Unit - 3

Numerical Differentiation

①

Numerical differentiation is concerned with the method of finding the successive derivatives of a func. at a given argument, using the given table of entries corresponding to a set of arguments, equally or unequally spaced.

Formulae for derivatives

1. Newton's forward difference interpolation formula :-

$$y = y_0 + p\Delta y_0 + \frac{p(p-1)}{\lfloor 2}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{\lfloor 3}\Delta^3 y_0 + \cdots$$

where $x = x_0 + ph$

by chain rule $\frac{dy}{dx} = \frac{dy}{dp}\cdot\frac{dp}{dh}$

$$\frac{dx}{dp} = h$$

$$\frac{dy}{dx} = \frac{1}{h}\left[\Delta y_0 + \frac{(2p-1)}{2}\Delta^2 y_0 + \frac{(3p^2-6p+2)}{6}\Delta^3 y_0 + \cdots\right] \quad (1)$$

eq① provides the value of dy/dx at any x which is not tabulated.

Now for tabulated values: at $x = x_0$, $p = 0$ then

$$\left(\frac{dy}{dx}\right)_{x=x_0} = \frac{1}{h}\left[\Delta y_0 - \frac{1}{2}\Delta^2 y_0 + \frac{1}{3}\Delta^3 y_0 - \frac{1}{4}\Delta^4 y_0 + \cdots\right]$$

2. Newton's backward difference interpolation formula :-

$$y = y_0 + p\Delta y_0 + \frac{p(p+1)}{\lfloor 2}\Delta^2 y_0 + \frac{p(p+1)(p+2)}{\lfloor 3}\Delta^3 y_0 + \cdots$$

$$\frac{dy}{dx} = \frac{1}{h}\left[\Delta y_0 + \frac{1}{2}\Delta^2 y_0 + \frac{1}{3}\Delta^3 y_0 + \frac{1}{4}\Delta^4 y_0 + \cdots\right]$$

Can find higher order also.

Tabular value
starting → N. forward
end → N. backward
centre → Central Methods

3. Lagrange's method :-

$$f(x) = \frac{(x-x_1)(x-x_2)\cdots(x-x_n)}{(x_0-x_1)(x_0-x_2)\cdots(x_0-x_n)} f(x_0) + \cdots + \cdots$$

find general polynomial first, then find derivative at the given point.

4. Newton's divided difference formula :-

$$f(x) = f(x_0) + (x-x_0)\Delta f(x_0) + (x-x_0)(x-x_1)\Delta^2 f(x_0) + \cdots + \cdots$$

then

$$f'(x) = \Delta f(x_0) + \{2x-(x_0+x_1)\}\Delta^2 f(x_0) + \cdots$$

Ex1. find $\frac{dy}{dx}$ at $x = 0.1$ from

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$
0.1	0.9975			
		−0.0075		
0.2	0.9900		−0.0049	
				0.0001
		−0.0124		
0.3	0.9776		−0.0048	
		−0.0172		
0.4	0.9604			

here $h = 0.1$, $y_0 = 0.9975$

$$\left[\frac{dy}{dx}\right]_{x=0.1} = \frac{1}{h}\left[\Delta y_0 - \frac{1}{2}\Delta^2 y_0 + \frac{1}{3}\Delta^3 y_0\right]$$

$$= \frac{1}{0.1}\left[-0.0075 - \frac{1}{2}(-0.0049) + \frac{1}{3}(0.0001)\right]$$

$$= -0.050167.$$

Ex2. find its acceleration at $t = 1.1$.

t	v	Δv	$\Delta^2 v$	$\Delta^3 v$	$\Delta^4 v$
1	43.1				
		4.6			
			−0.2		
$\boxed{1.1}$	$\boxed{47.7}$ v_0			0.1	
		4.4 Δv_0			0.1
			−0.1 $\Delta^2 v_0$		
1.2	52.1			0.2 $\Delta^3 v_0$	
		4.3			
			0.1		
1.3	56.4				
		4.4			
1.4	60.8				

$$\left[\frac{dv}{dt}\right]_{t=1.1} = \frac{1}{h}\left[\Delta v_0 - \frac{1}{2}\Delta^2 v_0 + \frac{1}{3}\Delta^3 v_0\right]$$

$$= \frac{1}{0.1}\left[4.4 - \frac{1}{2}(-0.1) + \frac{1}{3}(0.2)\right]$$

$$\left(\frac{dv}{dt}\right)_{t=1.1} = 45.1667.$$

Ex3. find $f'(1.1)$ and $f'(2.1)$

x	$f(x)$	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$
1	0	0.1280	0.298	0.018	0.06	−0.10 ($\nabla^5 y_p$)
1.2	0.1280	0.4260	0.316	0.078	−0.04 ($\nabla^4 y_p$)	
1.4	0.5540	0.7420	0.394	0.038 ($\nabla^3 y_p$)		
1.6	1.2960	1.1360	0.432 ($\nabla^2 y_p$)			
1.8	2.4320	1.5680 (∇y_p)				
2.0	4.000					

