ANALYTIC CONSTRUCTION OF PERIODIC ORBITS ABOUT THE COLLINEAR POINTS

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Abstract. A third-order analytical solution for halo-type periodic motion about the collinear points of the circular-restricted problem is presented. The three-dimensional equations of motion are obtained by a Lagrangian formulation. The solution is constructed using the method of successive approximations in conjunction with a technique similar to the Lindstedt-Poincaré method. The theory is applied to the Sun–Earth system.

1. Introduction

The subject of periodic solutions of the circular-restricted problem has received considerable attention in the past few decades. Investigations into periodic motion about the equilateral points were pursued vigorously in the 1960s; perhaps, in some instances, with a view toward future applications in the space program. Current interest concerning periodic motion in the neighborhood of the collinear points stems from work conducted in support of the International Sun–Earth Explorer Satellite, ISEE–3. Details of this mission can be found in Farquhar *et al.* (1979).

This paper deals with the analytical development of a local approximation for three-dimensional, halo-type periodic orbits about the collinear points. The equations of motion are developed from a Lagrangian mechanics approach as described in Richardson (1980). The successive approximation solution to these equations was automated via the author's version of an algebraic manipulation computer program developed by R. Dasenbrock (1973). In this way, the potential for inadvertent error was reduced while, at the same time, the capability for producing high-order approximations was available if the need arose.

Of the considerable volume of literature on the subject, the particular references that proved to be of the most value were the early works of Plummer (1901, 1903a, 1903b), Moulton's (1920) treatise, and the investigations of Pedersen (1933, 1935). Included also is the more recent work of Deprit (1965), Broucke (1968, 1969), and Farquhar and Kamel (1973). Finally, the discussions found in Szebehely (1967) proved quite indispensable.

2. Overview

The periodic nature of the solution can be seen by considering the linearized form of the equations. They are given as follows:

$$\ddot{x} - 2\dot{y} - (1 + 2c_2)x = 0 \quad (1a)$$

$$(c_2 = \text{constant} > 0)$$

$$\ddot{y} + 2\dot{x} + (c_2 - 1)y = 0 \quad (1b)$$

$$\ddot{z} + c_2 z = 0. \quad (1c)$$

The origin of the rotating x, y, z Cartesian system is located at one of the collinear libration points with the x–y plane coinciding with the plane of motion of the primaries, M_1 and M_2, as shown in Figure 1.1. The x-axis points along the line of syzygy away from the larger primary, herein assumed to be M_2. The z-axis completes the right-handed system.

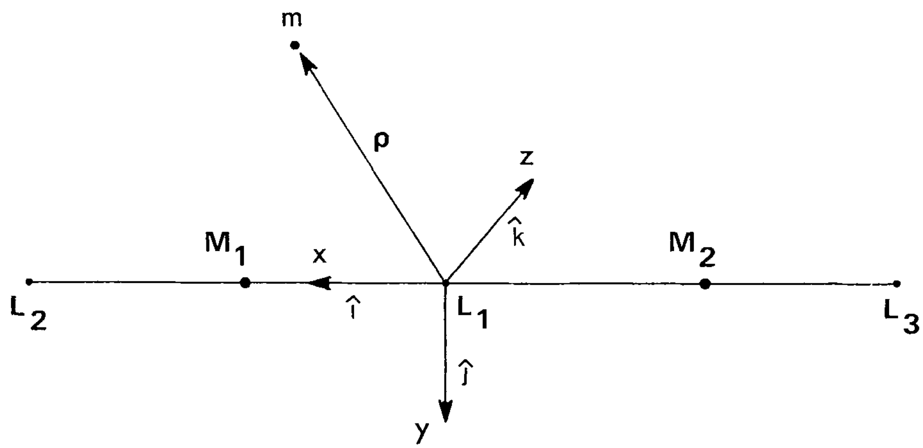

Fig. 1. Libration point geometry. The orientation of the synodic x, y, z coordinates is the same for each of the collinear points.

