3-Body Problem Punch List:

- Define (✓~~Frames~~, ✓~~Coordinates~~, Parameters, Algorithms, Tolerance)
- ✓ ~~Collect different methods / Interpretations~~
- Derive
- Write Code → (✓~~Lagrange Points~~, ✓~~Zero Velocity Contours~~, ✓~~Stability~~, ✓~~Trajectories~~, General Conic, ECEF Coordinates)
- ✓ ~~Test on solar system triplets~~
- ✓ ~~Share w/ Teams~~

Halo Reference
↳ Check

- Finalize

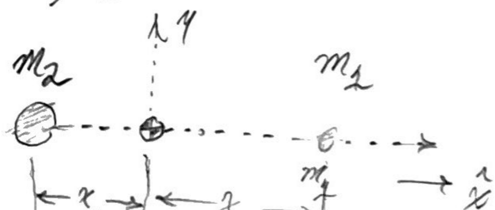
Methods:
- Richardson, "Analytic Construction..." (1979)
- Hanspeter Schaub, Carl Junkins → pp. 577-620
- Derek Paley, Jeremy Kasdin → pp. 324-328
- BMW → pp. 4-10

Lagrange's Quintic Equation:

Paley: $n^2 x (x - x_1)^2 (x + x_2)^2 - \mu_1 (x + x_2)^2 - \mu_2 (x - x_1)^2$

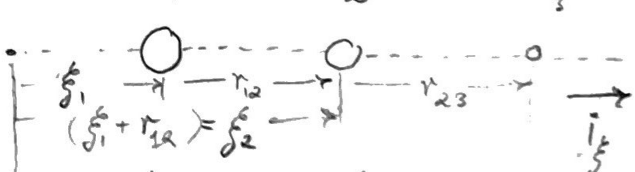
where: $n^2 = \frac{(\mu_1 + \mu_2)}{r^3}$

$m_1 x_1 = m_2 x_2$



Battin: $(m_1 + m_2)\chi^5 + (3m_1 + 2m_2)\chi^4 + (3m_1 + m_2)\chi^3 - (m_2 + 3m_3)\chi^2 - (2m_2 + 3m_3)\chi - (m_2 + m_3) = 0$

where: $\chi = \frac{r_{23}}{r_{12}}$



$\xi_1 + r_{12} + r_{23} = \xi_2 + r_{23} = \xi_3$

$\vec{r}_1 = \xi_1 i_\xi$

$\vec{r}_2 = (\xi_1 + r_{12}) i_\xi$

$\vec{r}_3 = (\xi_1 + r_{12} + r_{23}) i_\xi$

Restricted

$L_1: \frac{m_2}{m_1} = \frac{\rho_2 (3\rho^2 \rho_2 - 3\rho\rho_2^2 + \rho_2^3)}{(\rho^3 - \rho_2^3)(\rho - \rho_2)^2}$

$\omega^2 \sum_i^n m_i \vec{r}_i = \vec{C} \rightarrow \omega^2 = \frac{G(m_1 + m_2)}{r_{12}^3}$

Collinear Lagrange Points:

$$n^2\left\{(1)x^5 + (2x_2-2x_1)x^4 + (x_2^2-4x_1x_2+x_1^2)x^3\right\} + (-2n^2x_1x_2^2+2n^2x_1^2x_2-\mu_1-\mu_2)x^2$$
$$+ (n^2x_1^2x_2^2 - 2\mu_1x_2 + 2\mu_2x_1)x$$
$$+ (-\mu_1x_2^2 - \mu_2x_1^2)$$
$$= 0$$

- $a_5 = n^2$
- $a_4 = 2n^2(x_2 - x_1)$
- $a_3 = n^2(x_2^2 - 4x_1x_2 + x_1^2)$
- $a_2 = 2n^2(x_1^2x_2 - x_1x_2^2) - (\mu_1 + \mu_2)$
- $a_1 = n^2x_1^2x_2^2 - 2(\mu_1x_2 - \mu_2x_1)$
- $a_0 = -(\mu_1x_2^2 + \mu_2x_1^2)$

$\hookrightarrow$ $[L_1, L_2, L_3] \leftarrow$ if(isreal(roots($[a_5, a_4, a_3, a_2, a_1, a_0]$)))

L_1: $\mu_2\left[(\rho^3-\rho_2^3)(\rho-\rho_2)^2\right] - \mu_1\left[\rho_2^2(3\rho^2\rho_2 - 3\rho\rho_2^2+\rho_2^3)\right] = 0$

$(\rho^3-\rho_2^3)(\rho^2-2\rho_2\rho+\rho_2^2)$

$$\mu_2(\rho^5 - 2\rho_2\rho^4 + \rho_2^2\rho^3 - \rho_2^3\rho^2 + 2\rho_2^4\rho - \rho_2^5) - \mu_1(3\rho_2^3\rho^2 - 3\rho_2^4\rho + \rho_2^5) = 0$$

Again: $$\frac{\mu_2}{\mu_1}\cancel{\frac{m_2}{m_1}} = \frac{\rho_2^2(3\rho^2\rho_2 - 3\rho\rho_2^2+\rho_2^3)}{(\rho^3-\rho_2^3)(\rho-\rho_2)^2} = \frac{(3\rho_2^3\rho^2-3\rho_2^4\rho+\rho_2^5)}{(\rho^5-2\rho_2\rho^4+\rho_2^2\rho^3-\rho_2^3\rho^2+2\rho_2^4\rho-\rho_2^5)}$$

$(\rho^3-\rho_2^3)(\rho^2-2\rho_2\rho+\rho_2^2)$

$\rho^5-2\rho_2\rho^4+\rho_2^2\rho^3-\rho_2^3\rho^2+2\rho_2^4\rho-\rho_2^5$

Gather Like-Terms: Note: $\sum C_r^n(1-x)^r x^{n-r}$

$$\mu_2\left[\rho^5-2\rho_2\rho^4+\rho_2^2\rho^3-\rho_2^3\rho^2+2\rho_2^4\rho-\rho_2^5\right]-\mu_1\left[3\rho_2^3\rho^2-3\rho_2^4\rho+\rho_2^5\right]$$

order: $\left.\begin{matrix}\rho^5\\ \rho_2^5\end{matrix}\right\} \to [a_5, a_4, a_3, a_2, a_1, a_0]$

↳ $(\mu_2)\rho^5 + (-2\mu_2\rho_2)\rho^4 + (\mu_2\rho_2^2)\rho^3 + (-\mu_2\rho_2^3 - 3\mu_1\rho_2^3)\rho^2 + (2\mu_2\rho_2^4+3\mu_1\rho_2^4)\rho + (-\mu_2\rho_2^5 - \mu_1\rho_2^5) = 0$

$[-(\mu_2+\mu_1)]\rho_2^5 + [(2\mu_2+3\mu_1)\rho]\rho_2^4 + [-(\mu_2+3\mu_1)\rho^2]\rho_2^3 + [$

$\cancel{a_5 = \mu_2}$

$\cancel{a_4 = -2\mu_2\rho_2}$

$\cancel{a_3 = \mu_2\rho_2^2}$

$\cancel{a_2 = -(\mu_2\rho_2^3+3\mu_1\rho_2^3)}$

$\cancel{a_1 = 2\mu_2\rho_2^4+3\mu_1\rho_2^4}$

$\cancel{a_0 = -(\mu_2+\mu_1)\rho_2^5}$

→

$a_5 = -(\mu_2+\mu_1)$

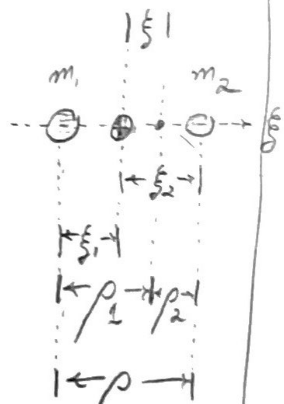
$a_4 = (2\mu_2+3\mu_1)\rho$

$a_3 = -(\mu_2+3\mu_1)\rho^2$

$a_2 = -\mu_2\rho^3$

$a_1 = -2\mu_2\rho^4$

$a_0 = \mu_2\rho^5$



$\rho_1 = \rho-\rho_2$, $\rho_1 = \xi-\xi_1$

$\rho_2 = -\xi+\xi_2$ $\Big\}\ \rho_1+\rho_2 = \xi_2-\xi_1$

$x = \frac{\rho}{\rho_1} - 1$

L2: $\frac{\mu_2}{\mu_1}\frac{\cancel{m_2}}{\cancel{m_1}} = \frac{\rho_2^2(3\rho^2\rho_2 + 3\rho\rho_2^2 + \rho_2^3)}{(\rho^3-\rho_2^3)(\rho+\rho_2)^2}$

Expand:

N, Num: $\mu_1\left[(3\rho^2)\rho_2^3 + (3\rho)\rho_2^4 + \rho_2^5\right]$

D, Denom.: $\mu_2\left[(\rho^3-\rho_2^3)(\rho^2+2\rho\rho_2+\rho_2^2)\right]$

$\mu_2\left[\rho^5 + 2\rho^4\rho_2 + \rho^3\rho_2^2 - \rho^2\rho_2^3 - 2\rho\rho_2^4 - \rho_2^5\right]$

cost: $f = \mu_2 D - \mu_1 N \longrightarrow f = \mu_2\left[\cancel{(-1)\rho_2^5} + \cancel{(-2\rho)\rho_2^4} + \cancel{(-\rho^2)\rho_2^3} + (\rho^3)\rho_2^2 + (2\rho^4)\rho_2 + \rho^5\right]$

$- \mu_1\left[\cancel{(1)\rho_2^5} + \cancel{(3\rho)\rho_2^4} + \cancel{(3\rho^2)\rho_2^3}\right]$

Battin Coefficients:

$a_5 = -(\mu_1+\mu_2)$

$a_4 = -(3\mu_1+2\mu_2)\rho$

$a_3 = -(3\mu_1+\mu_2)\rho^2$

$a_2 = \mu_2\rho^3$

$a_1 = 2\mu_2\rho^4$

$a_0 = \mu_2\rho^5$

L3: $\frac{m_1}{m_2} = \frac{\mu_1}{\mu_2} = \frac{\rho_1^2(3\rho^2\rho_1 + 3\rho\rho_1^2 + \rho_1^3)}{(\rho^3 - \rho_1^3)(\rho + \rho_1)^2}$

$\mu_2 N: \quad \mu_2\left[3\rho^2\rho_1^3 + 3\rho\rho_1^4 + \rho_1^5\right]$

$\mu_1 D: \quad \mu_1\left[(\rho^3 - \rho_1^3)(\rho^2 + 2\rho\rho_1 + \rho_1^2)\right]$

$\quad\quad \mu_1\left[\rho^5 + 2\rho^4\rho_1 + \rho^3\rho_1^2 - \rho^2\rho_1^3 - 2\rho\rho_1^4 - \rho_1^5\right]$

Cost: $f(\mu_1, \mu_2, \rho_1, \rho) = \dots$

$\dots = \mu_2 N - \mu_1 D$

$\dots = \mu_2\left[\rho_1^5 + (3\rho)\rho_1^4 + (3\rho^2)\rho_1^3\right]$

$- \mu_1\left[(-)\rho_1^5 + (-2\rho)\rho_1^4 + (-\rho^2)\rho_1^3 + (\rho^3)\rho_1^2 + (2\rho^4)\rho_1 + \rho^5\right.$

Battin Coefficients:

$$\boxed{\begin{aligned}
a_5 &= \mu_2 + \mu_1 \\
a_4 &= (2\mu_1 + 3\mu_2)\rho \\
a_3 &= (\mu_1 + 3\mu_2)\rho^2 \\
a_2 &= -\mu_1\rho^3 \\
a_1 &= -2\mu_1\rho^4 \\
a_0 &= -\mu_1\rho^5
\end{aligned}}$$