$x_0 = 1$, $\quad x = x_0 + ph$

$1.1 = 1 + p(0.2) \Rightarrow p = 0.5$

$$\left(\frac{dy}{dx}\right)_{1.1} = \frac{1}{h}\left[\Delta y_0 + \frac{(2p-1)}{2}\Delta^2 y_0 + \frac{(3p^2-6p+2)}{6}\Delta^3 y_0 + \left(\frac{2p^3-9p^2+11p-3}{12}\right)\Delta^4 y_0 + \frac{(5p^4-40p^3+105p-100p+24)}{120}\Delta^5 y_0\right]$$

$$\left(\frac{dy}{dx}\right)_{1.1} = 0.66724.$$

$$\left(\frac{dy}{dx}\right)_{2.1} = \frac{1}{h}\left[\nabla y_p + \frac{(2p+1)}{2}\nabla^2 y_p + \frac{(3p^2+6p+2)}{6}\nabla^3 y_p + \cdots\right]$$

$=$ put values

$x = x_p + ph$

$2.1 = 2 + p(0.2)$

~~$p = 0.2p \Rightarrow p = \frac{1}{0.2}$~~

$0.1 = p(0.2)$

$p = \frac{0.1}{0.2} = 0.5$

Ex4. (i) Using Newton's divided difference formula, find $f'(10)$ from

x :	3	5	11	27	34
$f(x)$:	-13	23	899	17315	35606

(ii)

x :	0	2	3	4	7	8
$f(x)$:	4	26	58	112	466	922

Use Lagrange's formula, find $f'(6)$.

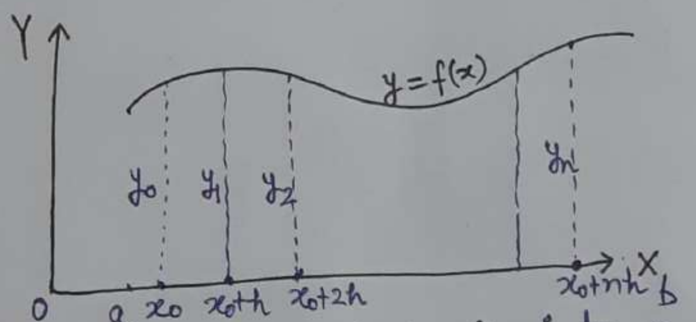
Numerical Integration

The process of evaluating a definite integral from a set of tabulated values of the integrand $f(x)$ is called numerical integration.

Newton - cotes Quadrature formula :-

$$\text{Let } I = \int_a^b f(x)\,dx$$

where $f(x)$ takes the values $y_0, y_1, y_2, --, y_n$ for $x = x_0, x_1, x_2, ---, x_n$.



Let us divide the interval (a,b) into n sub-intervals of width h so that $x_0 = a$, $x_1 = x_0+h$, $x_2 = x_0+2h$, $---$, $x_0+nh = b$.

Then

$$I = \int_{x_0}^{x_0+nh} f(x)\,dx \qquad \text{put } x = x_0 + rh, \quad dx = h\,dr$$

$$I = h\int_0^n \left\{ y_0 + r\Delta y_0 + \frac{r(r-1)}{\lfloor 2} \Delta^2 y_0 + \frac{r(r-1)(r-2)}{\lfloor 3} \Delta^3 y_0 + ------ \right\} dr$$

$$I = h\left[\quad \underline{\qquad\qquad} \quad \right]_0^n$$

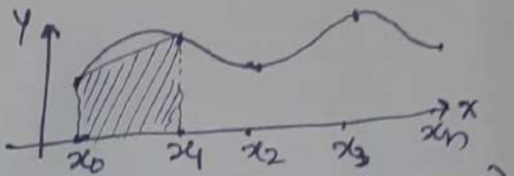
Newton's Cote's quadrature formula.
General formula.

we deduce the imp. quadrature rules by taking $n = 1, 2, 3, ------$

① Trapezoidal rule :- $(n=1)$.

$$\int_{x_0}^{x_0+nh} f(x)\,dx = \frac{h}{2}\left[(y_0+y_n) + 2\,(y_1+y_2+\cdots+y_{n-1})\right]$$

The area of each strip (trapezium) is found separately.



