Applied Data Science with Python

Linear Algebra

Learning Objectives

By the end of this lesson, you will be able to:

- Learn about the concepts of scalars and vectors and evaluate their linear independence
- Perform basic matrix operations
- Examine how to transpose a matrix and get its rank
- Apply the principles of linear algebra
- Examine the application of calculus in linear algebra

Business Scenario

ABC is a company that develops machine-learning models. It is dealing with high-dimensional data, which makes selecting the appropriate hyperparameters challenging.

To address this, the analysts in the company intend to employ linear algebra techniques to explore basic matrix operations, such as finding the dot product, sum, difference, rank, index, and determinant, to reduce the dimensions of the data. In addition, they aim to delve into the application of calculus in linear algebra and gain a deeper understanding of concepts such as scalars and vectors.

By doing so, they will be able to properly pick the correct hyperparameters and increase the performance of their machine-learning models.

Introduction to Linear Algebra

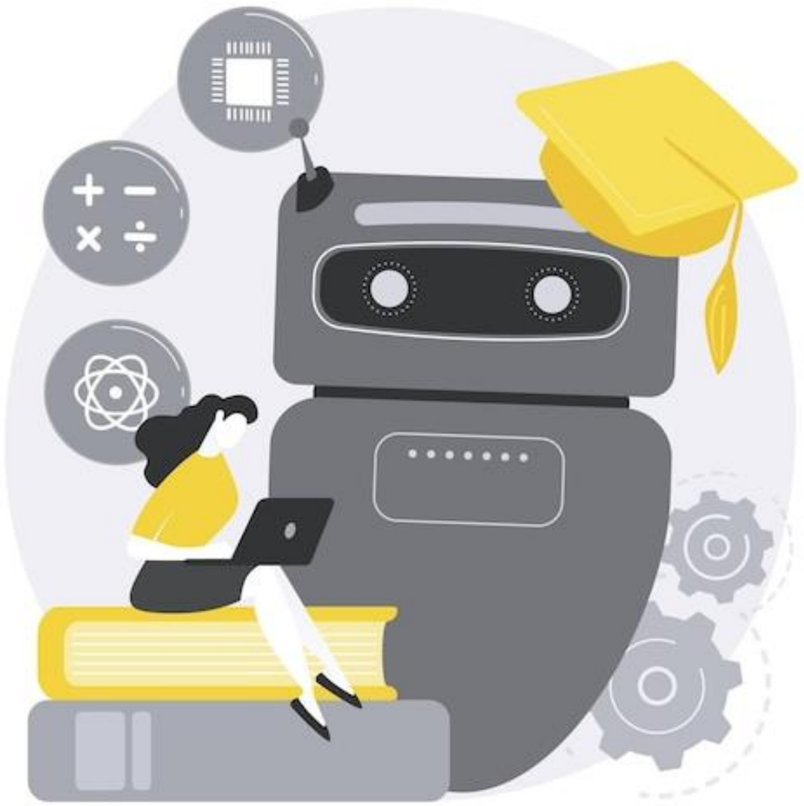
Discussion: Linear Algebra

Duration: 10 minutes

- What is linear algebra?
- What is the role of linear algebra in machine learning?

Introduction to Linear Algebra

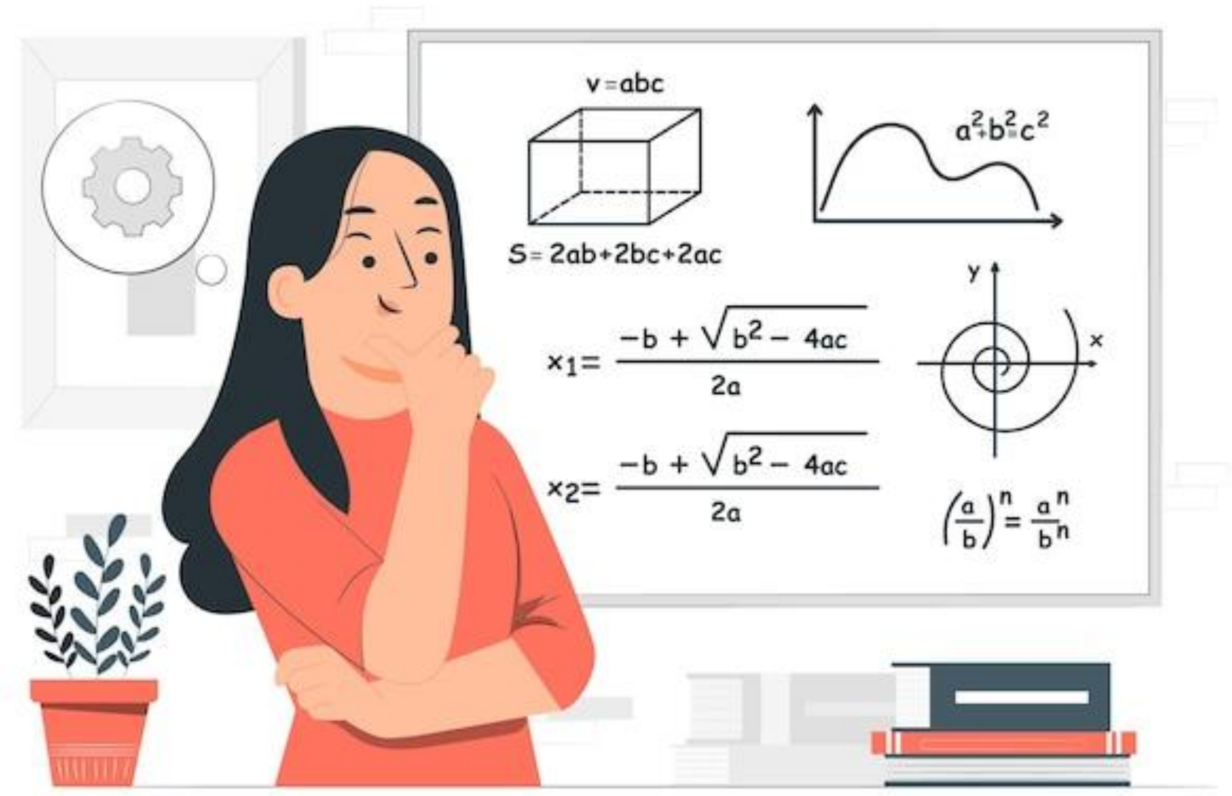
Linear algebra is essential for a deep understanding of machine learning.



It’s a branch of mathematics that deals with linear equations, vectors, and matrices.

Data in Linear Algebra

Data in linear algebra is represented by linear equations.



Linear equations are represented by matrices and vectors.

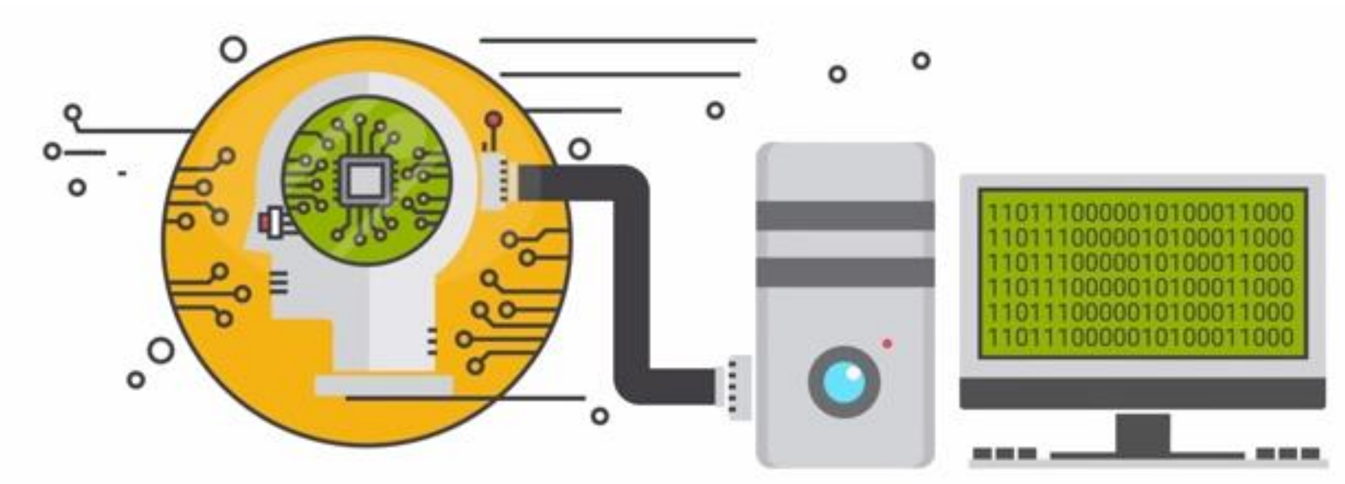
Data in Linear Algebra

Matrices and vectors simplify the process of representing large amounts of information.

A matrix consists of rows and columns of numbers, variables, or expressions.

Linear Algebra in Machine Learning

It is important to reduce the dimensions of data or choose the right hyperparameters when building machine learning models.