The solution of the characteristic equation for the x–y (in-plane) motion has two real and two imaginary roots. As is known, the two real roots are of opposite sign and therefore, arbitrarily chosen initial conditions will give rise, in general, to unbounded motion as time increases. If, however, the initial conditions are restricted so that only the non-divergent mode is allowed, the x–y solution can be expressed in the form

$$x = A_1 \cos \lambda t + A_2 \sin \lambda t \quad (2a)$$

$$y = -kA_1 \sin \lambda t + kA_2 \cos \lambda t. \quad (2b)$$

Since the z or out-of-plane linearized motion is simple harmonic, i.e.,

$$z = B_1 \sin \nu t + B_2 \cos \nu t, \quad (3)$$

then the linearized motion will become quasi-periodic inasmuch as the in-plane and out-of-plane frequencies, λ and ν respectively, are such that λ/ν is generally irrational. The projections of the motion onto the various coordinate planes produce Lissajoustype trajectories. A detailed discussion of various aspects of the linearized motion can be found in Szebehely (1967).

Halo-type periodic motion is obtained if the amplitudes of the in-plane and out-of-plane motions are of sufficient magnitude so that the non-linear contributions to the system produce eigenfrequencies that are equal. Motion for which λ/ν is rational but unequal to one is not considered. The linearized solution can then be expressed in the form:

$$x = -A_x \cos(\lambda t + \phi) \tag{4a}$$

$$y = kA_x \sin(\lambda t + \phi) \tag{4b}$$

$$z = A_z \sin(\lambda t + \psi). \tag{4c}$$

In these expressions, the amplitudes A_x and A_z are constrained by a certain non-linear algebraic relationship found as a result of the application of the perturbation method. The phases ϕ and ψ are also related to each other in a linear fashion. These constraint relationships between the in-plane and out-of-plane amplitudes and phases are the indirect result of the specification of non-arbitrary initial conditions.

Equations (4) form the first approximation for the successive approximations procedure for the third-order solution for periodic motion about the collinear points. The non-linear solutions are easily shown to be symmetric about the x–z plane. In addition, if the sign of the A_z amplitude is changed, then the solution bifurcates with the result that two solution branches are possible. The two branches are viewed as mirror reflections about the x–y plane. Similar results were obtained by Farquhar and Kamel (1973) in their pioneering study of quasi-periodic motion about the L_2 collinear point of the Earth-Moon system. Details are provided below.

3. Equations of Motion

The equations of motion for the small mass m moving in the vicinity of any of the libration points (L_1, L_2, L_3) are found from the Lagrangian L expressed in the form (Richardson, 1980)

$$L = \tfrac{1}{2}(\dot{\boldsymbol{\rho}} \cdot \dot{\boldsymbol{\rho}}) + GM_1\left[\frac{1}{|\mathbf{r}_1 - \boldsymbol{\rho}|} - \frac{\mathbf{r}_1 \cdot \boldsymbol{\rho}}{|\mathbf{r}_1|^3}\right] + GM_2\left[\frac{1}{|\mathbf{r}_2 - \boldsymbol{\rho}|} - \frac{\mathbf{r}_2 \cdot \boldsymbol{\rho}}{|\mathbf{r}_2|^3}\right], \tag{5}$$

where $\boldsymbol{\rho}$ is the position vector of m relative to the libration point:

$$\boldsymbol{\rho} = x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}. \tag{6}$$

The vectors $\mathbf{r}_1$ and $\mathbf{r}_2$ are the position vectors of the primaries M_1 and M_2 ($M_1 \leqslant M_2$) with respect to the libration point.