(sum of n trapeziums)

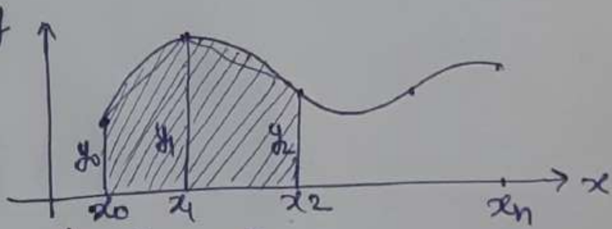
② Simpson's one-third rule $(n=2)$

$$\int_{x_0}^{x_0+nh} f(x)\,dx = \frac{h}{3}\left[(y_0+y_n) + 4\,(y_1+y_3+\cdots+y_{n-1}) + 2\,(y_2+y_4+\cdots y_{n-2})\right]$$

$$= \frac{h}{3}\left[(y_0+y_n) + 4(\text{odd}) + 2(\text{even})\right]$$

(we find the area of two strips at a time.)

(Even no. of subintervals)

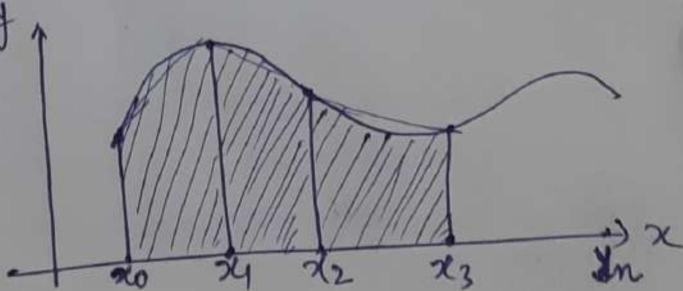


③ Simpson's three-eighth rule $(n=3)$

$$\int_{x_0}^{x_0+nh} f(x)\,dx = \frac{3h}{8}\left[(y_0+y_n) + 3\,(y_1+y_2+y_4+\cdots) + 2\,(y_3+y_6+\cdots)\right]$$

$$= \frac{3h}{8}\left[(y_0+y_n) + 3\,(\text{Rest values}) + 2\,(3\text{ multiple values})\right]$$

(No. of sub-intervals should be taken as multiple of 3.).



(If subintervals are multiple of 3)

(4) Boole's Rule $(n=4)$

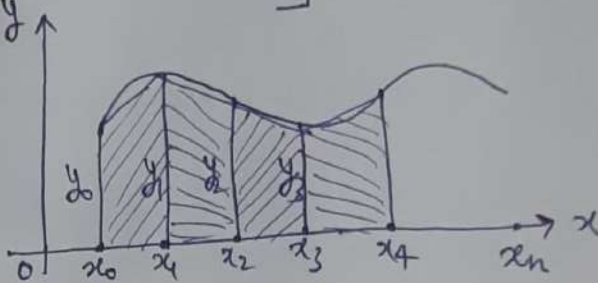
$y_0 \; y_1 \; y_2 \; y_3 \; y_4$
$7 \; 32 \; 12 \; 32 \; 7$

$7 \; 32 \; 12 \; 32 \; 14$ add 4
$y_4 \; y_5 \; y_6 \; y_7 \; y_8$

$$\int_{x_0}^{x_0+nh} f(x)\,dx = \frac{2h}{45}\Big[7y_0 + 32y_1 + 12y_2 + 32y_3 + 14y_4 + 32y_5 + 12y_6 + 32y_7 + 14y_8 + \cdots\Big]$$

No. of sub-intervals should be taken as a multiple of 4.

(If subintervals are multiple of 4).



(5) Weddle's Rule $(n=6)$

$y_0 \; y_1 \; y_2 \; y_3 \; y_4 \; y_5 \; y_6$
$1 \; 5 \; 1 \; 6 \; 1 \; 5 \; 1$ add 6
$1 \; 5 \; 1 \; 6 \; 1 \; 5 \; 1$
$y_6 \; y_7 \; y_8 \; y_9 \; y_{10} \; y_{11} \; y_{12}$

$$\int_{x_0}^{x_0+nh} f(x)\,dx = \frac{3h}{10}\Big[y_0 + 5y_1 + y_2 + 6y_3 + y_4 + 5y_5 + y_6 + y_6 + 5y_7 + y_8 + 6y_9 + y_{10} + 5y_{11} + \cdots\Big]$$

No. of sub-intervals should be taken as multiple of 6.

