will be a Thom class of $f^*\xi$.*

Proof. Let u be a Thom class of ξ and $a \in A$ any point. Consider the restriction $i_a^*(f^*u)$ of the pullback of u to the fiber over a. To show that this is a generator of $H^k(E_a, E_a \setminus \{0\}; R)$ first use that pullbacks commute with restriction:

$$i_a^*(f^*u) = (f \circ i_a)^*u = (i_{f(a)} \circ f)^*u = f^*(i_{f(a)}^*u)$$

$i^*_{f(a)}u$ is a generator by definition of u. Now the restriction of f

$$f\colon ((f^*E)_a, (f^*E)_a \setminus \{0\}) \to (E_{f(a)}, E_{f(a)} \setminus \{0\})$$

is an isomorphism, and thus $f^*\colon H^r(E_{f(a)}, E_{f(a)} \setminus \{0\}) \cong H^r((f^*E)_a, (f^*E)_a \setminus \{0\})$ sends generators to generators for all $r \in \mathbb{N}$. $\square$

Definition 3.9. Let ξ, η be vector bundles over a space B. Define the *cross-product* as in [Hat02, p. 214] to be the map

$$\begin{aligned} H^*(\mathrm{E}\xi, \mathrm{E}\xi^0) \otimes H^*(\mathrm{E}\eta, \mathrm{E}\eta^0) &\longrightarrow H^*(\mathrm{E}(\xi \oplus \eta), \mathrm{E}(\xi \oplus \eta)^0) \\ x \otimes y &\longmapsto \pi_\xi^* x \cup \pi_\eta^* y =: x \times y\ . \end{aligned}$$

Corollary 3.10. *The Thom class construction for coefficients in a field R is multiplicative in the following sense: For vector bundles $\xi\colon E \to B$, $\eta\colon E' \to B$ of rank k respectively l over a paracompact space B, and Thom classes $u(\xi) \in H^k(\mathrm{E}\xi, \mathrm{E}\xi^0; R)$, $u(\eta) \in H^l(\mathrm{E}\eta, \mathrm{E}\eta^0; R)$ the class*

$$u(\xi) \times u(\eta) \coloneqq \pi_\xi^* u(\xi) \cup \pi_\eta^* u(\eta) \in H^{k+l}(\mathrm{E}(\xi \oplus \eta), \mathrm{E}(\xi \oplus \eta)^0)$$

is a Thom class of $\xi \oplus \eta$.

Proof. Consider a fiber $b \in B$. As cup product and pullback commute with restriction, the cross-product also commutes with restriction, i.e. one has to show that

$$i_b^*(u(\xi) \times u(\eta)) = (i_b^* u(\xi)) \times (i_b^* u(\eta))$$

is a generator of $H^{k+l}(\mathrm{E}(\xi \oplus \eta)_b, \mathrm{E}(\xi \oplus \eta)_b \setminus \{0\}; R)$. On fibers one has that

$$\begin{aligned} (\mathrm{E}(\xi \oplus \eta)_b, \mathrm{E}(\xi \oplus \eta)_b \setminus \{0\}) &= (\mathrm{E}\xi_b \times \mathrm{E}\eta_b, (\mathrm{E}\xi_b \times (\mathrm{E}\eta_b \setminus \{0\})) \cup ((\mathrm{E}\xi_b \setminus \{0\}) \times \mathrm{E}\eta_b)) \\ &= (\mathrm{E}\xi_b, \mathrm{E}\xi_b \setminus \{0\}) \times (\mathrm{E}\eta_b, \mathrm{E}\eta_b \setminus \{0\} \end{aligned}$$

which makes the relative Künneth isomorphism theorem applicable. For the latter see e.g. [Hat02, Theorem 3.18]. By the naturality of that isomorphism there is the following commutative diagram that translates this problem to one on the cohomology of spheres (all cohomology rings with R-coefficients), similar to [Hat02, proof of 3.19, p. 221]:

$$\begin{tikzcd}
H^*(\mathrm{E}\xi_b, \mathrm{E}\xi_b \setminus \{0\}) \otimes H^*(\mathrm{E}\eta_b, \mathrm{E}\eta_b \setminus \{0\}) \arrow[r, "\cong"] \arrow[d, dash, "\wr"'] & H^*(\mathrm{E}(\xi \oplus \eta)_b, \mathrm{E}(\xi \oplus \eta)_b \setminus \{0\}) \arrow[d, dash, "\wr"] \\
H^*(\mathbb{R}^k, \mathbb{R}^k \setminus \{0\}) \otimes H^*(\mathbb{R}^l, \mathbb{R}^l \setminus \{0\}) \arrow[r, "\cong"] \arrow[d, dash, "\wr"'] & H^*(\mathbb{R}^{k+l}, \mathbb{R}^{k+l} \setminus \{0\}) \arrow[d, dash, "\wr"] \\
H^*(I^k, \partial I^k) \otimes H^*(I^l, \partial I^l) \arrow[r, "\cong"] \arrow[d, dash, "\wr"'] & H^*(I^{k+l}, \partial I^{k+l}) \arrow[d, dash, "\wr"] \\
H^*(S^k) \otimes H^*(S^l) \arrow[r, "\cong"] & H^*(S^{k+l})
\end{tikzcd}$$

Furthermore, the simple structure of the cohomology of spheres yields for the Künneth isomorphism in the desired degree

$$\begin{aligned} H^{k+l}(S^{k+l};R) &\cong \Big(H^*(S^k;R)\otimes H^*(S^l;R)\Big)_{k+l} \\ &:= \bigoplus_{r+s=k+l} H^r(S^k;R)\otimes H^s(S^l;R) = H^k(S^k;R)\otimes H^l(S^l;R) \end{aligned}$$

by leaving out zero-summands for the last equality. Thus, a generator $\iota_k \otimes \iota_l$ of $H^k(S^k;R)\otimes H^l(S^l;R)$, which is the tensor product of two generators, is mapped to a generator $\iota_{k+l} = \iota_k \times \iota_l$ of $H^{k+l}(S^{l+l};R)$. Using the isomorphisms above proves the claim. □

Corollary 3.11. *Every vector bundle ξ has a unique Thom class $u(\xi)$ in $\mathbb{Z}_2$-coefficients. Furthermore, for any map of paracompact spaces $f\colon A \to B$ and vector bundle $\xi\colon E \to B$ holds $u(f^*\xi) = f^*u(\xi)$.*

Proof (sketch). Existence: See [Hat02, Theorem 4D.10] or use [Die08, Prop. 17.9.3].

Uniqueness: Using a suitable Mayer-Vietoris sequence for gluing, and an inductive argument starting with the trivial bundle case, one can show: Any two classes in $H^k(E, E^0; R)$ whose restrictions coincide on all fibers will coincide. However, for $R = \mathbb{Z}_2$ there is exactly one possible choice for a unit $u(\xi)|_{E_b}$ of the group $H^k(E_b, E_b \setminus \{0\})^\times \cong \mathbb{Z}_2^\times = \{1\}$ over each point b. For details see e.g. [Die08, Theorem (17.9.4)].

Naturality: Clear from uniqueness and the naturality of Thom classes.

□

Remark 3.12. Using paracompactness of B and [Die08, Proposition 17.9.6], one concludes that $u(\xi) \in H^k(E, E^0; R)$ has to be a unit.

Having a good notion of Thom classes by now, we recall the Thom isomorphism relating the (co)homology of the total space with that of the base space of a vector bundle.

Theorem 3.13. *For any Thom class $u(\xi)$ of ξ, and any degree r there are the following isomorphisms, called the* Thom isomorphism *for cohomology respectively homology, that are natural with respect to pullbacks of vector bundles over paracompact spaces:*

$$\begin{aligned} t\colon H^r(B;R) &\longrightarrow H^{r+k}(E,E^0;R) & \qquad t\colon H_{r+k}(E,E^0;R) &\longrightarrow H_r(B;R) \\ x &\longrightarrow p^*(x)\cup u(\xi) & \alpha &\longrightarrow p_*(u(\xi)\cap\alpha)\ . \end{aligned}$$

Proof. Naturality directly follows from the naturality of the Thom class in Corollary 3.8, and naturality of the cup- respectively cap-product. The cohomology part then is a direct application of Leray's theorem [Hat02, Theorem 4D.8]. For the homology part see e.g. [Swi02, Theorem 14.6]. □

Lemma 3.14. *If B is connected, the Thom isomorphisms are adjoint in the following sense: For $r \in \mathbb{N}$, $x \in H^r(B)$, and $\alpha \in H_{r+k}(E, E^0)$ holds*

$$p_* \langle t(x),\, \alpha\rangle = \langle x,\, t(\alpha)\rangle \in H_0(B) \cong \mathbb{Z}_2$$

In order to proof Lemma 3.14, first recall the following properties of the cap product.

Remark 3.15. For a map of triples of spaces $f\colon (Y, Y'', Y') \to (X, X'', X')$, cohomology classes $a \in H^i(X, X')$ and $b \in H^j(X, X')$, homology classes $\gamma \in H_{i+j}(X, X' \cup X'')$ and $\beta \in H_j(Y, Y' \cup Y'')$, and a vector bundle $E \xrightarrow{p} B$ holds

$$\langle a \cup b,\, \beta\rangle = \langle b,\, a \cap \beta\rangle \in H_0(X, X'') \tag{3.4}$$
$$\langle a,\, f_*\beta\rangle = f_* \langle f^*a,\, \beta\rangle \in H_0(X, X'') \qquad \text{[Die08, Sec. 18.1.1]} \tag{3.5}$$

Proof of Lemma 3.14. With $u \coloneqq u(\xi)$ one calculates

$$\begin{aligned} p_* \langle t(x),\, \alpha\rangle &= p_* \langle p^*x \cup u,\, \alpha\rangle \\ &\overset{(3.4)}{=} p_* \langle p^*x,\, u \cap \alpha\rangle \\ &\overset{(3.5)}{=} \langle x,\, p_*(u \cap \alpha)\rangle = \langle x,\, t(\alpha)\rangle \in \mathbb{Z}_2 \end{aligned}$$ □

Now, we can focus on the specific case of normal bundles of manifolds. The following is a well-known result from intersection theory, which links the fundamental class of a manifold with that of a submanifold using the Thom isomorphism corresponding to the normal bundle of the embedding.

Lemma 3.16. *Let M^n and W^{n+k} be manifolds, and $\iota\colon M \to W$ be an embedding with normal bundle ν_ι of rank $k > 0$. The normal bundle gives rise to an embedding $e\colon \mathrm{E}\nu_\iota \to W$ of its total space as a tubular neighborhood $e(\mathrm{E}\nu_\iota)$ of $i(M)$ into W (see [Die08, Sec. 15.6]). The quotient map*

$$c\colon W \to W/\left(W \setminus e(\mathrm{E}\nu_\iota)\right) \cong \mathrm{D}\nu_\iota/\mathrm{S}\nu_\iota$$

that collapses every point outside of $e(\mathrm{E}\nu_\iota)$ to the infinity point fulfills

$$\begin{array}{ccccccc} H_{n+k}(W) & \xrightarrow{c_*} & H_{n+k}(\mathrm{D}\nu_\iota/\mathrm{S}\nu_\iota) & \xrightarrow{\mathrm{incl}_*} & \widetilde{H}_{n+k}(\mathrm{D}\nu_\iota/\mathrm{S}\nu_\iota) & \xrightarrow[\cong]{t_*} & H_n(M) \\ [W] & \longmapsto & & & & & t(\mathrm{incl}_* c_*[W]) = [M] \end{array}$$

where incl: $(\mathrm{D}\nu_\iota/\mathrm{S}\nu_\iota, \emptyset) \to (\mathrm{D}\nu_\iota/\mathrm{S}\nu_\iota, \{\infty\})$ *is the canonical inclusion of pairs of spaces, and the isomorphism $\widetilde{H}_{n+k}(\mathrm{D}\nu_\iota/\mathrm{S}\nu_\iota) \cong H_{n+k}(\mathrm{E}\nu_\iota, \mathrm{E}\nu_\iota{}^0)$ of the neighborhood deformation retract pair $(\mathrm{E}\nu_\iota, \mathrm{E}\nu_\iota{}^0)$ was used. This holds for any choice of i and e.*

Proof. First note that by the long exact sequence of the pair $(\mathrm{DE}\nu_\iota/\mathrm{SE}\nu_\iota, \{\infty\})$, the map $\mathrm{incl}_*\colon H_r(\mathrm{D}\nu_\iota/\mathrm{S}\nu_\iota) \to \widetilde{H}_r(\mathrm{D}\nu_\iota/\mathrm{S}\nu_\iota)$ is an isomorphism in every degree $r > 0$. The proof will be conducted in two steps, first proving the connected case, then the general one.

Connected case: Assume that M is connected. Then

$$H_{n+k}(\mathrm{D}\nu_\iota/\mathrm{S}\nu_\iota) \cong \widetilde{H}_{n+k}(\mathrm{D}\nu_\iota/\mathrm{S}\nu_\iota) \cong H_n(M) \cong \mathbb{Z}_2 = \{[M], 0\}$$

by the Thom isomorphism, and since M is connected and closed. Thus, one only has to show that $c_*[W]$ is non-zero. The trick now is to use that manifolds locally look like some Euclidean space, and that this property is mostly preserved by the collapse. Restricted to $e(\mathrm{E}\nu_\iota)$ the collapse map looks like

$$c|_{e(\mathrm{E}\nu_\iota)}\colon e(\mathrm{E}\nu_\iota) \xrightarrow[\sim]{e} \mathrm{E}\nu_\iota \cong \mathrm{D}\nu_\iota \setminus \mathrm{S}\nu_\iota \cong (\mathrm{D}\nu_\iota/\mathrm{S}\nu_\iota) \setminus \{\infty\}$$

i.e. it is a homeomorphism onto $(\mathrm{D}\nu_\iota/\mathrm{S}\nu_\iota) \setminus \{\infty\}$. Therefore, by excision there is for any point $p \in (\mathrm{D}\nu_\iota/\mathrm{S}\nu_\iota) \setminus \{\infty\}$ the following commutative diagram on homology

$$\begin{tikzcd}
{[W]} \arrow[d, mapsto] & \in & H_{n+k}(W) \arrow[rr, "\mathrm{incl}_* c_*"] \arrow[d, "\mathrm{incl}_*"] & & \widetilde{H}_{n+k}(\mathrm{D}\nu_\iota/\mathrm{S}\nu_\iota) \arrow[d, "\mathrm{incl}_*"] \\
{[W]|_{c^{-1}(p)}} & \in & H_{n+k}(W, W \setminus c^{-1}(p)) \arrow[r, "c_*", "\cong"'] \arrow[d, dash, "\mathrm{exc.}"', "\cong"] & H_{n+k}(\mathrm{D}\nu_\iota/\mathrm{S}\nu_\iota, (\mathrm{D}\nu_\iota/\mathrm{S}\nu_\iota) \setminus p) \arrow[r, phantom] & {} \arrow[d, dash, "\mathrm{exc.}"', "\cong"] \\
& & H_{n+k}(\mathbb{R}^{n+k}, \mathbb{R}^{n+k} \setminus *) & & H_{n+k}(\mathbb{R}^{n+k}, \mathbb{R}^{n+k} \setminus *)
\end{tikzcd}$$

By definition of the fundamental class $[W]$, the class $[W]|_{c^{-1}(p)}$ in the diagram is a generator, and thus also is $c_*\left([W]|_{c^{-1}(p)}\right) = (\mathrm{incl}_* c_*[W])\,|_p$. However, then $\mathrm{incl}_* c_*[W] \in \widetilde{H}_{n+k}(\mathrm{D}\nu_\iota/\mathrm{S}\nu_\iota)$ cannot be zero as was to be shown.

General case In case $M = \coprod_i M_i$ is the disjoint sum of its connected components M_i, $i \in I$ for some index set I, note the following:

- $\mathrm{E}\nu_\iota = \coprod_i \mathrm{E}\nu_{\iota_i}$, where $\iota_i := \iota|_{M_i}$.
- Thus, $\mathrm{D}\nu_\iota/\mathrm{S}\nu_\iota = \bigvee_i \mathrm{D}\nu_{\iota_i}/\mathrm{S}\nu_{\iota_i}$ using the collapse maps

  $$c_i\colon W \xrightarrow{c} W/\left(W \setminus e(\mathrm{E}\nu_\iota)\right) \xrightarrow{\mathrm{proj}} W/\left(W \setminus e(\mathrm{E}\nu_{\iota_i})\right)$$

  for the disjoint parts, and $c = \bigvee_i c_i$.
- Thus, $\widetilde{H}_r(\mathrm{D}\nu_\iota/\mathrm{S}\nu_\iota) = \bigoplus_i \widetilde{H}_r(\mathrm{D}\nu_{\iota_i}/\mathrm{S}\nu_{\iota_i})$ for all degrees r, $[M] = ([M_i])_i$, and $c_* = \prod_i c_{i*}$.

With this one sees directly from the definition of the fundamental class of a manifold that $\mathrm{incl}_* c_*[W] = [M]$ if and only if for all connected component manifolds M_i holds $\mathrm{incl}_* c_{i*}[W] = [M_i]$, which is true by the first case. □

3.2.2 A Construction of Stiefel-Whitney Classes

Now, the promised construction of the Stiefel-Whitney classes from Thom classes and Steenrod squares can be presented.

Theorem 3.17. *The Stiefel-Whitney classes can be given as*

$$\mathrm{Sq}^i(u(\xi)) = t(w_i(\xi)) = p^* w_i(\xi) \cup u(\xi)$$

for any vector bundle $\xi\colon E \to B$ over a paracompact space X. As the Thom isomorphism is a group homomorphism, one can formulate the above as

$$\mathrm{Sq}(u(\xi)) = t(w(\xi)) = p^* w(\xi) \cup u(\xi)$$

Proof. One has to check naturality of the expression and all further defining properties from Definition 2.13.

Naturality: Both Sq^i and t respectively also t^{-1} are natural.

$w_0 = 1$*:* $H^0(\mathrm{E}\gamma_0, \mathrm{E}\gamma_0{}^0) = \mathbb{Z}_2$, thus 1 is the only candidate for a Thom class, $\mathrm{Sq}^0(1) = \mathrm{id}(1) = 1$, and the Thom isomorphism sends 1 to 1 in this degree.

$w(\gamma_1) = 1 + x$*:* Recall that $H^*(\mathbb{R}\mathrm{P}^k) = \mathbb{Z}_2[x_k]/(x_k^{k+1})$. For $w_0(\gamma_1)$ the defining relation

$$p^* w_0(\gamma_1) \cup u(\gamma_1) = \mathrm{Sq}^0(u(\gamma_1)) \overset{(3.3)}{=} u(\gamma_1) \neq 0$$

directly gives $p^* w_0(\gamma_1) = 1$, and thus $w_0(\gamma_1) = 1 \in H^0(\mathbb{R}\mathrm{P}^1) \cong \mathbb{Z}_2$. For $w_1(\gamma_1)$ the defining relation looks like

$$t(w_1(\gamma_1)) = p^* w_1(\gamma_1) \cup u(\gamma_1) = \mathrm{Sq}^1(u(\gamma_1)) \overset{(3.2)}{=} u(\gamma_1)^2 ,$$

which, if non-zero, directly proves that $w_1(\gamma_1)$ is the unique non-zero element of degree 1 in $H^*(\mathbb{R}\mathrm{P}^1) = \mathbb{Z}_2[x]/(x^2)$, which is x as was desired. To see that $u(\gamma_1)^2$ is non-zero, observe that the tautological line bundle actually is the normal bundle of the embedding of $\mathbb{R}\mathrm{P}^1$ into $\mathbb{R}\mathrm{P}^2$ induced by the embedding of the circle as equator into the 2-sphere. This then yields by excision that

$$H^r(\mathrm{E}\gamma_1, \mathrm{E}\gamma_1{}^0) \cong H^r(\mathbb{R}\mathrm{P}^2, \mathbb{R}\mathrm{P}^2 \setminus \mathbb{R}\mathrm{P}^1) \cong H^r(\mathbb{R}\mathrm{P}^2, *) = \widetilde{H}^r(\mathbb{R}\mathrm{P}^2) .$$

Let $\mathrm{incl}\colon \mathbb{R}\mathrm{P}^2 \to (\mathbb{R}\mathrm{P}^2, *)$ be the inclusion, which is an isomorphism on cohomology groups in degree greater 0. Since $u(\gamma_1) \in \widetilde{H}^1(\mathbb{R}\mathrm{P}^2)$ is the unique non-zero element, $\mathrm{incl}^* u(\gamma_1) = x_2$ and $\mathrm{incl}^*(u(\gamma_1)^2) = x_2^2 \neq 0$. So $u(\gamma_1)^2 \neq 0$.

Multiplicativity: Consider vector bundles ξ, η over a paracompact space B. With $u(\xi \oplus \eta) = u(\xi) \cup u(\eta)$ and the fact that

$$p_\xi \circ \pi_\xi = p_{\xi\oplus\eta} = p_\eta \circ \pi_\eta \tag{3.6}$$

one gets

$$
\begin{aligned}
t(w_i(\xi \oplus \eta)) &= \mathrm{Sq}^i(u(\xi \oplus \eta)) \\
&\overset{3.10}{=} \mathrm{Sq}^i(\pi_\xi^* u(\xi) \cup \pi_\eta^* u(\eta)) \\
&\overset{\text{Cartan's formula}}{=} \sum_{r+s=i} \mathrm{Sq}^r(\pi_\xi^* u(\xi)) \cup \mathrm{Sq}^s(\pi_\eta^* u(\eta)) \\
&\overset{\text{Naturality}}{=} \sum_{r+s=i} \pi_\xi^* \mathrm{Sq}^r(u(\xi)) \cup \pi_\eta^* \mathrm{Sq}^s(u(\eta)) \\
&\overset{\text{Definition}}{=} \sum_{r+s=i} \pi_\xi^* t(w_r(\xi)) \cup \pi_\eta^* t(w_s(\eta)) \\
&\overset{\text{Definition}}{=} \sum_{r+s=i} \pi_\xi^* \left(p_\xi^* w_r(\xi) \cup u(\xi)\right) \cup \pi_\eta^* \left(p_\eta^* w_s(\eta) \cup u(\eta)\right) \\
&= \left(\sum_{r+s=i} \pi_\xi^* p_\xi^* w_r(\xi) \cup \pi_\eta^* p_\eta^* w_s(\eta) \right) \cup \left(\pi_\xi^* u(\xi) \cup \pi_\eta^* u(\eta)\right) \\
&\overset{\text{(3.6), Definition}}{=} \left(\sum_{r+s=i} p_{\xi\oplus\eta}^* w_r(\xi) \cup p_{\xi\oplus\eta}^* w_s(\eta) \right) \cup u(\xi) \times u(\eta) \\
&\overset{\text{Group Hom., 3.10}}{=} p_{\xi\oplus\eta}^* \left(\sum_{r+s=i} w_r(\xi) \cup w_s(\eta) \right) \cup u(\xi \oplus \eta) \\
&\overset{\text{Definition}}{=} t \left(\sum_{r+s=i} w_r(\xi) \cup w_s(\eta) \right) .
\end{aligned}
$$

Applying t^{-1} yields the result. $\square$

3.2.3 Equivalent Definitions of Wu Classes

Let M be a compact, n-dimensional manifold. In the following, two approaches to define the Wu characteristic classes are pursued: an explicit one specially for normal bundles of manifolds using Poincaré duality, and a more general one, which makes clear that the Wu classes indeed are characteristic classes of vector bundles (i.e. natural).

Definition 3.18. The ith *Wu class* $v_i(M)$ of M for $0 \leq i \leq n$ is defined as the cohomology class in $H^i(M)$ that is uniquely determined by

$$
\begin{array}{ccccccc}
H^i(M) & \cong & H_{n-i}(M) & \cong & \mathrm{Hom}_{\mathbb{Z}_2}(H^{n-i}(M), \mathbb{Z}_2) & \cong & \mathbb{Z}_2 \\
y & \longmapsto & & & & & \langle x \cup y,\, [M] \rangle \\
v_i(M) & \longmapsto & & & & & \langle \mathrm{Sq}^i(x),\, [M] \rangle
\end{array}
$$

where the first isomorphism from the left is Poincaré duality and the second is the universal coefficient theorem for the field $\mathbb{Z}_2$. Equivalently, for any cohomology class $x \in H^{n-i}(M)$ of fixed degree $n-i$ holds

$$x \cup v_i(M) = \mathrm{Sq}^i(x) \in H^n(M) \cong \mathbb{Z}_2$$

Mind the fixed degree of x—the above will not be true for other degree cohomology classes in general!

Remark 3.19. Some immediate consequences of the definitions of the Wu classes of M are

- $v_0(M) = 1$
- $v_i(M) = 0$ for $i > \frac{n}{2}$, because $\mathrm{Sq}^i(x) = 0$ if the degree of x is lower than i.

As for the Stiefel-Whitney classes, one has a notion of a total class.

Definition 3.20. The *total Wu class* of M is defined as the sum $\sum_{i\geq 0} v_i(M)$. The *total dual Wu class* $\overline{v}(M) =: \sum_{i\geq 0} \overline{v}_i(M)$ and the dual Wu classes $\overline{v}_i(M)$ of M are defined by

$$v(M) \cup \overline{v}(M) = 1$$

or equivalently

$$\begin{aligned} 1 &= \overline{v}_0(M) \cup v_0(M) = \overline{v}_0(M)\ , && \text{and} \\ 0 &= \sum_{r+s=i} v_r(M) \cup \overline{v}_s(M) && \text{in degree } 0 \leq i \leq n. \end{aligned}$$

The following more general definition of Wu classes will turn out to be equivalent in a certain sense to the one above in Definition 3.18. For the formulation recall the antipode of the Steenrod algebra (see Definition 3.6).

Definition 3.21. Let $\xi\colon E \xrightarrow{p} B$ be a vector bundle over a paracompact space B. The ith *Wu class* $v_i(\xi)$ of ξ for $0 \leq i \leq n$ is defined as the cohomology class in $H^i(B)$ that is uniquely determined by

$$\chi(\mathrm{Sq}^i)(u(\xi)) = t(v_i(\xi)) = p^* v_i(\xi) \cup u(\xi)$$

The *total Wu class* of ξ is defined as usual as $v(\xi) \coloneqq \sum_{i\geq 0} v_i(\xi)$, and satisfies accordingly $\chi(\mathrm{Sq})(u(\xi)) = t(v(\xi))$.

