

## Chapter 7.1

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# Chapter 7.1

## Mean of a Discrete Random Variable

## Motivation

The Probability Mass Function provides a global overview of the behavior of a random variable.

Numerical summaries such as mean and variance can help us understand the *typical* behavior of the random variable.

## Example 7.1

Suppose a high school student has taken 10 courses and received 5 A's, 4 B's and 1 C. Using the traditional numerical scale where an A is worth 4, B is worth 3 and a C is worth 2, what is this student's overall grade point average (GPA)?

$$\begin{aligned} \text{GPA} &= \frac{5 \times 4 + 4 \times 3 + 1 \times 2}{10} \\ \text{of } \overset{\text{value}}{A} &= 4 \times \frac{5}{10} + 3 \times \frac{4}{10} + 2 \times \frac{1}{10} \\ &= \boxed{3.4} \end{aligned}$$



The grade point average in example 7.1 is a weighted sum of the numerical worth of a grade:

$$GPA = \sum (\text{grade})(\text{proportion of times student gets that grade}).$$

The mean of a discrete random variable is a similar weighted sum with the probabilities serving as weights.

## Mean of a discrete random variable

**Definition 7.1** Let  $X$  be a discrete random variable with PMF  $f$ . The mean (also called **expected value**) of  $X$  is denoted as  $\mu$  or  $E[X]$  and is defined by

$$E[X] = \sum_x x \cdot P(X = x) = \sum_x x \cdot f(x).$$

The sum is taken over all the possible values of  $X$ . When the possible values are infinite, we require that the sum is well defined and is finite. If  $E[X] = \pm\infty$ , we simply say that  $E[X]$  does not exist.

## Example 7.2

Find  $E[X]$  where  $X$  is the outcome when we roll a fair die.

Write down PMF.

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Write down PMF.

$x$	1	2	3	4	5	6
$f(x)$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$

$$E[X] = 1 \times \frac{1}{6} + 2 \times \frac{1}{6} + 3 \times \frac{1}{6} + 4 \times \frac{1}{6} + 5 \times \frac{1}{6} + 6 \times \frac{1}{6}$$
$$= 3.5$$



It is not necessary for the mean/expected value to be a possible value for  $X$ .

“Expected” means in the average over many performances of the experiment, not expected on the next trial.

## Mean of a Binomial Random variable

**Theorem 7.1** Let  $X \sim \text{Binom}(n, \pi)$ . Then

$$E[X] = n\pi.$$

Before diving into the proof, let's be mindful of a couple of useful facts:

- Fact 1: For any  $x \geq 1$ , we have the result:

$$x \binom{n}{x} = n \binom{n-1}{x-1}.$$

## Mean of a Binomial Random variable

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$$E[X] = n\pi.$$

Before diving into the proof, let's be mindful of a couple of useful facts:

- Fact 2: The binomial PMF sums to 1:

$$\sum_{x=0}^n \binom{n}{x} \pi^x (1-\pi)^{n-x} = 1.$$

## Mean of a Binomial Random variable: proof

$$\begin{aligned} E[X] &= \sum_{x=0}^n x \cdot f(x) = \sum_{x=0}^n x \binom{n}{x} \pi^x (1-\pi)^{n-x}, \\ &= \sum_{x=1}^n x \binom{n}{x} \pi^x (1-\pi)^{n-x}, \\ &= \sum_{x=1}^n n \binom{n-1}{x-1} \pi^x (1-\pi)^{n-x}, \quad \text{by fact 1} \\ &= n \pi \sum_{x=1}^n \binom{n-1}{x-1} \pi^{x-1} (1-\pi)^{n-x}, \\ &= n \pi \underbrace{\sum_{y=0}^{n-1} \binom{n-1}{y} \pi^y (1-\pi)^{n-1-y}}_{\text{PMF of a } \textit{Binom}(n-1, \pi)}, \quad y = x-1 \\ &= n \pi. \quad \text{by fact 2} \end{aligned}$$

## Linearity of Expected Values

Suppose in the die roll example, we are interested in calculating  $E[X + 3]$ .

It seems plausible that the answer should be  $E[X] + 3$ , since we are just adding 3 to the number that results from rolling the die.

The definition below formalizes this idea and is stated for any arbitrary linear transformation of  $X$ .