In such cases, notations and the formalization of linear algebra can help describe and execute complex operations used in machine learning.

Essential Parts of Linear Algebra

Understanding of the following linear algebra principles is useful for machine learning practitioners

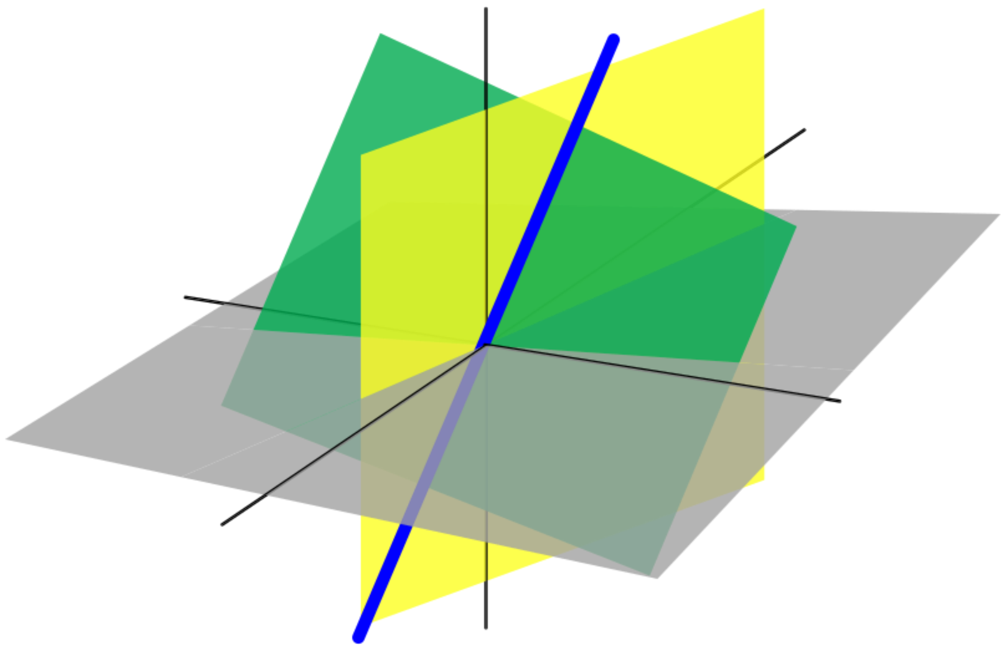
Notation

Operations

Matrix factorization

Notation

Clear notation can simplify the understanding of algorithms presented in papers and books.

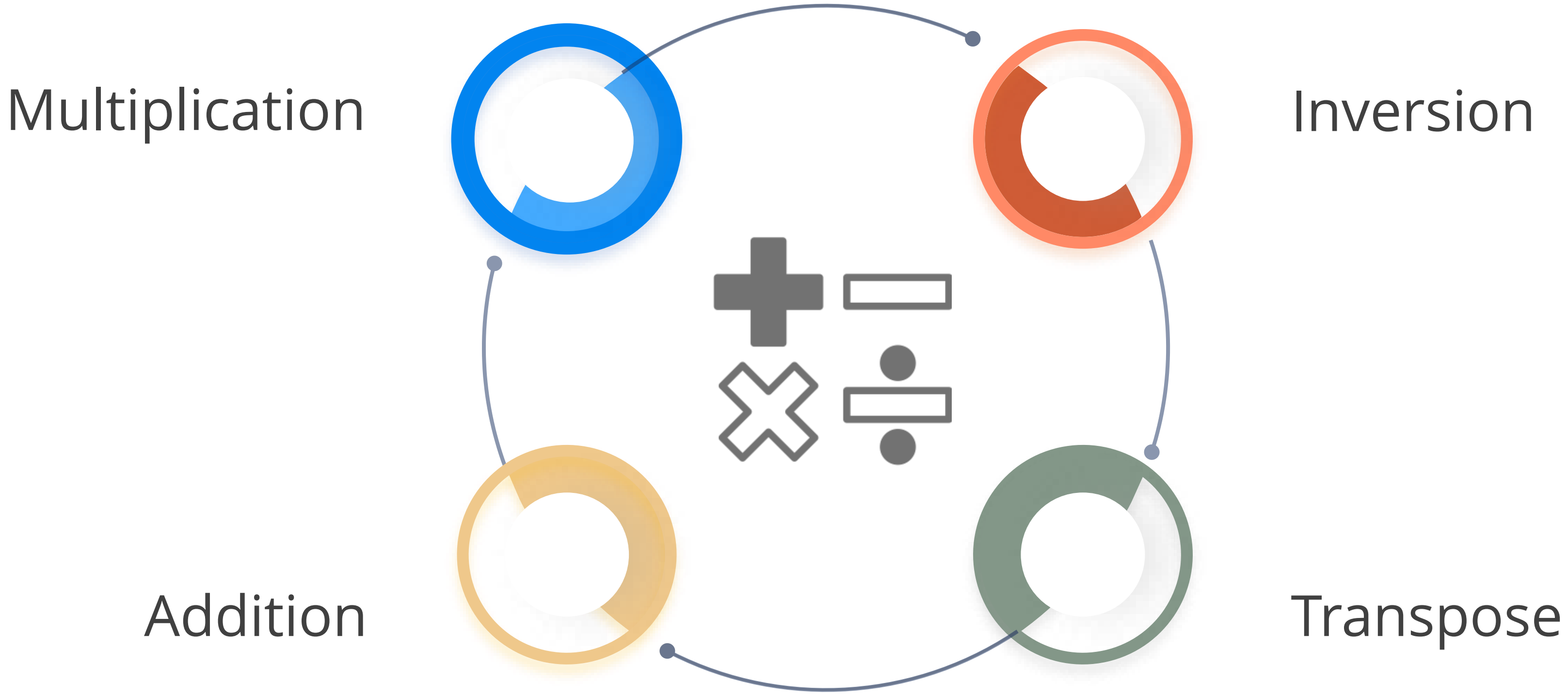


This is true even while reading Python code.

Operations

It is helpful to understand operations when working with vectors and matrices.

Some common operations:



Matrix Factorization

Matrix factorization is essential for machine learning, and it is the decomposition of a matrix into product of two or three matrices.

Regression algorithms can be simplified using matrix decomposition methods like:

- Singular value decomposition (SVD)
- QR decomposition

Discussion: Linear Algebra

Duration: 10 minutes

- What is linear algebra?

Answer: Linear algebra, a fundamental branch of mathematics, focuses on linear equations, vectors, and matrices. It forms an integral part of understanding machine learning at a deep level.

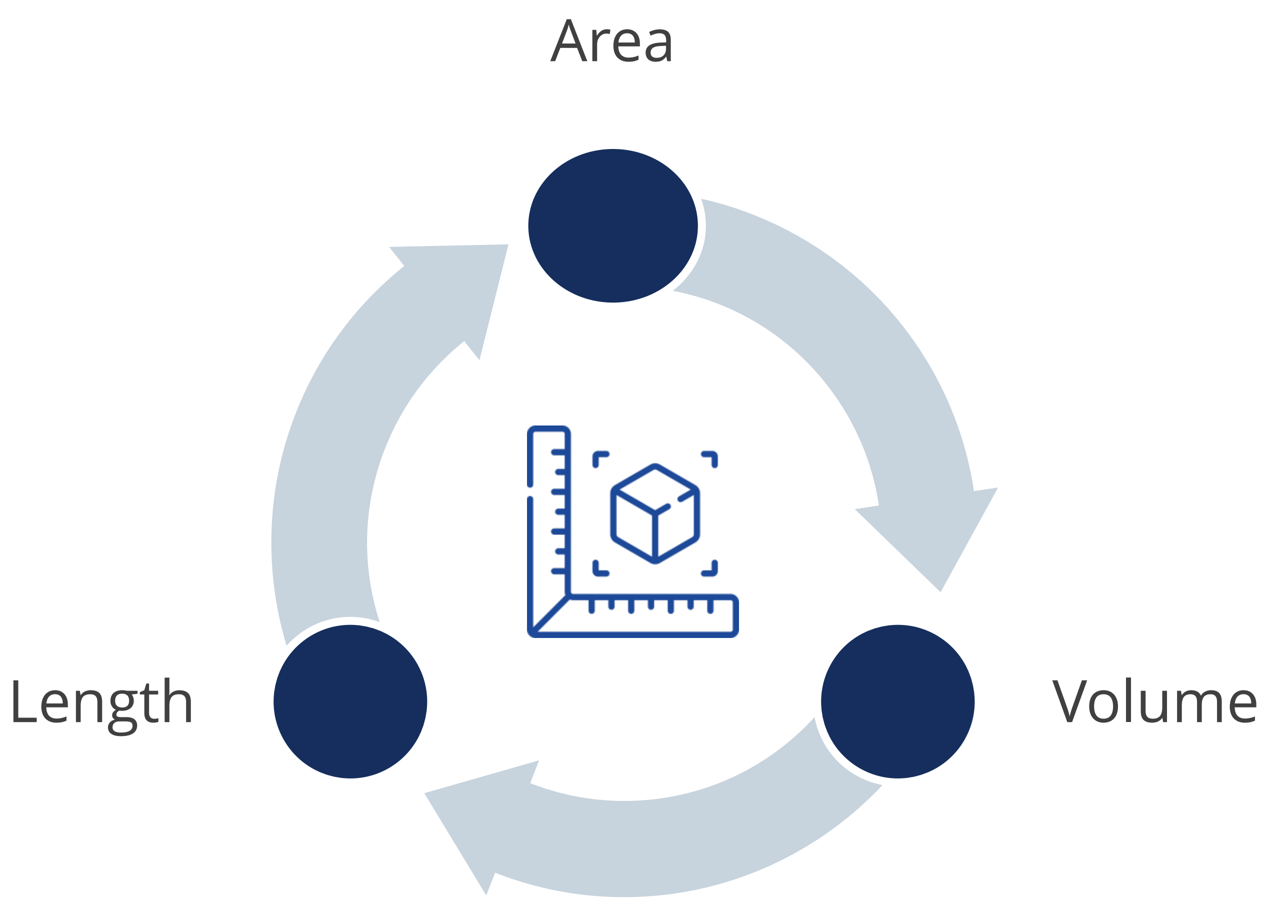
- What is the role of linear algebra in machine learning?

