Set 2 Homework, Analysis of Algorithms

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- P 52: 3.1-1, 3.1-2
- P 60: 3.2-1, 3.2-2 and Problems: 3-1, 3-3, 3-4
- P 107 Problems: 4-1, 4-2, 4-4

Chapter 3

3.1-1 Prove: $max(f(n), g(n)) = \Theta(f(n) + g(n))$

By theorem 3.1, in order for a func to be big-Theta, it should be both big-O and big-Omega.

$$
\begin{aligned}
&\mathcal{O}: \\
&max(f(n), g(n)) \leq f(n) + g(n) \\
&max(f(n), g(n)) = O(f(n) + g(n)) \\
&\Omega: \\
&2 * max(f(n), g(n)) >= f(n) + g(n) \\
&max(f(n), g(n)) >= f(n) + g(n) * 1/2 \\
&max(f(n), g(n)) = \Omega(f(n) + g(n))
\end{aligned}
$$

3.1-2 Prove: $(n + a)^b = Theta(n^b)$

Similarly to 3.1-1, we need to prove that the RHS is both big-O and big-Omega of the LHS.

$$
\begin{aligned}
&\mathcal{O}: \\
&Show: (n + a)^b \leq n^b * c && \text{for some constant c} \\
&Where: b > 0 \\
&Cases: \\
&a \leq 0: \\
&(n - a))^b < n^b \\
&(n - a))^b = O(n^b) \\
&a > 0: \\
&n + a \leq n * a \\
&(n + a)^b \leq (n * a)^b = n^b * a^b \\
&(n + a)^b \leq n^b * a^b \\
&(n + a)^b = O(n^b) && \text{with constant } a^b \text{ for } a > 0
\end{aligned}
$$

$\Omega:$
$a \geq 0:$
$(n+a) \geq n$
$(n+a)^b \geq n^b$
$(n+a)^b = \Omega(n^b)$
$a < 0:$
$(n-a) \geq n \cdot -a$
$(n-a)^b \geq (n \cdot -a)^b$
$(n-a)^b \geq n^b \cdot -a^b$
$(n-a)^b = \mathcal{O}(n^b)$ with constant a^b for $a < 0$

3.2-1

$Show:$
If f(n) and g(n) are monotonically increasing, then so are:
$f(n) + g(n):$
$f(n) \leq f(m)$
$g(n) \leq g(m)$
$f(n) + g(n) \leq f(m) + g(m)$
$f(g(n)):$
$f(n) \leq f(m)$
$g(n) \leq g(m)$
$f(g(n)) \leq f(g(m))$
$*$ Let: $g(n) = p$ and $g(m) = q$
$*$ We know that $p \leq q$ because it was stated that $g(n) \leq g(m)$
$*$ We already said $f(n) \leq f(m)$ for all $n \leq m$, and that $p \leq q$
$*$ Thus $f(p) \leq f(q)$, that is $f(g(n)) \leq f(g(m))$

$Show:$
If $f(n)$ and $g(n)$ are nonnegative, then:
$f(n) \cdot g(n)$ is monotonically increasing
$Definitions:$
$*\, f(n) \leq f(m)$ forall $n \leq m$
$*\, g(n) \leq g(m)$ forall $n \leq m$
$*\, f(n) > 0$ forall n
$*\, g(n) > 0$ forall n
$Conclusions:$
$*$ Since $f(n)$ and $g(n)$ are monotonically increasing and only positive, then they will only be positively increasing.
$*\, f(n) \cdot g(n) \leq f(m) \cdot g(m)$ forall $n \leq m$
$*$ This holds true because increasing positive integers multiplied will still be increasing.

3.2-2

$$Prove:$$
$$a^{log(b,c)} = c^{log(b,a)}$$
I assume we can use the equations above this one.
$$Definition: q = b^y <=> log(b,q) = y$$
$$a^{log(b,c)} = c^{log(b,a)}$$
$$= log(c, a^{log(b,c)}) = log(b,a)$$
$$= log(b,c) * log(c,a) = log(b,a)$$
$$= log(c,a) = log(b,a)/log(b,c)$$
$$= log(c,a) = log(c,a)$$
This used equations on p56 above the equation we proved.

3-1 The following is a lemma that I'll use for this problem:

a. Prove: $k \geq d \rightarrow p(n) = \mathcal{O}(n^k)$

$$Show: \sum_{i=0}^{d} a_i n^i \leq c \cdot n^k \text{ for some constant c}$$
Let $a_m = max(a_i)$
$$\sum_{i=0}^{d} a_i n^i \leq (a_m d) \cdot n^d \leq (a_m d) \cdot n^k$$
$$\sum_{i=0}^{d} a_i n^i = \mathcal{O}(n^k) \qquad \text{with constant } (a_m \cdot d)$$

b. Prove: $k \leq d \rightarrow p(n) = \Omega(n^k)$

$$Show: \sum_{i=0}^{d} a_i n^i \geq c \cdot n^k \text{ with some constant c}$$
$$\sum_{i=0}^{d} a_i n^i \geq n^d \geq n^k$$
$$\sum_{i=0}^{d} a_i n^i = \Omega(n^k) \qquad \text{with constant 1}$$

c. Prove: $k = d \rightarrow p(n) = \Theta(n^k)$
See proof in (a) and (b); by Theorem 3.1, n^d is also Θ.

$$Show: \sum_{i=0}^{d} a_i n^i \geq c \cdot n^d \text{ with some constant c}$$
$$Also: \sum_{i=0}^{d} a_i n^i \leq e \cdot n^d \text{ with some constant e}$$
$$\sum_{i=0}^{d} a_i n^i \leq (a_m d) \cdot n^d$$
$$\sum_{i=0}^{d} a_i n^i \geq n^d$$

d. Prove: $k > d \rightarrow p(n) = o(n^k)$

$$Show: \sum_{i=0}^{d} a_i n^i < c \cdot n^k \text{ with some constant c}$$
$$\sum_{i=0}^{d} a_i n^i \leq (a_m d) \cdot n^d < (a_m d) \cdot n^k$$

e. Prove: $k < d \rightarrow p(n) = \omega(n^k)$

$$Show: \sum_{i=0}^{d} a_i n^i > c \cdot n^k \text{ with some constant c}$$
$$\sum_{i=0}^{d} a_i n^i \geq n^d > n^k$$

3-3

3-4

Chapter 4

4-1

4-2

4-4