Remark 3.22. Compare this to the possible definition of the Stiefel-Whitney classes in Theorem 3.17. Similarly, naturality with respect to pullbacks of vector bundles follows immediately from the definition, making the Wu classes characteristic classes.

Recall that the definition of Stiefel-Whitney classes of manifolds utilizes the canonical tangent bundle structure. In contrast to that, the Wu classes of manifolds utilize normal bundles, as will be clear from the promised equivalence below.

Theorem 3.23. *Let M be a compact manifold of dimension n, and let $\nu_\iota\colon \mathrm{E}\nu_\iota \xrightarrow{p} M$ be any normal bundle of rank k of an embedding $\iota\colon M \hookrightarrow \mathbb{R}^{n+k}$. Then*

$$v_i(M) = v_i(\nu_\iota) \in H^i(M)$$

Proof. To proof Theorem 3.23, the defining property of $v_i(M)$ will be checked on $v_i(\nu_\iota)$, i.e. one has to show that for any $x \in H^{n-i}(M)$ holds

$$\langle x \cup v_i(\nu_\iota),\, [M]\rangle = \left\langle \mathrm{Sq}^i(x),\, [M]\right\rangle$$

Note, that this is simply the ith degree of the equation

$$\langle x \cup v(\nu_\iota),\, [M]\rangle = \langle \mathrm{Sq}(x),\, [M]\rangle \tag{3.7}$$

which will be proven below. Beforehand, recall the following:

By 3.14: $p_* \langle t(z),\, \alpha\rangle = \langle z,\, t(\alpha)\rangle$ for $r \in \mathbb{N}$, $z \in H^r(M)$, $\alpha \in H_{r+k}(\mathrm{E}\nu_\iota, \mathrm{E}\nu_\iota{}^0)$.

By 3.16: $t(\mathrm{incl}_* c_*[S^{n+k}]) = [M]$ where $c\colon S^{n+k} \to \mathrm{D}\nu_\iota/\mathrm{S}\nu_\iota$ is the collapse map of a tubular embedding of the normal bundle ν_M.

By (3.1) *and* (3.3)*:* The total Steenrod square $\mathrm{Sq}\colon H^m(S^m) \to H^*(S^m)$ is the identity on $H^m(S^m)$, i.e. $\mathrm{Sq}^i\colon H^m(S^m) \to H^{m+i}(S^m)$ is zero for $i \neq 0$.

By Definition 3.1: The total Steenrod Square Sq is natural and a ring homomorphism.

By (3.5)*:* For any map of spaces $f\colon X \to Y$ and co-/homology classes a and β in the corresponding co-/homology groups holds $\langle a,\, f_*\beta\rangle = f_* \langle f^*a,\, \beta\rangle$

For the proof fix some $i \leq n$, some arbitrary $x \in H^{n-i}(M)$, as well as a collapse map $c\colon S^{n+k} \to \mathrm{D}\nu_\iota/\mathrm{S}\nu_\iota$ as in Lemma 3.16. For simplicity, denote $v \coloneqq v(\nu_\iota)$, $S \coloneqq S^{n+k}$, $u \coloneqq u(\nu_\iota)$ and $c_* \coloneqq (\mathrm{incl} \circ c)_*$ respectively $c^* \coloneqq (\mathrm{incl} \circ c)^*$.

First reformulate $\langle x \cup v(\nu_\iota),\, [M]\rangle$ using the cohomology of the $(n+k)$-sphere:

$$\begin{aligned}
\langle x \cup v,\, [M]\rangle &\overset{3.16}{=} \langle x \cup v,\, t\left(c_*[S]\right)\rangle \\
&\overset{3.14}{=} p_* \left\langle t\left(x \cup v\right),\, c_*[S]\right\rangle \\
&\overset{(3.5)}{=} p_* c_* \left\langle c^* t\left(x \cup v\right),\, [S]\right\rangle \\
&\overset{(3.1) \text{ and } (3.3)}{=} p_* c_* \left\langle \mathrm{Sq}\left(c^* t\left(x \cup v\right)\right),\, [S]\right\rangle \\
&\overset{\text{Naturality}}{=} p_* c_* \left\langle c^* \mathrm{Sq}\left(t\left(x \cup v\right)\right),\, [S]\right\rangle \\
&\overset{(3.5)}{=} p_* \left\langle \mathrm{Sq}\left(t\left(x \cup v\right)\right),\, c_*[S]\right\rangle
\end{aligned} \tag{3.8}$$

Having introduced Sq on the left hand side, one observes a certain commutativity of the Thom isomorphism, and the total Steenrod square:

$$\begin{aligned}
\mathrm{Sq}\left(t\left(x \cup v\right)\right) &= \mathrm{Sq}\left(\left(p^* x \cup p^* v\right) \cup u\right) \\
&= \mathrm{Sq}\left(p^* x \cup \left(p^* v \cup u\right)\right) \\
&\overset{\text{Def. } t}{=} \mathrm{Sq}\left(p^* x \cup t(v)\right) \\
&\overset{\text{Naturality, Ring Hom.}}{=} p^*\mathrm{Sq}(x) \cup \mathrm{Sq}(t(v)) \\
&\overset{\text{Def. } v(M)}{=} p^*\mathrm{Sq}(x) \cup \mathrm{Sq}(\chi(\mathrm{Sq})(u)) \\
&\overset{\text{Def. } \chi}{=} p^*\mathrm{Sq}(x) \cup u \\
&\overset{\text{Def. } t}{=} t(\mathrm{Sq}(x))
\end{aligned} \tag{3.9}$$

Inserting (3.9) into (3.8) from above easily yields the claim in (3.7) that proves the theorem:

$$\begin{aligned}
\langle x \cup v,\, [M]\rangle &\overset{(3.8)}{=} p_* \left\langle \mathrm{Sq}\left(t\left(x \cup v\right)\right),\, c_*[S]\right\rangle \\
&\overset{(3.9)}{=} p_* \left\langle t(\mathrm{Sq}(x)),\, c_*[S]\right\rangle \\
&\overset{3.14}{=} \left\langle \mathrm{Sq}(x),\, t\left(c_*[S]\right)\right\rangle \\
&\overset{3.16}{=} \left\langle \mathrm{Sq}(x),\, [M]\right\rangle
\end{aligned}$$

□

3.2.4 Wu's Theorem

Wu's theorem, the goal of this section, gives a close relation of the Wu classes from above with the Stiefel-Whitney classes using Steenrod squares. Besides, it also immediately proves that Wu classes as defined in Theorem 3.23 are actually characteristic classes.

Theorem 3.24 (Wu)**.** *A closed manifold gives rise to the following two equalities*

$$\overline{w}(M) = \mathrm{Sq}\left(\overline{v}(M)\right) \quad \textit{respectively} \quad \overline{w}_k(M) = \sum_{i \geq 0} \mathrm{Sq}^i\left(\overline{v}_{k-i}(M)\right) \quad \textit{and}$$

$$w(M) = \mathrm{Sq}\left(v(M)\right) \quad \textit{respectively} \quad w_k(M) = \sum_{i \geq 0} \mathrm{Sq}^i\left(v_{k-i}(M)\right) \quad \text{(Wu's formula)}$$

that are equivalent using $\overline{w}(M) \cup w(M) = 1$ *and*

$$\mathrm{Sq}(\overline{v}(M)) \cup \mathrm{Sq}(v(M)) = \mathrm{Sq}(\overline{v}(M) \cup v(M)) = \mathrm{Sq}(1) = 1\ .$$

The proof uses the alternative characterization

$$\chi(\mathrm{Sq})(u(\nu_\iota)) = t(v(\nu_\iota))$$

of the Wu classes from Theorem 3.23, and directly follows from the following Lemma.

Lemma 3.25. *For a vector bundle $\xi\colon \mathrm{E}\xi \xrightarrow{p} \mathrm{B}\xi$ over a paracompact space holds*

$$\mathrm{Sq}(v(\xi)) \cup w(\xi) = 1 \ .$$

Proof of Wu's formula. Lemma 3.25 states for a closed manifold M^n and any embedding $\iota\colon M \to \mathbb{R}^{n+k}$ that

$$\mathrm{Sq}(v(M)) \cup \overline{w}(M) = \mathrm{Sq}(v(\nu_M)) \cup w(\nu_\iota) \overset{3.25}{=} 1 \ .$$

Cupping with $w(M) = w(\mathrm{T}M)$ on both sides yields the claim as $w(\mathrm{T}M) \cup w(\nu_\iota) = 1$ by Remark 2.15.b). □

Proof of Lemma 3.25. For simplicity use the shortenings $u \coloneqq u(\xi)$, $v \coloneqq v(\xi)$, and $w \coloneqq w(\xi)$. Then calculate

$$\begin{aligned}
t(1) &= u \\
&\overset{\text{Def. 3.6}}{=} \mathrm{Sq}\left(\chi(\mathrm{Sq})(u)\right) \\
&\overset{\text{Def. 3.21}}{=} \mathrm{Sq}\left(t(v)\right) \\
&\overset{\text{Def. } t}{=} \mathrm{Sq}\left(p^* v \cup u\right) \\
&\overset{\text{Cartan's formula, Naturality}}{=} p^*\mathrm{Sq}(v) \cup \mathrm{Sq}(u) \\
&\overset{3.17}{=} p^*\mathrm{Sq}(v) \cup t(w) \\
&\overset{\text{Def. } t}{=} p^*\mathrm{Sq}(v) \cup (p^* w \cup u) \\
&= p^*\left(\mathrm{Sq}(v) \cup w\right) \cup u \\
&\overset{\text{Def. } t}{=} t\left(\mathrm{Sq}(v) \cup w\right)
\end{aligned}$$

Applying the inverse of the Thom isomorphism to both sides yields the equality which was to be shown. □

3.3 Massey's Theorem

Massey's main theorem on the Stiefel-Whitney classes of manifolds [Mas60, Theorem I.] gives a concrete statement, in which degrees the dual Stiefel-Whitney classes may be non-zero in general. As of Corollary 2.21 this is directly related to the immersion problem respectively an important obstruction for it.

Theorem 3.26 (Massey)**.** *Let M be a compact, n-dimensional manifold. Given an integer q with $0 < q < n$ such that $\overline{w}_{n-q}(M) \neq 0$, there is a sequence of integers $h_1 \geq \cdots \geq h_q \geq 0$ of length q that fulfills*

$$n = \sum_{i=1}^{q} 2^{h_i}$$

Note that the minimal length of a representation of n by powers of two is its binary representation of length $\alpha(n)$. As an immediate consequence:

Corollary 3.27. *For any manifold, all its dual Stiefel-Whitney classes of degree greater than $n - \alpha(n)$ must be zero.*

The following subsections are dedicated to the proof of Massey's Theorem. Let M be a compact, n-dimensional manifold throughout the proof. The latter consists of several steps:

Step 1: Show that for any q, admissible iterated Steenrod square Sq^I, and cohomology class $x \in H^q(M)$ of degree q such that $\mathrm{Sq}^I(x)$ is non-trivial there exists some representation of the form

$$\deg\left(\mathrm{Sq}^I(x)\right) = 2^k \cdot \left(2^{k_1} + \cdots + 2^{k_{q-1}} + 1\right) \ .$$

(All results during this step hold for any space X.)

Step 2: Find some iterated Steenrod square candidate to which the above degree formula is applicable, i.e. which is non-trivial in degree

$$\mathrm{Sq}^I \colon H^q(M) \to H^n(M) \ .$$

Applying Step 1 to the Steenrod square Sq^I from Step 2 and some $x \in H^q(M)$ with $\mathrm{Sq}^I(x) \neq 0$ immediately yields the result because

$$n = \deg\left(\mathrm{Sq}^I(x)\right) = \underbrace{2^{k_1+k} + \cdots + 2^{k_{q-1}+k} + 2^k}_{q \text{ summands}} \ .$$

3.3.1 Step 1: A Degree Formula for Iterated Steenrod Squares

For this step M may be any space. Step 1 requires to proof the following claim.

Lemma 3.28 (Step 1)**.** *Let $q \geq 0$ be an integer, and $I \in \mathbb{N}^{l(I)}$ an admissible sequence of integers. Further, let $x \in H^q(M)$ be a cohomology class of degree q such that $\mathrm{Sq}^I(x)$ is non-trivial. Then there exists $k \in \mathbb{N}$ and a sequence of integers $0 \leq k_1 \leq \cdots \leq k_{q-1}$ of length $q-1$ such that the degree of $\mathrm{Sq}^I(x)$ can be represented as the dissection*

$$\deg \mathrm{Sq}^I(x) = \deg x + d(I) = 2^k \cdot \left(1 + 2^{k_1} + \cdots + 2^{k_{q-1}}\right) \ .$$

In order to split the proof into several cases, recall that $\mathrm{Sq}^I(x) = 0$ for $e(I) > \deg x$ by Remark 3.5.iii). This leaves the two cases $e(I) < \deg x$ and $e(I) = \deg x$. Inductively applying the following Lemma by Serre restricts the proof of Lemma 3.28 to the first case where $e(I) < q$.

Lemma 3.29 (Serre). *Every admissible sequence $I \in \mathbb{N}^{l(I)}$ of excess $e(I) > 0$ admits an admissible sequence J with $e(J) < e(I)$, together with some $k \in \mathbb{N}$ such that for any cohomology class $x \in H^{e(I)}(M)$ holds*

$$\mathrm{Sq}^I(x) = \left(\mathrm{Sq}^J(x)\right)^{2^k} \qquad \textit{respectively} \qquad \deg\left(\mathrm{Sq}^I(x)\right) = 2^k \cdot \deg\left(\mathrm{Sq}^J(x)\right) \ .$$

Before proving Lemma 3.29 one can finish the argumentation for the case $e(I) < \deg x$.

Proof of Lemma 3.28. Let $q \in \mathbb{N}$ and $I = (i_1, \dots, i_l)$ be an admissible sequence such that $e(I) < q$. Assume there is a cohomology class $x \in H^q(M)$ such that $\mathrm{Sq}^I(x) \neq 0$. Set

$$\begin{aligned} \alpha_0 &= q - 1 - e(I) \geq 0 && \text{which is positive as } e(I) < q, \\ \alpha_r &= i_r - 2i_{r+1} && \text{for } 1 \leq r < l(I), \text{ and} \\ \alpha_{l(I)} &= i_{l(I)} \ . \end{aligned}$$

It is an easy exercise that the excess of I can be rewritten as $e(I) = \sum_{r=1}^{l(I)} \alpha_r$, so

$$\sum_{r=0}^{l(I)} \alpha_r = \alpha_0 + e(I) \overset{\text{Def.}}{=} q - 1 \ . \tag{3.10}$$

Just as easily one sees

$$i_s = \sum_{r=0}^{s} 2^r \alpha_{s+r} \tag{3.11}$$

The above definitions then directly imply the following reformulation of $d(I)$ in terms of α_i:

$$\begin{aligned} d(I) &:= \sum_{s=1}^{l(I)} i_s \\ &\overset{(3.11)}{=} \sum_{s=1}^{l(I)} \sum_{r=0}^{s} 2^r \alpha_{s+r} \\ &\overset{\text{Reorder}}{=} \sum_{j=1}^{l(I)} \left(\sum_{m=0}^{j-1} 2^m \right) \alpha_j = \sum_{j=1}^{l(I)} (2^j - 1)\alpha_j = \sum_{j=1}^{l(I)} 2^j \alpha_j - \sum_{j=1}^{l(I)} \alpha_j \\ &= \sum_{j=1}^{l(I)} 2^j \alpha_j - e(I) \ . \end{aligned} \tag{3.12}$$

All put together yields

$$\begin{aligned}
\deg\left(\mathrm{Sq}^I(x)\right) &= \deg(x) + d(I) \\
&= 1 + \deg(x) - 1 + d(I) \\
&\overset{(3.12)}{=} 1 + q - 1 - e(I) + \sum_{j=1}^{l(I)} 2^j \alpha_j \\
&\overset{\text{Def.}}{=} 1 + \alpha_0 + \sum_{j=1}^{l(I)} 2^j \alpha_j \\
&= 1 + \sum_{j=0}^{l(I)} 2^j \alpha_j = 1 + \left(\underbrace{2^0 + \cdots + 2^0}_{\alpha_0 \text{ times}} + \cdots + \underbrace{2^{l(I)} + \cdots + 2^{l(I)}}_{\alpha_{l(I)} \text{ times}} \right)
\end{aligned}$$

which is one plus a sum of exactly $\sum_{j=0}^{l(I)} \alpha_j \overset{(3.10)}{=} q-1$ powers of two as was to be shown. The k_j are in this case

$$k_j = \begin{cases} 0 & 0 < j \leq \alpha_0 \\ 1 & \alpha_0 < j \leq \alpha_1 \\ \vdots & \\ l(I) & \alpha_{l(I)-1} < j \leq \alpha_{l(I)} \end{cases}$$

□

Proof of Lemma 3.29. A version of the following proof can be found in [Ser12, p. 159, Lemma 1, converse part]. First note that any admissible sequence $I := (i_1, \ldots, i_l) \in \mathbb{N}^l$ of excess $e(I) > 0$ can be written as

$$I = (2^{k-1} i_k, \ldots, 2 i_k, i_k, i_{k+1}, \ldots, i_l)$$

with $l > k \geq 1$ chosen maximal, i.e. $i_k > 2 i_{k+1}$. If I is admissible, the subsequence $J := (i_{k+1}, \ldots, i_l)$ will be admissible as well. In order to see that such J and k fulfill the requirements from Lemma 3.29 one only needs to show

claim. For I and J as above, and $x \in H^{e(I)}(X)$ a cohomology class of a space X holds

i) $\mathrm{Sq}^I(x) = \left(\mathrm{Sq}^J(x)\right)^{2^k}$, and

ii) $e(I) < e(J)$.

For the first part one simply has to check that $\deg(\mathrm{Sq}^J(x)) = i_k$, as the statement then inductively follows from property (3.2) that $\mathrm{Sq}^i(y) = y^2$ for any cohomology class with

$i = \deg(y)$. So calculate

$$\begin{aligned}
\deg(\mathrm{Sq}^J(x)) &= d(J) + \deg(x) \\
&= d(J) + e(I) \\
&= d(J) + \left(\sum_{r=1}^{l(I)-1} (i_r - 2i_{r+1}) \right) + i_{l(I)} \\
&= d(J) + \left(\sum_{r=1}^{k-1} \underbrace{(i_r - 2i_{r+1})}_{=0} \right) + (i_k - 2i_{k+1}) + \left(\sum_{r=k+1}^{l(I)-1} (i_r - 2i_{r+1}) \right) + i_{l(I)} \\
&= d(J) + (i_k - 2i_{k+1}) + e(J) \\
&= d(J) + i_k - 2i_{k+1} + 2i_{k+1} - d(J) \\
&= i_k
\end{aligned}$$

Comparing the excesses yields the second part:

$$e(I) - e(J) = \left(\sum_{r=1}^{k-1} \underbrace{(i_r - 2i_{r+1})}_{=0} \right) + \underbrace{(i_k - 2i_{k+1})}_{>0 \text{ by def. of k}} > 0 \qquad \square$$

3.3.2 Step 2: Candidates for the Degree Formula

This section is dedicated to the search of an iterated Steenrod square Sq^I of degree $n-q$ that is non-trivial in degree $\mathrm{Sq}^I : H^q(M) \to H^n(M)$ in order to complete Step 2 of the proof of Theorem 3.26.

The candidate is the multiplication map

$$H^q(M) \longrightarrow H^n(M) \qquad x \longmapsto x \cup \overline{w}_{n-q}(M)$$

which is non-trivial if and only if $\overline{w}_{n-q}(M) \neq 0$, since the cup-product is non-degenerated by Poincaré duality [Hat02, Proposition 3.38]. It remains to see that this map actually comes from a sum of iterated Steenrod squares, one of which then must be non-trivial and of the correct degree. But this easily follows from the next main Lemma, the proof of which essentially uses Wu's theorem and the properties of the Wu classes.

Lemma 3.30. *Let $0 < k < n$ and $x \in H^k(M)$ be a cohomology class. Then*

$$x \cdot \overline{w}_{n-k}(M) = \sum_{i>0} \mathrm{Sq}^i(x) \cdot \overline{w}_{n-k-i}(M)$$

Again, before proving the above lemma, let us see how this contributes to the final result.

Proof of Theorem 3.26. The trick to complete Step 1 is to show that for $\overline{w}_{n-q}(M) \neq 0$, and k, x as in the Lemma above

$$x \cdot \overline{w}_{n-q}(M) = \sum_{I \in A} \mathrm{Sq}^I(x) \ ,$$

where A is some collection of sequences of degree $(n-q)$. Because then for at least one $I \in A$, $\mathrm{Sq}^I(x)$ is non-trivial, as is needed to apply Lemma 3.28 which then yields the desired decomposition of $n = \deg(\mathrm{Sq}^I(x))$.

The above lemma serves for a descending induction on the maximum degree $n-q-r$ of the dual Stiefel-Whitney classes occurring in the sum describing $x \cdot \overline{w}_{n-q}(M)$. More precisely, assume that

$$x \cdot \overline{w}_{n-q}(M) = \sum_{I \in A} \mathrm{Sq}^I(x) + \sum_{j=r}^{n-q-1} \mathrm{Sq}^{I_j}(x) \cdot \overline{w}_{n-q-j}(M) \ .$$

Then Lemma 3.30 reduces the maximum degree $n-q-r$ of dual Stiefel-Whitney classes with the induction rule

$$\mathrm{Sq}^I(x) \cdot \overline{w}_{n-k-j}(M) \overset{3.30}{=} \sum_{i>0} \mathrm{Sq}^i \circ \mathrm{Sq}^I(x) \cdot \overline{w}_{n-k-(i+j)}(M)$$

as follows:

$$\begin{aligned}
\sum_{j=r}^{n-q-1} \mathrm{Sq}^{I_j}(x) \cdot \overline{w}_{n-q-j}(M) &= \sum_{j=r}^{n-q-1} \sum_{i>0} \mathrm{Sq}^i(\mathrm{Sq}^{I_j}(x)) \cdot \overline{w}_{n-q-j-i}(M) \\
&= \sum_{j=r}^{n-q-1} \sum_{i=1}^{n-q-j} \mathrm{Sq}^{(i)\oplus I_j}(x) \cdot \overline{w}_{n-q-j-i}(M) \\
&= \sum_{r+1 \leq i+j \leq n-q} \mathrm{Sq}^{(i)\oplus I_j}(x) \cdot \overline{w}_{n-q-j-i}(M) \\
&= \sum_{i+j=n-q} \mathrm{Sq}^{(i)\oplus I_j}(x) \cdot \overline{w}_0(M) + \sum_{r+1 \leq i+j \leq n-q-1} \mathrm{Sq}^{(i)\oplus I_j}(x) \cdot \overline{w}_{n-q-j-i}(M) \\
&= \sum_{i+j=n-q} \mathrm{Sq}^{(i)\oplus I_j}(x) + \sum_{r+1 \leq i+j \leq n-q-1} \mathrm{Sq}^{(i)\oplus I_j}(x) \cdot \overline{w}_{n-q-j-i}(M) \ .
\end{aligned}$$

Going down respectively up to $r = n-q$, one obtains that $x \cdot \overline{w}_{n-q}(M)$ is a sum of iterated Steenrod squares evaluated on x as was to be shown. $\square$

Proof of Lemma 3.30. As above let $0 < k < n$ and $x \in H^k(M)$. For simplicity write $w_i := w_i(M)$, $\overline{w}_i := \overline{w}_i(M)$, $v_i = v_i(M)$ and $\overline{v}_i = \overline{v}_i(M)$. In order to translate from Stiefel-Whitney classes to Steenrod squares and back again recall the following results.

By Theorem 3.24 (Wu): $\sum_{j=0}^{s} \mathrm{Sq}^j(\overline{v}_{s-j}) = \overline{w}_{s-j}$ for any $s \leq n$.

By Definition 3.18 of v: $v_i \cdot y = \mathrm{Sq}^i(y)$ for any $i \leq n$ and $y \in H^{n-i}(M)$.

By Definition 3.20 of $\overline{v}$: $\overline{v}_d = \sum_{i=1}^{d} v_i \cdot \overline{v}_{d-i}$ for $d > 0$.

The full calculation is then

$$\begin{aligned}
x \cdot \overline{w}_{n-k} &\overset{3.24}{=} x \cdot \sum_{i=0}^{n-k} \mathrm{Sq}^i(\overline{v}_{n-k-i}) \\
&\overset{\text{Def. }\mathrm{Sq}^0}{=} x \cdot \left(\overline{v}_{n-k} + \sum_{i=1}^{n-k} \mathrm{Sq}^i(\overline{v}_{n-k-i}) \right) \\
&\overset{\text{Def. }\overline{v}}{=} x \cdot \left(\left(\sum_{i=1}^{n-k} v_i \cdot \overline{v}_{n-k-i} \right) + \left(\sum_{i=1}^{n-k} \mathrm{Sq}^i(\overline{v}_{n-k-i}) \right) \right) \\
&= \sum_{i=1}^{n-k} \left(v_i \cdot x \cdot \overline{v}_{n-k-i} + x \cdot \mathrm{Sq}^i(\overline{v}_{n-k-i}) \right) \\
&\overset{\text{Def. }v}{=} \sum_{i=1}^{n-k} \left(\mathrm{Sq}^i(x \cdot \overline{v}_{n-k-i}) + x \cdot \mathrm{Sq}^i(\overline{v}_{n-k-i}) \right) \\
&\overset{\text{Cartan's formula, Def.}}{=} \sum_{i=1}^{n-k} \left(\left(\sum_{r=0}^{i} \mathrm{Sq}^r(x) \cdot \mathrm{Sq}^{i-r}(\overline{v}_{n-k-i}) \right) + \mathrm{Sq}^0(x) \cdot \mathrm{Sq}^i(\overline{v}_{n-k-i}) \right) \\
&= \sum_{i=1}^{n-k} \left(\sum_{r=1}^{i} \mathrm{Sq}^r(x) \cdot \mathrm{Sq}^{i-r}(\overline{v}_{n-k-i}) \right) \\
&\overset{\text{Reorder}}{=} \sum_{r=1}^{n-k} \mathrm{Sq}^r(x) \cdot \left(\sum_{j=0}^{n-k-r} \mathrm{Sq}^j(\overline{v}_{n-k-(j+r)}) \right) \\
&\overset{3.24}{=} \sum_{r=1}^{n-k} \mathrm{Sq}^r(x) \cdot \overline{w}_{n-k-r} \qquad \square
\end{aligned}$$

This finished the proof of Massey's theorem.

