Unit-2

Types of operator :-

i) Shifting Operator :- It is denoted by 'E' & defined as

$$Ef(x) = f(x+h) \quad , \text{where } y = f(x)$$

where, h = step length & generally, $[h = 1]$

$$\boxed{\begin{aligned} E f(x+h) &= f(x+h+h) = f(x+2h) \\ E^2 f(x) &= E\left(E f(x)\right) \\ &= E\left(f(x+h)\right) \\ &= f(x+2h) \\ \text{so on, } E^n f(x) &= f(x+nh) \end{aligned}}$$

Q Find the relation between E & D (ii) (differiential operator)

↳ we know that,

the definition of shifting operator is

$$Ef(x) = f(x+h)$$

$$Ef(x) = f(x) + h f'(x) + \frac{h^2}{2!} f''(x) + \ldots$$

$$Ef(x) = f(x) + h\, Df(x) + \frac{h^2}{2!} D^2 f(x) + \ldots$$

$$Ef(x) = f(x)\left[1 + hD + \frac{h^2}{2!} D^2 + \ldots\right]$$

$$E = e^{hD} \qquad \left[e^x = 1 + x + \frac{1}{2!} x^2 + \ldots\right]$$

where, D is differential operator, $\frac{d}{dx}$

(iii) Forward difference operator :-

It is denoted by 'Δ' & defined as

$$\Delta f(x) = f(x+h) - f(x)$$

- also known as I^{st} Forward difference

Now, the value of 2^{nd} forward difference is

$$\Delta^2 f(x) = \Delta(\Delta f(x)) = \Delta(f(x+h) - f(x))$$

$$= \Delta f(x+h) - \Delta f(x)$$
$$= f(x+2h) - 2f(x+h) + f(x)$$

① Relation b/w Forward difference operator & Shifting operator

$$\Delta f(x) = f(x+h) - f(x)$$
$$\Delta f(x) = Ef(x) - f(x)$$
$$\Delta f(x) = (E-1) f(x)$$

$$\Rightarrow \quad \Delta = E - 1$$

iv) Backward difference operator :-

Denoted by '∇' & defined as

$$\nabla f(x) = f(x) - f(x-h) \quad \rightarrow \text{i.e } I^{st} \text{ Backward difference}$$

Nabla or Del

Now, value of 2^{nd} Backward diff. is

$$\nabla^2 f(x) = \nabla(f(x) - f(x-h))$$
$$= f(x) - f(x-h) - f(x-h) + f(x-2h)$$
$$= f(x) - 2f(x-h) + f(x-2h)$$

Note :- We know that, f^n; $y = f(x)$, if $x = 0$ & $h = 1$, then $\Delta y_x = y_{x+h} - y_x$

$$\Delta y_0 = y_1 - y_0$$

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Similiarly, $\Delta y_2 = y_3 - y_2$

$\nabla y_0 = y_0 - y_{-1}$, $\nabla y_3 = y_3 - y_2$

$E^2 y_{10} = y_{12}$

Q Relation b/w ∇ & E.

we know,

$$\nabla f(x) = f(x) - f(x-h)$$
$$\nabla f(x) = f(x) - E^{-1}(f(x))$$
$$\nabla = 1 - \frac{1}{E}$$

or $\boxed{E^{-1} + \nabla = 1}$

Q Relation b/w Δ & ∇.

$$\nabla f(x) = f(x) - f(x-h)$$
$$\cancel{\nabla f(x)} \quad \nabla = 1 - E^{-1}$$

we know, $\Delta = E - 1$

$$\Delta = E\left(1 - \frac{1}{E}\right)$$
$$\boxed{\Delta = E\nabla}$$
$$\Delta = (1+\Delta)\nabla$$
$$\Rightarrow \nabla = \frac{\Delta}{1+\Delta}$$

Forward Difference Table :-

Let $y = f(x)$ be a given function, in which independent variable x is called argument & dependent variable y is called Entry.

