~~ntiation

Unit-4

Numerical Solution of ordinary Differential Equation

Consider the ordinary diffⁿ eqⁿ

$$\frac{dy}{dx} = f(x,y)$$

with $y(x_0) = y_0$

Many analytical techniques exists for solving such eqⁿ. (2nd order only.)
But, majority of d.e. in physical problems cannot be solved analytically. Thus it becomes necessity to discuss their solⁿ by numerical method.

In numerical methods, we ~~don't~~ find the numerical values of the dependent variable for certain values of independent variable.

Solution

Single step

* This method which require only the numerical value y_i in order to compute next value y_{i+1}

→ Taylor's series

→ Picard's method

Multi-step

* This method which require not only the numerical value y_i but at least one of the past value $y_{i-1}, y_{i-2}, --$ to evaluate next value y_{i+1}

→ Euler

→ Modified Euler

→ R-K method

→ Milne's method.

form of solⁿ

A series for y in terms of powers of x from which the value of y can be obtained by direct substitution

→ Picard's

A set of tabulated values of x and y.

→ (Rest Methods belong here)

Note:- ① In Euler's and R-K method, the interval range h should be kept small hence they can be applied for tabulating y only over a limited range.

② To get functional values over a wide range Milne's method may be used, which require starting values usually obtained by Picard's, Taylor's or RK-method.

① Picard's method

consider the diff^n eq^n

$$\frac{dy}{dx} = f(x,y) \; ; \; y(x_0) = y_0 \quad \text{——(1)},$$

Integrating eq^n (1) b/w x_0 to x

$$\int_{y_0}^{y} dy = \int_{x_0}^{x} f(x,y)\,dx$$

$$y - y_0 = \int_{x_0}^{x} f(x,y)\,dx$$

$$y = y_0 + \int_{x_0}^{x} f(x,y)\,dx$$

Now the first approximation is

$$y^{(1)} = y_0 + \int_{x_0}^{x} f(x, y_0)\,dx$$

second app. $y^{(2)} = y_0 + \int_{x_0}^{x} f(x, y^{(1)})\,dx$

third app $y^{(3)} = y_0 + \int_{x_0}^{x} f(x, y^{(2)})\,dx$

$\vdots$

n^{th} approx $\boxed{y^{(n)} = y_0 + \int_{x_0}^{x} f(x, y^{(n-1)})\,dx}$ with $y(x_0) = y_0$.

... lation ... at a given argument,

Ex1. Use Picard's method to solve y at $x=0.2$. 13

$$\frac{dy}{dx} = x-y \quad \text{with } y(0)=1$$

Sol → $f(x,y) = (x-y)$, $x_0=0$, $y_0=1$

1st approx

$$y^{(1)} = y_0 + \int_{x_0}^{x} f(x, y_0)\, dx$$

$$y^{(1)} = 1 + \int_0^x f(x,1)\, dx$$

$$= 1 + \int_0^x (x-1)\, dx = 1 - x + \frac{x^2}{2} \Rightarrow y^{(1)}(0.2) = 0.82$$

2nd app.

$$y^{(2)} = y_0 + \int_0^x f(x, y^{(1)})\, dx$$

$$= 1 + \int_0^x \left\{ x - 1 + x - \frac{x^2}{2} \right\} dx$$

$$y^{(2)} = 1 - x + x^2 - \frac{x^3}{6}, \quad y^{(2)}(0.2) = 0.83867$$

3rd appr

$$y^{(3)} = y_0 + \int_0^x f(x, y^{(2)})\, dx$$

$$= 1 + \int_0^x \left\{ x - 1 + x - x^2 + \frac{x^3}{6} \right\} dx$$

$$= 1 - x + x^2 - \frac{x^3}{3} + \frac{x^4}{24}, \quad y^{(3)}(0.2) = 0.83740$$

4th app.

$$y^{(4)} = y_0 + \int_0^x f(x, y^{(3)})\, dx$$

$$= 1 + \int_0^x \left(x - 1 + x - x^2 + \frac{x^3}{3} - \frac{x^4}{24} \right) dx$$

$$= 1 - x + x^2 - \frac{x^3}{3} + \frac{x^4}{12} - \frac{x^5}{120}$$

$$y^{(4)}(0.2) = 0.83746.$$

Q2. If $\frac{dy}{dx} = \frac{y-x}{y+x}$, find $y(0.1)$ with $y(0)=1$.