3.4 Best Possible Result

Recall that a closed n-manifold can only immerse into $\mathbb{R}^{n+k}$ if all Stiefel-Whitney classes $\overline{w}_i(\nu_M)$ of its normal bundle are zero in degrees greater than k. Massey's Theorem now states that this condition is met for $k = n - \alpha(n)$, and thus for all manifolds the above obstruction to the immersion property vanishes. From this arose the idea for the immersion conjecture.

Naturally, there occurs the question whether Massey's $n - \alpha(n)$ is the best possible (i.e. smallest) result for arbitrary closed n-manifolds, making the immersion conjecture the best guess of a general possible immersion codimension. The answer is yes, proved by the following counterexamples of manifolds not immersing into $\mathbb{R}^{2n-(\alpha(n)+1)}$.

Theorem 3.31. *Denote by* $\mathbb{RP}^i$ *the* i*th real projective space.*

i) For $n = 2^i$, *the Stiefel-Whitney class* $\overline{w}_{n-\alpha(n)}(\mathbb{RP}^{2^i})$ *is not zero.*

ii) For $n \in \mathbb{N}$ *with binary expansion* $n = \sum_{r=1}^{q} 2^{i_r}$, $i_1 > \cdots > i_q$, *the Stiefel-Whitney class* $\overline{w}_{n-\alpha(n)}(\prod_{r=1}^{q} \mathbb{RP}^{2^r})$ *is not zero.*

Proof. Compare also [CT89, p. 87]. The total Stiefel-Whitney class of $\mathbb{RP}^n$ for arbitrary $n \in \mathbb{N}$ is

$$w(\mathbb{RP}^n) = (1 + x_n)^{n+1} \in H^*(\mathbb{RP}^n) \cong \mathbb{Z}_2[x_n]/(x_n^{n+1})$$

(see e.g. [Die08, Example (19.4.1)]). For $n = 2^i$ this takes the form

$$\begin{aligned} w(\mathbb{RP}^{2^i}) &= (x+1)^{2^i+1} = 1 + x + x^{2^i} \ , \quad \text{thus} \\ \overline{w}(\mathbb{RP}^{2^i}) &= \sum_{r=0}^{2^i-1} x^r \ , \quad \text{especially} \\ \overline{w}_{n-\alpha(n)}(\mathbb{RP}^{2^i}) &= x^{2^i-1} \neq 0 \ , \quad \text{since } \alpha(2^i) = 1, \end{aligned}$$

which proves the first statement. For the second statement where $n = \sum_{r=1}^{q} 2^{i_r}$ note that

$$n - \alpha(n) = \left(\sum_{r=1}^{q} 2^{i_r}\right) - q = \sum_{r=1}^{q} \left(2^{i_r} - 1\right) \ . \tag{3.13}$$

Using multiplicativity of the Stiefel-Whitney classes one gets

$$\begin{aligned} \overline{w}(\prod_{r=1}^{q} \mathbb{RP}^{2^{i_r}}) &= \prod_{r=1}^{q} \overline{w}(\mathbb{RP}^{2^{i_r}}) = \prod_{r=1}^{q} \left(\sum_{i=1}^{2^{i_r}-1} x_{2^{i_r}}^i\right) \qquad \text{and by combinatorics} \\ \overline{w}_{n-\alpha(n)}(\prod_{r=1}^{q} \mathbb{RP}^{2^{i_r}}) &= \prod_{r=1}^{q} x_{2^{i_r}}^{2^{i_r}-1} \neq 0 \end{aligned}$$

in $\mathbb{Z}_2\left[x_{2^{i_1}}, \ldots, x_{2^{i_q}}\right] / \left(x_{2^{i_1}}^{2^{i_1}+1}, \ldots, x_{2^{i_q}}^{2^{i_q}+1}\right) \cong H^*\left(\prod_{r=1}^{q} \mathbb{RP}^{2^{i_r}}\right)$. $\square$

4 The Immersion Conjecture up to Cobordism

The overall goal of this chapter is to prove the following theorem of R. L. Brown following his paper [Bro71], which essentially states that the immersion conjecture is true up to the cobordism relation.

Theorem 4.1 (Brown)**.** *Every closed n-manifold is cobordant to an n-manifold that immerses into* $\mathbb{R}^{2n-\alpha(n)}$.

As one easily sees that this property is stable under the ring operations (Lemma 4.39), the main idea for the proof is to find manifolds fulfilling the conjecture whose cobordism classes form a generating set of the cobordism ring. As the latter has the form of a polynomial algebra $\mathbb{Z}_2[\sigma_i \mid i \neq 2^r - 1]$, a set of elements $([G^i] \mid i \neq 2^r - 1)$ will already be a generating set if all of the $[G^i]$ are indecomposable. Thus, the candidate generating elements that will be constructed in Section 4.3 need to be tested for:

i) indecomposability, which is the more lengthy part as it requires the preliminary results from Subsection 4.1.4 respectively Subsection 4.2.3, and

ii) the property to fulfill the immersion conjecture.

The odd generators are going to be constructed using twisted products, which are introduced in Subsection 4.2.1.

This chapter is structured into some preliminary work on finding a criterion to easily detect indecomposable elements of the cobordism ring in Section 4.1, the twisted product construction and its properties—especially concerning indecomposability—in Section 4.2, and the final proof with the construction of the generating set in Section 4.3, where all preliminary work is merged.

For clarity of presentation, both a couple of results from Thom's paper [Tho07] within the review in Subsection 4.1.3 will merely be referenced without proof. The reader is assumed to be familiar with symmetric polynomials.

Call a manifold *indecomposable* if it represents an indecomposable element of the cobordism ring.

4.1 Detecting Indecomposable Elements of the Cobordism Ring

In order to find representatives for a set of algebraically independent generators of the polynomial cobordism ring, one needs a way to detect indecomposable elements. Indecomposable in this context means to not be expressible as a sum of products of lower degree elements.

This section will follow an approach of Thom [Tho07, Chapters IV.5 and IV.6], in which a certain characteristic class serves as indicator. This characteristic class is constructed out of Stiefel-Whitney classes using certain functions on symmetric polynomials (see Subsection 4.1.1), and becomes particularly useful on tangent bundles of manifolds (see Subsection 4.1.2). The final indication lemma is then stated and proved in Subsection 4.1.4.

4.1.1 Special Properties of Symmetric Polynomials

This subsection examines a special kind of polynomials that obey a product rule similar to Cartan's formula whenever evaluated on elements of the form of a total Stiefel-Whitney class.

As a consequence, one can express certain combinations of Stiefel-Whitney numbers of product manifolds in terms of ones of their factors, which will be investigated in detail in Subsection 4.1.2. From this, Subsection 4.1.4 will deduce a simple criterion for a manifold to be cobordant to a product of manifolds.

Beforehand, mind the following notation of partitions needed for symmetrising polynomials.

Definition 4.2. Let $k, l \in \mathbb{N}$ be integers.

- A *partition* $\mathscr{P} = (i_1, \dots, i_l)$ of k is an unordered sequence of integers such that $k = \sum_{r=0}^{l} i_r$. Two partitions only differing by zeros are considered equal. In other words, a partition is an equivalence class of sequences in $\bigoplus_\infty \mathbb{N}$ under the relation $\mathscr{P} \sim \sigma(\mathscr{P})$ for any permutation σ.
- The notation I^l for a sequence of integers will mean a sequence of length l. Write $I^l \in \mathscr{P}$ for a sequence of length l in the equivalence class of the partition $\mathscr{P}$.
- Denote by $\Pi(k)$ the set of partitions of k.
- Write $()$ for the unique partition of 0.
- Call a sequence or partition *non-dyadic*, if none of its entries is of the form $2^m - 1$.

- The concatenation of sequences, and analogously partitions, will be denoted by

$$\begin{aligned} \mathbb{N}^{l_1} \times \mathbb{N}^{l_2} &\xrightarrow{-\oplus-} \mathbb{N}^{l_1+l_2} \\ \Pi(k_1) \times \Pi(k_2) &\xrightarrow{-\oplus-} \Pi(k_1+k_2) \\ (i_1,\dots,i_r),(j_1,\dots,j_s) &\longmapsto (i_1,\dots,i_r,j_1,\dots,j_s) \end{aligned}$$

Also as preparation, recall some properties of symmetric polynomials and agree on a shorthand for symmetrised monomials.

Lemma/Definition 4.3. Let $n \in \mathbb{N}$ and $\mathbb{Z}_2[t_1,\dots,t_n]$ be the graded polynomial algebra in n variables over the fields $\mathbb{Z}_2$, each t_i of degree one.

i) Let $\mathcal{S}^n_* := \mathbb{Z}_2[t_1,\dots,t_n]^{\Sigma_n} \subset \mathbb{Z}_2[t_1,\dots,t_n]$ be the graded subalgebra of symmetric polynomials in n variables.

ii) $\mathcal{S}^n_*$ has a basis indexed by partitions $\mathscr{P}$ consisting of symmetrised monomials, i.e. elements of the form
$$\mathrm{Sym}_n t^{\mathscr{P}} := \sum_{I^n \in \mathscr{P}} t^I \in \mathcal{S}^n_k$$
where $t^{(i_1,\dots,i_n)} := t_1^{i_1} \cdots t_n^{i_n}$.

Proof. See also [Tho07, footnote 2, p. 154]. Linear independence is clear as monomials t^I and $t^{I'}$ are linearly independent if $I \neq I'$, and sequences belonging to different partitions must be unequal. For the generating property note that every symmetric polynomial p is of the form $\sum_{I^n \in A} t^I$. This however can be written as a sum of symmetrised monomials by descending induction: For a symmetric polynomial
$$p = \sum_{\mathscr{P} \in B} \mathrm{Sym}_n t^{\mathscr{P}} + \sum_{I^n \in A} t^I ,$$
and $I' \in A$, one must have $s(I') \in A$ for $s \in \Sigma_n$, since
$$p - \sum_{\mathscr{P} \in B} \mathrm{Sym}_n t^{\mathscr{P}} = \sum_{I^n \in A} t^I$$
is still symmetric. Hence one can write
$$\begin{aligned} p &= \sum_{\mathscr{P} \in B} \mathrm{Sym}_n t^{\mathscr{P}} + \sum_{s \in \Sigma_n} t^{s(I')} + \sum_{I^n \in A \setminus \{s(I') \mid s \in \Sigma_n\}} t^I \\ &= \sum_{\mathscr{P} \in B \cup \{[I']\}} \mathrm{Sym}_n t^{\mathscr{P}} + \sum_{I^n \in A \setminus \{s(I') \mid s \in \Sigma_n\}} t^I , \end{aligned}$$
and thus inductively decrease $\#A$ to zero, expressing p as a sum of symmetrised polynomials. $\square$

iii) $\mathcal{S}_*^n$ is generated as algebra by the algebraically independent, elementary symmetric polynomials in n variables

$$\sigma_i^n := \mathrm{Sym}_n t^{\mathscr{P}_i} \in \mathcal{S}_i^n \qquad \text{for } \mathscr{P}_i = (1, \dots, 1) \in \Pi(i)$$

for $1 \leq i \leq n$. E.g. $\sigma_1^n = \sum_{r=1}^n t_r$, $\sigma_2^n = \sum_{1 \leq r < s \leq n} t_r t_s$.

Proof. This is the well-known fundamental theorem on symmetric polynomials, see e.g. [Bos13, Chap. 4.4, Satz 1]. □

iv) As a simple calculation shows that the elementary symmetric polynomials in n variables fulfill

$$1 + \sum_{i=1}^n \sigma_i^n = \prod_{r=1}^n (1 + t_n) \tag{4.1}$$

Now the desired polynomials can be defined. The following definition is according to [Mil57, p. 90].

Definition 4.4. Let $k \in \mathbb{N}$, $\mathscr{P} \in \Pi(k)$, and let $\mathbb{Z}_2[\alpha_1, \dots, \alpha_k]$ be the polynomial ring in k variables where α_i has degree i.

i) Define the homogeneous polynomial $s_{\mathscr{P}} \in \mathbb{Z}_2[\alpha_1, \dots, \alpha_k]$ of degree k by

$$s_{\mathscr{P}}(\sigma_1^n, \dots, \sigma_k^n) = \mathrm{Sym}_n t^{\mathscr{P}} \in \mathcal{S}_k^n$$

for some $n \geq k$. This means, $s_{\mathscr{P}}$ gives the representation of $\mathrm{Sym}_n t^{\mathscr{P}}$ in terms of the generating set $(\sigma_i^n)_i$. Mind, that this

a) is well-defined, as the elementary symmetric polynomials are algebraically independent generators, and

b) does not depend on n as long as $k \leq n$.

ii) Further, for a graded, commutative ring $A_* = \bigoplus_{i \geq 0} A_i$ write elements as

$$a = \sum_i a_i := (a_0, a_1, a_2, \dots)\ , \quad \text{and}$$

iii) define the evaluation of $s_{\mathscr{P}}$ on such an element a as

$$s_{\mathscr{P}}(a) := s_{\mathscr{P}}(a_1, \dots, a_k) \in A_k\ ,$$

i.e. skip a_0 and all higher a_i.

Example 4.5. The first such polynomials over $\mathbb{Z}_2$ are (see [Mil57, p. 90]):

$k = 0:\quad s_{()} = 1$

$k = 1:\quad s_{(1)} = \alpha_1$

$k = 2:\quad s_{(2)} = \alpha_1^2 \qquad s_{(1,1)} = \alpha_2$

$k = 3:\quad s_{(3)} = \alpha_1^3 + \alpha_1\alpha_2 + \alpha_3 \qquad s_{(1,2)} = \alpha_1\alpha_2 + \alpha_3 \qquad s_{(1,1,1)} = \alpha_3$

Remark 4.6. If $k \leq r \leq n \in \mathbb{N}$ and one is given an element $a = 1 + \sum_{i \geq 1} a_i$ in a graded commutative algebra A_*, which can be written as

$$a = 1 + \sum_{i=1}^{r} \sigma_i^n(f_1, \ldots, f_n)$$

for some degree one elements $f_i \in A_*$, then for any partition $\mathscr{P} \in \Pi(k)$ one gets

$$\begin{aligned} s_{\mathscr{P}}(a) &= s_{\mathscr{P}}\left(\sigma_1^n(f_1, \ldots, f_n), \ldots, \sigma_k^n(f_1, \ldots, f_n)\right) \\ &= \left(s_{\mathscr{P}}\left(\sigma_1^n, \ldots, \sigma_k^n\right)\right)(f_1, \ldots, f_n) \\ &= \mathrm{Sym}_n f^{\mathscr{P}} \in A_k \ . \end{aligned}$$

This trick will be very useful when dealing with Stiefel-Whitney classes.

As promised, these translation-polynomials between the basis $\mathrm{Sym}_n t^{\mathscr{P}}$ of $\mathcal{S}_k^n$ and the generators σ_i^n fulfill the following interesting property concerning multiplication of elements that have the form of a total Stiefel-Whitney class.

Lemma 4.7. *Let $k \in \mathbb{N}$, and let $A_* = \bigoplus_{i \geq 0} A_i$ be a graded commutative ring. For $a, b \in A_*$ with $a_0 = 1 = b_0$, and any partition $\mathscr{P} \in \Pi(k)$ holds*

$$s_{\mathscr{P}}(a \cdot b) = \sum_{\mathscr{P}_1 \oplus \mathscr{P}_2 = \mathscr{P}} s_{\mathscr{P}_1}(a) \cdot s_{\mathscr{P}_2}(b) \ .$$

Proof. A proof can also be found in [Mil57, Theorem 33, p. 91f]. We will prove the lemma for a special which generalizes to the lemma. Beforehand, recall that for $r \geq 1$ and any element $f = \sum_{i \geq 0} f_i$ in a commutative graded ring A_* holds

$$s_{\mathscr{P}}\left(\sum_{i \geq 0} f_i\right) = s_{\mathscr{P}}\left(\sum_{i=0}^{k+r} f_i\right) \ . \tag{4.2}$$

Now let A_* be the subring of $\mathbb{Z}[t_1, \ldots, t_{2k}]$ that is generated by the algebraically independent elements

$$a_i := \sigma_i^k(x_1, \ldots, x_k) \qquad \text{and} \qquad b_i := \sigma_i^k(y_1, \ldots, y_k)$$

for $1 \leq i \leq k$ and with $x_i = t_i$ and $y_i = t_{2i}$. Then, for any other graded commutative ring $\bar{A}_*$, and elements $\bar{a} = 1 + \sum_{i \geq 1} \bar{a}_i$, $\bar{b} = 1 + \sum_{i \geq 1} \bar{b}_i \in \bar{A}_*$, the ring homomorphism defined by

$$\phi\colon A_* \longrightarrow \bar{A}_* \ , \qquad a_i \longmapsto \bar{a}_i \ , \qquad b_i \longmapsto \bar{b}_i$$

is both well-defined, since a_i, b_i are algebraically independent generators, and surjective onto the subring of $\bar{A}_*$ generated by $\bar{a}_i, \bar{b}_i$ with $1 \leq i \leq k$. Hence, if we assume the statement is proven for $a = 1 + \sum_{i=1}^k a_i$ and $b = 1 + \sum_{i=1}^k b_i$ in A_*, one has

$$
\begin{aligned}
s_{\mathscr{P}}(\bar{a} \cdot \bar{b}) &\overset{(4.2)}{=} s_{\mathscr{P}}\left(\left(1 + \sum_{i=1}^{k} \bar{a}_i\right) \cdot \left(1 + \sum_{i=1}^{k} \bar{b}_i\right)\right) = s_{\mathscr{P}}(\phi(a) \cdot \phi(b)) \\
&= \phi(s_{\mathscr{P}}(a \cdot b)) \\
&\overset{\text{Assumption}}{=} \phi\left(\sum_{\mathscr{P}_1 \oplus \mathscr{P}_2 = \mathscr{P}} s_{\mathscr{P}_1}(a) \cdot s_{\mathscr{P}_2}(b)\right) \\
&= \sum_{\mathscr{P}_1 \oplus \mathscr{P}_2 = \mathscr{P}} s_{\mathscr{P}_1}(\phi(a)) \cdot s_{\mathscr{P}_2}(\phi(b)) = \sum_{\mathscr{P}_1 \oplus \mathscr{P}_2 = \mathscr{P}} s_{\mathscr{P}_1}\left(\sum_{i=1}^{k} \bar{a}_i\right) \cdot s_{\mathscr{P}_2}\left(\sum_{i=1}^{k} \bar{b}_i\right) \\
&\overset{(4.2)}{=} \sum_{\mathscr{P}_1 \oplus \mathscr{P}_2 = \mathscr{P}} s_{\mathscr{P}_1}(\bar{a}) \cdot s_{\mathscr{P}_2}(\bar{b}) \ ,
\end{aligned}
$$

which is the statement of the lemma for $\bar{a}, \bar{b}$. Therefore, it suffices to show the lemma for a and b as above.

In order to do so, observe that

$$
\begin{aligned}
a &:= \prod_{r=1}^{k}(1 + t_r) \overset{(4.1)}{=} 1 + \sum_{i=1}^{k} \sigma_i^k(x_1, \ldots, x_k) \ , \text{ and} \\
b &:= \prod_{r=k+1}^{n}(1 + t_r) \overset{(4.1)}{=} 1 + \sum_{i=1}^{k} \sigma_i^k(y_1, \ldots, y_k) \ , \text{ then} \\
a \cdot b &= \prod_{r=1}^{n}(1 + t_r) \overset{(4.1)}{=} 1 + \sum_{i=1}^{2k} \sigma_i^n(t_1, \ldots, t_{2k}) \ ,
\end{aligned}
$$

which nicely fits into the defining relation for $s_{\mathscr{P}}$. Now calculate

$$
\begin{aligned}
s_{\mathscr{P}}(a \cdot b) &:= \mathrm{Sym}_n t^{\mathscr{P}} \\
&\overset{\text{Def.}}{=} \sum_{I^n \in \mathscr{P}} t_1^{i_1} \cdots t_n^{i_n} \\
&= \sum_{I^n \in \mathscr{P}} x^{(i_1, \ldots, i_k)} \cdot y^{(i_{k+1}, \ldots, i_n)} \\
&= \sum_{J_1^k \oplus J_2^k \in \mathscr{P}} x^{J_1} \cdot y^{J_2} \\
&\overset{\text{Group by equiv.}}{=} \sum_{\mathscr{P}_1 \oplus \mathscr{P}_2 = \mathscr{P}} \sum_{\substack{J_1^k \in \mathscr{P}_1 \\ J_2^k \in \mathscr{P}_2}} x^{J_1} \cdot y^{J_2} \\
&= \sum_{\mathscr{P}_1 \oplus \mathscr{P}_2 = \mathscr{P}} \left(\sum_{J_1^k \in \mathscr{P}_1} x^{J_1}\right) \cdot \left(\sum_{J_2^k \in \mathscr{P}_2} y^{J_2}\right)
\end{aligned}
$$

$$\stackrel{\text{Def.}}{=} \sum_{\mathscr{P}_1 \oplus \mathscr{P}_2 = \mathscr{P}} \left(\mathrm{Sym}_k x^{\mathscr{P}_1}\right) \cdot \left(\mathrm{Sym}_k y^{\mathscr{P}_2}\right)$$
$$\stackrel{\text{Def.}}{=} \sum_{\mathscr{P}_1 \oplus \mathscr{P}_2 = \mathscr{P}} s_{\mathscr{P}_1} \left(\sigma_1^k(x_1, \ldots, x_k), \ldots\right) \cdot s_{\mathscr{P}_2} \left(\sigma_1^k(y_1, \ldots, y_k), \ldots\right)$$
$$\stackrel{\text{Def.}}{=} \sum_{\mathscr{P}_1 \oplus \mathscr{P}_2 = \mathscr{P}} s_{\mathscr{P}_1}(a) \cdot s_{\mathscr{P}_2}(b) \qquad \square$$

Example 4.8. The most important partition of an integer k will be the trivial one $(k) \in \Pi(k)$. In this case Lemma 4.7 says

$$s_{(k)}(a \cdot b) = s_{(k)}(a) + s_{(k)}(b) \; .$$

4.1.2 Stiefel-Whitney Numbers of Product Manifolds

In order to apply the special polynomials out of the preceding section, as well as their product property from Lemma 4.7, to the Stiefel-Whitney numbers of (product) manifolds, first start with Stiefel-Whitney classes.

So, let $M^n = M_1^{n_1} \times M_2^{n_2}$ all be closed manifolds of the noted dimension throughout this section.

Recall that

i) the cohomology ring $H^*(M)$ is a graded ring,

ii) the total Stiefel-Whitney number of a manifold is of the form

$$w(M_i) = 1 + w_1(M_i) + \cdots + w_n(M_i) \; , \quad \text{and that}$$

iii) by the Künneth isomorphism, we have

$$\begin{aligned} H^*(M_1) \otimes H^*(M_2) &\xrightarrow{\cong} H^*(M) \\ c_1 \otimes c_2 &\longmapsto c_1 \times c_2 := \mathrm{proj}_1^* c_1 \cup \mathrm{proj}_2^* c_2 \end{aligned}$$

and $w(M) = w(M_1) \times w(M_2)$ by Axiom 3 of the Stiefel-Whitney classes.

Thus, one can apply the multiplication rule 4.7 to $w(M)$, which immediately yields:

Corollary 4.9. *For $M = M_1 \times M_2$ manifolds as above one gets for any partition $\mathscr{P} \in \Pi(n)$:*

$$\begin{aligned} s_{\mathscr{P}}(w(M)) &= s_{\mathscr{P}}\left(w(M_1) \times w(M_2)\right) \\ &= \sum_{\mathscr{P}_1 \oplus \mathscr{P}_2 = \mathscr{P}} s_{\mathscr{P}_1}(w(M_1)) \times s_{\mathscr{P}_2}(w(M_2)) \\ &= \sum_{\substack{\mathscr{P}_1 \oplus \mathscr{P}_2 = \mathscr{P} \\ \mathscr{P}_1 \in \Pi(n_1) \\ \mathscr{P}_2 \in \Pi(n_2)}} s_{\mathscr{P}_1}(w(M_1)) \times s_{\mathscr{P}_2}(w(M_2)) \end{aligned}$$

Proof. For the last equality note that by definition of $s_{\mathscr{P}}$ for any partition $\mathscr{P}$ of some $k \in \mathbb{N}$, the element $s_{\mathscr{P}}(w(W))$ lies in $H^k(M)$, hence must be zero if $k > \dim W$. Therefore, for any combination of partitions $\mathscr{P}_1 \in \Pi(k_1)$, $\mathscr{P}_2 \in \Pi(k_2)$ where

$$k_1 + k_2 = n = n_1 + n_2 \quad \text{with } k_i \neq n_i,$$

the product $s_{\mathscr{P}_1}(w(M_1)) \cdot s_{\mathscr{P}_2}(w(M_2))$ will have a zero factor and can be skipped. □

In order to pass to Stiefel-Whitney numbers instead of classes, use the following notation.

Definition 4.10. Let W be a closed manifold and $\mathscr{P} \in \Pi(\dim W)$. Then write

$$\begin{aligned} s_{\mathscr{P}}(W) &:= s_{\mathscr{P}}(w(W)) \\ s_{\mathscr{P}}[W] &:= \langle s_{\mathscr{P}}(W), [W]\rangle \ . \end{aligned}$$

Now the product rule from Lemma 4.7 translates to

Corollary 4.11. *For closed manifolds M_1, and M_2 with dimensions n_1 and n_2 one has*

$$s_{\mathscr{P}}[M_1 \times M_2] = \sum_{\substack{\mathscr{P}_1 \oplus \mathscr{P}_2 = \mathscr{P} \\ \mathscr{P}_1 \in \Pi(n_1) \\ \mathscr{P}_2 \in \Pi(n_2)}} s_{\mathscr{P}_1}[M_1] \cdot s_{\mathscr{P}_2}[M_2] \quad \in \quad \mathbb{Z}_2$$

and as a special case

$$s_{(n_1+n_2)}[M_1 \times M_2] = 0 \ . \tag{4.3}$$

In particular, if $s_{(\dim W)}[W] \neq 0$ for a closed manifold W, W is no product manifold.

