Numerical Analysis / Method

→ It deals with numerical data/numbers.

(i) **Exact Number**

→ These number are also known as terminating

eg → 1, 2, 3, 3.5, etc.

(ii) **Approximate Number**

→ These are also known as non terminating or not close to exact.

eg → π (3.14285) (3.14159), $\frac{1}{3} = 0.3333$

Error

Let 'X' is a true value and 'X′' is approximate value then error is defined as

$$\text{Error (E)} = X - X'$$

example ÷ if x is 5 as a solution of a given eqn & [x − 5 = 0] it's approximate solution is 4.999, then error is 5 − 4.999 = 0.001

Types of Error

(i) Inherent Error → Already present in the question. It can be minimize by taking actual data or by using high precision.

ii) Round off error → arise due to rounding of the digits. eg → if $X = 5$ & $X' = 4.976$ is approx solution.

iii) Trunate Error → obtain by cutting of a finite term from infinite term. If $X = e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$, $X' = e^x = 1 + x + \frac{x^2}{2!}$

iv) Absolute Error

v) Relative Error → $E_R = \frac{|X - X'|}{X} = \frac{E_a}{X}$

vi) Percentage Error → $E_P = E_R \times 100\% = \frac{|X - X'|}{X} \times 100\%$

Absolute Error ÷ If X is true value/solⁿ & X′ is approx. value/solⁿ then absolute is defined as, $E_a = |X - X'|$

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General Formula of Error

Let us consider y is f^n

$$y = f(x_1, x_2) \quad -(i)$$

then

$$y + \delta y = f(x_1 + \delta x_1, x_2 + \delta x_2) \quad -(ii)$$

now expanding right hand side using Taylor's expansions, $\left[y = f(x+h, z+k), \; y = f(x, z) + \left(h\frac{\partial f}{\partial x} + k\frac{\partial f}{\partial y}\right) + \ldots \right]$

$$y + \delta y = f(x_1, x_2) + \left(\frac{\partial f}{\partial x_1}\delta x_1 + \frac{\partial f}{\partial x_2}\delta x_2\right) + \ldots \quad (iii)$$

where $\delta x_1, \delta x_2, \delta y$ are error in y

since all the error are very small

$\therefore$ neglecting the highest power of error in R.H.S

$$y + \delta y = f(x_1, x_2) + \frac{\partial f}{\partial x_1}\delta x_1 + \frac{\partial f}{\partial x_2}\delta x_2$$

put eqn (i) here,

$$\cancel{y} + \delta y = \cancel{y} + \frac{\partial f}{\partial x_1}\delta x_1 + \frac{\partial f}{\partial x_2}\delta x_2$$

$$\left[\delta y = \frac{\partial y}{\partial x_1}\delta x_1 + \frac{\partial y}{\partial x_2}\delta x_2 \right] \quad -(iv)$$

This is called error or absolute error.

Now,

Relative Error, $E_r \Rightarrow \frac{\delta y}{y} = \left(\frac{\partial y}{\partial x_1}\right)\left(\frac{\delta x_1}{y}\right) + \left(\frac{\partial y}{\partial x_2}\right)\left(\frac{\delta x_2}{y}\right)$

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Q Compute the % error in time period $T = 2\pi\sqrt{\frac{l}{g}}$ for $l = 1m$ & error in measurement of l is 0.01

$$\cancel{\left[\begin{array}{l} T = 2 \times 3.14 \sqrt{\frac{1}{9.8}} \\ T = 6.28 \sqrt{} \end{array}\right]}$$, δl (error) $= 0.01$

$$T = 2\pi\sqrt{\frac{l}{g}} \quad \text{-(i)}$$

taking log both side, we get

$$\log T = \log 2\pi + \frac{1}{2}\log l - \frac{1}{2}\log g \quad \text{- (ii)}$$

$$\frac{1}{T}\frac{\delta T}{\delta l} = \frac{1}{2l}$$

$$\frac{\delta T}{T} = \frac{\delta l}{2l}$$

$$E_r = \frac{0.01}{2 \times 1}$$

$$E_r = \frac{100}{10^3 \times 2}$$

$$E_r = 5 \times 10^{-3}$$

$$\% \text{ Error} = 5 \times 10^{-3} \times 10^2 = 0.5\,\%$$

Q If $u = \frac{4x^2y^3}{z^4}$ & errors in x, y, z is 0.001, compute Relative maximum error when $x = y = z = 1$

$$\cancel{\boxed{\frac{\partial u}{\partial x} = 4\left[\frac{\partial^2 x}{\partial x^2}\delta x + \partial^3 y\right]}}$$

$$\delta u = \frac{\partial u}{\partial x}\delta x + \frac{\partial u}{\partial y}\delta y + \frac{\partial u}{\partial z}\delta z$$

$$\delta u = \left[\frac{8xy^3}{z^4} + \frac{12x^2y^2}{z^4} - \frac{16x^2y^3}{z^5}\right]10^{-3} = [8 + 12 + 16]\,10^{-3}$$

Remove -ve sign [Max. Relative error]

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$\delta u = 36 \times 10^{-3}$

$E_a = 0.036$

Maximum Relative Error, $E_r = \frac{\delta u}{(u)_{x=y=z=1}} = \frac{0.036}{4} = 0.09$

Error of addition of number

Let us consider 'X' $= x_1 + x_2$ —(i), then absolute error of eqn (i) is

$$\Delta X = \Delta x_1 + \Delta x_2 \quad \text{—(ii)}$$

Now, the relative error E_r is

$$E_r = \frac{\Delta X}{X} = \frac{\Delta x_1}{X} + \frac{\Delta x_2}{X} \quad \text{—(iii)}$$