L1:

$$\frac{\partial U}{\partial \xi} = \frac{\partial J}{\partial \rho_1}\frac{\partial \rho_1}{\partial \xi} + \frac{\partial J}{\partial \rho_2}\frac{\partial \rho_2}{\partial \xi}$$

$$\frac{\partial U}{\partial \rho_1} = \frac{\partial J}{\partial \rho_2} = 0$$

$$\rho_1 = \xi - \xi_1 \rightarrow \frac{\partial \rho_1}{\partial \xi} = 1$$

$$\rho_2 = -\xi + \xi_2 \rightarrow \frac{\partial \rho_2}{\partial \xi} = -1$$

$$\frac{\partial U}{\partial \xi} = \frac{d^2\xi}{dt^2} - 2\omega\frac{d\eta}{dt} = \frac{\partial U}{\partial \rho_1} - \frac{\partial U}{\partial \rho_2}$$

$$\frac{\partial \rho_1}{\partial \eta} = \frac{\eta}{\rho_1}, \quad \frac{\partial \rho_2}{\partial \eta} = \frac{\eta}{\rho_2} \longrightarrow \text{if } \eta = 0, \text{ collinear solutions obtain}$$

$$\frac{1}{\rho_1}\frac{\partial U}{\partial \rho_1}(\xi - \xi_1) - \left(-\frac{1}{\rho_2}\frac{\partial U}{\partial \rho_2}(\xi - \xi_2)\right) = 0$$

$$\frac{\partial U}{\partial \rho_1} = \frac{\partial U}{\partial \rho_2},$$

$$U(\rho_1, \rho_2) = \frac{Gm_1}{2}\left(\frac{\rho_1^2}{\rho^3} + \frac{2}{\rho_1}\right) + \frac{Gm_2}{2}\left(\frac{\rho_2^2}{\rho^3} + \frac{2}{\rho_2}\right)$$

$$\frac{\partial U}{\partial \rho_1} = \frac{Gm_1}{2}\left(\frac{2\rho_1}{\rho^3} - \frac{2}{\rho_1^2}\right) = \mu_1\left(\frac{\rho_1^3 - \rho^3}{\rho^3\rho_1^2}\right)$$

$$\frac{\partial U}{\partial \rho_2} = Gm_2\left(\frac{\rho_2}{\rho^3} - \frac{1}{\rho_2^2}\right) = \mu_2\left(\frac{\rho_2^3 - \rho^3}{\rho^3\rho_2^2}\right)$$

$$\mu_1\left(\frac{\rho_1^3 - \rho^3}{\rho^3\rho_1^2}\right) = \mu_2\left(\frac{\rho_2^3 - \rho^3}{\rho^3\rho_2^2}\right)$$

$$\frac{\mu_2}{\mu_1} = \frac{\cancel{\rho^3}\rho_2^2}{\cancel{\rho^3}\rho_1^2}\left(\frac{\rho_1^3 - \rho^3}{\rho_2^3 - \rho^3}\right) \longrightarrow \text{as } f(\rho_2), \; \rho_1 = \rho - \rho_2$$

$$= \frac{\rho_2^2\left[(\rho - \rho_2)^3 - \rho^3\right]}{(\rho - \rho_2)^2(\rho_2^3 - \rho^3)}$$

$$(\rho^2 - 2\rho\rho_2 + \rho_2^2)(\rho - \rho_2)$$

$$= \cancel{\rho^3} - \cancel{\rho^2\rho_2} - \cancel{2\rho^2\rho_2} + 2\rho\rho_2^2 + \rho\rho_2^2 - \rho_2^3$$

$$= \cancel{\rho^3} - 3\rho^2\rho_2 + 3\rho\rho_2^2 - \rho_2^3$$

$$\boxed{\text{L1:} \quad \frac{\mu_2}{\mu_1} = \frac{\rho_2^2(3\rho^2\rho_2 - 3\rho\rho_2^2 + \rho_2^3)}{(\rho^3 - \rho_2^3)(\rho - \rho_2)^2}} \quad \square$$

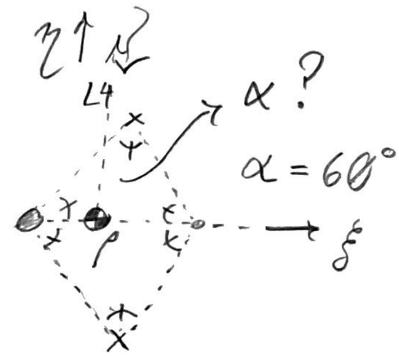
$L4, L5$:

$$\frac{\partial J}{\partial \xi} = \frac{\partial J}{\partial \rho_1}\frac{\partial \rho_1}{\partial \xi} + \frac{\partial J}{\partial \rho_2}\frac{\partial \rho_2}{\partial \xi}$$

$$\frac{\partial J}{\partial \eta} = \frac{\partial J}{\partial \rho_1}\frac{\partial \rho_1}{\partial \eta} + \frac{\partial J}{\partial \rho_2}\frac{\partial \rho_2}{\partial \eta}$$

$L4/L5$:

$$\frac{\partial J}{\partial \rho_1} = \frac{\partial J}{\partial \rho_2} = 0, \quad \rho_1 = \rho_2 = \rho$$



$$\frac{\partial J}{\partial \xi} = \frac{d^2\xi}{dt^2} - 2\omega\frac{d\eta}{dt}$$

$$\frac{\partial J}{\partial \eta} = \frac{d^2\eta}{dt^2} + 2\omega\frac{d\xi}{dt}$$

$\left(\frac{dv}{dt} = \dot{v}_\xi + \omega \times v_\xi \right.$

$$\boxed{\begin{aligned}\ddot{\xi} &= -2\omega\dot{\eta}\\ \ddot{\eta} &= 2\omega\dot{\xi}\end{aligned}} \rightarrow F = m_3(\ddot{\xi}\, i_\xi + \ddot{\eta}\, i_\eta)$$

$$= m_3(-2\omega\dot{\eta}\, i_\xi + 2\omega\dot{\xi}\, i_\eta)$$

$$\boxed{F_3 = 2m_3\omega\left[-\dot{\eta}\, i_\xi + \dot{\xi}\, i_\eta\right]}$$

$L4(\xi, \eta)$

$L5(\xi, -\eta)$

where: $\xi = \rho\cos(60°)$

$\eta = \rho\sin(60°)$

Lagrangian Analysis:

inertial velocity : ${}^{\mathcal{I}}v = \omega r \quad \therefore \quad KE = \frac{1}{2} m \, {}^{\mathcal{I}}v^2$

$= \sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2} \qquad = \frac{1}{2} m (\omega r)^2$

$= \frac{1}{2} m r^2 \omega^2$

$= \frac{1}{2} I \dot{\theta}^2$

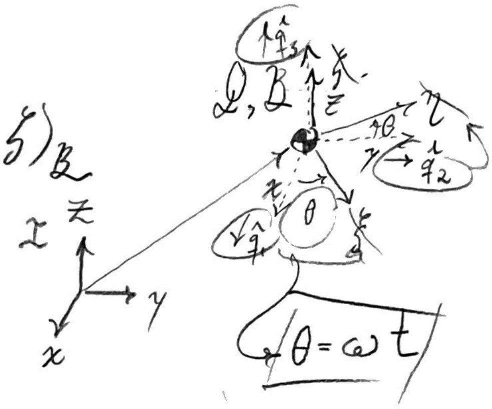
① Energy:

Potential: $U = \dfrac{G m_1 m_2}{\rho} + \dfrac{G m_1 m}{\rho_1} + \dfrac{G m_2 m}{\rho_2}$

Kinetic: $T = \underbrace{\frac{1}{2} m_1 r_1^2 \omega^2}_{\text{Body 1}} + \underbrace{\frac{1}{2} m_2 r_2^2 \omega^2}_{\text{Body 2}} + \frac{1}{2} m (\dot{x}^2 + \dot{y}^2 + \dot{z}^2)$

② Coordinates:

$(x, y, z)_{\mathcal{I}} \longrightarrow (q_1, q_2, q_3)_{\mathcal{Q}} = (\xi, \eta, \zeta)_{\mathcal{B}}$



$x = q_1 \cos(\theta) - q_2 \sin(\theta)$

$y = q_1 \sin(\theta) + q_2 \cos\theta$

$z = \zeta = q_3$

↳ where $\theta = \omega t$:

$$\boxed{\begin{aligned} x &= q_1 \cos(\omega t) - q_2 \sin(\omega t) \\ y &= q_1 \sin(\omega t) + q_2 \cos(\omega t) \\ z &= q_3 \end{aligned}} \longrightarrow R_z = \begin{bmatrix} \cos(\omega t) & -\sin(\omega t) & 0 \\ \sin(\omega t) & \cos(\omega t) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

③ Lagrangian / Hamiltonian:

$L = KE - PE = T - U$

$H = \sum_i^n \dot{q}_i p_i - L \quad \left(H(\vec{p}, \vec{q}, t) = \sum_i^n p_i \dot{q}^i - L(\vec{q}, \dot{\vec{q}}, t) \right)$ → "Einstein" notation

$$\frac{\partial L}{\partial \vec{q}} - \frac{d}{dt} \frac{\partial L}{\partial \dot{\vec{q}}} = 0$$

Over ⟶

↳ $\dfrac{\partial}{\partial \vec{q}}(L) = \dfrac{\partial}{\partial \vec{q}}(T) - \dfrac{\partial}{\partial \vec{q}}(U)$

$L = \frac{1}{2}\left[\dot{q}_1^2 + \dot{q}_2^2 + \dot{q}_3^2 + 2\omega(q_1 \dot{q}_2 - q_2 \dot{q}_1) + \omega^2 (q_1^2 + q_2^2)\right] + U$

$H = \frac{1}{2}\left[p_1^2 + p_2^2 + p_3^2 - 2\omega(q_1 p_2 - q_2 p_2)\right] - U$

$\left\} \; U = G\left[\frac{m_1}{\rho_1} + \frac{m_2}{\rho_2}\right.\right.$

$$\begin{bmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m \end{bmatrix} \begin{matrix} \ddot{q}_1 \\ \ddot{q}_2 \\ \ddot{q}_3 \end{matrix} + \begin{bmatrix} 0 & -2\omega & 0 \\ -2\omega & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{matrix} \dot{q}_1 \\ \dot{q}_2 \\ \dot{q}_3 \end{matrix} + \begin{bmatrix} -\omega^2 & 0 & 0 \\ 0 & -\omega^2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{matrix} q_1 \\ q_2 \\ q_3 \end{matrix} = \vec{F}$$

Lagrange's EOMs:

① $$\frac{d^2 q_1}{dt^2} - 2\omega \frac{dq_2}{dt} = \omega^2 q_1 + \frac{\partial U}{\partial q_1} \qquad \ddot{q}_1 - 2\omega \dot{q}_2 - \omega^2 q_1 = \frac{\partial U}{\partial q_1}$$

② $$\frac{d^2 q_2}{dt^2} - 2\omega \frac{dq_1}{dt} = \omega^2 q_2 + \frac{\partial U}{\partial q_2} \qquad \ddot{q}_2 - 2\omega \dot{q}_1 - \omega^2 q_2 = \frac{\partial U}{\partial q_2}$$

③ $$\frac{d^2 q_3}{dt^2} = \frac{\partial U}{\partial q_3} \qquad \ddot{q}_3 = \frac{\partial U}{\partial q_2}$$

