Renewal Processes with discrete Lifetime Distribution

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Outline

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What is Renewal Theory?

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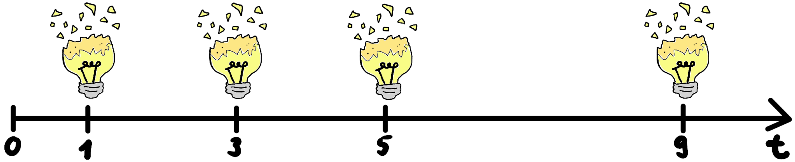

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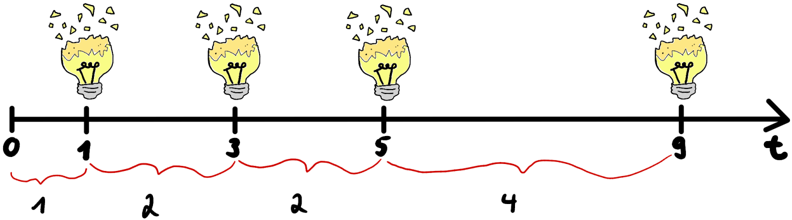
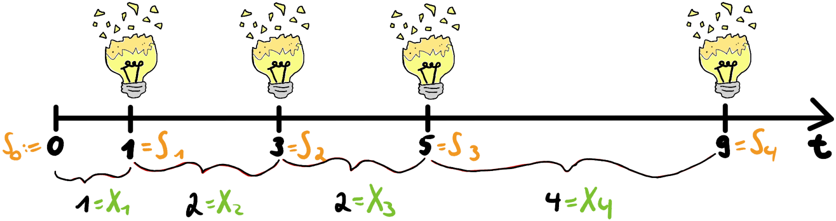


What is Renewal Theory?



Inter arrival times an their Distribution

2.2 Definition:

The common distribution of the sequence of i.i.d. random variables $(X_i)_{i\in\mathbb{N}}$ is referred to as *waiting time distribution of the renewal process*. It is depicted by $f = (f_n)_{n\in\mathbb{N}_0}$ with $f_n := \mathbb{P}(X_1 = n)$ for all $n \in \mathbb{N}_0$. The *cumulative distribution function* is denoted by $F(n) := \mathbb{P}(X_1 \leq n)$.

Some important Sequences

(i) renewal process:

$$S_n := \begin{cases} 0, & n = 0 \\ \sum_{i=1}^{n} X_i, & \text{else} \end{cases} \quad , n \in \mathbb{N}_0$$

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(ii) renewal counting process:

$$N(n) := \sum_{k=1}^{\infty} \mathbb{1}_{\{S_k \leq n\}} = \sum_{k=1}^{\infty} \mathbb{1}_{[0,n]}(S_k) \quad , n \in \mathbb{N}_0$$

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(iii) indicator sequence:

$$Z_n := \begin{cases} 1, & \text{if } n = S_m \text{ for some } m \geq 0 \\ 0, & \text{otherwise} \end{cases} = \sum_{m=0}^{\infty} \mathbb{1}_{\{S_m = n\}}$$

Renewal Function

(i)

$$\mathbb{E}N(n) = \sum_{k=1}^{\infty} \mathbb{P}(N(n) \geq k) = \sum_{k=1}^{\infty} \mathbb{P}(S_k \leq n) = \sum_{k=1}^{\infty} F^{*k}(n)$$

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(ii)

$$\Psi(n) := \mathbb{E}(N(n)+1) = \sum_{k=0}^{\infty} \mathbb{P}(S_k \leq n) = \sum_{k=0}^{\infty} \sum_{l=0}^{n} f_l^{*k}$$

Example

$$\mathbb{E}N(n) = F(n) + \sum_{k=0}^{n} f_k \, \mathbb{E}N(n-k)$$

Definition

2.11 Definition:

Let $(b_n)_{n\in\mathbb{N}_0}$ be a sequence with $\sum_{n=0}^{\infty} |b_n| < \infty$, $(f_n)_{n\in\mathbb{N}_0}$ with $f_0 < 1$ the common distribution of the i.i.d. non-negative random variables $X_1, \ldots, X_n$ and $(g_n)_{n\in\mathbb{N}_0}$ unknown variables, then we call

$$g_n = b_n + \sum_{k=0}^{n} f_k \, g_{n-k} \tag{1.1}$$

a *renewal equation in discrete time*.