(If subintervals are multiple of 6).

Ex1. Evaluate $\int_0^6 \frac{dx}{1+x^2}$ by using

$h = \frac{b-a}{\text{no. of intervals}}$

(i) Trapezoidal
(ii) Simpson's 1/3
(iii) Weddle's rule

x	0	1	2	3	4	5	6
$y=\frac{1}{1+x^2}$	1	0.5	0.2	0.1	0.0588	0.0385	0.027
	y_0	y_1	y_2	y_3	y_4	y_5	y_6

Sol → (i) $x_0 + nh = 6$, $h=1$ $\qquad h = \frac{0-6}{6} = 1$
$0 + nh = 6$

$$\int_0^6 \frac{dx}{1+x^2} = \frac{h}{2}\left[(y_0+y_6) + 2(y_1+y_2+y_3+y_4+y_5)\right]$$

$$= \frac{1}{2}\left[(1+0.027) + 2\left[0.5+0.2+\cdots+0.0385\right]\right]$$

$$= 1.4108$$

(ii) $$\int_0^6 \frac{dx}{1+x^2} = \frac{h}{3}\left[(y_0+y_6) + 4(y_1+y_3+y_5) + 2(y_2+y_4)\right]$$

$$= \frac{1}{3}\left[--\right] = 1.3662$$

(iii) $$\int_0^6 \frac{dx}{1+x^2} = \frac{3h}{8}\left[(y_0+y_6) + 3(y_1+y_2+y_4+y_5) + 2y_3\right]$$

$$= 1.3571$$

(iv) $$\int_0^6 \frac{dx}{1+x^2} = \frac{3h}{10}\left[y_0 + 5y_1 + y_2 + 6y_3 + y_4 + 5y_5 + y_6\right]$$

$$= 1.3735.$$

Also, $$\int_0^6 \frac{dx}{1+x^2} = \left[\tan^{-1}x\right]_0^6 = 1.4056$$

Ex2. Use Trapezoidal rule to integrate $\int_0^2 e^{x^2} dx$ taking the no. 10 intervals.

Sol → $y = e^{x^2}$, $h = 0.2$, $n = 10$ $\qquad h = \frac{2-0}{10} = 0.2$

x	0	0.2	0.4	0.6	0.8	1.0	1.2	1.4	1.6	1.8	2.0
y	1	1.0408	1.1735	1.4333	1.8964	2.1782	4.2206	7.0993	12.9358	25.5337	54.5981
	y_0	y_1	y_2	y_3	y_4	y_5	y_6	y_7	y_8	y_9	y_{10}

$$\int_0^1 e^{x^2} dx = \frac{h}{2}\left[(y_0+y_{10}) + 2(y_1+y_2+y_3+y_4+y_5+y_6+y_7+y_8+y_9)\right]$$

$$= \frac{0.2}{2}\left[(1+\ ) + \cdots\right]$$

$$= 17.0621$$

Ex3. The velocity v (km/min) of a moped which starts from rest, is given at fixed intervals of time t (min) as follows:-

t	2	4	6	8	10	12	14	16	18	20
v	10	18	25	29	32	20	11	5	2	0

Sol → If s (Km) be the distance covered in t (min), then

$$\frac{ds}{dt} = v. \qquad h = \frac{20-2}{9} = \frac{18}{9} = 2$$

$$|s|_{t=0}^{20} = \int_0^{20} v\,dt = \frac{h}{3}\left[X + 4.0 + 2.E\right]$$

$$= \frac{2}{3}\left[(v_0+v_{10}) + (v_1+\cdots v_9) + (v_2+v_4+\cdots v_8)\right]$$

$$= \frac{2}{3}\left[\ \cdots\right]$$

$$= 309.33 \text{ Km.}$$

Ex4.

s :	0	2.5	5	7.5	10.0	12.5	15.0	17.5	20.0
v :	16	19	21	22	20	17	13	11	9

Estimate the time taken by the particle to traverse the distance of 20 m, using Boole's rule.

Sol →
$$\frac{ds}{dt} = v$$

or $\frac{dt}{ds} = \frac{1}{v} = y.$ $\qquad h = \frac{20-0}{8} = 2.5$

$$\Rightarrow \left|t\right|_{s=0}^{20} = \int_0^{20} y\, ds$$

here $h = 2.5$ and $n = 8$.

