The algorithmic Search for the optimal Number of Imputations

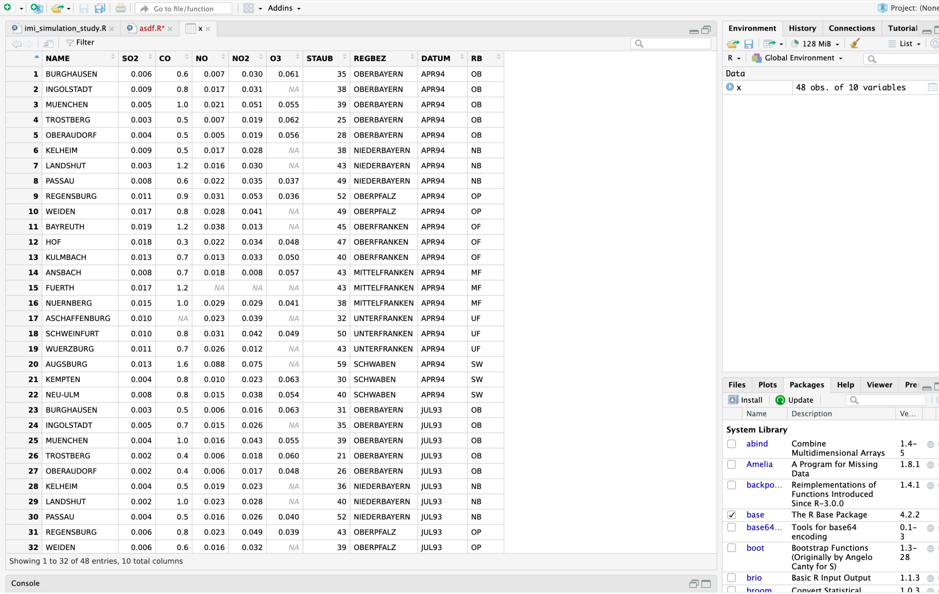
Gedeon Alexander Vogt

27.03.2023

Outline

Outline

The Problem



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Showing 1 to 32 of 48 entries, 10 total columns

Environment: Data — x: 48 obs. of 10 variables

Possible Solutions

(i) Drop rows containing NAs

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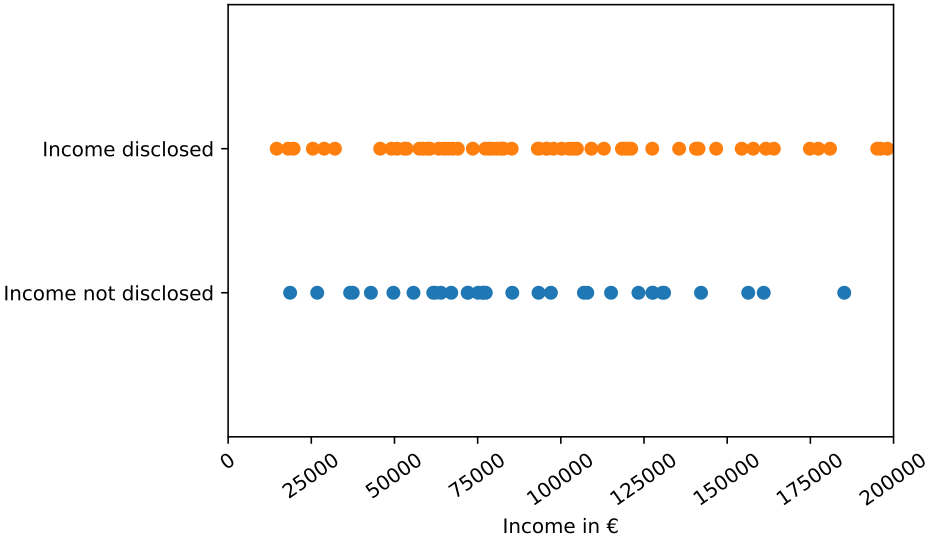
(iii) Multiple imputation