Hamilton's Canonical Equations:

$$\frac{dq_1}{dt} = p_1 + \omega q_2$$

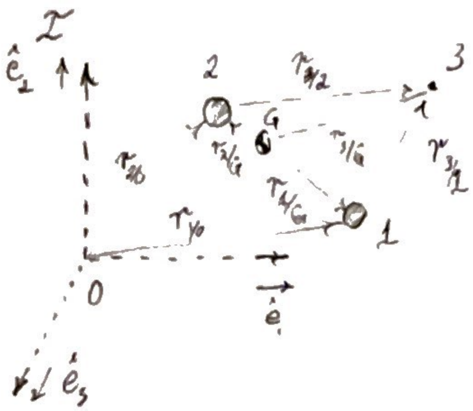
$$\frac{dq_2}{dt} = p_2 - \omega q_1 \quad \rightarrow \quad \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \dot{q}_3 \end{bmatrix} = \mathbb{1} \vec{p} + \begin{bmatrix} 0 & \omega & 0 \\ -\omega & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \vec{q}$$

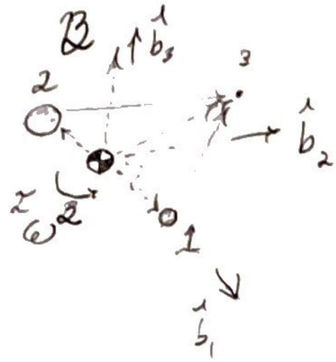
$$\frac{dq_3}{dt} = p_3$$

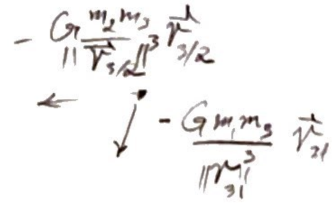
$$\frac{dp_1}{dt} = \omega p_2 + \frac{\partial U}{\partial q_1}$$

$$\frac{dp_2}{dt} = -\omega p_1 + \frac{\partial U}{\partial q_2} \quad \rightarrow \quad \begin{bmatrix} f_1 \\ f_2 \\ f_3 \end{bmatrix} = \begin{bmatrix} 0 & \omega & 0 \\ -\omega & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \vec{p} + \begin{bmatrix} D_{q_1} \\ D_{q_2} \\ D_{q_3} \end{bmatrix} U$$

$$\frac{dp_3}{dt} = \frac{\partial U}{\partial q_3}$$







$m_3 << m_2, m_1$

$\mathcal{I} = (G, \hat{e}_1, \hat{e}_2, \hat{e}_3)$

$\mathcal{B} = (G, \hat{b}_1, \hat{b}_2, \hat{b}_3)$

$\hat{b}_2 = \hat{b}_3 \times \hat{b}_1$

$^{\mathcal{I}}\vec{\omega}^{\mathcal{B}} = n\hat{e}_3$

$(\xi, \eta, \zeta)_{\mathcal{B}}$

$$\left.\begin{aligned}
\vec{r}_{3/G} &= \xi\hat{b}_1 + \eta\hat{b}_2 + \zeta\hat{b}_3 \\
^{\mathcal{I}}\vec{\omega}^{\mathcal{B}} &= n\hat{e}_3 \\
^{\mathcal{B}}\vec{a}_{3/G} &= \ddot{\xi}\hat{b}_1 + \ddot{\eta}\hat{b}_2 \\
\vec{r}_{3/1} &= (\xi - \xi_1)\hat{b}_1 + \eta\hat{b}_2 \\
\vec{r}_{3/2} &= (\xi + \xi_2)\hat{b}_1 + \eta\hat{b}_2
\end{aligned}\right\}$$

$$\cancel{m_3}\,{}^{\mathcal{I}}\vec{a}_{3/G} = -\frac{G m_1 \cancel{m_3}}{\|\vec{r}_{3/2}\|^3}\vec{r}_{3/1} - \frac{G m_2 \cancel{m_3}}{\|\vec{r}_{3/2}\|^3}\vec{r}_{3/2}$$

$$\left\{ {}^{\mathcal{B}}\vec{a}_{3/G} + 2\,{}^{\mathcal{I}}\vec{\omega}^{\mathcal{B}} \times {}^{\mathcal{B}}\vec{v}_{3/G} + {}^{\mathcal{I}}\vec{\omega}^{\mathcal{B}} \times ({}^{\mathcal{I}}\vec{\omega}^{\mathcal{B}} \times \vec{r}_{3/G}) = -\frac{\mu_1}{\|\vec{r}_{3/1}\|^3}\vec{r}_{3/1} - \frac{\mu_2}{\|\vec{r}_{3/2}\|^3}\vec{r}_{3/2} \right\}$$

$$\left[\ddot{\xi}\hat{b}_1 + \ddot{\eta}\hat{b}_2\right] + 2\left[n\hat{e}_3 \times (\dot{\xi}\hat{b}_1 + \dot{\eta}\hat{b}_2)\right] + \left[n\hat{e}_3 \times (n\hat{e}_3 \times (\xi\hat{b}_1 + \eta\hat{b}_2))\right]$$

$$= (\ddot{\xi}\hat{b}_1 + \ddot{\eta}\hat{b}_2) + 2\left[n\dot{\xi}\hat{b}_2 - n\dot{\eta}\hat{b}_1\right] + \left[n\hat{e}_3 \times (n\xi\hat{b}_2 - n\eta\hat{b}_1)\right]$$

$$\dots + \left[-n^2\xi\hat{b}_1 - n^2\eta\hat{b}_2\right]$$

$$= (\ddot{\xi} - 2n\dot{\eta} - n^2\xi)\hat{b}_1 + (\ddot{\eta} + 2n\dot{\xi} - n^2\eta)\hat{b}_2$$

$(1)\cdot 2\dot{\xi}) + ((2)\cdot 2\dot{\eta})$

$$2\left[\dot{\xi}\ddot{\xi} + \dot{\eta}\ddot{\eta} - n^2(\xi\dot{\xi} + \eta\dot{\eta})\right] = -\frac{2\mu_1((\xi-\xi_1)\dot{\xi} + \eta\dot{\eta})}{r_1^3} - \frac{2\mu_2((\xi+\xi_2)\dot{\xi} + \eta\dot{\eta})}{r_2^3}$$

EOM:

$$(1)\quad \ddot{\xi} - 2n\dot{\eta} - n^2\xi = -\frac{\mu_1}{\|\vec{r}_{3/1}\|^3}(\xi - \xi_1) - \frac{\mu_2}{\|\vec{r}_{3/2}\|^3}(\xi + \xi_2)$$

$$(2)\quad \ddot{\eta} + 2n\dot{\xi} - n^2\eta = -\frac{\mu_1}{\|\vec{r}_{3/1}\|^3}\eta - \frac{\mu_2}{\|\vec{r}_{3/2}\|^3}\eta$$

Surfaces of Zero Relative Velocity

$$\left.\begin{aligned} \rho_1^2 &= (\xi - \xi_1)^2 + \eta^2 \\ \rho_2^2 &= (\xi - \xi_2)^2 + \eta^2 \\ (\zeta &\doteq 0) \\ \rho \equiv r_{12} &= \xi_2 - \xi_1 \\ m_1 \xi_1 + m_2 \xi_2 &= 0 \end{aligned}\right\}$$

$$\xi_1 = \xi - \rho_1 + \eta$$

$$\begin{aligned} m_1\rho_1^2 + m_2\rho_2^2 &= m_1\left((\xi-\xi_1)^2 + \eta^2\right) \\ &\quad + m_2\left((\xi-\xi_2)^2 + \eta^2\right) \\ &= m_1\left[\xi^2 - 2\xi\xi_1 + \xi_1^2 + \eta^2\right] \\ &\quad + m_2\left[\xi^2 - 2\xi\xi_2 + \xi_2^2 + \eta^2\right] \\ &= (m_1+m_2)(\xi^2+\eta^2) + m_1\xi_1^2 + m_2\xi_2^2 \\ &\quad \cancel{-2(m_1\xi_1 + m_2\xi_2)\xi} \rightarrow 0, \text{ com corollary} \\ &= (m_1+m_2)(\xi^2+\eta^2) + \left(\frac{m_1 m_2}{m_1+m_2}\right)\rho^2 \end{aligned}$$

$$\therefore \quad \xi^2 + \eta^2 = \frac{m_1\rho_1^2 + m_2\rho_2^2}{m_1+m_2} - \underbrace{\frac{m_1 m_2}{(m_1+m_2)^2}\rho^2}_{\text{Const.}}$$

Jacobi's Integral:

$$\begin{aligned} v_{rel}^2 &= \omega^2(\xi^2+\eta^2) + \frac{2Gm_1}{\rho_1} + \frac{2Gm_2}{\rho_2} - C \\ &= \omega^2\left[\frac{m_1\rho_1^2 + m_2\rho_2^2}{m_1+m_2} - \frac{m_1 m_2}{(m_1+m_2)^2}\rho^2\right] + 2G\left[\frac{m_1}{\rho_1} + \frac{m_2}{\rho_2}\right] - C \\ &= \omega^2\left[\frac{m_1\rho_1^2 + m_2\rho_2^2}{m_1+m_2}\right] + 2G\left[\frac{m_1}{\rho_1} + \frac{m_2}{\rho_2}\right] - C^* \rightarrow C^* = C + \frac{m_1 m_2}{(m_1+m_2)^2}\rho^2 \end{aligned}$$

$$v_{rel}^2 = 2J - C^* \longrightarrow \text{For zero relative velocity:}$$

$$\boxed{2J_{(\rho_1,\rho_2)} = C^*}$$

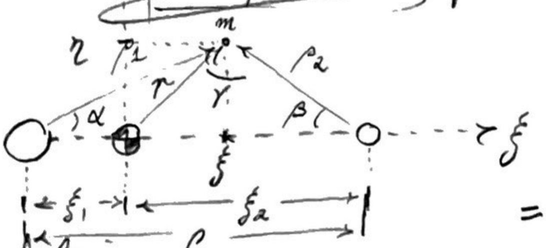
$$\left(J(\rho_1,\rho_2) = \frac{Gm_1}{2}\left(\frac{\rho_1^2}{\rho^3} + \frac{2}{\rho_1}\right) + \frac{Gm_2}{2}\left(\frac{\rho_2^2}{\rho^3} + \frac{2}{\rho_2}\right)\right)$$

$$J(\rho_1, \rho_2) = \frac{Gm_1}{2}\left(\frac{\rho_1^2}{\rho^3} + \frac{2}{\rho_1}\right) + \frac{Gm_2}{2}\left(\frac{\rho_2^2}{\rho^3} + \frac{2}{\rho_2}\right)$$

Let: $\boxed{x_1 = \frac{\rho_1}{\rho} > 0, \quad x_2 = \frac{\rho_2}{\rho} > 0.}$

$$\longrightarrow 2J(x_1, x_2) = \frac{Gm_1}{\cancel{2}}\left(\rho x_1^2 + \frac{2}{\rho x_1}\right) + \frac{Gm_2}{\cancel{2}}\left(\rho x_2^2 + \frac{2}{\rho x_2}\right)$$

$$= Gm_1\left(\frac{\rho^2 x_1^3 + 2}{\rho x_1}\right) + Gm_2\left(\frac{\rho^2 x_2^3 + 2}{\rho x_2}\right)$$



$$= \frac{Gm_1(\rho^2 x_1^3 + 2)\rho x_2 + Gm_2(\rho^2 x_2^3 + 2)\rho x_1}{\rho^2 x_1 x_2}$$

Law of Cosines:

$\rho_2^2 = \rho_1^2 + \rho^2 - 2\rho_1\rho\cos(\alpha)$

$\cos(\alpha) = \frac{(\rho_1^2 + \rho^2) - \rho_2^2}{2\rho_1\rho}$

$\cos(\beta) = \frac{(\rho_2^2 + \rho^2) - \rho_1^2}{2\rho_2\rho}$

$\tan(\alpha) = \frac{\eta}{\xi}$

$$= (Gm_1\rho^2 x_1^3 + 2Gm_1)\rho x_2 + (Gm_2\rho^2 x_2^3 + 2Gm_2)\rho x_1$$

$$= x_1^3 + \frac{2\cancel{Gm_1\rho x_2}}{\cancel{Gm_1\rho^2\rho x_2}} + \frac{Gm_2\rho^2 x_2^3 \rho x_1}{Gm_1\rho^2\rho x_2} + \frac{2Gm_2\rho x_1}{Gm_1\rho^2\rho x_2}$$