~~Add~~

Error in subtraction of number :-

Let, $X = x_1 - x_2$ —(i)

then, absolute error of eqn (i) is

$$\Delta X = \Delta x_1 - \Delta x_2$$

Now, Relative error E_r is

$$E_r = \frac{\Delta X}{X} = \frac{\Delta x_1}{X} - \frac{\Delta x_2}{X}$$

Note: Maximum absolute error is equal to,

$$\text{Max } E_a = |\Delta X| \leq |\Delta x_1| + |\Delta x_2|$$

then, $\text{Max. } E_r = \left|\frac{\Delta X}{X}\right| \leq \left|\frac{\Delta x_1}{X}\right| + \left|\frac{\Delta x_2}{X}\right|$

Q If $u = 2v^6 - 5v$. Find % error in u at $v = 1$, if $\delta v = 0.05$.

$E_r = 12\Delta v - 5\Delta v = (12 - 5)\,0.05 = \frac{8 \times 5}{100} = 0.40$

~~% error = 0.85 × 100 = 85%~~

Error in multiplication of numbers :-

Let us consider $X = x_1 x_2$ —(i)

$$\Delta X = \frac{\partial x}{\partial x_1}\frac{\Delta x_1}{X} + \frac{\partial X}{\partial x_2}\frac{\Delta x_2}{X} \quad \text{—(ii)}$$

$$\frac{\Delta X}{X} = \frac{\Delta x_1}{x_1} + \frac{\Delta x_2}{x_2}$$

Maximum Relative Error is

$$\left|\frac{\Delta X}{X}\right| \leq \left|\frac{\Delta x_1}{x_1}\right| + \left|\frac{\Delta x_2}{x_2}\right|$$

Error in division of number :-

Let us consider $X = \frac{x_1}{x_2}$

$$\Delta X = \frac{\partial X}{\partial x_1}\Delta x_1 + \frac{\partial X}{\partial x_2}\Delta x_2 \quad \text{—(1)}$$

$$\frac{\Delta X}{X} = \frac{\partial X}{\partial x_2}\frac{\Delta x_1}{X} + \frac{\partial X}{\partial x_2}\frac{\Delta x_2}{X} \quad \text{—(11)}$$

$$= \frac{\Delta x_1}{x_1} - \frac{\Delta x_2}{x_2}$$

Max Relative Error is

$$\left|\frac{\Delta X}{X}\right| \leq \left|\frac{\Delta x_1}{x_1}\right| + \left|\frac{\Delta x_2}{x_2}\right|$$

Q Error in measurement of area of circle is not allowed to exceed 0.1%. How exactly should diameter be measured?

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$\Rightarrow$ given,

$$A = \pi r^2 = \frac{\pi d^2}{4} \quad -(i) \qquad , \qquad \frac{\delta A}{A} \times 100 = 0.1\,\% \quad -(ii)$$

now, taking log both side of eq (i),

$$\log A = \log \frac{\pi}{4} + 2 \log d$$

now, diff. wrt to d

$$\frac{1}{A}\frac{\delta A}{\delta d} = 0 + 2\frac{1}{d}$$

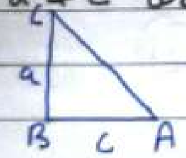
$$\frac{\delta A}{A} = \frac{2}{d}\,\delta d$$

$$\frac{\delta A}{A} \times 100 = 2\frac{\delta d}{d} \times 100$$

$$0.1 = 2\frac{\delta d}{d} \times 100$$

$$\Rightarrow \frac{\delta d}{d} \times 100 = 0.05\,\%$$

Q In $\triangle ABC$, $a = 6cm$, $\angle B = 90^\circ$ & $c = 15cm$. Find maximum possible error in computed value of A. If error in measurement of a, & c are 1mm & 2mm.



$\angle B = 90^\circ$, $a = 6cm$, $c = 15cm$

$\delta a = 1mm = 0.1cm$

$\delta c = 0.2cm$

slope of A, $\tan A = \frac{a}{c}$, $A = \tan^{-1}\frac{a}{c}$

absolute error in A is,

$$\Delta A = \frac{\partial A}{\partial a}\Delta a + \frac{\partial A}{\partial c}\Delta c$$

$$= \frac{1}{1+\frac{a^2}{c^2}} \times \frac{1}{c} \times \Delta a + \frac{1}{1+\frac{a^2}{c^2}}\left(\frac{-a}{c^2}\right)\Delta c$$

$$= \pm \frac{c}{a^2+c^2}\Delta a - \frac{a}{a^2+c^2}\Delta c$$

Max Error, we will remove the -ve sign, we get

$$|\Delta A| = \frac{15}{36+225} \times 0.1 + \frac{6}{36+225} \times 0.2$$

$$= \frac{1.5 + 1.2}{261} = \frac{2.7}{261} = 0.0103$$

Error in a Series

It can be evaluated by using remainder of 'm' terms of taylor expansion i.e

$$f(x) = f(a + \overline{x-a}) = f(a) + \frac{(x-a)}{1!} f'(a) + \frac{(x-a)^2}{2!} f''(a) + \dots + \frac{(x-a)^{m-1}}{(m-1)!} f^{(m-1)}(a) + R_n(x)$$

where, $R_n(x)$ = Error in a series

$$R_n(x) = \frac{(x-a)^m}{m!} f^m(\theta), \quad a<\theta<x$$

If this series is convergent (and at some point)

if $R_m(x) \Rightarrow 0$ as $m \Rightarrow \infty$

if we approximate fn '$f(x)$' by first m terms of the series then,

Maximum Error is given by $\Rightarrow R_n(x)$

Q Find the no. of terms of expansional series such that there sum gives the value of e^x correct to 6 decimal places at $x=1$.

