Set 2 Homework, Analysis of Algorithms

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- P 52: 3.1-1, 3.1-2
- P 60: 3.2-1, 3.2-2 and Problems: 3-1, 3-3, 3-4
- P 107 Problems: 4-1, 4-2, 4-4

Chapter 3

3.1-1 Prove: $max(f(n), g(n)) = \Theta(f(n) + g(n))$
By theorem 3.1, in order for a func to be big-Theta, it should be both big-O and big-Omega.

$$
\begin{aligned}
&\mathcal{O}:\\
&max(f(n), g(n)) \leq f(n) + g(n)\\
&max(f(n), g(n)) = O(f(n) + g(n))\\
&\Omega:\\
&2 * max(f(n), g(n)) >= f(n) + g(n)\\
&max(f(n), g(n)) >= f(n) + g(n) * 1/2\\
&max(f(n), g(n)) = \Omega(f(n) + g(n))
\end{aligned}
$$

3.1-2 Prove: $(n + a)^b = Theta(n^b)$
Similarly to 3.1-1, we need to prove that the RHS is both big-O and big-Omega of the LHS.

$$
\begin{aligned}
&\mathcal{O}:\\
&Show : (n + a)^b \leq n^b * c && \text{for some constant c}\\
&Where : b > 0\\
&Cases :\\
&a \leq 0 :\\
&(n - a))^b < n^b\\
&(n - a))^b = O(n^b)\\
&a > 0 :\\
&n + a \leq n * a\\
&(n + a)^b \leq (n * a)^b = n^b * a^b\\
&(n + a)^b \leq n^b * a^b\\
&(n + a)^b = O(n^b) && \text{with constant } a^b \text{ for } a > 0
\end{aligned}
$$

$\Omega :$
$a \geq 0 :$
$(n+a) \geq n$
$(n+a)^b \geq n^b$
$(n+a)^b = \Omega(n^b)$
$a < 0 :$
$(n-a) \geq n \cdot -a$
$(n-a)^b \geq (n \cdot -a)^b$
$(n-a)^b \geq n^b \cdot -a^b$
$(n-a)^b = \mathcal{O}(n^b)$ with constant a^b for $a < 0$

3.2-1

$Show :$
If f(n) and g(n) are monotonically increasing, then so are:
$f(n) + g(n) :$
$f(n) \leq f(m)$
$g(n) \leq g(m)$
$f(n) + g(n) \leq f(m) + g(m)$
$f(g(n)) :$
$f(n) \leq f(m)$
$g(n) \leq g(m)$
$f(g(n)) \leq f(g(m))$
$*$ Let: $g(n) = p$ and $g(m) = q$
$*$ We know that $p \leq q$ because it was stated that $g(n) \leq g(m)$
$*$ We already said $f(n) \leq f(m)$ for all $n \leq m$, and that $p \leq q$
$*$ Thus $f(p) \leq f(q)$, that is $f(g(n)) \leq f(g(m))$

$Show :$
If $f(n)$ and $g(n)$ are nonnegative, then:
$f(n) \cdot g(n)$ is monotonically increasing
$Definitions :$
$*$ $f(n) \leq f(m)$ forall $n \leq m$
$*$ $g(n) \leq g(m)$ forall $n \leq m$
$*$ $f(n) > 0$ forall n
$*$ $g(n) > 0$ forall n
$Conclusions :$
$*$ Since $f(n)$ and $g(n)$ are monotonically increasing and only positive, then they will only be positively increasing.
$*$ $f(n) \cdot g(n) \leq f(m) \cdot g(m)$ forall $n \leq m$
$*$ This holds true because increasing positive integers multiplied will still be increasing.

3.2-2

$Prove:$

$a^{log(b,c)} = c^{log(b,a)}$

I assume we can use the equations above this one.