By Boole's Rules, we have

$$\left|t\right|_{s=0}^{20} = \frac{2h}{45}\left[\cdots\right]$$

$$= \frac{1}{9}(12.11776) = 1.35$$

$t = 1.35$ sec.

Ex5. A solid of revolution is formed by rotating about x-axis, the area b/w the x-axis, the lines $x=0$ & $x=1$ and a curve through the pts. with the following co-ordinates:

x :	0	0.25	0.50	0.75	1
y :	1	0.9896	0.9589	0.9089	0.8415

Estimate the volume of the solid formed using Simpson's rule

Sol → $h = 0.25$, $y_0 = y_1, y_2$ etc

$$\text{volume} = \int_0^1 \pi y^2 dx = \pi \frac{h}{3}\left[(y_0^2 + y_4^2) + 4(y_1^2 + y_3^2) + 2y_2^2\right]$$

$$= \frac{0.25}{3}\pi\left[- - \right] \qquad h = \frac{1-0}{4} = 0.25$$

$$= 0.2618(10.7687)$$

$$= 2.8192.$$

Q→ Solve the integral by dividing the integral in 11 ordinates; $\int_0^{\pi/2} \sin x \, dx$

$[-\cos x]_0^{\pi/2}$

$0+1 = 1$

Sol →. $h = \dfrac{\frac{\pi}{2} - 0}{10}$ $n = 10$

$h = \dfrac{\pi}{20}$

x : 0 $\frac{\pi}{20}$ $\frac{2\pi}{20}$ $\frac{3\pi}{20}$ $\frac{4\pi}{20}$ $\frac{5\pi}{20}$ $\frac{6\pi}{20}$ $\frac{7\pi}{20}$ $\frac{8\pi}{20}$ $\frac{9\pi}{20}$ $\frac{10\pi}{20}$

$f(x)$:

~~x : $\frac{11\pi}{20}$ $\frac{12\pi}{20}$ $\frac{13\pi}{20}$ $\frac{14\pi}{20}$ $\frac{15\pi}{20}$ $\frac{16\pi}{20}$ $\frac{17\pi}{20}$ 18~~

$f(x)$:

Apply simpson's $\frac{1}{3}$rd rule.

Q→ Evaluate $\int_{0.2}^{1.4} (\sin x - \log_e x + e^x) \, dx$ using Weddle's Rule. correct to 4 decimal places.

Sol

Ans 4.051

(1)

Q→ find $\int_0^6 \frac{e^x}{1+x} dx$ by Simpson's $\frac{3}{8}^{th}$ rule.

70.1652

Q→ Evaluate $\int_0^4 \frac{dx}{1+x^2}$ using Boole's rule

by taking $h = 0.5$.

compare the results with the actual value & indicate the err in both.

$y_0 \to y_8$

exact $= 1.326373$

1.325818

% err $= -0.0419\%$

Q→ A tank is discharging water through an orifice at a depth of x m below the surface of the water whose area is A m^2. following are the values

A :	1.257	1.39	1.52	1.65	1.809	1.962	2.123	2.295
x :	1.5	1.65	1.8	1.95	2.1	2.25	2.4	2.55

$\left(\frac{1}{3}^{rd}\right)$

A :	2.462	2.650	2.827
x :	2.7	2.85	3

Using the formula $(0.018)T = \int_{1.5}^{3} \frac{A}{\sqrt{x}} dx$, Calculate T, the time in sec. for the level of the water to drop from 3 m to 1.5 m above the orifice. ($T = 110$ sec)

Q→ A reservoir discharging water through sluices at a depth h below the water surface, has a surface area A for various values of h as given

h (m)	10	11	12	13	14
A (m²)	950	1070	1200	1350	1530

(2)

If t denote time in min, the rate of fall of the surface is $\frac{dh}{dt} = -\frac{48}{A}\sqrt{h}$.

Estimate the time taken for the water level to fall from 14 to 10 m above the sluices.

Sol $\rightarrow$.

$$t = -\frac{1}{48}\int_{14}^{10} \frac{A}{\sqrt{h}}\, dh = \frac{1}{48}\int_{10}^{14} \frac{A}{\sqrt{h}}\, dh.$$

$\left(\frac{1}{3}rd\right)$

$h : \rightarrow$

$\frac{A}{\sqrt{h}} : \rightarrow$

$$t = 29.0993$$