Analysis of Algorithms Final

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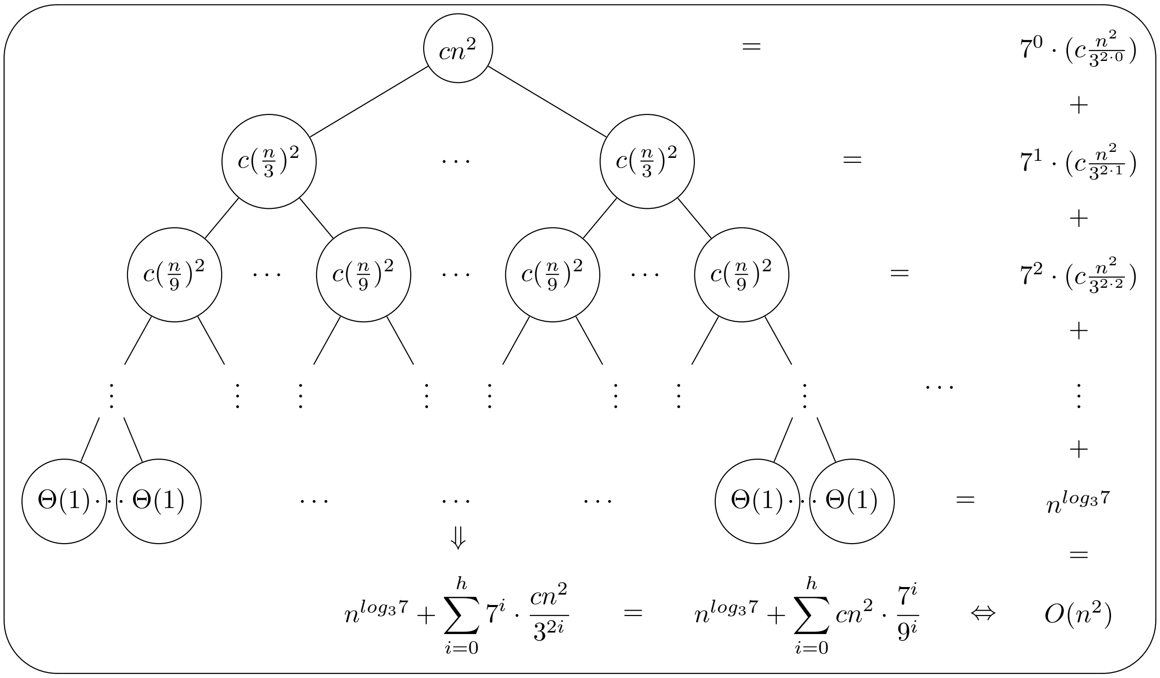
1 ***(a)*** $height = log_3 n$



(b)

$$\sum_{i=0}^{log_3 n} \frac{7^i cn^2}{3^{i2}} + n^{log_3 7}$$
$$< cn^2 \sum_{i=0}^{\infty} \frac{7^i}{9^i} + n^{log_3 7}$$
$$= cn^2 \frac{1}{1 - 7/9} + n^{log_3 7}$$
$$= 3cn^2 + n^{log_3 7}$$
$$< 4cn^2$$
$$= \mathcal{O}(n^2)$$

(c) Induction for $\mathcal{O}$

Inductive Hypothesis: $T(n) \geq cn^2$

Induction: $T(n) \geq 7c\frac{n^2}{3^2} + dn^2$

$= cn^2 \frac{7}{9} + dn^2$

$\geq cn^2$ with $c \leq 4d$ and $n \geq 0$

Induction for Ω

$$\text{Inductive Hypothesis: } T(n) \leq cn^2$$
$$\text{Induction: } T(n) \leq 7c\frac{n^2}{3^2} + dn^2$$
$$= cn^2\frac{7}{9} + dn^2$$
$$\leq cn^2 \quad \text{with } c \geq 5d \text{ and } n \geq 0$$

(d)

$$a = 7, \; b = 3$$
$$f(n) = n^2 = \Omega(n^{log_3 7+\epsilon})$$
$$lim_{n\to\infty}\frac{n^2}{n^{log_3 7+\epsilon}} < \infty \text{ (it is polynomially larger)}$$
Regularity condition:
$$7\frac{n^2}{3^2} \leq cn^2$$
$$= \frac{7}{9}n^2 \leq cn^2 \quad \text{for } c \geq 7/9 \text{ and } c \leq 1 \text{ and } n \geq 0$$
(Passes)