Answer: In the context of machine learning, linear algebra plays a key role in data dimension reduction and optimal hyperparameter selection. These steps are crucial for building effective machine-learning models.

Scalars and Vectors

Scalars

A scalar is a measurable quantity that is entirely characterized by its magnitude.



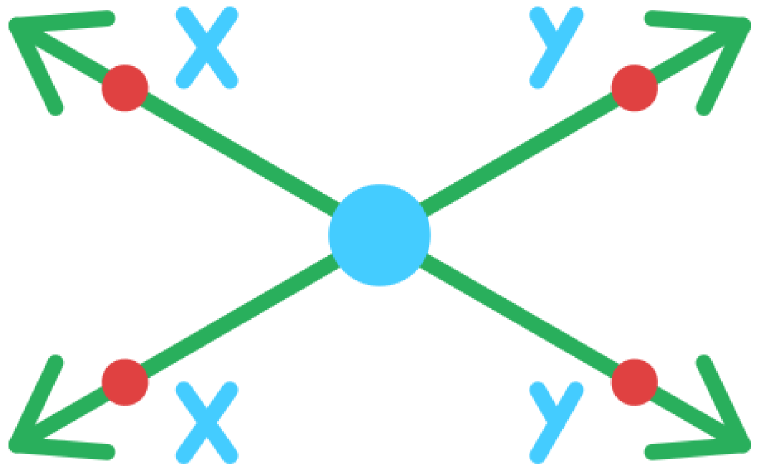
Vectors

A vector is an object that has both magnitude and direction.

Example: A wind velocity of 15 km/h towards the northeast has both speed and direction.

Vectors

Linear algebra studies vectors.



It is often represented by an arrow with the same direction as the quantity and a length proportional to the magnitude of the quantity.

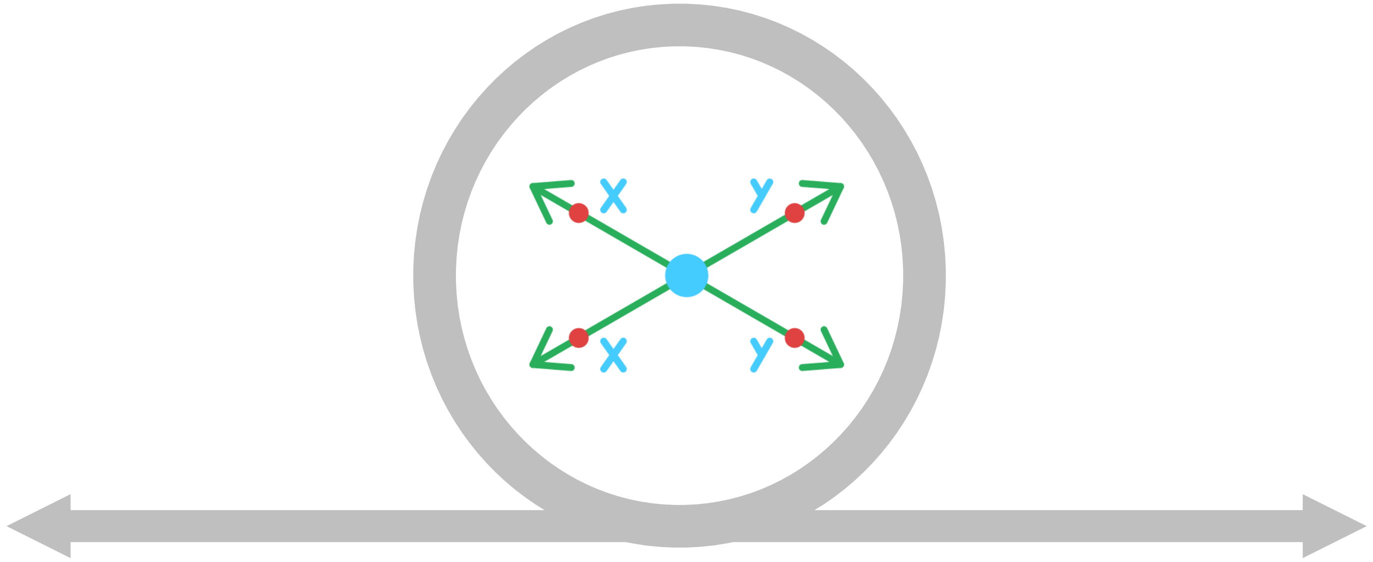
Vectors are ordered lists of finite numbers.

Vectors are the most fundamental mathematical objects in machine learning.

Vectors are used to represent attributes of entities such as income and test scores.

Vectors

Vectors are represented by:



$\mathbf{x}$

$\vec{x}$

Bold lowercase letter like **x**

Lowercase letter with an arrow on top like $\vec{x}$

Vectors

Vectors can be multiplied or added together to create another object of the same type.

Adding vectors	Multiplying a vector by a constant
First vector **x** = speed of vehicle A Second vector **y =** speed of vehicle B Third vector **z = x + y**	**Multiply 2 by X to obtain 2X**

The result will always be a vector.

Dot Product of Two Vectors

Dot Product of Two Vectors

The dot product of two vectors is the sum of the products of the corresponding elements of the two vectors.

$$x.y = x_1y_1+x_2y_2+x_3y_3+\ldots\ldots\ldots+x_ny_n$$

The result is a scalar. The dot product of two vectors is also referred to as the scalar product of two vectors.

Dot Product of Two Vectors

In vector algebra, if two vectors are given as:

$\vec{x} = [x_1, x_2, x_3, x_4, \ldots\ldots, x_n]$

$\vec{y} = [y_1, y_2, y_3, y_4, \ldots\ldots, y_n]$

Then, the dot product is:

$x.y = x_1y_1 + x_2y_2 + x_3y_3 + \ldots\ldots\ldots + x_ny_n$

$$\vec{x}.\vec{y} = \Sigma_{i=1}{}^{n}\, x_i \cdot y_i$$

Dot Product of Two Vectors

The dot product of two vectors is given by multiplying their corresponding elements and adding the individual results.

$$\vec{x} = \begin{bmatrix} 5 \\ 6 \\ -7 \end{bmatrix}$$

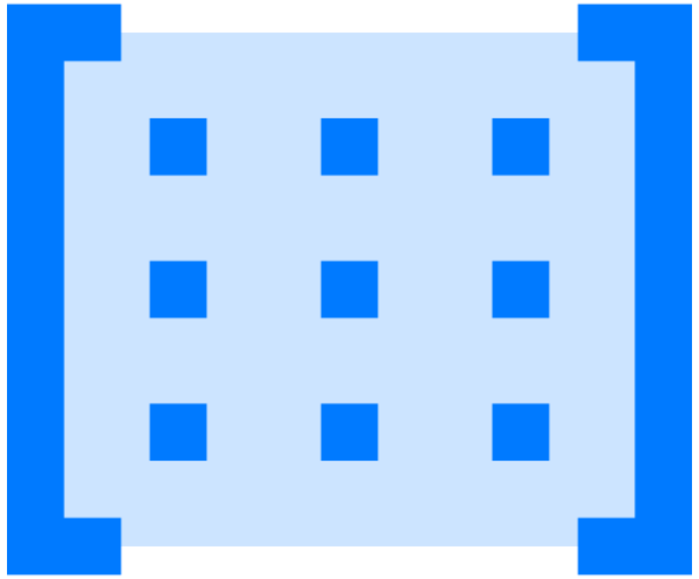
$$\vec{y} = \begin{bmatrix} -1 \\ 2 \\ -5 \end{bmatrix}$$

$$\vec{x} \,.\, \vec{y} = \begin{bmatrix} 5*-1 \\ 6*2 \\ -7*-5 \end{bmatrix} = -5+12+35 = 42$$

Therefore, $\vec{x} \,.\, \vec{y}$ = (5*-1) + (6*2) + (-7*-5) = -5+12+35 = 42

Matrix Representation of Dot Product

The dot product of vectors can be easily computed if the vectors are represented by row or column matrices.



