Data fitting with Cubic spline

In the interpolation, we have seen that a single polynomial has been fitted to the given data. If the given set of points belong to the polynomial then the interpolation methods give the best approximation to the polynomial, otherwise a rough approximation to the polynomial are obtained.

Two cubics one through A_i and A_{i+1}, and other through A_{i+1} and A_{i+2} are drawn such that the slopes and curvatures of two curves match at A_{i+1}. Such curves are called cubic splines and fitting such curves are called spline fitting.

Cubic spline interpolation

Let we have the data points of observations (x_0, y_0), (x_1, y_1), (x_2, y_2), (x_3, y_3) ,..............., (x_n, y_n) such that $a = x_0 < x_1 < x_2 < x_3 .... < x_n = b$. Then, a spline function $S(x)$ is defined as

$$S(x) = \frac{1}{6h}\left[(x_i - x)^3 M_{i-1} + (x - x_{i-1})^3 M_i + (x_i - x)(6y_{i-1} - h^2 M_{i-1}) + (x - x_{i-1})(6y_i - h^2 M_i)\right]$$

with

$$h^2\left[M_{i-1} + 4M_i + M_{i+1}\right] = 6(y_{i-1} - 2y_i + y_{i+1}).$$

The spline function $S(x)$ satisfies the following properties:

(i) $S(x_i) = y_i$

(ii) $S(x)$, $S'(x)$, $S''(x)$ are continuous on $[a, b]$.

(iii) $S(x)$ is a cubic polynomial in each sub-interval $[x_{i-1}, x_i]$ for $i = 1, 2, 3, ..., n$.

End conditions

There are several ways in which the end conditions may be imposed so that the different types of cubic splines are obtained:

(i) If $M_0 = M_n = 0$, then the spline is called natural cubic spline.

(ii) If $M_0 = M_n$, $M_1 = M_{n+1}$, $y_0 = y_n$, $y_1 = y_{n+1}$, $h_1 = h_{n+1}$, then the spline is called periodic spline.

(iii) $S'(a) = y'(a) = y_0'$ and $S'(b) = y'(b) = y_n'$ then the spline is called non-periodic spline.

Working rule to obtain cubic spline

Step1: For the interval (x_{i-1}, x_i), the cubic spline is

$$S(x)=\frac{1}{6h}\left[(x_i-x)^3M_{i-1}+(x-x_{i-1})^3M_i+(x_i-x)(6y_{i-1}-h^2M_{i-1})+(x-x_{i-1})(6y_i-h^2M_i)\right]$$

Step2: If not given, choose $M_0=M_3=0$ (for the interval $0\le x\le 3$)

Step3: For $i=1,2,3,\dots\dots\dots,n$, choose values of M_1 and M_2 such that

$$h^2\left[M_{i-1}+4M_i+M_{i+1}\right]=6(y_{i-1}-2y_i+y_{i+1})$$

exists for two sub-intervals $0\le x\le 1$ and $1\le x\le 2$ respectively.

Step4: Find $S(x)$ for different sub-intervals.

Ques: The following values of x and y are given:

x	1	2	3	4
y	1	2	5	11

Find the cubic spline and evaluate $y(1.5)$ and $y'(3.5)$.

Solution: Here, the points x_0, x_1, x_2, x_3 are equispaced with $h=1$.

For the interval (x_{i-1}, x_i), the cubic spline is

$$S(x)=\frac{1}{6h}\left[(x_i-x)^3M_{i-1}+(x-x_{i-1})^3M_i+(x_i-x)(6y_{i-1}-h^2M_{i-1})+(x-x_{i-1})(6y_i-h^2M_i)\right]$$

.........................(1)

with $M_0=M_3=0$ and $i=1,2$.

Then, we have

$$h^2\left[M_{i-1}+4M_i+M_{i+1}\right]=6(y_{i-1}-2y_i+y_{i+1})$$ ....(2)

Putting the value of $h=1$, $M_0=M_3=0$ and $i=1,2$, we have

$$M_0 + 4M_1 + M_2 = 6(y_0 - 2y_1 + y_2) \quad \Rightarrow \quad 0 + 4M_1 + M_2 = 6(1 - 2(2) + 5)$$

$$\Rightarrow \quad 4M_1 + M_2 = 12 \qquad \text{.......................(3)}$$

and

$$M_1 + 4M_2 + M_3 = 6(y_1 - 2y_2 + y_3) \Rightarrow M_1 + 4M_2 + M_3 = 6(2 - 2(5) + 11)$$

$$\Rightarrow \quad M_1 + 4M_2 = 18 \qquad \text{.......................(4)}$$

Solving Eq. (3) and (4), we have

$M_1 = 2$ and $M_2 = 4$.

For the interval $1 \le x \le 2$, put $i = 1$ in Eq. (1),

$$S(x) = \frac{1}{6h}\left[(x_1 - x)^3 M_0 + (x - x_0)^3 M_1 + (x_1 - x)(6y_0 - h^2 M_0) + (x - x_0)(6y_1 - h^2 M_1)\right]$$

$$= \frac{1}{6}\left[(2 - x)^3(0) + (x - 1)^3(2) + (2 - x)(6 - 0) + (x - 1)(12 - 2)\right]$$

$$= \frac{1}{3}\left[x^3 - 3x^2 + 5x\right]$$

For the interval $2 \le x \le 3$, put $i = 2$ in Eq. (1),

$$S(x) = \frac{1}{6h}\left[(x_2 - x)^3 M_1 + (x - x_1)^3 M_2 + (x_2 - x)(6y_1 - h^2 M_1) + (x - x_1)(6y_2 - h^2 M_2)\right]$$

$$= \frac{1}{6}\left[(3 - x)^3(2) + (x - 2)^3(4) + (3 - x)(12 - 2) + (x - 2)(30 - 4)\right]$$

$$= \frac{1}{3}\left[x^3 - 3x^2 + 5x\right]$$

For the interval $3 \le x \le 4$, put $i = 3$ in Eq. (1),

$$S(x) = \frac{1}{6h}\left[(x_3 - x)^3 M_2 + (x - x_2)^3 M_3 + (x_3 - x)(6y_2 - h^2 M_2) + (x - x_2)(6y_3 - h^2 M_3)\right]$$

$$= \frac{1}{6}\left[(4 - x)^3(4) + (x - 3)^3(0) + (4 - x)(30 - 4) + (x - 3)(66 - 0)\right]$$

$$= \frac{1}{3}\left[-2x^3 + 24x^2 - 76x + 81\right]$$

Thus, the cubic splines are

$$S(x) = \begin{cases} \frac{1}{3}\left[x^3 - 3x^2 + 5x\right], & 1 \le x \le 2 \\ \frac{1}{3}\left[x^3 - 3x^2 + 5x\right], & 2 \le x \le 3 \\ \frac{1}{3}\left[-2x^3 + 24x^2 - 76x + 81\right], & 3 \le x \le 4 \end{cases}$$

Now,

$$y(1.5) = \frac{1}{3}\left[(1.5)^3 - 3(1.5)^2 + 5(1.5)\right] \quad \Rightarrow \quad y(1.5) = \frac{11}{8}$$

To evaluate $y'(3.5)$, we have

$$y(x) = \frac{1}{3}\left[-2x^3 + 24x^2 - 76x + 81\right]$$

$$y'(x) = \frac{1}{3}\left[-6x^2 + 48x - 76\right]$$

$$y'(3.5) = 6.16666$$