Analysis of Algorithms Midterm

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1 *(a)*

$$
\begin{aligned}
&T(n) = 3T(n/4) + n\ log\ n \\
&a = 3, b = 4, f(n) = n\ log\ n \\
&f(n) = \Omega(n^{log_4 3+e}) \quad \text{case 3} \\
&(n\ log\ n)/(n^{log_4 3}) \quad \text{(it is polynomially larger)} \\
&\text{Regularity condition:} \\
&3(n/4)log(n/4) \leq c \cdot n\ log\ n \\
&n(3/4)(log\ n - log\ 4) \leq c \cdot n\ log\ n \\
&n(3/4)(log\ n - 2) \leq c \cdot n\ log\ n \\
&n(3/4)(log\ n) - (3/2) \leq c \cdot n\ log\ n \\
&(3/4) \cdot n(log\ n) - (3/2) \leq 1 \cdot n\ log\ n \\
&\text{(Passes)} \\
&T(n) = \Theta(n\ log\ n)
\end{aligned}
$$

(b)

$$
\begin{aligned}
&T(n) = 3T(n/3) + n/3 \\
&a = 3, b = 3, f(n) = n/3 \\
&f(n) = \Theta(n^{log_3 3}) \quad \text{case 2} \\
&T(n) = \Theta(n)
\end{aligned}
$$

2 *(a)* $T(n) = 3T(n/4) + n\ log\ n$

```
                  n log n
          /          |          \
(n/4)log(n/4)  (n/4)log(n/4)  (n/4)log(n/4)
...                 ...            ...
```

Height of tree is $h = log_4 n$, and width of leaves is $n^{log_4 3}$

$$T(n) = \Theta(n^{log_4 3}) + \sum_{i=1}^{h} 3^i((n/4^i)\ log\ (n/4^i))$$

$T(n) = 3T(n/3) + n/3$

```
                      n/3
          /            |          \
     (n/3)/3        (n/3)/3     (n/3)/3
...                   ...          ...
```

Height of tree is $h = log_3 n$, and width of leaves is $n^{log_3 3}$

$$T(n) = \Theta(n) + \sum_{i=1}^{h}(3^i((n/3^i)/3))$$

(b)

$$\begin{aligned}
&(b)\\
&T(n) = 3T(n/4) + n\ log\ n\\
&= \Theta(n^{log_4 3}) + \sum_{i=1}^{h} 3^i((n/4^i)\ log\ (n/4^i))\\
&= \Theta(n^{log_4 3}) + \sum_{i=1}^{h} n \cdot 3^i/4^i \cdot log\ (n/4^i))\\
&= \Theta(n^{log_4 3}) + \sum_{i=1}^{h} n \cdot log(n/4^i) \cdot 3^i/4^i\\
&= \Theta(n^{log_4 3}) + \sum_{i=1}^{h}(n\ log\ n - n\ log\ 4^i) \cdot 3^i/4^i\\
&= \Theta(n^{log_4 3}) + \sum_{i=1}^{h} 3^i/4^i \cdot n\ log\ n - 3^i/4^i \cdot n\ log\ 4^i\\
&= \Theta(n^{log_4 3}) + \sum_{i=1}^{h} 3^i/4^i \cdot n\ log\ n - \sum_{i=1}^{h} 3^i/4^i \cdot n\ log\ 4^i\\
&= \Theta(n^{log_4 3}) + n\ log\ n \sum_{i=1}^{h} 3^i/4^i - \sum_{i=1}^{h} 3^i/4^i \cdot n\ log\ 4^i\\
&= \Theta(n^{log_4 3}) + n\ log\ n \sum_{i=1}^{h}(3/4)^i - n\sum_{i=1}^{h}(3/4)^i \cdot log\ 4^i\\
&= \Theta(n^{log_4 3}) + n\ log\ n \sum_{i=1}^{h}(3/4)^i - n\sum_{i=1}^{h}(3/4)^i \cdot log\ 4^i\\
&< \Theta(n^{log_4 3}) + n\ log\ n \sum_{i=1}^{\infty}(3/4)^i - n\sum_{i=1}^{h}(3/4)^i \cdot log\ 4^i\\
&= \Theta(n^{log_4 3}) + n\ log\ n \left(\frac{1}{1-(3/4)}\right) - n\sum_{i=1}^{h}(3/4)^i \cdot log\ 4^i\\
&= \Theta(n^{log_4 3}) + n\ log\ n \cdot 4 - n\sum_{i=1}^{h}(3/4)^i \cdot log\ 4^i\\
&= \Theta(n^{log_4 3}) + \Theta(n\ log\ n) - n\sum_{i=1}^{h}(3/4)^i \cdot log\ 4^i\\
&= \Theta(n^{log_4 3}) + \Theta(n\ log\ n) - n\sum_{i=1}^{h}\frac{3^i\ log\ 4^i}{4^i}\\
&= \Theta(n^{log_4 3}) + \Theta(n\ log\ n) - n\sum_{i=1}^{h}\left(\frac{3\ log\ 4}{4}\right)^i\\
&< \Theta(n^{log_4 3}) + \Theta(n\ log\ n) - n\sum_{i=1}^{\infty}\left(\frac{3\ log\ 4}{4}\right)^i
\end{aligned}$$