The values of y are y_0, y_1, y_2, y_3, y_4 & y_5 when $x = x_0$, $x = x_1 = x_0 + h$, $x = x_2 = x_0 + 2h$ ..., $x = x_5 = x_0 + 5h$

Then Forward Difference table is

x	y	$\Delta y = \Delta f(x) = f(x+h) - f(x)$	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$
x_0	y_0	$y_1 - y_0 = \Delta y_0$	$\Delta^2 y_0$	$\Delta^3 y_0$	$\Delta^4 y_0$	$\Delta^5 y_0$
x_1	y_1	$y_2 - y_1 = \Delta y_1$	$\Delta^2 y_1$	$\Delta^3 y_1$	$\Delta^4 y_1$	
x_2	y_2	$y_3 - y_2 = \Delta y_2$	$\Delta^2 y_2$	$\Delta^3 y_2$		
x_3	y_3	$y_4 - y_3 = \Delta y_3$	$\Delta^2 y_3$			
x_4	y_4	$y_5 - y_4 = \Delta y_4$				
x_5	y_5					

Q Construct forward diff. table from the following table data

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
5	9962	-114	-75	2	-1
10	9848	-189	-73	1	2
15	9659	-262	-72	3	
20	9397	-334	-69		
25	9063	-403			
30	8660				

$\Delta^2 y_{10} = -73$

$\Delta^4 y_5 = -1$

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Backward Difference table :-

Let $y = f(x)$ be given fn in which x is independent variable called as argument & dependent variable y is called as entry.
values are $y_0, y_1, y_2 \ldots y_5$ are $x = x_0, x_1 = x_0 + h$
$\therefore x = x_5 = x_0 + 5h$.

x	y	∇y	$\nabla^2 y$	$\nabla^3 y$	$\nabla^4 y$	$\nabla^5 y$
x_0	y_0					
x_1	y_1	∇y_1	$\nabla^2 y_2$	$\nabla^3 y_3$		
x_2	y_2	∇y_2	$\nabla^2 y_3$	$\nabla^3 y_4$	$\nabla^4 y_4$	$\nabla^5 y_5$
x_3	y_3	∇y_3	$\nabla^2 y_4$		$\nabla^4 y_5$	
x_4	y_4	∇y_4	$\nabla^2 y_5$	$\nabla^3 y_5$		
x_5	y_5	∇y_5				

Q

x	y	∇y	$\nabla^2 y$	$\nabla^3 y$	$\nabla^4 y$	$\nabla^5 y$
10	1	+0.3010		-0.1249		
20	1.3010	+0.[illegible]	-.125		0.0738	-0.0508
30	1.4771	+0.1250	-.051	-0.0511		
40	1.6021	.0969		-0.0211	0.023	
50	1.6990					

$$\nabla^3 y_{40} = 0.0738, \quad \nabla^4 y_{50} = -0.0508$$

Newton Forward $\div$ $y(x) = y_p = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \ldots$

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here, $p = \frac{x - x_0}{h}$

Newton's ~~Forw~~ Backward $\div$

$$y(x) = y_p = y_n + p\nabla y_n + \frac{p(p+1)}{2!}\nabla^2 y_n + \ldots$$

here, $p = \frac{x - x_n}{h}$

V.I ① When p is any real value $p = \frac{x - x_n}{h}$ & h is steplength & x_n is last value.

From the following table estimate the no. of students who obtained the marks b/w 40 & 45.

Marks :	30-40	40-50	50-60	60-70	70-80
No. of students :	31	42	51	35	31

First we generate the cummulative data of following data

Marks (Less than)	No. of student (y)	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
40	31				
		42			
50	73		9		
		51		-25	
60	124		-16		37
		35		12	
70	159		-4		
		31			
80	190				

here, $h = 10$, $x = 45$, $x_0 = 40$

$$p = \frac{45 - 40}{10} = 0.5$$

Newton Forward diff. is

$$y(x) = y_p = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0$$

$$+ \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 + \frac{p(p-1)(p-2)(p-3)}{4!}\Delta^4 y_0 \quad \text{①}$$

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$$y(45) = y_{0.5} = 31 + 21 + \left[\frac{1}{2} \times \frac{-1}{2} \times \frac{1}{2} \times 9\right] + \left[\frac{1}{2} \times \frac{-1}{2} \times \frac{-3}{2} \times \frac{1}{\cancel{2}\times 3} \times -25\right]$$

$$+ \frac{1}{2} \times \frac{-1}{2} \times \frac{-3}{2} \times \frac{-5}{2} \times 37 \times \frac{1}{4 \times \cancel{3} \times 2}$$

$$y(45) = 52 - \frac{9}{8} - \frac{25}{16} - \frac{37 \times 5}{32 \times 4}$$

$$= 52 - 1.125 - 1.5625 - \cancel{23.125}\ 1.445$$

$$= 52 - \cancel{25.8125}\ 4.1325$$

$$= 47.867$$

$$y(45) \simeq 48$$

128 and ~~ma~~ 40 marks obtain by 31

then marks b/w 40 & 45 is obtained by

$= 48 - 31$

$= 17$

Q Find a cubic polynomial which takes the values

	x	$f(x)=y$	Δy	$\Delta^2 y$	$\Delta^3 y$
$x_0 \rightarrow$	0	1	1	-2	12
	1	2	-1	10	
	2	1	9		
	3	10			