The first matrix is transposed to get a row vector.

It is then multiplied with the second column matrix to get the dot product.

Matrix Representation of Dot Product

Example:

$$\vec{x} = \begin{bmatrix} x1 \\ x2 \\ x3 \end{bmatrix} \qquad \vec{y} = \begin{bmatrix} y1 \\ y2 \\ y3 \end{bmatrix} \qquad \overline{xT} = \begin{bmatrix} x1 & x2 & x3 \end{bmatrix}$$

$$\vec{x} \,.\, \vec{y} = \begin{bmatrix} x1 & x2 & x3 \end{bmatrix} \begin{bmatrix} y1 \\ y2 \\ y3 \end{bmatrix} = x1y1 + x2y2 + x3y3$$

Dot product of $\vec{x}$ and $\vec{y}$ is $\vec{x} . \vec{y}$ = x1y1 + x2y2 + x3y3.

Python Code for Dot Product

An example of Python code for the dot product can be seen below:

```
def dot_product(x,y):
       return sum(i*j for i,j in
zip(x,y,strict = True))
```

Ensures that both the vectors have the same length

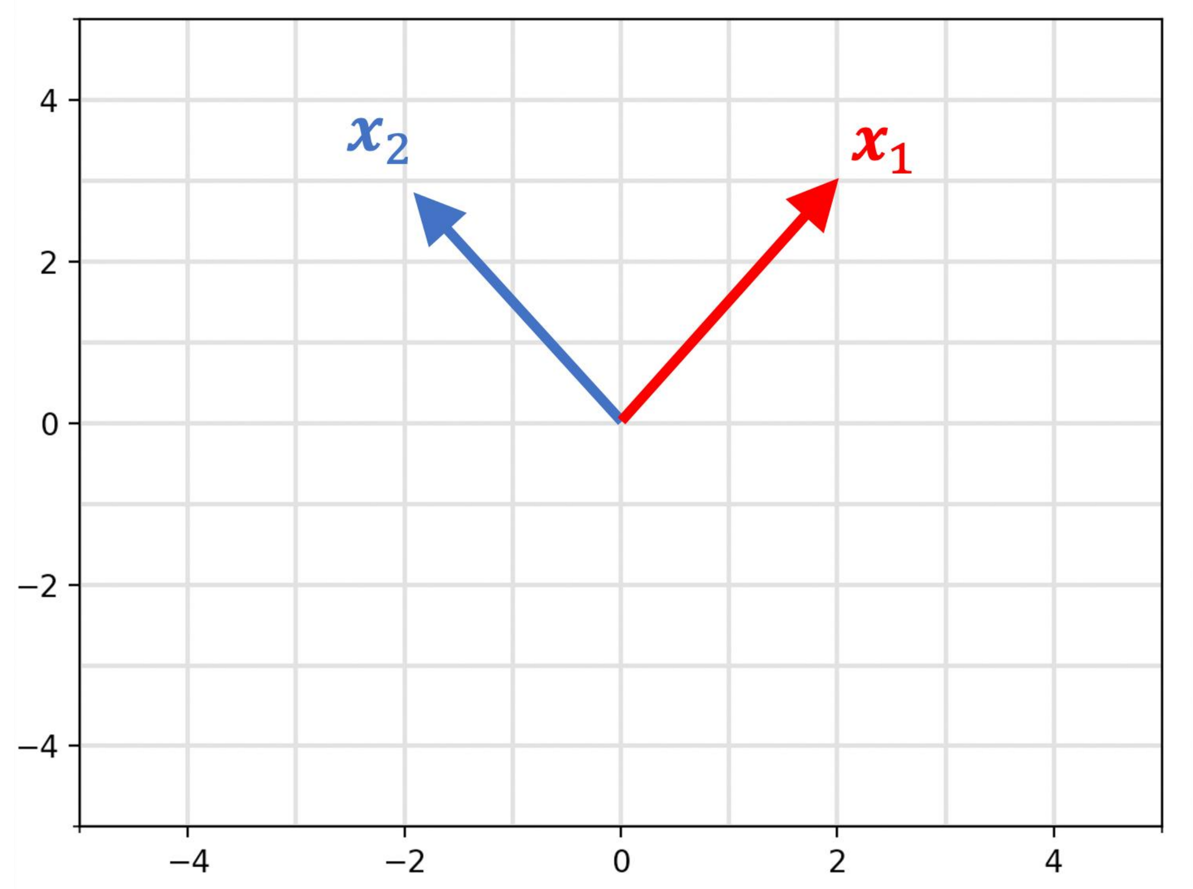
```
dot_product([3,2,6],[1,7,-2])

# Output:
5
```

Linear Independence of Vectors

Linear Independence of Vectors

A set of vectors is said to be linearly independent if no vector in the set can be expressed as a linear combination of the other vectors in the same set.



Otherwise, the vectors are linearly dependent.

Linear Independence of Vectors

A set of vectors $V_1, V_2, V_3, \ldots\ldots, V_n$ is linearly dependent if there are scalars $C_1, C_2, C_3, \ldots\ldots, C_n$, at least one of which is not zero.

$$C_1V_1 + C_2V_2 + C_3V_3 + \ldots\ldots\ldots + C_nV_n = 0$$

A set of vectors that doesn't satisfy the above condition is called linearly independent.

Numerical example: These three vectors are linearly dependent, as at least one in this case is non-zero.

$$\begin{pmatrix} 2 \\ 6 \end{pmatrix} - 2\begin{pmatrix} 1 \\ 3 \end{pmatrix} + 0\begin{pmatrix} 4 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Norm of a Vector

Discussion: Norm of a Vector

Duration: 10 minutes

- What is the norm of a vector?
- How is a matrix defined?

Norm of a Vector

The norm of a vector is its length and is denoted by $\|v\|$.

$$\|v\| = \sqrt{v \cdot v}$$

The norm of a vector can be calculated using the dot product with itself. Then take the square root of the result.

Norm of a Vector

A numerical example of the norm of a vector can be seen below:

$$V = \begin{bmatrix} 2 \\ 5 \\ -3 \end{bmatrix}$$

$$v.v = 2*2 + 5*5 + (-3*-3) = 4 + 25 + 9 = 38$$

$$\|v\| = \sqrt{v \cdot v} = \sqrt{38} = 6.16$$

Python Code for Norm of a Vector

The norm of a vector, by definition, is the square root of its dot product with itself.

```
import math
Import numpy as np

def norm_vector(v):
   numpy.linalg.norm
   dot_product = sum(i*i for i in v)
   return math.sqrt(dot_product)
```

The elements of the vector are fed into the system.

The system then calculates the square of every element.

New elements are added to get the dot product.

The system displays the result's square root (norm) as the output.

Matrix

Matrix

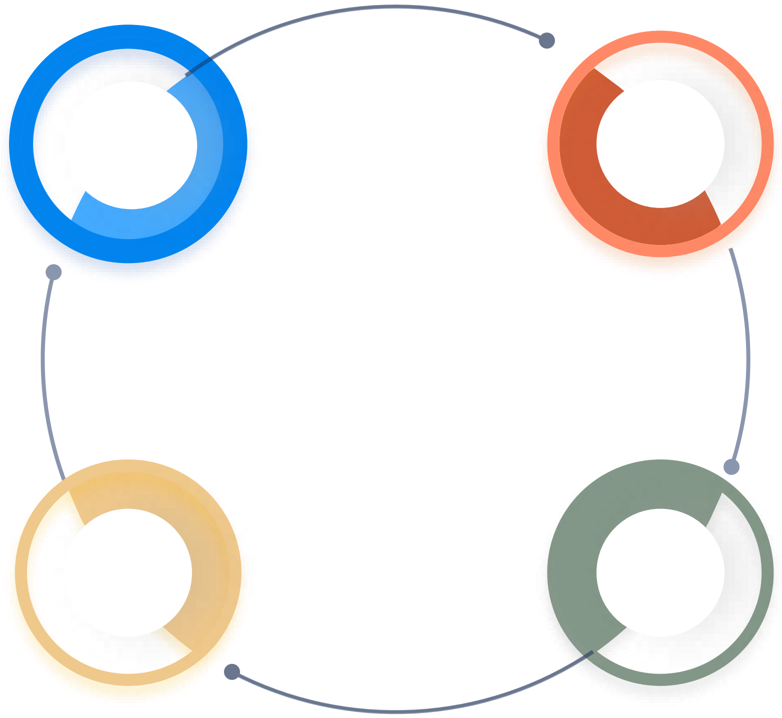
A matrix is a rectangular array of numbers or expressions arranged into columns and rows.

$$\begin{pmatrix} 1 & 2 & 2 \\ 2 & 5 & 7 \\ 2 & 7 & 9 \end{pmatrix} \qquad \begin{pmatrix} 7 & 4 & 2 \\ 6 & 8 & 1 \\ 3 & 7 & 8 \end{pmatrix}$$

It is used to represent a mathematical object or a property of the object.

