Chapter 3

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Chapter 3

Equally Likely Rule and Counting methods

Review of last week

- **Probability** is a set function which satisfies certain axioms:
  - A1: $P(E) \geq 0$
  - A2: $P(S) = 1$
  - A3: The probability of a *disjoint* union is the sum of the probabilities.
- There are many corollaries to the axioms:
  - Rule of complements $P(A^c) = 1 - P(A)$
  - Addition rule $P(A \cup B) = P(A) + P(B) - P(A \cap B)$.

$P(A \cup B) = P(A) + P(B)$ so long as $A \cap B = \emptyset$

The Equally Likely Rule

Equally Likely Rule Suppose our sample space consists of n equally likely outcomes $s_1, s_2, \ldots, s_n$. Then

$$P(E) = \frac{|E|}{|S|} = \frac{|E|}{n} = \frac{\text{number of elements in } E}{\text{number of elements in } S}.$$

Proof: Let p be the probability of any outcome. Then:

$$\begin{aligned} P(S) &= P(s_1 \cup s_2 \cup \cdots \cup s_n) \\ &= P(s_1) + P(s_2) + \cdots + P(s_n) \qquad \text{axiom 2} \\ &= p + p + p + \cdots + p = np = 1. \qquad \text{axiom 3} \end{aligned}$$

$$\Rightarrow p = \frac{1}{n}.$$

ex: 2 fair coins

$S = \{(HH), (HT), (TH), (T,T)\}$

$E =$ at least 1 H

$= \{(HT), (TH), (H,H)\}$

$P(E) = \frac{\# E}{\# S} = \boxed{\frac{3}{4}}$

The Equally Likely Rule

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$$P(E) = \frac{|E|}{|S|} = \frac{|E|}{n} = \frac{\text{number of elements in } E}{\text{number of elements in } S}.$$

Proof: The event E consists of some outcomes from S. Again, by axiom 3, we require

$$\begin{aligned} P(E) &= \sum_{s_i \in E} P(s_i) = \sum_{s_i \in E} p, \\ &= p \cdot \sum_{s_i \in E} 1, \\ &= p \cdot |E| = \frac{1}{n}|E|. \end{aligned}$$

Example 3.1:

A local TV station advertises two news casting positions. If two women (W_1, W_2) and two men (M_1, M_2) apply, the "experiment" of hiring two coanchors generates a sample space with 6 outcomes:

$$S = \{(W_1, W_2), (W_1, M_1), (W_1, M_2), (W_2, M_1), (W_2, M_2), (M_1, M_2)\} .$$

Let E denote the event that *at least one woman is hired*. Calculate $P(E)$ assuming all six outcomes in S are equally likely.

$\# S = 6$

$E = \{ (W_1, W_2), (W_1, M_1), (W_1, M_2), (W_2, M_1), (W_2, M_2) \}$

$P(E) = 5/6$ by the equally likely rule.

- When all the outcomes in a sample space are equally likely, calculating a probability is as easy as counting.
- In complicated situations, or for large sample spaces, the counting may itself be challenging.
- Combinatorics is an area of mathematics primarily concerned with counting. In combinatorics, the **rule of product** or **multiplication principle** is a basic counting principle and is stated below.

Multiplication principle

Lemma 3.1 Multiplication Principle for Counting If a job consists of k separate tasks performed in series, the *ith* one of which can be done in n_i ways, then the entire job can be done in $n_1 \times n_2 \times \cdots \times n_k$ ways.

A deli has a lunch special which consists of a sandwich and a drink. They offer the following choices:

Sandwich: chicken salad, tuna salad and veggie

Drink: tea, coffee, coke, diet coke

How many different lunch specials are there?

Which sandwich? $\quad$ Which drink?

$n_1 = 3 \quad \times \quad n_2 = 4 \quad = \quad \boxed{12}$

Multiplication principle

How many integers between 100 and 999? How many with distinct digits?

$999 - 100 + 1 = 900$

Redo using Multiplication principle: $\underset{\text{1st digit}}{\underline{9}} \cdot \underset{\text{2nd digit}}{\underline{10}} \cdot \underset{\text{3rd digit}}{\underline{10}} = 900$ ✓

How many with distinct digits?

$\underset{\text{1st digit}}{\underline{9}} \cdot \underset{\text{2nd digit}}{\underline{9}} \cdot \underset{\text{3rd digit}}{\underline{8}} = \boxed{648}$

Example 3.2:

In 1824, Louis Braille (1809-1852) invented the standard alphabet for the blind. It uses a six dot matrix where some of the dots are raised. For instance, the letter "b" is

The configuration with no raised dots is useless. How many letters are in the Braille alphabet?