The GM_1 and GM_2 terms on the right-hand sides of Equation (5) can be viewed as *perturbing potentials* due to the primaries M_1 and M_2. These quantities can be developed in a power series in the non-dimensional distance ratios $|\boldsymbol{\rho}|/|\mathbf{r}_1|$ and $|\boldsymbol{\rho}|/|\mathbf{r}_2|$ hereinafter denoted by ρ/r_1 and ρ/r_2. The validity of this development is determined by the magnitude of these ratios. As long as ρ/r_1 and ρ/r_2 are both less than unity, the expansion of the Lagrangian in this fashion is considered legitimate.

In order to facilitate subsequent developments, a system of normalized units is adopted. The unit of distance is taken as follows:

$$r_1 = 1\,, \quad \text{for motion about} \quad L_1 \text{ or } L_2$$

$$r_2 = 1\,, \quad \text{for motion about} \quad L_3\,.$$

The units of mass and time are determined from Kepler's third law

$$G(M_1 + M_2) = n_1^2 a_1^3 = 1\,,$$

where n_1 and a_1 are the orbital mean motion and mean distance of M_1 in its orbit about M_2. Define the constant μ as the ratio

$$\mu \triangleq \frac{M_1}{M_1 + M_2},$$

then $GM_1 = \mu$, and $GM_2 = 1 - \mu$. Also, define the dimensionless quantity γ_L as the ratio

$$\gamma_L \triangleq \frac{r_1}{a_1} = n_1^{2/3}\,, \qquad \text{(motion about } L_1 \text{ or } L_2).$$

$$\gamma_L \triangleq \frac{r_2}{a_1} = n_1^{2/3}\,, \qquad \text{(motion about } L_3).$$

Finally, if a new independent variable s is introduced through the relationship

$$s = n_1 t\,,$$

then the Lagrangian can be expressed as

$$L = \tfrac{1}{2}(\rho^* \cdot \rho^*) + \sum_{n=2}^{\infty} c_n \rho^n P_n(x/\rho)\,, \tag{7}$$

where asterisks (*) denote differentiation with respect to s, and $P_n(x/\rho)$ is the nth Legendre polynomial of the first kind with argument x/ρ. The constants c_n are given by the expressions:

$$c_n = \frac{1}{\gamma_L^3}\left[(\pm 1)^n \mu + (-1)^n \frac{(1-\mu)\gamma_L^{n+1}}{(1 \mp \gamma_L)^{n+1}}\right], \qquad (L_1 \text{ or } L_2) \tag{8a}$$

$$c_n = \frac{1}{\gamma_L^3}\left[1 - \mu + \frac{\mu \gamma_L^{n+1}}{(1 + \gamma_L)^{n+1}}\right], \qquad (L_3)\,. \tag{8b}$$

In these expressions and in all future developments, whenever double signs appear, *the upper sign applies to trajectories near the L_1 point, and the lower sign applies to trajectories in the vicinity of L_2.*

The application of Lagrange's equations produces the following expressions for the full three-dimensional equations of motion:

$$x^{**} - 2y^{*} - (1 + 2c_2)x = \sum_{n=2}^{\infty} (n + 1)c_{n+1}\rho^n P_n(x/\rho), \tag{9a}$$

$$y^{**} + 2x^{*} + (c_2 - 1)y = \sum_{n=3}^{\infty} c_n y\rho^{n-2} \tilde{P}_n(x/\rho), \tag{9b}$$

$$z^{**} + c_2 z = \sum_{n=3}^{\infty} c_n z\rho^{n-2} \tilde{P}_n(x/\rho), \tag{9c}$$

where the quantity $\tilde{P}_n(x/\rho)$ denotes the sum

$$\tilde{P}_n(x/\rho) \triangleq \sum_{k=0}^{[(n-2)/2]} (3 + 4k - 2n)P_{n-2k-2}(x/\rho). \tag{10}$$

The [ ] brackets denote the integer part of $(n - 2)/2$.

A high-order successive approximation solution to these equations is a lengthy and tedious algebraic process. Most of the work can be accomplished with a minimal likelihood of error by use of an algebraic manipulation computer program. The above equations lend themselves to easy implementation in such programs as successively higher-orders in the development are readily constructed using the Legendre-polynomial recursion relations.