Proof. The statement immediately follows from the previous Corollary 4.9 with the following two facts:

i) The generator $[M_1 \times M_2] \in H^{n_1+n_2}(M_1 \times M_2)$ corresponds to the generator

$$[M_1] \otimes [M_2] \in H^{n_1}(M_1) \otimes H^{n_2}(M_2) \cong H^{n_1+n_2}(M_1 \times M_2)$$

under the Künneth isomorphism.

ii) For cohomology classes $c_1 \in H^{n_1}(M_1)$, $c_2 \in H^{n_2}(M_2)$ holds

$$\langle c_1 \otimes c_2, [M_1 \times M_2]\rangle = \langle c_1 \otimes c_2, [M_1] \otimes [M_2]\rangle = \langle c_1, [M_1]\rangle \cdot \langle c_2, [M_2]\rangle \in \mathbb{Z}_2$$

by the universal property of the tensor product. □

Equation 4.3 is the desired obstruction for a manifold to be a product or—as will be explained in the next subsections—to be cobordant to a product. Finally, this statement will be an invaluable tool for detecting manifolds that are not only not cobordant to a product manifold but whose cobordism class is indecomposable.

4.1.3 Review: The Cobordism Ring Structure

Recall that two closed manifolds of the same dimension n are (unoriented) cobordant if their disjoint union is the boundary of an $(n+1)$-dimensional manifold. This is an equivalence relation amongst n-manifolds, and the set of equivalence classes forms an Abelian group η_n of order two with the disjoint sum as addition and the n-sphere as zero element. The Cartesian product turns the graded $\mathbb{Z}_2$-module $\eta_* := \bigoplus_{n\geq 0} \eta_n$ into an $\mathbb{Z}_2$-algebra called the (unoriented) cobordism ring. Denote the cobordism equivalence class of a manifold M by $[M]$.

Most remarkably, the cobordism relation is homotopy invariant, which will become clear from the property described in Theorem 4.14. Further, the structure of this algebra is well-known to be as follows.

Theorem 4.12 (Thom)**.** *There is an isomorphism of graded $\mathbb{Z}_2$-algebras*

$$\eta_* \cong \mathbb{Z}_2[\sigma_i \mid i \neq 2^r - 1] .$$

Proof. Compare [CT89, Thm. 1.23], and [Tho07, Theorem IV.9]. See [Tho07, Theorem IV.12], or Subsection 4.1.4, Step 3, for a proof using Theorem 4.13 below. Alternatively see [Sto68, Chap. VI]. □

During the proof of the above theorem, Thom constructs special manifolds that form a basis of the cobordism ring, and are uniquely characterized by the below properties. For the formulation recall that a sequence or partition is called *non-dyadic*, if none of its entries is of the form $2^r - 1$.

Theorem 4.13. *There exists a basis of the cobordism ring represented by manifolds $V_{\mathscr{P}}$ that in each degree k is indexed by non-dyadic partitions $\mathscr{P}$ of k. Further, the $V_{\mathscr{P}}$ are uniquely characterized by*

$$s_{\mathscr{P}'}(w(V_{\mathscr{P}})) = \delta_{\mathscr{P},\mathscr{P}'} \in H^k(V_{\mathscr{P}}) \cong \mathbb{Z}_2$$

for any non-dyadic partitions $\mathscr{P}, \mathscr{P}' \in \Pi(k)$, where δ is the usual Kronecker delta.

Proof. See [Tho07, Section IV.5, proof of Theorem IV.9]. □

In order to relate the results on the Stiefel-Whitney numbers of manifolds—respectively certain linear combinations of them—from before with cobordism classes, one needs the following more general connection. Thom rather directly deduces this from the existence of the above basis.

Theorem 4.14 (Thom)**.** *Two closed manifolds are cobordant if and only if all of their Stiefel-Whitney numbers coincide.*

Proof (sketch). The proof that manifolds with coinciding Stiefel-Whitney numbers are cobordant was conducted by Thom [Tho07, Theorem IV.10]. To see that cobordant manifolds have the same Stiefel-Whitney numbers, let M^n be a null-bordant closed manifold, i.e. assume $M^n = \partial W$ for a closed manifold W. Now, consider any sequence $(i_1, \ldots, i_l) =: I$ with $\sum_{r=1}^{l} i_r = n$, and the corresponding Stiefel-Whitney number

$$\left\langle w_1(M)^I,\, [M] \right\rangle \coloneqq \left\langle w_1(M)^{i_1} \cdots w_l(M)^{i_l},\, [M] \right\rangle$$

of M. One calculates using the long exact sequence of cohomology respectively homology of the pair $i\colon M \hookrightarrow W$ (abbreviated les) with boundary map ∂:

$$\begin{aligned} w(M) &= w(\mathrm{T}M) = w(\mathrm{T}M \oplus \varepsilon) = w(\mathrm{T}W|_M) \overset{\text{Def.}}{=} w(i^*\mathrm{T}W) = i^* w(W) \\ [M] &\overset{\text{les}}{=} \partial[W] \ . \end{aligned}$$

With the fact $i_* \circ \partial \overset{\text{les}}{=} 0$ this yields for the number from above

$$\left\langle w(M)^I,\, [M] \right\rangle = \left\langle i^* w(W)^I,\, \partial[W] \right\rangle = i_* \left\langle w(W)^I,\, i_*\partial[W] \right\rangle = 0 \ . \qquad \square$$

4.1.4 A Criterion for Indecomposability

Now, we focus on the ultimate goal of the current section, which is to deduce the following indecomposability criterion. This originates as a corollary from Thom's proof of the multiplicative structure of the cobordism ring (see [Tho07, Section IV.5]).

Theorem 4.15. *A closed n-manifold M represents an indecomposable element of the cobordism ring η_*, if and only if*

$$s_{(n)}[M] \neq 0 \in \mathbb{Z}_2 \ .$$

Proof of Theorem 4.15. Using the main theorems 4.13 and 4.14 from Subsection 4.1.3, as well as the main corollaries 4.9 and 4.11 from Subsection 4.1.2, enables to obtain the desired result in the following steps.

Step 1: As the classes $[V_{\mathscr{P}}]$ form a basis of the cohomology ring by Theorem 4.13, any manifold M^n is cobordant to a unique linear combination, i.e. disjoint sum,

$$[M] = \coprod_{\substack{\mathscr{P} \in \Pi(n) \\ \text{non-dyadic}}} \alpha_{\mathscr{P}} [V_{\mathscr{P}}] \ , \quad \alpha_{\mathscr{P}} \in \mathbb{Z}_2,$$

of the classes $[V_{\mathscr{P}}]$. Now the Stiefel-Whitney numbers are determined by the cobordism class according to Theorem 4.14, and are additive with respect to disjoint sums. So, one gets for any Stiefel-Whitney number $w^I[M]$ of M

$$w^I[M] = \sum_{\substack{\mathscr{P} \in \Pi(n) \\ \text{non-dyadic}}} \alpha_{\mathscr{P}} w^I[V_{\mathscr{P}}] \ , \quad \text{especially} \quad s_{\mathscr{P}'}[M] = \sum_{\substack{\mathscr{P} \in \Pi(n) \\ \text{non-dyadic}}} \alpha_{\mathscr{P}} s_{\mathscr{P}'}[V_{\mathscr{P}}] \overset{\text{Def.}}{=} \alpha_{\mathscr{P}'} \ .$$

Thus, $[V_{\mathscr{P}}]$ is a summand of $[M]$ if and only if $s_{\mathscr{P}}[M]$ is non-zero. In other words, M is cobordant to

$$\coprod_{\substack{\mathscr{P}\in\Pi(n) \text{ non-dyadic} \\ s_{\mathscr{P}}(W)\neq 0}} V_{\mathscr{P}} \ .$$

Step 2: For partitions $\mathscr{P}_1'$ of n_1, and $\mathscr{P}_2'$ of n_2 and any partition $\mathscr{P}'$ holds

$$\begin{aligned} s_{\mathscr{P}'}(V_{\mathscr{P}_1'} \times V_{\mathscr{P}_2'}) &\overset{4.9}{=} \sum_{\substack{\mathscr{P}_1\oplus\mathscr{P}_2=\mathscr{P}' \\ \mathscr{P}_1\in\Pi(n_1) \\ \mathscr{P}_2\in\Pi(n_2)}} s_{\mathscr{P}_1}(V_{\mathscr{P}_1'}) \cdot s_{\mathscr{P}_2}(V_{\mathscr{P}_2'}) \overset{\text{Def.}}{=} \sum_{\substack{\mathscr{P}_1\oplus\mathscr{P}_2=\mathscr{P}' \\ \mathscr{P}_1\in\Pi(n_1) \\ \mathscr{P}_2\in\Pi(n_2)}} \delta_{\mathscr{P}_1,\mathscr{P}_1'} \cdot \delta_{\mathscr{P}_2,\mathscr{P}_2'} = \delta_{\mathscr{P}',\mathscr{P}_1'\oplus\mathscr{P}_2'} \\ &\overset{\text{Def.}}{=} s_{\mathscr{P}'}(V_{\mathscr{P}_1'\oplus\mathscr{P}_2'}) \ . \end{aligned}$$

Thus, by Step 1 the basis representations of $[V_{\mathscr{P}_1'}] \times [V_{\mathscr{P}_2'}] = [V_{\mathscr{P}_1'} \times V_{\mathscr{P}_2'}]$ and $[V_{\mathscr{P}_1'\oplus\mathscr{P}_2'}]$ coincide, wherefore they must be equal.

Step 3: By Step 2, all basis elements $[V_{\mathscr{P}}]$ can be written as a product of lower degree basis elements, except for those corresponding to a trivial partition (k), $k \in \mathbb{N}$. Furthermore, such a basis element $[V_{(k)}]$ cannot be decomposable, as otherwise $s_{(k)}[V_{(k)}] = 0$ by Equation 4.3 in Corollary 4.11, which contradicts the definition in Theorem 4.13.

Altogether, the basis elements represented by a k-dimensional manifold $V_{(k)} \in \eta_k$ for $k \neq 2^m - 1$ are indecomposable—hence algebraically independent—generators of the cobordism ring. This is a proof of Theorem 4.12 using Theorem 4.13.

Step 4: By Step 3, the cobordism class of a manifold W of dimension k is an indecomposable element of η_* if and only if its unique representation by basis elements $[V_{\mathscr{P}}]$ contains as a summand the unique k-dimensional indecomposable basis element $[V_{(k)}]$, i.e. if and only if $s_{(k)}[W] \neq 0$ by Step 1. □

This directly yields the following example which will be a key point in constructing a candidate generating set of the cobordism ring.

Example 4.16. For $k \in \mathbb{N}$ even, the projective space $\mathbb{R}\mathrm{P}^k$ represents an indecomposable element of the cobordism ring.

Proof. If one applies Remark 4.6 to

$$w(\mathbb{R}\mathrm{P}^k) = (1+x)^{k+1} = \prod_{i=1}^{k+1}(1+x) = 1 + \sum_{j=1}^{k+1} \sigma_j^{k+1}(x, \ldots, x)$$

where x is the generator in degree one of $H^*(\mathbb{R}\mathrm{P}^k) \cong \mathbb{Z}_2[x]/(x^{k+1})$, one gets

$$s_{(k)}(\mathbb{R}\mathrm{P}^k) \overset{4.6}{=} \sum_{i=1}^{k+1} x^k = (k+1)x^k$$
$$s_{(k)}[\mathbb{R}\mathrm{P}^k] = \left\langle (k+1)x^k,\, [\mathbb{R}\mathrm{P}^k] \right\rangle = k+1 \equiv 1 \mod 2 \,. \qquad \square$$

4.2 Twisted Products

The candidates for a generating set needed for the proof of Brown's Theorem 4.1 will be inductively constructed using the so-called twisted product construction explained below. The main advantage of this tool is—besides quite a couple of handy preservation properties—the fact that a twisted product is indecomposable if and only if its factor is and the dimension was chosen correctly (Theorem 4.33). The latter will be the main result of this section, and is discussed in Subsection 4.2.3.

4.2.1 Definition

The following definition is according to [CT89, p. 83] respectively compare [Bro71, §4, Def. $P(m, X)$].

Definition 4.17. Let X be a space and $k \in \mathbb{N}$ an integer. Define the *twisted product of X by S^k*, denoted $D_k(X)$, to be the orbit space of the properly discontinuous $\mathbb{Z}_2$-action on $S^k \times (X \times X)$ given by

$$\mathbb{Z}_2 \curvearrowright S^k \times (X \times X) \,, \qquad [1] * (s, (p_1, p_2)) \coloneqq (-s, (p_2, p_1)) \,,$$

which combines the antipodal action $[1] * s \coloneqq -s$ on S^k and twisting on $X \times X$. For a map $f\colon X \to Y$ of spaces, define

$$D_k(f) \coloneqq (\mathrm{id} \times f \times f/\sim)\colon D_k(X) \longrightarrow D_k(Y) \,, \quad [s, (p_1, p_2)] \longmapsto [s, (f(p_1), f(p_2))] \,.$$

Example 4.18. Major examples needed later are

- $D_k(*) = \mathbb{R}\mathrm{P}^k$, and
- $D_0(M) = M \times M$.

First, gather some rather immediate, convenient properties. It is especially noteworthy how well the twisted product behaves concerning manifolds and fiber bundles.

Remark 4.19. Let X be a space and $k \in \mathbb{N}$.

i) $D_k(-)$ is a functor on the category of topological spaces preserving injectivity.

ii) $D_k(-)$ preserves fiber bundles, i.e. for a fiber bundle $\xi\colon \mathrm{E}\xi \to X$ with fiber F the twisted product $D_k(\xi)\colon D_k(\mathrm{E}\xi) \to D_k(X)$ is again a fiber bundle with fiber $F \times F$. This comes from the fiber bundle

$$F \times F \longrightarrow S^k \times (\mathrm{E}\xi \times \mathrm{E}\xi) \longrightarrow S^k \times (X \times X)$$

where all maps are maps of $\mathbb{Z}_2$-spaces. As a special case, $D_k(X)$ admits a fiber bundle

$$X \times X \longrightarrow D_k(X) \longrightarrow \mathbb{R}\mathrm{P}^k = S^k/\sim \tag{4.4}$$

with fiber $X \times X$ which comes from the trivial fiber bundle $X \to *$. Further, let $\eta\colon \mathrm{E}\eta \to X$ be another fiber bundle, and $f\colon X' \to X$ a map. One has:

a) $D_k(-)$ respects pullbacks, i.e.

$$D_k(f^*\xi) = (D_k(f))^* (D_k(\xi))$$

b) $D_k(-)$ respects Whitney sums of vector bundles, i.e.

$$D_k(\xi \oplus \eta\colon \mathrm{E}(\xi \oplus \eta) \to X) = D_k(\xi) \oplus D_k(\eta) \ .$$

To see this, observe that the following is a well-defined commutative pullback diagram of vector bundles:

$$\begin{tikzcd}
D_k(\mathrm{E}(\xi \oplus \eta)) \arrow[r] \arrow[d, "D_k(\xi\oplus\eta)"] & D_k(\mathrm{E}(\xi \times \eta)) \arrow[r] \arrow[d, "D_k(\xi\times\eta)"] & D_k(\mathrm{E}\xi) \times D_k(\mathrm{E}\eta) \arrow[d, "D_k(\xi)\times D_k(\eta)"] \\
D_k(X) \arrow[r, "D_k(\Delta)"] & D_k(X \times X) \arrow[r, "\widetilde{\Delta}"] & D_k(X) \times D_k(X)
\end{tikzcd}$$

where $\widetilde{\Delta}\colon\ [s,(x_1,y_1),(x_2,y_2)] \longmapsto ([s,x_1,x_2]\,,[s,y_1,y_2])$.

iii) $D_k(-)$ preserves closed smooth manifolds, i.e. for a closed smooth manifold M^n, $D_k(M)$ is a again a $(2m+k)$-dimensional closed smooth manifold. This is because the proper discontinuity comes from the antipodal $\mathbb{Z}_2$-action, and makes the projection

$$S^k \times (X \times X) \xrightarrow{\pi} D_k(X) := \Big(S^k \times (X \times X)\Big)/\sim$$

a two-leaved covering space. Further:

a) $D_k(-)$ preserves immersions.

b) $\mathrm{T}D_k(M) \cong \mathrm{proj}^*\mathrm{T}\mathbb{R}\mathrm{P}^k \oplus D_k(\mathrm{T}M)$, i.e. the tangent space of $D_k(M)$ can be obtained from $D_k(\mathrm{T}M)$ by adding the missing tangent space part of the sphere:

$$\begin{aligned}
\mathrm{T}D_k(M) &\cong \mathrm{T}\Big(S^k \times (M \times M)\Big)/\sim \\
&\cong \quad \mathrm{T}S^k \times \mathrm{T}M \times \mathrm{T}M/\sim \xrightarrow{\cong} \mathrm{proj}^*\mathrm{T}\mathbb{R}\mathrm{P}^k \oplus D_k(\mathrm{T}M) \\
&\quad [(s,v),(m_1,v_1),(m_2,v_2)] \longmapsto (([s],v),[s,(m_1,v_1),(m_2,v_2)])
\end{aligned}$$

where proj: $D_k(M) \to \mathbb{RP}^k$ is the projection. The first isomorphism is due to the covering space property, and the last is easily seen to be a well-defined isomorphism of vector bundles. Further note that for a map of manifolds $f\colon M \to N$, the differential map $\mathrm{D}D_k(f)$ on tangent spaces will be the identity on the first summand.

As a last inside into the definition have a look at the twisted product of real spaces.

Lemma 4.20. *Let $k, n \in \mathbb{N}$, and by γ_k the tautological line bundle over $\mathbb{RP}^k$. The fiber bundle $D_k(\mathbb{R}^n) \to \mathbb{RP}^k$ is isomorphic to the vector bundle*

$$(n \cdot \gamma_k) \oplus \varepsilon^n = (\gamma_k \oplus \cdots \oplus \gamma_k) \oplus \varepsilon^n$$

Proof. Compare also [Bro71, Prop. 4.3, p 1107]. A well-defined vector bundle isomorphism is for example

$$\begin{aligned} D_k(\mathbb{R}^n) &\xrightarrow{\sim} (S^k \times (\mathbb{R}^n \times \mathbb{R}^n)/\approx) \cong (S^k \times \mathbb{R}^n/\approx) \times \mathbb{R}^n \cong \mathrm{E}((n\cdot\gamma_k) \oplus \varepsilon^n) \\ [s, v_1, v_2] &\longmapsto [s, v_1 + v_2, v_1 - v_2] \end{aligned}$$

where $\approx$ is the equivalence relation identifying (s, v_1, v_2) and $(-s, -v_1, v_2)$ respectively (s, v) and $(-s, -v)$. Here it was used that γ_k is by construction

$$\mathrm{E}\gamma_k = (S^k \times \mathbb{R}/\approx) \longrightarrow \mathbb{RP}^k \qquad [s, v] \longmapsto [s]\ ,$$

and hence $\mathrm{E}(n\gamma_k) \cong (S^k \times \mathbb{R}^n/\approx)$. □

4.2.2 The Cohomology Ring of Twisted Products

Besides the above direct properties, there is a fairly easy description of the cohomology ring of a twisted product relating it to the cohomology ring of its factor. This can be revealed inductively using a diagram of long exact sequences which relates the cohomology of two degrees of twisted products and known sequences of spheres. Use the following notation.

Definition 4.21. Let X be a space and $k \in \mathbb{N}$. Note that by the Künneth isomorphism $H^*(X^2) \cong H^*(X) \otimes H^*(X)$, and recall that all exact sequences of $\mathbb{Z}_2$-vector spaces split. Define by

- $\pi_k\colon S^k \times X^2 \to D_k(X)$ to be the quotient map from the definition,
- proj: $D_k(X) \to \mathbb{RP}^k$ to be the fiber bundle map,
- $T\colon S^k \times X^2 \to S^k \times X^2$, $(s, p, q) \mapsto (-s, q, p)$,
- $N \coloneqq \ker(x \otimes y \mapsto x \otimes y + y \otimes x) = (a \otimes b + b \otimes a \mid a, b \in H^*(X \times X))_{\mathbb{Z}_2}$,
- $d\colon H^*(X) \to H^*(X \times X)$, $d(a) \coloneqq a \otimes a$, and $D \coloneqq \mathrm{im} d = \{a \otimes a \in H^*(X \times X)\}$,

- $s_k \in H^k(S^k) \cong \mathbb{Z}_2$ to be the generator, and
- $c_k \coloneqq \mathrm{proj}^* x \in H^1(D_k(X))$, where $x \in H^*(\mathbb{RP}^k) \cong \mathbb{Z}_2[x]/(x^{k+1})$ is the unique generator.

The indices k are omitted if they are obvious from the context.

Remark 4.22. Note that $N + D$ is closed under multiplication and addition, and—as a first hint on the cohomology structure—proj admits a section $[s] \mapsto [s, p, p]$ for any point $p \in X$, thus making $H^*(\mathbb{RP}^k)$ a direct summand of $H^*(D_k(X))$ of the form $\mathbb{Z}_2[c]/(c^{k+1})$. From the commutative diagram factorization

$$\begin{tikzcd} S^k \times X^2 \arrow[r, "\mathrm{proj}_{S^k}"] \arrow[d, "\pi"] & S^k \arrow[d, "\pi"] \\ D_k(X) \arrow[r, "D_k(X \to *)"] & \mathbb{RP}^k \end{tikzcd}$$

and with $\pi^* \colon \mathbb{RP}^k \to S^k$ being zero, one gets $\pi^*(c) = 0$.

Theorem 4.23. *Let X be a space and $k \in \mathbb{N}$. Then the cohomology ring of $D_k(X)$ has the form*

$$H^*(D_k(X)) \cong \Big(\mathbb{Z}_2[c, s]/(c^{k+1}, s^2, cs)\Big) \otimes (N + D)$$

with c of degree 1, s of degree k, and the additional properties

i) $c \otimes N = 0 = s \otimes D$,

ii) $\mathrm{proj}^* x = c \otimes d(1)$, *hence*

$$\mathrm{proj}^* \colon H^*(\mathbb{RP}^k) \cong \mathbb{Z}_2[c]/(c^{k+1}) \subset H^*(D_k(X)) \ , \quad \textit{and}$$

iii) $\pi^*(s \otimes n) = s \otimes n$, $\pi^*(1 \otimes (n + d(a))) = 1 \otimes (n + d(a))$ *for $n \in N$ and $a \in H^*(X)$, hence*

$$\pi^* \colon (1 \otimes (N + D)) + (s \otimes N) \cong (1 \otimes (N + D)) \oplus (s_k \otimes N) \ .$$

For readability skip $1 \otimes -$ and $- \otimes d(1)$ in element notation wherever it the meaning is clear from the context. Further note that D is multiplicatively, but not additively closed, whereas $(N + D) \subset H^*(X^2)$ is a subring via $d(a) + d(b) = d(a + b) + (a \otimes b + b \otimes a)$.

A proof of Theorem 4.23 can be found at the end of the section.

4.2.2.1 Stiefel-Whitney Classes of Twisted Products of Vector Bundles

The results of this section will yield a splitting principle applicable to Stiefel-Whitney classes of twisted products of vector bundles.

Some immediate consequences of the above structure theorem are:

Corollary 4.24. *Let X again be a space and $k \in \mathbb{N}$.*

i) For any section $s_p \colon \mathbb{R}\mathrm{P}^k \to D_k(X)$, $q \mapsto [q,p,p]$, of the fiber bundle described in (4.4), and $n_1, n_2 \in N$, $0 \neq a \in H^*(X)$ holds

$$s_p^*(c) = x \;, \qquad \text{and} \qquad s_p^*(c \otimes d(a) + 1 \otimes n_1 + s \otimes n_2) = 0 \;. \tag{4.5}$$

ii) $D_k(-)$ preserves injectivity on cohomology.

Proof. The section property is clear from the properties of proj^*. For the other statement, consider a map $f \colon X \to Y$ which is injective on cohomology and induces the map

$$F \colon S^k \times X^2 \longrightarrow S^k \times Y^2 \;, \qquad [s,(x_1,x_2)] \longmapsto [s,(f(x_1),f(x_2))] \;.$$

Since every element in $H^*(\mathbb{R}\mathrm{P}^k) \otimes D$ can uniquely be written as $\sum_{i=0}^k c^i \cdot d(a_i)$, one only has to check injectivity of $D_k(f)^*$ on the two parts

$$\Big(\mathbb{Z}_2[c]/(c^{k+1}) \otimes 1\Big) \qquad \text{and} \qquad ((1 \otimes (N+D)) + (s \otimes N)) \;.$$

First part: Since $D_k(f)$ is a morphism of vector bundles over $\mathbb{R}\mathrm{P}^k$,

$$D_k(f)^*(c) = D_k(f)^*(\mathrm{proj}^*(x)) = \mathrm{proj}^*(x) = c \in H^*(D_k(X)) \;,$$

so $D_k(f)^*$ is injective on $H^*(\mathbb{R}\mathrm{P}^k) \otimes 1 \subset H^*(D_k(Y))$.

Second part: Obviously $F^* \colon H^*(S^k \times Y^2) \to H^*(S^k \times X^2)$ will be injective. The isomorphism property of π^* hence implies injectivity of $D_k(f)^*$ on $(1 \otimes D) + (1 \otimes N) + (s_k \otimes N) \subset H^*(D_k(Y))$. □

Remark 4.25. Let ξ be a vector bundle over a space X. Recall that by the splitting principle [Die08, Theorem (19.3.9)], $w(\xi)$ is pulled back to the product $\prod_i w(\xi_i)$ of total Stiefel-Whitney classes of some line bundles ξ_i, along a map $f \colon Y \to X$ which is injective on cohomology. Since $D_k(-)$ preserves Whitney sums and injectivity on cohomology, $w(D_k(\xi))$ will injectively pull back along $D_k(f)$ to the product $\prod_i w(D_k(\xi_i))$.

Thus, to get from the Stiefel-Whitney classes of a vector bundle to the ones of its kth twisted product, a good approach is to investigate how $D_k(-)$ acts on the Stiefel-Whitney classes of line bundles.

