Problem Set 2a
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Intro

Welcome to my third problem set, my next attempt at LaTeX... **See the next page for proofs!**

The problems:

1. Prove the following using the definition of sequence convergence.

   (a) Let $a_n = 7 - \frac{1}{\sqrt{n}}$. Show that $a_n \to 7$ as $n \to \infty$.

      i. Noting a_n to be a sum of sequences of reals, we define:
      $b_n = 7$, $b_n \to b$ as $n \to \infty$,
      $c_n = \frac{1}{\sqrt{n}}$, $c_n \to c$ as $n \to \infty$.

      Showing that b_n and c_n converge to their respective limits will allow us to apply limit law (2) from Theorem 2.10 in "Real Analysis Notes."

      A. Proof: b_n converges to $b = 7$.
      (Let: $n \in \mathbb{N}$)
      By convergence: $\exists b \in \mathbb{R}$ s.t. $\forall \epsilon > 0 \; \exists N \in \mathbb{N}$ s.t. $\forall n > N$, $|b_n - b| < \epsilon$
      Choose: $\epsilon = \frac{1}{N}$ (anticipating Archimedean principle/density of reals)
      Substitute: $|b_n - b| < \epsilon \to |(7) - b| < (1/N)$
      Asserting: $|7 - b| < 1/N \iff |7 - b| \leq 0$
      $\implies 0 \leq b - 7 \leq 0 \therefore b = 7$ and, by Archimedean principle, $\frac{1}{n} < (\frac{1}{N} = \epsilon)$ and $|7 - (7)| < \frac{1}{n} < \epsilon$ holds and b_n converges to $b = 7$.

      B. Proof: c_n converges to $c = 0$.
      (Let: $n \in \mathbb{N}$)
      By convergence: $\exists c \in \mathbb{R}$ s.t. $\forall \epsilon > 0 \; \exists N \in \mathbb{N}$ s.t. $\forall n > N$, $|c_n - c| < \epsilon$
      Choose: $\epsilon = \frac{1}{\sqrt{N}}$,
      and substitute: $|(\frac{1}{\sqrt{n}}) - c| < (\frac{1}{\sqrt{N}})$
      $\implies -(\frac{1}{\sqrt{N}}) < \frac{1}{\sqrt{n}} - c < \frac{1}{\sqrt{N}}$
      $\implies c > \frac{1}{\sqrt{n}} - \frac{1}{\sqrt{N}}$ and $c < \frac{1}{\sqrt{n}} + \frac{1}{\sqrt{N}}$
      which (for arbitrary N) holds i.f.f. $c = 0$. c_n converges to $c = 0$ and we may now apply (2).

      ii. Applying (2):
      $a_n = b_n - c_n \implies \lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n - \lim_{n\to\infty} c_n$
      $... = b - c = (7) - (0) = 7 \diamond$

2. Suppose that (a_n) and $(b_n))$ are convergent sequences of real numbers with limits $a_n \to a$ and $b_n \to b$, and let $c \in \mathbb{R}$. Prove the following:

   (a) $(a_n - b_n) \to a - b$

      i. That the sequences converge tells us they are composed of reals, which affords us closure/completeness and the ability to perform arithmetic operations on their terms, i.e.:
      $(a_n) - (b_n) = (a_1 - b_1, a_2 - b_2, ..., a_n - b_n)$.

      Now, suppose $c_n = a_n - b_n$ and $c_n \to c$.

         A. Restating convergence:
         $\exists c \in \mathbb{R}$ s.t. $\forall \epsilon > 0\ \exists N \in \mathbb{N}$ s.t. $\forall n > N$, $|c_n - c| < \epsilon = \frac{1}{N}$
         and substituting:
         $|(a_n - b_n) - (a - b)| < \frac{1}{N}$
         $\implies |(a_n - a) + (b_n - b)| \leq |a_n - a| + |b_n - b| < \frac{1}{2N} + \frac{1}{2N}$
         which holds for arbitrary N given the convergence of a_n and b_n. (Via Triangle Inequality, we have revealed a combined statement of convergence.)
         $\diamond$

3. Complete the following proofs started in the lectures:

   (a) Prove the decreasing case of the Monotone Convergence Theorem (MCT).

      i. To prove the decreasing case of MCT, we must prove two properties for a monotone-decreasing series: A. Divergence to infinity, and B. Convergence to the infimum of the set of values generated by a_n.

         A. Divergence to infinity is defined as follows:
         $\forall M \leq 0\ \exists N$ s.t. $n > N \implies a_n \leq M$
         Recalling the definition of a monotone-decreasing series:
         $a_n \geq a_{n+1}$,
         and the definition of divergence: $\exists \epsilon > 0$ s.t. $\forall N\ \exists n > N$
         $|a_n - a| \geq \epsilon$,
         we see that a_n diverges when the Archimedean principle is applied s.t. $\epsilon = \frac{1}{n} \implies |a_n - a| \geq \frac{1}{n}$ for any choice of a.

         $\therefore \forall M \leq 0\ \exists N \in \mathbb{N}$ s.t. $\exists n > N \implies a_n \leq a_N \leq M$
         $\implies a_n \leq M$ and a_n diverges to (negative) infinity.

         B. a_n is bounded from below.
         By completeness axiom, $\exists \beta = \inf\left(\{a_n : n \in \mathbb{N}\}\right)$. By definition of infimum, $\exists a_N$ s.t. $\beta - \epsilon \geq a_N \geq \beta$
         for $n > N$, monotonicity yeilds:
         $\beta \leq a_n \leq a_N \leq \beta - \epsilon$
         $\beta \leq a_n \leq \beta + \epsilon \implies 0 \leq a_n - \beta \leq \epsilon$
         $\therefore a = \beta = \inf(\{a_n : n \in \mathbb{N}\})$
         $\diamond$