→ given that,

Taylor expansion of e^x is

$f(x)=e^x$

$e^0=1$

$$e^x = 1 + x + \frac{x^2}{2!} + \dots + R_n(x) \quad -(i)$$

$f(x)=e^x$

$f(\theta)=e^\theta$

$f^n(\theta)=e^{n\theta}$

where $R_n(x) = \frac{(x-a)^n}{n!} f^n(\theta) = \frac{x^n}{n!} f^n(\theta) = \frac{x^n}{n!} e^{n\theta}$

here, $a=0$

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$R_n(x) = \frac{x^n}{n!} e^{\theta}$, $0 < \theta < x$ — (ii)

Maximum Ab. Error of $R_n(x)$ at $\theta = x$ is

$$R_n(x) = \frac{x^n}{n!} e^{x} \quad \text{—(iii)}$$

Max. Relative Error, $\frac{R_n(x)}{f(x)} = \frac{x^n}{n!}$ —(iv)

Max. Relative error at $x = 1$

Max. R Error $= \frac{1}{n!}$

we know that if a no. or series connect to n decimal places then

$$\frac{1}{n!} < \frac{1}{2} \times 10^{-6}$$

$$n! > 2 \times 10^{6}$$

Q The Fn f(x) is equal to $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots$

Compute the no. of terms required to estimate $\cos \frac{\pi}{4}$ so that result is correct to 2 significant digit.

→ $f(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots + R_n(x)$

$R_n(x) = (-1)^n \frac{x^{2n}}{2n!} \cos\theta$ $\quad 0 < \theta < x$

Max. Absolute Error for $\theta = x$

$$R_n(x) = \left| (-1)^n \frac{x^{2n}}{2n!} \cos x \right| = \frac{x^{2n}}{2n!} \cos x$$

Max. Relative Error, $\frac{Rn(x)}{f(x)} = \frac{x^{2n}}{2n!}$

Max. Relative Error at $\left(x = \frac{\pi}{4}\right) = \frac{(\pi/4)^{2n}}{2n!}$

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we know that if a no. or series connect to 2 significant digit then

$$\frac{(\pi/4)^{2n}}{2n!} < \frac{1}{2} \times 10^{-2}$$

$$\frac{2n!}{\left(\frac{\pi}{4}\right)^{2n}} > 200$$

$$2n! > 200 \times \left(\frac{\pi}{4}\right)^{2n}$$

$$2n! > 200 \times (0.785)^{2n}$$

when $[n=3]$, true

Q $f(x) = \tan^{-1}(x)$ can be expanded as

$$\tan^{-1}(x) = x - \frac{x^3}{3} + \frac{x^5}{5} - \dots + \frac{(-1)^{n-1} x^{2n-1}}{2n-1} + \dots$$

Compute the no. of terms required for $\tan^{-1}(x)$ correct to 8 significant digit at $x = 1$

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Solution of Algebric & Transcendental Eq ns

Polynomials → $f(x) = x^3 + 2x^2 + 7x + 1$

Algebric Equation ÷

Let $f(x)$ is a polynomial of degree '3' then, it is said to be algebric eqn if, $f(x) = 0$

eg → $f(x) = 0$

$x^3 + 2x^2 + 7x + 1 = 0$

Transcendental Eqn ÷

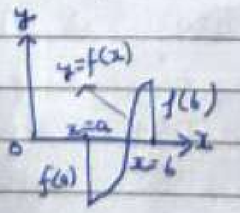
If the f^n $f(x)$ is combination of algebric terms along with trignometric, logarithmic, exponetionel,

eg →

$f(x) = e^x - \sin x = 0$

Intermediate value Property →

If a f^n $f(x)$ is continuous in close interval $[a,b]$ & $f(a), f(b)$ having different sign i.e $f(a)f(b) < 0$ then, $f(x)$ has atleast one ~~root~~ b/w $x=a$ & $x=b$.



Bisection Method ÷

This method is based on the repeated application of intermediate value property

Let $f(x)$ is continuous in $[a,b]$ & $f(a)$ is -ve & $f(b)$ is +ve then the first approxi. to the root is $x_1 = \frac{a+b}{2}$

It is also known as first iteration i.e x_1.

If $f(x_1)$ is +ve & $f(a) = $ -ve then

2nd approx. to root is $x_2 = \frac{a+x_1}{2}$ (2nd iteration)

if $f(x_2)$ is +ve & $f(a)$ is -ve then

3rd approx. to root is $x_3 = \frac{a+x_2}{2}$ (3rd iteration)

we will repeat this process, until we found the root of desired accuracy

Xo Find a root of the eqn $x^3 - 4x - 9 = 0$ using Bisection method correct to 4 decimal place.