Plot: $\rho x_1 = \rho_1$

$\rho x_2 = \rho_2$

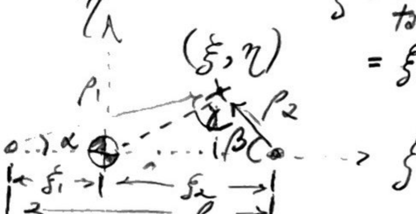
$\eta = \rho_1\sin(\alpha)$

$= \rho_2\sin(\beta)$

$\xi = \frac{\eta}{\tan(\alpha)} - \xi_1$

$= \xi_2 - \frac{\eta}{\tan(\beta)}$

$$x_1^3 + \frac{2}{\rho^2} + \frac{m_2}{m_1}x_2^2 x_1 + 2\frac{m_2}{m_1}\frac{x_1}{\rho^2 x_2}$$



$$= x_1^3 + \frac{m_2}{m_1}\left(\frac{\rho^2 x_2^3}{\rho^2 x^2} + \frac{2}{\rho^2 x_2}\right)x_1 + \frac{2}{\rho^2}$$

$\rho^2 = \rho_1^2 + \rho_2^2 - 2\rho_1\rho_2\cos(\alpha)$

$$= x_1^3 + \dots x_1 + \frac{2}{\rho^2}$$

$\cos\gamma = \frac{\rho^2 - (\rho_1^2 + \rho_2^2)}{-2\rho_1\rho_2}$

$\therefore$ $\boxed{x_1^3 + Ax_1 + 2 = 0_\square}$, $A = \rho\frac{C^*}{Gm_1} - \frac{m_2}{m_1}\left(x_2^2 + \frac{2}{x_2}\right)$

$\cos\gamma = \frac{\rho_1^2 + \rho_2^2 - \rho^2}{2\rho_1\rho_2}$

If $A \geq 3$, $3 = \rho\frac{C^*}{Gm} - \frac{m_2}{m_1}\left(x_2^2 + \frac{2}{x_2}\right)$

$\boxed{x_2^3 - Bx_2 + 2 \leq 0,}$ $B = 3 + \frac{\rho}{Gm_2}(C^* - 3\rho^2\omega^2)$

Law of Sines:

$\eta = \rho_1\sin(\alpha) = \rho_2\sin(\beta)$

$\xi = \frac{\eta}{\tan\alpha} - |\xi_1| = ||\xi_2|| - \frac{\eta}{\tan(\beta)}$

Law of Cosines:

$\cos(\alpha) = \frac{(\rho_1^2 + \rho^2) - \rho_2^2}{2\rho_1\rho}$, $\cos(\beta) = \frac{(\rho_2^2 + \rho^2) - \rho_1^2}{2\rho_2\rho}$

Algorithm:

- C^*:
  - Set $C^* > 3\rho^2\omega^2$
- B:
  - Set $B = 3 + \frac{\rho}{\mu_2}(C^* - 3\rho^2\omega^2)$
    > Check $B < 0$?
- Solve:

  $x^3 - Bx + 2 = 0 \longrightarrow x_2 = \text{roots}([1, 0, -B, 2])$
- Sort:

  if $(\text{isreal}(x_2)\ \&\&\ (x_2 > 0))$

  $x_2 \rightarrow \min, \max$
- Locus:

$\forall\, x_2 \in (x_{2,min}, x_{2,max})$

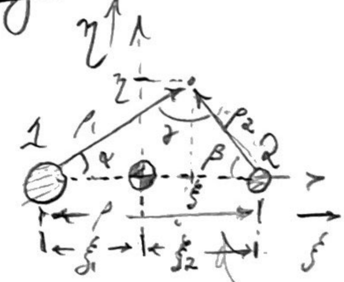
- $A = \frac{\rho C^*}{\mu_1} - \frac{\mu_2}{\mu_1}\left(x_2^2 + \frac{2}{x_2}\right)$
- if $(A < 3)$?

  > Solve $x_1^3 - Ax_1 + 2 = 0$.

  ↳ $(x_1, x_2) \xrightarrow{\rho} (\rho_1, \rho_2)$; by $\begin{cases} \rho_1 = \rho x_1 \\ \rho_2 = \rho x_2 \end{cases}$

  $(\rho_1, \rho_2) \longrightarrow (\xi, \eta)$; by $\begin{cases} \eta = \rho_1 \sin(\alpha) = \rho_2 \sin(\beta) \\ \xi = \|\xi_1\| - \frac{\eta}{\tan(\alpha)} \end{cases}$

fig. 1:



Jacobi Integral:

$$2\int \dot{\xi}\, d\dot{\xi} + 2\int \dot{\eta}\, d\dot{\eta} - 2n^2\int \xi\, d\xi - 2n^2\int \eta\, d\eta = -2\mu_1\int \frac{dr_1}{r_1^2} - 2\mu_2\int \frac{dr_2}{r_2^2}$$

$$\dot{\xi}^2 + \dot{\eta}^2 - n^2\xi^2 - n^2\eta^2 = \frac{2\mu_1}{r_1} + \frac{2\mu_2}{r_2}$$

$$= \boxed{\dot{\xi}^2 + \dot{\eta}^2 - n^2(\xi^2 + \eta^2) = 2\left(\frac{\mu_1}{r_1} + \frac{\mu_2}{r_2}\right) - C}$$

$$\hookrightarrow v_{rel}^2 = \omega^2 r^2 + \frac{2\mu_1}{r_1} + \frac{2\mu_2}{r_2} - C$$

Equilibrium:

$$\cancel{\ddot{\xi}}^{\,0} - \cancel{2n\dot{\eta}}^{\,0} - n^2\xi = -\frac{\mu_1(\xi - \xi_1)}{((\xi - \xi_1)^2 + \eta^2)^{3/2}} - \frac{\mu_2(\xi + \xi_2)}{((\xi + \xi_2)^2 + \eta^2)^{3/2}}$$

$$\cancel{\ddot{\eta}}^{\,0} + \cancel{2n\dot{\xi}}^{\,0} - n^2\eta = -\frac{\mu_1\eta}{((\xi - \xi_1)^2 + \eta^2)^{3/2}} - \frac{\mu_2\eta}{((\xi + \xi_2)^2 + \eta^2)^{3/2}}$$

$$-n^2\xi = -\frac{\mu_1(\xi - \xi_1)}{((\xi - \xi_1)^2 + \eta^2)^{3/2}} + \frac{\mu_2(\xi + \xi_2)}{((\xi + \xi_2)^2 + \eta^2)^{3/2}}$$

$$n^2\cancel{\eta} = \cancel{\eta}\left[\frac{\mu_1}{(\dots)^{3/2}} + \frac{\mu_2}{(\dots)^{3/2}}\right]$$

$$n^2 x(x^2 - 2xx_1 + x_1^2)(x^2 + 2xx_2 + x_2^2) - \mu_1(x^2 + 2xx_2 + x_2^2) - \mu_2(x^2 - 2xx_1 + x_1^2) = 0$$

$$n^2 x(x^4 + 2x^3x_2 + x^2x_2^2 - 2x^3x_1 - 4x^2x_1x_2 - 2xx_1x_2^2 + x^2x_1^2 + 2xx_1^2x_2 + x_1^2x_2^2)$$
$$-\mu_1(x^2 + 2xx_2 + x_2^2) - \mu_2(x^2 - 2xx_1 + x_1^2) = 0$$

$$n^2x^5 + (2n^2x_2 - 2n^2x_1)x^4 + (n^2x_2^2 - 4n^2x_1x_2 + n^2x_1^2)x^3 + (-2n^2x_1x_2^2 + 2n^2x_1^2x_2 - \mu_1 - \mu_2)x^2$$
$$+ (n^2x_1^2x_2^2 - 2\mu_1x_2 + 2\mu_2x_1)x - (\mu_1x_2^2 + \mu_2x_1^2) = 0$$

$$n^2\left\{(1)x^5 + (2x_2 - 2x_1)x^4 + (x_2^2 - 4x_1x_2 + x_1^2)x^3\right\} + (-2n^2x_1x_2^2 + 2n^2x_1^2x_2 - \mu_1 - \mu_2)x^2 + (n^2x_1^2x_2^2 - 2\mu_1x_2 + 2\mu_2x_1)x$$
$$- (\mu_1x_2^2 + \mu_2x_1^2) = 0$$

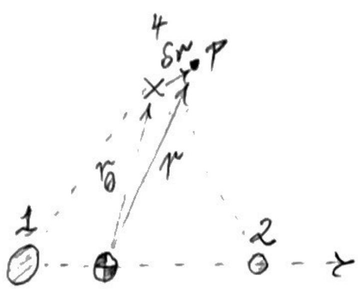
Stability of Lagrange Points

$\vec{r} = \vec{r}_e + \vec{\delta r}, \quad \rightarrow \quad (\delta r)^2 \rightarrow 0,$

$\vec{v} = \delta\vec{v}. \qquad (\delta v)^2 \rightarrow 0,$

$\delta r \delta v \rightarrow 0,$

$\delta v \delta r \rightarrow 0.$



$r = (\xi_0 + \delta\xi) i_\xi + (\eta_0 + \delta\eta) i_\eta$

$\rho = (\xi_0 - \xi_1 + \delta\xi) i_\xi + (\eta_0 + \delta\eta) i_\eta$

$\hookrightarrow \rho = r - (\xi_1 i_\xi + 0 i_\eta)$

$\frac{\partial \rho}{\partial r} = 1 i_\xi + 1 i_\eta$

$$\boxed{\frac{d\vec{x}}{dt} = M\vec{x}} \quad (8.41) \xrightarrow{\text{Characteristic}} \boxed{[M - \lambda \mathbb{1}] = \vec{0}}$$

$\hookrightarrow \vec{x} = \begin{bmatrix} \vec{\delta r} \\ \vec{\delta v} \end{bmatrix} \quad (8.42)$

$\hookrightarrow \dim(M) = 6$

$\hookrightarrow$ • 6^{th}-Order poly.

• roots:
- stable:
  > $Re(\lambda) < 0$
- unstable:
  > $Re(\lambda) \geq 0$

Equilateral Libration Points:

$$\underline{f}(\vec{r}) = -\frac{Gm_1}{\|\vec{\rho}_1\|^3}\vec{\rho}_1 - \frac{Gm_2}{\|\vec{\rho}_2\|^3}\vec{\rho}_2,$$

Expansion:

$$= f(r_0) + \left.\frac{\partial f}{\partial r}\right|_{r=r_0},$$

$$\vec{\delta r} + \ldots \equiv \vec{f}_0 + \vec{F}_0 \delta r + \ldots \longrightarrow F_0 = \nabla f(r)\Big|_{r_0}$$

As for $\vec{\rho}_2$,

$\rightarrow \frac{\partial \rho_1}{\partial r} = \frac{\partial \vec{\rho}_1}{\partial \vec{\rho}_1} = \mathbb{1},$

$$F = \frac{\partial \vec{f}}{\partial \vec{r}} = \frac{Gm_1}{\rho_1^5}(3\vec{\rho}_1\vec{\rho}_1^T - \rho_1^2\mathbb{1}) + \frac{Gm_2}{\rho_2^5}(3\vec{\rho}_2\rho_2^T - \rho_2^2\mathbb{1}),$$

$\frac{\partial \|\vec{\rho}_1\|}{\partial \vec{r}} = \frac{\partial \|\vec{\rho}_1\|}{\partial \vec{\rho}_1} = \frac{1}{\rho_1}\vec{\rho}_1^T$

(inverse unit?)

