Chapter 2.7

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- 1 Give a semantic proof of Exercise 6.13 (Theorem on constants). $\varphi(x_1, \cdots, x_n)$ is a formula in L. $c_1, \cdots, c_n$ are constant symbols not in L. $\forall x_1, \cdots, \forall x_n \varphi(x_1, \cdots, x_n)$ is tableau provable iff $\varphi(c_1, \cdots, c_n)$ is valid. Now apply completeness.

  We'll do this in both directions (because of the *iff*).

  - $\forall x_1 \cdots \forall x_n \varphi(x_1, \cdots, x_n) \to \varphi(c_1, \cdots, c_n)$.
    If $\forall x_1 \cdots \forall x_n \varphi(x_1, \cdots, x_n)$ is valid, then it is true in all structures A for L. We can extend L with new constants (call it L'). Again, since $\forall x_1 \cdots \forall x_n \varphi(x_1, \cdots, x_n)$ is true in all structures, there are individuals $t_1, \cdots, t_n$ in any structure such that $\varphi(t_1, \cdots, t_n)$ is true. And since $c_1, \cdots, c_n$ are new we can make them map: $c_1 \to t_1, c_2 \to t_2, \cdots$
  - $\varphi(c_1, \cdots, c_n) \to \forall x_1 \cdots \forall x_n \varphi(x_1, \cdots, x_n)$
    If $\varphi(c_1, \cdots, c_n)$ is valid, since these constants are not in L, they stand for any values in the domain specified in the structure. As there is no constraints on which values these may be, in different structures they may be mapped to different values, (ie they map to any ground terms, which are any constants, which is the meanin of $\forall$.) So it is true in all structures, so it is valid. So $\forall x_1 \cdots \forall x_n \varphi(x_1, \cdots, x_n)$.

- 6 This is the deduction theorem which we proved before for propositional logic.

  - $\Sigma \models \varphi$
  - $\models \bigwedge \Sigma \to \varphi$
  - $\Sigma \vdash \varphi$
  - $\vdash \bigwedge \Sigma \to \varphi$

  I won't do them in full detail, since it is very similar to th propositional case.

  $\Sigma \models \varphi$ says that we assume the truth of each formula in Σ. That means that we are only interested in structures that make each formula true. If each of these formulas is true and if φ is true (since these are the only structures we are interested in), then $\bigwedge \Sigma \to \varphi$. (Trivially, if we have a structure where all the formulas are not true, then these two are equivalent as well.)

  $\Sigma \vdash \varphi$ says that we add the atomic tableau $T\psi$ for each formula ψ in Σ. So we have a proof of φ which includes $T\psi$ for each formula $\psi \in \Sigma$. If we look at the atomic tableau for $\wedge$ (2a) we see that we add $T\psi$ and $T\varphi$ for each $\psi \wedge \varphi$, which is the same as we do in the definition of the costruction of a CST fro S (Defn 6.9).

  Now we can just use completeness and/or soundness (the shorter way) to connect the two pairs OR we can argue semantically, The soundnes proof is give in Theorem 7.2 and the completeness in Theorem 7.8.

  So, suppose we assume that we have a tableau proof of φ from Σ, but $\Sigma \not\models \varphi$. If that is the case, then there must be a structure that makes each formula in Σ true, but not φ:

$\Sigma \models \neg\varphi$. But then in the tableau (which has a root of $F\varphi$ and so $T\neg\varphi$), there must be a path that agrees with the signed entry. But we assumed that we had a tableau proof, meaning each path was contradictory. So this contradiction says that $\Sigma \models \varphi$.

Now suppose we assume that $\Sigma \models \varphi$. If it is valid, it is true in all strucures. Suppose there is no proof, then there is a finished tableau with a non-contradictory path P with a root $F\varphi$. But if we have a non-contradictory path, then we can create a structure which agrees with each entry on the non-contradictory path. But we just said that φ is true in all models, so we have our desired contradiction.

The only trick here is that the finished tableau may have an infinite path, and the non-contradictory path may be infinite.