Here we have given,

$f(x) = 0$

$x^3 - 4x - 9 = 0$ —(1)

$f(a) = $ -ve, let $a = 2$

$f(b) = $ +ve, let $b = 3$

Ist appr. is $x_1 = \frac{2+3}{2} = 2.5$

$f(x_1 = 2.5 = $ -ve$)$ & $b = $ +ve

2nd approx. is $x_2 = \frac{2.5+3}{2} = \cancel{2.25}\ \frac{5.5}{2} = 2.75$

$\cancel{\text{3rd approx. is } x_3 = \frac{2.25+2}{2} = \frac{4.25}{2} = 2.125}$

$\cancel{\text{4rd approx. is } x_4 = \frac{2.125+2}{2} = \frac{4.125}{2} = 2.0625}$

$\cancel{[\text{root is } 2.0625]}$

$f(x_2) = $ +ve, $a = $ -ve

$x_3 = \frac{2.75+2}{2} = \frac{4.75}{2} = 2.375$

$f(x_3) = $ -ve, $f(x_2) = $ +ve

$x_4 = \frac{2.375+2.75}{2} = 2.5625$

3rd Approximation, $x_3 = \frac{x_1+x_2}{2} = \frac{2.5+2.75}{2} = 2.625$

4th Approximation, $x_4 = \frac{x_2+x_3}{2} = \frac{2.625+2.75}{2} = 2.6875$

similarly

$x_5 = 2.71875$, $x_6 = 2.70313$, $x_7 = 2.71094$

$x_8 = 2.70703$, $x_9 = 2.70508$, $x_{10} = 2.70605$

$x_{11} = 2.70654$, $x_{12} = 2.70642$, $x_{13} = 2.70648$

upto 4 decimal place,

$\boxed{\text{one root} = 2.70648}$

Q Find a real root of the eqn $x^3 - x = 1$ correct to 2 decimal places in $[\cancel{0}, 1, 2]$.

$f(a) = -ve$, $f(b) = +ve$

Ist Approx $x_1 = \frac{2+1}{2} = 1.5$

$f(x_1) = +ve$

2nd Approx. $x_2 = \frac{1.5+1}{2} = \frac{2.5}{2} = 1.25$

$f(x_2) = -ve$

3rd Approx, $x_3 = \frac{x_1+x_2}{2} = \frac{1.25+1.5}{2} = 1.375$

$f(x_3) = +ve$

4th Approx, $x_4 = \frac{x_2+x_3}{2} = \frac{1.25+1.375}{2} = 1.3125$

$f(x_4) = -ve$

5th Approx, $x_5 = \frac{x_3+x_4}{2} = 1.34375$

$f(x_5) = +ve$

6th Appro, $x_6 = \frac{1.3125+1.34375}{2} = 1.328125$

$f(x_6) = +ve$

$x_7 = 1.3203125$, x_8 $\Rightarrow$ Root is 1.32

1.32

Q Find a root of the eqn $x - \cos x = 0$ correct to 3 decimal places.

$f(0) = 0 - \cos 0 = -1 = -ve$

$f(1) = 1 - \cos 1 = 1 - 0.5 = 0.5 = +ve$

$a = 0, b = 1$

interval is $[0, 1]$

$x_1 = [0.7, 0.8]$ $f(a) = -ve, f(b) = +ve$

$x_1 = \frac{0.7 + 0.8}{2} = 0.75$

$f(x_1) = +ve$

$x_2 = \frac{0.75 + 0.7}{2} = 0.725$, $f(x_2) = -ve$

$x_3 = \frac{0.75 + 0.725}{2} = 0.7375$, $f(x_3) = -ve$

$x_4 = \frac{0.75 + 0.7375}{2} = 0.74375$, $f(x_4) = +ve$

$x_5 = \frac{0.7375 + 0.74375}{2} = 0.740625$, $f(x_5) = +ve$

$x_6 = \frac{0.7375 + 0.740625}{2} = 0.7390625$, $f(x_6) = -ve$

$x_7 = \frac{0.740625 + 0.7390625}{2} = 0.73984375$, $f(x_7) = +ve$

$x_8 = \frac{x_6 + x_7}{2} = 0.73945$

[Root is 0.739]

Q By using bisection method, find value of $\sqrt{2}$ correct to 3 decimal places.

let $x = \sqrt{2}$

$f(x) = x^2 - 2 = 0$

$a = 1.3, \; b = 1.5$

$f(a) = -ve \quad f(b) = +ve$

$x_1 = 1.4, \; f(x_1) = +ve$

$x_2 = \frac{1.4 + 1.3}{2} = 1.35, \; f(x_2) = +ve$

$x_3 = 1.35 +$

Q $f(x) = 1 + \log_{10} x - \frac{x}{2} = 0$ [2 decimal places]

Secant Method :-

Let x_0 & x_1 are the limit of interval, then general formula for successive approximation/iteration of second method is

$$x_{n+1} = x_n - \left[\frac{x_n - x_{n-1}}{f(x_n) - f(x_{n-1})}\right] . f(x_n)$$

where $n = 1, 2, 3, \ldots$

putting $n = 1$, we get

Ist iteration, $x_2 = x_1 - \left[\frac{x_1 - x_0}{f(x_1) - f(x_0)}\right] . f(x_1)$

Q Find the root of the eqn $x^3 - x - 1 = 0$ correct to 2d close interval [1, 2] by using secant Method.

⇒ here, $x_0 = 1, \; x_1 = 2$

Ist iteration, $x_2 = 2 - \left[\frac{2-1}{5+1}\right] . 5$

$$x_2 = 2 - \left[\frac{5}{6}\right] = \frac{12-5}{6} = \frac{7}{6} = 1.1666$$

$x_3 = 1.166 - \left[\frac{1.166 - 2}{-5 + 0.58}\right] (0.58) = 1.25286$

$\boxed{x_3 = 1.166 + \left[\frac{0.834}{0.4192}\right]}$

$x_4 = 1.2528 - \left[\frac{1.2528 - 1.166}{-0.2879 + 0.5813}\right] . (-0.2879)$

$= 1.2528 + \left[\frac{0.0868}{0.2934}\right] \times 0.2879$

$= 1.2528 + 0.0855$

$= 1.3379$

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Q $f(x) = x^2 - 2x - 3\cos x$, correct to 3 decimal places by bisection & secent method.

