Gauge Fields, Knots and Gravity Notes

Part 1: Electromagnetism

Summer 2019

1 Maxwell's Equations

1.1 Gauge theories, the Standard Model, and Quantum Gravity

We begin with Maxwell's theory of electromagnetism which paved the way for tremendous progress in understanding the basic particles and forces that constitute our world. Einstein, Weyl, Kaluza, and Klein were unsuccessful in creating a unified description of the forces of nature using ideas from geometry: physics at small distances (or equivalently, high energies) are dominated by *quantum theory*.

The *Standard Model* concerns three forces: electromagnetism and the weak and strong nuclear forces. These are **gauge fields**, meaning that they are described by equations closely modelled after Maxwell's equations. We can think of forces as being carried by particles: electromagnetism (photons), weak force (W and Z particles), and strong force (gluons).

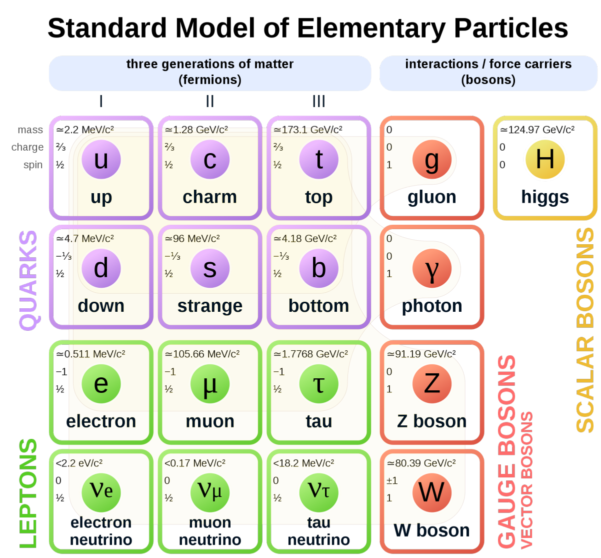

Figure 1: Source: Wikipedia

Quarks (influenced by strong force) and leptons are charged. Most of the matter we see is made up of first-generation fermions: proton (up, up, down), neutron (down, down, up).

Where does gravity fit into the picture? quantum gravity effects may become significant at distances scales comparable to the **Planck length**,

$$l_p = (\hbar\kappa/c^3)^{1/2},$$

where $\hbar$ is Planck's const, κ is Newton's gravitational constant, and c is the speed of light.

Gauge theories involve the study of knots in 3-dimensional space, which is why we're here!

1.2 Unpacking differential geometry in Maxwell's equations

Maxwell's Equations

$$\nabla \cdot \vec{B} = 0 \tag{1}$$

$$\nabla \times \vec{E} + \frac{\partial \vec{B}}{\partial t} = 0 \tag{2}$$

$$\nabla \cdot \vec{E} = \rho \tag{3}$$

$$\nabla \times \vec{B} - \frac{\partial \vec{E}}{\partial t} = \vec{J} \tag{4}$$

Lorentz force law

$$\vec{F} = q(\vec{E} + \vec{v} \times \vec{B}) \tag{5}$$

The dependence of sign on the right hand rule convention is an important clue as to the mathematical structure of Maxwell's equations. Also, notice that there are two natural pairs: equations 1 & 2, and 3 & 4.

The minus sign in Ampere's Law shows a symmetry called **duality**. If we introduce a complex valued vector field then duality amounts to the transformation:

$$\vec{\mathcal{E}} = \vec{E} + i\vec{B}$$
$$\vec{\mathcal{E}} \to -i\vec{\mathcal{E}}$$

and Maxwell's equations **in vacuum** become:

$$\nabla \cdot \vec{\mathcal{E}} = 0, \qquad \nabla \times \vec{\mathcal{E}} = i\frac{\partial \vec{\mathcal{E}}}{\partial t}$$

The symmetry of and does not extend to non-vacuum Maxwell's equations.

1.3 Connections to special relativity

Special relativity bridges the symmetries of space (translations and rotations) and time (translations) with Lorentz transformations. These coordinate transformations can be expressed in terms of **rapidity**, defined such that $\tanh\phi = v$. To see the transformation matrix, click here. Rapidity can be thought of as the hyperbolic angle that differentiates two frames of reference in relative motion, each frame being associated with distance and time coordinates.

Maxwell's equations are invariant under Lorentz transformations, i.e. if we have a solution to Maxwell's equations and do a Lorentz transformation on the coordinates together with a certain transformation on $\vec{E}, \vec{B}, \rho$ and $\vec{J}$, we again have a solution.

new version of Maxwell's equations

$$dF = 0 \tag{6}$$

$$\star\, d \star F = J \tag{7}$$

Where:

$$\begin{aligned} F &= \text{electromagnetic field} \\ J &= \text{current} \end{aligned}$$

d and $\star$ are a slick way of summarizing all the curls, divergences, and time derivatives. Equation 6 is equivalent to equations 1 and 2, and equation 7 is equivalent to equations 3 and 4.

2 Manifolds