$$
\begin{aligned}
&= \Theta(n^{log_4 3}) + \Theta(n\ log\ n) - n\left(\frac{1}{1-(3\ log\ 4/4)}\right) \\
&\approx \Theta(n^{log_4 3}) + \Theta(n\ log\ n) - n \cdot 0.15 \\
&= \Theta(n^{log_4 3}) + \Theta(n\ log\ n) - \Theta(n) \\
&= \Theta(n\ log\ n)
\end{aligned}
$$

So the final guess is actually $\mathcal{O}(n\ log\ n)$ because I had to use A.6 twice.

$$
\begin{aligned}
&(b) \\
&T(n) = 3T(n/3) + n/3 \\
&= \Theta(n) + \sum_{i=1}^{h}(3^i(\frac{n/3^i}{3})) \\
&= \Theta(n) + \sum_{i=1}^{h}\frac{(3^i n)/3^i}{3} \\
&= \Theta(n) + \sum_{i=1}^{h}\frac{n}{3} \\
&= \Theta(n) + h/3 \cdot n \\
&= \Theta(n) + \Theta(n) \\
&= \Theta(n)
\end{aligned}
$$

(c) For (a), where $T(n) = \Theta(n\ log\ n)$:
Inductive hypothesis: $T(n) \leq c \cdot n\ log\ n$ for $\mathcal{O}$ and $T(n) \geq c \cdot n\ log\ n$ for Ω. Induction for $\mathcal{O}$:

$$
\begin{aligned}
&T(n) \leq 3\ (n/4)\ log\ (n/4) + n\ log\ n \\
&= (3/4)n(log\ n - log\ 4) + n\ log\ n \\
&= 3/4 \cdot n\ log\ n - 3/4 \cdot n\ log\ 4 + n\ log\ n \\
&\leq 3/4 \cdot n\ log\ n + n\ log\ n \\
&= \frac{7}{4} \cdot n\ log\ n
\end{aligned}
$$

$$
\begin{aligned}
&T(n) \geq 3\ (n/4)\ log\ (n/4) + n\ log\ n \\
&= (3/4)n(log\ n - log\ 4) + n\ log\ n \\
&= 3/4 \cdot n\ log\ n - 3/4 \cdot n\ log\ 4 + n\ log\ n \\
&\geq 3/4 \cdot n\ log\ n + n\ log\ n \\
&= n\ log\ n
\end{aligned}
$$

For (b), where $T(n) = \Theta(n)$:
Inductive hypothesis: $T(n) \leq c \cdot n\ log\ n$ for $\mathcal{O}$ and $T(n) \geq c \cdot n\ log\ n$ for Ω. Induction for $\mathcal{O}$:

$$
\begin{aligned}
T(n) &= 3\ (n/4)\ log\ (n/4) + n\ log\ n \\
&= (3/4)n(log\ n - log\ 4) + n\ log\ n \\
&= 3/4 \cdot n\ log\ n - 3/4 \cdot n\ log\ 4 + n\ log\ n \\
&\leq 3/4 \cdot n\ log\ n - 3/4 \cdot n\ log\ n + n\ log\ n \\
&= n\ log\ n
\end{aligned}
$$

(d)

3 ***(a)***

(b)

(c)

(d)