Chapter 1

Monday, September 18, 2023 2:42 PM

Chapter 1

Set Theory

What is probability?

- **Probability** is a number between 0 and 1 (inclusive) that describes how likely it is that something will occur.
- As an area of mathematics, **probability** is the study of randomness, a particular form of uncertainty.
    - the possible outcomes are known; however
    - they are unpredictable, but
    - there is a well-defined rule for choosing among them.
- Probability theory uses the language of sets, hence we begin a brief review of concepts from set theory.

Sample space

Definition 1.1 The set S, of all possible outcomes of a random process is called the sample space. It is also known as the outcome set.

Random process	Sample space
Toss a coin once	$S = \{H; T\}$
Toss a coin twice	$S = \{(H, H); (H, T); (T, H); (T, T)\}$

ordered pair (Tuple)

Example 1.1

For each of the following "experiments", describe the sample space.

a. A local TV station advertises two newscasting positions. If two women (W_1, W_2) and two men (M_1, M_2) apply, list all the possible teams of two co-anchors that can be formed?

$S = \{(W_1, W_2); (W_1, M_1); (W_1, M_2);$

two co-anchors that can be formed?

$S = \{ (W_1, W_2);\ (W_1, M_1);\ (W_1, M_2);$
$(W_2, M_1);\ (W_2, M_2);\ (M_1, M_2) \}$

Example 1.1

sports announcer

weather forecaster

For each of the following "experiments", describe the sample space.

b. A local TV station is seeking to hire a sports announcer and a weather forecaster. If two women (W_1, W_2) and two men (M_1, M_2) apply, list all the possible teams that can be formed?

$S = \{ (W_1, W_2);\ (W_2, W_1);\ (W_1, M_1);\ (M_1, W_1);$
$(W_1, M_2);\ (M_2, W_1);\ (W_2, M_1);\ (M_1, W_2);$
$(W_2, M_2);\ (M_2, W_2);\ (M_1, M_2);\ (M_2, M_1) \}$

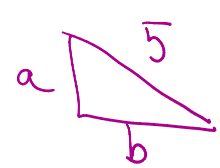
Example 1.1

For each of the following "experiments", describe the sample space.

c. Toss a coin repeatedly until the first head shows.

$S = \{ H ; (TH) ; (TTH) ; \ldots \}$

Example 1.1



For each of the following "experiments", describe the sample space.

d. Let S be the set of right triangles with a 5" hypotenuse and whose height and length are a and b, respectively. Characterize the outcomes in S.

$S = \{ (a, b) \in \mathbb{R}^+ : a^2 + b^2 = 5^2 \}$

in S.

$S = \{ (a, b) \in \mathbb{R}^+ : a^2 + b^2 = 5^2 \}$

Event

Definition 1.2 An event E is a collection of outcomes in S. That is, E is a subset of S, which is written in short hand as $E \subseteq S$.

Experiment	Sample space	Event
Toss two dice	$S = \left\{ \begin{array}{cccc} (1,1) & (1,2) & \cdots & (1,6) \\ (2,1) & (2,2) & \cdots & (2,6) \\ \vdots & \vdots & \vdots & \vdots \\ (6,1) & (6,2) & \cdots & (6,6) \end{array} \right\}$	E : Sum of numbers is 3 $\{(1,2),(2,1)\}$

Example 1.2:

Consider the experiment of choosing coefficients for the quadratic equation

$$ax^2 + bx + c = 0.$$

Characterize the values of a, b, c associated with the event E: Equation has complex roots.

$S = \{(a, b, c) \in \mathbb{R}\}$

$E = \{(a, b, c) \in \mathbb{R} : b^2 - 4ac < 0\}$

Set Operations

Given any two events A and B we have the following operations:

UNION: The union of A and B, written $A \cup B$ is the set of elements that belong to either A or B or both:

and/or

$$A \cup B \quad = \quad \{x : x \in A \text{ or } x \in B\}.$$

belong to either A or B or both:

and/or

$$A \cup B \quad = \quad \{x : x \in A \text{ or } x \in B\}\,.$$

The symbol $\in$ means "in" or "a member of". Similarly $\notin$ means "not in" or "not a member of".

Set Operations

Given any two events A and B we have the following operations:

INTERSECTION: The intersection of A and B, written $A \cap B$ is the set of elements that belong to both A and B:

$$A \cap B \quad = \quad \{x : x \in A \text{ and } x \in B\}\,.$$

Set Operations

Definition 1.3 Two events A and B are disjoint (or mutually exclusive) if $A \cap B = \emptyset$ where $\emptyset$ denotes the void or empty set (consisting of no elements).