Matrix Operations

Matrix operations include arithmetic operations such as addition, subtraction, and multiplication of matrices.



Multiplication of matrices

Addition of matrices

Scalar multiplication of matrices

Subtraction of matrices

Matrix operations are useful for combining two or more matrices into a single matrix.

Broadcasting Rules

Familiarizing oneself with broadcasting rules is crucial before delving into matrix operations.

- Broadcasting handles arrays with different shapes during arithmetic operations.
- It enables the smaller array to be broadcast across the larger array for shape compatibility.
- Broadcasting vectorizes array operations, improving computational efficiency by utilizing C loops.

Adding Matrices

If $X[a_{ij}]$ and $Y[b_{ij}]$ are *mXn* matrices, their sum X+Y is the *mXn* matrix obtained by adding the corresponding elements.

$$X+Y = [a_{ij}+b_{ij}]$$

Two matrices of different orders cannot be added.

Matrix Addition

Example: Adding two 3X3 matrices X and Y

$$X = \begin{pmatrix} 2 & -1 & 5 \\ 10 & 3 & 1 \\ -7 & 0 & 3 \end{pmatrix}$$

$$Y = \begin{pmatrix} 3 & 2 & 9 \\ -7 & -7 & 6 \\ 1 & 2 & -4 \end{pmatrix}$$

$$X + Y = \begin{pmatrix} 2+3 & -1+2 & 5+9 \\ 10+(-7) & 3+(-7) & 1+6 \\ -7+1 & 0+2 & 3+(-4) \end{pmatrix} = \begin{pmatrix} 5 & 1 & 14 \\ 3 & -4 & 7 \\ -6 & 2 & -1 \end{pmatrix}$$

Matrix Addition

The Python implementation for the addition of two matrices can be seen below:

```
def matrix_addition(x,y):
xrows = len(x)
xcols = len(x[0])
yrows = len(y)
ycols = len(y[0])
if xrows!=yrows or xcols!=ycols:
     print("Sum is not defined as the
matrices have different orders")
else:
     z = x
     for i in range(xrows):
         for j in range(xcols):
             z[i][j] = z[i][j] + y[i][j]
     return z
```

```
matrix_addition([[1,2,5],[3,4,1]],[[5,1,2
],[9,3,4]])

# Output:
[[6,3,7],[12,7,5]]
```

The code takes two matrices of the same order and adds them. If the matrices are of different order, it prints an error code indicating the same.

Broadcasting Rules for Matrix Addition

One can use broadcasting rules and NumPy to perform matrix addition.

```
import numpy as np

# Create matrices with different shapes.
a = np.array([[1, 2, 3],
              [4, 5, 6]])

b = np.array([10, 20, 30])

# Perform matrix subtraction using
broadcasting.
result = a + b
print(result)
```

```
# Output:
                [[11 22 -33]
                 [14 25 36]]
```

Scalar Multiplication

Scalar multiplication is the product of a real number and a matrix.

$$CX = c[a_{ij}] = [ca_{ij}]$$

If X is an *mXn* matrix and c is a scalar, then cX is the *mXn* matrix obtained by multiplying every element of X with c.

Example

$$\mathbf{X} = \begin{pmatrix} 4 & 0 & 3 \\ -7 & 1 & 2 \end{pmatrix}$$

$$\mathbf{2X} = \begin{pmatrix} 8 & 0 & 6 \\ -14 & 2 & 4 \end{pmatrix} \quad \mathbf{\tfrac{1}{2}\,X} = \begin{pmatrix} 2 & 0 & 1.5 \\ -3.5 & 0.5 & 1 \end{pmatrix} \quad \mathbf{-X} = \begin{pmatrix} -4 & 0 & -3 \\ 7 & -1 & -2 \end{pmatrix}$$

Scalar Multiplication

The Python code snippet for scalar multiplication can be seen below:

```
def scalar_multiplication(c,X):
cX = X
for i in range(len(X)):
     for j in range(len(X[0])):
         cX[i][j] = c*cX[i][j]
return cX
```

```
scalar_multiplication(-3,[[2,6,-
1],[2,8,0],[9,8,7]])

# Output:
[[-6,-18,3],[-6,-24,0],[-27,-24,-21]]
```

It takes a scalar value and a matrix as input and gives the resultant matrix.

Subtraction of Two Matrices

Subtraction of matrices involves element-wise subtraction.

If, $X[a_{ij}]$ and $Y[b_{ij}]$ are *mXn* matrices, then:

$$X\text{-}Y = [a_{ij} - b_{ij}]$$

Two matrices of different orders cannot be subtracted.

Subtraction of Two Matrices

Example: Subtracting two 3X2 matrices:

$$X = \begin{pmatrix} 5 & -9 \\ 0 & 2 \\ 0 & 4 \end{pmatrix}$$

$$Y = \begin{pmatrix} 3 & 0 \\ 6 & -5 \\ 2 & 5 \end{pmatrix}$$

$$X - Y = \begin{pmatrix} 5 - 3 & -9 - 0 \\ 0 - 6 & 2 - (-5) \\ 0 - 2 & 4 - 5 \end{pmatrix} = \begin{pmatrix} 2 & -9 \\ -6 & 7 \\ -2 & -1 \end{pmatrix}$$

Subtraction of Two Matrices

The code here takes two matrices and checks if they are of the same order.

```
def matrix_subtraction(x,y):
xrows = len(x)
xcols = len(x[0])
yrows = len(y)
ycols = len(y[0])
if xrows!=yrows or xcols!=ycols:
     print("Difference is not defined as
the matrices have different orders")
else:
     z = x
     for i in range(xrows):
         for j in range(xcols):
             z[i][j] = z[i][j] - y[i][j]
     return z
```

```
matrix_subtraction([[5,2,3],[3,4,-
9]],[[5,3,2],[8,2,4]])

# Output:
[[0,-1,1],[-5,2,-13]]
```

If they are, then it performs a subtraction operation; otherwise, it prints an error message.

Broadcasting Rules for Matrix Subtraction

The broadcasting rules for matrix subtraction are like those for matrix addition.

```
import numpy as np

# Create matrices with different shapes.
a = np.array([[1, 2, 3],
              [4, 5, 6]])

b = np.array([10, 20, 30])

# Perform matrix subtraction using
broadcasting.
result = a - b
print(result)
```

```
# Output:
               [[-9 -18 -27]
                [-6 -15 -24]]
```

Matrix Multiplication

To obtain the product of two matrices, multiply the elements of the rows of the first matrix with the corresponding elements of the columns of the second matrix.

$$\begin{pmatrix} 1 & 2 & 2 \\ 2 & 5 & 7 \\ 2 & 7 & 9 \end{pmatrix} \qquad \begin{pmatrix} 7 & 4 & 2 \\ 6 & 8 & 1 \\ 3 & 7 & 8 \end{pmatrix}$$

Two matrices can be multiplied only if the number of columns in the first matrix is equal to the number of rows in the second matrix.

Matrix Multiplication

If X is an *mXn* matrix and Y is an *nXr* matrix, then their product Z = X Y is an *mXr* matrix.

$$z_{ij} = x_{i1}y_{1j} + x_{i2}y_{2j} + \dots\dots\dots + x_{in}y_{nj}$$

Example: Multiplication of a 2X3 matrix with a 3X2 matrix

$$\mathbf{X} = \begin{pmatrix} 2 & 0 & 5 \\ 8 & -3 & 5 \end{pmatrix}$$

$$\mathbf{Y} = \begin{pmatrix} 5 & 1 \\ 2 & 2 \\ -3 & -8 \end{pmatrix}$$

$$\mathbf{XY} = \begin{pmatrix} (2*5) + (0*2) + (5* -3) & (2*1) + (0*2) + (5* -8) \\ (8*5) + (-3*2) + (5* -3) & (8*1) + (-3*2) + (5* -8) \end{pmatrix} = \begin{pmatrix} -5 & -38 \\ 19 & -38 \end{pmatrix}$$

Matrix Multiplication

The code below takes two matrices and multiplies them if the number of columns in the first matrix is equal to the number of rows in the second matrix.

```
def matrix_multiplication(x,y):
xrows = len(x)
xcols = len(x[0])
yrows = len(y)
ycols = len(y[0])
if xcols!=yrows:
     print("Product is not defined.")
else:
      z = [ [ 0 for i in range(ycols) ] for j in
range(xrows) ]
      for i in range(xrows):
         for j in range(ycols):
              total = 0
              for ii in range(xcols):
                  total += x[i][ii] * y[ii][j]
              z[i][j] = total
      return z
```

```
matrix_multiplication([[1,2,5],[3,4,
1]],[[5,1,2],[9,3,4],[1,5,3]])

# Output:
[[28, 32, 25], [52, 20, 25]]
```

Matrix Multiplication

One can also use the NumPy functions to perform matrix multiplication.

```
import numpy as np

# Create matrices a and b.
a = np.array([[1, 2],
              [3, 4]])

b = np.array([[5, 6],
              [7, 8]])

# Matrix multiplication using numpy.matmul()
result = np.matmul(a, b)

# Print the result.
print(result)
```

```
# Output:
              [[19 22]
               [43 50]]
```

Dot Product of Matrices

The dot product of two matrices is shown below:

$$\begin{bmatrix} a & b \end{bmatrix} \bullet \begin{bmatrix} X \\ y \end{bmatrix} = \begin{bmatrix} ax + by \end{bmatrix}$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \bullet \begin{bmatrix} X \\ y \end{bmatrix} = \begin{bmatrix} ax + by \\ cx + dy \end{bmatrix}$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \bullet \begin{bmatrix} w & x \\ y & z \end{bmatrix} = \begin{bmatrix} aw + by & ax + bz \\ cw + dy & cx + dz \end{bmatrix}$$

Dot Product of Matrices

Using the NumPy function in Python, one can get the dot product as shown below:

```
import numpy as np

# Create vectors a and b.
a = np.array([1, 2, 3])
b = np.array([4, 5, 6])

# Calculate the dot product using numpy.dot().
dot_product = np.dot(a, b)

# Print the dot product.
print(dot_product)
```

```
# Output:
32
```

Discussion: Norm of a Vector

Duration: 10 minutes

- What is the norm of a vector?