Unique Solution

2.13 Theorem: (unique solution of renewal equations)
If $b_n \geq 0$ for $n \in \mathbb{N}$ and $\sum_{n=0}^{\infty} b_n \leq \infty$, then the discrete-time renewal equation (1.1) has the unique solution

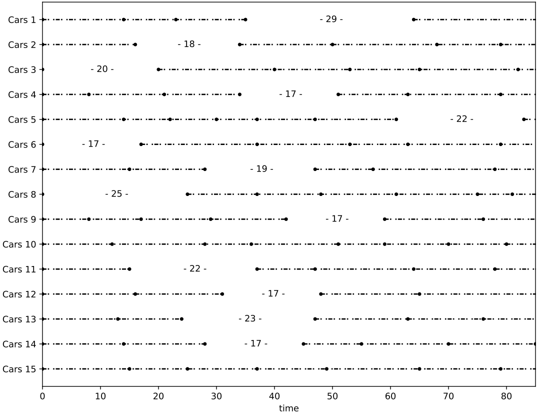
$$g_n = (u * b)_n \quad , n \in \mathbb{N}.$$

What is u?

$$u_n := \mathbb{P}(Z_n = 1) = 1 \cdot \mathbb{P}(Z_n = 1) + 0 \cdot \mathbb{P}(Z_n = 0)$$
$$= \mathbb{E}(Z_n) = 1 \cdot \sum_{k=0}^{\infty} \mathbb{P}(S_k = n) = \sum_{k=0}^{\infty} f_n^{*k}$$

Example I

Question: A pedestrian arriving at time 0 crosses the lane as soon as he sees a time interval $x > 0$ between two consecutive cars. How long must he wait, on average?



Example II

Step 1:
Consider a defective renewal process with $\sum_{n=0}^{\infty} f_n = p$, $p \in (0,1)$ and $f_0 = 0$. Define the lifetime of the process by

$$T := S_N = \sum_{i=1}^{N} X_i,$$

where $S_{N+1} = S_{N+2} = \ldots = \infty$.

Question: What is the distribution of N?

Solution:

Example II

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Solution: $\mathbb{P}(N = n) = p^n(1 - p)$

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Determine the distribution of T.

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$$\underbrace{\mathbb{P}(T = n)}_{=:g_n} = \underbrace{(1-p)\mathbb{1}_{\{n=0\}}}_{=:b_n} + \sum_{k=1}^{n} \mathbb{P}(T = n-k) f_k \overset{f_0=0}{=} b_n + \sum_{k=0}^{n} g_{n-k} f_k .$$

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Bring $\mathbb{P}(T = n)$ in the form of a renewal equation:

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Apply Theorem (2.13) and obtain the unique solution

$$\mathbb{P}(T = n) = (1-p)u_n.$$

Example IV

Step 3:

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- $x > 0$ the required time that the pedestrian is able to cross the street

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Define new distribution:

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Solution:

$$\mathbb{E}T = \sum_{n=0}^{\infty} n\mathbb{P}(T = n) = \sum_{n=0}^{\infty} n(1-p)u_n$$

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Age Replacement Policy I

The problem: The failure of a cooling unit damages a computer. It is cheaper to replace it beforehand.

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Question: What is the optimal cycle in which the cooling unit should be replaced?