$\Rightarrow$ Bisection:

$\rightarrow$ $f(x=1.7) = -0.12 = -ve$

$f(x=1.8) = 0.32 = +ve$

$a = 1.7,\ b = 1.8$

$x_1 = \frac{1.7+1.8}{2} = 1.75$

$f(x_1) = 0.87 = +ve$

$x_2 = \frac{1.7+1.75}{2} = 1.725$, $f(x_2) = -0.013 = -ve$

$x_3 = \frac{1.75+1.725}{2} = 1.7375$, $f(x_3) = 0.041 = +ve$

$x_4 = \frac{1.725+1.7375}{2} = 1.73125$, $f(x_4) = 0.014 = +ve$

$x_5 = \frac{1.725+1.73125}{2} = 1.728125$, $f(x_5) = 0.0002 = +ve$

$x_6 = \frac{1.725+1.728125}{2} = 1.7265625$, $f(x_6) = -0.006 = -ve$

$x_7 = \frac{1.728125+1.7265625}{2} = 1.72734375$, $f(x_7) = -ve$

$x_8 = \frac{x_5+x_7}{2} = 1.727734375$, $f(x_8) = -ve$

$x_9 = \frac{x_5+x_8}{2} = 1.72792968$

root is 1.727

⇒ Secant method :

here, $x_0 = 1.7$, $x_1 = 1.8$

Ist, $x_2 = 1.8 - \left[\frac{1.8 - 1.7}{0.3216 + 0.1234}\right] (0.3216)$

$x_2 = 1.8 - \left[\frac{0.03216}{0.445}\right]$

$x_2 = 1.8 - 0.0722$

$x_2 = 1.7277$

2nd, $x_3 = 1.7277 - \left[\frac{1.7277 - 1.8}{-0.0016 - 0.32}\right] (-0.0016)$

$x_3 = 1.7277 - \left[\frac{-0.0723}{-0.3216}\right] (-0.0016)$

$x_3 = 1.7277 + 0.00035$

$x_3 = 1.72805$

$x_4 = 1.72805 - \left[\frac{1.72805 - 1.7277}{-0.0001 + 0.0016}\right] (-0.0001)$

$x_4 = 1.72805 + \left[\frac{0.00035}{0.0015}\right] (0.0001)$

$x_4 = 1.72707$

Ans → 1.727

Newton - Raphson Method :-

let $x_1 = x_0 + h$ ① is the exact root of eqn $f(x)=0$
then, by taylor's theorem.

$$f(x_0+h) = f(x_0) + h f'(x_0) + \frac{h^2}{2!} f''(x_0) + \dots = 0 \quad \text{(ii)}$$

If h is very small, then

$$f(x_0) + h f'(x_0) = 0$$

$$\Rightarrow \quad h = -\frac{f(x_0)}{f'(x_0)}$$

put 'h' in eqn ①, we get

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

this is the I^{st} approximation of NR method.
Similarly,

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

$$\vdots$$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

where x_0 is the root which is sufficiently close to the exact root.

. the rate of conversion of NR method is faster than bisection, secant Method & iteration method.

Q Find the real root of eqn $x^2-2x-3\cos x$ correct to 3 decimal places by NR Method.

$$f(x) = x^2-2x-3\cos x$$

$$f'(x) = 2x-2+3\sin x$$

$$f(x=1.7) = -0.12 = -ve, \quad f(x=1.8) = 0.22 = +ve$$

Now, root lie b/w 1.7 & 1.8

Now, we know that NR method for successive iteration is

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \quad , \quad x_{n+1} = x_n - \frac{(x_n^2 - 2x_n - 3\cos x_n)}{2x_n - 2 + 3\sin x_n}$$

I^{st} iteration, $x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$

$$x_{n+1} = \frac{2x_n^2 - 2x_n + 3x_n \sin x_n - x_n^2 + 2x_n + 3\cos x_n}{2x_n - 2 + 3\sin x_n}$$

$$x_{n+1} = \frac{x_n^2 + 3\cos x_n + 3x_n \sin x_n}{2x_n - 2 + 3\sin x_n}$$

let $x_0 = 1.72$, then

I^{st} iteration, for $n=0$ is

$$x_1 = \frac{7.615}{4.406} = 1.728$$

2^{nd} iteration, x_2 for $n=1$

$$x_2 = \frac{7.636}{4.419} = 1.727$$

3^{rd} iteration,

$$x_3 = \frac{7.633}{4.417} = 1.72809$$

4^{th} iteration, $x_4 = 1.727$

Q $x_0 = 1.78$

$$x_1 = \frac{7.768}{4.494} = 1.728$$

$$x_2 = 1.727$$

Q Find the real root of the eqn $x - \cos x = 0$ correct to 3 decimal places using NR M.

interval is 0.7 & 0.8

$f(0.7) = -0.06 = -ve$

$f(0.8) = 0.10 = +ve$

let $x_0 = 0.78$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n - \cos x_n}{1 + \sin x_n}$$

$$x_{n+1} = \frac{x_n + x_n \sin x_n - x_n + \cos x_n}{1 + \sin x_n}$$

$$x_{n+1} = \frac{x_n \sin x_n + \cos x_n}{1 + \sin x_n}$$

.561

1.236

1.673

$$x_1 = \frac{1.259}{1.703} = 0.739$$

$x_2 = 0.739$, $x_3 = 0.7390$

Iteration Method : (Successive Approximation Method)

$f(x) = 0$ —(i)

it can be written as

$x = \phi(x)$ —(ii)

then the successive iteration is

$x_{n+1} = \phi(x_n)$ —(iii)

where $n = 0, 1, 2, 3, \ldots$

if $n=0$, then

I[st] iteration is, $x_1 = \phi(x_0)$ —(iv)

where x_0 is called intial approximation which is obtained from the interval

In other word, x_0 is root lies b/w $x=a$ & $x=b$.