For L4: $\vec{r}_0 = \frac{1}{2}\left[(\xi_1 + \xi_2) i_\xi + \sqrt{3}\rho\, i_\eta\right],$

$$\vec{\rho}_1 = \frac{1}{2}\rho\left(i_\xi + \sqrt{3}\, i_\eta\right), \qquad \vec{\rho}_2 = \frac{1}{2}\rho\left(-i_\xi + \sqrt{3}\, i_\eta\right)$$

$$F_0 = \frac{Gm_1}{4\rho^3}\begin{bmatrix} -1 & 3\sqrt{3} & 0 \\ 3\sqrt{3} & 5 & 0 \\ 0 & 0 & -4 \end{bmatrix} + \frac{Gm_2}{4\rho^3}\begin{bmatrix} -1 & -3\sqrt{3} & 0 \\ -3\sqrt{3} & 5 & 0 \\ 0 & 0 & -4 \end{bmatrix}$$

$$= \frac{\omega^2}{4}\begin{bmatrix} -1 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & -4 \end{bmatrix} + \frac{\omega^2}{4}\left(\frac{m_1 - m_2}{m_1 + m_2}\right)\begin{bmatrix} 0 & 3\sqrt{3} & 0 \\ 3\sqrt{3} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$\frac{\partial \rho_1}{\partial \xi} = \frac{1}{2}\rho, \quad \frac{\partial \rho_1}{\partial \eta} = \frac{\sqrt{3}}{2}\rho; \quad \frac{\partial \rho_2}{\partial \xi} = -\frac{\rho}{2}, \quad \frac{\partial \rho_2}{\partial \eta} = \frac{\sqrt{3}}{2}$

State-Space Representation:

$$\left.\begin{aligned} \delta v &= \delta \dot{r} \\ \delta \dot{v} &= F_0\, \delta r \end{aligned}\right\} \qquad \begin{bmatrix} \delta \dot{\vec{r}} \\ \delta \dot{\vec{v}} \end{bmatrix} = \begin{bmatrix} \vec{0} & \mathbb{1} \\ F_0 - \Omega\Omega & -2\Omega \end{bmatrix} \begin{matrix} \delta \vec{r} \\ \delta \vec{v} \end{matrix} \rightarrow \Omega = \begin{bmatrix} 0 & -\omega & 0 \\ \omega & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\left[M - \lambda \mathbb{1}\right] = \left(\lambda^2 + \omega^2\right)\left[\lambda^4 + \omega^2 \lambda^2 + \frac{27}{4}\omega^4\left(\frac{m_1 m_2}{(m_1 + m_2)^2}\right)\right]$$

$$\Downarrow$$

$$\text{i.f.f.} \quad \cancel{\omega^4}^{\,1} - 27\cancel{\omega^4}^{\,1}\frac{m_2 m_2}{(m_1 + m_2)^2} \geq 0,$$

L4 orbit is stable.

$$\hookrightarrow \quad \frac{1}{27} \geq \frac{m_1 m_2}{(m_1 + m_2)^2} = \frac{m_1 m_2}{m_1^2 + 2m_1 m_2 + m_2^2}$$

$$m_1^2 + \cancel{2m_1 m_2} + m_2^2 \geq \cancel{27}^{\,25} m_1 m_2$$

$$\frac{m_1^2 + m_2^2}{m_1 m_2} \geq 25$$

$$\boxed{\text{L4:} \quad \frac{m_1}{m_2} + \frac{m_2}{m_1} \geq 25}$$

Collinear Libration Points:

$$\xi = \xi_0 + \delta\xi, \quad \eta = 0 + \delta\eta \quad \rightarrow \quad J = \frac{\omega^2}{2}\left[(\xi + \delta\xi)^2 + (\eta + \delta\eta)^2\right] + \frac{Gm_1}{\rho_1} + \frac{Gm_2}{\rho_2}$$

$$\frac{\partial J}{\partial \xi} = J_\xi + J_{\xi\xi}\,\delta\xi + J_{\eta\xi}\,\delta\eta + \ldots$$

$$\frac{\partial J}{\partial \eta} = J_\eta + J_{\xi\eta}\,\delta\xi + J_{\eta\eta}\,\delta\eta + \ldots$$

$$J_\xi = \omega^2(\xi + \delta\xi) - Gm_1 \quad \text{return}$$

$$M - \lambda \mathbb{1} = \begin{bmatrix} -\lambda & 0 & 1 & 0 \\ 0 & -\lambda & 0 & 1 \\ J_{\xi\xi} & J_{\eta\xi} & -\lambda & 2\omega \\ J_{\xi\eta} & J_{\eta\eta} & -2\omega & -\lambda \end{bmatrix} \rightarrow$$

Characteristic:

$$\lambda^4 + (4\omega^2 - J_{\xi\xi} - J_{\eta\eta})\lambda^2 + J_{\xi\xi}J_{\eta\eta} - J_{\xi\eta}^2 = 0$$

$$\hookrightarrow \frac{\partial^2 J}{\partial \xi^2} = \frac{\partial^2 J}{\partial \rho_1^2}\left(\frac{\partial \rho_1}{\partial \xi}\right)^2 + \frac{\partial J}{\partial \rho_1}\frac{\partial^2 \rho_1}{\partial \xi^2} + \frac{\partial^2 J}{\partial \rho_2^2}\left(\frac{\partial \rho_2}{\partial \xi}\right)^2 + \frac{\partial J}{\partial \rho_2}\frac{\partial^2 \rho_2}{\partial \xi^2}$$

tability cont'd)

$\rho_1 \frac{\partial \rho_1}{\partial \xi} = (\xi - \xi_1)$, $\quad \frac{\partial^2 \rho_1}{\partial \xi^2} = \frac{\partial}{\partial \xi}\left(\frac{2(\xi-\xi_1)}{2\sqrt{(\xi-\xi_1)^2}}\right) = 0$

$\rho_2 \frac{\partial \rho_2}{\partial \xi} = \xi - \xi_2$,

$\left(\frac{\partial \rho_1}{\partial \xi}\right)^2 = \left(\frac{2(\xi-\xi_1)}{2\sqrt{(\xi-\xi_1)^2 + \cancel{0+0}}}\right)^2 = 1$

$\left(\frac{\partial \rho_1}{\partial \xi}\right)^{\cancel{2}\,\to 1} + \rho_1\left(\frac{\partial^2 \rho_1}{\partial \xi^2}\right)^{\to 0} = 1$, vice-versa for ρ_2

$$\frac{\partial^2 J}{\partial \xi^2} = \frac{\partial^2 J}{\partial \rho_1^2}(\ldots)^{2\,\to 1} + \cancel{\frac{\partial J}{\partial \rho_1}\frac{\partial^2 \rho_1}{\partial \xi^2}}^{\to 0} + \frac{\partial^2 J}{\partial \rho_2^2}(\ldots)^{2\,\to 1} + \cancel{\frac{\partial J}{\partial \rho_2}\left(\frac{\partial^2 \rho_2}{\partial \xi^2}\right)}^{\to 0}$$

$$= \frac{\partial^2 J}{\partial \rho_1^2} + \frac{\partial^2 J}{\partial \rho_2^2}$$

$$= Gm_1\left(\frac{1}{\rho^3} + \frac{2}{\rho_1^3}\right) + Gm_2\left(\frac{1}{\rho^3} + \frac{2}{\rho_2^3}\right) \geq 0 \rightsquigarrow \text{cannot be stable}$$

$$\frac{\partial^2 J}{\partial \eta^2} = \frac{1}{\rho_1}\frac{\partial J}{\partial \rho_1} + \frac{1}{\rho_2}\frac{\partial J}{\partial \rho_2}$$

Points

L1: $\frac{\partial^2 J}{\partial \eta^2} = \left(\frac{1}{\rho_1} + \frac{1}{\rho_2}\right)\frac{\partial J}{\partial \rho_1} = Gm_1\rho_1\left(\frac{1}{\rho_1} + \frac{1}{\rho_2}\right)\left(\frac{1}{\rho^3} - \frac{1}{\rho_1^3}\right) < 0$

$\to < 0$; $\rho_1 < \rho,\ \rho_1 + \rho_2 = \rho$

L2: $\frac{\partial^2 J}{\partial \eta^2} = \left(\frac{1}{\rho_1} - \frac{1}{\rho_2}\right)\frac{\partial J}{\partial \rho_1} = Gm_1\rho_1\left(\frac{1}{\rho_1} - \frac{1}{\rho_2}\right)\left(\frac{1}{\rho^3} - \frac{1}{\rho_1^3}\right) < 0$

$\hookrightarrow \rho_2 < \rho_1 \therefore < 0$

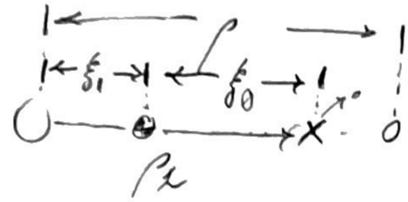
L3: $\frac{\partial^2 J}{\partial \eta^2} = \left(\frac{1}{\rho_2} - \frac{1}{\rho_1}\right)\frac{\partial J}{\partial \rho_2} = Gm_1\rho_2\left(\frac{1}{\rho_2} - \frac{1}{\rho_1}\right)\left(\frac{1}{\rho^3} - \frac{1}{\rho_2^3}\right) < 0$

$\hookrightarrow \rho_2 > \rho_1 \therefore > 0 \to \rho < \rho_2 \therefore < 0$

Summary:

L_1, L_2, L_3 do not meet Nyquist stability criterion.

State - Space Representation of Collinear Points:



L1: $\vec{\rho}_1 = (\xi - \xi_1 + \delta\xi)\, i_\xi + \delta\eta\, i_\eta$

$\vec{\rho}_2 = (\xi - \xi_2 + \delta\xi)\, i_\xi + \delta\eta\, i_\eta$

F_0 ?

$$F = \frac{\partial \vec{f}}{\partial \vec{r}} = \frac{Gm_1}{\rho_1^5}\left(3\vec{\rho}_1\vec{\rho}_1^{\,T} - \rho_1^2 \mathbb{1}\right) + \frac{Gm_2}{\rho_2^5}\left(3\vec{\rho}_2\vec{\rho}_2^{\,T} - \rho_2^2 \mathbb{1}\right)$$

$$\vec{\rho}_1\vec{\rho}_1^{\,T} = \begin{bmatrix} (\xi - \xi_1 + \delta\xi) \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} (\xi - \xi_1 + \delta\xi) & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} \overbrace{(\xi - \xi_1 + \delta\xi)^2}^{\rho_1^2} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 3\rho_1^2 & \cdots \\ \vdots & \ddots \end{bmatrix} - \begin{bmatrix} \rho_1^2 & & \\ & \rho_1^2 & \\ & & \rho_1^2 \end{bmatrix}$$

$$F_0 = \frac{Gm_1}{\rho_1^{\cancel{5}\,3}} \begin{bmatrix} 2\cancel{\rho_1^2} & 0 & 0 \\ 0 & -\cancel{\rho_1^2} & 0 \\ 0 & 0 & -\cancel{\rho_1^2} \end{bmatrix}$$

$$\boxed{\begin{aligned} F_0 &= \frac{\mu_1}{\rho_1^3}\begin{bmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} + \frac{\mu_2}{\rho_2^3}\begin{bmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \\ &= \left(\frac{\mu_1}{\rho_1^3} + \frac{\mu_2}{\rho_2^3}\right)\begin{bmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \end{aligned}}$$

Plug in to ODE 45...

$$M = \begin{bmatrix} \bar{0} & \mathbb{1} \\ F_0 - \Omega\Omega & -2\Omega \end{bmatrix}$$

$$\Omega = \begin{bmatrix} 0 & -\omega & 0 \\ \omega & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