LETTERS CAN BE THOUGHT OF MAKING 6 DECISIONS

$\underset{\text{1st dot}}{\underline{2}} \quad \underset{\text{2nd dot}}{\underline{2}} \quad \cdots \quad \underset{\text{6th dot}}{\underline{2}} = 2^6$

1^{st} dot 2^{nd} dot 6^{th} dot

Since 1 configuration has all dots lowered, we have $2^6 - 1$ total letters.

Example 3.3

A statistics student needs to take three STAT electives (403, 425, 435) in their final four quarters (Fall, Winter, Spring, Summer). In how many ways can they plan their schedule, assuming they don't want to take more than one statistics elective in a quarter? (All three electives are offered each quarter)

Two decisions:

Which 3 quarters?

F, W, Sp
F, W, Su
F, Sp, Su
W, Sp, Su

Which course sequence?

403, 425, 435
403, 435, 425
425, 403, 435
425, 435, 403
435, 403, 425
435, 425, 403

$n_1 = 4$ $n_1 = 6$

24 COURSE SCHEDULES

- We could have used the multiplication rule to count all the rearrangements of the three electives 425, 403, 435:

$$\textit{Choices} : \underset{1}{\underline{3}} \times \underset{2}{\underline{2}} \times \underset{3}{\underline{1}} = 6$$

- But doing so for selecting which 3 quarters (out of 4) to study statistics

$$\textit{Choices} : \underset{1}{\underline{4}} \times \underset{2}{\underline{3}} \times \underset{3}{\underline{2}} = 24$$

  would lead to an overestimate.
- This is because this form of counting considers the choice "Fall, Winter, Spring" as different from "Winter, Fall, Spring". In other words, it keeps track of the position an item occupies.

Permutations and partial permutations

Position (or order) is a relevant characteristic when we are interested in rearrangements (or **permutations**) of a finite collection of items.

The number of rearrangements of k items can be found by a simple application of the multiplication rule:

$$\textit{Choices} : \underset{1}{\underline{k}} \times \underset{2}{\underline{k-1}} \times \underset{3}{\underline{k-2}} \cdots \times \underset{k-2}{\underline{3}} \times \underset{k-1}{\underline{2}} \times \underset{k}{\underline{1}} = k!$$

We can also use the multiplication rule to count the number of rearrangements of only k items selected from n objects - called a *partial permutation* or ordered subset:

$$\textit{Choices} : \underset{1}{\underline{n}} \times \underset{2}{\underline{n-1}} \times \underset{3}{\underline{n-2}} \cdots \times \underset{k}{\underline{n-(k-1)}} = \frac{n!}{(n-k)!}$$

Ex: A, B, C

rearrangements

A B C $\quad \underline{3} \cdot \underline{2} \cdot \underline{1} = 3!$

A C B

B C A

B A C

C A B

C B A

Ex: # rearrangements of 2 items selected from 4 : A, B, C, D

$\underline{4} \cdot \underline{3} = 12$

A B

B A

A C

$4 \times 3 = \frac{4!}{2!} = \frac{4 \times 3 \times 2!}{2!}$

Choices : $\underset{1}{n} \times \underset{2}{n-1} \times \underset{3}{n-2} \cdots \times \underset{k}{n-(k-1)} = \frac{n!}{(n-k)!}$

$n! = n \times (n-1) \times (n-2) \times \cdots \times (n-(k-1)) \times (n-k)!$

A B
B A
A C
C A
A D
D A
B C
C B
B D
D B
C D
D C

$4 \times 3 = \frac{4!}{2!} = \frac{4 \times 3 \times 2!}{2!}$

Combinations

- Position (or order) obviously does not matter when all we are interested in are unordered subsets - called a combination - of k items that can be formed from n.
- Since from any subset of k items, we can create $k!$ rearrangements or ordered subsets, we have the result:

$$\begin{aligned}\#\text{unordered subsets} &= \frac{1}{k!} \times \#\text{ ordered subsets}\\ &= \boxed{\frac{1}{k!} \times \frac{n!}{(n-k)!}}.\end{aligned}$$

Comment

- As an example, consider the following groupings of 3 items taken from 5: a, b, c, d, e.