If x_0 is initial approximation, then for iteration method, the condition

$$|\phi'(x_0)| < 1$$

Q Find the root of eqn $x - \cos x = 0$ correct to 3 decimal places by using iteration method.

→ $f(x) = x - \cos x$ —(i)

$x = \cos x$

$x = \phi(x) = \cos x$ —(ii)

Now, diff eqn(ii) w.r.t, x, we get

$\phi'(x) = -\sin x$ —(iii)

$f(0.7) = -ve$, $f(0.8) = +ve$

now, root lies between 0.7 & 0.8

let $x_0 = 0.8$ (say)

I[st] iteration, at $n=0$, $x_1 = \phi(x_0) \Rightarrow x_1 = \cos x_0 = \cos(0.8) = \cancel{747}$

$$|\phi'(x_0)| = |-\sin 0.8| = 0.71 < 1$$

$$x_1 = \cos 0.8 = 0.697$$

$x_2 = 0.766$

$x_3 = 0.720$

$x_4 = 0.751$

$x_5 = 0.731$

$x_6 = 0.744$

$x_7 = 0.735$

$x_8 = 0.741$

$x_9 = 0.737$

$x_{10} = 0.740$

$x_{11} = 0.738$

$x_{12} = 0.739$

$x_{13} = 0.739$

$\frac{1}{x \log_{10}}$

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Q Find the root of eqn $x^2\ 2x - \log_{10} x = 7$ correct to 4 decimal place.

$f(x) = 2x - \log_{10} x - 7$

$x = \frac{7 + \log_{10} x}{2}$

$\phi(x) = \frac{7 + \log_{10} x}{2}$

$\phi'(x) = \frac{1}{2x}[\log_{10} e]$ $\quad |\phi'(x_0)| = \left|\frac{1}{2x_0}\log_{10} e\right| = 0.05 < 1$

$f(3) = -ve, \ f(4) = +ve$

$f(3.7) = -ve, \ f(3.8)\ +ve$, let $x_0 = 3.8$ (say)

I^{st} iteration, $x_1 = 3.7898$

$x_2 = 3.7893$

$x_3 = 3.7892$

$x_4 = 3.7892$

Q Find the root of eqn $\cos x = 3x - 1$ correct to 3 decimal place

$f(x) = \cos x - 3x + 1$

$x = \frac{\cos x + 1}{3}$

$\phi(x) = \frac{\cos x + 1}{3}$

$\phi'(x) = \frac{1}{3}[-\sin x]$ $\quad |\phi'(x_0)| = 0.18 < 1$

$f(0) = +ve, \ f(1) = -ve$

$f(0.7) = -ve, \ f(0.6) = +ve$

let, $x_0 = 0.6$

$x_1 = \frac{\cos x_0 + 1}{3} = 0.608$

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$x_2 = 0.606$
$x_3 = 0.607$
$x_4 = 0.607$

Q. $f(x)$ $x = (15)^{1/3}$ (4 decimal place)

$f(x) = x^3 - 15 = 0$

$$x = \phi(x) = \frac{15}{x^2}$$

$$\phi'(x) = \frac{-30}{x^3}$$

$f(2.4) = -ve$, $f(2.5) = +ve$, let $x_0 = 2.5$ (say)

$$|\phi'(x_0)| = \left|\frac{-30}{x_0^3}\right| = 1.92 > 1$$ not satisfy

so, $x^2 = \frac{15}{x} \Rightarrow x = \sqrt{\frac{15}{x}} = \phi(x)$

$$\phi'(x) = \frac{-\sqrt{15}}{2x^{3/2}} \qquad |\phi'(x)| = 0.48 < 1$$

$x_1 =$ Order of Convergence :-

In the order of convergence is calculated by

$$e_{n+1} = k e_n^p \quad \text{or} \quad \frac{e_{n+1}}{e^p} = k$$

where k is constant, p is order of convergence and e is the error

$x_1 = \alpha + e_1$
$x_2 = \alpha + e_2$
$\vdots$
$x_n = \alpha + e_n$

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Order of Convergence of Newton Raphson Method :-

Let $f(x) = 0$ is an equation and its exact root is α, then $f(\alpha) = 0$ (i) & n^{th} iteration is differ from α by e_n then $x_n = \alpha + e_n$ — (ii)

where e is very small quantity or error.