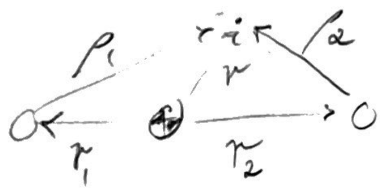
$$\Omega^2 = \begin{bmatrix} -\omega^2 & 0 & 0 \\ 0 & -\omega^2 & 0 \\ 0 & 0 & 0 \end{bmatrix} = -\omega^2 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

$$L_1: \quad \frac{m_2}{m_1} = \rho_2^2\frac{(3\rho^2\rho_2 - 3\rho\rho_2^2 + \rho_2^3)}{(\rho^3 - \rho_2^3)(\rho - \rho_2)^2} \qquad \rho_1 = \rho - \rho_2$$

$$\Downarrow$$

$$L_2: \quad \frac{m_2}{m_1} = \frac{\rho_2^3(3\rho^2\rho_2 + 3\rho\rho_2^2 + \rho_2^3)}{(\rho^3 - \rho_2^3)(\rho + \rho_2)^2} \qquad \rho_1 = r - r_1$$

$$L_3: \quad \frac{m_1}{m_2} = \rho_1^2(3\rho^2$$



$$\frac{m_2}{m_1} = \rho_2\frac{^2(3\rho\rho_2^2 - 3\rho\rho_2^2 + \rho_2^3)}{(\rho^3 - \rho_2^3)(\rho - \rho_2)^2}$$

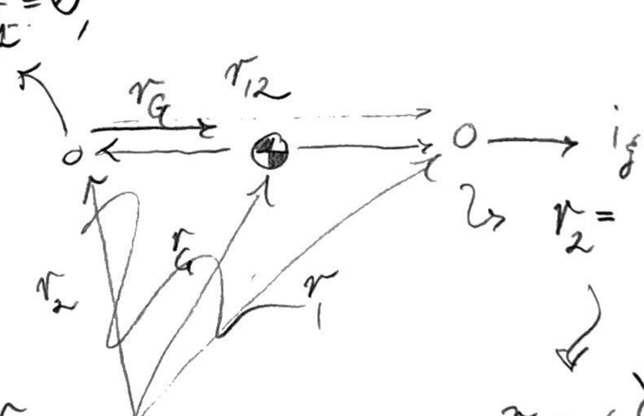
$$m_2(\rho^3 - \rho_2^3)(\rho^2 - 2\rho_2\rho + \rho_2^2) = m_1\rho_2^2(3\rho\rho_2^2 - 3\rho\rho_2^2 + \rho_2^3)$$

$$m_2\left[\rho^5 - 2\rho_2\rho^4 + \rho^3\rho_2^2 - \rho_2^3\rho^2 + 2\rho_2^4\rho - \rho_2^5\right] = m_1\left[3\rho^2\rho_2^3 - 3\rho\rho_2^4 + \rho_2^5\right]$$

$$m_2\rho^5 - 2m_2\rho_2\rho^4 + m_2\rho_2^2\rho^3 + (-3m_1\rho_2^3 - m_2\rho_2^3)\rho^2 + (2m_2\rho_2^4 + 3m_1\rho_2^4)\rho - (m_2 + m_1)\rho_2^5 = 0$$

$$\vec{r}_G = \frac{1}{M}\sum m_i r_i$$

let $r_1 = 0$,



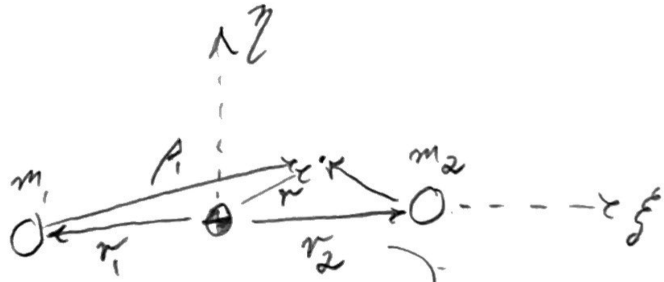
$$\leadsto r_2 = r_{12}\, i_\xi$$

$$\Downarrow$$

$$r_1 = r_{12} - r_G \qquad r_G = \frac{\cancel{G}}{(\mu_1 + \mu_2)}\left(\frac{\mu_2}{\cancel{G}}\right) r_{12}$$

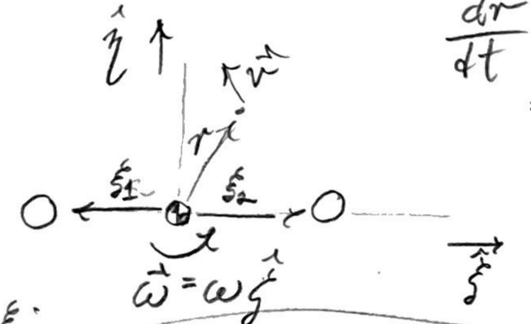
$$\omega^2 r_{12}^3 = G(m_1 + m_2) = \mu_1 + \mu_2$$

$$\frac{d^2\vec{r}}{dt^2} + 2\vec{\omega}\times\frac{dr}{dt} + \vec{\omega}\times(\vec{\omega}\times\vec{r}) + \frac{Gm_1}{\rho_1^3}\vec{\rho}_1 + \frac{Gm_2}{\rho_2^3}\vec{\rho}_2 = \vec{0}$$



$$\vec{\rho}_1 = r - r_1$$
$$\vec{\rho}_2 = r - r_2$$

$$\frac{d\vec{r}}{dt} = \dot{\vec{\rho}} + \vec{\omega}\times\vec{r}$$
$$= \dot{\rho} + \omega\xi i_\eta - \omega\eta i_\xi$$



$$\vec{\omega}\times\frac{d\vec{r}}{dt}$$
$$= -\omega\frac{d\eta}{dt} i_\xi + \omega\frac{d\xi}{dt} i_\eta$$

$$\vec{r} = \xi i_\xi + \eta i_\eta + \zeta i_\zeta$$

$$\vec{\omega}\times(\vec{\omega}\times r)$$
$$= -\omega^2[\xi i_\xi + \eta i_\eta]$$

$$\rho_1^2 = (\xi - \xi_1)^2 + \eta^2 + \zeta^2$$
$$\rho_2^2 = (\xi - \xi_2)^2 + \eta^2 + \zeta^2$$

$$J = \frac{\omega^2}{2}(\xi^2 + \eta^2) + \frac{Gm_1}{\rho_1} + \frac{Gm_2}{\rho_2}$$

$$\frac{d^2\vec{r}}{dt^2} + 2\vec{\omega}\times\frac{d\vec{r}}{dt} = \left[\frac{\partial J}{\partial\vec{r}}\right]^T$$

$$\frac{d^2\vec{r}}{dt^2}\cdot\frac{d\vec{r}}{dt} = \frac{1}{2}\frac{d}{dt}\left(\frac{dr}{dt}\cdot\frac{dr}{dt}\right) - \frac{\partial J}{\partial\vec{r}}\frac{d\vec{r}}{dt} = \frac{dJ}{dt}$$

$$\boxed{v_{rel}^2 = \omega^2(\xi^2 + \eta^2) + \frac{2Gm_1}{\rho_1} + \frac{2Gm_2}{2} - C}$$

$$J(\rho_1,\rho_2) = \frac{Gm_1}{2}\left(\frac{\rho_1^2}{\rho^3} + \frac{2}{\rho_1}\right) + \frac{Gm_2}{2}\left(\frac{\rho_2^2}{\rho^3} + \frac{2}{\rho_2}\right)$$

$$2J(\rho_1,\rho_2) - C^* = 0$$

per $J = \frac{\omega^2}{2}(\xi^2+\eta^2) + \frac{Gm_1}{\rho_1} + \frac{Gm_2}{\rho_2}$

$$\rho_1^2 = (\xi-\xi_1)^2 + \eta^2$$

$$\rho_2^2 = (\xi-\xi_2)^2 + \eta^2$$

$$f_1 = G\frac{m_1}{\rho_1^2}, \quad f_2 = G\frac{m_2}{\rho_2^2}, \quad f_i = -m_i\omega^2 r_i$$

ξ_η

$$\omega^2(m_1\rho_1 + m_2\rho_2) + G\left(\frac{m_1}{\rho_1^2} + \frac{m_2}{\rho_2^2}\right) = 0$$

$$J(\rho_1,\rho_2) = \omega^2(m_1) - \frac{Gm_1}{\rho_1^3} + \omega^2(m_2) - G\left(\frac{m_2}{\rho_3^3}\right)$$

$$= \omega^2(m_1+m_2) - G\left(\frac{m_1}{\rho_1^3} + \frac{m_2}{\rho_3^3}\right) = 0$$

$$\rho_2^2 = \sqrt{\rho_1^2 + \rho^2 - 2\rho_1\rho\cos\theta_1}$$

$$\rho_1^2 = \sqrt{\rho_2^2 + \rho^2 - 2\rho_2\rho\cos\theta_2}$$

$$v_{rel}^2 = \omega^2(\xi^2+\eta^2) + \frac{2Gm_1}{\rho_1} + \frac{2Gm_2}{\rho_2} - C$$

$$x_1^3 - Ax_1 + 2 = 0$$

$$x_1 = \frac{\rho_1}{\rho} > 0$$

$$x_2 = \frac{\rho_2}{\rho} > 0$$

$$\cos\theta_1 = \frac{\xi_1}{\rho_1} \qquad \cos\theta_2 = \frac{\xi_2}{\rho_2}$$

$$\tan\theta_1 = \frac{\eta}{\xi_1}$$

$$\eta = \xi_1\tan\theta_1$$

$$\rho_1 = \xi_1\sec\theta_1$$

$$\left(\frac{\rho_1}{\rho}\right)^3 - \left(\frac{\rho C^*}{Gm_1} - \frac{m_2}{m_1}\left(\left(\frac{\rho_2}{\rho}\right)^2 + \frac{2}{\left(\frac{\rho_2}{\rho}\right)}\right)\right)\left(\frac{\rho_1}{\rho}\right)$$

$$\rho_1^3 + \frac{m_2}{m_1}\rho_1\rho_2^2 + \left[\left(\frac{m_2}{m_1}\right)\left(\frac{\rho_1}{\rho_2}\right) + 2 - \frac{\rho_1 C^*}{Gm_1}\right]\rho^3 = 0$$

$$\left(\frac{\rho_1}{\rho}\right)^3 - \left[\frac{\rho C^*}{Gm_1} - \frac{m_2}{m_1}\left(\left(\frac{\rho_2}{\rho}\right)^2 + \frac{2}{\left(\frac{\rho_2}{\rho}\right)}\right)\right]\left(\frac{\rho_1}{\rho}\right) + 2 = 0$$

$$\rho_1^3 - \left[\frac{\rho^3 C^*}{Gm_1}\rho_1 - \frac{m_2}{m_1}\left(\rho_2^2\rho_1 + \frac{\rho_1}{\rho_2}\rho^3\right)\right] + 2\rho^3 = 0$$

$$\rho_1^3 + \frac{m_2}{m_1}\rho_1\rho_2^2 - \frac{\rho_1 C^*}{Gm_1}\rho^3 + \frac{m_2}{m_1}\frac{\rho_1}{\rho_2}\rho^3 + 2\rho^3 = 0$$

$$\rho_1^3 + \frac{m_2}{m_1}\rho_1\rho_2^2 + \left[-\rho_1\frac{C^*}{Gm_1} + \frac{m_2}{m_1}\frac{\rho_1}{\rho_2} + 2\right]\rho^3 = 0$$

