Problem Set 1b
RAI - AS 110.405(88) - SP2026

Tom West

February 1, 2026

Intro

Welcome to my second problem set, my next attempt at LaTeX... **See the next page for proofs!**

The problems:

1. Let A and B be sets. Prove the following:

   (a) $A \cup B = A \iff B \subseteq A$

      i. Let: C is a set, $a \in A, b \in B, c \in C$

         A. $A \cup B = A \implies \forall b \ \exists a = b \implies B \subseteq A$

         B. $B \subseteq A \implies \forall a \ \exists b = a$
            Let: $C = A \cup B$
            $C = \{z | z = a \vee z = b\}$
            $= \{z | z = b = a \vee z = a\}$
            $= \{z | z = a \in A\} = A$
            $\therefore C = A$ and $1a$ is true. $\diamond$

   (b) $A \cap B = A \iff A \subseteq B$

      i. Let: C is a set, $a \in A, b \in B, c \in C$

         A. $A \cap B = A \implies \forall a \ \exists b = a$

         B. $A \subseteq B \implies \forall a \exists b = a$
            Let: $A \cap B = C$
            $C = \{c | c \in A \wedge c \in b\}$
            $... = \{c | c \in A\}$ (since $A \subseteq B$)
            $\therefore C = A$ and $1b$ is true. $\diamond$

   (c) $A \setminus B = A \iff A \cap B = \emptyset$

      i. Let: C is a set, $a \in A, b \in B, c \in C$

         A. $A \setminus B = A \implies \forall b \ \nexists a = b \implies A \cap B = \emptyset$

         B. $A \cap B = \emptyset \implies \forall a \ \nexists b = a$
            Let: $C = A \setminus B$
            $C = \{c | c \in A \wedge c \notin B\}$
            $\therefore C = A$ and $1c$ is true. $\diamond$

(d) $A \setminus B = \emptyset \iff A \subseteq B$

i. Let: C is a set, $a \in A, b \in B, c \in C$

A. $A \setminus B = \emptyset \implies \forall a\ \exists b = a \implies A \subseteq B$

B. $A \subseteq B \implies \forall a\ \exists b = a$
Let: $C = A \setminus B$
$C = \{c | c \in A \land c \notin B\} =$
$\therefore C = \emptyset$ and $1d$ is true. $\diamond$

2. Let X and Y be sets and $A \subseteq X$ and $B \subseteq Y$ be subsets. For a given function $f : X \mapsto Y$, define the *image* of A under f to be the set $f(A) \coloneqq \{f(x) : x \in A\} \subseteq Y$. Define the *preimage* of B under f to be the set $f^{-1}(B) \coloneqq \{x : f(x) \in B\} \subseteq X$.

(a) Prove that $f(f^{-1}(B)) \subseteq B$. (Hint: Let $x \in f(f^{-1}(B))$ be an arbitrary element. Unpack what this means to show that $x \in B$

(b) Give an example where $f(f^{-1}(B)) \neq B$.

(c) Prove that $A \subseteq f(f^{-1}(A))$.

(d) Give an example where $A \neq f(f^{-1}(A))$.

3. (a) Prove that $\sqrt{3}$ is irrational. Your argument will be similar to the one given in Lecture 1.2 that $\sqrt{2}$ is irrational.

   i. (Remembering that a product of odd numbers is odd and a product of a mixture i
   Let: $\mathbb{Q} = \{\frac{a}{b} | a, b \in \mathbb{Z}, b \neq 0\},\ k, p, q \in \mathbb{Z},\ x \in \mathbb{Q},\ x = \frac{p}{q} = \sqrt{3}$
   Removing all factors of 2 to form odd p, p_1, and odd q, q_1,
   s.t. $x^2 = 2^k(\frac{p_1}{q_1})$ (where a quotient of odd numbers must itself be odd).
   Now, $2^k(\frac{p_1}{q_1})^2 = 3 \rightarrow 2^k \cdot p_1^2 = q_1^2 \cdot 3 = q_1^2 \cdot (2+1)$
   $2 \cdot (2^{2k-1} \cdot p_1^2 - q_1^2) = q_1^2$
   $\implies 2 \cdot (...) =$ "even," which contradicts $q_1^2 =$ "odd"
   $\therefore \nexists x \in \mathbb{Q}$ s.t. $x = \sqrt{3}$ $\diamond$

(b) Try to adapt your argument from *(a)* in order to show that $\sqrt{4}$ is irrational (which is not true). Explain what goes wrong.

   i. Skipping redundant steps to arrive at the conflict:
   $2^2k(p_1/q_1)^2 = 4 \rightarrow 2^{2k-1}(p_1/q_1) = 2 \implies$ "even" = "even," which is NOT a contra

(c) By the same argument as in *(a)*, one can show that $\sqrt{5}$ is irrational. Prove that $\sqrt{3}+\sqrt{5}$ is rational if and only if $\sqrt{3}-\sqrt{5}$ is rational; use this and the above to conclude that they both must be irrational.

