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# Renewal Processes with discrete Lifetime Distribution

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# Contents

# 1 Preface

Renewal theory is an area of probability theory with applications in fields such as operations research, engineering, and actuarial science. It focuses on the study of processes that reset or "renew" after certain events occur at random intervals. This theory is particularly useful in modeling and analyzing phenomena such as equipment failures, system repairs, and lifetimes of components in reliability theory. This thesis, titled "Renewal Processes with Discrete Lifetime Distributions", specifically addresses renewal processes where the time between successive events follows a discrete distribution.

The thesis begins with an introduction to the fundamentals of discrete renewal processes in Chapter 2. In addition to that, this chapter covers key concepts, such as renewal equations and provides an introduction to Lifetime Processes, which are taken up again later in Chapter 6.

In Chapter 3 the thesis further explores a longer application-related example, where the aim is to find a cost-optimized cycle for the preventive replacement of components. This always makes sense if replacing components that are still functioning is more cost-effective than replacing those that are already broken (e.g. if the failure of a component causes damage to the higher-level system and thus drives up the cost of replacement).

Limit theorems, presented in Chapter 4, play a crucial role in understanding the long-term behavior of renewal processes. This section explores classical results as the elementary renewal theorem and the renewal theorem in the recurrent and defective case. It concludes with the renewal reward theorem, which is used particularly in actuarial mathematics and relates a renewal process to a sequence of damage amounts.

In Chapter 5, delayed renewal processes are considered. These are processes where the first event occurs after a delay, affecting the overall timing and analysis of the process. Both general delayed renewal processes and stationary renewal processes are discussed.

The study of lifetime processes, explored in the final Chapter 6, results in a theorem that targets the limit behavior of the distribution of the three lifetime processes that were introduced in the first chapter. Subsequently, a closed form for the tail distribution of the forward recurrence time is determined by using results from the Chapters 4 and 5.

By focusing on discrete lifetime distributions, this thesis aims to provide a comprehensive understanding of renewal processes and highlighting their practical applications by providing examples throughout the paper.

# 2 Discrete Renewal Processes

We start this section with a simple example to get a better understanding of how renewal theory can be applied to real-world problems. Consider the light bulb of a desk lamp. It is an utility item and comes with a certain limited expected life time of for instance 1000 hours. Not all light bulbs will break after exactly 1000 hours. Some will live a little bit longer and some will need a replacement some time earlier. In fact, a light bulb could burn out at any time which implies a continuous distribution for the lifetime. A popular distribution to model those lifetimes is the exponential distribution which leads to the so called Poisson process.

After a random amount of time, the light bulb burns out and needs a replacement. We always assume that the needed amount of time for replacing the bulb is zero and therefore the transition from one bulb to its successor is fluid. Depending on the context the bulb's life time is also considered as waiting time (e.g. waiting for a bus) or inter arrival time (e.g. counting taxis at an airport). The act of replacing is also called renewal, since the life of the observed object is reset and starts over again.

When we denote $X_i$ as the waiting time between $(i-1)$-th and $i$-th light bulb, we obtain a sequence of waiting times. We assume that all waiting times are equally distributed and independent from each other. The goal of the renewal theory is to make some statements about the behavior of the sequence, for instance, how many renewals take place in a certain time interval in expectation and how does that number change when we do not start the observation from the beginning but some time later?

Now suppose that the functionality of the light bulb is not steadily monitored but only checked once a day. Thus the observations are not continuous anymore, they are discrete and require a different, discrete distribution which will ultimately lead to slightly different statements that we can make.

## 2.1 Fundamentals

We adapt the following definition provided by Barbu and Limnios (2008, p. 18) by allowing the random variables to take the value zero. Note that it will be necessary to exclude the zero from time to time in the following.

**2.1 Definition:**
Consider a sequence of i.i.d. random variables $(X_i)_{i\in\mathbb{N}}$, $X_i \in \mathbb{N}_0$ for all $i \in \mathbb{N}$. The corresponding summation process

$$S_n := \begin{cases} 0, & n = 0 \\ \sum_{i=1}^{n} X_i, & \text{else} \end{cases} \quad , n \in \mathbb{N}_0$$

is called *renewal process* and every $S_n$ is called *arrival time* or *renewal time*.

Barbu and Limnios (2008, p. 19) suggest to consider $(S_n)_{n\in\mathbb{N}_0}$ as the successive instants when the renewals take place while $(X_i)_{i\in\mathbb{N}}$ depict the inter arrival times, therefore like already mentioned above the time difference between two renewals. In the case that $X_i =$

$\infty$ for some $i \in \mathbb{N}$, we get that $S_i = S_{i+1} = ... = \infty$. Thus, the property of a monotonically increasing sequence only holds when all $X_i < \infty$.

Since the discrete renewal processes are subject of this paper, the inter arrival times take on discrete non-negative values. Thus, we map each natural number including the zero to a probability. This results in the common distribution of $(X_i)_{i\in\mathbb{N}}$, $X_i \geq 0$. We define it as follows.

### 2.2 Definition:

The common distribution of $(X_i)_{i\in\mathbb{N}}$ is referred to as *waiting time distribution of the renewal process.* It is depicted by $f = (f_n)_{n\in\mathbb{N}_0}$ with $f_n := \mathbb{P}(X_1 = n)$ for all $n \in \mathbb{N}_0$. The *cumulative distribution function* is denoted by $F(n) := \mathbb{P}(X_1 \leq n)$.

At this point it is important to emphasize a particular phenomenon if we assume $X_i > 0$. Thus, it is $f_0 = 0$. In contrast to the continuous case, the minimum waiting time for discrete renewal processes is therefore 1 time unit. This leads to the fact, that in a time interval of $n$ units, $n$ renewals can occur at most (Babu and Limnios, 2008, p. 20).

Because of his packed time schedule the janitor who checks the functionality of the light bulbs only has time to come by every three days. Thus, no renewal will ever be recorded between those three days. Hence the distribution of the inter arrival times only concentrates on a certain lattice and the probability is otherwise zero. This brings us to a particular case of discrete distributions that are called lattice distribution (see Petrov, 1975).

### 2.3 Definition:

The discrete renewal distribution $(f_n)_{n\in\mathbb{N}_0}$ is called *lattice* (resp., *non-lattice*) if there exists an integer $d > 0$ such that the probability distribution concentrates on a set of points of the form $a + dn$ with $a \in \mathbb{R}$. The number $d$ is called the *span of the renewal distribution.* The largest $d$ that satisfies those requirements is called *period.* In the case of $a = 0$, we obtain an *arithmetic* distribution. Furthermore, we call a distribution *periodic* if the $d > 1$ (resp., *aperiodic*).

Given this definition it can be stated that discrete renewal distributions are a particular case of lattice distributions. In the following, we assume a span of $d = 1$.

Define $\overline{f} := \sum_{n=0}^{\infty} f_n$. This is the probability that at least one renewal will occur at some point (Barbu and Limnios, 2008, p. 19), i.e. the probability that $X_1$ is finite. Conversely, the probability that the lifetime of a bulb is unlimited is $1-\overline{f}$. Based on this, we now define a number of expressions that we will later use for some theorems (Barbu and Limnios, 2008, p. 19).

### 2.4 Definition:

A renewal process is called *recurrent* if $\overline{f} = \mathbb{P}(X_1 < \infty) = 1$ and *transient* if $\overline{f} = \mathbb{P}(X_1 < \infty) < 1$. In the case that the renewal process is transient, the random variables $X_i$, $i \in \mathbb{N}$ are referred to as *improper* or *defective.* We denote the defect as $\alpha := P(X_1 = \infty)$.

As mentioned at the beginning, we would like to be able to count renewals later on. Therefore, there is another process in addition to the renewal process the so-called renewal

| n | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | ... |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $X_n$ | - | 1 | 2 | 2 | 4 | 3 | 2 | 5 | 2 | 1 | ... |
| $S_n$ | 0 | 1 | 3 | 5 | 9 | 12 | 14 | 19 | 21 | 22 | ... |
| N(n) | 0 | 1 | 1 | 2 | 2 | 3 | 3 | 3 | 3 | 4 | ... |
| $Z_n$ | 1 | 1 | 0 | 1 | 0 | 1 | 0 | 0 | 0 | 1 | ... |

Table 1: Comparison of the four sequences (waiting times sequence, renewal time sequence, renewal counting sequence, indicator sequence) using an example with concrete values

counting process. It is also an stochastic process which is right continuous, monotonically increasing and has discontinuities of height one (Barbu and Limnios, 2008, p. 24).

### 2.5 Definition:

The stochastic process $(N(n))_{n\in\mathbb{N}_0}$ defined by

$$N(n) := \sum_{k=1}^{\infty} \mathbb{1}_{\{S_k \leq n\}} = \sum_{k=1}^{\infty} \mathbb{1}_{[0,n]}(S_k) \quad , n \in \mathbb{N}_0$$

and is called *renewal counting process.*

The renewal counting process can also be expressed with the help of supremum and infimum. It can be written as

$$N(t) = \sup\{n \in \mathbb{N}_0;\, S_n \leq t\} = \inf\{n \in \mathbb{N}_0;\, S_{n+1} > t\}.$$

In addition to that, the respective process is closely linked to the renewal process by the following lemma formulated by William Feller in 1949.

**2.6 Lemma:** (Feller's Lemma (1949))

$$N(t) \geq n \iff S_n \leq t \quad , \forall n \in \mathbb{N}_0 \, \forall t > 0.$$

Another sequence $(Z_n)_{n\in\mathbb{N}}$ that is worth considering, since we will use it in upcoming theorems, indicates whether a renewal happens at time $n$. Like in Barbu and Limnios (2008, p. 20) we define it as

$$Z_n := \begin{cases} 1, & \text{if } n = S_m \text{ for some } m \geq 0 \\ 0, & \text{otherwise} \end{cases} = \sum_{m=0}^{\infty} \mathbb{1}_{\{S_m = n\}}.$$

As mentioned above, in the case of $X_i > 0$ there are at most $n$ renewals in an interval of length $n$. Therefore, $(Z_n)_{n\in\mathbb{N}}$ has to be finite and it is only necessary to sum the sequence elements for $m = 0, ..., n$. Furthermore, it is worth noting that $Z_0 = 1$ since it is $S_0 = 0$ by definition. Hence, time zero is a renewal time.

To get a better understanding how those four sequences are intertwined, consider table 2.1. The sequence $(X_n)_{n\in\mathbb{N}}$ gives the waiting time between renewals and $(S_n)_{n\in\mathbb{N}}$ sums the waiting times up so that we obtain a certain time for each renewal. Up to this point, the indices of both sequences do not correspond to units of time. The sequence $(Z_n)_{n\in\mathbb{N}}$ changes that fact by considering its indices as point in time and indicates the occurrence of a renewal by an one. This makes especially sense in this setting of discrete renewal processes since the waiting times are discrete, the renewal times are as well and therefore the indices can be perfectly interpreted as time units. Note that every time $Z_n = 1$ the value of $N(n)$ increases by one, i.e. $N(n) = N(n-1) + 1$. Hence, Barbu and Limnios (2008, p. 24) suggest that the link between both can be written as

$$N(n) = \sum_{k=1}^{n} Z_k.$$

Keep in mind, that the summation starts at $k = 1$ instead of 0 which means, that the renewal counting process only counts new renewals and disregards the initial one which is indicated by $Z_0$.