Answer: The norm signified by $\|v\|$ denotes the length of a vector. The computation of this value involves taking the square root of the result from the dot product of the vector with itself.

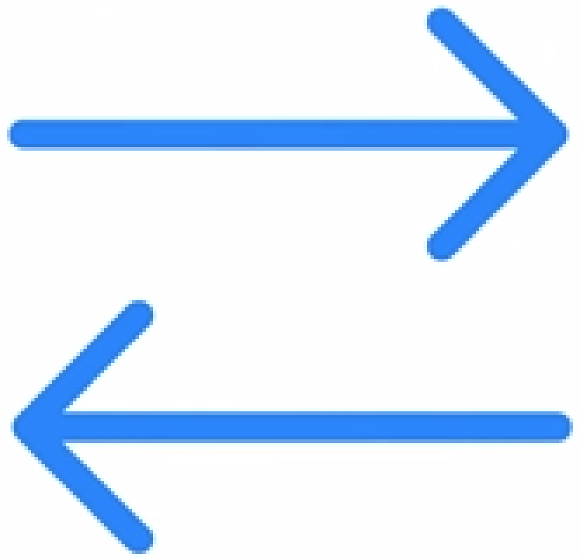
- How is a matrix defined?

Answer: A matrix is a rectangular array of numbers or expressions arranged into columns and rows. It is used to represent a mathematical object or a property of the object.

Transpose of a Matrix

Transpose of a Matrix

A matrix's transpose is obtained by swapping its rows and columns. It is basically the same matrix with flipped axes.



Example: The transpose of an m n matrix X is an n m matrix X^T obtained by interchanging the rows and columns of X.

The i^{th} column of X^T is the i^{th} row of X for all i.

Transpose of a Matrix

Example: X is a 2x3 matrix.

$$\mathbf{X} = \begin{pmatrix} 1 & 9 & -6 \\ 5 & 3 & -7 \end{pmatrix} t$$

$$\mathbf{X}^T = \begin{pmatrix} 1 & 5 \\ 9 & 3 \\ -6 & -7 \end{pmatrix}$$

The transpose of X is X^T, a 3x2 matrix.

Transpose of a Matrix

The Python code below takes a matrix of order *mXn* and finds its transpose.

```
#Using functions to find transpose of a
matrix
def matrix_transpose(x):
   xrows = len(x)
   xcols = len(x[0])
z = [ [ 0 for i in range(xrows) ] for j in
range(xcols) ]
   for i in range(xcols):
         for j in range(xrows):
               z[i][j] = x[j][i]

   return z

matrix_transpose([[1,2,5],[3,5,4]])
```

```
# Output:
[[1,3],[2,5],[5,4]]
```

Transpose of a Matrix

The Python code below takes a matrix of order ***mXn*** and finds its transpose.

```
# Using the NumPy function to find the
transpose

arr = np.array([[1, 2, 3],
                [4, 5, 6]])

transposed_arr = np.transpose(arr)

print("Original array:")
print(arr)

print("Transposed array:")
print(transposed_arr)
```

```
Output:

Original array:
[[1 2 3]
 [4 5 6]]

Transposed array:
[[1 4]
 [2 5]
 [3 6]]
```

Rank of a Matrix

Rank of a Matrix

The rank of a matrix is the maximum number of linearly independent columns (or rows) of a matrix.

Example:

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 4 \\ 3 & 0 & 5 \end{bmatrix}$$

To find the rank of a matrix, first convert it into the row echelon form.

Rank of a Matrix: Example

For a matrix to be in its echelon form, it must follow three rules:

x_1	x_2	x_3
0	x_4	x_5
0	0	x_6

- Any row of all zeros is below nonzero rows.
- All entries in a column below a leading entry are zeros.
- Each leading entry of a row is in a column to the right of the leading entry of the row above it.

Rank of a Matrix: Example

To reduce a matrix to its row echelon form, use the following elementary row operations:

- Interchange two rows
- Multiply a row by a nonzero constant
- Add a multiple of a row to another row

Rank of a Matrix: Example

Example:

$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 4 \\ 3 & 0 & 5 \end{pmatrix}$$

The output, after using elementary transformations, is shown below:

$$\left.\begin{array}{l} R_2 \rightarrow R_2 - 2R_1 \\ R_3 \rightarrow R_3 - 3R_1 \end{array}\right\}$$

Here, R represents a row.

$$\begin{pmatrix} 1 & 2 & 3 \\ 0 & -3 & -2 \\ 0 & -6 & -4 \end{pmatrix}$$

Rank of a Matrix: Example

Solution:

$$\begin{pmatrix} 1 & 2 & 3 \\ 0 & -3 & -2 \\ 0 & -6 & -4 \end{pmatrix}$$

$$R_3 \rightarrow R_3 - 2R_2$$

$$\begin{pmatrix} 1 & 2 & 3 \\ 0 & -3 & -2 \\ 0 & 0 & 0 \end{pmatrix}$$

The above matrix is in row echelon form.

Rank of a Matrix: Example

In a matrix, the leading entry of a row is the first nonzero entry in that row.

$$\begin{pmatrix} 3 & 0 & 5 & -3 & 2 \\ 0 & 2 & 1 & 9 & 0 \\ 0 & 0 & 0 & 7 & 5 \\ 0 & 0 & 0 & 0 & 0 \\ 5 & 2 & 6 & 1 & 10 \end{pmatrix}$$

Apply the elementary row operations to the matrix above to get its echelon form

Rank of a Matrix: Example

The step-wise conversion of the matrix to its echelon form is as follows:

Step 1: Multiply the first row by 1/3

$$\begin{pmatrix} 1 & 0 & -5/3 & 1 & 2/3 \\ 0 & 2 & 1 & 9 & 0 \\ 0 & 0 & 0 & 7 & 5 \\ 0 & 0 & 0 & 0 & 0 \\ 5 & 2 & 6 & 1 & 10 \end{pmatrix}$$

Step 2: Add -5 times the first row to the fifth row

$$\begin{pmatrix} 1 & 0 & -5/3 & 1 & 2/3 \\ 0 & 2 & 1 & 9 & 0 \\ 0 & 0 & 0 & 7 & 5 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 43/3 & -4 & 20/3 \end{pmatrix}$$

Rank of a Matrix: Example

The step-wise conversion of the matrix to its echelon form is as follows:

Step 3: Multiply the second row by half

$$\begin{pmatrix} 1 & 0 & -5/3 & 1 & 2/3 \\ 0 & 1 & 1/2 & 9/2 & 0 \\ 0 & 0 & 0 & 7 & 5 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 43/3 & -4 & 20/3 \end{pmatrix}$$

Step 4: Add -2 times the second row to the fifth row

$$\begin{pmatrix} 1 & 0 & -5/3 & 1 & 2/3 \\ 0 & 1 & 1/2 & 9/2 & 0 \\ 0 & 0 & 0 & 7 & 5 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 40/3 & -13 & 20/3 \end{pmatrix}$$

Rank of a Matrix: Example

The step-wise conversion of the matrix to its echelon form is as follows:

Step 5: Interchange the third row and the fifth row

$$\begin{pmatrix} 1 & 0 & -5/3 & 1 & 2/3 \\ 0 & 1 & 1/2 & 9/2 & 0 \\ 0 & 0 & 40/3 & -13 & 20/3 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 7 & 5 \end{pmatrix}$$

Step 6: Multiply the third row by 3/40

$$\begin{pmatrix} 1 & 0 & -5/3 & 1 & 2/3 \\ 0 & 1 & 1/2 & 9/2 & 0 \\ 0 & 0 & 1 & -39/40 & 1/2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 7 & 5 \end{pmatrix}$$

Rank of a Matrix: Example

The step-wise conversion of the matrix to its echelon form is as follows:

Step 7: Interchange the fourth and fifth row

$$\begin{pmatrix} 1 & 0 & -5/3 & 1 & 2/3 \\ 0 & 1 & 1/2 & 9/2 & 0 \\ 0 & 0 & 1 & -39/40 & 1/2 \\ 0 & 0 & 0 & 7 & 5 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Step 8: Multiply the fourth row by 1/7

$$\begin{pmatrix} 1 & 0 & -5/3 & 1 & 2/3 \\ 0 & 1 & 1/2 & 9/2 & 0 \\ 0 & 0 & 1 & -39/40 & 1/2 \\ 0 & 0 & 0 & 1 & 5/7 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Rank of a Matrix: Example

As the number of nonzero rows in the row echelon form is 4, the rank of the given matrix is 4.

$$\begin{pmatrix} 1 & 0 & -5/3 & 1 & 2/3 \\ 0 & 1 & 1/2 & 9/2 & 0 \\ 0 & 0 & 1 & -39/40 & 1/2 \\ 0 & 0 & 0 & 1 & 5/7 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Determinant of a Matrix and Identity Matrix

Discussion: Determinant of a Matrix and Identity Matrix

Duration: 10 minutes

- What is the determinant of a matrix?
- How is an identity matrix defined?