Hence Evalute $f(4)$

$$p = \frac{x-0}{1} = x$$

$$y(x) = 1 + x + \frac{x(x-1)(-2)}{2} + \frac{x(x-1)(x-2)\,12^{2}}{\cancel{6}_{1}}$$

$$y(x) = 1 + x + (x^2-1)(-1) + (x^2-1)(2x-4)$$

$$= 1 + x - x^2 + 1 + 2x^3 - 2x - 4x^2 + 4$$

$$= 2x^3 - 7x^2 + 6x + 1$$

$$y(4) = 2(4)^3 - 5 \times 16 - 24 + 2$$
$$= 128 - 80 + 2$$
$$= 50$$

Q. Calculate the value of $\tan 48^\circ 15'$ from following table. ($\to 48.25^\circ$)

x	$y = \tan x$	$10^5 y$	$10^5 \nabla y$
45	1	1	
46	1.03053	103053	
47	1.07237	107237	
48	1.11061	111061	
49	1.15037	115037	
50	1.19175	119175	

$$p = \frac{x - x_n}{h} = \frac{48.25 - 50}{1} = -1.75$$

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Q Evaluate using Newton Backward difference, Find the value $e^{-1.9}$

x :	1	1.25	1.50	1.75	2.00
e^{-x} :	0.3679	0.2865	0.2231	0.1738	0.1353

x	$e^{-x}=y$	∇y	$\nabla^2 y$	$\nabla^3 y$	$\nabla^4 y$
1	0.3679	-0.0814			
1.25	0.2865	-0.0634	0.0180	-0.0039	
1.50	0.2231	-0.0493	0.0141		0.0006
1.75	0.1738	-0.0385	0.0108	-0.0033	
2.00	0.1353				

we know that,

$$p = \frac{x - x_n}{h} = \frac{1.9 - 2}{0.25} = -0.4$$

$$y(1.9) = y_p = y_n + p\nabla y_n + \frac{p(p+1)}{2!}\nabla^2 y_n + \frac{p(p+1)(p+2)}{3!}\nabla^3 y_n + \frac{p(p+1)(p+2)(p+3)}{4!}\nabla^4 y_n$$

$$y(1.9) = 0.1353 + \left(\frac{-2}{5}\right) \times -0.385 + \frac{(-0.4)(0.6)}{2} \times 0.0108 + \frac{(-0.4)(0.6)(1.6)}{6}(-0.0033) + \frac{(-0.4)(0.6)(1.6)(2.6)}{24} \times 0.0006$$

$$y(1.9) = 0.1353 + 0.154 - 0.001296 + 0.0002112 - 0.00002496$$

$$y(1.9) = 1.495$$

Note :-

1> We'll use Newton Forward Diff. formula for interpolation if the value of x near the beginning of tabulated value.

11> We'll use Newton Backward Diff. formula for interpolation if value of x near the end of tabulated value.

Central Difference :- Interpolation

i> Gauss Forward interpolation formula

Let $y=f(x)$ be a given f^n, in which values of y are

$\ldots y_{-2}, y_{-1}, y_0, y_1, y_2, \ldots$

& values of x are $\ldots x = x_{-2} = x_0 - 2h$, $x = x_{-1} = x_0 - h$,

$x = x_0$, $x = x_1 = x_0 + h$, ...

here, steplength $= h$

then, Gauss Forward formula is

$$y(x) = y_p = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_{-1} + \frac{p(p-1)(p+1)}{3!}\Delta^3 y_{-1}$$

$$+ \frac{(p+1)(p)(p-1)(p-2)}{4!}\Delta^4 y_{-2}$$

$$+ \frac{(p+2)(p+1)(p)(p-1)(p-2)}{5!}\Delta^5 y_{-2} + \ldots$$

where $p = \dfrac{x - x_0}{h}$

[Note :- In Gauss forward Formula, the value of p lies b/w 0 & 1]

