Chapter 12

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Chapter 12

Mean, variance and higher moments

Mean and variance

Definition 12.1 The mean and variance of a continuous random variable are computed much like they are for discrete random variables, except that we replace summations with integration. Let X be a continuous random variable with PDF f. Then

- $E[X] = \mu = \int_{-\infty}^{\infty} x f(x) dx.$ (mean of X)

PDF

- $Var[X] = \sigma^2 = E[(X-\mu)^2] = \int_{-\infty}^{\infty} (x-\mu)^2 f(x) dx.$

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Recall: In the discrete case

$E(X) = \mu = \sum_{-\infty}^{\infty} x \cdot f(x)$

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$Var(X) = \sigma^2 = E((X-\mu)^2) = \sum_{-\infty}^{\infty} (x-\mu)^2 \cdot f(x)$

As with discrete distributions, the following simplifies the calculation of the variance.

$$Var[X] = E\left[X^2\right] - \mu^2.$$

$$= \int_{-\infty}^{\infty} x^2 \cdot f(x)\, dx - \mu^2$$

Example 12.1

Let $X \sim Unif(a, b)$. Then

$f(x) = \frac{1}{(b-a)}$ $a \leq x < b$

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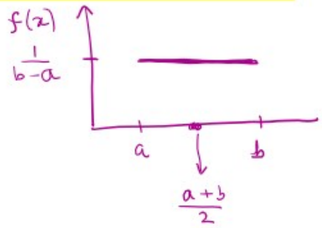
Let $X \sim Unif(a, b)$. Then

- $E[X] = \frac{(a+b)}{2}$. ✓
- $Var[X] = \frac{(b-a)^2}{12}$.

$f(x) = \frac{1}{(b-a)} \quad a \leqslant x < b$



- $E[X] = \mu = \int_{-\infty}^{\infty} x \cdot f(x)\, dx$

$= \int_a^b x \cdot \frac{1}{(b-a)}\, dx = \frac{1}{(b-a)} \int_a^b x\, dx = \frac{1}{(b-a)} \left. \frac{x^2}{2} \right|_a^b$

$= \frac{1}{(b-a)} \frac{(b^2 - a^2)}{2} = \frac{1}{(b-a)} \frac{(b-a).(b+a)}{2} = \boxed{\frac{(b+a)}{2}}$

(Recall: $b^2 - a^2 = (b-a)(b+a)$)

EX: $X \sim Unif(0, 10)$
The mean of X is 5

- $Var(X) = \sigma^2 = E(X^2) - \mu^2$

$E(X^2) = \int_{-\infty}^{\infty} x^2 . f(x)\, dx = \int_a^b x^2 . \frac{1}{(b-a)} dx = \frac{1}{(b-a)} \left. \frac{x^3}{3} \right|_a^b$

$= \frac{1}{3(b-a)} (b^3 - a^3) = \frac{1}{3(\cancel{b-a})} \cancel{(b-a)} \cdot (b^2 + ab + a^2)$

(Recall: $b^3 - a^3 = (b-a)(b^2 + ab + a^2)$)

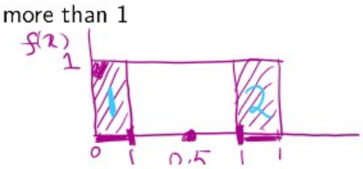
$\sigma^2 = \frac{b^2 + ab + a^2}{3} - \frac{(a+b)^2}{4} = \frac{(b-a)^2}{12}$

EX: $X \sim Unif(0, 10)$
The variance of X is $\frac{10^2}{12} = \frac{100}{12}$
The standard deviation of X is $\sqrt{\frac{100}{12}} = \sqrt{\frac{10^2}{2^2 \times 3}} = \frac{10}{2} \times \frac{1}{\sqrt{3}} = \frac{5}{\sqrt{3}}$

Example 12.2 contd.