Determinant of a Matrix

The determinant of a matrix is a scalar quantity that is a function of the elements of the matrix.

$$\begin{pmatrix} 3 & 0 & 5 & -3 & 2 \\ 0 & 2 & 1 & 9 & 0 \\ 0 & 0 & 0 & 7 & 5 \\ 0 & 0 & 0 & 0 & 0 \\ 5 & 2 & 6 & 1 & 10 \end{pmatrix}$$

Determinants are defined only for square matrices.

They are useful in determining the solution of a system of linear equations.

Determinant of a Matrix

Let $X = [a_{ij}]$ be an ***nXn*** matrix, where $n \geq 2$.

det X $= a_{11} \det A_{11} - a_{12} \det A_{12} + \ldots\ldots\ldots\ldots\ldots + (-1)^{1+n} a_{1n} \det A_{1n}$

$= \sum_i =1^n (-1)^{1+i} a_{1i} \det A_{1i}$

Determinant of a Matrix

Apply the definition of the determinant to the following 2x2 and 3x3 matrices:

$$X = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$$|X| = ad - bc$$

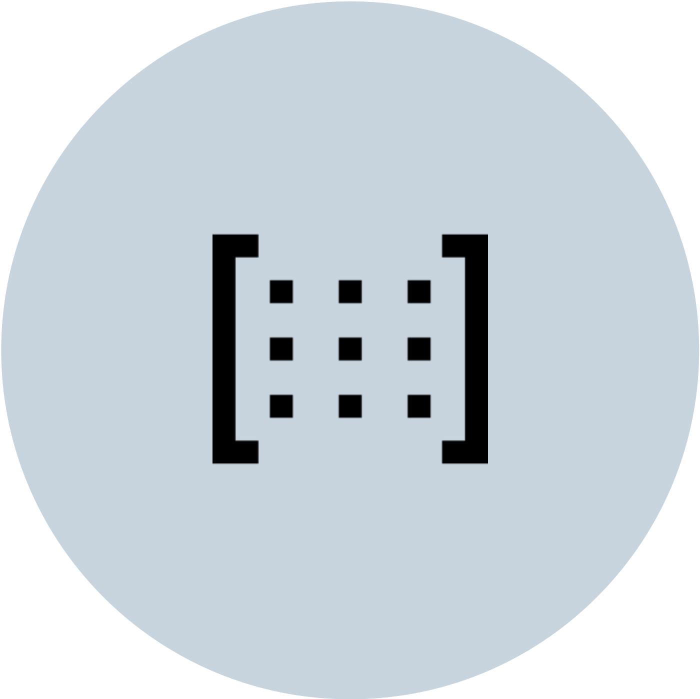
$$X = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$$

$$|X| = a \det\begin{pmatrix} e & f \\ h & i \end{pmatrix} - b \det\begin{pmatrix} d & f \\ g & i \end{pmatrix} + c \det\begin{pmatrix} d & e \\ g & h \end{pmatrix}$$

Substitute the expressions for a determinant of a 2x2 matrix in the above equation:

$$|X| = a\,(ei-fh) - b\,(di-fg) + c\,(dh-eg)$$

Determinant of a Matrix



- If the determinant of a square matrix is 0, then it is not invertible.
- If the determinant of a matrix is nonzero, the linear system it represents is linearly independent.
- When the determinant is zero, its rows are linearly dependent vectors.

Identity Matrix

An identity matrix (I) is a square matrix that, when multiplied with a matrix X, gives the same result as X.

I = Identity matrix

XI = IX = X

```
import numpy as np
# Create a 3x3 identity matrix
identity_matrix = np.identity(3)
print(identity_matrix)
```

In the system of numbers, this corresponds to the number 1.

Identity Matrix

The diagonal elements of I are all 1, and all its non-diagonal elements are 0.

Two-dimensional identity matrix

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Three-dimensional identity matrix

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Discussion: Determinant of a Matrix and Identity Matrix

Duration: 10 minutes

- What is the determinant of a matrix?

Answer: The determinant, a scalar quantity, is a function of a matrix's elements. However, determinants apply only to square matrices.

- How is an identity matrix defined?

Answer: An identity matrix (I), defined as a square matrix, results in the original matrix (X) when it multiplies X.

Inverse of a Matrix, Eigenvalues, and Eigenvectors

Inverse of a Matrix

An ***nXn*** matrix X has an inverse ***nXn*** matrix X^{-1}, such that:

$$X X^{-1} = X^{-1} X = I$$

I is the ***nXn*** identity matrix. If an X^{-1} exists for X, then X is described as invertible.

Inverse of a Matrix

Example:

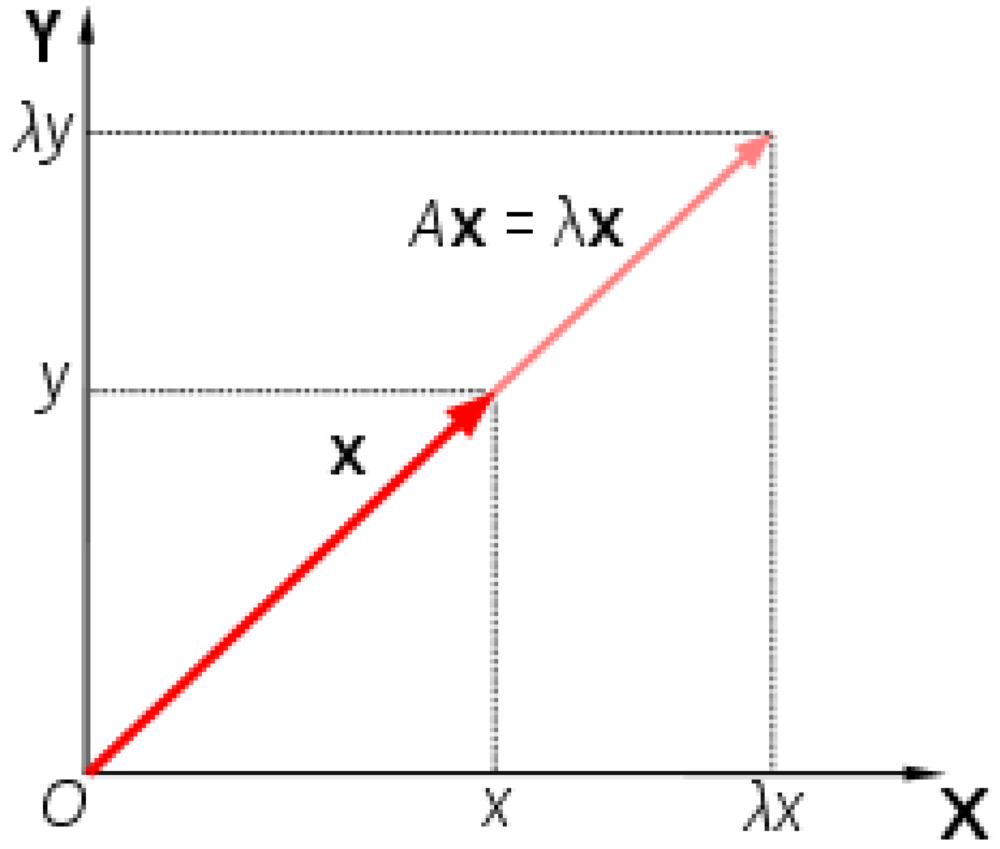
$$X = \begin{pmatrix} 2 & 5 \\ 1 & 3 \end{pmatrix} \qquad X^{-1} = \begin{pmatrix} 3 & -5 \\ -1 & 2 \end{pmatrix}$$

$$XX^{-1} = \begin{pmatrix} 2 & 5 \\ 1 & 3 \end{pmatrix}\begin{pmatrix} 3 & -5 \\ -1 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$X^{-1}X = \begin{pmatrix} 3 & -5 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} 2 & 5 \\ 1 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Eigenvalues and Eigenvectors

Vectors with a specific direction under a specific linear transformation are called eigenvectors.