$Definition: q = b^y <=> log(b,q) = y$

$a^{log(b,c)} = c^{log(b,a)}$

$= log(c, a^{log(b,c)}) = log(b,a)$

$= log(b,c) * log(c,a) = log(b,a)$

$= log(c,a) = log(b,a)/log(b,c)$

$= log(c,a) = log(c,a)$

This used equations on p56 above the equation we proved.

3-1 The following is a lemma that I'll use for this problem:

a. Prove: $k \geq d \rightarrow p(n) = \mathcal{O}(n^k)$

$$Show: \sum_{i=0}^{d} a_i n^i \leq c \cdot n^k \text{ for some constant c}$$

Let $a_m = max(a_i)$

$$\sum_{i=0}^{d} a_i n^i \leq (a_m d) \cdot n^d \leq (a_m d) \cdot n^k$$

$$\sum_{i=0}^{d} a_i n^i = \mathcal{O}(n^k) \qquad \text{with constant } (a_m \cdot d)$$

b. Prove: $k \leq d \rightarrow p(n) = \Omega(n^k)$

$$Show: \sum_{i=0}^{d} a_i n^i \geq c \cdot n^k \text{ with some constant c}$$

$$\sum_{i=0}^{d} a_i n^i \geq n^d \geq n^k$$

$$\sum_{i=0}^{d} a_i n^i = \Omega(n^k) \qquad \text{with constant 1}$$

c. Prove: $k = d \rightarrow p(n) = \Theta(n^k)$

See proof in (a) and (b); by Theorem 3.1, n^d is also Θ.

$$Show: \sum_{i=0}^{d} a_i n^i \geq c \cdot n^d \text{ with some constant c}$$

$$Also: \sum_{i=0}^{d} a_i n^i \leq e \cdot n^d \text{ with some constant e}$$

$$\sum_{i=0}^{d} a_i n^i \leq (a_m d) \cdot n^d$$

$$\sum_{i=0}^{d} a_i n^i \geq n^d$$

d. Prove: $k > d \rightarrow p(n) = o(n^k)$

$$Show: \sum_{i=0}^{d} a_i n^i < c \cdot n^k \text{ with some constant c}$$

$$\sum_{i=0}^{d} a_i n^i \leq (a_m d) \cdot n^d < (a_m d) \cdot n^k$$

e. Prove: $k < d \rightarrow p(n) = \omega(n^k)$

$$Show: \sum_{i=0}^{d} a_i n^i > c \cdot n^k \text{ with some constant c}$$

$$\sum_{i=0}^{d} a_i n^i \geq n^d > n^k$$

3-3 From largest to smallest:

$$2^{2^n}$$
$$(n+1)!$$
$$n!$$
$$e^n$$
$$n \cdot 2^n$$
$$2^n$$
$$(3/2)^n$$
$$(lgn)^{lgn}$$
$$(lgn)!$$
$$n^3$$
$$n^2$$
$$nlgn, lg(n!)$$
$$n$$
$$2^{\sqrt{2lgn}}$$
$$(lgn)^2$$
$$lgn$$
$$\sqrt{lgn}$$
$$lglgn$$
$$2^{lg \cdot n}$$
$$(lgn)*$$
$$n^{1/lgn}$$

Some more of them may be in equivalence classes...

3-4 **a.** False by counterexample: n and n^2

b. False by counterexample: n and n^2

c. True: $f(n) \leq g(n)$ and $lg(f(n)) \leq lg(g(n))$

d. True: $f(n) \leq g(n)$ and $2^{f(n)} \leq 2^{g(n)}$

e. True: $f(n) \leq f(n)^2$

f. True by transpose symmetry.

g. False: $n^2 > c * (n/2)^2$

h. True:

$$f(n) + o(n) \leq 2 \cdot f(n)$$
$$f(n) + o(n) \geq 1/2 \cdot f(n)$$

Chapter 4

4-1 **a.**

b.

c.

d.

e.

f.

g.

4-2 **a.**

b.

4-4

a.

b.

c.

d.