In the following, we would like to be able to add random variables together to examine the distribution of their sum. Therefore, we introduce the convolution product for discrete random variables which can be understood as functions from $\mathbb{N}_0$ to $\mathbb{R}$. Similar to Barbu and Limnios (2008, p. 20) we define it the following:

### 2.7 Definition:

Let $f, g : \mathbb{N}_0 \to \mathbb{R}$. The discrete convolution product of $f$ and $g$ is the function $f * g : \mathbb{N}_0 \to \mathbb{R}$ defined by

$$(f * g)(n) := \sum_{k=0}^{n} f(n-k)g(k), \quad n \in \mathbb{N}_0.$$

In the case of $X$ and $Y$ being independent with their respective distributions $f$ and $g$ that possess the above defined properties, the convolution product $f * g$ is the distribution of $X + Y$. By induction it can be shown, that the distribution of any arbitrary sum of independent random variables $X_1 + ... + X_n$, with $n \in \mathbb{N}$ is the convolution product of their distribution. If all independent random variables share a common distribution $f$, the distribution of their sum is the $n$-fold convolution product (Barbu and Limnios, 2008, pp. 20-21).

### 2.8 Definition:

Let $f, g : \mathbb{N}_0 \to \mathbb{R}$. The $n$-fold convolution of $f$ is the function

$$f^{*n} : \mathbb{N}_0 \to \mathbb{R}, \quad k \mapsto \underbrace{(f * ... * f)}_{n\text{-times}}(k)$$

for $n \geq 2$. The edge cases are handled as $f^{*0}(k) := \mathbb{1}_{\{0\}}(k)$ and $f^{*1}(k) := f(k)$.

Applying this concept to our topic provides a distribution for our arrival time $S_n$ which is the sum of the i.i.d. non-negative integer-valued random variables that represent the waiting times. That distribution is simply the $n$-fold convolution product of their distribution, i.e. $f^{*n}$.

In the following we want to derive our first theorem which will provide a condition for a renewal process to be transient and in the latter event, a probability that a renewal will ever occur. In order to do so, we define the probability that a renewal takes place at instance $n$ as $u_n := \mathbb{P}(Z_n = 1)$ with $u_0 := 1$. This probability can also be expressed with convolution products by

$$\begin{aligned} u_n =& \mathbb{P}(Z_n = 1) = 1 \cdot \mathbb{P}(Z_n = 1) + 0 \cdot \mathbb{P}(Z_n = 0) \\ =& \mathbb{E}(Z_n) = 1 \cdot \sum_{k=0}^{\infty} \mathbb{P}(S_k = n) = \sum_{k=0}^{\infty} f_n^{*k}. \end{aligned}$$

(Barbu and Limnios, 2008, p. 21) and therefore

$$u_n = \sum_{k=0}^{\infty} f_n^{*k}. \tag{2.1}$$

Again, if we only consider positive waiting times, i.e. $X_i > 0$, we only sum for $k = 0, ..., n$, due to $S_k \geq k$. In addition to that, we can conclude that

$$\mathbb{P}(S_1 = 0) = \mathbb{P}(\sum_{i=1}^{1} X_i = 0) = \mathbb{P}(X_1 = 0) = 0.$$

If $Z_n = 1$ it means that by definition there exists an $m \in \{1, ..., n\}$ such that $S_m = n$. Assume that $S_1 = n$ is given, then we have already found such $m$, i.e. $m = 1$. For this reason it is $\mathbb{P}(Z_n = 1 \,|\, S_1 = n) = 1$. In addition to that, it is

$$\begin{aligned} \mathbb{P}(Z_n = 1 \,|\, S_1 = k) &= \mathbb{P}(S_m = n \text{ for some } m \geq 0 \,|\, S_1 = k) \\ &= \mathbb{P}(k + \sum_{i=2}^{m} X_i = n \text{ for some } m \geq 0) \\ &= \mathbb{P}(\sum_{i=1}^{m-1} X_{i+1} = n - k \text{ for some } m \geq 0) \\ &\overset{\text{iid}}{=} \mathbb{P}(S_{m-1} = n - k \text{ for some } m \geq 0) = \mathbb{P}(Z_{n-k} = 1). \end{aligned} \tag{2.2}$$

Using these facts, we can conclude the following

$$\begin{aligned} u_n = \mathbb{P}(Z_n = 1) &= \sum_{k=0}^{n} \mathbb{P}(S_1 = k)\, \mathbb{P}(Z_n = 1 \,|\, S_1 = k) \\ &= \mathbb{P}(S_1 = n)\, \mathbb{P}(Z_n = 1 \,|\, S_1 = n) \,+\, \sum_{k=1}^{n-1} \mathbb{P}(S_1 = k)\, \mathbb{P}(Z_n = 1 \,|\, S_1 = k) \end{aligned}$$

$$
\begin{aligned}
&+ \underbrace{\mathbb{P}(S_1 = 0)}_{=0 \text{ bc } X_i > 0} \mathbb{P}(Z_n = 1 \,|\, S_1 = 0) \\
&= \mathbb{P}(S_1 = n) + \sum_{k=1}^{n-1} \mathbb{P}(S_1 = k) \, \mathbb{P}(Z_n = 1 \,|\, S_1 = k) \\
&\overset{\text{eq. (2.2)}}{=} \mathbb{P}(X_1 = n) + \sum_{k=1}^{n-1} \mathbb{P}(X_1 = k) \, \mathbb{P}(Z_{n-k} = 1) \\
&= f_n + \sum_{k=1}^{n-1} f_k \, u_{n-k}
\end{aligned}
$$

and obtain the equation

$$
u_n = f_n + \sum_{k=1}^{n-1} f_k \, u_{n-k} \tag{2.3}
$$

which is the discrete time version of what we will later call a renewal equation (Barbu and Limnios, 2008, p. 21). Doing one more step we get

$$
u_n = \sum_{k=1}^{n} f_k \, u_{n-k} = (f * u)_n
$$

where latter is again the convolution product.

We can now formulate a necessary and sufficient condition to check if a renewal process is transient (Barbu and Limnios, 2008, p. 22).

**2.9 Theorem:**

A renewal chain is transient iff $\overline{u} := \sum_{n=0}^{\infty} u_n < \infty$. If this is the case, the probability that a renewal will ever occur is given by $\overline{f} = (\overline{u} - 1)/\overline{u}$.

This theorem can be proved by using generating functions, which transform the convolution product into ordinary products. Both sequences, $(f_n)_{n \in \mathbb{N}_0}$ and $(u_n)_{n \in \mathbb{N}_0}$, possess generating functions which are related, i.e. they can be expressed by using the respective other.

**2.10 Example:** (Bernoulli distribution part 1)

Consider i.i.d. random variables $X_1, X_2, ...$ that are Bernoulli distributed, i.e. $f_1 := p$ and $f_0 := 1 - p = q$ with $\mathbb{E}X_1 = p$. According to Cramer and Kamps (2020, pp. 209-210) the sum of those variables is distributed like $\sum_{i=1}^{n} X_i \sim bin(n, p)$ and therefore, the $n$-fold convolution product is given by (Cramer and Kamps, 2020, p. 173)

$$
f_k^{*n} = \binom{n}{k} p^k (1 - p)^{n-k}.
$$

With equation (2.1) we obtain

$$u_n = \sum_{n=0}^{\infty} f_k^{*n} = \sum_{n=0}^{\infty} \binom{n}{k} p^k (1-p)^{n-k} = \frac{p^k}{(1-p)^k} \sum_{n=0}^{\infty} \binom{n}{k} \underbrace{(1-p)^n}_{<1}$$
$$= \frac{p^k}{(1-p)^k} \frac{(1-p)^k}{(1-(1-p))^{k+1}} = \frac{p^k}{p^{k+1}} = \frac{1}{p} = \frac{1}{\mu}. \tag{2.4}$$

Furthermore, it is

$$\overline{u} = \sum_{n=0}^{\infty} u_n = \sum_{n=0}^{\infty} \frac{1}{p} = \infty,$$

therefore the renewal process is not transient (see Theorem (2.9)). This means, that $\mathbb{P}(X_1 < \infty) = 1$ which implies a recurrent renewal process (Definition (2.4)).

We call $\mathbb{E}N(n)$ the *renewal function*, which gives the expected number of renewals in the interval $[0, n]$. $(N(n))_{n \geq 0}$ is the renewal counting process as defined above. The renewal function can be written as the sum of convolution products:

$$\mathbb{E}N(n) = \sum_{k=1}^{\infty} \mathbb{P}(N(n) \geq k) \overset{\text{La. (2.6)}}{=} \sum_{k=1}^{\infty} \mathbb{P}(S_k \leq n) = \sum_{k=1}^{\infty} F^{*k}(n)$$

Note that $S_k \geq k$ and thus $\mathbb{P}(S_k \leq n) = 0$ for all $k > n$ if $X_i > 0$. Barbu and Limnios (2008, p. 25) suggest, that sometimes for technical reasons it can be an advantage to not use the renewal function, that does not consider the renewal at the origin, but the function

$$\Psi(n) := \mathbb{E}(N(n) + 1) = \sum_{k=0}^{\infty} \mathbb{P}(S_k \leq n) = \sum_{k=0}^{\infty} \sum_{l=0}^{n} f_l^{*k} \quad , n \in \mathbb{N}.$$

This function can also be written using the above defined sequence $(u_n)_{n \in \mathbb{N}}$ (Barbu and Limnios, 2008, p. 26):

$$\Psi(n) = \mathbb{E}(N(n) + 1) = \mathbb{E}\left( \sum_{k=0}^{n} Z_k \right) = \sum_{k=0}^{n} u_k. \tag{2.5}$$

## 2.2 Renewal Equations

With equation (2.3) we have seen a first example for a renewal equation. Another vivid example is given by Pinsky and Karlin (2011, p. 379). Consider again the example of light bulbs mentioned at the beginning where we start with a new bulb which is replaced every time it burns out. The corresponding renewal equation is

$$\mathbb{E}N(n) = F(n) + \sum_{k=0}^{n} f_k \, \mathbb{E}N(n-k), \tag{2.6}$$

which is solved by $\mathbb{E}N(n)$. In the case that the first bulb expires at some time $k \leq n$, we obtain its renewal and the average of $\mathbb{E}N(n-k)$ in the interval $[0,n]$. In other words, we obtain the first term $F(n)$ by conditioning on the first renewal being later than $n$ and the second summand, which is basically a convolution product by conditioning on the first occurrence of the first renewal and shifting the starting point so that this first renewal epoch is now considered the beginning (Resnik, 2002, p. 198). The equation, (2.6), as well as equation (2.3) are only specific examples for renewal equations. In general we define (Pinsky and Karlin, 2011, p. 380)