Q Apply central formula to obtain $f(32)$ from

$f(25) = 0.2707$, $f(35) = 0.3386$, $f(30) = 0.3027$,
$f(40) = 0.3794$

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$
25	0.2707	0.0320		
$x_0 \leftarrow 30$	0.3027	0.0359	0.0039	0.0010
35	0.3386	0.408	0.0049	
40	0.3794			

we know that,

$h = 5$, $\quad p = \frac{x - x_0}{h} = \frac{32-30}{5} = \frac{2}{5} = 0.4$

$$y(32) = 0.3027 + (0.4) \times (0.0359) + \frac{(0.4)(-0.6)}{2} \times (0.0039)$$
$$+ \frac{(0.4)(1.4)(-0.6)}{6} \times (0.0010)$$

$$y(32) = 0.3027 + 0.01436 - 0.000468 - 0.000056$$

$$\boxed{y(32) = \cancel{0.4457}\ 0.3165}$$

Gauss Backward Formula :

It can be written as

$$y(x) = y_p = y_0 + p \Delta y_{-1} + \frac{p(p+1)}{2!} \Delta^2 y_{-1} + \frac{(p-1)(p)(p+1)}{3!} \Delta^3 y_{-2}$$
$$+ \frac{(p+2)(p+1)(p)(p-1)}{4!} \Delta^4 y_{-2} + \ldots$$

where $p = \frac{x - x_0}{h}$

[$\because$ p must be lies b/w -1 to 0.]

Q Find the value of $\cos 51^\circ 42'$ by using gauss backward formula from the following data.

x :	50°	51°	52°	53°	54°
$\cos x$:	0.6428	0.6293	0.6157	0.6018	0.5878

sol^n

$h = 1$, $x_0 = 52^\circ$, $51^\circ 42' = 51.7^\circ$

$p = \frac{51.7 - 52}{1} = -0.3$

x	$y = \cos x$	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
x_{-2} 50	0.6428				
		-0.0135			
x_{-1} 51	0.6293		-0.0001		
		-0.0136		-0.0002	
x_0 52	0.6157				0.0004
		-0.0139	-0.0003		
x_1 53	0.6018			0.0002	
		-0.0140	-0.0001		
x_2 54	0.5878				

$$y(51.7^\circ) = 0.6153 + (-0.3) \times (-0.0136) + \frac{(-0.3)(0.7)}{2} \times (-0.0003)$$
$$+ \frac{(0.3)(0.7)(-1.3)}{6} \times (-0.0002)$$
$$+ \frac{(-0.3)(0.7)(-1.3)(1.7)}{-12} (0.0004)$$

$$y(51.7^\circ) = 0.6153 + 0.00408 + 3.15 \times 10^{-5} - 9.1 \times 10^{-6} + 1.542 \times 10^{-5}$$

$$\boxed{y(51.7)^\circ = 0.6197}$$

Interpolation for Unequal Intervals :-

Method 1> Legrange's Formula

1936

Q

year :	1901	1911	1921	1931	1941	1951
Population :	12	15	20	27	39	52

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$
x_0 1901	12					
		3				
x_1 1911	15		2			
		5		0		
x_2 1921	20		2		3	
		7		3		-10
x_3 1931	27		5		-7	
		12		-4		
x_4 1941	39		1			
		13				
x_5 1951	52					

$$p = \frac{x - x_0}{h} = \frac{1936 - 1941}{10} = -0.5$$

$$y(1936) = 39 + (0.5)(12) + \frac{(0.5)(-0.5)(1)}{2}$$

$$+ \frac{(0.5)(-1.5)(0.5)}{6} \times (-4)$$

In table, $\Delta^4 y$ & $\Delta^5 y$ is not given i.e we neglect them

$$y(1936) = 39 - 6 \quad 0.125 - 0.25$$

$$\boxed{y(1936 = 32.625}$$

Dividend Difference :-

Let $y = f(x)$ be given f^n & value of y are $y_0, y_1, \ldots$ corresponding to $x = x_0, x = x_1, \ldots$ then I^{st} dividend difference