4. Construction of Periodic Solutions

As explained in Section 2, halo-type periodic solutions are obtained by assuming that the amplitudes A_x and A_z of the linearized solution are large enough so that the non-linear contributions to the system produce eigenfrequencies which are equal. With this assumption, the linearized equations are rewritten as

$$x^{**} - 2y^{*} - (1 + 2c_2)x = 0, \tag{11a}$$

$$y^{**} + 2x^{*} + (c_2 - 1)y = 0, \tag{11b}$$

$$z^{**} + \lambda^2 z = 0, \tag{11c}$$

where λ^2 has replaced the coefficient c_2 in Equation (9c). The correction Δ, expressed by

$$\Delta \triangleq \lambda^2 - c_2,$$

must be added to the right-hand side of the z equation when higher-order approximations are constructed. The new z equation then reads

$$z^{**} + \lambda^2 z = \sum_{n=3}^{\infty} c_n z\rho^{n-2} \tilde{P}_n(x/\rho) + \Delta z. \tag{12}$$

It is assumed the correction constant Δ obeys the order-of-magnitude relation

$$\Delta = O(A_z^2).$$

Accordingly, the Δz contribution to the solution will first appear in the expressions for the third-order corrections.

Finally, in order to help remove any secular terms which appear as a result of the successive approximation procedure, a new independent variable τ is introduced via the relation

$$\tau = \omega s\,. \tag{13}$$

The quantity ω is the frequency correction written in general as

$$\omega = 1 + \sum_{n\geqslant 1} \omega_n\,, \qquad \omega_n < 1\,,$$

where the ω_n are assumed to be of order $O(A_x^n)$. Each of the ω_n is chosen during the development of the solution to remove the secular terms as they appear.

Equations (9a), (9b), and (12) are then rewritten in terms of the new independent variable τ. The result through third-order in x, y, z is

$$\omega^2 x'' - 2\omega y' - (1 + 2c_2)x = \tfrac{3}{2}c_3(2x^2 - y^2 - z^2) + 2c_4 x(2x^2 - 3y^2 - 3z^2) + O(4) \tag{14}$$

$$\omega^2 y'' + 2\omega x' + (c_2 - 1)y = -3c_3 xy - \tfrac{3}{2}c_4 y(4x^2 - y^2 - z^2) + O(4) \tag{15}$$

$$\omega^2 z'' + \lambda^2 z = -3c_3 xz - \tfrac{3}{2}c_4 z(4x^2 - y^2 - z^2) + \Delta z + O(4). \tag{16}$$

The third-order successive approximation solution to these equations is a lengthy process. The generating solution is the linearized solution expressed by Equations (4) with t replaced by τ and k given by $k = (\lambda^2 + 1 + 2c_2)/2\lambda$. Most of the secular terms can be removed by proper specification of the ω_n. From the second-order equations, it is found that ω_1 is not needed, and consequently

$$\omega_1 = 0\,.$$

Removal of the secular terms in the third-order x and y equations provides the proper expression for ω_2:

$$\omega_2 = s_1 A_x^2 + s_2 A_z^2\,, \tag{17}$$

where the constants s_1 and s_2 are lengthy expressions involving the linearized frequency λ and are reproduced here in Appendix I.

The secular terms which appear in the third-order z equation cannot be removed in the usual manner of the Lindstedt-Poincaré method as ω_1 and ω_2 of the frequency correction have been utilized previously. It therefore becomes necessary to specify amplitude and phase-angle constraint relationships in order to complete the non-secular development. The amplitude relationship is easily deduced and can be expressed as

$$l_1 A_x^2 + l_2 A_z^2 + \Delta = 0\,, \tag{18}$$

where l_1 and l_2 are constants whose expressions are given in Appendix I. The phase-angle constraint relationship is