**2.11 Definition:**

Let $(b_n)_{n\in\mathbb{N}_0}$ be a sequence with $\sum_{n=0}^{\infty}|b_n| < \infty$, $(f_n)_{n\in\mathbb{N}_0}$ with $f_0 < 1$ the common distribution of the i.i.d. non-negative random variables $X_1, ..., X_n$ and $(g_n)_{n\in\mathbb{N}_0}$ unknown variables, then we call

$$g_n = b_n + \sum_{k=0}^{n} f_k\, g_{n-k} \tag{2.7}$$

a *renewal equation in discrete time.*

**2.12 Example:**

Another example is the so called *basic renewal equation* (Bremaud, 2017, pp. 442-443). Assume $X_i > 0$, $i = 1, 2, ...$ and define the Dirac sequence as $(\delta_n)_{n\in\mathbb{N}}$ by $\delta_0 = 1$ and $\delta_n = 0$ for $n \geq 1$. When we take equation (2.7) and set $b_n = \delta_n$ we obtain the above mentioned basic renewal equation. Its solution is called *fundamental solution.* We label this solution as $(h_n)_{n\in\mathbb{N}}$ with $h_0 = 1$ and

$$h_n = \sum_{k=0}^{n} f_k\, h_{n-k}$$

for $n \geq 1$. The special aspect of this solution is its rather simple and vivid interpretation we obtain in the case of $X_i > 0$. We can interpret $h_n$ as the probability that $n$ is a renewal time (Pinsky and Karlin, 2011, p. 380). We can prove this by showing that $g_n$ is the solution of the renewal equation with $b_n = \delta_n$, i.e. $h_n = g_n$. As $\delta_0 = 1$ we obtain $h_0 = g_0 = 1$. According to Bremaud (2017, p. 443) we get for $n \geq 1$:

$$\begin{aligned}
h_n &= \sum_{k=0}^{n-1} \mathbb{P}(n \text{ is renewal, last renewal strictly before n is k}) \\
&= \sum_{i=0}^{\infty} \sum_{k=0}^{n-1} \mathbb{P}(X_{i+1} = n-k,\, k = S_i) \\
&\overset{\text{s.i.}}{=} \sum_{i=0}^{\infty} \sum_{k=0}^{n-1} \mathbb{P}(S_{i+1} = n-k)\, \mathbb{P}(k = S_i) \\
&= \sum_{k=0}^{n-1} \mathbb{P}(S_{i+1} = n-k) \left( \sum_{i=0}^{\infty} \mathbb{P}(k = S_i) \right)
\end{aligned}$$

$$= \sum_{k=0}^{n-1} \mathbb{P}(S_{i+1} = n-k)\, g_k$$
$$= \sum_{k=0}^{n-1} g_k\, f_{n-k} \overset{f_0=0}{=} \sum_{k=0}^{n} f_k\, g_{n-k}.$$

To conclude the assertion we need uniqueness of the solution which is provided by Barbu and Limnios (2008, p. 23) or Pinsky and Karlin (2011, p. 381). This applies not only to this particular instance but to renewal equations in general.

**2.13 Theorem:** (unique solution of renewal equations)
If $b_n \geq 0$ for $n \in \mathbb{N}$ and $\sum_{n=0}^{\infty} b_n \leq \infty$, then the discrete-time renewal equation (2.7) has the unique solution

$$g_n = (u * b)_n \quad , n \in \mathbb{N}.$$

**2.14 Example:** (Bernoulli distribution part 2)
Let $(b_n)_{n\in\mathbb{N}_0}$ be the Dirac sequence and the renewal process be defined as in Example (2.10). Then, it is $\sum_{n=0}^{\infty} b_n = 1 < \infty$. According to Theorem (2.13) the solution of the renewal equation is given by

$$g_n = (u * b)_n = \sum_{k=0}^{n} u_{n-k} b_k = u_n \delta_0 + \sum_{k=1}^{n} u_{n-k} \underbrace{\delta_k}_{=0} \overset{\text{eq. (2.4)}}{=} u_n = \frac{1}{p}.$$

It remains to be shown that we found a valid solution. We do so by using induction. For $n = 0$ we get

$$\delta_0 + f_0 \frac{1}{p} = 1 + (1-p)\frac{1}{p} = \frac{p + 1 - p}{p} = \frac{1}{p}$$

and for $n = 1$

$$\sum_{k=0}^{1} f_k \frac{1}{p} = ((1-p) + p)\frac{1}{p} = \frac{1}{p}.$$

Therefore, the induction step is given by

$$\sum_{k=0}^{n+1} f_k \frac{1}{p} = \underbrace{f_{n+1}}_{=0} \frac{1}{p} + \underbrace{\sum_{k=0}^{n} f_k \frac{1}{p}}_{=1/p} = \frac{1}{p}$$

which verifies the solution.

The results from Example (2.12) are summarized by Pinsky and Karlin (2011, p. 381) in the following lemma:

**2.15 Lemma:**

If $(g_n)_{n\in\mathbb{N}_0}$ is the solution of equation (2.7) and $(h_n)_{n\in\mathbb{N}_0}$ the solution of the basic renewal equation, then

$$g_n = \sum_{k=0}^{n} b_{n-k} h_k.$$

Consider the following example by Bremaud (2017, p. 444).

**2.16 Example:**

Let $(X_n)_{n\in\mathbb{N}}$ be an i.i.d. sequence with $X_1 \sim geom(p)$ for $p \in (0,1)$. Recall that the geometric distribution is defined by $\mathbb{P}(X_1 = k) = p\,(1-p)^{k-1}$, for $k \in \mathbb{N}$ (Gallager, 2013, p. 17). Use the Dirac sequence for $(b_n)_{\in\mathbb{N}}$ and obtain as fundamental solution

$$h_0 = \delta_0 + \underbrace{\sum_{k=1}^{0} \mathbb{P}(X_1 = k)\, h_{0-k}}_{=0} = \delta_0 = 1.$$

We claim that for $n \geq 1$ the fundamental solution is given by $h_n = p$. We prove this by complete induction. We start the induction at $n = 1$:

$$h_1 = \delta_1 + \sum_{k=1}^{1} p\,(1-p)^{k-1}\, h_{1-k} = p\,(1-p)^0 \underbrace{h_0}_{=1} = p.$$

We assume that for a fixed but arbitrary $n \in \mathbb{N}$ the statement applies for all $0, ..., n-1$. Hence, our induction step is given by

$$\begin{aligned}
h_n &= \delta_n + \sum_{k=1}^{n} p(1-p)^{k-1} h_{n-k} \\
&= \sum_{k=1}^{n-1} p^2(1-p)^{k-1} \; + \; p(1-p)^{n-1} h_0 \\
&= \sum_{k=0}^{n-2} p^2(1-p)^{k-1} \; + \; p(1-p)^{n-1} \\
&= p^2 \frac{1-(1-p)^{n-1}}{1-(1-p)} \; + \; p(1-p)^{n-1} \\
&= p \; - \; p(1-p)^{n-1} \; + \; p(1-p)^{n-1} \; = \; p,
\end{aligned}$$

which proves this statement. Recall, that we have previously shown that the fundamental solution can be interpreted as the probability that $n$ is a renewal time, i.e. $u_n$.

For an arbitrary sequence, $(b_n)_{n\in\mathbb{N}}$ that satisfies the prerequisites of Definition (2.11), we can write the general solution as in Bremaud (2017, p. 444) like

$$g_n = b_n + p\sum_{k=0}^{n-1} b_k.$$

Before we move on with the next section we want to discuss one last example for a renewal equation and how to find its solution. This time we consider the lifetime of a defective renewal sequence (Bremaud, 2017, p. 442).

**2.17 Example:**

We define the lifetime of a defective renewal sequence as

$$L := \inf\{S_k \,|\, k \in \mathbb{N}_0,\, X_{k+1} = \infty\}.$$

Thus, $L$ is the the smallest renewal time before the subsequent waiting time is infinity. Since the sets $\{X_1 > n\}$ and $\{X_1 \leq n\}$ are disjoint, we get the following identity

$$\mathbb{1}_{\{L>n\}} = \mathbb{1}_{\{L>n\}}\mathbb{1}_{\{X_1>n\}} + \mathbb{1}_{\{L>n\}}\mathbb{1}_{\{X_1\leq n\}}. \tag{2.8}$$

Assume that $L > n$ and $X_1 > n$. If $k^*$ is the index which satisfies $L$, it can be concluded that $S_{k^*} > n$. This implies that $X_{k^*+1} = \infty$ and $X_1, ..., X_{k^*} < \infty$. Therefore $n < X_1 < \infty$. If we start with $n < X_1 < \infty$, we can immediately conclude that $X_1 > n$. Furthermore, since $X_1$ is finite and larger than $n$, it must be true that $S_{k^*} > n$. Therefore we get the identity

$$\{L > n,\, X_1 > n\} = \{n < X_1 < \infty\}. \tag{2.9}$$

Now shift the renewal process by subtracting $S_1$ and define $\hat{L} := \inf\{S_k - S_1 \,|\, k \geq 1,\, X_{k+1} = \infty\}$. Recall that $S_1 = X_1$. With the equivalent transformations $L > n \Leftrightarrow S_{k^*} > n \Leftrightarrow S_k - S_1 > n - S_1 \Leftrightarrow \hat{L} > n - X_1$ we then obtain

$$\{L > n,\, X_1 \leq n\} = \{\hat{L} > n - X_1,\, X_1 \leq n\} \tag{2.10}$$

and use the up to this point collected equations for

$$\mathbb{P}(L > n) \overset{\text{eq. (2.8)}}{=} \underbrace{\mathbb{P}(n < X_1 < \infty)}_{\text{eq. (2.9)}} + \underbrace{\mathbb{P}(\hat{L} > n - X_1,\, X_1 \leq n)}_{\text{eq. (2.10)}}. \tag{2.11}$$

Since $L$ and $\hat{L}$ are equally distributed and because of $\hat{L}$ being independent of $X_1$ we have

$$\mathbb{P}(\hat{L} > n - X_1,\, X_1 \leq n) = \sum_{k=0}^{n} \mathbb{P}(\hat{L} > n - k)\mathbb{P}(X_1 = k) = \sum_{k=0}^{n} f_k \mathbb{P}(\hat{L} > n - k). \tag{2.12}$$

When we define $g_n := \mathbb{P}(L > n)$ and $b_n := \mathbb{P}(n < X_1 < \infty)$ we get

$$\begin{aligned} g_n = \mathbb{P}(L > n) &\overset{\text{eq. (2.8)}}{=} \mathbb{P}(n < X_1 < \infty) + \mathbb{P}(\hat{L} > n - X_1,\, X_1 \leq n) \\ &\overset{\text{eq. (2.12)}}{=} b_n + \sum_{k=0}^{n} f_k \mathbb{P}(\hat{L} > n-k) = b_n + \sum_{k=0}^{n} f_k g_{n-k}. \end{aligned}$$

Therefore, we have found a solution for the renewal equation in question, which is also unique according to Theorem (2.13).