Sol → $y^{(1)} = 1 + \int_0^x \left(\frac{1-x}{1+x}\right) dx = 1 + \int_0^x \left(\frac{2}{1+x} - 1\right) dx$

$= 1 - x + 2\log(1+x)$.

$y^{(2)} = 1 + x - 2\int_0^x \frac{x\,dx}{1+2\log(1+x)}$ which is difficult to integrate.

Hence only 1st approx. $\Rightarrow y^{(1)}(0.1) = 1.09062$.

Q3. If $\frac{dy}{dx} = y + x$ s.t. $y=1$ when $x=0$.
obtain a solⁿ up to fifth approx.

$\left\{1 + x + x^2 + \frac{x^3}{3} + \frac{x^4}{12} + \frac{x^5}{60} + \frac{x^6}{720}\right\}$.

2. Taylor's method

Consider the first order eqⁿ $\frac{dy}{dx} = f(x,y)$.

with $y(x_0) = y_0$ or $y(x_p) = y_p$.

then Taylor's series expansion about the pt. x_p is given by

$$y(x) = y(x_p) + (x-x_p)\,y'(x_p) + \frac{(x-x_p)^2}{\lfloor 2} y''(x_p) + \cdots$$

find all the values of the right hand side and solve.

Q1. Find by Taylor's series method, the value of y at $x = 0.1$ and $x = 0.2$ to five places of decimals from

$$\frac{dy}{dx} = x^2 y - 1 \quad ; \quad y(0) = 1.$$

Sol → $f(x,y) = x^2 y - 1$, $y(x_p) = y_p$

$x_p = 0,\ y_p = 1$

Now. $y' = f(x,y) = x^2 y - 1 = \frac{dy}{dx} = y'$

$f'(x,y)$ ie. y'

②

$$y' = x^2y - 1 \qquad \text{at } x_p = 0 \Rightarrow \quad y' = -1$$

$$y'' = 2xy + x^2y' \qquad \Rightarrow y'' = 0$$

$$y''' = 2y + 2xy' + 2xy' + x^2y'' \Rightarrow y''' = 2$$

$$= 2y + 4xy' + x^2y''$$

$$y^{iv} = 2y' + 4y' + 4xy'' + 2xy'' + x^2y'''$$

$$= 6y' + 6xy'' + x^2y''' \qquad \Rightarrow y^{iv} = -6$$

etc

Hence put in Taylor's series, we have

$$y(x) = y(x_p) + (x - x_p).y'(x_p) + \frac{(x-x_p)^2}{\lfloor 2} y''(x_p) + \cdots$$

$$= 1 + (x-0).(-1) + \frac{(x-0)^2}{\lfloor 2}(0) + \frac{(x-0)^3}{\lfloor 3}(2) + \frac{(x-0)^4}{\lfloor 4}(-6) + \cdots$$

$$y(x) = 1 - x + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$$

at $x = 0.1$

$y(0.1) = 0.90033$

$y(0.2) = 0.80227$

Q2. Solve $y' = x + y$; $y(0) = 1$ by Taylor's series. find y at $x = 0.1$ and 0.2

$y(0.1) = 1.1103$

$y(0.2) = 1.2427$

Q3. Employ Taylor's method to obtain approximate value of y at $x = 0.2$ for $\frac{dy}{dx} = 2y + 3e^x$; $y(0) = 0$. Compare the numerical solⁿ obtained with the exact solⁿ.