Suppose $X \sim Unif(\overset{a}{0}, \overset{b}{1})$. Calculate the probability that X is more than 1 standard deviation from the mean.

$f(x) = 1 \quad 0 \leqslant x < 1$

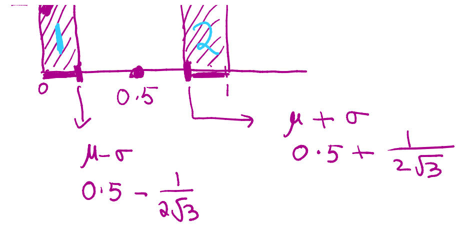


$\mu = E(X) = \frac{a+b}{2} = \frac{0+1}{2} = 0.5$

$f(x) = 1 \quad 0 \leq x < 1$

$\mu = E(X) = \frac{a+b}{2} = \frac{0+1}{2} = 0.5$

$\sigma = \sqrt{Var(X)} = \sqrt{\frac{(b-a)^2}{12}} = \sqrt{\frac{1}{12}} = \frac{1}{2\sqrt{3}}$



Want

$P(X < (\mu-\sigma) \cup X > (\mu+\sigma)) = P(|X-\mu| > \sigma)$

For $\mu = \frac{1}{2}$, $\sigma = \frac{1}{2\sqrt{3}}$

shaded area in 1 shaded area in 2

$P\left(X < \frac{1}{2} - \frac{1}{2\sqrt{3}} \cup X > \frac{1}{2} + \frac{1}{2\sqrt{3}}\right) = P\left(X < \frac{1}{2} - \frac{1}{2\sqrt{3}}\right) + P\left(X > \frac{1}{2} + \frac{1}{2\sqrt{3}}\right)$

$= \left(\frac{1}{2} - \frac{1}{2\sqrt{3}}\right) \cdot 1 + \left(1 - \left(\frac{1}{2} + \frac{1}{2\sqrt{3}}\right)\right) \cdot 1$

$= \left(\frac{1}{2} - \frac{1}{2\sqrt{3}}\right) \cdot 2 = 1 - \frac{1}{\sqrt{3}} = \boxed{.4226}$

Recall: Chebychev's inequality says

$P(|X-\mu| > k\sigma) \leq \frac{1}{k^2}$

if $k = 1$, then all we know is the probabily is less than or equal to 1

Mean and variance of an exponential distribution

Lemma 12.1 Suppose $X \sim Exp(\lambda)$. Then

- $E[X] = \frac{1}{\lambda}$.
- $Var[X] = \frac{1}{\lambda^2}$.

$SD(X) = \frac{1}{\lambda}$

Proof of Lemma 12.1

$$E[X] = \int_{-\infty}^{\infty} xf(x)dx = \int_{0}^{\infty} x \cdot \lambda e^{-\lambda x} dx.$$

We will use the technique of integration by parts

$$\int u \cdot dv = u \cdot v - \int v \cdot du$$

to prove the first result. To use this formula, we need to identify u and dv, and then compute du and v. Note that v is simply the integral of dv.

Proof of Lemma 12.1 contd.

Defining $u = x$ and $dv = \lambda e^{-\lambda x} dx$, we have

$$du = dx$$

and

$$v = \int \lambda e^{-\lambda x} dx = \lambda \cdot \frac{-e^{-\lambda x}}{\lambda} = -e^{-\lambda x}.$$

Therefore

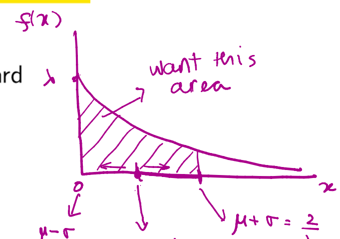
$$\begin{aligned} E[X] &= -xe^{-\lambda x}\Big|_0^\infty + \int_0^\infty e^{-\lambda x} dx \\ &= 0 + \frac{1}{\lambda}, \\ &= \frac{1}{\lambda}. \end{aligned}$$

Proof of Lemma 12.1 contd.