At last, we want to give an example which illustrates to what extent the solving of renewal equations can also be used in application-oriented problems, which at first glance may seem to have nothing to do with renewal theory. To do this, we look at the Exercise 2.4 by Barbu and Limnios (2008) and Exercise 18.3.1 by Bremaud (2017).

**2.18 Example:**

Consider a defective renewal process with waiting time distribution $(f_n)_{n\in\mathbb{N}_0}$, $f_0 = 0$, such that $\sum_{n=0}^{\infty} f_n = p$ with $p \in (0,1)$. The lifetime of the process is defined by

$$T := S_N = \sum_{i=1}^{N} X_i,$$

where $S_{N+1} = S_{N+2} = ... = \infty$. The goal is to provide the distribution for the process lifetime $T$. In the first step, we note that $N$ is a random variable whose distribution we intend to determine. It is easy to see that $\mathbb{P}(X_1 < \infty) = p$ and therefore $\mathbb{P}(X_1 = \infty) = 1 - p$. In order for $N$ to take the value $n \in \mathbb{N}$ it has to be $X_i < \infty$ for $i = 1, ..., N$ and $X_{N+1} = \infty$. This can be modelled by the geometrically distribution with parameter $p$. Therefore, it is for $n \in \mathbb{N}$

$$\mathbb{P}(N = n) = p^n(1-p). \tag{2.13}$$

Now we move on to the distribution of $T$. First of all, consider the case of $n = 0$. Since $X_i > 0$ by definition it has to be $N = 0$. It can be concluded that

$$\mathbb{P}(T = 0) = \mathbb{P}(S_N = 0) = \mathbb{P}(N = 0) \overset{\text{eq. (2.13)}}{=} p^0(1-p) = 1 - p. \tag{2.14}$$

For $n > 0$ we obtain

$$\begin{aligned} \mathbb{P}(T = n) &= \mathbb{P}(S_N = n) = \mathbb{P}(X_1 + \sum_{i=2}^{N} X_i = n) \\ &= \sum_{k=1}^{n} \mathbb{P}(X_1 + \sum_{i=2}^{N} X_i = n \,|\, X_1 = k)\mathbb{P}(X_1 = k) \end{aligned} \tag{2.15}$$

$$= \sum_{k=1}^{n} \mathbb{P}(\sum_{i=2}^{N} X_i = n-k)\mathbb{P}(X_1 = k)$$

$$= \sum_{k=1}^{n} \mathbb{P}(\sum_{i=1}^{N} X_i = n-k)\mathbb{P}(X_1 = k) = \sum_{k=1}^{n} \mathbb{P}(T = n-k) f_k.$$

Combining equations (2.14) and (2.15) we get the renewal equation

$$\underbrace{\mathbb{P}(T = n)}_{=:g_n} = \underbrace{(1-p)\mathbb{1}_{\{n=0\}}}_{=:b_n} + \sum_{k=1}^{n} \mathbb{P}(T = n-k) f_k \overset{f_0=0}{=} b_n + \sum_{k=0}^{n} g_{n-k} f_k.$$

Since $\sum_{n=0}^{\infty} b_n = b_0 < \infty$ Theorem (2.13) is applicable. Therefore, there exists a unique solution for $g_n$ given by

$$g_n \overset{\text{Th. (2.13)}}{=} (u * b)_n = \sum_{k=0}^{n} u_{n-k} b_k = u_n b_0,$$

which leads to the process lifetime distribution of

$$\mathbb{P}(T = n) = (1-p)u_n. \tag{2.16}$$

We want to apply this result in the following example. Consider an elderly pedestrian who wants to get from one side of a busy street to the other. As the pedestrian is not the fastest anymore, he is only able cross the street when there is a time window between the passing cars, which is bigger than $x \in \mathbb{N}$. The question we would like to answer is: How long does the pedestrian has to wait in expectation?

Let the cars arrive at times $S_0, S_1, S_2, ...$ and $(S_n)_{n \in \mathbb{N}_0}$ be a recurrent renewal process with inter arrival times defined as usual. Furthermore, let the pedestrian arrive at time $S_0 = 0$. Now, let us say that the process is at life as long as $X_i < x$. Define a new distribution by

$$f'_n := \begin{cases} f_n, & n \leq x \\ 0, & n > x \end{cases}.$$

Therefore, it is $p := \sum_{n=0}^{\infty} f_n = \sum_{n=0}^{x} f_n$ with $p \in (0,1)$. In addition to that we define $T$ like above with $S_{N+k}$ for all $k \in \mathbb{N}$ if $X_{N+1} > x$ and $X_i \leq x$ for all $i = 1, ..., N$. This is the exact setting from the beginning of this example. Therefore, the expected waiting time for the pedestrian until a suitable time window comes up is given by

$$\mathbb{E}T = \sum_{n=0}^{\infty} n\mathbb{P}(T = n) \overset{\text{eq. (2.16)}}{=} \sum_{n=0}^{\infty} n(1-p)u_n,$$

with $u_n = \sum_{k=0}^{n} f_n'^{*k}$.

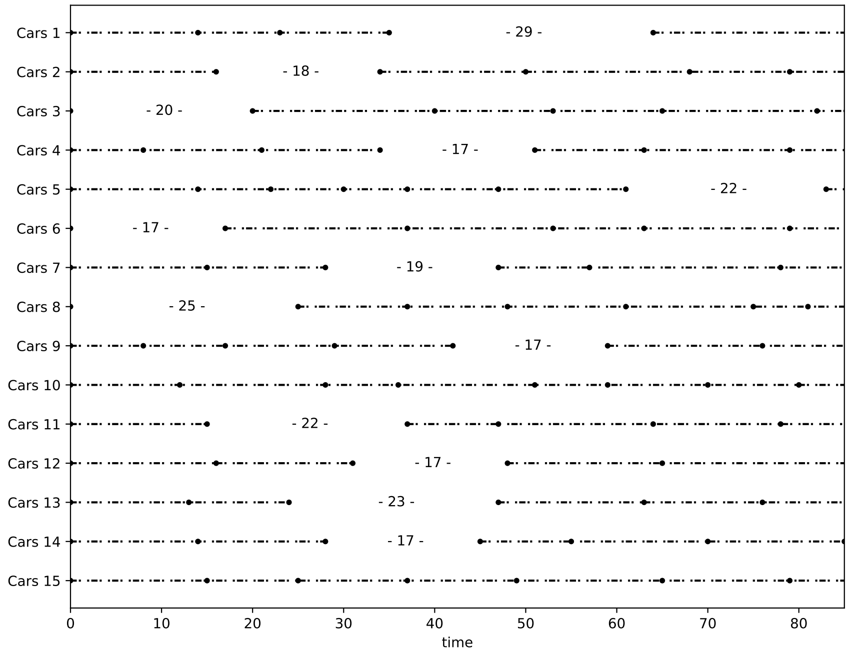


Figure 1: Simulation of arriving cars with i.i.d. inter arrival times that follow a Pois(15) distribution. The minimal required time window for a pedestrian to cross the street is $x = 16$ which results in an average waiting time of 33.8. The number in each row provides the length of the first appropriate time window and the dashed line indicates the time windows with too short a length.

## 2.3 Introduction of Lifetime Processes

Last but not least, we want to introduce some additional stochastic processes (Barbu and Limnios, 2008, p. 24). Those processes are called lifetime processes and will be taken up again in later chapters.

**2.19 Definition:**

Given is a renewal process $(S_n)_{n \in \mathbb{N}}$:

1. We call $U(n) := n - S_{N(n)}$ *age* at time $n$ or *backward recurrence time.*
2. We call $V(n) := S_{N(n)+1} - n$ the *residual lifetime* at time $n$ or the *forward recurrence time.*
3. We call $L(n) := X_{N(n)+1} = S_{N(n)+1} - S_{N(n)} = U(n) + V(n)$ the *total lifetime* at time $n$.

As their names suggest, the backward recurrence time provides the duration from the most recent renewal to time $n$ and the forward recurrence time the duration until the next occurring renewal. In addition to that, the total lifetime $L(n)$ is nothing other than the waiting time $X_{N(n)+1}$.

Using this definition and the Feller's Lemma, we can deduce the following corollary with some interesting facts.

### 2.20 Corollary:

It is $x > 0$:

1. $\mathbb{P}(U(n) \geq x) = \mathbb{P}(S_{N(n)} \leq n - x)$
2. $U(n) \geq x \Leftrightarrow$ there are no renewals in $[n - x,\, n]$
3. $V(n - x) > x \Leftrightarrow$ there are no renewals in $[n - x,\, n]$

# 3 The Conversion from continuous to discrete Renewal Processes

Let us revisit the example of the light bulbs at the beginning. Those were not serviced on a continuous basis but only at certain times, e.g. once a day. Now consider the case where it is cheaper to replace light bulbs that are still working than those that have already burned out. This raises the question of the intervals at which it makes sense to replace the light bulbs as a precaution. We want to get to the bottom of this problem in the following example from Nakagawa (2011, pp. 86-89) and find an optimum maintenance cycle. Here, we assume $\mathbb{P}(X_1 = 0) = 0$ which is equivalent to $X_i > 0$, $i = 1, 2, ....$

**3.1 Example:** (age replacement policy)
Given is a waiting times sequence $(X_k)_{k\in\mathbb{N}}$ and its associated discrete waiting time distribution $(f_k)_{k\in\mathbb{N}}$, with its usual definition. We define $Z_j := \min\{X_j, N\}$, $j \in \mathbb{N}$, where $N \in \mathbb{N}$. This means that we want to replace the light bulbs at the latest after N time units, or even before if necessary. $Z_k$ is also an i.i.d. random variable for which we have to determine the distribution $\mathbb{P}(Z_1 \leq k)$. If $N \geq k$ it must be $X_1 \leq k$ so that $Z_1 \leq k$. Therefore, $\mathbb{P}(X_1 \leq k) = \sum_{i=0}^{k} \mathbb{P}(X_1 = i) = \sum_{i=0}^{k} f_i$. In the case that $N < k$ it follows that $\min\{X_1, N\} < k$ which leads to the fact that $\mathbb{P}(Z_k \leq k) = 1$. In total, we therefore obtain

$$\mathbb{P}(Z_1 \leq k) = \begin{cases} \sum_{i=0}^{k} f_i, & N \geq k \\ 1, & N < k \end{cases}. \tag{3.1}$$