$y(0.2) = 0.8110$

exact

$$\frac{dy}{dx} - 2y = 3e^x$$

$$\text{I.F.} = e^{-\int 2\,dx} = e^{-2x}$$

solⁿ

$$ye^{-2x} = \int 3e^x . e^{-2x}\,dx + c$$

$$y = -3e^x + ce^{2x}$$

at $x = 0$, $y = 0 \Rightarrow c = 3 \Rightarrow y(x) = 3(e^{2x} - e^x)$

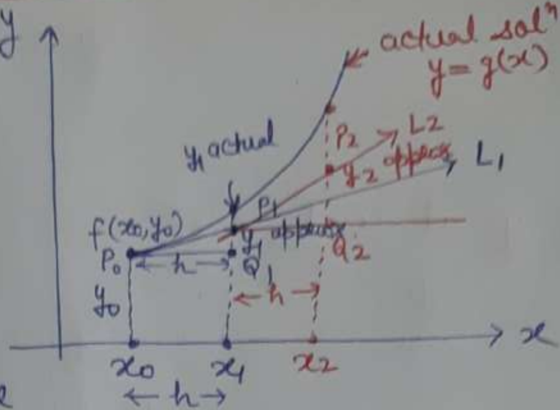
$y(0.2) = 0.8112$

3. Euler's method

Consider a diff^n eq^n as

$$\frac{dy}{dx} = f(x,y)$$

with $y(x_0) = y_0$.



Aim :- With the help of initial condition and diff^n eq^n, find the approximate value which is nearer to the actual value or sol^n.

hence $y_1 = y_0 + P_1 Q_1$

$P_1 Q_1 = h\, f(x_0, y_0)$

by $\tan\theta = \frac{P_1 Q_1}{h}$

$\left(\frac{dy}{dx}\right)_{(x_0,y_0)} = \frac{P_1 Q_1}{h}$

first approx $\underset{y(x_0+h)}{y_1} = y_0 + h\, f(x_0, y_0)$

second approx $\underset{y(x_0+2h)}{y_2} = y_1 + P_2 Q_2$

$= y_1 + h\, f(x_1, y_1)$

third approx $y_3 = y_2 + P_3 Q_3$

$= y_2 + h\, f(x_2, y_2)$

General or $(n+1)^{th}$ appr $\boxed{y_{n+1} = y_n + h\, f(x_n, y_n)}$

Ex1. Using Euler's method, find an approximate value of y corresponding to $x = 0.2$; given that $\frac{dy}{dx} = x + y$ and $y = 1$ when $x = 0$. [$h = 0.1$] / ~~h=0.1~~

Sol → $f(x,y) = x + y$, $x_0 = 0$, $y_0 = 1$, $h = 0.2$, $y(0.2) = ?$

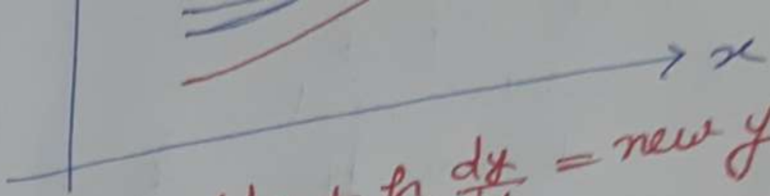
Make table

new y = old $y + h\left(\frac{dy}{dx}\right)$

	x	y	$\frac{dy}{dx} = x+y$
x_0	0	1	1
x_1	0.1	1.1	1.2
x_2	0.2	1.22	$\boxed{1.42}$

$1 + 0.1(1) = 1.1$

$1.1 + 0.1(1.2) = 1.22$



Q) find y at $x=1$, then

$\Downarrow$

x	y	$x+y=\frac{dy}{dx}$	old $y + h\frac{dy}{dx}$ = new y
0	1	1	$1 + 0.1(1) = 1.1$
0.1	1.10	1.20	$1.10 + 0.1(1.20) = 1.22$
0.2	1.22	1.42	$1.22 + 0.1(1.42) = 1.36$
0.3	1.36	1.66	$1.36 + 0.1(1.66) = 1.53$
0.4	1.53	1.93	$1.53 + 0.1(1.93) = 1.72$
0.5	1.72	2.22	1.94
0.6	1.94	2.54	2.19
0.7	2.19	2.89	2.48
0.8	2.48	3.29	2.81
0.9	2.81	3.71	3.18
1.0	$\boxed{3.18}$		

Q2. Given $\frac{dy}{dx} = \frac{y-x}{y+x}$ with $y(0) = 1$. find $y(0.1)$.
$(h = 0.02)$.