(I) $[x_0, x_1] = \underset{x_1}{\Delta} y_0 = f(x_0, x_1) = \dfrac{f(x_1) - f(x_0)}{x_1 - x_0}$ —(1)

$$[x_1, x_2] = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

(II) 2^{nd} dividend difference :-

$$[x_0, x_1, x_2] = \underset{x_1 \; x_2}{\Delta^2} y_0 = \frac{f(x_1, x_2) - f(x_0, x_1)}{x_2 - x_0}$$

(III) 3^{rd} dividend difference :-

$$\underset{x_1, x_2, x_3}{\Delta^3} y_0 = [x_0, x_1, x_2, x_3] = \frac{f(x_1, x_2, x_3) - f(x_0, x_1, x_2)}{x_3 - x_0}$$

Newton Dividend difference Formula :-

* It is used for unequal intervals & defined as

$$y(x) = f(x) = y_0 + (x - x_0) \Delta y_0 + (x - x_0)(x - x_1)(x - x_2) \Delta^2 y_0$$
$$+ (x - x_0)(x - x_1)(x - x_2) \Delta^3 y_0 + \ldots$$

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Construct the D D Table

① Find the value of $f(x)$ from the following data

x	$y=f(x)$	Δy_0	$\Delta^2 y_0$	$\Delta^3 y_0$	$\Delta^4 y_0$
x_0 -4	1245				
		6.85			
x_1 -1	33		-8.7125		
		-28		3.11875	
x_2 0	5				1.12
			10		
x_3 2	9	2		13	
			88		
x_4 5	1335	442			

① Find value of $f(x)$

x	y	Δy_0	$\Delta^2 y_0$	$\Delta^3 y_0$	$\Delta^4 y_0$
-4	1245				
		-404	94		
-1	33			-14	3
		-28			
0	5		10		
		2		13	
2	9				
		442	88		
5	1335				

$$f(x) = 1245 + (x+4)(-404) + (x+4)(x+1)(94)$$
$$+ (x+4)(x+1)(x)(-14)$$
$$+ (x+4)(x+1)(x)(x-2)(3)$$

$$= 1245 + (x+4)[-404 + 94x + 94]$$

$$+ (x+4)(x+1)(x)[-14 + 3x - 6]$$

$$= 1245 + (x+4)[-310 + 94x] + (x+4)(x+1)(x)[3x-20]$$

$$= 1245 + 94x^2 - 310x + 376x - 1240 + (x^3 + 2x^2 + 4x)(3x-20)$$

$$= 94x^2 + 66x + 5 + 3x^4 - 20x^3 + 6x^3 - 40x^2 + 12x^2 - 80x$$

$$f(x) = 3x^4 - 5x^3 + 06x^2 - 14x + 5$$

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Q Find the missing term from the following data

	x	y	Δy_0	$\Delta^2 y_0$	$\Delta^3 y_0$	$\Delta^4 y_0$
x_0	1	14	1			
x_1	2	15	-5	-2	$\frac{y_3+6}{12}$	$\frac{78-5y_3}{12}$
x_2	4	5	y_3-5	$y_3/3$	$\frac{21-y_3}{3}$	
x_3	5	y_3	$9-y_3$	7		
x_4	6	9				

x_0	y	Δy_0	$\Delta^2 y_0$	$\Delta^3 y_0$
1	14	1	-2	$3/4$
2	15	-5	$7/4$	
4	5	2		
6	9			

$\frac{y_3}{3}+2$

$\frac{y_3+6}{3\times4}$

$7-\frac{y_3}{3}$

$\frac{21-y_3}{3}-\frac{y_3+6}{12}$

$\frac{84-4y_3-y_3-6}{12}$

$\frac{78-5y_3}{12}$

$x^2-x-2x+2$

x^2-3x+2

$-2x^2+6x-4$

$$f(x) = 14 + (x-1) + (x-1)(x-2)(-2) + (x-1)(x-2)(x-4)\frac{3}{4}$$

$$f(x) = 14 + (x-1) - 2x^2+6x-4 + (x^2-3x+2)(x-4)\frac{3}{4}$$

$$f(x) = 14 + x - 1 - 2x^2 + 6x - 4 + x^3 - 4x^2 - 3x^2 + 12x$$

$$f(x) = 14 - 2x - 1 = 13 - 2x$$

f at $x = 5$

$f(5) = 13 - 10$

$f(5) = 3$

$y_3 = 3$