Possible Solutions – Drop Rows containing NAs



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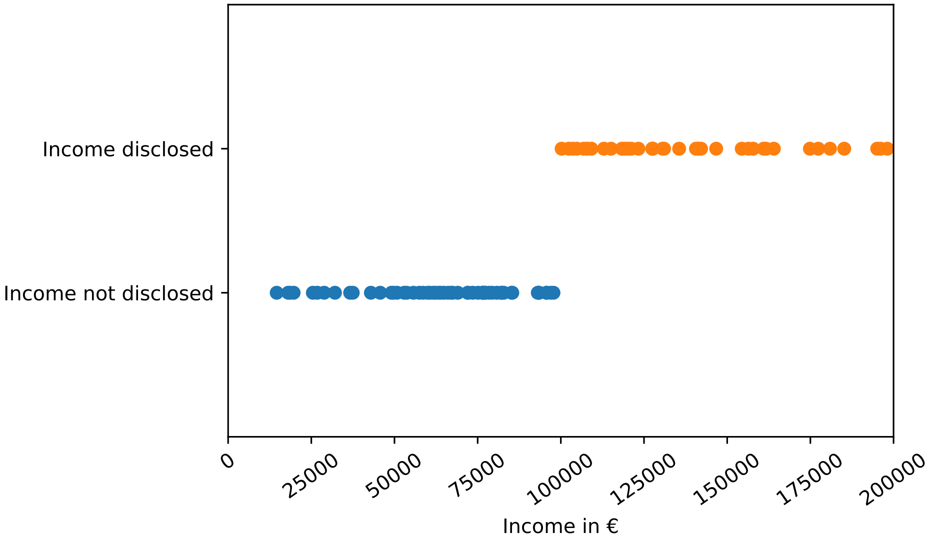
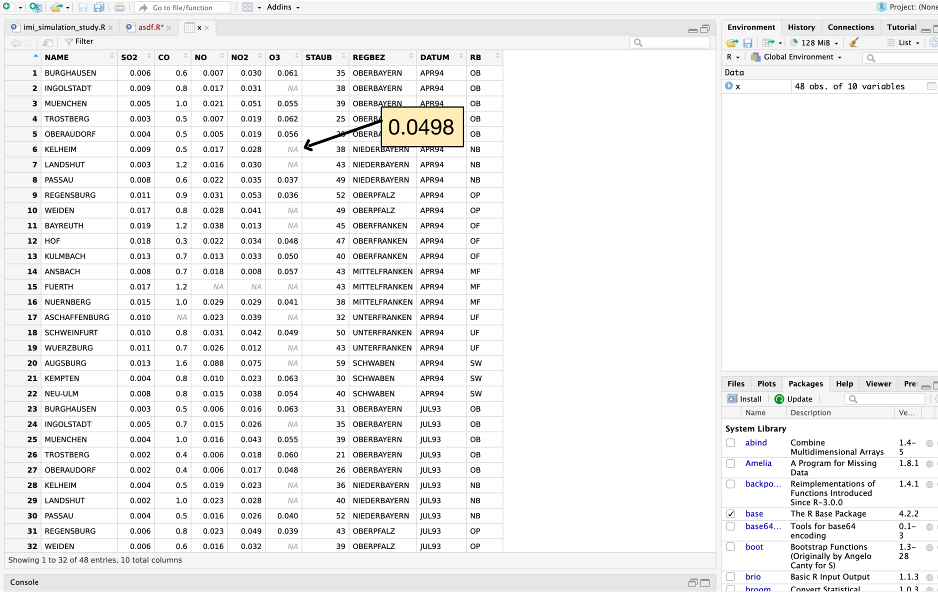


Possible Solutions – Single Imputation



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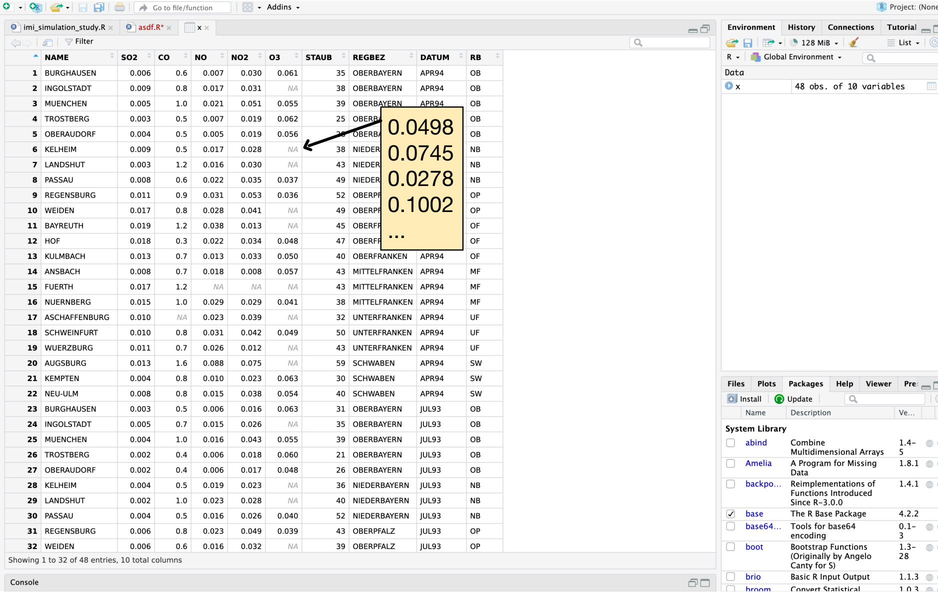