Sol →

x	y	dy/dx
0	1	1
0.02	1.02	0.9615
0.04	1.0392	0.926
0.06	1.0577	0.893
0.08	1.0756	0.862
0.10	1.0928	

old $y + h\frac{dy}{dx}$ = new y

$1 + 0.02(1) = 1.02$

$1.02 + 0.02(0.9615) = 1.0392$

$1.0392 + 0.02(0.926) = 1.0577$

1.0756

1.0928

4. Modified Euler's Method

In Euler's method;

$$y_1 = y_0 + h f(x_0, y_0)$$

Now we find a better approximation $y_1^{(1)}$ as

1st approx $\quad y_1^{(1)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(\underbrace{x_0 + h}_{x_1}, y_1)\right]$

or

$$y_1^{(1)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(E)})\right]$$

$$y_1^{(2)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(1)})\right]$$

$$y_1^{(3)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(2)})\right]$$

This is repeated till two consecutive value of y agree. This is to be taken as the value of y_1.

2nd approx $\quad y_2 = y_1 + h f(x_1, y_1)$

$$y_2^{(1)} = y_1 + \frac{h}{2}\left[f(x_1, y_1) + f(x_2, y_2^{(E)})\right]$$

$$y_2^{(2)} = y_2 + \frac{h}{2}\left[f(x_1, y_1) + f(x_2, y_2^{(1)})\right]$$

repeat till similar values.

Then we proceed to calculate y_3 and so on.

Ex1:- Use Modified Euler's method, find an app. value of y when $x = 0.3$, given that $\frac{dy}{dx} = x + y$ & $y = 1$ when $x = 0$.

Sol→

$$y(0.3) = ?$$

$$h = 0.1$$

$x_0 = 0$

$x_1 = x_0 + h = 0.1$

$x_2 = x_0 + 2h = 0.2$

$x_3 = x_0 + 3h = 0.3$

x	$x+y=\frac{dy}{dx}$	Mean slope	old y + (0.1)[Mean slope] = new y
0	$0+1=1$ old y	–	$1+(0.1)(1)=1.10$
0.1	$0.1+1.1=1.2$	$\frac{1}{2}(1+1.2)=1.1$	$1+(0.1)(1.1)=1.11$
0.1	$0.1+1.11=1.21$	$\frac{1}{2}(1+1.21)=1.105$	$1+(0.1)(1.105)=1.1105$
0.1	$0.1+1.1105=1.2105$	$\frac{1}{2}(1+1.2105)=1.1052$	$1+(0.1)(1.1052)=1.1105$

Since last two values are same, (equal.) hence $y_1 = 1.1105$

x	$x+y=\frac{dy}{dx}$	Mean slope	old y + (0.1)[Mean slope] = new y
0.1	1.2105 slope	–	$1.1105+(0.1)(1.2105)=1.2316$
0.2	1.4316	$\frac{1}{2}(1.2105+1.4316)=1.3$[illegible]	$1.1105+(0.1)(1.3$[illegible]$)=1.2426$
0.2	$0.2+1.2426=1.4426$	$\frac{1}{2}(1.2105+1.4426)=1.3266$	$1.1105+(0.1)(1.3266)=1.2432$
0.2	$0.2+1.2432=1.4432$	$\frac{1}{2}(1.2105+1.4432)=1.3268$	$1.1105+(0.1)(1.3268)=1.2432$

y_2 or $y(0.2)=1.2432$

x	$x+y=\frac{dy}{dx}$	Mean slope	old y + (0.1)[Mean slope] = new y
0.2	1.4432	–	$1.2432+(0.1)(1.4432)=1.3875$
0.3	$0.3+1.3875=1.6875$	$\frac{1}{2}[1.4432+1.6875]=1.5654$	$1.2432+(0.1)(1.5654)=1.3997$
0.3	$0.3+1.3997=1.6997$	$\frac{1}{2}(1.4432+1.6997)=1.5715$	$1.2432+(0.1)(1.5715)=1.4003$
0.3	$0.3+(1.4003)=1.7003$	$\frac{1}{2}(1.4432+1.7003)=1.5718$	$1.2432+(0.1)(1.5718)=1.4004$
0.3	$0.3+1.4004=1.7004$	$\frac{1}{2}(1.4432)+(1.7004)=1.5718$	$1.2432+(0.1)(1.5718) \Rightarrow 1.4004$

hence y_2 or $y(0.2)=1.4004$ approx

1.32905

Q2. Use Modified Euler method & find $y(0.2)$ & $y(0.4)$ for $\frac{dy}{dx} = y + e^x$; $y(0) = 0$, $h = 0.2$