The goal is to determine an optimal $N^*$ which indicates the largest waiting time after which we want to replace a bulb. In order to do so, we need an objective function which we want to minimize. In this case, the expected cost rate is an appropriate choice. For that we need the expected value of $Z_1$ which is given by

$$\begin{aligned}
\mathbb{E}Z_1 &= \sum_{k=1}^{\infty} k\mathbb{P}(Z_1 = k) \overset{\text{eq. (3.1)}}{=} \sum_{k=1}^{N} kf_k + N \sum_{k=N+1}^{\infty} f_k \\
&= \sum_{k=1}^{N} k\Big(\mathbb{P}(X_1 \geq k) - \mathbb{P}(X_1 \geq k+1)\Big) + N \sum_{k=N+1}^{\infty} \Big(\mathbb{P}(X_1 \geq k) - \mathbb{P}(X_1 \geq k+1)\Big) \\
&= \sum_{k=1}^{N} k\mathbb{P}(X_1 \geq k) - \sum_{k=2}^{N+1} (k-1)\mathbb{P}(X_1 \geq k) + N\Big(\sum_{k=N+1}^{\infty} \mathbb{P}(X_1 \geq k) - \sum_{k=N+2}^{\infty} \mathbb{P}(X_1 \geq k)\Big) \\
&= \sum_{k=1}^{N} k\mathbb{P}(X_1 \geq k) - \sum_{k=2}^{N+1} k\mathbb{P}(X_1 \geq k) + \sum_{k=2}^{N+1} \mathbb{P}(X_1 \geq k) + N\mathbb{P}(X_1 \geq N+1) \\
&= \mathbb{P}(X_1 \geq 1) - (N+1)\mathbb{P}(X_1 \geq N+1) + \mathbb{P}(X_1 \geq N+1) + \sum_{k=2}^{N} \mathbb{P}(X_1 \geq k)
\end{aligned}$$

$$
\begin{aligned}
&+ N\mathbb{P}(X_1 \geq N+1) \\
&= \sum_{k=1}^{N} \mathbb{P}(X_1 \geq k) - (N+1)\mathbb{P}(X_1 \geq N+1) + (N+1)\mathbb{P}(X_1 \geq N+1) \\
&= \sum_{k=1}^{N} \mathbb{P}(X_1 \geq k) = \sum_{k=1}^{N}\sum_{j=k}^{\infty} \mathbb{P}(X_1 = j).
\end{aligned}
$$

Using this fact and equation (3.1) the expected cost rate is given by

$$
C(N) := \frac{c_1\mathbb{P}(X_1 \leq N) + c_2\mathbb{P}(X_1 > N)}{\mathbb{E}Z_1} = \frac{c_1 \sum_{k=1}^{N} f_k + c_2 \sum_{k=N+1}^{\infty} f_k}{\sum_{k=1}^{N}\sum_{j=k}^{\infty} f_j},
$$

where $c_1$ is the costs to replace a failed unit and $c_2$ is the cost to replace a non-failed unit at time $N$. In order to have an incentive to even think about a planned replacement, we assume $c_2 < c_1$.

As already mentioned, our goal is to minimize $C(N)$. Usually we would set the derivative of $C(N)$ with respect to $N$ to zero. But since we are in a discrete setting, this is not possible. As an alternative we consider instead

$$
\frac{C(N+1) - C(N)}{(N+1) - N} = C(N+1) - C(N).
$$

This is the difference quotient for the smallest possible step size of 1. Due to the discrete nature of $C(N)$ the value 0 does not necessarily have to be assumed. Instead, we set the expression greater to zero. Therefore, it can be shown that

$$
C(N+1) - C(N) \geq 0 \quad \Leftrightarrow \quad \underbrace{h_{N+1}\sum_{k=1}^{N}\sum_{j=k}^{\infty} f_j - \sum_{j=1}^{N} f_j}_{=:L(N)} \geq \frac{c_2}{c_1 -_c 2}, \tag{3.2}
$$

with $h_j := f_j / \sum_{i=j}^{\infty} f_i$. We call $h_j$ the discrete failure rate of the probability function $f_j$ and assume it to be strictly increasing with a mean $\mu < \infty$. In order to verify the existence of a unique $N^*$ it has to be shown that $L(N+1) - L(N) > 0$ and $L_\infty := \lim_{N\to\infty} L(N) < \infty$. We begin with

$$
\begin{aligned}
L(N+1) - L(N) &= L(N+1) - L(N) + f_{N+1} - f_{N+1}\frac{\sum_{k=N+1}^{\infty} f_k}{\sum_{k=N+1}^{\infty} f_k} \\
&= L(N+1) + f_{N+1} - h_{N+1}\sum_{k=1}^{N}\sum_{j=k}^{\infty} f_j + \sum_{j=1}^{N} f_j - h_{N+1}\sum_{j=N+1}^{\infty} f_j \\
&= L(N+1) + \sum_{j=1}^{N+1} f_j - h_{N+1}\sum_{k=1}^{N}\sum_{j=k}^{\infty} f_j - h_{N+1}\sum_{j=N+1}^{\infty} f_j
\end{aligned}
$$

$$
\begin{aligned}
&= h_{N+2} \sum_{k=1}^{N+1} \sum_{j=k}^{\infty} f_j - \sum_{j=1}^{N+1} f_j + \sum_{j=1}^{N+1} f_j - h_{N+1} \sum_{k=1}^{N+1} \sum_{j=k}^{\infty} f_k \\
&= \underbrace{(h_{N+2} - h_{N+1})}_{>0 \text{ by assump.}} \sum_{k=1}^{N+1} \sum_{j=k}^{\infty} f_j > 0.
\end{aligned}
$$

To show the second assertion we need the boundedness of $h_j$ to follow with the monotony the convergence of $h_\infty := \lim_{N\to\infty} h_{N+1} < \infty$. We have

$$
0 < h_j = \frac{f_j}{\sum_{k=j}^{\infty} f_k} = \frac{f_j}{f_j + \underbrace{\sum_{k=j+1}^{\infty} f_j}_{\geq 0}} \leq \frac{f_j}{f_j} = 1.
$$

Therefore it can be concluded that

$$
\begin{aligned}
\lim_{N\to\infty} L(N) &= \lim_{N\to\infty} \left( h_{N+1} \sum_{k=1}^{N} \sum_{j=k}^{\infty} f_j - \sum_{j=1}^{N} f_j \right) = \lim_{N\to\infty} h_\infty \lim_{N\to\infty} \sum_{k=1}^{N} \sum_{j=k}^{\infty} f_j - \lim_{N\to\infty} \sum_{j=1}^{N} f_j \\
&= h_\infty \sum_{k=1}^{\infty} \mathbb{P}(X_1 \geq k) - \sum_{j=1}^{\infty} f_j = h_\infty \, \mathbb{E} X_1 - 1 = h_\infty \, \mu - 1.
\end{aligned}
$$

Using this and equation (3.2) we obtain

$$
L_\infty > \frac{c_2}{c_1 - c_2} \Leftrightarrow h_\infty \, \mu - 1 > \frac{c_2}{c_1 - c_2} \Leftrightarrow h_\infty \, \mu > \frac{c_2 + c_1 - c_2}{c_1 - c_2} \Leftrightarrow h_\infty > \frac{c_1}{\mu \, (c_1 - c_2)} =: K.
$$

Because of $L(N+1) > L(N)$, $L(N)$ intersects $c_2/(c_1 - c_2)$ exactly once. A unique minimum at $N^*$ follows because of $L_\infty > c_2/(c_1 - c_2)$. Then, $N^* := \min\{N \in \mathbb{N} \mid C(N) > c_2/(c_1 - c_2)\}$ with $(c_1 - c_2) \, h_{N^*} \leq C(N^*) < (c_1 - c_2) \, h_{N^*+1}$.

On the other hand, if $h_\infty \leq K$ it is $N^* = \infty$ which means, that no planned replacement is recommended. Same in the case of $K \geq 1$, since $h_j \in (0, 1]$.

Lastly, let us take a look at the matter using an explicit numerical example. Consider the following negative binomial distribution $f_k := k p^2 q^{k-1}$ with $q := 1 - p$ and $p \in (0, 1)$. The expected value is $\mu = (1 + q)/p$. Before we start to determine the minimum, we have to check the condition that $h_j$ is indeed strictly monotonically increasing. For the denominator we have

$$
\begin{aligned}
\sum_{i=j}^{\infty} i p^2 q^{i-1} &= \sum_{i=0}^{\infty} (i + j) p^2 q^{i-1+j} = \sum_{i=0}^{\infty} i p^2 q^{i-1+j} + \sum_{i=0}^{\infty} j p^2 q^{i-1+j} \\
&= p^2 q^{j-1} \sum_{i=0}^{\infty} i q^i + j p^2 q^{j-1} \sum_{i=0}^{\infty} q^i = p^2 q^{j-1} \frac{q}{(1-q)^2} + j p^2 q^{j-1} \frac{1}{1-q}
\end{aligned}
$$