AX= λX

Where,

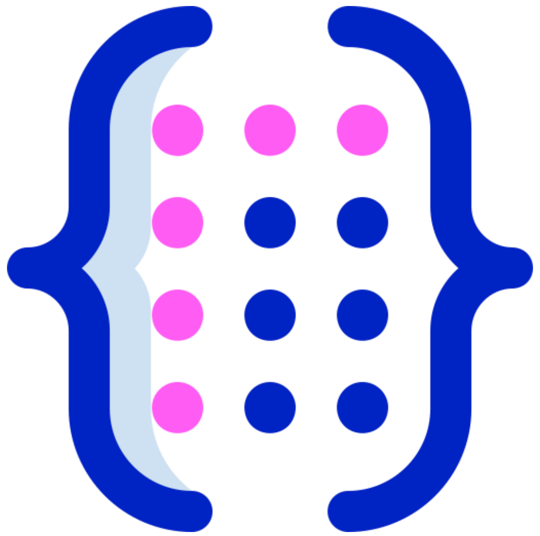
A is a matrix
λ is eigenvector
X is eigenvalue

The scaling factor of eigenvectors is called the eigenvalue.

Eigenvalues and Eigenvectors

When X, an ***nXn*** matrix, is multiplied with vector A, it:

Scales the vector



Rotates the vector

Eigenvalues and Eigenvectors

When X acts on a certain set of vectors, it results in scaling the vector and not changing the direction of the vector.

These vectors are called eigenvectors and make linear transformations simple to grasp.

The amount by which these vectors stretch or compress is called the corresponding eigenvalue.

Eigenvalues and Eigenvectors

If X is an ***nXn*** matrix:

A scalar λ is called an eigenvalue of X if there is a nonzero vector A such that $XA = \lambda A$.

This vector A is called an eigenvector of X corresponding to λ.

Eigenvalues and Eigenvectors

Example: Find the eigenvalues and eigenvectors

Step 1: Consider the X and A values as shown below:

$$\mathbf{X} = \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix} \qquad \mathbf{A} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Step 2: Calculate the XA value, then take out the common value from it

$$\mathbf{XA} = \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 4 \\ 4 \end{pmatrix} = 4 \begin{pmatrix} 1 \\ 1 \end{pmatrix} = 4A$$

Calculus in Linear Algebra

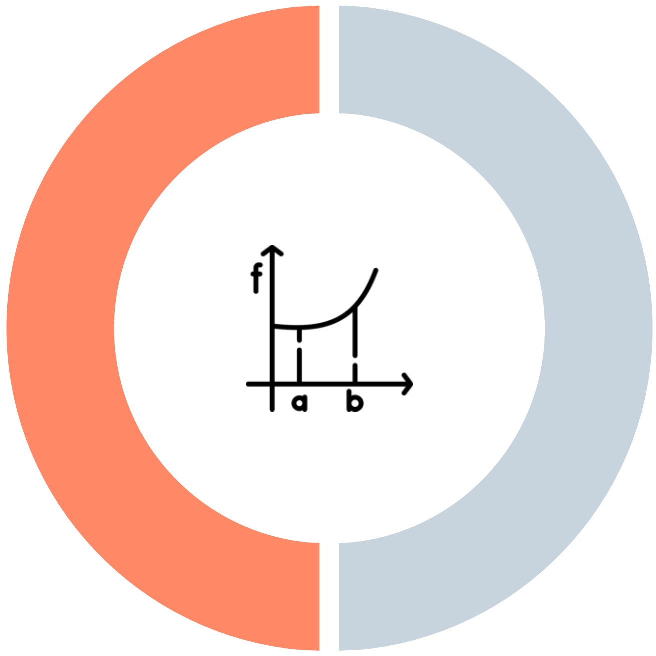
Calculus in Linear Algebra

Calculus is the branch of mathematics that studies continuous changes in quantities, and it deals with finding derivatives and integrals of functions.

- It measures quantities such as the slopes of objects or curves.
- It deals with finding derivatives and integrals of functions.

Calculus in Linear Algebra

Calculus can be divided broadly into two sections:



Differential Calculus

It is used to find instantaneous rates of change and slopes of curves.

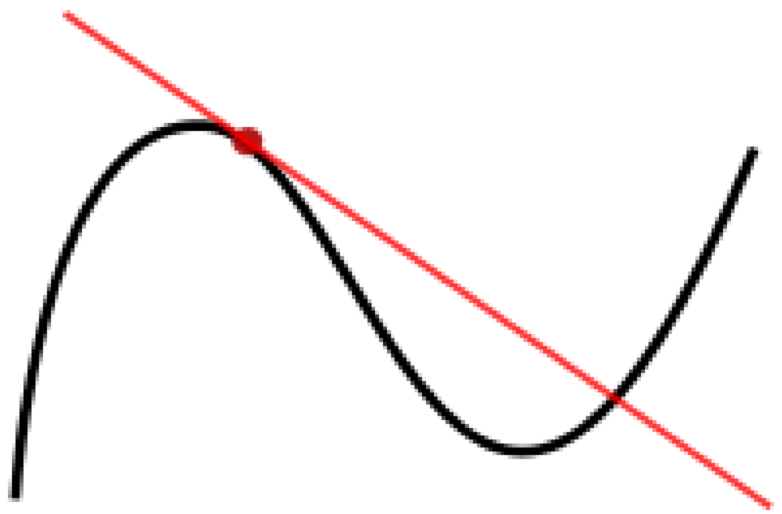
Integral Calculus

It is used to find an accumulation of quantities and the areas under or between curves.

Calculus is necessary for developing an intuition for machine learning algorithms.

Differential Calculus

Differential calculus principles are used in significant machine learning techniques, such as gradient descent.



Gradient descent is vital in the back propagation of neural networks to measure how the output of a function changes when the input is changed by small amounts.

Applications of Differential Calculus

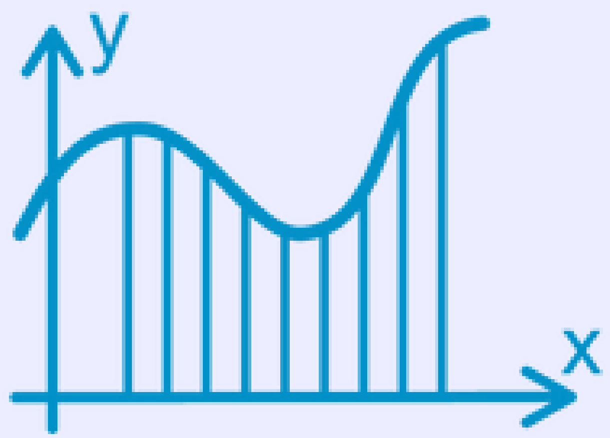
Other applications of differential calculus in machine learning algorithms are:

Finding the maximum margin in support vector machines

Discovering the maximum in the expectation-maximization algorithm

Integral Calculus

Integral calculus is commonly used to determine the probability of events.



Example:

To find the posterior in a Bayesian model or to bound the error in a sequential decision as per the Neyman-Pearson Lemma

Key Takeaways

- Linear algebra is a branch of mathematics that deals with linear equations.
- Matrices and vectors in linear algebra simplify the process of representing large amounts of information and are useful to machine learning practitioners.
- Matrix operations include addition, scalar multiplication, subtraction, and matrix multiplication.
- One can determine the rank, transpose, determinant, and inverse of a matrix.
- Eigen vectors are vectors that are fixed in direction under a given linear transformation, and their scaling factor is called the Eigen value.

Knowledge Check

Knowledge Check 1

If A = [2,-3,7] and B=[-4,2,-4], find the dot product of the vectors A and B?

A. 42

B. -42

C. 12

D. 22

Knowledge Check 1

If A = [2,-3,7] and B=[-4,2,-4], find the dot product of the vectors A and B?

A. 42

B. -42

C. 12

D. 22

The correct answer is **B**

$A.B = 2\,(-4) + (-3)2 + 7\,(-4) = -8 - 6 - 28 = -42$

Knowledge Check

2

What are the essential parts of linear algebra?

A. Notation

B. Operations

C. Matrix multiplication

D. Matrix factorization

Knowledge Check

2

What are the essential parts of linear algebra?

A. Notation

B. Operations

C. Matrix multiplication

D. Matrix factorization

The correct answer are **A,B and D**

Notation, operations, and matrix factorization are the essential parts of linear algebra.

Knowledge Check

3

What is/are true about scalars and vectors?

A. Scalar quantities can be described by specifying only their magnitude

B. Distance is example of vector quantity

C. Vectors are represented using lowercase letter with an arrow on top like $x^{\rightarrow}$

D. Vector quantities require both magnitude and direction

Knowledge Check

3

What is/are true about scalars and vectors?

A. Scalar quantities can be described by specifying only their magnitude

B. Distance is example of vector quantity

C. Vectors are represented using lowercase letter with an arrow on top like $x^{\rightarrow}$

D. Vector quantities require both magnitude and direction

The correct answer is **A,C and D**

Scalar quantities can be described by specifying only their magnitude. Vectors are represented using lowercase letter with an arrow on top and require both magnitude and direction.

Thank You