Corollary 4.26. *Let $\xi\colon E \to X$ be a line bundle, let $w(\xi) = 1 + \alpha$ be its total Stiefel-Whitney class, and let $k \in \mathbb{N}$. Define*

$$e\colon H^*(X) \longrightarrow N \subset H^*(X \times X)\ , \qquad e(a) \coloneqq 1 \otimes a + a \otimes 1\ .$$

Then, along the isomorphism from Theorem 4.23,

$$w(D_k(\xi)) = 1 + (c_k \otimes d(1) + 1 \otimes e(\alpha)) + 1 \otimes d(\alpha) = 1 + c_k + e(\alpha) + d(\alpha)\ ,$$

respectively $w_1(\xi) = c_k + e(\alpha)$, $w_2(\xi) = d(\alpha)$.

Proof. See also [Bro71, Prop. 7.4, p. 1113]. For $k = 0$ this is simply the product rule for the total Stiefel-Whitney class because $D_0(\xi) = \xi \times \xi$ and $c = 0$. Thus, assume $k \geq 0$. With $\deg w_i(D_k(\xi)) \leq \operatorname{rk} D_k(\xi) = 2$, the total Stiefel-Whitney class of the two-dimensional vector bundle $D_k(\xi)$ must by Theorem 4.23 be of the general form

$$\begin{aligned} w(D_k(\xi)) &= 1 + \sum_{i=1}^{k} c^i \otimes d(a_i) + s \otimes n' + 1 \otimes (n + d(a)) \\ &= 1 + \underbrace{c \otimes d(a') + c^2 \otimes d(a'')}_{\text{check section}} + \underbrace{s \otimes n' + 1 \otimes (n + d(a))}_{\text{check } \pi}\ , \end{aligned}$$

for some $n, n' \in N$ and $a, a', a'' \in H^*(X)$, with $\deg a \leq 0 \geq \deg a''$, $\deg n' \leq 1 \geq \deg a$, and $\deg n \leq 2$. In order to determine the unknown elements, note that by Theorem 4.23

- π^* is an isomorphism on $s \otimes N + 1 \otimes (D + N)$, and
- for any point $p \in X$ and section $s_p\colon \mathbb{RP}^k \to D_k(X)$, s_p^* is an isomorphism on $\{c^i \otimes d(b) \mid b \in H^0(X), 1 \leq i \leq k\}$.

So the following remains to check:

$\pi^* w(D_k(\xi))$*:* $D_k(\xi)$ is the quotient of the bundle $\varepsilon \times \xi \times \xi\colon S^k \times (E \times E) \to S^k \times (X \times X)$, where

$$w(\mathrm{id} \times \xi \times \xi) = 1 \cdot w(\xi) \cdot w(\xi) = 1 + 1 \otimes e(\alpha) + 1 \otimes d(\alpha)\ .$$

As π is a covering map, $\pi^* D_k(\xi) = \varepsilon \times \xi \times \xi$, and thus $\pi^* w(D_k(\xi)) = w(\mathrm{id} \times \xi \times \xi)$. By 4.23.iii), π^* is the identity on elements of this form, so this yields $n' = 0$, $n = e(\alpha)$, and $a = \alpha$.

$s_p^* w(D_k(\xi))$*:* Consider a section $s_p\colon [s] \mapsto [s, p, p]$, $p \in X$, of the bundle $D_k(X) \to \mathbb{RP}^k$. Since $s_p = D_k((* \mapsto p \in X))$, the pullback of $D_k(\xi)$ along s_p yields

$$\begin{aligned} s_p^* D_k(\xi) &= D_k(* \mapsto p)^* \,(D_k(\xi)) \\ &= D_k((* \mapsto p)^* \xi) \\ &= D_k(\varepsilon^1) \overset{4.20}{=} \gamma_k \oplus \varepsilon^1\colon D_k(\mathbb{R}) \to \mathbb{RP}^k\ , \quad [s, v_1, v_2] \mapsto [s]\ . \end{aligned}$$

Then

$$s_p{}^* w(D_k(\xi)) = w(s_p{}^* D_k(\xi)) = w(\gamma \oplus \varepsilon) = 1 + x \in H^*(\mathbb{RP}^k)\ ,$$

.and thus $a' = 1$, $a'' = 0$. ◻

4.2.2.2 A Proof of the Structure Theorem

The rest of this section is dedicated to the proof of Theorem 4.23 on the cohomology structure of twisted products.

The essential step is to relate a twisted product to

1. its lower dimensional counterpart (analogous to the embedding of a projective space into a higher dimensional one), and
2. well-known sequences of spheres and disks.

This can be done using an alternative pushout construction, which is explained below. The proofs are omitted since all facts are easy to check.

Having constructed a diagram of pairs of spaces with a corresponding one of cohomology groups, the more tedious part then is to finalize the proof with an inductive diagram chase, which is given for completeness but may be skipped by the reader.

Facts 4.27. *Let X again be a space and $k \in \mathbb{N}_{\geq 1}$.*

i) The twisted product $D_k(X)$ is the pushout $(D^k \times X^2) \cup_T (S^{k-1} \times X^2)$ of

$$S^k \times X^2 \xleftarrow{T} S^k \times X^2 \xrightarrow{\text{incl}} D^k \times X^2$$

i.e. it is the product $D^k \times X^2$ of the closed k-disk with X^2 with the identification $(s, p, q) = T((s, p, q)) \coloneqq (-s, q, p)$ on all boundary points in $\partial D^k \times X^2 = S^{k-1} \times X^2$.

ii) The pushouts

$$D_k(X) = (D^k \times X^2) \cup_T (S^{k-1} \times X^2) \qquad \textit{and}$$
$$D_{k-1}(X) = (S^{k-1} \times X^2) \cup_T (S^{k-1} \times X^2)$$

merge to the commutative pushout diagram

$$\begin{tikzcd} S^{k-1} \times X^2 \arrow[r, equal] \arrow[d, "T"] & S^{k-1} \times X^2 \arrow[r, tail, "\text{incl}"] \arrow[d, "\pi"] & D^k \times X^2 \arrow[d] \\ S^{k-1} \times X^2 \arrow[r, "\pi"] & D_{k-1}(X) \arrow[r, tail, "\text{incl}"] & D_k(X) \end{tikzcd}$$

making $(D_k(X), D_{k-1}(X))$ a neighborhood deformation retract pair using the stability of cofibrations under pushout. Furthermore, for a smaller disk $D^k_+ \subsetneq D^k$ there is an induced excision

$$(D^k \times X^2, S^{k-1} \times X^2) \rightarrowtail (D_k(X), \overline{D_{k-1}(X)})$$

where $\overline{D_{k-1}(X)} \coloneqq ((D^k \setminus D^k_+) \times X^2) \cup_T (S^{k-1} \times X^2)$ is the embedded $D_{k-1}(X)$ with a collar, i.e. a neighborhood deformation retract of $D_{k-1}(X)$ in $D_k(X)$.

iii) The excision

$$(D^k_+, S^{k-1}_+) \amalg (D^k_-, S^{k-1}_-) \hookrightarrow (S^k, S^{k-1} \times I)$$

of an upper and a lower polar cap into the sphere relative to its bloated equator, is compatible with the excision from above in the sense that the following diagram commutes:

$$\begin{tikzcd}
(D^k_+, S^{k-1}_+) \amalg (D^k_-, S^{k-1}_-) \arrow[r, hook, "excis."] \arrow[d, "\pi"] & (S^k, S^{k-1} \times I) \arrow[r, dash, "\simeq"] \arrow[d, "\pi"] & (S^k, S^{k-1}) \arrow[d, "\pi"] \\
(D^k, S^{k-1}) \times X^2 \arrow[r, hook, "excis."] & (D_k(X), \overline{D_{k-1}(X)}) \arrow[r, dash, "\simeq"] & (D_k(X), D_{k-1}(X))
\end{tikzcd}$$

This now nicely fits into a larger diagram of pairs of spaces, which induces a diagram of long exact sequences of cohomology.

Facts 4.28. *For a space X and $k \in \mathbb{N}_{\geq 1}$ the following diagram commutes*

$$\begin{tikzcd}
(D^k_+, S^{k-1}_+) \times X^2 \arrow[r, hook, "\tau"] & \coprod_{+,-} (D^k, S^{k-1}) \times X^2 \arrow[r, "\pi"] \arrow[d, hook, "excision"] & (D^k_+, S^{k-1}_+) \times X^2 \arrow[d, hook, "excision"] & \\
 & (S^k, S^{k-1} \times I) \times X^2 \arrow[r, "\pi"] \arrow[d, dash, "\simeq"'] & \left(D_k(X), \overline{D_{k-1}(X)}\right) \arrow[d, dash, "\simeq"'] & \\
 & (S^k, S^{k-1}) \times X^2 \arrow[r, "\pi_k"] & (D_k(X), D_{k-1}(X)) \arrow[r, "\mathrm{proj}"] & (\mathbb{RP}^k, \mathbb{RP}^{k-1}) \\
D^k_+ \times X^2 \arrow[uuu, hook, "\tilde{j}"] \arrow[r, hook, "\iota"] & (S^k \times X^2) \arrow[u, hook, "\hat{j}"] \arrow[r, "\pi_k"] & D_k(X) \arrow[u, hook, "j"] \arrow[r, "\mathrm{proj}"] & \mathbb{RP}^k \arrow[u, hook] \\
S^{k-1} \times X^2 \arrow[u, hook, "\tilde{i}"] \arrow[r, equal] & S^{k-1} \times X^2 \arrow[u, hook, "\hat{i}"] \arrow[r, "\pi_{k-1}"] & D_{k-1}(X) \arrow[u, hook, "i"] \arrow[r, "\mathrm{proj}"] & \mathbb{RP}^{k-1} \arrow[u, hook]
\end{tikzcd}$$

with $\pi \circ \tau = \mathrm{id}$, resulting in the commutative diagram of exact cohomology sequences

$$
\begin{tikzcd}
\vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] \\
H^l((D^k, S^{k-1}) \times X^2) \arrow[d, "\tilde{j}^*"] & H^l((S^k, S^{k-1}) \times X^2) \arrow[l, "\tau^*"'] \arrow[d, "\hat{j}^*"] & H^l((D^k, S^{k-1}) \times X^2) \arrow[l, "{\pi_k}^*"'] \arrow[d, "j^*"] \\
H^l(D^k \times X^2) \arrow[d, "\tilde{i}^*"] & H^l(S^k \times X^2) \arrow[l, "\iota^*"'] \arrow[d, "\hat{i}^*"] & H^l(D_k(X)) \arrow[l, "{\pi_k}^*"'] \arrow[d, "i^*"] \\
H^l(S^{k-1} \times X^2) \arrow[r, equal] \arrow[d, "\tilde{\delta}"] & H^l(S^{k-1} \times X^2) \arrow[d, "\hat{\delta}"] & H^l(D_{k-1}(X)) \arrow[l, "{\pi_{k-1}}^*"'] \arrow[d, "\delta"] \\
H^{l-1}((D^k, S^{k-1}) \times X^2) \arrow[d] & H^{l-1}((S^k, S^{k-1}) \times X^2) \arrow[l, "\tau^*"'] \arrow[d] & H^{l-1}((D^k, S^{k-1}) \times X^2) \arrow[l, "{\pi_k}^*"'] \arrow[d] \\
\vdots & \vdots & \vdots
\end{tikzcd}
$$

where $\tau^*\pi^*$ is the identity.

The useful thing about the cohomology diagrams arising from the above diagram of pairs of spaces is that most of the columns and sideways maps are well-known. Some facts are listed below.

Facts 4.29. *Let X again be a space and $k \in \mathbb{N}$.*

i) The cohomology sequences of the pairs $(D^k, S^{k-1}) \times X^2$ and $(S^k, S^{k-1}) \times X^2$ are well-known from the sequences of the pairs (D^k, S^{k-1}) and (S^k, S^{k-1}). For $k > 1$ they are given for $l \in \mathbb{N}$ in the following commutative diagram:

$$
\begin{tikzcd}
 & (1\otimes f, s_{k-1}\otimes g) \arrow[r, mapsto] & (s_k\otimes g, s_k\otimes g) & \\
(1\otimes f, s_k\otimes g) \arrow[r, mapsto] & (1\otimes f, 0) & (s_k\otimes f, s_k\otimes g) \arrow[r, mapsto] & (0, s_k\otimes(f+g)) \\
H^{l-1}(S^k\times X^2) \arrow[r, "\hat{i}^*"] & H^{l-1}(S^{k-1}\times X^2) \arrow[r, "\hat{\delta}"] & H^l((S^k, S^{k-1})\times X^2) \arrow[r, "\hat{j}^*"] & H^l(S^k\times X) \\
\shortparallel & \shortparallel & \shortparallel & \\
1\otimes H^{l-1}(X^2) & 1\otimes H^{l-1}(X^2) & s_k\otimes H^{l-k}(X^2) & \\
\bigoplus & \bigoplus & \bigoplus & \\
s_k\otimes H^{l-k-1}(X^2) & s_{k-1}\otimes H^{l-k}(X^2) \arrow[ddddd, equal] & s_k\otimes H^{l-k}(X^2) & \\
(1\otimes f, s_k\otimes g) \arrow[d, mapsto, "\iota^*"] & & (s_k\otimes f, s_k\otimes g) \arrow[d, mapsto, "\tau^*"] & \\
f & & s_k\otimes f & \\
H^{l-1}(X^2) & & s_k\otimes H^{l-k}(X^2) & \\
\shortparallel & & \shortparallel & \\
H^{l-1}(D^k\times X^2) \arrow[r, "\tilde{i}^*"] & H^{l-1}(S^{k-1}\times X^2) \arrow[r, "\tilde{\delta}"] & H^l((D^k, S^{k-1})\times X^2) \arrow[r, "\tilde{j}^*", "=0"'] & H^l(D^k\times X) \\
f \arrow[r, mapsto] & (1\otimes f, 0) & & \\
 & (1\otimes f, s_{k-1}\otimes g) \arrow[r, mapsto] & s_k\otimes g &
\end{tikzcd}
$$

For $k = 1$ one has the modifications

$$
\begin{aligned}
(1\otimes f, s_0\otimes g) &\xmapsto{\hat{\delta}} (s_1\otimes(f+g), s_1\otimes(f+g)) \\
(1\otimes f, s_1\otimes g) &\xmapsto{\hat{i}^*} (1\otimes f, s_0\otimes f) \\
f &\xmapsto{\tilde{i}^*} (1\otimes f, s_0\otimes f) \\
(1\otimes f, s_0\otimes g) &\xmapsto{\tilde{\delta}} s_1\otimes(f+g)
\end{aligned}
$$

ii) Furthermore, it is easily seen that

$$
\begin{aligned}
\pi^*\colon H^l((D^k, S^{k-1})\times X^2) &\to H^l((S^k, S^{k-1})\times X^2) \\
s_k\otimes x\otimes y &\mapsto (s_k\otimes x\otimes y, s_k\otimes y\otimes x)
\end{aligned}
$$

iii) It is known respectively easily seen that

$$
\begin{tikzcd}
 & x^k \arrow[r, mapsto] \arrow[ddl, bend right, mapsto] & x^k, \quad x \arrow[r, mapsto] & x \\
 & H^*(\mathbb{R}\mathrm{P}^k, \mathbb{R}\mathrm{P}^k - 1) \arrow[r] \arrow[d, "\mathrm{proj}^*"] & H^*(\mathbb{R}\mathrm{P}^k) \arrow[r] & H^*(\mathbb{R}\mathrm{P}^{k-1}) \\
s_k\otimes 1\otimes 1 & H^*((D^k, S^{k-1})\times X^2) & &
\end{tikzcd}
$$

Some further immediate results are:

Corollary 4.30. *Let X be a space, $k \in \mathbb{N}$, $a, b \in H^*(X)$, $f \in H^*(X^2)$, and let $u \in H^*(D_k(X))$.*

i) $\pi^* j^* = \hat{j}^* \pi^* \colon s_k \otimes (a \otimes b) \mapsto s_k \otimes (a \otimes b + b \otimes a)$

ii) $\delta = (\tau^* \pi^*)^{-1} \tilde{\delta} \pi^* = \tilde{\delta} \pi^*$

iii) $j^*(s_k \otimes 1 \otimes 1) = c^k := \mathrm{proj}^*(x^k)$

iv) $i^*(c_k) = i^*(c_{k-1})$

v) $j^*(s_k \otimes f) \cdot u = j^*\left(s_k \otimes (f \cdot \iota^* \pi^*(u))\right)$

Proof. For v) first observe that the ring

$$H^*(D_k(X), D_{k-1}(X)) = H^*((D^k, S^{k-1}) \times X^2) = s_k \otimes H^*(X^2)$$

is both a module over $H^*(D_k(X))$ and over $H^*(D^k \times X^2) = H^*(X^2)$. It is known that the $H^*(X^2)$-module structure looks like

$$(s_k \otimes a_1 \otimes b_1) \cdot (a_2 \otimes b_2) = s_k \otimes \big((a_1 \otimes b_1) \cdot (a_2 \otimes b_2)\big) = s_k \otimes (a_1 a_2) \otimes (b_1 b_2) \ .$$

Now the base change between the two module structures is the one along $\iota^* \pi^*$, and this means that for any $u \in H^*(D_k(X))$ and $s_k \otimes f \in s_k \otimes H^*(X^2)$

$$(s_k \otimes f) \cdot u = (s_k \otimes f) \cdot \iota^* \pi^*(u) = s_k \otimes (f \cdot \iota^* \pi^*(u)) \ .$$

As a last step note that

$$j^* \colon H^*(D_k(X), D_{k-1}(X)) \to H^*(D_k(X))$$

is a morphism of $H^*(D_k(X))$-modules, i.e.

$$j^*(f \cdot g) = j^*(f) \cdot g \qquad \text{for } f \in H^*(D_k(X), D_{k-1}(X)) \text{ and } g \in H^*(D_k(X)).$$ □

We are going to inductively prove the following reformulation of Theorem 4.23.

Theorem 4.31. *For a space X and $k \in \mathbb{N}$, $H^*(D_k(X))$ is the group*

$$\begin{aligned} H^*(D_k(X)) &\cong \bar{c}^{(k)} \otimes D \oplus \left(\bigoplus_{i=1}^{k-1} \bar{c}^{(i)} \otimes D \right) \oplus (1 \otimes (N + D)) \oplus (s \otimes N) \\ &\cong \bar{c}^{(k)} \otimes D \oplus \left(\bigoplus_{i=1}^{k-1} \bar{c}^{(i)} \otimes D \right) \oplus (1 \otimes D) \oplus (1 \otimes N) \oplus (s \otimes N) \end{aligned}$$

with the additive relation $\bar{c}^{(i)} \otimes d(a) + \bar{c}^{(i)} \otimes d(b) = \bar{c}^{(i)} \otimes d(a+b)$ for $a, b \in H^*(X)$ and $1 \leq i \leq k$, which is equipped with a multiplication determined by:

i) The following restriction of π^* *is a ring isomorphism onto its image*

$$\begin{aligned}\pi^*\colon (1\otimes D)\oplus(1\otimes N)\oplus(s_k\otimes N) &\longrightarrow (1\otimes(D+N))\oplus(s_k\otimes N)\subset H^*(S^k\times X^2)\\ 1\otimes d+1\otimes n_1+s\otimes n_2 &\longmapsto 1\otimes(d+n_1)+s_k\otimes n_2\ .\end{aligned}$$

ii) $(\bar{c}^{(i)}\otimes d(a))\cdot(\bar{c}^{(j)}\otimes d(b))=\bar{c}^{(i+j)}\otimes d(ab)$ *for* $1\leq i,j$ *and* $i+j\leq k-1$*, so* $\bar{c}^{(i)}\otimes d(1)=(\bar{c}^{(1)}\otimes d(1))^i$*.*

iii) $(\bar{c}^{(i)}\otimes d(a))\cdot(1\otimes d(b))=\bar{c}^{(i)}\otimes d(ab)$ *for* $1\leq i\leq k-1$*, so* $\bar{c}^{(i)}\otimes D=(\bar{c}^{(1)}\otimes 1)^i\cdot(1\otimes D)$*.*

iv) $(\bar{c}^{(k)}\otimes d(a))\cdot(1\otimes d(b))=\bar{c}^{(k)}\otimes d(ab)$*, so* $\bar{c}^{(k)}\otimes D=(\bar{c}^{(1)}\otimes 1)^i\cdot(1\otimes D)$*.*

v) $\mathrm{proj}^*(x^k)=\bar{c}^{(k)}\otimes d(1)$ *and* $\mathrm{proj}^*(x)=\bar{c}^{(1)}\otimes d(1)$*, so* $(\bar{c}^{(1)}\otimes d(1))^k=\bar{c}^{(k)}\otimes d(1)$ *and*

$$\ker\pi^*=(\bar{c}^{(k)}\otimes D)\oplus\left(\bigoplus_{i=1}^{k-1}\bar{c}^{(i)}\otimes D\right)=\sum_{i=1}^{k}c^i\cdot(1\otimes D)\ .$$

vi) $c\cdot(1\otimes N+s\otimes N)=0$*, so* $(\bar{c}^{(i)}\otimes D)\cdot(1\otimes N+s\otimes N)=0$ *for* $1\leq i\leq k$*.*

Remark 4.32*.* Note that the demanded multiplication properties of the $\mathbb{Z}_2$-vector space in the theorem do already fully qualify a multiplication, as all cases of combinations of components are covered ($1\leq i,j\leq k$):

$(1\otimes(D+N)\oplus s\otimes N)\cdot(1\otimes(D+N)\oplus s\otimes N)$: i)

$(\bar{c}^{(i)}\otimes D)\cdot(\bar{c}^{(j)}\otimes D)$: By ii) and iii) $(\bar{c}^{(i)}\otimes D)=(\bar{c}^{(1)}\otimes 1)\cdot(1\otimes D)$, which can be simplified with $(\bar{c}^{(1)}\otimes 1)=c_k:=\mathrm{proj}^*x_k$ from v) to $(\bar{c}^{(i)}\otimes D)=c^i\cdot(1\otimes D)$. Analogously with iv) and v) $(\bar{c}^{(k)}\otimes D)=c^k\cdot(1\otimes D)$.

$(\bar{c}^{(i)}\otimes D)\cdot(1\otimes D)$: iii) resp. iv) for the case $i=k$

$(\bar{c}^{(i)}\otimes D)\cdot(1\otimes N)=0$: vi)

$(\bar{c}^{(i)}\otimes D)\cdot(s\otimes N)=0$: vi)

Proof of Theorem 4.31 respectively Theorem 4.23. This is roughly geared to the proof in [Bro71, Theorem 7.1]. With the above reformulation, the proof is a straight forward induction on k. I.e. split up the $\mathbb{Z}_2$-vector space $H^*(D_k(X))$ into direct summands of the form in the theorem, which are then either known from the case $k-1$, or calculable, and then check the needed multiplication and isomorphism properties. The split looks as follows:

$$\begin{aligned}\mathrm{im}j^*&\cong\ker(\pi^*|_{\mathrm{im}j^*})\oplus(\mathrm{im}j^*/\ker(\pi^*|_{\mathrm{im}j^*}))\\&\cong j^*(\ker(\pi^*j^*))\oplus\mathrm{im}(\pi^*|_{\mathrm{im}j^*})\\&\cong j^*(\ker(\pi^*j^*))\oplus\mathrm{im}(\pi^*j^*)\\ \mathrm{im}i^*&=\ker\delta=\ker(\tilde{\delta}\pi^*_{k-1})\\&=\ker(\tilde{\delta}|_{\mathrm{im}\pi^*_{k-1}})\oplus\ker\pi^*_{k-1}\end{aligned}$$

$$
\begin{aligned}
H^*(D_k(X)) &\cong \mathrm{im} j^* \oplus (H^*(D_k(X))/\mathrm{im} j^*) \\
&= \mathrm{im} j^* \oplus (H^*(D_k(X))/\ker i^*) \\
&\cong \mathrm{im} j^* \oplus \mathrm{im} i^* \\
&\cong j^*(\ker(\pi^* j^*)) \oplus \mathrm{im}(\pi^* j^*) \oplus \ker(\tilde{\delta}|_{\mathrm{im}\pi^*_{k-1}}) \oplus \ker \pi^*_{k-1}
\end{aligned}
$$

In the end this is supposed to look like

$$
\begin{aligned}
j^*(\ker(\pi^* j^*)) &= c_k^k \cdot (\pi^*)^{-1}(1 \otimes D) \cong c^k \otimes D \\
\ker \pi^*_{k-1} &= \textstyle\bigoplus_{i=1}^{k-1} c^i_{k-1} \otimes D \\
\ker(\tilde{\delta}|_{\mathrm{im}\pi^*_{k-1}}) &= 1 \otimes (N + D) \\
\mathrm{im}(\pi^* j^*) &= s_k \otimes N
\end{aligned}
$$

so the correspondents to the symbols from the theorem's notation are $\bar{c}^{(i)} = c^i_{k-1}$ for $i \le k-1$, $\bar{c}^{(k)} = c^k_k$, $s = s_k$.

Induction step

We will start with the induction step, so assume for a space X and $k > 1$ that Theorem 4.31 holds in the case $k-1$. Begin with identifying the direct summands of the vector space, carefully tracking where the induction assumption

$\mathrm{im}(\pi^* j^*) = s_k \otimes N$: Recall from Corollary 4.30.i) that

$$
\begin{aligned}
\pi^* j^* \colon s_k \otimes H^*(X^2) &\to 1 \otimes H^*(X^2) \oplus s_k \otimes H^*(X^2) \\
s_k \otimes (a \otimes b) &\mapsto s_k \otimes (a \otimes b + b \otimes a)
\end{aligned}
$$

and hence $\mathrm{im}(\pi^* j^*) = s_k \otimes N$. Obviously π^* maps this part of $H^*(D_k(X))$ isomorphically to $s_k \otimes N \subset H^*(S^k \times X^2)$, already inducing multiplication.

$\ker(\tilde{\delta}|_{\mathrm{im}\pi^*_{k-1}}) \cong 1 \otimes (D + N)$: By the assumption on $k-1$

$$
\mathrm{im}(\pi^*_{k-1}) = (1 \otimes (D + N)) \oplus (s_k \otimes N) \in H^*(S^{k-1} \times X^2)\,.
$$

With $\ker(\tilde{\delta}) = 1 \otimes H^*(X^2)$, one gets $\ker(\tilde{\delta}|_{\mathrm{im}\pi^*_{k-1}}) = 1 \otimes (D + N)$. Since π^*_{k-1} is the identity on $1 \otimes (D + N) \in H^*(D_{k-1}(X))$ and $\tilde{i}^*$ is the identity on $1 \otimes (D + N)$ in $H^*(S^k \times X^2)$, π^*_k and i^* are both the identity on $\ker(\tilde{\delta}|_{\mathrm{im}\pi^*_{k-1}})$, inducing ring structure on this subring.