Age Replacement Policy II

- $(X_k)_{k\in\mathbb{N}}$
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- $Z_j := \min\{X_j, N\}$, $j \in \mathbb{N}$ and $N \in \mathbb{N}$
- we obtain:

$$\mathbb{P}(Z_1 \leq k) = \begin{cases} \sum_{i=0}^{k} f_i, & N \geq k \\ 1, & N < k \end{cases}.$$

The goal: Determine the optimal N, N^*.

Age Replacement Policy III

Expected cost rate:

$$C(N) := \frac{c_1\mathbb{P}(X_1 \leq N) + c_2\mathbb{P}(X_1 > N)}{\mathbb{E}Z_1}$$

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c_1: costs if cooling unit fails
c_2: costs if cooling unit got replaced before failure

Age Replacement Policy IV

- $\frac{C(N+1)-C(N)}{(N+1)-N} = C(N+1) - C(N)$

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- $C(N+1) - C(N) \geq 0 \quad \Leftrightarrow \quad \underbrace{h_{N+1} \sum_{k=1}^{N} \sum_{j=k}^{\infty} f_j - \sum_{j=1}^{N} f_j}_{=:L(N)} \geq \frac{c_2}{c_1 - c_2}$

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  with $h_j := f_j / \sum_{i=j}^{\infty} f_i$.
- In order to verify the existence of a unique N^* it has to be shown that $L(N+1) - L(N) > 0$ and $L_\infty := \lim_{N\to\infty} L(N) < \infty$
- To show the second assertion we need the boundedness of h_j

Age Replacement Policy V

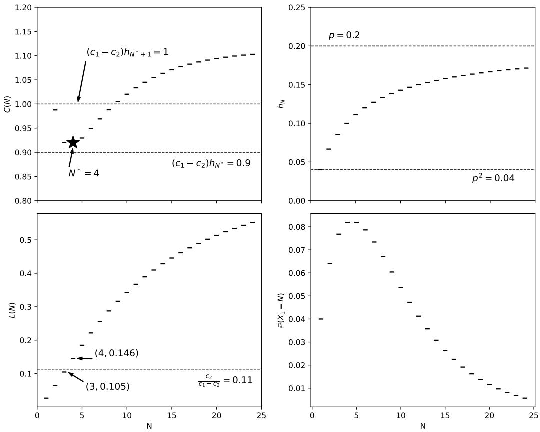

Figure: Exemplary age replacement policy with negative binomially distributed waiting times with $p = 0.2$ (e.g. $q = 0.8$) and replacement costs of $c_1 = 10$ and $c_2 = 1$.

Outline

Strong Law of large Numbers for Renewal Counting Processes

4.2 Theorem:

For a recurrent renewal process $(S_n)_{n\in\mathbb{N}}$ we have

$$\lim_{n\to\infty} \frac{N(n)}{n} = \frac{1}{\mu} \text{ a.s.}$$

with $\mu := \mathbb{E}X_1$.

Stopping Time

4.4 Definition:

It is $t > 0$, then

$$\tau(t) = \inf\{n \in \mathbb{N} \mid S_n > t\}, \quad \inf \varnothing := \infty$$

is called *first passage time*.

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4.5 Definition:

A random variable $\tau \in \mathbb{N}$ is called *stopping time* for $(X_n)_{n\in\mathbb{N}}$, if

$$\{\tau = n\} \in \underbrace{\sigma(X_1, ..., X_n)}_{\sigma\text{-algebra}} \quad \forall n \in \mathbb{N}.$$

Elementary Renewal Theorem

4.6 Theorem:

For a recurrent renewal process $(S_n)_{n\in\mathbb{N}}$ and τ being a stopping time we have

$$\frac{\mathbb{E}\tau(t)}{t} \xrightarrow{t\to\infty} \frac{1}{\mu}$$

with $1/\infty := 0$.

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with $1/\infty := 0$.