Case 3: $\Theta(n^2)$

2 ***(a)*** I'll use the four conditions listed in our book on page 379. It took me a bit to realize how this was not a greedy choice problem. Tricky!

i. Our inital choice can be a division that we make of C, where $C_{choice} < C$ is optimal and $C - C_{choice}$ is our subproblem.
ii. The optimal choice given to us would be the sum C_{choice}, which is less than C so divides it at some point.
iii. The subproblem that ensues is $C - C_{choice}$ which is the remaining sum we have yet to find an optimum on.
iv. Suppose we have come to an optimal solution with suboptimal C_{choice} and $C - C_{choice}$. If we "cut" away our two suboptimal subproblems and replace them with more optimal ones, then we would supposedly increase the optimality of the whole problem. But this is a contradiction of our suppostion that we had an optimal solution.

(b) In haskell:

```
vs = [1,5,10,25,100,200]

change 0 _ = 0
change c [1] = c
change c v
 | last v <= c = min (change c (init v)) (1 + change (c-(last v)) v)
 | last v > c  = change c (init v)

-- >> change 7 vs
-- >> 3
-- >> change 42 vs
-- >> 5
```

With memoization:

```
-- Generate our memoization matrix.
-- There's probably a prettier way to do it, perhaps with list comp.
matrix c v = map (\(n,cs) -> (map (\c -> (n,c)) cs)) (zip [1..n] (replicate n [c,c-1..1]))
 where n = length v
-- Map our change function over the matrix
change_matrix c v = map (map (\(x,y) -> ch y (take x v))) (matrix c v)
 where
 ch 0 _ = 0
 ch c [1] = c
 ch c v
  | last v <= c = min (mchange c (init v)) (1 + mchange (c-(last v)) v)
  | last v > c  = mchange c (init v)
  where n = length v

-- Get the cell in the matrix for which we used n coins on c sum
mchange c v = (change_matrix c v) !! (length v - 1) !! 0

-- >> mchange 7 vs
-- >> 3
-- >> mchange 42 vs
-- >> 5
```

That was fun. It seems to work correctly but I'm not entirely sure it memoizes rather than recomputes the matrix every time. I've read before about using a fixed point function to factor out memoization in haskell that looked really neat but I probably don't have time to figure it out.

(c) For each coin $v(n)$ we have to consider $min(f(C-v(n),n), f(C-v(n-1),n), \ldots, f(C-v(1),n))$ where f is our choice function. That amounts to n total choices per coin.

(d) Since for each coin we must consider n subproblems, and the worst case is C total coins, our complexity would then be $\Theta(Cn)$ total choices overall.

Here is a greedy version of the algorithm I made before realizing it isn't optimal. Might as well leave it here.

Let C be our goal sum and $v(n)$ be the coin of n^{th} value.

$NCoins(C,1) = C$

$NCoins(C,n) = NCoins(C \ \% \ v(n), n-1) + \lfloor C/v(n) \rfloor$

3 ***(a)*** I'll use the conditions on page 423 of our book:

i. Suppose there are two leaves, x and y, having both the least weights and the largest depths. To show that this is a greedy choice problem, we can show that the tree with x and y as given is an optimal tree (labeled T).
By contradiction let's suppose that these two leaves with the two largest depths are not x and y so do not have the least weights. Instead they are b and d which produces the supposed optimal T'.
Now let's swap one of the least weights, x, with one of the lowest leaves, b, to get a new tree T''. Let $d(b) - d(x)$ from T' be $diff$. The depth of x in T'' is increased by $diff$ while the depth of b is decreased by $diff$.
The cost of our new tree is then: $cost(T'') = cost(T') + diff(cost(x) - cost(b))$. By supposition however, $cost(x) \leq cost(b)$, so $diff(cost(x) - cost(b)) \leq 0$. That means by swapping x and b to put x in one of the two lowest leaves, we get T'' which is more optimal than T'. This is more optimal so contradicts our supposition. We can do the same procedure on y and d.

ii. Optimal substructure property: Given a set of trees T, we must choose t', which is an optimal new tree that merges two trees within T. We do this repeatedly until there is one tree, so each choice or step is the subproblem.

(b) We begin with a set of single-leaf trees, each containing the single characters mapped to their weight (the number of occurences in the given string). We then go bottom-up (and either left-to-right or right-to-left), find the two trees with the least weights, and merge them as two leaves in a new tree with the sum of the two subtrees' weights as the weight of the root.