Example 1.3

A coin is tossed four times and the resulting sequence of heads and/or tails is recorded. Define the events:

E: exactly two heads appear
F: heads and tails alternate
G: first two tosses are heads

Which events, if any, are mutually exclusive? F & G

G: first two tosses are heads

Which events, if any, are mutually exclusive? F & G

E&F : (H, T, H, T) in common X

E&G : (H, H, T, T) " " X

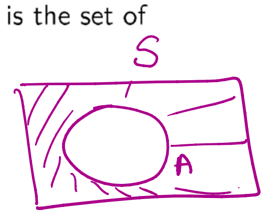
Set Operations

A'

Given any two events A and B we have the following operations:

COMPLEMENTATION: The complement of A, written A^c is the set of elements that are in S but not in A:

$$A^c = \{x \in S : x \notin A\}.$$



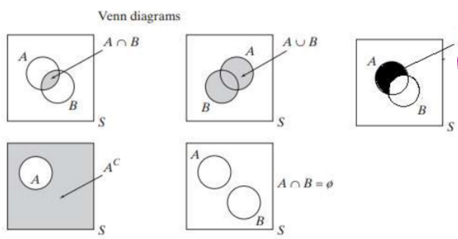
Set Operations

Given any two events A and B we have the following operations:

SET DIFFERENCE: The set difference of B and A, denoted by $B \setminus A$ (or $B - A$) is

$$
\begin{aligned}
B \setminus A &= (x \in S : x \in B \; and \; x \notin A\}, \\
&= B \cap A^c.
\end{aligned}
$$

- The following figure shows the Venn diagrams for a set union, intersection, complement and set difference.

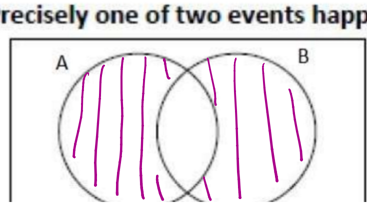


$A \cap B^c$

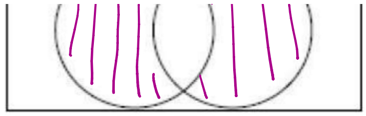
Example 1.4:

Shade the events corresponding to the following descriptions. Then write them in terms of set operations.

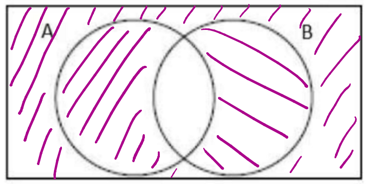
Precisely one of two events happen



$(A \cup B) \setminus (A \cap B)$



At most one of two events happen



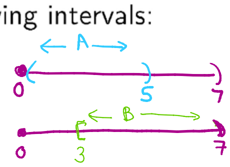
• $(A \cup B)^C \cup ((A \cup B) \setminus (A \cap B))$

• $(A \cap B)^C$

Example 1.5

Let events A, B and sample space S be defined as the following intervals:

$$S = \{x : 0 \leq x < 7\}$$
$$A = \{x : 0 < x < 5\}$$
$$B = \{x : 3 \leq x < 7\}$$



Characterize the following events:

a. $A^c = \{0\} \cup \{x : 5 \leq x < 7\}$

b. $A \cap B : \{x : 3 \leq x < 5\}$ (must be in both A & B)

c. $A \cup B = \{x : 0 < x < 7\}$ (everything in A and/or B)

d. $A \cap B^c = \{x : 0 < x < 3\}$

e. $A^c \cup B = \{0\} \cup \{x : 3 \leq x < 7\}$

f. $A^c \cap B^c = \{0\}$

- Unions and intersections can be defined for arbitrary collections of sets. For example, if $A_1, A_2, A_3, \ldots$ is a collection of sets, all defined on a sample space S, then

$$
\begin{aligned}
\bigcup_{i=1}^{\infty} A_i &= \{x \in S : x \in A_i, \text{ some } i\}, \\
\bigcap_{i=1}^{\infty} A_i &= \{x \in S : x \in A_i, \text{ for all } i\}
\end{aligned}
$$

Properties of set operations

Theorem 1.1 For any three events E, F, and G defined on a sample space S, we have the following identities:

