Chapter 2

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Chapter 2

The Probability Function

Theoretical Probability Calculation

When you say you "know" that the toss of a fair coin has a 50% of being heads, you are reasoning that:

- the outcome is either a head or a tail
- outcomes are equally likely
- the two probabilities must add to 100%

Theoretical Probability Calculation

The basic idea is to combine

- some general properties that should be true of all probability situations (called **axioms**) with
- some additional assumptions about the situation at hand

To use this method, we need axioms.

Axioms of Probability

Definition 2.1 (Axioms of probability) Let S be a sample space for a random experiment. A probability assignment for S is a function P mapping events to the real line such that:

A1 $P(E) \geq 0$ for any event E

A2 $P(S) = 1$

A3 The probability of a *disjoint* union is the sum of probabilities. "COUNTABLE ADDITIVITY"

- $P(E_1 \cup E_2) = P(E_1) + P(E_2)$, provided $E_1 \cap E_2 = \phi$ E_1 & E_2 are disjoint
- $P(E_1 \cup E_2 \cup \cdots \cup E_k) = P(E_1) + P(E_2) + \cdots + P(E_k)$ provided $E_i \cap E_j = \phi$ whenever $i \neq j$.
- $P(\bigcup_{i=1}^{\infty} E_i) = \sum_{i=1}^{\infty} P(E_i)$ provided $E_i \cap E_j = \phi$ whenever $i \neq j$.

Axioms are not concerned with interpretations of probability. There are, however, two common interpretations given to probability calculations:

- $P(A)$ is the long run proportion of times event A occurs in repetitions of the experiment. (Frequentist)
- $P(A)$ measures an observer's strength of belief that A is true. (Bayesian)

Corollaries of the axioms

Theorem 2.1 If P is a probability function and A is any set in S then:

a. $P(A^c) = 1 - P(A)$ (Rule of Complements)

Pf: Complement Law states

$A \cup A^c = S$

$\Rightarrow P(A \cup A^c) = P(S) = 1$ (A.2)

By A3 $P(A \cup A^c) = P(A) + P(A^c)$

(since A & A^c are disjoint)

$\therefore P(A^c) = 1 - P(A)$

(since A & A^c are disjoint)

$\therefore\ P(A^c) = 1 - P(A)$

Corollaries of the axioms

Theorem 2.1 If P is a probability function and A is any set in S then:

b. $P(A) \leq 1$

By A1 we know $P(A^c) \geqslant 0$

By (a) " " $P(A^c) = 1 - P(A) \geqslant 0$

$\therefore\ P(A) \leqslant 1$

Corollaries of the axioms

Theorem 2.1 If P is a probability function and A is any set in S then:

c. $P(\emptyset) = 0$

$\emptyset$ is the identity element for "$\cup$" operation

$$S \cup \emptyset = S$$

$$P(S \cup \emptyset) = P(S) = 1 \quad (A2)$$

Using A3: $P(S \cup \emptyset) = P(S) + P(\emptyset)$

[S & $\emptyset$ are disjoint]

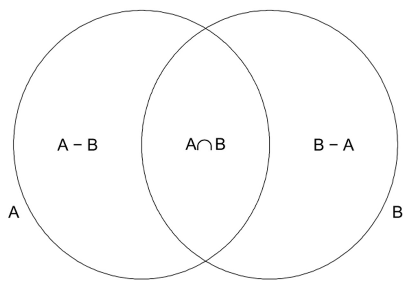
$$S \cap \emptyset = \emptyset$$

since S is identity element for "$\cap$"

Corollaries of the axioms

Theorem 2.2 (Addition rule, two event version) If P is a probability function, and A and B are any sets in S

$$P(A \cup B) = P(A) + P(B) - P(A \cap B).$$



Proof of Theorem 2.2

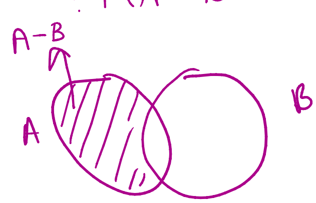
We begin by writing:

$$A \cup B = (A - B) \cup (A \cap B) \cup (B - A).$$

Now we can use axiom [A3] to write:

$$\begin{aligned} P(A \cup B) &= P\left((A - B) \cup (A \cap B) \cup (B - A)\right), \\ &= P(A - B) + P(A \cap B) + P(B - A), \end{aligned}$$