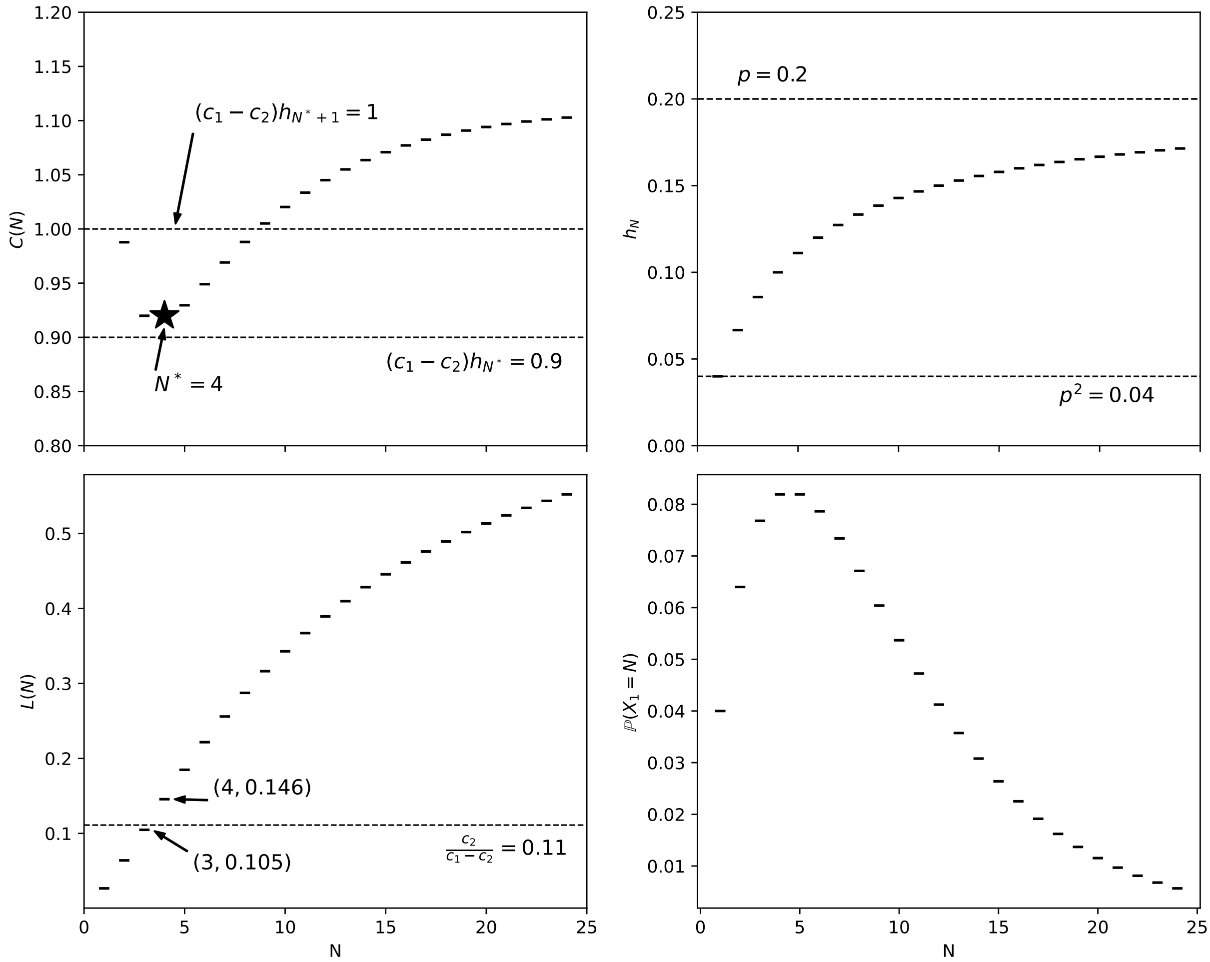


Figure 2: Exemplary age replacement policy with negative binomially distributed waiting times with $p = 0.2$ (e.g. $q = 0.8$) and replecament costs of $c_1 = 10$ and $c_2 = 1$.

$$= \frac{p^2 q^j + jp^2 q^{j-1} - jp^2 q^j}{(1-q)^2} = \frac{p^2 q^j + jp^2 q^{j-1} - jp^2 q^j}{p^2} = q^j + jq^{j-1} - jq^j.$$

Therefore, it can be concluded that

$$h_j = \frac{jp^2 q^{j-1}}{q^j + jq^{j-1} - jq^j} = \frac{jp^2}{q + j - jq} = \frac{jp^2}{q + j - j + jp} = \frac{jp^2}{q + jp}.$$

To show that $h_j$ is strictly monotonically increasing in $j$ we show that the derivative is strictly greater than zero:

$$\frac{d}{dj} h_j = \frac{p^2(q + jp) - jp^3}{(q + jp)^2} = \frac{p^2 q + jp^3 - jp^3}{(q + jp)^2} = \frac{p^2 q}{(q + jp)^2} > 0.$$

Now be $c_1 = 10$, $c_2 = 1$ and $p = 0.2$. Therefore, it is $\mu = 9$. The top right of Figure 3.1 shows besides the monotony of $h_j$ its boundedness between $p^2$ and $p$. In addition to that, the bottom left shows that the first $N$ that satisfies the condition in equation (3.2) is $N = 4$. Hence, it is $N^* = 4$ which is clearly consistent with the desired minimum of $C(N)$ that can easily be seen in the top left of Figure 3.1.

# 4 Limit Theorems

In this section we analyse the asymptotic behavior of different quantities related to the renewal process that we have got to know in the previous chapters.

## 4.1 Classic results in Renewal Theory

We start with the following lemma (Barbu and Limnios, 2008, p. 26):

**4.1 Lemma:**
Given is a renewal process $(S_n)_{n\in\mathbb{N}}$ with waiting times $X_i > 0$, $i = 1, 2, ...$ and a recurrent renewal counting process $(N(n))_{n\in\mathbb{N}}$. Then it is

1. $\lim_{n\to\infty} S_n = \infty$ a.s.
2. $\lim_{n\to\infty} N(n) = \infty$ a.s.

Since it holds true that $S_n \geq n$, the first claim is quite intuitive since we found a divergent lower bound. For the second claim it can be argued that $\lim_{k\to\infty} \mathbb{P}(S_k \geq k) = 0$, since the renewal chain is recurrent and therefore $S_n < \infty$ for any $n \in \mathbb{N}$. The result follows with the Feller's Lemma (2.6).

After that, we want to take a closer look at the asymptotic behavior of a recurrent renewal counting process with the strong law of large numbers and the central limit theorem for renewal counting processes (Barbu and Limnios, 2008, p. 26) which also exist in a similar form for general, i.e. also for continuous, renewal processes.

**4.2 Theorem:** (strong law of large numbers for renewal counting processes)
For a recurrent renewal process $(S_n)_{n\in\mathbb{N}}$ we have

$$\lim_{n\to\infty} \frac{N(n)}{n} = \frac{1}{\mu} \text{ a.s.}$$

with $\mu := \mathbb{E}X_1$.

The idea to prove this theorem is to use the inequality $S_{N(n)} \leq n \leq S_{N(n)+1}$ to show that $n/N(n)$ converges to $\mu$ as $n$ tends to infinity by applying the sandwich lemma. Consider the following theorem by Barbu and Limnios (2008, p. 27).

**4.3 Theorem:** (central limit theorem for renewal counting processes)
Consider a positive recurrent renewal process $(S_n)_{n\in\mathbb{N}}$ with $\mu < \infty$ and $0 < \sigma^2 := Var(X_1) < \infty$. Then

$$\frac{N(n) - n/\mu}{\sqrt{n\sigma^2/\mu^3}} \xrightarrow{n\to\infty} \mathcal{N}(0,1).$$

Before we discuss the next standard result in renewal theory we want to introduce the concept of stopping times.

### 4.4 Definition:

It is $t > 0$, then

$$\tau(t) = \inf\{n \in \mathbb{N} \,|\, S_n > t\}, \quad \inf \varnothing := \infty$$

is called *first passage time.*

This is basically the index of the renewal time which exceeds the point $t$ for the first time. Using the first passage time we define the stopping time as

### 4.5 Definition:

A random variable $\tau \in \mathbb{N}$ is called *stopping time* for $(X_n)_{n\in\mathbb{N}}$, if

$$\{\tau = n\} \in \underbrace{\sigma(X_1, ..., X_n)}_{\sigma\text{-algebra}} \quad \forall n \in \mathbb{N}.$$

This means, that the stopping time is only dependent on $X_1, ..., X_n$ and independent of $X_{n+1}, X_{n+2}, ...$. Resnik (2002, p. 44) suggests to think of $\sigma(X_1, ..., X_n)$ as the set of the events where we can decide whether they have happened or not solely based on the knowledge of the values of $X_1, ..., X_n$. In addition to that, he provides a comparison with a gambler who can only make the decision if he wants to stop gambling based on the past events and not on the upcoming ones.

### 4.6 Theorem: (elementary renewal theorem)

For a recurrent renewal process $(S_n)_{n\in\mathbb{N}}$ and $\tau$ being a stopping time we have

$$\frac{\mathbb{E}\tau(t)}{t} \xrightarrow{t\to\infty} \frac{1}{\mu}$$

with $1/\infty := 0$.

We would like to find stopping times using known quantities, so that we can apply Theorem (4.6) in the context of this paper. Consider $\tau(t) = N(t) + k$. Then

$$\begin{aligned}
\{\tau(t) = n\} = \{N(t) + k = n\} &= \{N(t) = n - k\} \\
&= \{S_{n-k} \leq t \leq S_{n-k+1}\} \\
&= \left\{ \sum_{i=1}^{n-k} X_i \leq t, \sum_{i=1}^{n-k+1} X_i > t \right\}.
\end{aligned}$$

It can be observed that $\{\tau = n\}$ only depends on $X_1, ..., X_{n-k+1}$. According to Definition (4.5), $\tau$ is stopping time iff $\{\tau = n\}$ solely depends on $X_1, ..., X_n$. Therefore, for $k > 1$ is $\tau$ a stopping time. Recall that we have defined $\Psi(n) = \mathbb{E}(N(n) + 1)$. Thus, we immediately get the following corollary (Barbu and Limnios, 2008, p. 28).

### 4.7 Corollary:

For a recurrent renewal process $(S_n)_{n\in\mathbb{N}}$ we have

$$\lim_{n\to\infty} \frac{\Psi(n)}{n} = \frac{1}{\mu}.$$

**4.8 Example:** (Bernoulli distribution part 3)
Again, consider a renewal process as defined in Example (2.10). Equation (2.5) provides

$$\Psi(n) = \sum_{k=0}^{n} u_k \overset{\text{eq. (2.4)}}{=} \frac{n+1}{p}.$$

Therefore, we can conclude

$$\frac{\Psi(n)}{n} = \frac{n+1}{np} = \frac{1+1/n}{p} \xrightarrow{n\to\infty} \frac{1}{p} = \frac{1}{\mu},$$

which is consistent with Corollary (4.7).

For the next theorems, we have to distinguish between periodic and aperiodic distributions. Recall that we have a periodic distribution if the support of the distribution, i.e. the points that have a probability greater null, is concentrated on a multiple, i.e. $d > 1$, of the natural numbers: $0d, 1d, 2d, \ldots$. In literature, the continuous versions of the following theorems also take this distinction. This emphasises the strong connection between the asymptotic behavior of different aspects of renewal processes with the periodicity (Barbu and Limnios, 2008, p. 29).

**4.9 Theorem:** (renewal theorem)
It is $u_n := \mathbb{P}(Z_n = 1)$ and $u_0 := 1$. Then

1. For a recurrent aperiodic renewal process $(S_n)_{n\in\mathbb{N}}$ we have

$$\lim_{n\to\infty} u_n = \frac{1}{\mu}$$

2. For a periodic recurrent renewal process of period $d > 1$ $(S_n)_{n\in\mathbb{N}}$ we have

$$\lim_{n\to\infty} u_{nd} = \frac{d}{\mu}$$

and $u_k = 0$ for all $k$ not multiple of $d$.

The first part of the result was proved by Erdös, Feller and Pollard in 1949. Consider the following simple example taken from Mitov and Omey (2014, p. 55) which shows the validity of the first part of Theorem (4.9) with the help of generating functions.