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– increased running time

Outline

Prior- & Posterior Distribution

$$\pi(\vartheta|Y_{obs}, D) \equiv \pi(\vartheta|Y_{obs}) = constant \times \pi(\vartheta) \times f(Y_{obs}|\vartheta)$$

Expected Value & Variance

3.3 Proposition: (Approximated Expected Value and Variance)

The expected values of ϑ can be approximated as

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where $\bar{\vartheta}$ is the corresponding MI estimator. Similarly we can approximate

$$Var_{\vartheta}(\vartheta | Y_{obs}) \approx \frac{1}{D} \sum_{d=1}^{D} Var_{\vartheta}(\vartheta | Y_{mis}^{(d)}, Y_{obs}) + \frac{1}{D-1} \sum_{d=1}^{D} [E_{\vartheta}(\vartheta | Y_{mis}^{(d)}, Y_{obs}) - \bar{\vartheta}]^2.$$

Within- and Between Variability

3.4 Definition: (Within- and Between Variability)

The summands of Prop. (3.3) can be denoted as

$$\hat{W} := \frac{1}{D} \sum_{d=1}^{D} Var_{\vartheta}(\vartheta | Y_{mis}^{(d)}, Y_{obs})$$

$$\hat{B} := \frac{1}{D-1} \sum_{d=1}^{D} [E_{\vartheta}(\vartheta | Y_{mis}^{(d)}, Y_{obs}) - \bar{\vartheta}]^2 = \frac{1}{D-1} \sum_{d=1}^{D} (\hat{\vartheta}_d - \bar{\vartheta})^2.$$

Asymptotic Distribution and finite Imputations Correction

3.5 Corollary:

Let $(\hat{\vartheta}_{(d)})_{n \in \mathbb{N}}$ be a sequence of random variables, that are i.i.d. with $\sigma^2 = Var(\hat{\vartheta}_{(1)}) = \hat{B} + \hat{W} < \infty$ and $\mu = E(\hat{\vartheta}_{(1)}) = \vartheta$. If $\hat{\vartheta}_{(d)}$ is a random vector and $\hat{\vartheta}_{(1)}, \hat{\vartheta}_{(2)}, \hat{\vartheta}_{(3)}, \ldots$ i.i.d., then for $D \longrightarrow \infty$, $\bar{\vartheta}$ is distributed as:

$$\bar{\vartheta} \overset{as}{\sim} N(\vartheta, \hat{B} + \hat{W}).$$

Asymptotic Distribution and finite Imputations Correction

3.6 Proposition: (Total Variance for finite D)

The total variance for finite D can be written as

$$\hat{V}_D := (1 + D^{-1})\,\hat{B} + \hat{W}.$$

For $D \longrightarrow \infty$ we get $\hat{V} := \hat{B} + \hat{W}$.

The algorithmic Search for the optimal Number of Imputations

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The Algorithm

1. **Start.** Select an initial number of imputed datasets, D_0, $\bar{\vartheta}_{D_0} = \sum_{i=1}^{D_0} \hat{\vartheta}_i / D_0$
2. **Update.** For $D > D_0$,

$$\bar{\vartheta}_{D+1} = \frac{D\,\bar{\vartheta}_D + \hat{\vartheta}_{D+1}}{D+1}$$

3. **Distance.** Compute: $d_{D+1} = d(\bar{\vartheta}_{D+1}, \bar{\vartheta}_D)$ using an appropriate distance.
4. **Stopping rule.** $d_j < \varepsilon$ for $j = D+1, ..., D+k_0$

(Nassiri et al., 2020)

Outline

Simulation Set Up

Settings:

- n-dimensional random vector: $\mathbf{Y}_i \sim N(\mu \mathbf{1}_n, \sigma^2 I_n + \tau J_n)$
- 100 random draws
- create missing data with *mice*
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(i) Compound-Symmetry (estimated parameters: μ, σ^2, τ)

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The Models:

(i) Compound-Symmetry (estimated parameters: μ, σ^2, τ)

(ii) Logistic Regression (estimated parameters: β_1, β_2, β_3)

Simulation Results

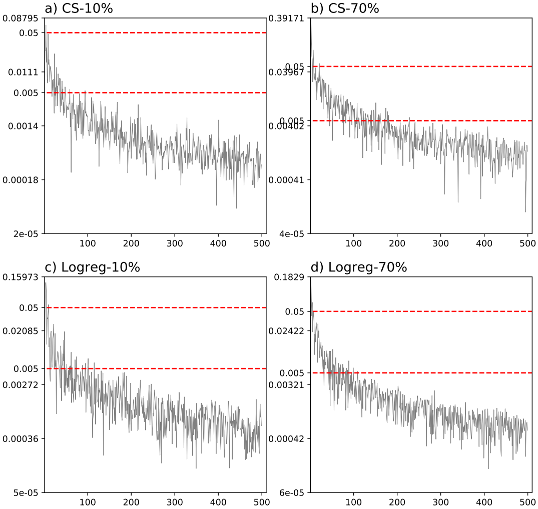

Figure: Convergence rates of the distances for CS and Logreg for $\sigma^2 = 0.25$, $\varrho = 0.1,\ k_0 = 5$ and $\beta = (0.2, -2, 0.5)^T$.

Simulation Results

Model	ϱ	ε	Mean	SD	$\mu(\beta_1)$ MAD	$\sigma^2(\beta_2)$ MAD	$\tau(\beta_3)$ MAD
			$k_0 = 1$				
CS-10%	0.1	0.005	25.55	10.11	0.02	0.02	0.01
		0.05	4.98	1.66	0.02	0.01	0.01
	0.9	0.005	12.86	4.79	0.14	0.02	0.29
		0.05	3.7	1.01	0.12	0.01	0.26
CS-70%	0.1	0.005	53.58	16.97	0.03	0.02	0.01
		0.05	10.28	3.52	0.03	0.02	0.01
	0.9	0.005	25.42	9.33	0.13	0.01	0.26
		0.05	5.72	2.49	0.12	0.02	0.25
Logreg-10%	0.1	0.005	22.57	9.01	0.11	0.14	0.06
		0.05	4.82	1.6	0.10	0.13	0.05
	0.9	0.005	22.16	9.26	0.11	0.15	0.11
		0.05	4.93	1.79	0.1	0.16	0.11
Logreg-70%	0.1	0.005	22.21	8.57	0.1	0.14	0.06
		0.05	4.91	1.46	0.1	0.15	0.05
	0.9	0.005	24.45	9.92	0.1	0.17	0.11
		0.05	4.65	1.48	0.1	0.15	0.11

(a) Validation steps: $k_0 = 1$

Model	ϱ	ε	Mean	SD	$\mu(\beta_1)$ MAD	$\sigma^2(\beta_2)$ MAD	$\tau(\beta_3)$ MAD
			$k_0 = 5$				
CS-10%	0.1	0.005	65.36	19.83	0.02	0.01	0.01
		0.05	7.73	2.67	0.02	0.01	0.01
	0.9	0.005	42.42	17.87	0.12	0.01	0.29
		0.05	5.36	2.43	0.12	0.01	0.23
CS-70%	0.1	0.005	152.23	27.11	0.02	0.01	0.01
		0.05	19.06	4.47	0.02	0.01	0.01
	0.9	0.005	91.55	26.09	0.11	0.02	0.24
		0.05	12.94	4.85	0.12	0.01	0.28
Logreg-10%	0.1	0.005	57.61	18.04	0.1	0.14	0.06
		0.05	6.52	2.42	0.1	0.12	0.06
	0.9	0.005	57.88	20.02	0.11	0.14	0.1
		0.05	7.56	3.43	0.11	0.16	0.12
Logreg-70%	0.1	0.005	60.93	17.99	0.1	0.14	0.05
		0.05	7.04	2.94	0.11	0.14	0.06
	0.9	0.005	60.7	22.02	0.11	0.16	0.11
		0.05	7.49	2.85	0.11	0.15	0.1

(b) Validation steps: $k_0 = 5$

Figure: Mean, SD and their mean absolute deviation from the true parameter (β_i corresponds to the Logreg model and μ, σ^2, τ the CS model) for selected D given $\sigma^2 = 0.25$ and different values for ε and ϱ using the Mahalanobis distance with $S = \hat{V}$.

Thank you for your attention!

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