4.7 Corollary:

For a recurrent renewal process $(S_n)_{n\in\mathbb{N}}$ we have

$$\lim_{n\to\infty} \frac{\Psi(n)}{n} = \frac{1}{\mu}.$$

Elementary Renewal Theorem: Example

Consider i.i.d. random variables $X1, X2, ...$ that are Bernoulli distributed, i.e. $f_1 := p$ and $f_0 := 1 - p = q$ with $\mathbb{E}X_1 = p$.

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$$\Psi(n) = \sum_{k=0}^{n} u_k \overset{\text{eq. (2.5)}}{=} \frac{n+1}{p}.$$

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Therefore, we can conclude

$$\frac{\Psi(n)}{n} = \frac{n+1}{np} = \frac{1+1/n}{p} \xrightarrow{n\to\infty} \frac{1}{p} = \frac{1}{\mu},$$

which is consistent with Corollary (4.7).

Renewal Theorem

4.9 Theorem:

It is $u_n := \mathbb{P}(Z_n = 1)$ and $u_0 := 1$. Then

1. For a recurrent aperiodic renewal process $(S_n)_{n\in\mathbb{N}}$ we have
$$\lim_{n\to\infty} u_n = \frac{1}{\mu}$$
2. For a periodic recurrent renewal process of period $d > 1$ $(S_n)_{n\in\mathbb{N}}$ we have
$$\lim_{n\to\infty} u_{nd} = \frac{d}{\mu}$$
and $u_k = 0$ for all k not multiple of d.

Renewal Theorem: Example I

Preliminary remarks:

- probability generating function $\tilde{f}(z) := \mathbb{E}[z^X] = \sum_{n=0}^{\infty} f_n z^n$

Renewal Theorem: Example I

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- probability generating function $\tilde{f}(z) := \mathbb{E}[z^X] = \sum_{n=0}^{\infty} f_n z^n$
- $u(z) := \sum_{n=0}^{\infty} u_n z^n = \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} f_n^{*k} z^n = \sum_{k=0}^{\infty} \underbrace{\tilde{f}^k(z)}_{<1} = \frac{1}{1-\tilde{f}(z)}$

Renewal Theorem: Example II

Consider $(X_n)_{n\in\mathbb{N}}$ with common distribution

$$f_n := \mathbb{P}(X = n) = \begin{cases} p, & \text{if } n = 1 \\ q = 1 - p, & \text{if } n = 2 \end{cases}$$

Renewal Theorem: Example II

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Goal: determine $\lim_{n\to\infty} u_n$

Renewal Theorem: Example III

Step 1: Probability generating function

$$\tilde{f}(z) = \mathbb{P}(X = 1)z^1 + \mathbb{P}(X = 2)z^2 = pz + qz^2$$

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Step 2: Generating function of u

$$u(z) = \frac{1}{1 - \tilde{f}(z)} = \frac{1}{1+q} \underbrace{\frac{1}{1-z}}_{=:g_1(z)} + \frac{q}{1+q} \underbrace{\frac{1}{1+qz}}_{=:g_2(z)}$$

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Step 3: Derivatives of g_1 and g_2

$$g_1^{(n)} = \frac{n!}{(1-z)^{n+1}} \quad \text{and} \quad g_2^{(n)} = \frac{n!(-q)^n}{(1+qz)^{n+1}}$$

Renewal Theorem: Example IV

Step 4: According to According to Johnson et al. (1993, p. 59)

$$u_n = \frac{u^{(n)}(0)}{n!} = \ldots = \frac{1}{1+q} + \frac{q}{1+q}(-q)^n$$

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Step 5: Determine the limit

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Step 5: Determine the limit

$$\lim_{n\to\infty} u_n = \frac{1}{1+q} = \frac{1}{\mu}$$

Result is consistent with the renewal theorem!

Key Renewal Theorem

4.14 Theorem:

Consider a recurrent renewal process $(S_n)_{n\in\mathbb{N}}$ and a real sequence $(b_n)_{n\in\mathbb{N}}$.