## Linearity of Expected Values

**Lemma 7.2** Let  $X$  be a discrete random variable, let  $a$  and  $b$  be constants and let  $Y = aX + b$ . Then  $Y$  is a discrete random variable and

$$E[Y] = aE[X] + b.$$

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$E(\text{linear fn of } X) = \text{same linear fn of } E(X)$

$y$  is a linear fc of  $X$ .

$X$  = temp in  $^{\circ}\text{F}$

$y$  = temp in  $^{\circ}\text{C}$

$$y = \underbrace{\frac{5}{9}}_a \cdot X - \underbrace{\frac{160}{9}}_b$$

Suppose  $E(X) = 60^{\circ}\text{F}$

what is  $E(Y)$

$$E(Y) = \frac{5}{9}E(X) - \frac{160}{9} = \frac{5}{9} \cdot 60 - \frac{160}{9}$$

## Linearity of Expectation: proof

Suppose  $X$  takes values  $x_1, x_2, x_3, \dots$  and  $f(x_1), f(x_2)$  and so on, are the probabilities.

Then  $Y$  takes values  $ax_1 + b, ax_2 + b, ax_3 + b, \dots$  and

$$P(Y = ax_i + b) = P(X = x_i) = f(x_i).$$

Therefore, the expected value of  $Y$  can now be found as follows:

$$\begin{aligned} E[Y] &= \sum_x (ax + b) \cdot f(x), \\ &= a \sum_x x \cdot f(x) + b \sum_x f(x), \\ &= aE[X] + b. \end{aligned}$$

## Example 7.3

A typical day's production of a certain electronic component is twelve. The probability that one of these components needs rework is 0.11. Each component needing rework costs \$100. What is the average daily cost for defective components?

in a day    ... working

# components  
that need  
reworking in a  
day is  
 $0.11 \times 12 = 1.32$   
total cost of

probability that one of these components needs rework is 0.11. Each component needing rework costs \$100. What is the average daily cost for defective components?

$$X = \# \text{ components} \text{ in a day} \text{ that need reworking}$$

$$X \sim \text{Binom}(n=12, \pi=0.11)$$

(assuming trials are independent)

$$\text{What is } E(X) = 12 \times 0.11 = 1.32$$

Define  $Y = \text{daily cost of reworking defective components}$

$Y = 100 \times X$  is a linear fn of  $X$  with  $a=100, b=0$

total cost of  
reworking  
 $1.32 \times 100 = 132$

$$\begin{aligned} E(Y) &= E[100 \times X] = 100 E[X] \\ &= 100 \times 12 \times 0.11 \\ &= \$132 \end{aligned}$$

$\therefore$  expected <sup>daily</sup> cost of reworking the defective items  
is \$132

Lemma 7.2 and example 7.4 illustrate that we do not need to find the PMF of  $Y$  in order to find its expectation.

In fact, this idea applies to any transformation of  $X$ , not just a linear one, and is stated next.

What if  $Y$  is a non-linear function of  $X$

$$\text{ex: } Y = X^2$$

$$E[Y] = E[X^2] \neq (E[X])^2$$

new random variable

## Law of Unconscious Probabilist

**Lemma 7.3** Let  $X$  be a discrete random variable with PMF  $f$  and let  $t(X)$  be a transformation of  $X$  for some function  $t$ . Then  $Y = t(X)$  is a discrete random variable and

$$E[Y] = E[t(X)] = \sum_x t(x)f(x).$$

## Example 7.4

Suppose  $X$  is a discrete random variable with PMF as shown in the following table.

$x$	-2	-1	0	1	2
probability	0.2	0.1	0.4	0.1	0.2

Use Lemma 7.3 to calculate  $E[X^2]$ .

$$Y = X^2$$

Find  $E[Y]$ .  
 possible values for  $Y$  :  $y = 0, 1, 4$

$$x=0 \quad x=-1 \text{ or } x=+1$$

$$x=-2 \text{ or } x=2$$

$$f_Y(y) = P(Y = y)$$

$$\begin{array}{c|ccc} y & 0 & 1 & 4 \\ \hline p(Y=y) & 0.4 & 0.2 & 0.4 \end{array}$$

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$$E[Y] = 0 \times 0.4 + 1 \times 0.2 + 4 \times 0.4$$

$$= 0.2 + 1.6 = \boxed{1.8}$$