$$\psi = \phi + \frac{n\pi}{2}, \qquad n = 1, 3. \tag{19}$$

Using these results, the complete third-order solution for periodic motion about any of the collinear points is found to be

$$x = a_{21}A_x^2 + a_{22}A_z^2 - A_x \cos\tau_1 + (a_{23}A_x^2 - a_{24}A_z^2)\cos 2\tau_1$$
$$+ (a_{31}A_x^3 - a_{32}A_xA_z^2)\cos 3\tau_1, \tag{20a}$$
$$y = kA_x \sin\tau_1 + (b_{21}A_x^2 - b_{22}A_z^2)\sin 2\tau_1$$
$$+ (b_{31}A_x^3 - b_{32}A_xA_z^2)\sin 3\tau_1, \tag{20b}$$
$$z = \delta_n A_z \cos\tau_1 + \delta_n d_{21}A_xA_z(\cos 2\tau_1 - 3)$$
$$+ \delta_n(d_{32}A_zA_x^2 - d_{31}A_z^3)\cos 3\tau_1, \tag{20c}$$

where the switch function δ_n is given by

$$\delta_n = 2 - n, \qquad n = 1, 3, \tag{21}$$

and the a_{ij}, b_{ij}, and d_{ij} are constants (Appendix I), with τ_1 defined as

$$\tau_1 \triangleq \lambda\tau + \phi.$$

Without any loss of generality, A_x and A_z are to have values such that

$$A_x > 0.$$
$$A_z \geqslant 0.$$

The solution bifurcation manifests itself through the phase-angle constraint relationship, Equation (19). The two solution branches are obtained from Equations (20) by setting the switch function according to Equation (21). It is seen that this is equivalent to altering the sign of the A_z amplitude. Representative plots for both branches are given in the next section.

The relative size of the periodic orbits that result from Equations (20) is obtained from the amplitude-constraint relationship. From this expression, it is seen that the minimum permissible value for A_x occurs for $A_z = 0$ and is given by the equation

$$A_{x\,\mathrm{minimum}} = \sqrt{|\Delta/l_1|}. \tag{22}$$

In the case of periodic motion about the L_1 or L_2 points of the Earth–Sun system, the value of A_x is about 14% of the normalized distance r_1. This amounts to a minimum amplitude of approximately 200000 kilometers.

5. Sun–Earth Halo Orbits

Plots of the third-order solution for motion about the collinear points of the Sun–Earth* system are shown in Figures 2, 3, and 4. Both $n = 1$ and $n = 3$ solution branches are presented. In accordance with Farquhar and Kamel (1973), the solution branches are denoted as Class I, corresponding to $n = 1$, and Class II, corresponding to $n = 3$. The plots are representative of the general nature of the third-order solution as it is seen when projected onto the synodic x–y, x–z and y–z coordinate planes. Values of the constants used in the solution equations are given in Table I. (p. 253).