Sol →

x	$\frac{dy}{dx} = y + e^x$	Meanslope	old y + h(Meanslope) = new y
0	1 slope	-	$0 + (0.2)(1) = 0.2$
0.2	$0.2 + e^{0.2} = 1.4214$	$\frac{1}{2}(1 + 1.4214) = 1.2107$	$0 + (0.2)(1.2107) = 0.2421$
0.2	$0.2421 + e^{0.2} = 1.4635$	$\frac{1}{2}(1 + 1.4635) = 1.2317$	$0 + 0.2(1.2317) = 0.2463$
0.2	$0.2463 + e^{0.2} = 1.4677$	$\frac{1}{2}(1 + 1.4677) = 1.2338$	$0 + 0.2(1.2338) = 0.2468$ ✓
0.2	$0.2468 + e^{0.2} = 1.4682$	$\frac{1}{2}(1 + 1.4682) = 1.2341$	$0 + 0.2(1.2341) = 0.2468$ ✓ old y

hence y_1 or $y(x_1)$ or $y(x_0 + h)$, $y(0.2) = 0.2468$

x	$\frac{dy}{dx}$	Meanslope	new y
0.2	1.4682	—	$0.2468 + (0.2)(1.4682) = 0.5404$
0.4	$0.5404 + e^{0.4} = 2.0322$	$\frac{1}{2}(1.4682 + 2.0322) = 1.7502$	$0.2468 + (0.2)(1.7502) = 0.5968$
0.4	$0.5968 + e^{0.4} = 2.0887$	$\frac{1}{2}(1.4682 + 2.0887) = 1.7784$	$0.2468 + (0.2)(1.7784) = 0.6025$
0.4	$0.6025 + e^{0.4} = 2.0943$	$\frac{1}{2}(1.4682 + 2.0943) = 1.78125$	$0.2468 + (0.2)(1.78125) = 0.6030$
0.4	$0.6030 + e^{0.4} = 2.0949$	$\frac{1}{2}(1.4682 + 2.0949) = 1.7815$	$0.2468 + (0.2)(1.7815) = 0.6031$
0.4	$0.6031 + e^{0.4} = 2.0949$	$\frac{1}{2}(1.4682 + 2.0949) = 1.7816$	$0.2468 + (0.2)(1.7816) = 0.6031$

hence y_2 or $y(x_0 + 2h) = y(0.4) = 0.6031$ approx.

5. Runge-Kutta Method (RK-method)

These methods agree with Taylor's series sol^n upto the term in h^r where r differs from method to method and is called the order of that method.

first order RK method :— Euler's method

second order RK method :— Modified Euler's method.

Third order RK method :— Runge's method.

Fourth order RK method :— This method is most commonly used and is often referred to as RK method only.

Working Rule :— $\frac{dy}{dx} = f(x,y)$; $y(x_0) = y_0$.

To find the increment k of y corresponding to increment h of x is

$$K_1 = hf(x_0, y_0)$$

$$k_2 = h f\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right)$$

$$k_3 = h f\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right)$$

$$k_4 = h f(x_0 + h, y_0 + k_3).$$

then compute $k = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$.

which gives the required approx. value as $y_1 = y_0 + k$.

Q1. Apply RK method of 4^{th} order to find approx. value of y when $x = 0.2$, $\frac{dy}{dx} = x+y$, $y(0)=1$.

Sol →. $x_0 = 0$, $y_0 = 1$, $h = 0.2$, $f(x,y) = x+y$

$$K_1 = hf(x_0, y_0)$$
$$= (0.2)[0+1] = 0.2$$

$$K_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{K_1}{2}\right)$$
$$= h\left\{\left(0+\frac{0.2}{2}\right) + \left(1+\frac{0.2}{2}\right)\right\}$$
$$= (0.2)\{0.1 + 1.1\} = 0.2 \times 1.2$$
$$= .24$$

$$K_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{K_2}{2}\right)$$
$$= (0.2)f\left(0.1, 1 + \frac{.24}{2}\right)$$
$$= (0.2)f(0.1, 1.12)$$
$$= 0.2(1.22)$$
$$= .2440$$

$$K_4 = hf(x_0 + h, y_0 + K_3)$$
$$= (0.2)f(0.2, 1 + .244)$$
$$= (0.2)f(0.2, 1.244)$$
$$= 0.2(1.444)$$
$$= .2888$$