The variance is found similarly using integration by parts twice to evaluate $\int_0^\infty x^2 \lambda e^{-\lambda x} dx$.

Example 12.3

Suppose $X \sim Exp(\lambda)$. Find the probability that X is within 1 standard deviation of the mean.



$f(x) = \lambda e^{-\lambda x} \, dx$

$E(X) = \mu = \frac{1}{\lambda}$ (lemma 12.1)

$\sigma = \frac{1}{\lambda}$

Want $P(X > \mu - \sigma$ and $X < \mu + \sigma) = P(|X-\mu| < \sigma)$

$P(X > \frac{1}{\lambda} - \frac{1}{\lambda}$ and $X < \frac{1}{\lambda} + \frac{1}{\lambda}) = P(X > 0$ and $X < \frac{2}{\lambda})$

$= P(0 < X < \frac{2}{\lambda})$

$= P(0 < \lambda < \frac{2}{\lambda})$

$$= \int_0^{2/\lambda} f(x)\, dx = \int_0^{2/\lambda} \lambda e^{-\lambda x} dx = \cancel{\lambda} \cdot \frac{e^{-\lambda x}}{-\cancel{\lambda}} \Big|_0^{2/\lambda}$$

$$= -e^{-\lambda \cdot \frac{2}{\lambda}} - (-e^{-\lambda \cdot 0})$$

$$= e^{-2} + 1 = \boxed{.8647}$$

There is a 86.47% probability that an exponential random variable will lie w/in 1 standard deviation of its mean.
Note: this result holds regardless of the value for λ

Algebra of expected values

We have already proved the following claims for discrete random variables. They are also true for continuous random variables.

Lemma 12.2 Let X be a continuous random variable with PDF f, and a and b are numbers. Then

- $E[t(X)] = \int_{-\infty}^{\infty} t(x) \cdot f(x) dx$ law of unconscious probalist
- $E[aX + b] = aE[X] + b$ linearity of expectation
- $Var[aX + b] = a^2 Var[X]$

Please try problem 4 on Problem 7. Rmd

4. $f(x) = 1 \quad a \leq x \leq a+1$ (a is some number)

Find $E\left[\frac{1}{X}\right] = \int_{-\infty}^{\infty} \frac{1}{x} \cdot f(x)\, dx$ $\quad t(X) = \frac{1}{X}$

You will find $E\left[\frac{1}{X}\right] \neq \frac{1}{E[X]}$

Example 12.4 Challenging problem (can skip)

A parking garage charges a flat fee of \$10 for the first hour (or fraction thereof) and any additional time at a rate of \$8 per hour.

Suppose the time, X (in hours), that we park in this lot is an exponential random variable with $\lambda = 1$. Let the random variable Y denote the cost (in dollars) that we will pay to park in the garage.

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a. How does Y relate to X?

Example 12.4

A parking garage charges a flat fee of \$10 for the first hour (or fraction thereof) and any additional time at a rate of \$8 per hour.

Suppose the time, X (in hours), that we park in this lot is an exponential random variable with $\lambda = 1$. Let the random variable Y denote the cost (in dollars) that we will pay to park in the garage.

b. What is our expected cost to park? That is, find $E[Y]$.

Higher moments

The expected values $E[X]$ and $E[X^2]$ are examples of **moments** of a random variable and its distribution. They are called the first and second moment about the origin.

The variance $E[(X-\mu)^2]$ is an example of a **central moment** or moment about the mean.

Higher moments describe additional features of the shape of a distribution. For instance,

$$E\left[(X-\mu)^3\right]$$

is zero for symmetric distributions and non-zero for asymmetric/skewed distributions, and therefore is often used as a measure of **skewness**. It is positive when the distribution is skewed to the right and negative when it is skewed to the left.