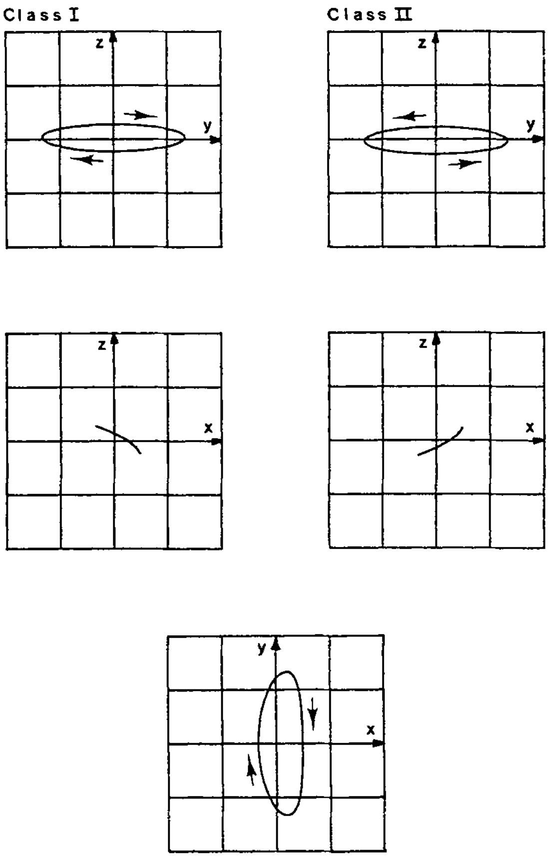

Fig. 2. L_1 orbits in the Sun–Earth system. ($A_z = 1.25 \times 10^5$ km). Each division represents 5×10^5 km with L_1 at the origin of coordinates.

The accuracy to which the third-order analytical solution represents actual halo-orbits was tested using a differential corrections scheme which produced these periodic orbits numerically. Discrepancies between values of the state variables were

* The actual value of μ includes the mass of the Moon.

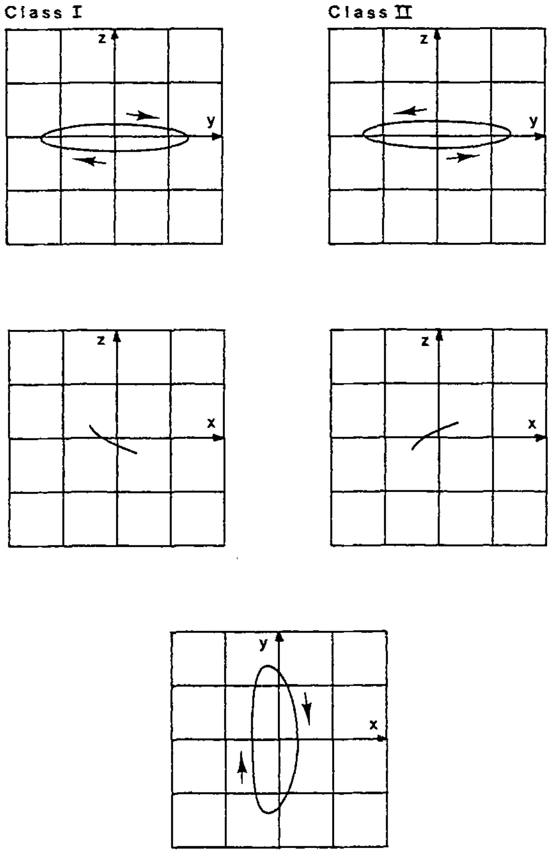

Fig. 3. L_2 orbits in the Sun–Earth system. ($A_z = 1.25 \times 10^5$ km). Each division represents 5×10^5 km with L_2 at the origin of coordinates.

determined for L_1 halo orbits of the Sun–Earth system corresponding to $A_z = 110\,000$ kilometers. The results showed that *maximum* variations in the state variables were less than 3%. All discrepancies were consistent with the order of magnitude of the truncation error in the third-order development. A detailed discussion of this comparison can be found in Richardson (1979).

Acknowledgments

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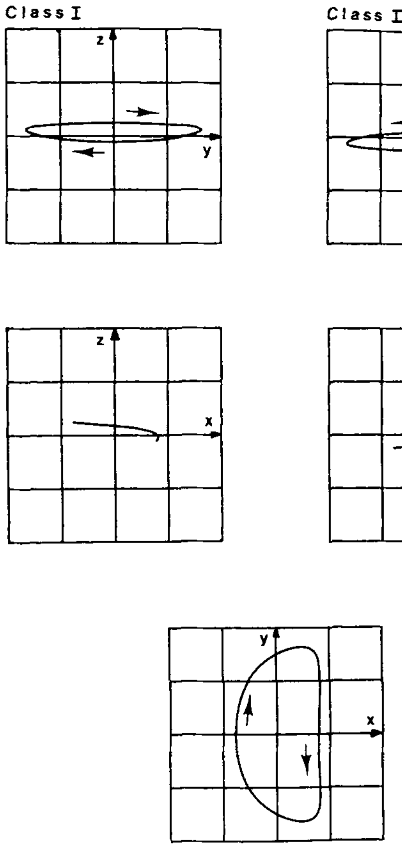

Fig. 4. L_3 orbits in the Sun–Earth system. ($A_z = 1.25 \times 10^7$ km). Each division represents 7.5×10^7 km with L_3 at the origin of coordinates.