$x_{n+1} = \alpha + e_{n+1}$ — (iii)

The successive iteration of NRM is

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$\alpha + e_{n+1} = \alpha + e_n - \left[\frac{f(\alpha + e_n)}{f'(\alpha + e_n)}\right]$$

using taylor's expansion & simplify, we get

$$e_{n+1} = e_n - \left[\frac{f(\alpha) + e_n f'(\alpha) + \frac{e_n^2}{2!} f''(\alpha) + \ldots}{f'(\alpha) + e_n f''(\alpha) + \frac{e_n^2}{2!} f'''(\alpha) + \ldots}\right]$$

neglect higher order [i.e very small, we get

$$e_{n+1} = e_n - \left[\frac{f(\alpha) + e_n f'(\alpha)}{f'(\alpha) + e_n f''(\alpha)}\right]$$

as $f(\alpha) = 0$,

$$e_{n+1} = e_n - \left[\frac{e_n f'(\alpha)}{f'(\alpha) + e_n f''(\alpha)}\right]$$

$$e_{n+1} = \frac{e_n^2 f''(\alpha)}{f'(\alpha) + e_n f''(\alpha)}$$

$$\frac{e_{n+1}}{e_n^2} = \frac{f''(\alpha)}{f'(\alpha) + e_n f''(\alpha)}$$

$$\frac{e_{n+1}}{e_n^2} = \frac{f''(\alpha)}{f'(\alpha)}\left[1 + \frac{e_n f''(\alpha)}{f'(\alpha)}\right]^{-1}$$

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$$\frac{e_{n+1}}{e_n^2} = \frac{f''(\alpha)}{f'(\alpha)}\left[1 - \underset{\text{(very small)}}{\text{neglected}}\right]$$

$$\frac{e_{n+1}}{e_n^2} = \frac{f''(\alpha)}{f'(\alpha)} = k$$

here, $p = 2$ [Quadratic]

Q Find the order of convergence of iteration method.

Let $f(x)=0$, its exact root is α,

then $f(\alpha)=0$ —(i), n^{th} iteration is differ from α by e_n

then $x_n = \alpha + e_n$ —(ii)

where e is very small error

$x_{n+1} = \alpha + e_{n+1}$ —(iii)

Successive iteration of iteration method is

$x_{n+1} = \phi(x_n)$

$[f(x)=x$
$\phi(x)=x$
$\phi(\alpha)=\alpha]$

$\alpha + e_{n+1} = \phi(\alpha + e_n)$

using taylor's expansion, we get

$\alpha + e_{n+1} = \phi(\alpha) + e_n\phi'(\alpha)$ → remove small value

as $\phi(\alpha) = \alpha$

$\alpha + e_{n+1} = e_n\phi'(\alpha) + \alpha$

$\frac{e_{n+1}}{e_n} = \phi'(\alpha) = k$

here, order of convergence, $p = 1$ [linear]

$[\text{if } \phi'(\alpha) \neq 0]$

Note: if $\phi'(\alpha) = 0$, then neglecting third onward terms of taylor expansion

$$\frac{e_{n+1}}{e_n^2} = \frac{1}{2}\phi''(\alpha)$$

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System of Linear Equation :-

System of linear equation is given by

$$\left.\begin{array}{l} a_{11}x + a_{12}y + a_{13}z = b_1 \\ a_{21}x + a_{22}y + a_{23}z = b_2 \\ a_{31}x + a_{32}y + a_{33}z = b_3 \end{array}\right\} \quad -(i)$$

The above eqn can be written as

$AX = B$ — (ii)

where , A = coefficient matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}_{3\times3}$$

X = variable column matrix

$$X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}_{3\times1} \quad \& \quad B = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}_{3\times1}$$

eqn (ii) can be written as, Augmented matrix as $(C) = [A|B]$
$= [A:B]$

$$C \Rightarrow \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

$$C = [A|B]$$

⇒ Method of solⁿ of linear equation :-

(i) Gauss Elimination method :-

In this method, by using elementary row transformation.

matrix A must be converted into upper triangular matrix.

2) Gauss Jordan Method:-

Using elementary row transformation, the matrix A must be converted into diagional matrix.

Q Solve the system of eqn $x+4y-z=5$, $x+y-6z=-12$, $3x-y-z=4$ by using gauss elimination & gauss jordan method.

(i) Gauss elimination method,

eqn can be written as

$AX=B$

Augmented matrix, $C = \left[\begin{array}{ccc|c} 1 & 4 & -1 & 5 \\ 1 & 1 & -6 & -12 \\ 3 & -1 & -1 & 4 \end{array}\right]$

$R_2 \to R_1 - R_2$, $R_3 \to R_3 - 3R_1$

$C = \left[\begin{array}{ccc|c} 1 & 4 & -1 & 5 \\ 0 & 3 & 5 & 17 \\ 0 & -13 & 2 & -11 \end{array}\right]$

$R_2 \to R_2/3$

$C = \left[\begin{array}{ccc|c} 1 & 4 & -1 & 5 \\ 0 & 1 & 5/3 & 17/3 \\ 0 & -13 & 2 & -11 \end{array}\right]$

$R_3 \to R_3 + 13R_2$

$C = \left[\begin{array}{ccc|c} 1 & 4 & -1 & 5 \\ 0 & 1 & 5/3 & 17/3 \\ 0 & 0 & 71/3 & \frac{+188}{3} \end{array}\right]$ —(iii)

$2+\frac{135}{3}$

$6+65$

$-11+\frac{(221)}{3}$

$-33+$

$y+\frac{5}{3}z = \frac{17}{3}$

$y = 1.253$

$8y = 17 \Rightarrow \boxed{\cancel{y = \frac{17}{8}}}$

$\frac{71}{3}z = \frac{188}{3} \Rightarrow \left[z = \frac{188}{71}\right]$, $x+4y-z=5$

$x = 2.65$

$z = 2.647$

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(ii) by Gauss Jordan,
from eq (III)

$$C = \left[\begin{array}{ccc|c} 1 & 4 & -1 & 5 \\ 0 & -1 & -5/3 & -17/3 \\ 0 & 0 & 71/3 & 188/3 \end{array}\right]$$