   i. $\sqrt{5}$ is irrational:

   Omitting redundant steps:
   $...2^k(p_1/q_1)^2 = (2+3) \rightarrow 2(2^{2k-1}p_1^2 - q_1^2) = 3q_1^2$
   left-hand side: even, right-hand side: odd $\cdot$ odd is odd
   $\implies$ even = odd, which is a contradiction!
   $\therefore \sqrt{5} \notin \mathbb{Q}$

   ii. $(\sqrt{3}+\sqrt{5}) \in \mathbb{Q} \iff (\sqrt{3}-\sqrt{5}) \in \mathbb{Q}$

      A. Define addition of rationals (with new x, p, q):
      $y, z \in \mathbb{Q}$ and $p, q, p', q' \in \mathbb{Z}$
      $x = \frac{p}{q}, y = \frac{p'}{q'} \in \mathbb{Q}$
      $x+y = \frac{p}{q}+\frac{p'}{q'} = \frac{pq'+p'q}{qq'} \in \mathbb{Q}$ by closure under addition/multiplication for integer

$\implies pq' - p'q \in \mathbb{Z} \therefore x - y = \frac{pq'-p'q}{qq'} \in \mathbb{Q}$

B. While it's a bit trivial to show the reverse...
$x - y = \frac{pq'-p'q}{qq'} \in \mathbb{Q}$(by field closure)
Let: $z = -y \in \mathbb{Q}$
$\forall x, z \;\; x + z \in \mathbb{Q} \in \mathbb{Q}$

C. (Also, consider the product of a rational's conjugate:)
$(x+y)(x-y) = x^2 - y^2$
For: $x = \sqrt{3}, y = \sqrt{5}, x^2 - y^2 = 3 - 5 \in \mathbb{Z} \subset \mathbb{Q}$
For closure, $x \cdot x$ and $y \cdot y$ must both be rational. $\diamond$

4. Prove that multiplicative identity in a field is unique.
(hint: Here's the setup for the proof. Let $\mathbb{F}$ be a field and suppose that $a \in \mathbb{F}$ and $b \in \mathbb{F}$ satisfy $a \cdot x = x \;\; \forall x$ and $b \cdot x = x \;\; \forall x$; show that $a = b$.)

5. Let $\mathbb{F}$ be an ordered field. Recall that the positive elements of $\mathbb{F}$ are a nonempty subset $P \subseteq \mathbb{F}$ satisfying:

   (i) If $a, b \in P$, then $a + b \in P$ and $a \cdot b \in P$

   (ii) If $a \in \mathbb{F}$ and $a \neq 0$, then either $a \in P$ or $-a \in P$, but not both.

   (a) Give an example of a nonempty subset, $P_1 \subseteq \mathbb{R}$ that satisfies (i) but not (ii).

      i. Let P_1 contain $c \in \mathbb{R}$ s.t. (ii) does not hold.
      $\implies c \in P \wedge -c \in P \wedge c \neq 0 \wedge c \in \mathbb{R} \therefore c \cdot -c = -c^2 \in P$, which is false for all real values other than zero.
      $\therefore P_1$ for which (ii) does not hold does not exist.
      $\diamond$

   (b) Give an example of a nonempty subset, $P_2 \subseteq \mathbb{R}$ that satisfies (ii) but not (i).

      i. Let P_2 contain $a, b \in P \subset \mathbb{R}$ s.t. (i) does not hold.
      $\implies a + b \notin P \vee a \cdot b \notin P$
      $\implies a + b < 0 \vee a \cdot b < 0$
      $\implies a < 0 \vee b < 0 \implies a \notin P \vee b \notin P. \therefore P_2$ for which (i) does not hold does not exis

   (c) Combining insights $5((a))i$ and $5((b))i$ reveals that neither subset can exist. (And you
   $\diamond$

6. Prove the Transitivity property of inequalities: If $x < y$ and $y < z$, then $x < z$. (Note: This may result may seem "obvious." If you can't figure out what to write, reference the definition of $x < y$ from Lecture 1.2 and include this definition in your proof.

   (a) Contradiction:
   $\exists \alpha : x < y \wedge y < \alpha \wedge x \not< \alpha$
   $\exists \epsilon > 0 : x - \alpha \geq \epsilon$
   $x - \alpha < y - \alpha \wedge y - \alpha < 0$
   $\implies x - \alpha < y - \alpha < 0 \implies \epsilon \leq x - \alpha < y - \alpha < 0 \implies \epsilon < 0.$
   $\therefore \nexists \alpha$ and $x < y < z \implies x < z \diamond$