? $P(A-B)$



Next, we use axiom [A3] repeatedly to find $P(A - B)$ and $P(B - A)$:

- $P(A) = P\left((A - B) \cup (A \cap B)\right) = P(A - B) + P(A \cap B)$, so

$$P(A - B) = P(A) - P(A \cap B).$$

$A = (A \cap B) \cup (A-B)$

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Next, we use axiom [A3] repeatedly to find $P(A - B)$ and $P(B - A)$:

- $P(B) = P((B - A) \cup (A \cap B)) = P(B - A) + P(A \cap B)$, so

$$P(B - A) = P(B) - P(A \cap B).$$

Proof of Theorem 2.2

Combining these, we have:

$$
\begin{aligned}
P(A \cup B) &= P\left((A-B) \cup (A \cap B) \cup (B-A)\right), \\
&= P(A-B) + P(A \cap B) + P(B-A), \\
&= \underbrace{P(A) - P(A \cap B)}_{P(A-B)} + P(A \cap B) + \underbrace{P(B) - P(A \cap B)}_{P(B-A)}, \\
&= P(A) + P(B) - P(A \cap B).
\end{aligned}
$$

Example 2.1

Let A and B be two events defined on a sample space S such that $P(A) = 0.3$, $P(B) = 0.5$, and $P(A \cup B) = 0.7$. Find:

a. $P(A \cap B) = 0.1$

By addition formula:

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$0.7 = 0.3 + 0.5 - P(A \cap B)$$

$$\Rightarrow \boxed{P(A \cap B) = 0.1}$$

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b. $P(A^c \cup B^c)$

By DeMorgan's second law

$$(A \cap B)^c = A^c \cup B^c$$

$$P(A^c \cup B^c) = P((A \cap B)^c) = 1 - P(A \cap B) \quad \text{(by Thm 2.1 a)}$$

$$= 1 - .1 = \boxed{.9}$$

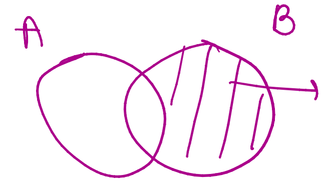
Example 2.1

Let A and B be two events defined on a sample space S such that

A B

Let A and B be two events defined on a sample space S such that $P(A) = 0.3$, $P(B) = 0.5$, and $P(A \cup B) = 0.7$. Find:

c. $P(A^c \cap B)$



$B - A = A^c \cap B$

$B = \underbrace{(B - A) \cup (B \cap A)}_{\text{disjoint}}$

$\therefore$ By A3:

$$P(B) = P(B-A) + P(B \cap A)$$

$$\therefore P(B-A) = P(B) - P(A \cap B) = 0.5 - 0.1 = \boxed{0.4}$$

- The addition rule

$$P(A \cup B) \quad = \quad P(A) + P(B) - P(A \cap B).$$

gives two useful identities:

- Union bound (aka Boole's inequality)

$$P(A \cup B) \leq P(A) + P(B)$$

$\because$ By Thm 2.1 a) $P(A \cap B) \geq 0$.

- Bonferroni's inequality

$$P(A \cap B) \geq P(A) + P(B) - 1$$

Pf: By Thm 2.1 b: $P(A \cup B) \leq 1$

$\Rightarrow P(A) + P(B) - P(A \cap B) \leq 1$

$\Rightarrow P(A \cap B) \geq P(A) + P(B) - 1$

Example 2.2

A new therapy for a disease will be approved if it is shown to be effective in two different studies. Suppose each study has a 95% probability of claiming that the treatment is effective. What can you say about the probability that the therapy is approved?

Event A: Study 1 finds therapy effective

Given: $P(A) = .95$

Event B: Study 2 finds therapy effective

Given $P(B) = .95$

Want $P(\text{Therapy is approved}) = P(A \cap B)$

By Bonferroni:

$$P(A \cap B) \geq P(A) + P(B) - 1$$

$$= 2 \times .95 - 1 = \boxed{.9}$$

I can say there is atleast a 90% chance that the therapy will be approved.