Now $$K = \frac{1}{6}(K_1 + 2K_2 + 2K_3 + K_4)$$
$$= 0.2428.$$

hence $$y_1 = y_0 + K$$
$$= 1 + 0.2428$$
$$= 1.2428$$

Q2. Use RK method, solve $\frac{dy}{dx} = \frac{y^2-x^2}{y^2+x^2}$ with $y(0)=1$ at $x=0.2$ & 0.4

Sol. $f(x,y) = \frac{y^2-x^2}{y^2+x^2}$, $x_0=0$, $y_0=1$, $h=0.1$

$K_1 = 0.2 f(0,1) = 0.2$

$K_2 = 0.2 f(0.1, 1.1) = .19672$

$K_3 = 0.2 f(0.1, 1.09836) = .1967$

$K_4 = 0.2 f(0.2, 1.1967) = .1891$

$K = \frac{1}{6}(0.2 + 2(.19672) + 2(.1967) + .1891)$

$= .19599$

$y_1 = y_0 + K = 1.196.$

To find $y(0.4)$ $h=0.2$

$x_1 = 0.2$, $y_1 = 1.196$, $h = 0.2$

$K_1 = h f(x_1, y_1) = .1891$

$K_2 = h f(x_1 + \frac{h}{2}, y_1 + \frac{K_1}{2}) = .1795$

$K_3 = h f(x_1 + \frac{h}{2}, y_1 + \frac{K_2}{2}) = .1793$

$K_4 = h f(x_1 + h, y_1 + K_3) = .1688$

$K = \frac{1}{6}(K_1 + 2K_2 + 2K_3 + K_4)$

$= 0.1792$

$y_2 = y_1 + K$

$= 1.196 + .1792$

$= 1.3752$

Q-1 $\frac{dy}{dx} = x + y^2$, $y(0) = 1$. find $y(0.2)$ in steps 0.1

(1.2736)

Q-1 $\frac{dy}{dx} = \frac{2xy + e^x}{x^2 + xe^x}$, $x_0 = 1$, $y_0 = 1$ $y(1.2)$ & $y(1.4)$

.1402 0.2705

6. Milne's Method

$$\frac{dy}{dx} = f(x,y)$$
$$y(x_0) = y_0.$$

first we get the approximate value of y_{n+1} by predictor formula and then improve ~~this~~ this using a corrector formula. Newton's forward Interpolation formula in terms of y' and p is

$$y' = y_0' + p\Delta y_0' + \frac{p(p-1)}{2}\Delta^2 y_0' + \frac{p(p-1)(p-2)}{6}\Delta^3 y_0' + \frac{p(p-1)(p-2)(p-3)}{24}\Delta^4 y_0' \quad \text{--- (1)}$$

where $p = \frac{x-x_0}{h}$

or $x = x_0 + ph$

Integrate (1) over x_0 to (x_0+4h)

$$\int_{x_0}^{x_0+4h} y'\,dx = h\int_0^4 y'\,dp \qquad \left\{ \because \begin{array}{l} x = x_0 + ph \\ dx = h\,dp \end{array} \right.$$

$$(y_4 - y_0) = h\int_0^4 \{\text{put from (1)}\}\,dp$$

$$= h\left\{4y_0' + 8\Delta y_0' + \frac{20}{3}\Delta^2 y_0' + \frac{8}{3}\Delta^3 y_0' + \frac{28}{90}\Delta^4 y_0'\right)$$

substitute the values of I, II, III differences, we get

$$y_4 - y_0 = h\left\{4y_0' + 8(E-1)y_0' + \frac{20}{3}(E-1)^2 y_0' + \frac{8}{3}(E-1)^3 y_0' + \cdots\right\}$$

$$= \frac{4h}{3}\{2y_1' - y_2' + 2y_3'\} + \cdots$$

$$\boxed{y_4 = y_0 + \frac{4h}{3}\{2y_1' - y_2' + 2y_3'\}} + \cdots \quad \text{--- (2)}$$

This is Milne's Predictor formula

or

$$y_4 = y_0 + \frac{4h}{3}(2f_1 - f_2 + 2f_3)$$

$f_1 = y_1'$

$f_2 = y_2'$

$f_3 = y_3'$

It is used to predict the values of y_4 when the value of y_0, y_1, y_2 and y_3 are known.