Appendix I

The full third-order successive approximations solution for periodic motion about the collinear points was expressed in Equations (3.8). The frequency co'rection and the amplitude-constraint relationship were given in Equations (3.5) and (3.6), respectively. Expressions for the factors and coefficients which comprise these results are listed below.

For the frequency correction:

$$s_1 = \frac{1}{2\lambda[\lambda(1 + k^2) - 2k]} \left\{\tfrac{3}{2}c_3[2a_{21}(k^2 - 2) - a_{23}(k^2 + 2) - 2kb_{21}] - \tfrac{3}{8}c_4(3k^4 - 8k^2 + 8)\right\},$$

$$s_2 = \frac{1}{2\lambda[\lambda(1+k^2) - 2k]}\{\tfrac{3}{2}c_3[2a_{22}(k^2-2)+a_{24}(k^2+2)+2kb_{22}+5d_{21}]$$
$$+ \tfrac{3}{8}c_4(12 - k^2)\},$$

where

$$k = \frac{1}{2\lambda}(\lambda^2 + 1 + 2c_2),$$

$$= \frac{2\lambda}{\lambda^2 + 1 - c_2}.$$

The linearized frequency λ is found from the solution to the following equation:

$$\lambda^4 + (c_2 - 2)\lambda^2 - (c_2 - 1)(1 + 2c_2) = 0.$$

For the amplitude-constraint relationship:

$$l_1 = a_1 + 2\lambda^2 s_1,$$

$$l_2 = a_2 + 2\lambda^2 s_2,$$

where

$$a_1 = -\tfrac{3}{2}c_3(2a_{21} + a_{23} + 5d_{21}) - \tfrac{3}{8}c_4(12 - k^2).$$

$$a_2 = \tfrac{3}{2}c_3(a_{24} - 2a_{22}) + \tfrac{9}{8}c_4.$$

For the solution equations and the above expressions:

$$a_{21} = \frac{3c_3(k^2 - 2)}{4(1 + 2c_2)}.$$

$$a_{22} = \frac{3c_3}{4(1 + 2c_2)}.$$

$$a_{23} = -\frac{3c_3\lambda}{4kd_1}[3k^3\lambda - 6k(k - \lambda) + 4].$$

$$a_{24} = -\frac{3c_3\lambda}{4kd_1}(2 + 3k\lambda).$$

$$a_{31} = -\frac{9\lambda}{4d_2}[4c_3(ka_{23} - b_{21}) + kc_4(4 + k^2)]$$
$$+ \left(\frac{9\lambda^2 + 1 - c_2}{2d_2}\right)[3c_3(2a_{23} - kb_{21}) + c_4(2 + 3k^2)].$$

$$a_{32} = -\frac{1}{d_2}\left\{\frac{9\lambda}{4}[4c_3(ka_{24} - b_{22}) + kc_4]\right.$$

$$+ \tfrac{3}{2}(9\lambda^2 + 1 - c_2)\left[c_3(kb_{22} + d_{21} - 2a_{24}) - c_4\right]\Big\}.$$