1. If the process is aperiodic and $\sum_{n=0}^{\infty} |b_n| < \infty$, then

$$\lim_{n\to\infty} \sum_{k=0}^{n} b_k u_{n-k} = \frac{1}{\mu} \sum_{n=0}^{\infty} b_n$$

2. If the process is periodic of period $d > 1$ and if for a certain positive integer l, $0 \leq l < d$, we have $\sum_{n=0}^{\infty} | b_{l+nd} | < \infty$, then

$$\lim_{n\to\infty} \sum_{k=0}^{l+nd} b_k u_{l+nd-k} = \frac{d}{\mu} \sum_{n=0}^{\infty} b_{l+nd},$$

where u_n is defined as before.

Defective Renewal Theorem: Intro

The problem: $\sum_{n=0}^{\infty} f_n < 1$

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The solution: Scaling the probabilities up by a factor of $\gamma > 1$ such that

$$\sum_{n=0}^{\infty} \gamma^n f_n = 1$$

Defective Renewal Theorem

4.18 Theorem:

Let $(f_n)_{n\in\mathbb{N}}$ be a defective and aperiodic renewal distribution. If there exists $\gamma > 1$ such that

$$\sum_{n=0}^{\infty} \gamma^n f_n = 1$$

and for the non negative sequence $(b_n)_{n\in\mathbb{N}}$ it is

$$\sum_{n=0}^{\infty} \gamma^n \mid b_n \mid < \infty,$$

then the asymptotic solution of the renewal equation (2.11) satisfies

$$\lim_{n\to\infty} \gamma^n g_n = \frac{\sum_{k=0}^{\infty} \gamma^k b_k}{\sum_{k=0}^{\infty} k\gamma^k f_k}.$$

Renewal Reward Theorem

4.21 Definition:

Given is a renewal process $(S_n)_{n\in\mathbb{N}_0}$ based on the i.i.d. sequence $(X_i)_{i\in\mathbb{N}}$, $X_i \in \mathbb{N}_0$ with renewal distribution $(f_n)_{n\in\mathbb{N}_0}$ and an i.i.d. sequence of random variables $(Z_i)_{i\in\mathbb{N}}$ with distribution $(g_n)_{n\in\mathbb{N}_0}$ and $(X_i, Z_i)_{i\in\mathbb{N}}$. The process $(\hat{Z}(n))_{n\in\mathbb{N}}$ defined by $\hat{Z}(n) := \sum_{i=1}^{N(n)} Z_i$ is called *renewal reward process*.

Renewal Reward Theorem

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4.22 Theorem:

Given is the situation from Definition (4.21) with $\mathbb{E}(X_1) < \infty$ and $\mathbb{E}|Z_1| < \infty$. Then applies:

1. $\lim_{n\to\infty} \frac{N(n)}{n} = \frac{1}{\mathbb{E}(X_1)}$
2. $\lim_{n\to\infty} \frac{\hat{Z}(n)}{n} = \frac{\mathbb{E}(Z_1)}{\mathbb{E}(X_1)}$

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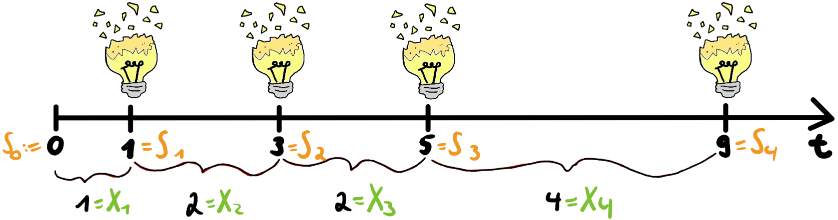
$S_0 := 0$
$1 = S_1$
$3 = S_2$
$5 = S_3$
$9 = S_4$
t
$1 = X_1$
$2 = X_2$
$2 = X_3$
$4 = X_4$