To obtain the corrector formula, we integrate (1) over the interval x_0 to (x_0+2h) (or $p=0$ to 2) and we get

$$y_2 - y_0 = h\left(2y_0' + 2\Delta y_0' + \frac{1}{3}\Delta^2 y_0' - \cdots\right)$$

Express in I, II, ~~III~~ differences in terms of $(E-1)$.

$$y_2 - y_0 = \frac{h}{3}\left(y_0' + 4y_1' + y_2'\right)$$

$$\boxed{y_2 = y_0 + \frac{h}{3}\left(y_0' + 4y_1' + y_2'\right)} \quad \text{--- (3)}$$

This is milne's corrector formula

or

~~$y_2 = y_0$~~.

Eqn (2) and (3) Can be expressed as

$$y_4 = y_0 + \frac{4h}{3}\left(2f_1 - f_2 + 2f_3\right) \quad \text{Predictor}$$

$$y_4 = y_2 + \frac{h}{3}\left(f_2 + 4f_3 + f_4\right) \quad \text{Corrector}$$

An improved of f_4 is then computed again & again by Milne's corrector method, find a still better value of y_4. we repeat this step untill y_4 remains unchanged.

Use it

Milne's Predictor-Corrector Method

Consider the d.e. $\frac{dy}{dx} = f(x,y)$

with I.C. $y(x_0) = y_0$

Aim :- find $y(x_n)$ (n must 4 atleast).

find $y(x_1), y(x_2), y(x_3)$ using any one of the following method

1. Picard's method
2. Euler
3. Modified Euler method
4. Taylor's series
5. R-K method

Then Calculate

$f_0 = f(x_0, y_0)$ $\quad$ $f_2 = f(x_2, y_2)$

$f_1 = f(x_1, y_1)$ $\quad$ $f_3 = f(x_3, y_3)$

By Milne's Predictor method,

$$y_4 = y_0 + \frac{4h}{3}(2f_1 - f_2 + 2f_3)$$

then find $f_4 = f(x_4, y_4)$

By Milne's corrector method,

$$y_4 = y_2 + \frac{h}{3}(f_2 + 4f_3 + f_4)$$

An improved of f_4 is then computed and again Milne's corrector method is applied to find a still better value of y_4, we repeat this step untill y_4 remains unchanged.

Q→ Using R-K method of 4^{th} order, to find y for $x = 0.1, 0.2, 0.3$ given that $\frac{dy}{dx} = xy + y^2$, $y(0) = 1$.
Continue the soln at $x = 0.4$ using Milne's method.