$\ker(\pi^*_{k-1}) \cong \bigoplus_{i=1}^{k-1} c^i_{k-1} \otimes D$: This holds by the induction assumption. Note that by construction i^* is injective on this direct summand which is a direct summand of $\mathrm{im} i^*$, hence the ring structure is inherited. In order to see that this summand lies in the kernel of π^*_k, one has to first see that $c_k = c_{k-1} \otimes d(1) \in H^*(D_k(X))$, respectively in the direct sum notation

$$
c_k = f + 1 \otimes d(a) + 1 \otimes n + s \otimes n' + \sum_{i=1}^{k-1} c^i_{k-1} \otimes d(a_i)
$$

all summands are zero except for $c_{k-1} \otimes d(1)$. So check all possible components:

- $f \in j^*(\ker \pi^* j^*)$ has degree greater k as will be seen below, so with $k > 1$ from the induction assumption has to be zero.
- $i^*(c_k) = c_{k-1}$, so $1 \otimes d(a) + 1 \otimes n = 0$.
- $\pi^*(c_k) = 0$, so $s \otimes n' = 0$.

Thus $\ker(\pi^*_{k-1}) \cong \sum_{k=1}^{k-1} c_k^i \cdot (1 \otimes D) \subset \ker \pi^*$.

$j^*(\ker(\pi^* j^*)) \cong c_k^k \otimes D$: Again recall Corollary 4.30.i) to directly obtain

$$\ker(\pi^* j^*) = s_k \otimes (N + D) \ .$$

So one has in general

$$\begin{aligned} j^*(\ker(\pi^* j^*)) &= j^*(s_k \otimes (N + D)) \\ &\overset{\text{Corollary 4.30.v)}}{=} j^*(s_k \otimes 1 \otimes 1) \cdot (\iota^* \pi^*)^{-1}(N + D) \\ &= c^k \cdot \big((\pi^*)^{-1}(1 \otimes (D + N))\big) \subset \ker \pi^* \ . \end{aligned}$$

Note that this lives, as was used above, in degree greater $k - 1$. Since by Corollary 4.30.i) $c^k \cdot \ker \pi^* = j^*(s_k \otimes \iota^* \pi^*(\ker \pi^*)) = j^*(s_k \otimes 0) = 0$ and the effect of π^* on all direct summands is known:

$$\begin{aligned} j^*(\ker(\pi^* j^*)) &= c^k \cdot \big((\pi^*)^{-1}(1 \otimes (D + N))\big) \\ &= c^k \cdot (\ker \pi^* + (1 \otimes (D + N))) \\ &= c^k \cdot (1 \otimes (D + N)) \end{aligned}$$

The next step is to use that by induction assumption and the form of $\tilde{\delta}$ holds

$$\ker j^* = \mathrm{im}\delta = \mathrm{im}\tilde{\delta}\pi^* = s_k \otimes N$$

hence $c^k \cdot (\pi^*)^{-1}(1 \otimes N) = j^*(s_k \otimes N) = 0$, $c^k \cdot -$ is injective on the set $(1 \otimes D)$, and

$$\begin{aligned} c^k \cdot (1 \otimes d(a)) + c^k \cdot (1 \otimes d(b)) &= c^k \cdot (1 \otimes (d(a) + d(b))) \\ &= c^k \cdot (1 \otimes (d(a + b) + a \otimes b + b \otimes a)) \\ &= c^k \cdot (1 \otimes d(a + b)) + c^k \cdot (a \otimes b + b \otimes a) \\ &= c^k \cdot (1 \otimes d(a + b)) \ . \end{aligned}$$

Altogether, this yields the desired isomorphism

$$j^*(\ker(\pi^* j^*)) = c_k^k \cdot (\pi^*)^{-1}(1 \otimes D) \cong c^k \otimes D \ .$$

Now it remains to check the multiplication properties:

i): Checked during construction.

ii), iii): These are the multiplication properties induced by i^* being injective on the corresponding summands, thus inheriting the multiplication properties from the induction assumption.

iv): It was shown that $c^k \otimes D \cong c^k \cdot (1 \otimes D)$.

v): This was shown during the identification of $\ker(\pi^*_{k-1})$ as part of $\ker \pi^*$.

vi): $c_k \cdot (1 \otimes N)$ is inherited via i^* from the assumed structure of $H^*(D_{k-1}(X))$ using $c_k = c_{k-1} \otimes d(1)$. The fact $c_k \cdot (s_k \otimes N) = 0$ follows from

$$\ker \pi^* j^* \cdot \mathrm{im} j^* \subset j^*(s_k \otimes (H^*(X) \cdot \iota^* \pi^*(\ker \pi^*))) = j^*(s_k \otimes 0) = 0$$

using $c_k \in \ker \pi^*$, and $(s_k \otimes N) \subset \mathrm{im} j^*$.

Induction Start

The case $k = 0$ is a bit simpler as $D_0(X) = X^2$. One still has $\mathrm{im}\pi^* j^* = s_1 \otimes N$, and, using the known formula

$$\pi^*(a \otimes b) = (1 \otimes a \otimes b, s_0 \otimes b \otimes a) \in H^*(S^0 \times X^2) = H^*(X^2)^2 \ ,$$

one gets $\delta(a \otimes b) = \tilde{\delta}\pi^*(a \otimes b) = s_1 \otimes (a \otimes b + b \otimes a)$, giving $\ker \delta = N + D$, and $\mathrm{im}\delta = \ker j^* = s_1 \otimes N$. Hence, analogously to above, one obtains $j^*(\ker(\pi^* j^*)) \cong c \otimes D$ and $c \cdot (1 \otimes N + s_1 \otimes N) = 0$. ☐

4.2.3 Indecomposability of Twisted Product Cobordism Classes

Now that the cohomology ring and Stiefel-Whitney classes of line bundles of a twisted product are well-known, one can investigate general Stiefel-Whitney numbers of twisted product manifolds. As promised, the following result will be the cornerstone when inductively defining the desired basis for the cobordism ring in the subsequent section.

Theorem 4.33. *Let M^n be a manifold and $k \in \mathbb{N}_{\geq 1}$. Then $D_k(M)$ represents an indecomposable class of the cobordism ring if and only if M does and $\binom{k+n-1}{n}$ is non-zero modulo two.*

Recall Theorem 4.15, saying a manifold M^n represents an indecomposable element if and only if $s_{(n)}[M]$ is non-zero, and recall that $D_k(-)$ preserves injectivity on cohomology by Theorem 4.23.ii). Then Theorem 4.33 is a direct consequence of the following Lemma.

Lemma 4.34. *Let M, n, k be as in Theorem 4.33 above. Then there is a map of spaces $f\colon X \to M$ which is injective on cohomology and fulfills*

$$D_k(f)^* s_{(2n+k)}(D_k(M)) = \binom{k+n-1}{n} \cdot c^k \cdot d\left(f^* s_{(n)}(M)\right) \in H^{2n+k}(D_k(X)) \ .$$

Proof of Lemma 4.34. Take f to be a reduction of $\mathrm{T}M$ to line bundles using the splitting principle [Die08, Theorem (19.3.9)]. I.e. choose a space X and a map $f\colon X \to M$ which is injective on cohomology and fulfills $f^*\mathrm{T}M = \xi_1 \oplus \cdots \oplus \xi_n$ for line bundles ξ_i over X, each with total Stiefel-Whitney class $w(\xi_i) = 1 + \alpha_i$. With the fiber bundle properties from Remark 4.19.ii)a) and the tangent space structure from Remark 4.19.iii)b) this yields on vector bundles:

$$\begin{aligned}
D_k(f)^* D_k(\mathrm{T}M) &\overset{4.19}{=} D_k(f^*\mathrm{T}M) = D_k(\bigoplus_{i\le n}\xi_i) = \bigoplus_{i\le n} D_k(\xi_i) \\
D_k(f)^* \mathrm{T}D_k(M) &\overset{4.19}{=} D_k(f)^*\left(\mathrm{T}\mathbb{R}\mathrm{P}^k \oplus D_k(\mathrm{T}M)\right) \\
&= D_k(f)^*\mathrm{T}\mathbb{R}\mathrm{P}^k \oplus D_k(f)^* D_k(\mathrm{T}M) = \mathrm{T}\mathbb{R}\mathrm{P}^k \oplus \bigoplus_{i\le n} D_k(\xi_i)
\end{aligned}$$

And on Stiefel-Whitney classes:

$$\begin{aligned}
D_k(f)^* w(D_k(\mathrm{T}M)) &= \prod_{i\le n} w(D_k(\xi_i)) \overset{4.26}{=} \prod_{i\le n}(1 + c + e(\alpha_i) + d(\alpha_i)) \\
D_k(f)^* w(\mathrm{T}D_k(M)) &= D_k(f)^* w(\mathrm{T}\mathbb{R}\mathrm{P}^k) \cdot \prod_{i\le n} w(D_k(\xi_i)) \\
&= (c+1)^{k+1} \cdot \prod_{i\le n}(1 + c + e(\alpha_i) + d(\alpha_i))
\end{aligned}$$

In order to work with these Stiefel-Whitney class expressions as symmetric polynomials, introduce variables u_i, v_i of degree one such that

$$\begin{aligned}
w_1(\xi_i) &= c + e(\alpha_i) = \sigma_1^2(u_i, v_i) = u_i + v_i \\
w_2(\xi_i) &= d(\alpha_i) = \sigma_2^2(u_i, v_i) = u_i v_i \\
w(\xi_i) &= 1 + c + e(\alpha_i) + d(\alpha_i) = 1 + \sum_{i\le 2}\sigma_i^2(u_i, v_i) = (1+u_i)(1+v_i)\ .
\end{aligned}$$

The key point now qualifying f for the proof, is that both f and thus by Theorem 4.23.ii) also $D_k(f)$ are injective on cohomology, and fulfill

i) with $w(f^*\mathrm{T}M) = \prod_{i=1}^n (1+\alpha_i)$ that

$$f^* s_{(n)}(M) = s_{(n)}(w(f^*\mathrm{T}M)) = \sum_{i=1}^n \alpha_i^n \tag{4.6}$$

ii) and with $w(f^*\mathrm{T}D_k(M)) = \prod_{i=1}^{k+1}(1+c) \cdot \prod_{i=1}^n (1+u_i)(1+v_i)$ that

$$\begin{aligned}
D_k(f)^* s_{(2n+k)}(D_k(M)) &= s_{(2n+k)}(w(f^*\mathrm{T}D_k(M))) \\
&= (k+1)c^{2n+k} + \sum_{i=1}^n (u_i^{2n+k} + v_i^{2n+k})\ .
\end{aligned} \tag{4.7}$$

In order to formulate $D_k(f)^* s_{(2n+k)}(D_k(M))$ in terms of c and $d(\alpha_i)$, use one of the Newton-Girard formulas saying:

Lemma 4.35 (Newton-Girard). *For integers $l, m \in \mathbb{N}$ holds*

$$\mathrm{Sym}_l t^m = \sum_{r_1+2r_2+\cdots+mr=m} (-1)^m \frac{m \cdot (r_1 + \cdots + r_m - 1)!}{(r_1)! \cdots (r_m)!} \cdot \prod_{i=1}^{m} (-\sigma_i^l)^{r_i} \, .$$

As a special case for $l = 2$ this becomes modulo 2

$$t_1^m + t_2^m = \sum_{r_1+2r_2=m} \frac{m \cdot (r_1 + r_2 - 1)!}{(r_1)!(r_2)!} \cdot (\sigma_1^2)^{r_1} \cdot (\sigma_2^2)^{r_2} = \sum_{r_1+2r_2=m} \{r_1 - 1, r_2\}(t_1 + t_2)^{r_1}(t_1 t_2)^{r_2}$$

where $\frac{(r_1+2r_2)(r_1+r_2-1)!}{(r_1)!(r_2)!} = \binom{r_1+r_2-1}{r_2} + 2\binom{r_1+r_2-1}{r_1}$ *and the notation* $\{p, q\} := \binom{p+q}{q}$ *was used.*

Proof. A proof of the main Newton-Girard formula can be found in [Sér00, Theorem 10.12.2]. This special case can immediately be obtained by iterated application of the formula. $\square$

Before applying this to $t_1 = u_i$, $t_2 = v_i$, $m = 2n + k$, and $l = 2$, recall the following special properties from Theorem 4.23 of the mentioned symbols in $H^*(D_k(X))$:

- $c^{k+1} = 0$,
- $c \cdot e(\alpha_i) = 0$, and
- $e(\alpha_i)^{r_1} d(\alpha_i)^{r_2} = 0$ for $r_1 + 2r_2 > 2n$, as $H^{*>2n}(M \times M) = 0$.

Then simplify in terms of c, $e(\alpha_i)$, $d(\alpha_i)$:

$$\begin{aligned}
u_i^{2n+k} + v_i^{2n+k} &= \sum_{r_1+2r_2=2n+k} \{r_1 - 1, r_2\}(u_i + v_i)^{r_1}(u_i v_i)^{r_2} \\
&\overset{\text{Def.}}{=} \sum_{r_1+2r_2=2n+k} \{r_1 - 1, r_2\}(c + e(\alpha_i))^{r_1} d(\alpha_i)^{r_2} \\
&\overset{c \cdot e(\alpha_i) = 0}{=} \sum_{r_1+2r_2=2n+k} \{r_1 - 1, r_2\} \left(c^{r_1} d(\alpha_i)^{r_2} + e(\alpha_i)^{r_1} d(\alpha_i)^{r_2}\right) \\
&\overset{c^{k+1} = 0,\ d(\alpha_i)^{n+s} = 0}{=} \{k - 1, n\} c^k d(\alpha_i)^n + \sum_{r_1+2r_2=2n+k} \{r_1 - 1, r_2\} e(\alpha_i)^{r_1} d(\alpha_i)^{r_2} \\
&= \{k - 1, n\} c^k d(\alpha_i)^n
\end{aligned} \tag{4.8}$$

Altogether it turns out that

$$\begin{aligned}
D_k(f)^* s_{(2n+k)}(D_k(M)) &\overset{(4.7)}{=} (k+1)c^{2n+k} + \sum_{i=1}^{n}(u_i^{2n+k} + v_i^{2n+k}) \\
&\overset{c^{k+1}=0}{=} \sum_{i=1}^{n}(u_i^{2n+k} + v_i^{2n+k}) \\
&\overset{(4.8)}{=} \sum_{i=1}^{n}\{k-1,n\}c^k d(\alpha_i)^n \\
&= \{k-1,n\}c^k d(\sum_{i=1}^{n}\alpha_i^n) \\
&\overset{(4.6)}{=} \{k-1,n\}c^k d(f^* s_{(n)}(M))
\end{aligned}$$

as was stated. □

Remark 4.36. By Theorem 4.33, one has to both choose

- the dimension combination, as well as
- the factor (see later)

of a twisted product correctly, in order to obtain the representative of an indecomposable cobordism class. For the correct dimension choice note that any integer m with binary expansion

$$m = 2^{i_0} + \cdots + 2^{i_l}\ , \qquad 0 \leq i_1 < \cdots < i_r$$

can be split at any splitting point $0 \leq l_k \leq l$ to

$$m = \underbrace{\left(\sum_{r=0}^{l_k} 2^{i_r}\right)}_{=:k} + 2 \cdot \underbrace{\left(\sum_{r=l_k+1}^{l} 2^{i_r - 1}\right)}_{=:n} =: k + 2n\ ,$$

where:

- $\alpha(m) = \alpha(k) + \alpha(n)$.
- For m not of the form $2^s - 1$, n will also not be of the form $2^s - 1$.

Such splits as described above are desirable because the binomial coefficient $\binom{k+n-1}{n}$ will never be zero modulo two by the following Lemma, as is required by the indecomposability criterion for twisted products in Theorem 4.33.

Lemma 4.37. *For $a, b \in \mathbb{N}$, the binomial coefficient $\binom{a+b}{b}$ will be non-zero modulo 2 if and only if $\alpha(a+b) = \alpha(a) + \alpha(b)$.*

Proof. This is a direct consequence of Lucas' well-known theorem which states

$$\binom{a+b}{b} \equiv \prod_{i=0}^{s} \binom{a_i+b_i}{b_i} \mod 2$$

where $a = \sum_{i=1}^{s} a_i 2^i$, $b = \sum_{i=1}^{s} b_i 2^i$ are the binary expansions of a and b, and $\binom{0}{1} := 0$. This expression will be non-zero, if and only if b_i is one for any i where $a_i + b_i$ is one. In other words, if and only if $\alpha(a+b) = \sum_{i=1}^{s}(a_i + b_i)$. □

Therefore:

Corollary 4.38. *For $m = k + 2n \in \mathbb{N}$ as above and M^n an indecomposable manifold, the m-dimensional twisted product $D_k(M)$ will be indecomposable.*

Proof. The previous Lemma 4.37 can be applied to the combination $k-1$, n from above, as one simply calculates

$$\begin{aligned}
\alpha(k-1+n) &= \alpha\left(\left(\sum_{r=0}^{l_k} 2^{i_r}\right) - 1 + \left(\sum_{r=l_k+1}^{l} 2^{i_r-1}\right)\right) \\
&= \alpha\left((2^{i_0}-1) + \left(\sum_{r=1}^{l_k} 2^{i_r}\right) + \left(\sum_{r=l_k+1}^{l} 2^{i_r-1}\right)\right) \\
&= \alpha\left(\left(\sum_{r=0}^{i_0-1} 2^{r}\right) + \left(\sum_{r=1}^{l_k} 2^{i_r}\right) + \left(\sum_{r=l_k+1}^{l} 2^{i_r-1}\right)\right) \\
&\overset{i_0 < i_1}{=} \alpha\left(\left(\sum_{r=0}^{i_0-1} 2^{r}\right) + \left(\sum_{r=1}^{l_k} 2^{i_r}\right)\right) + \alpha\left(\sum_{r=l_k+1}^{l} 2^{i_r-1}\right) \\
&= \alpha(k-1) + \alpha(n)
\end{aligned}$$

□

4.3 Brown's Theorem: Finding a Convenient Generating Set

Recall that the goal of this chapter is to prove R. L. Brown's theorem 4.1 which states that the immersion conjecture is true up to cobordism. Also recall, that if one can show that the conjecture is stable under the ring operations of the cobordism ring $\eta_* \cong \mathbb{Z}_2[\sigma_i \mid i \neq 2^r - 1]$, it suffices to find a generating set $([G^i] \mid i \neq 2^r - 1)$ of η_* which fulfills the conjecture. Thus, for the proof one needs to construct a set of manifolds $(G^i \mid i \neq 2^r - 1)$ that both fulfill the conjecture and represent indecomposable cobordism classes.

The major work required to check indecomposability of manifolds in the cobordism ring has been done in the previous chapters. For checking the immersion property of the selected candidates, several well-known results on immersions of projective spaces will be used.

Such generators in even dimension can easily be constructed as codimension-one submanifolds of odd-dimensional products of real projective spaces. The immersion property can in this case be checked using results of Sanderson [San64] which say that odd dimensional real projective spaces fulfill—and partly overfulfill—the immersion conjecture.

For odd dimensional generators the trick will be to take a twisted product of an even dimensional one, and use an immersion theorem by Mahowald and Milgram [Mil67] on the twisted product of an Euclidean space to conclude the immersion property.

4.3.1 Stability Properties of the Conjecture

The stability properties needed are the following.

Lemma 4.39. *Let $M_i^{n_i}$ be a closed manifold immersing into $\mathbb{R}^{2n_i-\alpha(n_i)}$ for $i = 1, 2$. Then both manifolds*

i) $M_1 \amalg M_2$ *for* $n_1 = n_2 = n$, *and*

ii) $M_1 \times M_2$ *for* n_1, n_2 *arbitrary*

immerse into $\mathbb{R}^{2(n_1+n_2)-\alpha(n_1+n_2)}$.

Proof. $M_1 \times M_2$ immerses into the real space of dimension

$$\begin{aligned}(2n_1 - \alpha(n_1)) + (2n_2 - \alpha(n_2)) &= 2(n_1 + n_2) - (\alpha(n_1) + \alpha(n_2)) \\ &\leq 2(n_1 + n_2) - (\alpha(n_1 + n_2))\end{aligned}$$

where the inequality is due to the number theoretic fact $\alpha(n_1 + n_2) \leq \alpha(n_1) + \alpha(n_2)$.

For $n_1 = n = n_2$ the images of the immersions

$$\iota_1 \colon M_1 \to \mathbb{R}^{2n-\alpha(n)} \qquad \text{and} \qquad \iota_2 \colon M_2 \to \mathbb{R}^{2n-\alpha(n)}$$

are compact. So by concatenation with translation they can be assumed to be disjoint, wherefore the disjoint union $\iota_1 \amalg \iota_2 \colon M_1 \amalg M_2 \to \mathbb{R}^{2n-\alpha(n)}$ is again an immersion. □

4.3.2 Generating Set

4.3.2.1 Even Dimensional Generators

The idea in finding even dimensional generators is to use the following immersion properties of real projective spaces, which have been investigated in detail throughout the last decades:

Lemma 4.40. *i)* $\mathbb{RP}^{2^i}$ *immerses into Euclidean space of dimension* $2 \cdot 2^i - 1$.

ii) $\mathbb{RP}^k$ *immerses into Euclidean space of dimension* $2k - 3$ *for* $k \geq 5$ *odd, i.e. overfulfills the immersion property in case* $\alpha(k) = 2$.

Proof. The first statement is Whitney's immersion theorem [Whi44]. The second one is a collection of results by Sanderson presented in [San64], more precisely Theorem (5.3) for the case $k > 8$, and Theorem (4.1) for immersions $\mathbb{RP}^5 \hookrightarrow \mathbb{R}^7$, and $\mathbb{RP}^7 \hookrightarrow \mathbb{R}^8$. $\square$

The trick now will be to utilize the above nice immersion property of odd dimensional projective spaces. More precisely, the following odd dimensional product of projective spaces overfulfills the immersion property by at least three, wherefore all (even-dimensional) submanifolds of codimension one fulfill the immersion property.

Definition 4.41. Let $m \in \mathbb{N}$ be even and $\alpha(m) > 1$ (i.e. m is not of the form 2^i) with minimal binary expansion $m = 2^{i_1} + \cdots + 2^{i_l}$, $k_r := 2^{i_r}$. Define

$$K^{m+1} := \left(\prod_{r=1}^{l-1} \mathbb{RP}^{k_r}\right) \times \mathbb{RP}^{k_l+1}$$

with cohomology ring

$$\begin{aligned} H^*(K^{m+1}) &= \left(\bigotimes_{r=1}^{k-1} H^*(\mathbb{RP}^{k_r})\right) \otimes H^*(\mathbb{RP}^{k_l+1}) \\ &= \mathbb{Z}_2[x_{k_1}, \ldots, x_{k_{l-1}}, x_{k_l+1}] / \left(x_{k_1}^{k_1+1}, \ldots, x_{k_{l-1}}^{k_{l-1}+1}, x_{k_l+1}^{k_l+2}\right) \end{aligned}$$

given by the Künneth isomorphism.

Lemma 4.42. *For m as above K^{m+1} immerses into $\mathbb{R}^{2m-\alpha(m)}$*

Proof. Observe that $m = 2^{i_1} + \cdots + 2^{i_{\alpha(m)}}$ with $0 < i_r < i_{r+1}$ and $k_l = 2^{i_{\alpha(m)}} \geq 5$ by assumption, so $\mathbb{RP}^{k_l+1}$ immerses into Euclidean space of dimension $2(k_l+1)-3 = 2k_l-1$ by the Lemma. Altogether this yields an immersion of K^{m+1} into Euclidean space of dimension

$$\sum_{r=1}^{k-1}(2k_r - 1) + 2k_l - 1 = 2\left(\sum_{r=1}^{k}\right) - l = 2m - \alpha(m)\ . \qquad \square$$

In order to define submanifolds of the above K^{m+1} which are additionally indecomposable in the cobordism ring, recall that the Steenrod problem on realizing homology classes by submanifolds is solved for homology classes in degree of codimension one, or more precisely:

Lemma 4.43. *Let M^n be a manifold. Then for every homology class $\alpha \in H_{n-1}(M)$ there is a submanifold N and an embedding $\iota\colon N \hookrightarrow M$ such that $\iota_*([N]) = \alpha$.*

Proof. See [Tho07, Theorem II.26]. $\square$

Thus the following candidates for even-dimensional generators are well-defined.

Definition 4.44. Let m be even. Define for

$m = 0$*:* $G^m = *$.

$\alpha(m) = 1$*:* $G^m = \mathbb{R}\mathrm{P}^m$.

$\alpha(m) > 1$*:* $\iota\colon G^m \subset K^{m+1}$ is the submanifold of codimension one realizing the Poincaré dual of $y = \sum_{r=1}^{l-1} x_{i_r} + x_{i_{l+1}} \in H^1(K^{m+1})$, i.e. $\iota_*[G^m] = y \cap [K^{m+1}]$.

Lemma 4.45. *Let $m = \sum_{r=1}^{l} k_r$ be even as in the definition above.*

i) G^m immerses into $\mathbb{R}^{2m-\alpha(m)}$, and

ii) G^m represents an indecomposable element of the cobordism ring.

Proof of Lemma 4.45. For $m = 0$ both claims are trivial.

G^m clearly immerses into $\mathbb{R}^{2m-\alpha(m)}$, either factoring over K^{n+1} in case $\alpha(m) > 1$ or by Whitney's immersion theorem in case $\alpha(m) = 1$.

For the indecomposability we want to use the criterion from Theorem 4.15, i.e. one has to show that $s_m[G^m] \neq 0$. In the case $\alpha(m) = 1$ this is Example 4.16. In the other case $\alpha(m) > 1$ we first need to identify the Stiefel-Whitney numbers of N in degrees greater 0 and express them as elementary symmetric polynomials in some variables. As one will see the factor $\mathbb{R}\mathrm{P}^{k_l+1}$ has no special role here, and the following holds for any product $\prod_{r=1}^{l} \mathbb{R}\mathrm{P}^{n_r}$ of real projective spaces with

i) $m + 1 = \sum_{r=1}^{l} n_r$,

ii) $n_r < m - 1$, and

iii) $\alpha(n_i + n_j) = \alpha(n_i) + \alpha(n_j)$ for $1 \leq i < j \leq l$, hence $\alpha(n_1 + \cdots + n_r) = \sum_{i=1}^{r} \alpha(n_i)$ for $2 \leq r \leq l$.