Definition

5.1 Definition:

Let $(X_i)_{i\in\mathbb{N}}$ be stochastically independent with X_1 having the distribution $b = (b_n)_{n\in\mathbb{N}_0}$ and X_i, $i \geq 2$, the distribution $f = (f_n)_{n\in\mathbb{N}_0}$. f and b are discrete probability distributions as defined in previous chapters. While f is called *common distribution*, b is referred to as the *initial distribution* (or *delayed distribution*). Then, $(S_n^D)_{n\in\mathbb{N}_0}$ defined as

$$S_n^D := \begin{cases} 0, & n = 0 \\ \sum_{i=1}^n X_i, & n \in \mathbb{N} \end{cases}$$

is called *delayed renewal process*. The corresponding renewal counting process is denoted as $(N^D(n))_{n\in\mathbb{N}_0}$ with $N^D(n) := \sum_{k=1}^\infty \mathbb{1}_{\{S_k \leq n\}}$.

Renewal Theorem for Delayed Renewal Processes

5.2 Theorem:

Consider a delayed recurrent renewal process $(S_n)_{n\in\mathbb{N}}$ with positive inter arrival times and an initial distribution $b = (b_n)_{n\in\mathbb{N}_0}$.

1. If the process is aperiodic, then

$$\lim_{n\to\infty} v_n = \frac{1}{\mu}\sum_{n=0}^{\infty} b_n.$$

2. If the chain is periodic of period $d > 1$, then for any positive integer l, $0 \le l < d$,

$$\lim_{n\to\infty} v_{l+nd} = \frac{d}{\mu}\sum_{n=0}^{\infty} b_{l+nd}.$$

Where $v_n := \sum_{k=0}^{\infty} \mathbb{P}(S_k = n)$.

Properties

This distribution is chosen so that v_n, which we have previously defined as the probability that a renewal will take place at time n, is independent of n.

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5.4 Proposition:

Let $(S_n)_{n\in\mathbb{N}}$ be a recurrent delayed renewal process with waiting times $(X_n)_{n\in\mathbb{N}_0}$ and $\mu := \mathbb{E}X_1 < \infty$.Then, $\mathbb{P}(X_1 = n) := 1/\mu\mathbb{P}(X_1 > n)$ is the unique choice for the initial distribution of the delayed renewal chain such that $v_n \equiv$ constant for all $n \in \mathbb{N}_0$. Moreover, this common constant is $1/\mu$. In this case $(S_n)_{n\in\mathbb{N}}$ is a stationary renewal process with the initial distribution being the stationary distribution.

Outline

Definition

2.19 Definition:

Given is a renewal process $(S_n)_{n\in\mathbb{N}}$:

1. We call $U(n) := n - S_{N(n)}$ *age* at time n or *backward recurrence time*.
2. We call $V(n) := S_{N(n)+1} - n$ the *residual lifetime* at time n or the *forward recurrence time*.
3. We call $L(n) := X_{N(n)+1} = S_{N(n)+1} - S_{N(n)} = U(n) + V(n)$ the *total lifetime* at time n.

Distribution of Lifetime Processes

Their distribution as a function of u_n:

- $\mathbb{P}(U(m) = n) = u_{m-n}\mathbb{P}(X_1 > n)$
- $\mathbb{P}(V(m) = n) = \sum_{j=0}^{m} \mathbb{P}(X_1 = m + n - j)u_j$
- $\mathbb{P}(L(m) = n) = \mathbb{P}(X_1 = n) \sum_{j=m-n+1}^{m} u_j$

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6.1 Theorem:

Assume that $0 < \mu < \infty$. Then

1. $\lim_{m\to\infty} \mathbb{P}(U(m) = n) = \frac{1}{\mu}\mathbb{P}(X_1 > n)$
2. $\lim_{m\to\infty} \mathbb{P}(V(m) = n) = \frac{1}{\mu}\mathbb{P}(X_1 \geq n)$
3. $\lim_{m\to\infty} \mathbb{P}(L(m) = n) = \frac{n}{\mu}\mathbb{P}(X_1 = n)$.

Thank you for your attention!

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