$m_1 = m_2 = m$

$\rho^2 = D^2 + \zeta^2$, $\omega^2 = \frac{Gm}{4D^3}$

$\tan\theta = \frac{\zeta}{D}$,

$\cos\theta = \frac{D}{\rho}$,

$\zeta = D\tan\theta$,

$\rho = D\sec\theta$

$$\left(\frac{d\zeta}{dt}\right)^2 = \frac{4Gm}{\rho} - C$$

$\zeta = \eta = 0$

$$\frac{dJ}{dt} = \frac{d}{dt}\left[(D^2+\zeta^2)\frac{Gm}{8D^3} + 2\left(\frac{Gm}{4D^3}\right)\right]$$

$$= \frac{d}{dt}\left[\cancel{\frac{Gm}{8D}} + 2\zeta\frac{Gm}{48D^3}\frac{d\zeta}{dt} - \cancel{\frac{Gm}{2D^4}D}\right]$$

$$\frac{dJ}{dt} = \frac{Gm}{4D^3}\zeta\frac{d\zeta}{dt}$$

$$\frac{d}{dt}\left(\frac{2Gm}{\rho}\right) = -\frac{2Gm}{\rho^2}\dot\rho$$

$$v_0^2 \longrightarrow C = \frac{4Gm}{\rho} - v_0^2,$$

$1 - \cos\theta$

$$\left(\frac{d\zeta}{dt}\right)^2 = v_0^2 - 16\omega^2 D^2\left(1 - \frac{D}{\rho}\right)$$

Let $B = \frac{v_0^2}{16\omega^2 D^2}$

$$\left(\frac{d\theta}{dt}\right)^2 = 16\omega^2\cos^4\theta\left[B - (1-\cos\theta)\right]$$

$B = 1 - \cos\theta_m$

$$\left(\frac{d\theta}{dt}\right)^2 = 16\omega^2\cos^4\theta\left[\cos\theta - \cos\theta_m\right]$$

Let $x = \cos\theta$,

$x_m = \cos\theta_m$

$$\left(\frac{d\theta}{dt}\right)^2 = \frac{1}{\omega^2}(2\omega x)^4\left[x - x_m\right]$$

$$\left(\frac{dx}{dt}\right)^2 = 16\omega^2 x^4(1-x^2)(x-x_m)$$

$$4\omega T = \int_{x_m}^{1}\frac{dx}{x^2\sqrt{P(x)}}, \quad P(x) = (1-x^2)(x-x_m)$$

$$\xi_1 = -\frac{r_{12}}{m_1+m_2+m_3}\left(m_2 + (1+\chi)m_3\right)$$

$$\frac{\omega^2}{G} r_{12}^3 \frac{\left[\frac{-r_{12}}{m_1+m_2+m_3}\left(m_2+(1+\chi)m_3\right)\right]}{r_{12}} - m_1 + \frac{1}{\chi^2} m_3 = 0$$

$$\omega^2 = -\frac{G}{r_{12}^3} \frac{\left(m_1 - 1/\chi^2 m_3\right)}{\left(m_2 + (1+\chi)m_3\right)}\left(m_1+m_2+m_3\right)$$

$$= \frac{G}{r_{12}^3}\left(\frac{m_1+m_2+m_3}{(1+\chi)^2}\right)\left(\frac{m_2(1+\chi)^2 + m_3}{m_2 + (1+\chi)m_3}\right)$$

$\downarrow$

Quintic Eqn. of Lagrange: $(m_1+m_2)\chi^5 + (3m_1+2m_2)\chi^4 + (3m_1+m_2)\chi^3 - (m_2+3m_3)\chi^2 - (2m_2+3m_3)\chi - (m_2+m_3) = 0$

Conic Sections:

$$f_i(t) = G\sum_{j=1}^{3}\frac{m_i m_j}{r_{ij}^3(t)}\left[r_j(t) - r_i(t)\right]$$

$$= \frac{1}{\rho^2} f_i(t_0)$$

$$m_i\left\{\left[\frac{d^2\rho}{dt^2} - \rho\left(\frac{d\theta}{dt}\right)^2\right]\mathbb{1} - \underbrace{\left[\frac{1}{\rho}\frac{d}{dt}\left(\rho^2\frac{d\theta}{dt}\right)\right]}_{0} J\right\} r_i(t_0)$$

$$= \frac{1}{\rho^2} f_i(t_0)$$

$\hookrightarrow f_i(t_0) = -m_i k^2 r_i(t_0)$

$$\frac{d^2\rho}{dt^2} - \rho\left(\frac{d\theta}{dt}\right)^2 = -\frac{k^2}{\rho^2}$$

$$R = \begin{bmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{bmatrix}, \quad \frac{dR}{dt} = \begin{matrix}-\dot\theta\sin\theta & -\dot\theta\cos\theta\\ \dot\theta\cos\theta & -\dot\theta\sin\theta\end{matrix} = -\dot\theta\begin{bmatrix}\sin & \cos\\ -\cos & \sin\end{bmatrix}$$

$$\Omega = R^T\frac{dR}{dt}$$

$$= \begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix}\dot\theta \quad \begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix} - \dot\theta$$

$$\Omega = -J\frac{d\theta}{dt}$$

$$f_i = -m_i \omega^2 r_i \, , \quad i = 1, 2, \ldots, n$$

$$\omega^2 (m_1 \vec{r}_1 + m_2 \vec{r}_2 + \ldots + m_n \vec{r}_n) = \vec{0}$$

$$\rightarrow \; -\cancel{m_1} \omega^2 r_1 = -G \frac{(m_2 \cancel{m_1})}{r_{12}^2}$$

$$\left(\frac{\omega^2}{G} - \frac{m_2}{r_{12}^3} - \frac{m_3}{r_{13}^3} \right) \vec{r}_1 + \frac{m_2}{r_{12}^3} \vec{r}_2 + \frac{m_3}{r_{13}^3} \vec{r}_3 = \vec{0}$$

$$\omega^2 = G \left(\frac{m_2}{r_{12}^3} \right)$$

$$\frac{m_1}{r_{21}^3} \vec{r}_1 + \left(\frac{\omega^2}{G} - \frac{m_1}{r_{21}^3} - \frac{m_3}{r_{23}^3} \right) \vec{r}_2 + \frac{m_3}{r_{23}^3} \vec{r}_3 = \vec{0}$$

$$\omega^2 = G \sum_i^n \frac{m_i}{r_{ji}^3}$$

$$m_1 \vec{r}_1 + m_2 \vec{r}_2 + m_3 \vec{r}_3 = \vec{0}$$

$$r_{12} \omega^2 = \frac{G(m_1 + m_2)}{r_{12}^2} \rightarrow m$$

$$\rightarrow I \ddot{\theta}$$

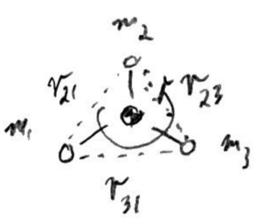
ρ_1

Equilateral Triangle:

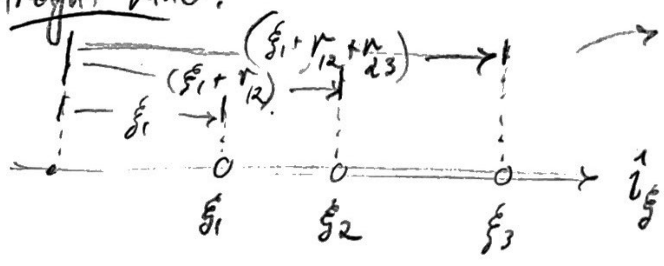
$$r_{12} = r_{13} = r_{21} = r_{23} = \rho$$

$$\rho \omega^2 = \frac{G(m_1 + m_2 + m_3)}{\rho^2}$$

$$\omega^2 = \frac{G}{\rho^3} (m_1 + m_2 + m_3)$$



Straight Line:



unknown:

$$\omega^2, \; \xi_1, \; r_{23}$$

$$\frac{\omega^2}{G} r_{12}^3 \frac{\xi_1}{r_{12}} + m_2 + \frac{1}{(1+\chi)^2} m_3 = 0$$

$$\frac{\omega^2}{G} r_{12}^3 \left(1 + \frac{\xi_1}{r_{12}} \right) - m_1 + \frac{1}{\chi^2} m_3 = 0$$

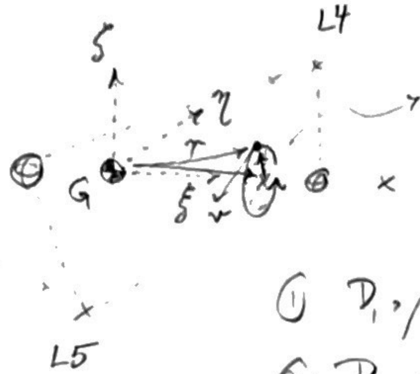
$$(m_1 + m_2 + m_3) \frac{\xi_1}{r_{12}} + m_2 + (1 + \chi) m_3 = 0$$

$$\chi = \frac{r_{23}}{r_{12}}$$

Rectilinear Oscillation of an Infinitessimal Mass

For equal masses: $m_1 = m_2 = m$

Let D, ρ be
(1) distance from G
(2) distance from mass



$\vec{h} = \vec{r} \times \vec{v}$

$= (I\omega)\, i_\zeta$

(1) $D_1, \rho_1 = \xi_1, \rho_1$

(2) $D_2, \rho_2 = \xi_2, \rho_2$

$D^2 = \left(\frac{D_1 + D_2}{2}\right)^2, \rho^2 = ?$

$\hookrightarrow \rho^2 = D^2 + \zeta^2$

$\omega^2 = \frac{Gm}{4D^3}$

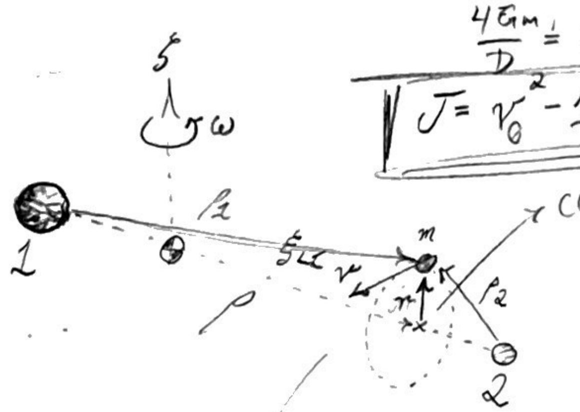
$\left(\frac{d\zeta}{dt}\right)^2 = \frac{4Gm}{\rho} - C$, Let: $v(0) = v_0,\ \zeta_0 = 0$

$v_0^2 = \frac{4Gm}{D} - C \rightarrow J = \left(\frac{d\zeta}{dt}\right)^2 = \frac{4Gm}{\rho} - \left[\frac{4Gm}{D} - v_0^2\right]$

$\frac{4Gm}{D} = 16D^2\omega^2, \quad \frac{4Gm}{\rho} = 16\frac{D^3}{\rho}\omega^2$

$$\boxed{J = v_0^2 - 16D^2\omega^2\left(1 - \frac{D}{\rho}\right)}$$

Circular Halo, $m_1 \neq m_2$



circular halo about L1

$\hookrightarrow \vec{r} = (\xi_{L1})\, i_\xi + \eta\, i_\eta + \zeta\, i_\zeta$

$\frac{\partial \vec{r}}{\partial t} = \frac{d\vec{r}}{dt} + \vec{\omega} \times \vec{r}$

$= \dot{\eta}\, i_\eta + \dot{\zeta}\, i_\zeta$

circular:
$\|\vec{r}\| = \rho = \text{const.}$

$\vec{r}(t) = f(\theta, t)$,

$\hookrightarrow \vec{r} = \xi_{L1} i_\xi + \rho\sin(\omega t)\, i_\eta + \rho\cos(\omega t)\, i_\zeta$

$\rho = \sqrt{\eta^2 + \zeta^2}$

$\frac{\partial \rho}{\partial t} = 0 = \frac{\partial}{\partial t}\left(\sqrt{\eta^2 + \zeta^2}\right)$

$0 = \frac{1}{\cancel{2\sqrt{\eta^2 + \zeta^2}}}\left(2\eta\dot{\eta} + 2\zeta\dot{\zeta}\right)$

Conservation:
$$\boxed{\eta\dot{\eta} + \zeta\dot{\zeta} = 0}$$

$\rho\sin(\omega t)\,[\rho\omega\cos(\omega t)]$
$+ \rho\cos(\omega t)\,[-\rho\omega\sin(\omega t)] = 0 \therefore$
$\rho^2\omega\, sc - \rho^2\omega\, cs = 0 \checkmark$