In our case take $n_l := k_l + 1$ and $n_r := k_r$ for $1 \leq k < l$. Now, to determine the Stiefel-Whitney numbers denote the normal bundle of the embedding $\iota\colon G^m \subset K^{n+1}$ by ν. As $\mathrm{T}G^m \oplus \nu_\iota = \iota^*\mathrm{T}K^{m+1}$ there is the relation of Stiefel-Whitney classes

$$\begin{aligned} w(\mathrm{T}G^m) \cdot w(\nu_\iota) &= \iota^* w(\mathrm{T}K^{m+1}) \\ &= \iota^* \left(w(\mathbb{R}\mathrm{P}^{k_l+1}) \cdot \prod_{r=1}^{l-1} w(\mathbb{R}\mathrm{P}^{k_r}) \right) \\ &= \iota^* \left((1 + x_{k_l+1})^{k_l+1} \cdot \prod_{r=1}^{l-1} (1 + x_{k_r})^{k_r} \right) . \end{aligned}$$

Note that ν is a line bundle, so one only has to find out $w_1(\nu)$. For this one needs the following generalization of Lemma 3.16 which gives the highest Stiefel-Whitney class of normal bundles of such embeddings.

Lemma 4.46. *Let M^n, W^{n+k} be compact manifolds, $k > 0$, $\iota\colon M \hookrightarrow W$ be an embedding with corresponding normal bundle ν_ι, $\infty \in W \setminus \iota(N)$, and Thom-Pontryagin collapse map on pairs of spaces corresponding to some tubular neighborhood embedding of ν_ι*

$$c\colon W \to W/\left(W \setminus \mathrm{E}\nu_\iota\right) \cong \mathrm{D}\nu_\iota/\mathrm{S}\nu_\iota \cong \mathrm{D}\nu_\iota/\mathrm{S}\nu_\iota \xrightarrow{\text{incl}} (\mathrm{D}\nu_\iota/\mathrm{S}\nu_\iota, \infty)\ .$$

Then

i) $\iota_*[M] = c^* u(\nu_\iota) \cap [W]$, giving a description of the Poincaré dual of $\iota_*[M]$, and

ii) $w_{\mathrm{rk}\,\nu_\iota}(\nu_\iota) = \iota^* c^* u(\nu_\iota)$, so the pushforward of the Poincaré dual of $\iota_*[N]$ along the embedding is the $\mathrm{rk}\,\nu$-Stiefel-Whitney class.

Proof. See also [Bre93, p. 371]. To directly obtain ii) directly from i), one can use the fact that the Stiefel-Whitney class $w_{\mathrm{rk}\,\nu_\iota}(\nu_\iota)$ in degree $\mathrm{rk}\,\nu_\iota$ is the Euler class of ν_ι, which is defined as $(0_{\nu_\iota})^*(u(\nu_\iota))$ (see e.g. [Bre93, Prop. 17.2]), because then one has with $0_{\nu_\iota} = c\iota$ that

$$w_{\mathrm{rk}\,\nu_\iota}(\nu_\iota) = (0_{\nu_\iota})^*(u(\nu_\iota)) = \iota^* c^* u(\nu_\iota)\ .$$

For i) recall from Lemma 3.16, that $[M] = p_*(u(\nu_\iota) \cap c^*[W])$, so

$$0_{\nu_\iota *}[M] = 0_{\nu_\iota *} p_*(u(\nu_\iota) \cap c^*[W]) = u(\nu_\iota) \cap c^*[W]\ . \tag{4.9}$$

Now observe that for the inclusion maps

$$W \xrightarrow{j} (W, W \setminus M) \xleftarrow[\text{exc.}]{i} (\mathrm{D}\nu_\iota, \mathrm{D}\nu_\iota \setminus M)$$

i is an excision, and $c_* = (i_*)^{-1} j_*$, $c^* = j^*(i^*)^{-1}$. Denote $u := u(\gamma_1)$ and the preimage of $u(\gamma_1)$ under the excision by $u' := (i^*)^{-1} u(\gamma_1) \in H^*(M, M \setminus N)$. Then calculate using the cap-product formula from Equation 3.5, and $\iota = i \circ 0_{\nu_\iota}$:

$$\begin{aligned} c^* u \cap [W] = j^* u' \cap [W] &\overset{(*)}{=} j_*(j^* u' \cap [W]) \\ &\overset{(3.5)}{=} u' \cap j_*[W] = u' \cap i_* c_*[W] \\ &\overset{(3.5)}{=} i_*(i^* u' \cap c_*[W]) \\ &\overset{(4.9)}{=} i_* 0_{\nu_\iota *}[M] \\ &= \iota_*[M]\ , \end{aligned}$$

where equality $(*)$ uses, that the map of triple of spaces $j\colon (W, \emptyset, \emptyset) \to (W, W \setminus N, \emptyset)$ is the identity when restricted to $(W, \emptyset) \to (W, \emptyset)$. $\square$

By the above Lemma

$$\begin{aligned} w_1(\nu) &= \iota^* y \\ w(\nu) = 1 + w_1(\nu) &= \iota^*(1+y) \overset{(*)}{=} \iota^*\left((1+y)^{-(2^s-1)}\right) = (\iota^*(1+y))^{-(2^s-1)} \end{aligned}$$

for some $2^s > n+1$, where equality (*) then comes from $y^{2^s} = 0$ and

$$(1+y)^{2^s-1} \cdot (1+y) = (1+y)^{2^s} = (1+y^{2^s}) = 1\ .$$

Hence, $w(\mathrm{T}G^m)$ can be expressed as

$$\begin{aligned} w(\mathrm{T}G^m) &= \iota^*(1+y)^{2^s-1} \cdot \iota^*\left(\prod_{r=1}^{l}(1+x_{n_r})^{n_r+1}\right) \\ &= \iota^*(1+\sigma_1(\underbrace{y,\dots,y}_{2^s-1},\underbrace{x_{n_1},\dots,x_{n_1}}_{n_1+1},\dots,\underbrace{x_{n_l},\dots,x_{n_l}}_{n_l+1})) \\ &= 1+\sigma_1(\iota^*y,\dots,\iota^*y,\iota^*x_{n_1},\dots,\iota^*x_{n_1},\dots,\iota^*x_{n_l},\dots,\iota^*x_{n_l})) \end{aligned}$$

which is a representation as a pullback of a sum of elementary symmetric polynomials in the given one dimensional variables. This yields

$$\begin{aligned} s_m(G^m) &= (2^s-1)\iota^*y^m + (n_1+1)\iota^*x_{n_1}^m + \cdots + (n_l+1)\iota^*x_{n_l}^m \\ &= \iota^*\left((2^s-1)y^m + (n_1+1)x_{n_1}^m + \cdots + (n_l+1)x_{n_l}^m\right) \\ &\overset{x_{n_r}^{n_r+1}=0,\ n_r+1<m}{=} \iota^*((2^s-1)y^m) \\ &\overset{2^s=0}{=} \iota^* y^m \end{aligned}$$

So

$$\begin{aligned} \iota_* s_m[G^m] &= \iota_* \langle s_m(G^m),\, [G^m]\rangle \\ &= \iota_* \langle \iota^* y^m,\, [G^m]\rangle \\ &= \langle y^m,\, \iota_*[G^m]\rangle \\ &\overset{\text{Def.}}{=} \left\langle y^m,\, y \cap [K^{m+1}]\right\rangle = \left\langle y^{m+1},\, [K^{m+1}]\right\rangle\ . \end{aligned}$$

As capping with $[K^{m+1}]$ is the Poincaré isomorphism, it suffices for the proof to show that y^{m+1}, hence the image of $s_m[G^m]$ under ι_*, is non-zero. But, by the relations of the x_r, combinatorics give that the only non-zero product $\prod_{r=1}^{l} x_{n_r}^{a_r}$ of degree $m+1$ is the one with $a_r = n_r$. With the multinomial theorem y^{m+1} reformulates to

$$y^{m+1} = (x_{n_1} + \cdots + x_{n_l})^{m+1} = \frac{(n_1+\cdots+n_l)!}{(n_1)!\cdots(n_l)!} \cdot \prod_{r=1}^{l} x_{n_r}^{n_r}$$

which is non-zero if and only if the binomial expression is. The latter however simplifies with descending induction using the easy formula

$$\frac{(a_1+\cdots+a_j)!}{(a_1)!\cdots(a_j)!} = \binom{a_1+\cdots+a_j}{a_j} \cdot \frac{(a_1+\cdots+a_{j-1})!}{(a_1)!\cdots(a_{j-1})!}$$

for $a_r \in \mathbb{N}$ to

$$\frac{(n_1 + \cdots + n_l)!}{(n_1)! \cdots (n_l)!} = \prod_{r=2}^{l} \binom{n_1 + \cdots + n_r}{n_r}$$

all factors of which are one by Lemma 4.37 due to the condition on the n_r. So altogether G^m represents an indecomposable element of the cobordism ring according to the indecomposability criterion in Theorem 4.15. $\square$

4.3.2.2 Odd Dimensional Generators

For the odd dimensional generators the twisted product construction comes into play. As already discussed, the twisted product of a manifold which is indecomposable in the cobordism ring will be so, too, if the product dimension was chosen correctly. So the selected choice of candidates will be twisted products of some G^n for n even, whose indecomposability was proven during the previous section. The trick here will be to select for a candidate in odd degree m a split of m of the form $m = k + 2n$ with n even(!).

Remark 4.47*.* The following is a convenient split in the above sense: Let m be odd and write the binary expansion as

$$m = \underbrace{2^{i_0} + \cdots + 2^{i_{l_k}}}_{=:k} + \underbrace{2^{i_{l_k+1}} + \cdots + 2^{i_{l_k+l_n}}}_{=:2n} = k + 2n$$

such that as usual $i_r < i_{r+1}$, and additionally $i_r = r$ for $0 \leq r \leq l_k$ and $i_{l_k+1} > l_k + 1$. In other words split the binary expansion of m at the first point where it skips a power of two and k to the first part and $2n$ to the second part. Since m is odd, this trick guarantees that n is even. Also, by Remark 4.36, if m is not of the form $2^s - 1$, n will not be either.

For the immersion property we will again utilize nice immersion properties of projective spaces, more precisely the following one by Mahowald and Milgram for the total space of line bundles over such:

Lemma 4.48 ([MM68, Theorem 4.1, p. 418])**.** *For $p, q \in \mathbb{N}$ odd and the total space of $(p+1)\gamma_q$ immerses in Euclidean space of dimension*

$$2q + p + 1 - \alpha(p + q + 1) + \alpha(p + 1)\ .$$

The trick for odd m now will be to find an immersion of the m-candidate into some $D_k(\mathbb{R}^s)$, which is the total space of $(s\gamma_k) \oplus \varepsilon^s$ by Corollary 4.20, and then apply the above lemma to a super-bundle $((s + s')\gamma_k) \oplus \varepsilon^s$ of the latter. At that point it will be essential that k from the above split of m is odd in order to make the lemma applicable in the first place.

With these preliminary considerations one can make a choice of generator candidates:

Definition 4.49. Let $m = k + 2n$ be odd and not of the form $2^s - 1 \in \mathbb{N}$ with the split from Remark 4.47 above. Define

$$G^m := D_k(G^n)$$

Lemma 4.50. *For m odd as in the above definition, G^m fulfills:*

i) G^m immerses into $\mathbb{R}^{2m-\alpha(m)}$.

ii) The cobordism classes of the G^m, $m \neq 2^s - 1$, are indecomposable.

Proof of Lemma 4.50.ii). The used split of m from Remark 4.47 is a special case of splits described in Remark 4.36, for all of which the twisted product preserves indecomposability by Corollary 4.38. Thus, as n from the split is even and G^n is indecomposable by Lemma 4.45, $G^m = D_k(G^n)$ is also an indecomposable element in the cobordism ring. □

Proof of Lemma 4.50.i). Here one needs that for $m = k + 2n$ as above G^n admits by Lemma 4.45 an immersion

$$\iota \colon G^n \hookrightarrow \mathbb{R}^{2n-\alpha(n)} \ .$$

As described in Remark 4.19.a), this induces an immersion of twisted products

$$\begin{array}{ccc} (S^{2^k} \times G^n \times G^n / \sim) & \xrightarrow{1 \times \iota \times \iota / \sim} & (S^{2^k} \times \mathbb{R}^{2n-\alpha(n)} \times \mathbb{R}^{2n-\alpha(n)} / \sim) \ . \\ \| & & \| \\ G^m & & D_k(\mathbb{R}^{2n-\alpha(n)}) \end{array}$$

The latter twisted product however is a vector bundle over $\mathbb{RP}^k$ as in Remark 4.19.ii). More precisely, by Lemma 4.20 it is the well-known vector bundle

$$((2n - \alpha(n))\gamma_k) \oplus \varepsilon^{2n-\alpha(n)}$$

where γ_k is the tautological line bundle over $\mathbb{RP}^k$. As $p = 2n - 1$ and $q = k$ are both odd, Lemma 4.48 is applicable to p and q, and yields an immersion of $(p+1)\gamma_q = 2n\gamma_k$ into Euclidean space of dimension

$$\begin{aligned} 2q + p + 1 - \alpha(q + p + 1) + \alpha(p + 1) &= 2k + 2n - 1 + 1 - \alpha(k + 2n) + \alpha(2n) \\ &= k + m - \alpha(m) + \alpha(n) \ . \end{aligned}$$

Hence, the total space of $(2n\gamma_k) \oplus \varepsilon^{2n-\alpha(n)}$ immerses into Euclidean space of codimension

$$(k + m - \alpha(m) + \alpha(n)) + 2n - \alpha(n) = (k + 2n) + m - \alpha(m) = 2m - \alpha(m) \ ,$$

altogether yielding an immersion

$$\begin{aligned} D_k(G^m) &\hookrightarrow D_k(\mathbb{R}^{2n-\alpha(n)}) = \mathrm{E}\left(((2n-\alpha(n))\gamma_k) \oplus \varepsilon^{2n-\alpha(n)}\right) \\ &\hookrightarrow \mathrm{E}\left((2n\gamma_k) \oplus \varepsilon^{2n-\alpha(n)}\right) \\ &\hookrightarrow \mathbb{R}^{2m-\alpha(m)} \end{aligned}$$

as was needed. $\square$

This finally gives Brown's theorem:

Theorem 4.51. *Lemmata 4.45 and 4.50 together yield a complete generating set of the cobordism ring, i.e.*

$$\eta_* \cong ([G^m] \mid m \neq 2^s - 1)_{\mathbb{Z}_2} = \mathbb{Z}_2\left[[G^m] \mid m \neq 2^s - 1\right] \ ,$$

each element of which fulfills the immersion property as was desired.

5 An Outline of the Proof of the Immersion Conjecture

This chapter gives an overview of the main steps towards a proof of the immersion conjecture. In the course of this it becomes clear how the key theorems discussed in this thesis—Theorem 3.26 by Massey and Theorem 4.1 by R. L. Brown—contributed to the development in this area. The proof discussed will be the one based on the work of E. H. Brown Jr., F. P. Peterson [BP79], which was finalized by R. L. Cohen in his dissertation [Coh85]. The outlines here are geared to Cohen's highly recommendable lecture notes [CT89] on this topic.

Proofs of the following statements are either sketched or omitted. For details the reader is advised to consult the latter reference if no other is given. Even though the proof of the immersion conjecture itself requires profound knowledge of the theory of spectra, basic knowledge should suffice to follow the outlines below. See [CT89, p. 77ff] for a very brief introduction, or [Swi02] for a more complete picture.

Notation. Normal Bundles will be considered to arise from immersions of manifolds into Euclidean space, and vector bundles will be identified with their classifying maps. For a vector bundle $\xi\colon X \to \mathrm{B}O$ denote its Thom spectrum by $\mathbf{T}\xi$, and for a map of spaces $f\colon Y \to X$ denote the induced map of Thom spaces resp. spectra by $Tf\colon \mathbf{T}(f^*\xi) \to \mathbf{T}\xi$. Denote by $\mathbf{K}(\mathbb{Z}_2)$ the Eilenberg-MacLane spectrum for $\mathbb{Z}_2$. For $k \in \mathbb{N}$ let

- γ_k be the universal bundle of $O(k)$,
- $MO(k) \cong \mathrm{D}\gamma/\mathrm{S}\gamma$ the corresponding Thom space,
- $\mathbf{MO}(k)$ the corresponding Thom spectrum,
- $\mathbf{MO} := \{MO(k) \mid k \in \mathbb{N}\}$ be the universal Thom spectrum, and
- $\mathbf{S}^k$ be the kth suspension of the sphere spectrum.

For a sequence $I = (i_1, \dots, i_l)$ of positive integers, denote by $|I| := \deg(\sigma^I)$ the degree of the monomial $\sigma^{i_1} \cdots \sigma^{i_l}$ in variables σ_i with $\deg(\sigma_i) = i$. Lastly, fix some $n \in \mathbb{N}$ throughout this chapter.

Recall from Chapter 2 that the statement in question is:

Theorem. *Every closed smooth n-manifold immerses into* $\mathbb{R}^{2n-\alpha(n)}$.

Furthermore, it was shown that this is equivalent to:

Theorem. *For any closed smooth n-manifold, the classifying map* $\nu_M \colon M \to \mathrm{B}O$ *factors over the inclusion* $\mathrm{B}O(n-\alpha(n)) \to \mathrm{B}O$.

This theorem essentially says that the universal $O(n-\alpha(n))$-vector bundle is a better choice for classifying normal bundles of n-manifolds than $\mathrm{B}O$. The idea for the proof of the immersion conjecture is to find a vector bundle $\mathrm{B}O/I_n \to \mathrm{B}O$ which

1. acts as universal bundle for normal bundles of n-manifolds, in the sense that all normal bundle maps factor over it, and which
2. is even better than $\mathrm{B}O(n-\alpha(n))$ in classifying normal bundles, i.e. it lifts to $\mathrm{B}O(n-\alpha(n))$.

In other words, we are looking for a space later called $\mathrm{B}O/I_n$ and a map

$$\rho_n \colon \mathrm{B}O/I_n \longrightarrow \mathrm{B}O(n-\alpha(n)) \longrightarrow \mathrm{B}O$$

over which all classifying maps of stable normal bundles factor. More precisely, for a fixed $n \in \mathbb{N}$ the proof encompasses the following main steps:

Step 1: Get an idea of the statement on cohomology, and determine up to a Thom isomorphism the ideal $I_n \subset H^*(\mathrm{B}O)$ of characteristic classes that vanish on all normal bundles of n-manifolds. This turns out to be a generalization of Massey's theorem on characteristic classes of normal bundles (Theorem 3.26).

Step 2: Find a spectrum $\mathbf{MO}/I_n$ with a map $\boldsymbol{\rho}_n \colon \mathbf{MO}/I_n \to \mathbf{MO}$ which acts as a universal spectrum for Thom spectra of normal bundles of n-manifolds, i.e. all maps $\mathbf{T}\nu_{M^n} \to \mathbf{MO}$ of a Thom spectrum of a normal bundle factor over the above map. Furthermore, $\boldsymbol{\rho}_n$ should factor over $\mathbf{MO}(n-\alpha(n)) \to \mathbf{MO}$.

Step 3: Find a sufficient criterion when a map of a spectrum into the Thom spectrum of a bundle *de-Thom-ifies*, i.e. sort of comes from a map of Thom spectra which is induced by a map of vector bundles.

Step 4: Construct a de-Thom-ification bundle $\mathrm{B}O/I_n \to \mathrm{B}O$ of $\mathbf{MO}/I_n \to \mathbf{MO}$ by de-Thom-ifying a suitable resolution, and show that this acts as a universal space for normal bundles of manifolds.

Step 5: Show that there is a space X^n and a map $X^n \to \mathrm{B}O/I_n \to \mathrm{B}O$, which factors over $\mathrm{B}O(n-\alpha(n))$, and admits some nice splitting and commutativity properties on the Thom spectra corresponding to the classifying maps.

Step 6: Show that the latter implies that the map $\mathrm{B}O/I_n \to \mathrm{B}O$ factors over the universal bundle $\mathrm{B}O(n-\alpha(n)) \to \mathrm{B}O$.

In the end, one has a space $\mathrm{B}O/I_n$ which admits for any n-manifold a commutative diagram

$$\begin{tikzcd} M \arrow[r] \arrow[drr, bend right, "\nu_M"'] & \mathrm{B}O/I_n \arrow[r] & \mathrm{B}O(n-\alpha(n)) \arrow[d, "\mathrm{incl}"] \\ & & \mathrm{B}O \end{tikzcd}$$

as was to be shown for the immersion conjecture.

The last two steps turn out to be quite tricky and are the topic of Cohen's dissertation [Coh85]. This is also the point where R. L. Brown's theorem from Chapter 4 comes into play. It is needed for some choices in the construction of X^n. The other results are due to E. H. Brown Jr. and F. P. Peterson.

In the following a couple of aspects of the above steps are filled with more details.

A Generalization of Massey's Theorem

In search of properties of such a universal bundle $\rho_n\colon X_n \to \mathrm{B}O$ for stable normal bundles of n-manifolds, one idea is that the "best" such one should fulfill

$$\ker(\rho_n^*) = \bigcap_{M \text{ mfd.}} \ker\left({\nu_M}^*\colon H^*(\mathrm{B}O) \to H^*(M)\right) =: I_n\ ,$$

i.e. all characteristic classes from $H^*(\mathrm{B}O)$ that vanish for normal bundles, already vanish for the bundle ρ_n. Note that by Massey's theorem the ideal $(w_i \mid i > n - \alpha(n))$ lies within I_n.

Remark. Recall that the Thom isomorphism extends to an isomorphism

$$t\colon H^*(\mathrm{B}O) \to H^*(\mathbf{MO})$$

(see e.g. [MS74]). Furthermore, note that $H^*(\mathbf{MO})$ is known to be a free module over the Steenrod algebra $\mathcal{A}$ of the form

$$H^*(\mathbf{MO}) = (\sigma_i \mid i \neq 2^s - 1)_{\mathcal{A}} \cong \mathcal{A} \otimes \mathbb{Z}_2[\sigma_i \mid i \neq 2^s - 1] \cong \bigoplus_{|I| \neq 2^s - 1} \mathcal{A}_{(*+|I|)}\sigma^I$$

where the σ_i are algebraically independent elements, $\deg(\sigma_i) = i$. Furthermore, the last isomorphism is one of $\mathcal{A}$-modules, and the sum is over arbitrary sequences I of positive integers (see e.g. [CT89, p. 82] or [Swi02, Chap. 20]).

Brown and Peterson investigated the image of I_n under the Thom isomorphism, which is characterized by the property that for any n-manifold M, $t(I_n) = \ker(T{\nu_M}^*)$. Their result was the following.

Theorem ([CT89, Theorem 2.3])**.** *One has the following equality of left-ideals of $\mathcal{A}$:*

$$J_r := \bigcap_{M^n \ \mathit{mfd.}} \{a \in \mathcal{A} \mid a(u(\nu_M)) = 0\} = \mathcal{A} \cdot \left\{\chi(\mathrm{Sq}^i) \;\middle|\; 2i > r\right\} \subset \mathcal{A} \,.$$

Furthermore, with notation as above,

$$\begin{aligned} H^*(\mathbf{MO})/t(I_n) &\cong \bigoplus_{n \geq |I| \neq 2^s - 1} \left(\mathcal{A}/J_{n-|I|}\right)_{*+|I|} \,, \qquad \textit{thus} \\ t(I_n) &= H^{*>n}(\mathbf{MO}) + \mathcal{A} \cdot \left\{\chi(\mathrm{Sq}^i)(\sigma^I) \;\middle|\; 2i > n - \deg(\sigma^I)\right\} \,. \end{aligned}$$

Note that Massey's theorem 3.26 can be obtained from the above statement as follows:

Remark. Note that the projection of

$$\left\{\chi(\mathrm{Sq}^I) \;\middle|\; I = (i_1, \ldots, i_l) \text{ admissible, } 2i_1 \leq r\right\}$$

is a $\mathbb{Z}_2$-vector space basis of $\mathcal{A}/J_r$. Taking a closer look at these elements, one notes that the largest dimension occurring is $r - \alpha(r)$. Therefore, for $i > r - \alpha(r)$, Sq^i is sent to zero under the projection of $\mathcal{A}/J_r$, respectively lies in J_r. Thus, Sq^i lies in $t(I_n)$ for $i > n - \alpha(n) = \max_k\{(n-k) - \alpha(n-k)\}$. This yields the statement of Massey's theorem, since for an n-manifold M one gets

$$t(w_i(\nu_M)) \overset{3.17}{=} \mathrm{Sq}^i(u(\nu_M)) \in \mathrm{Sq}^i(\mathrm{im}(\mathrm{T}\nu_M)^*) = (\mathrm{T}\nu_M)^*(\mathrm{Sq}^i \cdot H^*(\mathbf{MO})) = 0 \,.$$

See also [CT89, p. 93].

The Construction of $\mathbf{MO}/I_n$

The summands in $H^*(\mathbf{MO})/t(I_n)$ are isomorphic to the cohomology rings of the Brown-Gitler spectra. Thus, the form of $H^*(\mathbf{MO})/t(I_n)$ gives a good hint on how to proceed in the search of some universal spectrum for Thom spaces of normal bundles.

Remark. Recall that Thom calculated

$$\mathbf{MO} \cong \bigvee_{|I| \neq 2^s - 1} \mathbf{S}^{|I|} \wedge \mathbf{K}(\mathbb{Z}_2)$$

(see [CT89, p. 81f] for a good summary of the arguments).