**4.10 Example:**
We define the *probability generating function* $\tilde{f}(z) := \mathbb{E}[z^X] = \sum_{n=0}^{\infty} f_n z^n$ of the random variable X. When we multiply the series representation of $u_n$ (equation (2.1)) on both sides by $z^n$ we obtain

$$u_n = \sum_{k=0}^{\infty} f_n^{*k} \iff u_n z^n = \sum_{k=0}^{\infty} f_n^{*k} z^n,$$

and by summing over $n$ we get the generating function for $u_n$ which we label as $u(z)$:

$$u(z) := \sum_{n=0}^{\infty} u_n z^n = \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} f_n^{*k} z^n = \sum_{k=0}^{\infty}\sum_{n=0}^{\infty} f_n^{*k} z^n = \sum_{k=0}^{\infty} \underbrace{\tilde{f}^k(z)}_{<1} = \frac{1}{1-\tilde{f}(z)}. \tag{4.1}$$

Recall that $f_n^{*k} = \sum_{j=0}^{n} f_{n-j}^{*k-1} f_j^{*1}$ with $f_n^{*1} = f_n$ and $\sum_{n=0}^{\infty} f_n^{*k} z^n = \tilde{f}^k(z)$. Now, we define a distribution with only two supports:

$$f_n := \mathbb{P}(X = n) = \begin{cases} p, & \text{if } n = 1 \\ q = 1-p, & \text{if } n = 2 \end{cases},$$

with $0 < p < 1$. Therefore, the expected value of this distribution is given by $\mu = \mathbb{E}X = p + 2q = 1 + q$. The generating function for this probability distribution is

$$\tilde{f}(z) = \mathbb{P}(X = 1)z^1 + \mathbb{P}(X = 2)z^2 = pz + qz^2.$$

We can paste this result into equation (4.1) and get

$$\begin{aligned}
u(z) &= \frac{1}{1-\tilde{f}(z)} = \frac{1}{1-pz-qz^2} = \frac{1}{(p-p)+1-pz-qz^2} = \frac{1}{p-pz+q-qz^2} \\
&= \frac{1}{p(1-z)+q(1-z^2)} = \frac{1}{p(1-z)+q(1-z)(1+z)} = \frac{1}{(1-z)(p+q(1+z)} \\
&= \frac{1}{(1-z)(p+1-p+qz)} = \frac{1}{(1-z)(1+qz)} = \frac{1+q}{(1+q)(1-z)(1+qz)} \\
&= \frac{1+q+qz-qz}{(1+q)(1-z)(1+qz)} = \frac{1+qz}{(1+q)(1-z)(1+qz)} + \frac{q(1-z)}{(1+q)(1-z)(1+qz)} \\
&= \frac{1}{1+q}\frac{1}{1-z} + \frac{q}{1+q}\frac{1}{1+qz}
\end{aligned}$$

with $|z| \leq 1$.

According to Johnson et al. (1993, p. 59) we can recover the probability mass function by calculating derivatives of the probability generating function:

$$u_n = \frac{u^{(n)}(0)}{n!}. \tag{4.2}$$

The form in which we brought $u_n$ in the previous step will help us to find a closed form for the derivatives. In order to do so we define

$$g_1(z) := \frac{1}{1-z} \quad \text{and} \quad g_2(z) := \frac{1}{1+qz}.$$

By means of induction it can be shown that for $g_1$ and $g_2$ the derivatives are given by

$$g_1^{(n)} = \frac{n!}{(1-z)^{n+1}} \quad \text{and} \quad g_2^{(n)} = \frac{n!(-q)^n}{(1+qz)^{n+1}}.$$

**Base Case:** $n = 0$. $g_1^{(0)}(z) = \frac{0!}{(1-z)^{0+1}} = \frac{1}{1-z} = g_1(z)$, $g_2^{(0)}(z) = \frac{0!(-q)^0}{(1+qz)^{0+1}} = \frac{1}{1+qz} = g_2(z)$.
**Induction Hypothesis:** The respective derivatives hold true for some $n \in \mathbb{N}_0$.
**Induction Step:** $n \mapsto n+1$.

$$\begin{aligned}
g_1^{(n+1)}(z) &= \frac{d}{dz} g_1^{(n)}(z) \overset{\text{IH}}{=} \frac{n!(n+1)}{(1-z)^{n+2}} = \frac{(n+1)!}{(1-z)^{(n+1)+1}}, \\
g_2^{(n+1)}(z) &= \frac{d}{dz} g_2^{(n)}(z) \overset{\text{IH}}{=} -\frac{n!(n+1)(-q)^n q}{(1+qz)^{n+2}} = \frac{(n+1)!(-q)^{n+1}}{(1+qz)^{(n+1)+1}}.
\end{aligned}$$

According to the principle of induction, we thus obtain the assertion. Using equation (4.2) and the derivatives of $g_1$ and $g_2$ we get the following closed form for $u_n$:

$$\begin{aligned}
u_n &= \frac{1}{n!} \frac{1}{1+q} g_1^{(n)}(0) + \frac{1}{n!} \frac{q}{1+q} g_2^{(n)}(0) \\
&= \frac{1}{n!} \frac{1}{1+q} \frac{n!}{(1-0)^{n+1}} + \frac{1}{n!} \frac{q}{1+q} \frac{n!(-q)^n}{(1+0 \cdot q)^{n+1}} \\
&= \frac{1}{1+q} + \frac{q}{1+q}(-q)^n.
\end{aligned}$$

Since $q \in (0,1)$ the limit is

$$\lim_{n\to\infty} u_n = \frac{1}{1+q} = \frac{1}{\mu},$$

which is consisten with the result which we would get with the renewal theorem.

**4.11 Example:** (Bernoulli distribution part 4)
A renewal process with Bernoulli distributed inter arrival times (see Example (2.10)) is obviously recurrent and aperiodic. Therefore, it is

$$\lim_{n\to\infty} u_n \overset{\text{eq. (2.4)}}{=} \lim_{n\to\infty} \frac{1}{p} = \frac{1}{\mu},$$

which is consistent with the renewal theorem.

The following are two further brief examples taken from Mitov and Omey (2014, p. 64) in which we calculate the limit of $(u_t)_{t\in\mathbb{N}_0}$ by hand and show the consistency with the renewal theorem. We are going to take a look at waiting times that are Bernoulli and geometrically distributed.

**4.12 Example:**

Consider a renewal process with Bernoulli distributed waiting times such that $f_1 := \mathbb{P}(X_1 = 1) = p$ and $f_0 := \mathbb{P}(X_1 = 0) = 1 - p = q$ for $p \in (0,1)$. Then the generating function is given by

$$\tilde{f}(z) = \sum_{n=0}^{\infty} f_n z^n = f_0 z^0 + f_1 z = q + pz$$

and the expected value is $\mu = \mathbb{E}X_1 = p$ (Cramer and Kamps, 2020, pp. 230-231). We can use equation (4.1) and get for $|z| < 1$

$$u(z) \overset{\text{eq. (4.1)}}{=} \frac{1}{1 - \tilde{f}(z)} = \frac{1}{1 - (1 - p + pz)} = \frac{1}{p(1-z)} = \frac{1}{p} \sum_{n=0}^{\infty} z^n.$$

Therefore, the sequence elements of $(u_t)_{t \in \mathbb{N}_0}$ are given by $u_t = 1/p = 1/\mu$. Accordingly, the limit is $1/\mu$ which is consistent with the renewal theorem.

We continue with the following example from Mitov and Omey (2014, pp. 64-65).

**4.13 Example:**

Let the i.i.d. inter arrival times $X_1$ be geometrically distributed with $f_k := \mathbb{P}(X_1 = k) = pq^{k-1}$ for $k \in \mathbb{N}$ and $p \in (0,1)$. We obtain the following generating function:

$$\tilde{f}(z) = \sum_{k=1}^{\infty} f_k z^k = \sum_{k=1}^{\infty} pq^{k-1} z^k = \sum_{k=0}^{\infty} pq^k z^{k+1} = pz \sum_{k=0}^{\infty} \underbrace{(qz)}_{<1}{}^k = \frac{pz}{1-qz}, \quad |z| < 1.$$

According to Cramer and Kamps (2020, p. 233) it is $\mu = \mathbb{E}X_1 = 1/p$. Again with equation (4.1) the generating function of $(u_t)_{t \in \mathbb{N}_0}$ is given by

$$\begin{aligned} u(z) &= \frac{1}{1 - \frac{pz}{1-qz}} = \frac{1}{\frac{1-qz-pz}{1-qz}} = \frac{1-qz}{1-qz-pz} = \frac{1-qz}{1 - \underbrace{(q+p)}_{=1} z} \\ &= \frac{1-qz}{1-z} = \frac{p+q-qz}{1-z} = \frac{p}{1-z} + \frac{q(1-z)}{1-z} = \frac{p}{1-z} + q. \end{aligned}$$

We can follow

$$u(z) = \sum_{k=0}^{\infty} pz^k + q = p + q + \sum_{k=1}^{\infty} pz^k = 1 + \sum_{k=1}^{\infty} pz^k$$

which leads to the fact, that $u_0 = 1$ and $u_k = p$ for $k \in \mathbb{N}$. Therefore, it is $\lim_{k \to \infty} u_k = p = 1/\mu$ which is again consistent with the renewal theorem.

In the continuous setting there is Blackwell's Theorem (Blackwell, 1948 and Barbu and Limnios, 2008, p. 29) which makes a statement about how many renewals can be expected asymptotically in an interval of length $h > 0$. Again, the theorem exists for both the periodic and aperiodic recurrent renewal processes. For latter we get

$$\lim_{n\to\infty} [\Psi(n+h) - \Psi(n)] = \frac{h}{\mu}. \tag{4.3}$$

Using the renewal theorem we can derive the same result for discrete renewal processes (Barbu and Limnios, 2008, p. 29-30). In this case we assume $h \in \mathbb{N}$. It is

$$\Psi(n+h) - \Psi(n) \overset{\text{eq. (2.5)}}{=} \sum_{k=0}^{n+h} u_k - \sum_{k=0}^{n} u_k = \sum_{k=n+1}^{n+h} u_k. \tag{4.4}$$

According to Theorem (4.9) we get $\lim_{n\to\infty} u_k = 1/\mu$. Because of the finite number of summands, the sum is convergent and provides

$$\begin{aligned}
\lim_{n\to\infty} [\Psi(n+h) - \Psi(n)] &\overset{\text{eq. (4.4)}}{=} \sum_{k=n+1}^{n+h} u_k \\
&= \lim_{n\to\infty} [u_{n+1} + u_{n+2} + ... + u_{n+h}] \\
&= \lim_{n\to\infty} u_{n+1} + \lim_{n\to\infty} u_{n+2} + ... + \lim_{n\to\infty} u_{n+h} \\
&\overset{\text{Th. (4.9)}}{=} h\frac{1}{\mu}.
\end{aligned}$$

The result for periodic renewal processes is derived in the same way, assuming that $h$ is a multiple of the period $d$. Next, consider the following theorem (Barbu and Limnios, 2008, p. 30).

**4.14 Theorem:** (key renewal theorem)

Consider a recurrent renewal process $(S_n)_{n\in\mathbb{N}}$ and a real sequence $(b_n)_{n\in\mathbb{N}}$.