Sol→ To find $y(0.1)$

$K_1 = h f(x_0, y_0) = (0.1) f(0,1) = 0.1$

$K_2 = h f\left(x_0 + \frac{h}{2}, y_0 + \frac{K_1}{2}\right) = 0.1155$

$K_3 = h f\left(x_0 + \frac{h}{2}, y_0 + \frac{K_2}{2}\right) = 0.1172$

$K_4 = h f(x_0 + h, y_0 + K_3) = 0.1360$

$K = \frac{1}{6}(K_1 + 2K_2 + 2K_3 + K_4) = 0.1169$

$y_1 = y_0 + K = 1 + 0.1169 = 1.1169$

To find $y(0.2)$ $\quad \therefore x_1 = 0.1, y_1 = 1.1169, h = 0.1$

$K_1 = h f(x_1, y_1) = 0.1359$

$K_2 = h f\left(x_1 + \frac{h}{2}, y_1 + \frac{K_1}{2}\right) = 0.1581$

$K_3 = 0.1609$

$K_4 = 0.1888$

$K = \frac{1}{6}(K_1 + 2K_2 + 2K_3 + K_4) = 0.1605$

$y_2 = y_1 + K = 1.1169 + 0.1605 = 1.2774.$

To find $y(0.3)$ $\quad x_2 = 0.2, \; y_2 = 1.2774, \; h = 0.1$

$K_1 = 0.1887, \; K_2 = 0.2224, \; K_3 = 0.2275, \; K_4 = 0.2716$

$K = 0.2267$

$y_3 = 1.5041$

So we have $h = 0.1$ $\quad f(x, y) = xy + y^2$

$x_0 = 0 \quad y_0 = 1 \quad f_0 = f(x_0, y_0) = 1$

$x_1 = 0.1 \quad y_1 = 1.1169 \quad f_1 = f(x_1, y_1) = 1.3591$

$x_2 = 0.2 \quad y_2 = 1.2774 \quad f_2 = f(x_2, y_2) = 1.8869$

$x_3 = 0.3 \quad y_3 = 1.5041 \quad f_3 = f(x_3, y_3) = 2.7132$

By Milne's predictor method

$$y_4 = y_0 + \frac{4h}{3}(2f_1 - f_2 + 2f_3)$$

$$= 1.8344$$

$$\Rightarrow x_4 = 0.4,\ y_4 = 1.8344,\ f_4 = f(x_4, y_4) = 4.0988.$$

By Corrector method

$$y_4' = y_2 + \frac{h}{3}(f_2 + 4f_3 + f_4)$$

$$y_4^{(1)} = 1.8387$$

$$f_4 = f(x_4, y_4) = 4.1162$$

Again $y_4 = y_2 + \frac{h}{3}(f_2 + 4f_3 + f_4)$

$$\boxed{y_4^{(2)} = 1.8393}$$

$$f_4 = 4.1186$$

Again $y_4 = y_2 + \frac{h}{3}(f_2 + 4f_3 + f_4)$

$$\boxed{y_4^{(3)} = 1.8393}$$

Q2. find $y(2)$ if $\frac{dy}{dx} = \frac{1}{2}(x+y)$.

$$y(0) = 2$$

Given $y(0.5) = 2.636,\ y(1) = 3.595,\ y(1.5) = 4.968$

Sol → $x_0 = 0,\ x_1 = 0.5,\ x_2 = 1,\ x_3 = 1.5$

$y_0 = 2,\ y_1 = 2.636,\ y_2 = 3.595,\ y_3 = 4.968$

$$f(x,y) = \frac{1}{2}(x+y).$$

Sol^n → 6.873.

- uniquely

Q3. Use Milne's predictor-corrector method, obtain $y(0.4)$ from the given set of tabulated value of $\frac{dy}{dx} = y^2 - x^2$.

	x_0	x_1	x_2	x_3
x	0	0.1	0.2	0.3
y	1 y_0	0.11 y_1	1.25 y_2	1.42 y_3
f	1 f_0	1.22 f_1	1.52 f_2	1.92 f_3

$h = 0.1$

Sol:- Use predictor formula

$$y_4 = y_0 + \frac{4h}{3}(2f_1 - f_2 + 2f_3)$$

$$= 1 + \frac{4(0.1)}{3}(2\times1.22 - 1.52 + 2(1.92))$$

$$= 1.63466$$

then, $f_4 = f(x_4, y_4)$

$$= (y_4^2 - x_4^2) = 2.51211$$

Now apply corrector formula,

$$y_4^{(1)} = y_2 + \frac{h}{3}(f_2 + 4 f_3 + f_4)$$

$$= 1.25 + \frac{(0.1)}{3}[1.52 + 4(1.92) + 2.51211]$$

$y_4^{(1)} = 1.64040 \Rightarrow f_4^{(1)} = y_4^{(1)2} - x_4^2 = (1.64040)^2 - (0.4)^2$

Again; $y_4^{(2)} = y_2 + \frac{h}{3}(f_2 + 4f_3 + f_4^{(1)})$

$= 1.64103 \Rightarrow f_4^{(2)} = (1.64103)^2 - (0.4)^2$

Again; $y_4^{(2)} = y_2 + \frac{h}{3}(f_2 + 4f_3 + f_4^{(2)})$

$y_4^{(2)} = 1.64109 \simeq 1.6411 \Rightarrow f_4^{(3)} = $

Again $y_4^{(3)} \simeq 1.6411$

$\boxed{\text{hence } f(0.4) \text{ or } y_4 = 1.6411}$