Restating Jacobi Integral:

$$\boxed{4\omega x_m T = \sqrt{2}\, E(\sin(\theta_m/2)) + \frac{\pi}{2}\sqrt{\sec(\theta_m)}\left[1 - \Lambda_0(\epsilon, \sin(\theta_m/2))\right]}$$

$$J = v_0^2 - 16D^2\omega^2\left(1 - \frac{D}{\rho}\right)$$

↳ Introduce θ:

$$\zeta = D\tan\theta,$$
$$\rho = D\sec\theta,$$
$$B = \frac{v_0^2}{16\omega^2 D^2}$$
$$\dots = 1 - \cos\theta_m$$

$$\rightarrow \boxed{J = 16\omega^2\cos^4\theta\,[B - (1-\cos\theta)]}$$

↳ oscillatory iff $B \leq 1$:

- Let θ_m s.t. $\left.\frac{d\theta}{dt}\right|_{\theta_m} = 0$

$$\left.\frac{d\theta}{dt}\right|_{\theta_m} = \sqrt{\cancel{16\omega^2}\cos^4\theta_m\,[B - (1-\cos\theta_m)]} = 0$$

$$\cos^4(\theta_m)\,[B - (1-\cos\theta_m)] = 0$$

$$\cancel{\cos^4(\theta_m)}\,B = \cancel{\cos^4(\theta_m)}\,[1-\cos\theta_m]$$

$$\boxed{B = 1 - \cos\theta_m}$$

$$\cos\theta_m = 1 - B$$

$$\boxed{\theta_m = \arccos(1-B)}$$

EOM:

$$\left(\frac{d\theta}{dt}\right)^2 = 16\omega^2\cos^4\theta\,(\cos\theta - \cos\theta_m)$$

Parametric:

$$x = \cos\theta,$$
$$x_m = \cos\theta_m,$$

$$\boxed{\begin{aligned}\cos\epsilon &= \tan(\theta_m/2),\\ \sin\epsilon &= \sqrt{1-n}\,\sec(\theta_m/2)\\ \delta_2 &= \sqrt{\frac{n}{(1-n)(n-\sin^2(\theta_m/2))}}\\ &= \sqrt{2\sec(\theta_m)}\end{aligned}}$$

$$\frac{d\theta}{dt} = \frac{d}{dt}\left(\arccos(x)\right) = \frac{\dot{x}}{\sqrt{1-x^2}} \longrightarrow \left(\frac{\dot{x}}{\sqrt{1-x^2}}\right)^2 = 16\omega^2 x^4(x - x_m)$$

$$\boxed{\left(\frac{dx}{dt}\right)^2 = 16\omega^2 x^4(1-x^2)(x-x_m)}$$

deg. 4:

↳ $\frac{dx}{dt} = 4\omega x^2\sqrt{(1-x^2)(x-x_m)}$

Let $T = \frac{P}{4}$, $P = \frac{2\pi}{\nu} = \frac{4\pi^2}{\omega}$

$$\boxed{4\omega T = \int_{x_m}^{1} \frac{dx}{x^2\sqrt{P(x)}}}, \quad P(x) = (1-x^2)(x-x_m) \quad \int \frac{1}{x^2\sqrt{(1-x^2)(x-x_m)}}\,dx = \int 4\omega\, dt$$

↳ elliptic integral $\longrightarrow$ Legendre's Proof $\longrightarrow$ $4\sqrt{2}\,\omega x_m T = 2E(k) - K(k) + \Pi(2k^2, k, \tfrac{1}{2}\pi)$

$$n = B = 2\sin^2(\tfrac{1}{2}\theta_m) = 2k^2, \quad k^2 < n < 1$$

Heuman's Lambda Function:

$$\Lambda_0\left(\epsilon, \sin\left(\tfrac{\theta_m}{2}\right)\right) = \frac{2}{\pi}K\left(\sin\left(\tfrac{\theta_m}{2}\right)\right)\left[E\left(\epsilon, \sin\tfrac{\theta_m}{2}\right) - \mathcal{Q}\left(s\tfrac{\theta_m}{2}\right)F\left(\epsilon, \cos\tfrac{\theta_m}{2}\right)\right]$$

$$\Pi\left(n, \sin\left(\tfrac{\theta_m}{2}\right), \tfrac{\pi}{2}\right) = K\left(\sin\left(\tfrac{\theta_m}{2}\right)\right) + \frac{\pi}{2}\delta_2\left[1 - \Lambda_0\left(\epsilon, \sin\tfrac{\theta_m}{2}\right)\right]$$

$$x = A_1 \cos \lambda t + A_2 \sin \lambda t$$
$$y = -k A_1 \sin \lambda t + A_2 \cos \lambda t$$
$$z = B_1 \cos \vartheta t + B_2 \sin \vartheta t$$

Halo

$$x = -A_x \cos(\lambda t + \varphi)$$
$$y = \xi k A_x \sin(\lambda t + \varphi)$$
$$z = A_z \sin(\lambda t + \varphi)$$

$$2\pi \sqrt{\frac{a^3}{\mu}}$$

$$n = \sqrt{\frac{\mu}{a^3}}$$

$$s = n_1 t$$

$$\mathcal{L} = \frac{1}{2}(\rho^* \cdot \rho^*) + \sum_{n=2}^{\infty} c_n \rho^n P_n(x/\rho)$$

↳ n^{th} Legendre polynomial

$$c_n = \frac{1}{\gamma_L^3}\left[\mu + (-1)^n \frac{(1-\mu)\gamma_L^{n+1}}{(1-\gamma_L)^{n+1}}\right]$$

$$x^{**} - 2y^* - (1+2c_2)x = \sum_{n=2}^{\infty}(n+1)c_{n+1}\rho^n P_n(x/\rho)$$
$$y^{**} + 2x^* + (c_2 - 1)y = \sum_{n=3}^{\infty} c_n y \rho^{n-2} \tilde{P}_n(x/\rho)$$
$$z^{**} + c_2 z = \sum_{n=3}^{\infty} c_n z \rho^{n-2} \tilde{P}(x/\rho)$$

↳ $$\tilde{P}(x/\rho) \triangleq \sum_{k=0}^{[\frac{n-2}{2}]} (3 + 4k - 2n) P_{n-2k-2}(x/\rho)$$

$$(L_1, L_2, L_3)$$

$$\mathcal{L} = \frac{1}{2}(\dot{\vec{\rho}} \cdot \dot{\vec{\rho}}) + \mu_1 \left[\frac{1}{|\vec{r}_1 - \vec{\rho}|} - \frac{\vec{r}_1 \cdot \vec{\rho}}{|\vec{r}_1|^3}\right] + \mu_2 \left[\frac{1}{|\vec{r}_2 - \vec{\rho}|} - \frac{\vec{r}_2 \cdot \vec{\rho}}{|\vec{r}_2|^3}\right]$$

$$\vec{\rho} = x\hat{i} + y\hat{j} + z\hat{k}$$

normalize:

$r_L = 1$ for L_1, L_2

Kepler's 3rd Law:

$$\mu_1 + \mu_2 = n_1^2 a_1^3$$
$$\mu \triangleq \frac{M_{moon}}{M_{Earth} + M_{moon}}$$

$$\gamma_L = \frac{r_1}{a_1} = n^{2/3}$$

$$1 + \frac{r}{r_2} = \sqrt[3]{\cancel{\frac{\mu_1 + \mu_2}{a_1^3}}\ \cancel{\mu + 1 - \mu}}$$

↳ $a_1 = \cancel{\frac{1}{r}} + r_2$

$$\frac{1}{a_1} \cancel{\ldots} = \frac{1}{a_1} \checkmark$$

$$a_1 \cancel{= \frac{1}{1 + \frac{r_1}{r_2}} = \frac{r_2}{r_1 + r_2}}$$

Halo:

$$x^{**} - 2y^* - (1 + 2c_2)x = 0$$
$$y^{**} + 2x^* + (2c_2 - 1)y = 0$$
$$z^{**} + \lambda^2 z = 0,$$
$$\Delta \triangleq \lambda^2 - c_2,$$
$$z^{**} + \lambda^2 z = \sum_{n=3}^{\infty} c_n z \rho^{n-2} \tilde{P}_n(x/\rho) + \Delta z,$$
$$\Delta = \mathcal{O}(A_z^2)$$

$s = n_1 t$ $\qquad \omega^2 = \frac{\mu_1 + \mu_2}{a \cancel{x_{12}^3}} = \frac{\cancel{\mu} + 1 - \cancel{\mu}}{a^3} \rightsquigarrow \omega = \sqrt{\frac{1}{a^3}}$

$$\tau = \omega s$$
$$= \omega n_1 t$$
$$= \left(\sqrt{\frac{1}{a^3}}\right)\left(\sqrt{\frac{1}{a^3}}\right) t \approx \omega^2 t\ ?$$

$$\omega = 1 + \sum_{n \geq 1} \omega_n, \quad \omega_n < 1 \longrightarrow \quad \mathcal{O}(\omega_n) = \mathcal{O}(A_x^n)$$

$\downarrow$

$$\boxed{\begin{aligned} &\omega^2 x'' - 2\omega y' - (1 + 2c_2)x = \tfrac{3}{2} c_3 (2x^2 - y^2 - z^2) + 2c_4 x(2x^2 - 3y^2 - 3z^2) + \mathcal{O}(4) \\ &\omega^2 y'' + 2\omega x' + (c_2 - 1) y = -3c_3 xy - \tfrac{3}{2} c_4 y (4x^2 - y^2 - z^2) + \mathcal{O}(4) \\ &\omega^2 z'' + \lambda^2 z = -3c_3 xz - \tfrac{3}{2} c_4 z (4x^2 - y^2 - z^2) + \Delta z + \mathcal{O}(4) \end{aligned}}$$

$\downarrow$

Generating:

$$x = -A_x \cos(\lambda \tau + \varphi)$$
$$y = \left[\frac{(\lambda^2 + 1 + 2c_2)}{2\lambda}\right] A_x \sin(\lambda \tau + \varphi)$$
$$z = A_z \sin(\lambda \tau + \psi)$$

↳ Let $\omega_1 = 0$,

$$\boxed{\omega_2 = s_1 A_x^2 + s_2 A_z^2} \longrightarrow$$

$$s_1 = \frac{1}{2\lambda[\lambda(1+k^2) - 2k]} \left\{ \frac{3}{2} c_3 \left[2a_{21}(k^2 - 1) - a_{23}(k^2 + 2) - 2kb_{21}\right] - \frac{3}{8} c_4 (3k^4 - 8k^2 + 8) \right\} + 5d_{21}]$$

$$s_2 = \frac{1}{2\lambda[\lambda(1+k^2) - 2k]} \left\{ \frac{3}{2} c_3 \left[2a_{22}(k^2 - 2) + a_{24}(k^2 + 2) - 2kb_{22}\right] + \frac{3}{8} c_4 (12 - k^2) \right\}$$

(where, again, $k = \frac{(\lambda^2 + 1 + 2c_2)}{2\lambda}$
$= \frac{2\lambda}{\lambda^2 + 1 - c_2}$)

$$\boxed{l_1 A_x^2 + l_2 A_z^2 + \Delta = 0}$$

Frequency:

$$\boxed{\lambda^4 + (c_2 - 2)\lambda^2 - (c_2 - 1)(1 + 2c_2) = 0}$$

$$l_1 = a_1 + 2\lambda^2 s_1$$
$$l_2 = a_2 + 2\lambda^2 s_2,$$
$$a_1 = -\tfrac{3}{2} c_3 (2a_{21} + a_{23} + 5d_{21}) - \tfrac{3}{8} c_4 (12 - k^2)$$
$$a_2 = \tfrac{3}{2} c_3 (a_{24} - 2a_{22}) + \tfrac{9}{8} c_4$$