7. Let $\mathbb{F}$ be an ordered field and let $ab \in \mathbb{F}$.

   (a) Prove that $a \leq b \iff \forall \epsilon > 0, a < b + \epsilon$

      i. Forward case-by-case:
      Case 1: $a > b$
      Assume: $\forall \epsilon > 0,\ \ a < b + \epsilon$
      $a < a + \epsilon \implies a < a + \epsilon < b + \epsilon \implies a + \cancel{\epsilon} < b + \cancel{\epsilon}$
      $\implies a < b$ which is a contradiction.
      Case 2: $a = b \implies a < b + \epsilon$ which is true $\forall \epsilon > 0$
      Case 3: $a < b$ and holding $\epsilon > 0 \implies a - b < 0 < \epsilon$
      $\therefore a \leq b \implies \forall \epsilon > 0, a < b + \epsilon$

      ii. Backward case-by-case:
      Assume: $\forall \epsilon > 0 \exists a < b + \epsilon$
      $\implies \epsilon \cdot (a - b) < \epsilon^2$
      This does not establish the need for a ¡= b, however.
      Show: $\epsilon * \delta < \epsilon^2$
      Case 1: True for all $\delta < 0$.
      Case 2: True for all $\delta = 0$.
      Case 3: $\delta > 0$ is false by the Archimedean principle, whereby:
      $\forall \delta > 0\ \ \exists \epsilon > 0 : \epsilon \cdot \delta\ \not< \epsilon^2$
      $\therefore \forall \epsilon > 0\ \ \exists a < b + \epsilon \implies a \leq b \quad \diamond$

   (b) Use **??** to show that $a = b \iff \forall \epsilon > 0, |a - b| < \epsilon$.

      i. Backwards:
      Case 1: $|\delta| > 0$, i.e. $a \neq b$
      By Archimedean principle, $\forall \delta > 0\ \ \exists \epsilon > 0 : 0 < \epsilon < \delta \implies \epsilon \cdot \delta \not< \epsilon \cdot \epsilon$
      $\therefore \epsilon \cdot \delta < \epsilon \cdot \epsilon$
      Case 2: $\delta = 0$:
      $\forall \epsilon > 0,\ \ \epsilon \cdot 0 < \epsilon^2$

      ii. Forwards:
      $a = b \implies |a - b| = 0 \implies \forall \epsilon > 0, |a - b| < \epsilon \quad \diamond$

8. (a) Complete the proof of Corollary 1.12 in the notes, which we started in Lecture 1.3. (Note: If you're confused exactly what remains to be shown, just prove the corollary in its entirety.)

   i. Let $\Delta :=$ Triangle Inequality
      $\Delta : |x+y| \leq |x| + |y|$
      Let:$x = x'^2, y = y'^2$, Note: $|\alpha^2| = |\alpha| \cdot |\alpha| = |\alpha|^2$
      and $z = -y$ in $\Delta \implies |x-z| \leq |x| + |z|$
      $\therefore |x'|^2 + |y'|^2 = |x'^2| + |(-y')^2| \geq |x'^2 - y'^2| = ||x' + y'| \cdot |x' - y'|$
      Introduce: $x' = \sqrt{x''}$, $y' = \sqrt{y''}$
      $||\sqrt{x''} + \sqrt{y''}| \cdot |\sqrt{x''} - \sqrt{y''}||$
      $... = ||x''| - |y''|| \leq |(\sqrt{x''})^2| + |(-\sqrt{y'})^2| = |x''| + |y''|$
      $\therefore ||x''| - |y''|| \leq |x''| + |y''|$
      Combining the above relations, we may reach the reverse Δ through transitivity.

   (b) Complete the proof of the second bullet of Corollary 1.13 in the notes, which we started in Lecture 1.3.

9. Suppose that $A \subseteq B$ and that both A and B are bounded from above. Prove that $\sup(A) \leq \sup(B)$.

10. Is $\mathbb{N}$ complete? Justify your answer with a proof or counterexample

11. Suppose A is a nonempty set with finitely many elements. Complete the following steps to show using a *proof by induction* that A admits a maximal element $M \in A$ satisfying $x \leq M \;\; \forall x \in A$.

   (a) First prove the *base case*: if A contains only one element, then A admits a maximal element. This step is easy!

   (b) Next prove the *inductive case*: let $n \in \mathbb{N}$ and assume any set with n elements admits a maximal element. Prove that a set with *n+1* elements also admits a maximal element.

   (c) Explain why the work you did in (a) and (b) proves the claim.

12. Prove the infimum case (i.e. the second bullet point) of Theorem 1.21 in the notes.

13. Set $A = \{\frac{n}{n+1} : n \in \mathbb{N}\}$.

   (a) Prove that $\sup(A) = 1$ using the analytic description of a supremum given by Theorem 1.21 in the notes.

(b) Prove that $\sup(A) = 1/2$ using the analytic description of an infimum given by Theorem 1.21 in the notes.

14. Let $A \subseteq \mathbb{R}$ be nonempty and bounded from below. Define $-A := -x : x \in A$. Prove that $-A$ is nonempty and bounded from above, and moreover that $\sup(-A) = -\inf(A)$.