Definition. In the following denote by $\mathbf{T}_k$ the kth Brown-Gitler spectrum, thus especially $H^*(\mathbf{T}_k) \cong \mathcal{A}/J_k$ (see [BG73] or [CT89, p. 101] for definition and a construction). For $k \in \mathbb{N}$ let y_k be the generator of $H^*(\mathbf{T}_k) \cong \mathcal{A}/J_k$ as $\mathcal{A}$-module, i.e. $H^*(\mathbf{T}_k) = \mathcal{A} \cdot y_k$. Define

$$\boldsymbol{\rho}_n \colon \mathbf{MO}/I_n := \bigvee_{n \geq |I| \neq 2^s - 1} \mathbf{S}^{|I|} \wedge \mathbf{T}_{n-|I|} \longrightarrow \bigvee_{|I| \neq 2^s - 1} \mathbf{S}^{|I|} \wedge \mathbf{K}(\mathbb{Z}_2) \cong \mathbf{MO}$$

where the map is the wedge product given by the cohomology elements

$$y_k \colon \mathbf{T}_k \to \mathbf{K}(\mathbb{Z}_2) \,.$$

Theorem. $\boldsymbol{\rho}_n \colon \mathbf{MO}/I_n \to \mathbf{MO}$ *fulfills*

i) $\boldsymbol{\rho}_n^* = \mathrm{proj}\colon H^*(\mathbf{MO}) \to H^*(\mathbf{MO})/t(I_n) \cong H^*(\mathbf{MO}/I_n)$

ii) $\mathbf{MO}/I_n$ *together with* $\boldsymbol{\rho}_n$ *is a universal Thom spectrum for Thom spectra of normal bundles of n-manifolds. I.e. for every n-manifold there exists a lift* $\boldsymbol{f}_M$ *of* $T\nu_M$ *such that the following diagram commutes up to homotopy*

$$\begin{tikzcd} \mathbf{T}\nu_M \arrow[r, "\boldsymbol{f}_M"] \arrow[dr, bend right, "T\nu_M"'] & \mathbf{MO}/I_n \arrow[d, "\boldsymbol{\rho}_n"] \\ & \mathbf{MO} \end{tikzcd}$$

iii) $\boldsymbol{\rho}_n$ *factors up to homotopy over* $\mathbf{MO}(n-\alpha(n)) \to \mathbf{MO}$*.*

Outline of the proof. For a proof see [CT89, Lemma 2.28, Theorem 2.29]. Statement i) requires mainly the considerations from above, thus follows rather directly from the definition. For ii), one piece-wise constructs a lift on the wedge-product factors, i.e. for any sequence I with $n \geq d = |I| \neq 2^s - 1$, find a map $\mathrm{proj} \circ \boldsymbol{f}_M$ making the following diagram commute:

$$\begin{tikzcd} & & \Sigma^d\mathbf{T}_{n-d} \arrow[d, "\Sigma^d y_k"] \\ \mathbf{T}\nu_M \arrow[r, "T\nu_M"'] \arrow[urr, dashed, bend left, "\mathrm{proj}\circ f_M"] & \mathbf{MO} \arrow[r, "\mathrm{proj}"] & \Sigma^d\mathbf{K}(\mathbb{Z}_2) \end{tikzcd}$$

Lastly, iii) uses that $\mathbf{T}_k$ has modulo 2 the homotopy type of a CW complex of dimension $k - \alpha(k)$, which makes it possible to apply an obstruction argument to find the required lift. $\square$

This gives the desired intermediate result.

The Construction of $\mathrm{B}O/I_n$

Now that a nice universal spectrum for Thom spaces of normal bundles of n-manifolds is found, one has to somehow *de-Thom-ify* this result to a universal space of normal bundles with the desired properties. Let us first specify the meaning of de-Thom-ification.

Definition. For a classifying map $\xi\colon X \to \mathrm{B}O$, a map of spectra $\boldsymbol{\tau}\colon \mathbf{Z} \to \mathbf{T}\xi$ is said to *de-Thom-ify* through dimension k if there is

- a map of spaces $g\colon Y \to X$, and
- a k-connected map of spectra $\boldsymbol{\kappa}\colon \mathbf{T}(g^*\xi) \to \mathbf{Z}$,

such that the following diagram homotopy commutes

$$\begin{tikzcd}
\mathbf{Z} \arrow[r, "\boldsymbol{\tau}"] \arrow[d, "\boldsymbol{\kappa}"] & \mathbf{T}\xi \\
\mathbf{T}(g^*\xi) \arrow[ur, bend right, "Tg"'] &
\end{tikzcd} .$$

On the way a sufficient criterion for de-Thom-ification it is necessary to observe what additional structure a map of Thom spectra has if it is induced by a map on vector bundles. Indeed, such maps are linear with respect to an additional module structure (see [CT89, p. 107ff]).

Definition. On the tensor product $\mathcal{A} \otimes H^*(\mathrm{B}O)$ define the multiplication

$$(a \otimes x) \cdot (b \otimes y) := \sum_{i \in A} \big(a \circ b_i'\big) \otimes \big(\chi(b_i'')(x) \cdot y\big)$$

where b_i', b_i'' is such that $\sum_{i\in A} b_i' \otimes b_i''$ is the image of b under the diagonal map of $\mathcal{A}$ (see Definition 3.3).

Lemma. *Note that the multiplication defined above turns $\mathcal{A} \otimes H^*(\mathrm{B}O)$ into a ring. Then:*

i) For any classifying map $\xi\colon X \to \mathrm{B}O$, the corresponding $H^*(\mathrm{B}O)$-module structure on $H^*(X)$ induces the $(\mathcal{A} \otimes H^*(\mathrm{B}O))$-structure on $H^*(\mathbf{T}\xi)$

$$(a \otimes x) \cdot t(y) = a\,(t(\xi^* x \cup y)$$

via the Thom isomorphism.

ii) For any map of vector bundles, the induced map on Thom spaces is $(\mathcal{A}\otimes H^*(\mathrm{B}O))$-linear on cohomology.

In the spirit of the above observation, one gets even more additional structure around Thom maps induced by pullbacks of vector bundles. An essential result is the following de-Thom-ification criterion:

Theorem. *Let $k \in \mathbb{N}$ and $\xi\colon X \to \mathrm{B}O$ be k-connected. Assume there is a map of spectra $\boldsymbol{\tau}\colon \mathbf{Z} \to \mathbf{T}\xi$. Then $\boldsymbol{\tau}$ de-Thom-ifies through dimension $2k$ if and only if in dimensions lower or equals $2k$ the cohomology ring $H^*(\mathbf{C}_{\boldsymbol{\tau}})$ of the mapping cone of $\boldsymbol{\tau}$ is a free module over $\mathcal{A} \otimes H^*(\mathrm{B}O)$.*

Outline of the proof. See [CT89, Theorem 3.5]. For the necessity one should observe that a pullback $g\colon Y \to X$ of vector bundles can be modified to a certain 2-stage Postnikov tower

$$Y' \longrightarrow X \xrightarrow{\xi} \mathrm{B}O$$

without changing the homotopy type of $\mathbf{T}(g^*\xi)$ up to dimension $2k$. Then one can use a theorem by Brown and Peterson, which yields the needed result for the cohomology of the modified Thom mapping cone (see [CT89, Theorem 3.3]). For the other direction, a construction that rebuilds a system as the above one using the Thom isomorphism can be applied. $\square$

Brown and Peterson found a nice Adams resolution of spectra

$$\mathbf{MO}/I_n \to \ldots \to \mathbf{Z}_1 \to \mathbf{Z}_0 = \mathbf{MO}$$

which in each stage de-Thom-ifies through dimension n according to the above criterion. This finally gives:

Definition 5.1. Let

$$\ldots \to Y_1 \to Y_0 = \mathrm{B}O$$

be the result of the de-Thom-ification of the above resolution. Define $\mathrm{B}O/I_n$ as the n-skeleton of the inverse limit of the Y_i, and the map $\rho_n \colon \mathrm{B}O/I_n \to \mathrm{B}O$ by

$$\rho_n \colon \mathrm{B}O/I_n \to \ldots \to Y_1 \to Y_0 = \mathrm{B}O\ .$$

Theorem. *The special choice of the resolution equips $\rho_n \colon \mathrm{B}O/I_n \to \mathrm{B}O$ with the following properties:*

i) $T\rho_n = \boldsymbol{\rho}_n \colon \mathbf{T}\rho_n \simeq \mathbf{MO}/I_n \to \mathbf{MO}$.

ii) $\rho_n^* = \mathrm{proj} \colon H^*(\mathrm{B}O/I_n) \to H^*(\mathrm{B}O)/I_n \cong H^*(\mathrm{B}O/I_n)$.

iii) ρ_n acts as universal bundle for stable normal bundles of n-manifolds. I.e. for every n-manifold there exists a lift f_M of ν_M such that the following diagram commutes up to homotopy

$$\begin{tikzcd} M \arrow[r, "f_M"] \arrow[dr, bend right, "\nu_M"'] & \mathrm{B}O/I_n \arrow[d, "\rho_n"] \\ & \mathrm{B}O \end{tikzcd}$$

Outline of the proof. Statements i) and ii) follow from the choice of $\mathrm{B}O/I_n$ as de-Thom-ification of $\boldsymbol{\rho}_n \colon \mathbf{MO}/I_n \to \mathbf{MO}$, and application of the Thom isomorphism. For iii), consider the classifying map ν_M of a stable normal bundle of an n-manifold. This can inductively be lifted to the spaces Y_i, again using obstruction theory, which in this case only needs to be checked on the Thom spectrum level. However, for the latter a nice property of the chosen resolution is applicable. $\square$

This is already very close to the desired final result. However, note that Brown and Peterson could not directly conclude the needed factorization over $\mathrm{B}O(n-\alpha(n))$ from the definition (compare [BP79]). This still required quite a lot of work done by Cohen.

The Construction of the Lift $\mathrm{B}O/I_n \to \mathrm{B}O(n-\alpha(n))$

Cohen's approach to finding the factorization of $\rho_n \colon \mathrm{B}O/I_n \to \mathrm{B}O$ over $\mathrm{B}O(n-\alpha(n))$ is split into two main parts:

- Show an indirect version of the statement using a carefully constructed helper space X_n ([Coh85, Lemma B]), and
- prove that this already implies the factorization.

Definition. Let $P_n \coloneqq \mathrm{B}O/I_n \times_{\mathrm{B}O} \mathrm{B}O(n-\alpha(n))$ be the usual homotopy pullback of

$$\mathrm{B}O/I_n \xrightarrow{\rho_n} \mathrm{B}O \xleftarrow{\mathrm{incl}} \mathrm{B}O(n-\alpha(n)) \quad ,$$

i.e. defined by

$$P_n \coloneqq \left\{(x,y,\gamma) \in \mathrm{B}O/I_n \times \mathrm{B}O(n-\alpha(n)) \times \mathrm{B}O^I \;\middle|\; \gamma(0)=\rho_n(x), \gamma(1)=\mathrm{incl}(y)\right\} \ .$$

Cohen's Lemma B is as follows.

Theorem. *There exists a space and a map $p_n \colon X_n \to P_n$ respectively maps f_n, g_n fitting into the resulting homotopy commutative diagram*

$$\begin{tikzcd}
X_n \arrow[dr, "p_n"] \arrow[drr, bend left, "f_n"] \arrow[ddr, bend right, "g_n"'] & & \\
& P_n \arrow[r] \arrow[d] & \mathrm{B}O(n-\alpha(n)) \arrow[d, "\mathrm{incl}"] \\
& \mathrm{B}O/I_n \arrow[r, "\rho_n"] & \mathrm{B}O
\end{tikzcd}$$

satisfying the following properties. Let $\xi_n \coloneqq \rho_n \circ g_n = \mathrm{incl} \circ f_n$ be the vector bundle on X defined by the map into $\mathrm{B}O$.

i) There is a map $\boldsymbol{s}_n \colon \mathbf{MO}/I_n \to \mathbf{T}\xi_n$ of spectra, such that

$$\mathbf{MO}/I_n \xrightarrow{\boldsymbol{s}_n} \mathbf{T}\xi_n \xrightarrow{Tg_n} \mathbf{MO}/I_n$$

is homotopic to the identity, i.e. defines a split.

ii) Furthermore, $\boldsymbol{s}_n \circ Tg_n$ is close to the identity in the sense that the following diagram of spectra commutes up to homotopy

$$\begin{tikzcd}
\mathbf{T}\xi_n \arrow[r, "Tp_n"] \arrow[d, "Tg_n"'] & \mathbf{T}\xi_n \\
\mathbf{MO}/I_n \arrow[d, "\boldsymbol{s}_n"'] & \\
\mathbf{T}\xi_n \arrow[uur, bend right, "Tp_n"'] &
\end{tikzcd}$$

Cohen showed that Lemma B implies the existence of a lift of ρ_n to $\mathrm{B}O(n-\alpha(n))$ along incl. This was done by inductively constructing lifts along a slightly modified Postnikov tower of the map $\mathrm{incl}\colon \mathrm{B}O(n-\alpha(n)) \to \mathrm{B}O$. The induction step uses that the obstruction to such a lift is a map from a Thom spectrum to a spectrum which again admits a free $(\mathcal{A} \otimes H^*(\mathrm{B}O))$-module structure in lower dimensions. The latter is then used to get rid of the obstruction (compare [CT89, p. 118f]).

Outline of the proof of Lemma B. For more details see [CT89, Chap. III, §2]. We sketch the construction of X_n below. The idea is to dissect the map $\mathbf{MO}/I_n \to MO$ into its wedge summands in order to see that this actually factors over a wedge of Thom maps of vector bundles. $\xi_n\colon X_n \to \mathrm{B}O$ can be constructed from these vector bundles as disjoint sum of products in the correct dimension. Thus, first observe that

$$\mathbf{MO}/I_n \coloneqq \bigvee_{n \geq |I| \neq 2^s-1} \mathbf{S}^{|I|} \wedge \mathbf{T}_{n-\deg I} \longrightarrow \bigvee_{|I| \neq 2^s-1} \mathbf{S}^{|I|} \wedge \mathbf{K}(\mathbb{Z}_2) \cong \mathbf{MO}$$

on each wedge factor looks like

$$\mathbf{S}^{|I|} \wedge \mathbf{T}_{n-\deg I} \xrightarrow{M_{|I|} \wedge y_k} \mathbf{MO} \wedge \mathbf{MO} \longrightarrow \mathbf{MO}$$

where the second map is the multiplication of the ring spectrum MO. Now, both $M_{|I|}$ as well as y_k originate from classifying maps of vector bundles, which furthermore factor over $\mathrm{B}O(n-\alpha(n))$:

y_k: Using a specific construction shows that the kth Brown-Gitler spectrum is in fact the Thom spectrum of a vector bundle $\tilde{y}_k\colon B_k \to \mathrm{B}O$ over the Eilenberg-MacLane space $B_k \coloneqq K(\beta_k, 1)$, where β_k is the kth braid group (see [CT89, Chap. II, §2]). The homotopy type of B_k shows that $\tilde{y}_k$ factors over $\mathrm{B}O(k-\alpha(k))$ by obstruction theory.

$M_{|I|}$: The stable homotopy group $\pi_k^s(\mathbf{MO})$ is isomorphic to the kth cobordism group η_k via the Thom-Pontryagin isomorphism (see [Sto68, Chap. II]). The latter also says that a map $\mathbf{S}^k \to \mathbf{MO}$ represented by a manifold M^k representing under the above isomorphism factors up to homotopy as

$$\mathbf{S}^k \xrightarrow{\mathbf{c}} \mathbf{T}\nu_M \xrightarrow{T\nu_M} \mathbf{MO}$$

where $\boldsymbol{c}$ is the stable version of a collapse map. Furthermore, due to Theorem 4.1, one can always choose a representative M^k for which $\nu_M\colon M \to \mathrm{B}O$ factors over $\mathrm{B}O(n-\alpha(n))$.

The map $\xi_n\colon X_n \to \mathrm{B}O$ will look like

$$\coprod_{n \geq |I| \neq 2^s-1} M_{|I|} \times B_{n-|I|} \xrightarrow{\coprod \nu_{M_{|I|}} \times \tilde{y}_{n-|I|}} \mathrm{B}O$$

where $M_{|I|}$ is a manifold representing the map $M_{|I|}$, chosen such that it fulfills the immersion property. A choice like this will factor over $\mathrm{incl}\colon \mathrm{B}O(n-\alpha(n)) \to \mathrm{B}O$ as explained above. And a factorization over $\rho_n\colon \mathrm{B}O/I_n \to \mathrm{B}O$ is directly given if one observes that

- $\rho_i \times \rho_j\colon \mathrm{B}O/I_i \times \mathrm{B}O/I_j \to \mathrm{B}O$ factors over ρ_{i+j},
- $\tilde{y}_k$ factors over ρ_k, and that
- all stable normal bundles of k-manifolds factor over ρ_k by its universality property.

So, bundles of the form described above are candidates for ξ_n. They can also be shown to fulfill the splitting property i) using the well-known formula

$$\mathbf{T}(\nu_{M_i} \times \tilde{y}_j) = \mathbf{T}\nu_{M_i} \wedge \mathbf{T}\tilde{y}_j \ .$$

However, if one wants to obtain a vector bundle satisfying condition ii) using the above construction, one has to be careful when choosing the set of representatives

$$\left\{ M_{|I|} \mid n \geq |I| \neq 2^s - 1 \right\} \ .$$

Cohen's approach here is an induction on n: For $|I| = n$,

- the decomposable classes are represented as product manifold of the lower dimensional representatives, and
- the (unique) indecomposable class $\mathbf{S}^n \to MO$ of degree n is shown to originate from a split
$$\boldsymbol{j}\colon \mathbf{T}(\rho_n|_{C_n}) \vee \mathbf{S}^n \xrightarrow{\cong} \mathbf{MO}/I_n$$
where C_n is a nice subcomplex of $\mathrm{B}O/I_n$.

The neat properties of C_n and the induction assumption then imply ii) for this choice of ξ_n. $\square$

With this Cohen finalized the proof of the immersion conjecture.

Bibliography

[95] *Outside in.* Directed by Silvio Levy, Delle Maxwell and Tamara Munzner, With accompanying text ıt Making waves. University of Minnesota, Geometry Center, Minneapolis, MN; distributed by A K Peters, Ltd., Wellesley, MA, 1995, 22 min. ISBN: 1-56881-046-6.

[Ada93] Masahisa Adachi. *Embeddings and immersions.* Vol. 124. Translations of Mathematical Monographs. Translated from the 1984 Japanese original by Kiki Hudson. American Mathematical Society, Providence, RI, 1993, pp. x+183. ISBN: 0-8218-4612-4.

[BG73] Edgar H. Brown Jr. and Samuel Gitler. "A spectrum whose cohomology is a certain cyclic module over the Steenrod algebra". In: *Topology. An International Journal of Mathematics* 12 (1973), pp. 283–295. ISSN: 0040-9383.

[Bos13] S. Bosch. *Algebra.* Springer-Lehrbuch. Springer Spektrum, Berlin, Heidelberg, 2013. ISBN: 978-3-642-39567-3.

[BP79] E. H. Brown Jr. and F. P. Peterson. "A universal space for normal bundles of n-manifolds". In: *Commentarii Mathematici Helvetici* 54.3 (1979), pp. 405–430. ISSN: 0010-2571.

[Bre93] Glen E. Bredon. *Topology and geometry.* Vol. 139. Graduate Texts in Mathematics. Springer-Verlag, New York, 1993, pp. xiv+557. ISBN: 0-387-97926-3.

[Bro71] Richard L. W. Brown. "Immersions and embeddings up to cobordism". In: *Canadian Journal of Mathematics. Journal Canadien de Mathématiques* 23 (1971), pp. 1102–1115. ISSN: 0008-414X.

[Coh85] Ralph L. Cohen. "The immersion conjecture for differentiable manifolds". In: *Ann. of Math. (2)* 122.2 (1985), pp. 237–328. ISSN: 0003-486X.

[CT89] Ralph L. Cohen and Ulrike Tillmann. "Lectures on immersion theory". In: *Differential geometry and topology (Tianjin, 1986–87).* Vol. 1369. Lecture Notes in Math. Springer, Berlin, 1989, pp. 71–124.

[Die08] Tammo tom Dieck. *Algebraic topology.* EMS Textbooks in Mathematics. European Mathematical Society (EMS), Zürich, 2008, pp. xii+567. ISBN: 978-3-03719-048-7.

[Dol56] Albrecht Dold. "Erzeugende der Thomschen Algebra $\mathfrak{N}$". In: *Mathematische Zeitschrift* 65 (1956), pp. 25–35. ISSN: 0025-5874.

[EM02] Y. Eliashberg and N. Mishachev. *Introduction to the h-principle*. Vol. 48. Graduate Studies in Mathematics. American Mathematical Society, Providence, RI, 2002, pp. xviii+206. ISBN: 0-8218-3227-1.

[Hat02] Allen Hatcher. *Algebraic topology*. Cambridge University Press, Cambridge, 2002, pp. xii+544. ISBN: 0-521-79540-0.

[Hir59] Morris W. Hirsch. “Immersions of manifolds”. In: *Transactions of the American Mathematical Society* 93 (1959), pp. 242–276. ISSN: 0002-9947.

[Hir94] Morris W. Hirsch. *Differential topology*. Vol. 33. Graduate Texts in Mathematics. Corrected reprint of the 1976 original. Springer-Verlag, New York, 1994, pp. x+222. ISBN: 0-387-90148-5.

[Mas60] W. S. Massey. “On the Stiefel-Whitney classes of a manifold”. In: *American Journal of Mathematics* 82 (1960), pp. 92–102. ISSN: 0002-9327.

[May99] J. P. May. *A concise course in algebraic topology*. Chicago Lectures in Mathematics. University of Chicago Press, Chicago, IL, 1999, pp. x+243. ISBN: 0-226-51182-0.

[Mil57] J.W. Milnor. *Lectures on characteristic classes*. mimeographed notes by James Stasheff. Princeton University Press, Princeton, N. J., 1957, p. 144.

[Mil67] R. James Milgram. “Immersing projective spaces”. In: *Annals of Mathematics. Second Series* 85 (1967), pp. 473–482. ISSN: 0003-486X.

[MM68] M. Mahowald and R. James Milgram. “Embedding real projective spaces”. In: *Annals of Mathematics. Second Series* 87 (1968), pp. 411–422. ISSN: 0003-486X.

[MS74] John W. Milnor and James D. Stasheff. *Characteristic classes*. Annals of Mathematics Studies, No. 76. Princeton University Press, Princeton, N. J.; University of Tokyo Press, Tokyo, 1974, pp. vii+331.

[MT68] Robert E. Mosher and Martin C. Tangora. *Cohomology operations and applications in homotopy theory*. Harper & Row, Publishers, New York-London, 1968, pp. x+214.

[San64] B. J. Sanderson. “Immersions and embeddings of projective spaces”. In: *Proceedings of the London Mathematical Society. Third Series* 14 (1964), pp. 137–153. ISSN: 0024-6115.

[Sér00] Raymond Séroul. *Programming for mathematicians*. Universitext. Translated from the 1995 French original by Donal O’Shea. Springer-Verlag, Berlin, 2000, pp. xvi+429. ISBN: 3-540-66422-X.

[Ser12] Jean-Pierre Serre. “Cohomology modulo 2 of Eilenberg-MacLane complexes”. In: *Topological Library Part 3: Spectral Sequences in Topology*. Ed. by S. P. Novikov and I. A. Taimanov. Trans. French by V. P. Golubyatnikov. Vol. 50. Series on Knots and Everything. translation of [Ser53]. World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2012, pp. x+576. ISBN: 978-981-4401-30-2.

[Ser53] Jean-Pierre Serre. “Cohomologie modulo 2 des complexes d’Eilenberg-MacLane”. In: *Commentarii Mathematici Helvetici* 27 (1953), pp. 198–232. ISSN: 0010-2571.

[Sko08] Arkadiy B. Skopenkov. “Embedding and knotting of manifolds in Euclidean spaces”. In: *Surveys in contemporary mathematics*. Vol. 347. London Math. Soc. Lecture Note Ser. Cambridge Univ. Press, Cambridge, 2008, pp. 248–342.

[Spr05] David Spring. “The golden age of immersion theory in topology: 1959–1973. A mathematical survey from a historical perspective”. In: *American Mathematical Society. Bulletin. New Series* 42.2 (2005), pp. 163–180. ISSN: 0273-0979.

[Ste62] N. E. Steenrod. *Cohomology operations*. Lectures by N. E. Steenrod written and revised by D. B. A. Epstein. Annals of Mathematics Studies, No. 50. Princeton University Press, Princeton, N.J., 1962, pp. vii+139.

[Sto68] Robert E. Stong. *Notes on cobordism theory*. Mathematical notes. Princeton University Press, Princeton, N.J.; University of Tokyo Press, Tokyo, 1968, pp. v+354+lvi.

[Swi02] Robert M. Switzer. *Algebraic topology—homotopy and homology*. Classics in Mathematics. Reprint of the 1975 original [Springer, New York; MR0385836 (52 #6695)]. Springer-Verlag, Berlin, 2002, pp. xiv+526. ISBN: 3-540-42750-3.

[Tho07] René Thom. “Some “global” properties of differentiable manifolds”. In: *Topological Library Part 1: Cobordisms and Their Applications*. Ed. by S. P. Novikov and I. A. Taimanov. Trans. French by V.O. Manturov. Vol. 39. Series on Knots and Everything. translation of [Tho54] with M.M. Postnikov’s comments (1958). World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2007, pp. 131–209. ISBN: 978-981-270-559-4.

[Tho54] René Thom. “Quelques propriétés globales des variétés différentiables”. In: *Commentarii Mathematici Helvetici* 28 (1954), pp. 17–86. ISSN: 0010-2571.

[Whi44] Hassler Whitney. “The singularities of a smooth n-manifold in $(2n-1)$-space”. In: *Ann. of Math. (2)* 45 (1944), pp. 247–293. ISSN: 0003-486X.

[Wu50] Wen-tsün Wu. “Classes caractéristiques et i-carrés d’une variété”. In: *C. R. Acad. Sci. Paris* 230 (1950), pp. 508–511.

Eigenständigkeitserklärung

Hiermit erkläre ich, Gesina Schwalbe, geboren am 29.01.1996 in Aachen, dass

- die vorgelegten Druckexemplare und die vorgelegte elektronische Version der Arbeit identisch sind,
- ich die Arbeit selbständig verfasst und keine anderen als die angegebenen Quellen und Hilfsmittel benutzt habe und
- ich die Arbeit nicht bereits an einer anderen Hochschule zur Erlangung eines akademischen Grades eingereicht habe.

Desweiteren bestätige ich hiermit, dass ich von den in § 26 Abs. 6 Prüfungsordnung[1] vorgesehenen Rechtsfolgen Kenntnis habe.

Regensburg, den October 4, 2018 Gesina Schwalbe

[1] Prüfungs- und Studienordnung für den Masterstudiengang Mathematik an der Universität Regensburg (Fassung vom 14.10.2015)