A	partial permutations of the letters a,b,c,d, e taken 3 at a time **abc**, acb, bac, bca, cab, cba , **abd**, adb, bad, bda, dab, dba , **abe**, aeb, bae, bea, eab, eba , **acd**, adc, cda, cad, dac, dca , **ace**, aec, cae, cea, eac,eca , **ade**, aed, dae, dea, ead, eda , **bcd**, bdc, cbd, cdb, dcb, dbc , **bce**, bec, cbe, ceb, ebc, ecb , **bde**, bed, deb, dbe, edb, ebd , **cde**, ced, dce, dec, edc, ecd
B	groupings or *combinations* of the letters a, b, c, d,e taken 3 at a time **abc**, **abd**, **abe**, **acd**, **ace**, **ade**, **bcd**, **bce**, **bde**, **cde**

- For each element in B there are six (3!) distinct elements in A corresponding to the number of permutations or re-arrangements of the three letters in that element.
- Hence to count the number of elements in B, we can simply find the number of elements in A and divide by 3!

The binomial coefficient

Let n and k be integers with $0 \leq k \leq n$. We define $\binom{n}{k}$ as the number of different groupings of size k that can be formed from n items when the order of selection is irrelevant. Then

$$^{n}C_{k} = \binom{n}{k} = \frac{1}{k!} \cdot \frac{n!}{(n-k)!},$$

The notation $\binom{n}{k}$ is called a **binomial coefficient** and is read as "n choose k". The R function to compute binomial coefficients is ‘choose(n, r)‘.

By convention

$$\binom{n}{0} = 1, \quad \binom{n}{n} = 1, \quad \binom{n}{k} = 0, \; k < 0 \; or \; k > n$$

Example 3.4

From a group of 5 adults and 7 children:

a. how many different carpools of 2 adults and 3 kids can be formed?

WE ARE MAKING 2 DECISIONS IN SERIES

$$\underbrace{\binom{5}{2}}_{\text{Adults?}} \cdot \underbrace{\binom{7}{3}}_{\text{kids?}}$$

$\binom{5}{2} \cdot \binom{7}{3}$ carpools. $= \boxed{350}$

Example 3.4

From a group of 5 adults and 7 children:

b. how many different carpools of 2 adults and 3 kids can be formed if 2 of the kids are fighting and refuse to carpool together?

LET'S COUNT # CARPOOLS WITH BOTH THE FIGHTING KIDS IN IT.

$$\underbrace{\binom{5}{2}}_{\text{Adults}} \cdot \underbrace{\binom{5}{1}}_{\text{kids}} = \boxed{50}$$

$\therefore$ 300 CARPOOLS $(350 - 50)$ THAT DO NOT HAVE

∴ 300 CARPOOLS (350 − 50) THAT DO NOT HAVE

THE TWO KIDS WHO ARE FIGHTING.

Calculating a combinatorial probability

- In a combinatorial setting, making the transition from an enumeration to a probability is easy.
- If there are n ways to perform a certain operation and a total of m of those satisfy some stated condition—call it E —then by the equally likely rule, $P(E)$ is defined to be the ratio m/n.
- This assumes, of course, that each of the n possible ways are equally likely.

Example 3.5

From a standard well-shuffled deck of 52 playing cards (consisting of 4 suits – ♣, ♡, ♠, ♢ – each with 13 cards), you are dealt five cards.

a. The sample space S associated with this "experiment" consists of all groupings of five cards from the deck. How many elements are in S?

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$|S|$ = # of elements in S

$$= \binom{52}{5} = \frac{52!}{5!\,47!}$$

Example 3.5

From a standard well-shuffled deck of 52 playing cards (consisting of 4 suits – ♣, ♡, ♠, ♢ – each with 13 cards), you are dealt five cards.

b. Let E denote the event that the five cards are all of one suit. How many outcomes are in E?

$|E| : \binom{4}{1} \times \binom{13}{5}$

To count $|E|$, we make 2 decisions

$$\underbrace{\binom{4}{1}}_{\text{which suit?}} \times \underbrace{\binom{13}{5}}_{\text{which 5 cards from that suit?}}$$

Example 3.5

From a standard well-shuffled deck of 52 playing cards (consisting of 4 suits – ♣, ♡, ♠, ♢ – each with 13 cards), you are dealt five cards.

- **c.** Assume that each of the elements in S are equally likely. Calculate $P(E)$.

$$P(E) = \frac{|E|}{|S|} \quad \text{by equally likely rule}$$

$$= \frac{\binom{4}{1} \cdot \binom{13}{5}}{\binom{52}{5}}$$