$R_3 \to 3R_3/71$

$$C = \left[\begin{array}{ccc|c} 1 & 4 & -1 & 5 \\ 0 & -1 & -5/3 & -17/3 \\ 0 & 0 & 1 & 188/71 \end{array}\right]$$

$R_2 \to R_2 + \frac{5}{3}R_3$

$\frac{-5}{3} + \frac{5}{3}$

$$C = \left[\begin{array}{ccc|c} 1 & 4 & -1 & 5 \\ 0 & -1 & 0 & -267/213 \\ 0 & 0 & 1 & 188/71 \end{array}\right]$$

$\frac{-17}{3} + \frac{5}{3}\frac{(188)}{71}$

$R_2 \to -R_2$

$$C = \left[\begin{array}{ccc|c} 1 & 4 & -1 & 5 \\ 0 & 1 & 0 & 267/213 \\ 0 & 0 & 1 & 188/71 \end{array}\right]$$

940

$\frac{940-1207}{213}$

$R_1 \to R_1 + R_3$

$\frac{-267}{213}$

$$C = \left[\begin{array}{ccc|c} 1 & 4 & 0 & 543/171 \\ 0 & 1 & 0 & 267/213 \\ 0 & 0 & 1 & 188/71 \end{array}\right]$$

$5 + \frac{188}{71}$

3.175

$R_1 \to R_1 - 4R_2$

$$C = \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1.937 \\ 0 & 1 & 0 & 1.253 \\ 0 & 0 & 1 & 2.645 \end{array}\right]$$

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#V.I Gauss Seidal Method :-

Let us consider the system of Linear equation is

$$\left.\begin{array}{l} a_1x + b_1y + c_1z = d_1 \\ a_2x + b_2y + c_2z = d_2 \\ a_3x + b_3y + c_3z = d_3 \end{array}\right\} - ①$$

In this method, we will check the following condition

$|a_i| \geq |b_i| + |c_i|$

if this is true, then we will find value of 'x' in given eqn

if $|b_i| \geq |a_i| + |c_i|$

if true, then we will find value of 'y' in given eqn

similarly for z,

if $|c_i| \geq |b_i| + |a_i|$, we get value of 'z'.

The initial approximation (x_0) in Gauss Seidal Method is

$x = x_0, y = y_0, z = z_0$ which is equal to $x_0 = y_0 = z_0 = 0$

→ The first iteration is obtain at origin.

→ The 2^{nd} iteration is obtain by using value of I^{st} iteration and so on, we will repeat these step until we get desired value of x, y, z.

Q solve the eqn

$$8x - 3y + 2z = 20 \quad -(i)$$
$$4x + 11y - z = 33 \quad -(ii)$$
$$6x + 3y + 12z = 36 \quad -(iii)$$

from eqn (i),

$$x = \frac{1}{8}(20 + 3y - 2z) \quad -(iv)$$

from eqn (ii), $y = \frac{1}{11}(33 + z - 4x) \quad -(v)$

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$$z = \frac{1}{12}(36 - 3y - 6x) \quad -(vi)$$

for I^{st} iteration,
putting $y = y_0, z = z_0 \ (=0)$ in eq (iv)

$$x_1 = \frac{20}{8} = 2.5$$

similarly for eqn (v), $z = z_0 = 0$ & $x = x_1 = 2.5$

$$y_1 = \frac{1}{11}(33 - 10) = \frac{23}{11} = 2.09$$

for z_1, putting $y = y_1 = 2.09$ & $x = x_1 = 2.5$

$$z_1 = \frac{1}{12}(36 - 6.27 - 15) = 1.2275$$

In I^{st} iteration we get, $x_1 = 2.5, y_1 = 2.09, z_1 = 1.2275$

for 2^{nd} iteration,
putting $y = y_1 = 2.09$, $z = z_1 = 1.2275$ in eqn (iv)

$$x_2 = \frac{1}{8}(20 + 6.27 - 2.455) = 2.976$$

putting $z = z_1 = 1.2275$, $x = x_2 = 2.976$ in eqn (v)

$$y_2 = \frac{1}{11}(33 + 1.2275 - 11.904) = 2.029$$

similarly, $z_2 = 1.007$

3^{rd} iteration,	4^{th} iteration,	5^{th} iteration
$x_3 = 3.005$	$x_4 = 2.999$	$x_5 = 3$
$y_3 = 1.997$	$y_4 = 2.000$	$y_5 = 2$
$z_3 = 0.998$	$z_4 = 1.000$	$z_5 = 1$

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Q Solve the system of eqn $2x+y+6z=9$
$8x+3y+2z=13$
$x+5y+z=7$

Q $20x+y-2z=17$
$3x+20y-z=-18$
$2x+3y+20z=25$

↳ $x = \frac{1}{20}[17+2z-y]$ - (1)

$y = \frac{1}{20}[z-3x-18]$ - (11)

$z = \frac{1}{20}[25+3y-2x]$ - (111)

I^{st} iteration,
$x_1 = 0.85$, $y_1 = -1.0275$ $z_1 = 1.010$

2^{nd} iteration,
$x_2 = 1.002$, $y_2 = -0.999$, $z_2 = 0.999$

3^{rd} iteration,
$x_3 = 0.999$, $y_3 = -0.999$, $z_3 = 1.000$

4^{th} iteration,
$x_4 = 0.999$, $y_4 = -0.999$, $z_3 = 1.000$

$[x=1, y=-1, z=1]$