$$b_{21} = -\frac{3c_3\lambda}{2d_1}(3k\lambda - 4).$$

$$b_{22} = \frac{3c_3\lambda}{d_1}.$$

$$b_{31} = \frac{3}{8d_2}\{8\lambda[3c_3(kb_{21} - 2a_{23}) - c_4(2 + 3k^2)]$$

$$+ (9\lambda^2 + 1 + 2c_2)[4c_3(ka_{23} - b_{21}) + kc_4(4 + k^2)]\}.$$

$$b_{32} = \frac{1}{d_2}\{9\lambda[c_3(kb_{22} + d_{21} - 2a_{24}) - c_4]$$

$$+ \tfrac{3}{8}(9\lambda^2 + 1 + 2c_2)[4c_3(ka_{24} - b_{22}) + kc_4]\}.$$

$$d_{21} = -\frac{c_3}{2\lambda^2}.$$

$$d_{31} = \frac{3}{64\lambda^2}(4c_3a_{24} + c_4).$$

$$d_{32} = \frac{3}{64\lambda^2}[4c_3(a_{23} - d_{21}) + c_4(4 + k^2)].$$

With

$$d_1 = \frac{3\lambda^2}{k}[k(6\lambda^2 - 1) - 2\lambda].$$

$$d_2 = \frac{8\lambda^2}{k}[k(11\lambda^2 - 1) - 2\lambda].$$

TABLE I

Values of the constants for halo orbits in the Sun–Earth system

$n_1 = 1.99099 \times 10^{-7}$ rad sec^{-1} $\mu = 3.04036 \times 10^{-6}$
$a_1 = 1.49598 \times 10^8$ km $A_z = 125000$ km

constant	L_1 Orbits (period = 177.704 days)	L_2 Orbits (period = 180.145 days)	L_3 Orbits (period = 365.255 days)
γ_L	1.00109×10^{-2}	1.00782×10^{-2}	9.99998×10^{-1}
λ	2.08645	2.05701	1.00000
k	3.22927	3.18723	2.00000
Δ	2.92214×10^{-1}	2.90785×10^{-1}	2.66029×10^{-6}
c_2	4.06107	3.94052	1.00000
c_3	3.02001	−2.97984	1.00000
c_4	3.03054	2.97026	1.00000
s_1	-8.24661×10^{-1}	-7.44452×10^{-1}	-1.59141×10^{-6}
s_2	1.21099×10^{-1}	1.25047×10^{-1}	6.29433×10^{-6}
l_1	-1.59656×10^{1}	-1.48288×10^{1}	-1.57717×10^{-5}
l_2	1.74090	1.67369	1.40258×10^{-5}
a_1	−8.78563	−8.52882	-1.25889×10^{-5}
a_2	6.86546×10^{-1}	6.15466×10^{-1}	1.43702×10^{-6}
d_1	3.11184×10^{2}	2.93192×10^{2}	1.20002×10^{1}
d_2	1.58787×10^{3}	1.49800×10^{3}	7.20008×10^{1}
a_{21}	2.09270	−2.05304	5.00000×10^{-1}
a_{22}	2.48298×10^{-1}	-2.51646×10^{-1}	2.50000×10^{-1}
a_{23}	-9.05965×10^{-1}	8.96284×10^{-1}	-4.99999×10^{-1}
a_{24}	-1.04464×10^{-1}	1.06600×10^{-1}	-2.49999×10^{-1}
a_{31}	7.93820×10^{-1}	7.80646×10^{-1}	3.75000×10^{-1}
a_{32}	8.26854×10^{-2}	8.36960×10^{-2}	1.25000×10^{-1}
b_{21}	-4.92446×10^{-1}	4.91357×10^{-1}	-2.50000×10^{-1}
b_{22}	6.07465×10^{-2}	-6.27190×10^{-2}	2.49998×10^{-1}
b_{31}	8.85701×10^{-1}	8.55305×10^{-1}	2.91666×10^{-1}
b_{32}	2.30198×10^{-2}	2.04354×10^{-2}	-1.24999×10^{-1}
d_{21}	-3.46865×10^{-1}	3.52118×10^{-1}	-4.99999×10^{-1}
d_{31}	1.90439×10^{-2}	1.88290×10^{-2}	2.53854×10^{-7}
d_{32}	3.98095×10^{-1}	3.94028×10^{-1}	3.74999×10^{-1}

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