1. Commutative laws

- $E \cup F = F \cup E$
- $E \cap F = F \cap E$

2. Associative laws

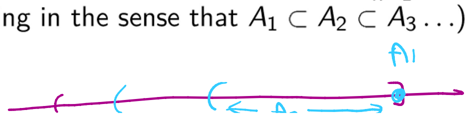
- $E \cup (F \cup G) = (E \cup F) \cup G$
- $E \cap (F \cap G) = (E \cap F) \cap G$

3. Distributive laws

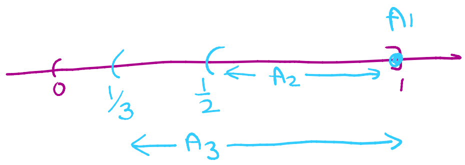
- $E \cap (F \cup G) = (E \cap F) \cup (E \cap G)$
- $E \cup (F \cap G) = (E \cup F) \cap (E \cup G)$

Example 1.6

Let $S = (0, 1]$ and $A_k = (1/k, 1]$. What is $\bigcup_{k=1}^{\infty} A_k$? (Note: A_k are nested and increasing in the sense that $A_1 \subset A_2 \subset A_3 \ldots$)



and increasing in the sense that $A_1 \subset A_2 \subset A_3 \ldots$)



$$\bigcup_{k=1}^{\infty} A_k = [1] \cup [\tfrac{1}{2}, 1] \cup (\tfrac{1}{3}, 1] \cup \ldots \cup (\tfrac{1}{k}, 1] \cup \ldots$$
$$= (0, 1]$$

Variation:

$S = [0, 1]$, $A_k = (\frac{1}{k}, 1]$

$\bigcup_{k=1}^{\infty} A_k = (0, 1]$ still since there is no finite k for which $1/k = 0$.

Properties of set operations

Theorem 1.1 For any three events E, F, and G defined on a sample space S, we have the following identities:

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- $E \cup F = F \cup E$
- $E \cap F = F \cap E$

2. Associative laws

- $E \cup (F \cup G) = (E \cup F) \cup G$
- $E \cap (F \cap G) = (E \cap F) \cap G$

3. Distributive laws

- $E \cap (F \cup G) = (E \cap F) \cup (E \cap G)$
- $E \cup (F \cap G) = (E \cup F) \cap (E \cup G)$

By comparison:

Addition $x + y \cdot z \neq (x+y) \cdot (x+z)$ ✗

Multiplication $x \cdot (y+z) = x \cdot y + x \cdot z$ ✓

Addition does not distribute over multiplication!

Multiplication distributes over addition

Properties of set operations

Theorem 1.1 For any three events E, F, and G defined on a sample space S, we have the following identities:

4. Identity laws

- $E \cap S = E$
- $E \cup \phi = E$

5. Complement laws

- $E \cup E^c = S$
- $E \cap E^c = \phi$

Properties of set operations

Theorem 1.2 (DeMorgan's Laws) For any two events E and F defined on a sample space S:

- $(E \cup F)^c = E^c \cap F^c$
- $(E \cap F)^c = E^c \cup F^c$

Proof of DeMorgan's first law

First we show $(E \cup F)^c \subseteq E^c \cap F^c$. Consider an element $x \in (E \cup F)^c$.

$$
\begin{aligned}
x \in (E \cup F)^c &\Rightarrow x \notin (E \cup F), \\
&\Rightarrow x \notin E \text{ AND } x \notin F \\
&\Rightarrow x \in E^c \text{ AND } x \in F^c \\
&\Rightarrow x \in (E^c \cap F^c).
\end{aligned}
$$

Since this holds for any arbitrary element x in $(E \cup F)^c$, we can say

$$(E \cup F)^c \subseteq E^c \cap F^c.$$

Proof of Demorgan's first law

Next we show $E^c \cap F^c \subseteq (E \cup F)^c$. Consider an element $y \in E^c \cap F^c$.

$$\begin{aligned} y \in E^c \cap F^c &\Rightarrow x \in E^c \textit{ AND } x \in F^c, \\ &\Rightarrow x \notin E \textit{ AND } x \notin F, \\ &\Rightarrow x \notin (E \cup F), \\ &\Rightarrow x \in (E \cup F)^c. \end{aligned}$$

Again, since y is any arbitrary element of $E^c \cap F^c$, we have the result

$$E^c \cap F^c \subseteq (E \cup F)^c$$

and hence

$$(E \cup F)^c = E^c \cap F^c$$

- De Morgan's laws commonly apply to text searching using Boolean operators AND, OR, and NOT.

  The queries

  - Search A: NOT (cars OR trucks)
  - Search B: (NOT cars) AND (NOT trucks)

  return the same results