1. If the process is aperiodic and $\sum_{n=0}^{\infty} |b_n| < \infty$, then
$$\lim_{n\to\infty} \sum_{k=0}^{n} b_k u_{n-k} = \frac{1}{\mu} \sum_{n=0}^{\infty} b_n$$
2. If the process is periodic of period $d > 1$ and if for a certain positive integer $l$, $0 \le l < d$, we have $\sum_{n=0}^{\infty} \mid b_{l+nd} \mid < \infty$, then
$$\lim_{n\to\infty} \sum_{k=0}^{l+nd} b_k u_{l+nd-k} = \frac{d}{\mu} \sum_{n=0}^{\infty} b_{l+nd},$$
where $u_n$ is defined as in equation (2.1).

Now we want to investigate the asymptotic behavior of the unique solution of the renewal equation (2.7) (Bremaud, 2017, p. 444 and Pinsky and Karlin, 2011, p. 383).

**4.15 Corollary:**

Let $(b_n)_{n\in\mathbb{N}}$ be a sequence with $\sum_{n=0}^{\infty}|b_n| < \infty$ and $(g_n)_{n\in\mathbb{N}}$ the unique solution of the renewal equation (2.7). In addition to that, assume $0 < f_1 < 1$.

1. Is $(f_n)_{n\in\mathbb{N}}$ an aperiodic and recurrent probability distribution, then

$$\lim_{n\to\infty} g_n = \frac{\sum_{k=0}^{\infty} b_k}{\sum_{k=1}^{\infty} k f_k},$$

   where the right-hand side is null if the series in the denominator is divergent.

2. Is $(f_n)_{n\in\mathbb{N}}$ a periodic and recurrent probability distribution, with period $d > 1$, then for all $r \in [0, d-1]$ it is

$$\lim_{n\to\infty} g_{r+nd} = d\frac{\sum_{k=0}^{\infty} b_{r+kd}}{\sum_{k=1}^{\infty} k f_k}.$$

Since $\sum_{k=1}^{\infty} k f_k$ is nothing else but the expected value of the distribution $f$, we obtain limits that have a similar structure to those in the renewal theorem.

If we consider the basic renewal equation, where $(b_n)_{n\in\mathbb{N}}$ is the Dirac sequence, we get

$$\lim_{n\to\infty} g_n = \frac{1}{\mu},$$

since $\sum_{k=0}^{\infty} \delta_k = 1$. This implies, that the asymptotic probability of a renewal taking place in a certain interval is equal to the reciprocal of $\mu$, i.e. the expected life time.

We want to apply Theorem (4.15) in the following example (Pinsky and Karlin, 2011, p. 383).

**4.16 Example:**

Consider the forward recurrence time $V(n) = S_{N(n)+1} - n$. We define

$$g_n(m) := \mathbb{P}(V(n) = m) = \mathbb{P}(S_{N(n)+1} = m + n).$$

Since $N(n)$ counts the number of renewals before the time $n$, $S_{N(n)} + 1$ can be interpreted as the time at which the first renewal after time $n$ takes place. Therefore, $g_n$ is equal to the probability that $S_{N(n)+1}$ takes on the value $m + n$.

Our goal is to use $g_n$ to establish a renewal equation where we can apply Theorem (4.15) on and eventually get a probability mass function. To do so we condition on $X_1$ and obtain

$$\mathbb{P}(V(n) = m \,|\, X_1 = k) = \begin{cases} g_{n-k}(m), & \text{if } 0 \le k \le n \\ 1, & \text{if } k = m + n \\ 0, & \text{otherwise} \end{cases}. \tag{4.5}$$

The first case results from the fact that for $0 \le k \le n$ it is

$$\mathbb{P}(V(n) = m \mid X_1 = k) = \mathbb{P}(\sum_{i=2}^{N(n)+1} X_i = m+n-k) = \mathbb{P}(\sum_{i=1}^{N(n-k)+1} X_i = m+n-k) = g_{n-k}(m).$$

For the second case we use that if $X_1 = m + n$ it follows that $N(n) = 0$ since

$$N(n) = \max\{l \in \mathbb{N}_0 \mid S_l \le n,\, X_1 = m+n\} = \max\{l \in \mathbb{N}_0 \mid m+n+\sum_{i=2}^{l} X_i \le n)\}.$$

Therefore, we can conclude

$$\mathbb{P}(V(n) = m \mid X_1 = m+n) = \mathbb{P}(X_1 = m+n \mid X_1 = m+n) = 1.$$

For the last case assume that $k > n$ and $k \neq n+m$. Thus, there exists an $\alpha > 0$ such that $k = n + \alpha$ and therefore $n + \alpha = n + m$. Hence, it can be concluded that $\alpha \neq m$. Like above it is again $N(n) = 0$. Using these facts we obtain

$$\mathbb{P}(S_{N(n)+1} = m+n \mid X_1 = k) = \mathbb{P}(X_1 = m+n \mid X_1 = k) = \mathbb{P}(n+\alpha = m+n) = 0,$$

which verifies the distribution given in equation (4.5). Using this conditional probability distribution we can bring $g_n$ in the form of a renewal equation by applying the law of total probability.

$$\begin{aligned} g_n(m) = \mathbb{P}(V(n) = m) &\overset{\text{LotP}}{=} \sum_{k=0}^{\infty} \mathbb{P}(V(n) = m \mid X_1 = k) f_k \\ &\overset{\text{eq. (4.5)}}{=} f_{m+n} + \sum_{k=0}^{n} g_{n-k}(m) f_k + \sum_{k=n+1}^{\infty} \mathbb{P}(V(n) = m \mid X_1 = k) f_k \\ &= f_{m+n} + \sum_{k=0}^{n} g_{n-k}(m) f_k. \end{aligned} \tag{4.6}$$

We use Theorem (4.15) and set $b_k = f_{m+n}$. We receive

$$\begin{aligned} \lim_{n\to\infty} g_n &= \frac{\sum_{k=0}^{\infty} f_{m+k}}{\sum_{k=1}^{\infty} k f_k} \\ &= \frac{\sum_{k=0}^{\infty} \mathbb{P}(X_1 = m+k)}{\mathbb{E}(X_1)} \\ &= \frac{\mathbb{P}(X_1 \ge m)}{\mathbb{E}(X_1)}. \end{aligned}$$

The last step is to verify that this is indeed a probability mass function. Since the nominator and denominator are both non negative and therefore the fraction, we need to examine if the probabilities add up to 1. We can easily see that

$$\sum_{m=1}^{\infty} \frac{\mathbb{P}(X_1 \geq m)}{\mathbb{E}(X_1)} = \frac{\sum_{m=1}^{\infty} \mathbb{P}(X_1 \geq m)}{\mathbb{E}(X_1)} = \frac{\mathbb{E}(X_1)}{\mathbb{E}(X_1)} = 1,$$

thus all requirements are satisfied.

## 4.2 Defective Renewal Theorem

For this section we assume a defective renewal distribution, i.e. $\mathbb{P}(X_1 < \infty) < 1$. Like before, we want to investigate the asymptotic behavior of the renewal equation's solution (Bremaud, 2017, p. 446).

**4.17 Theorem:**

Let $(f_n)_{n\in\mathbb{N}}$ be a defective renewal distribution and the sequence $(b_n)_{n\in\mathbb{N}}$ with $b_n \geq 0$ for all $n \in \mathbb{N}$ satisfies

$$\lim_{n\to\infty} b_n =: b_\infty < \infty.$$

Then the asymptotic solution of the renewal equation (2.11) satisfies

$$\lim_{n\to\infty} g_n = \frac{b_\infty}{\alpha},$$

with $\alpha = \mathbb{P}(X_1 = \infty)$ being the defect of $(f_n)_{n\in\mathbb{N}}$.

Note that we are not assuming $\sum_{n=0}^{\infty} |b_n| < \infty$ like it was the case in Definition (2.11) but the more relaxed condition that $(b_n)_{n\in\mathbb{N}}$ converges.

The next theorem shows what requirements have to be satisfied to 'repair' a defective renewal distribution and being able to apply Theorem (4.15). Because the distribution is defective the probabilities will not add up to 1, hence $\sum_{n=0}^{\infty} f_n < 1$. The idea is to scale those probabilities up by some factor $\gamma$ to obtain a proper probability function (Bremaud, 2017, p. 447).

**4.18 Theorem:** (defective renewal theorem)

Let $(f_n)_{n\in\mathbb{N}}$ be a defective and aperiodic renewal distribution. If there exists $\gamma > 1$ such that

$$\sum_{n=0}^{\infty} \gamma^n f_n = 1 \tag{4.7}$$

and for the non negative sequence $(b_n)_{n\in\mathbb{N}}$ it is

(4.8)
$$\sum_{n=0}^{\infty} \gamma^n \mid b_n \mid < \infty,$$

then the asymptotic solution of the renewal equation (2.11) satisfies

$$\lim_{n\to\infty} \gamma^n g_n = \frac{\sum_{k=0}^{\infty} \gamma^k b_k}{\sum_{k=0}^{\infty} k\gamma^k f_k}.$$

We can define $\tilde{f}_n := \gamma^n f_n$, $\tilde{b}_n := \gamma^n b_n$ and $\tilde{g}_n := \gamma^n f_n$. This provides the new renewal equation

$$\tilde{g}_n = \tilde{b}_n + \sum_{k=1}^{n} \tilde{f}_k \tilde{g}_{n-k}.$$

The new sequence $(\tilde{f}_n)_{n\in\mathbb{N}}$ owns all characteristics of a probability distribution, i.e. non negativity and the sum of the probabilities add up to one. Therefore, $(\tilde{f}_n)_{n\in\mathbb{N}}$ is an aperiodic and recurrent probability distribution. Theorem (4.15) is thus applicable.

In the following two examples we want to show in which context Theorem (4.18) can be used (Bremaud, 2017, pp. 447-448).

**4.19 Example:** (convergence rate in the defective case)

We assume a defective and aperiodic renewal distribution $(f_n)_{n\in\mathbb{N}_0}$. Furthermore we have a non-negative sequence $(b_n)_{n\in\mathbb{N}}$ with $\lim_{n\to\infty} b_n =: b_\infty < \infty$ and we assume the existence of $\gamma > 1$ such that the equations (4.7) and (4.8) hold.

According to Theorem (4.17) the limit of the renewal equation's solution is given by

$$\lim_{n\to\infty} g_n = \frac{b_\infty}{\alpha} =: g_\infty.$$

Since we want to obtain the convergence rate in the defective case, we want to rewrite the renewal equation the following way:

$$\begin{aligned}
g_n - g_\infty &= b_n + \sum_{k=0}^{n} f_k g_{n-k} - g_\infty \\
&= b_n - g_\infty + \sum_{k=0}^{n} f_k (g_{n-k} - g_\infty + g_\infty) \\
&= b_n - g_\infty + g_\infty \sum_{k=0}^{n} f_k + \sum_{k=0}^{n} f_k (g_{n-k} - g_\infty) \\
&= b_n - g_\infty (1 - \sum_{k=0}^{n} f_k) + \sum_{k=0}^{n} f_k (g_{n-k} - g_\infty)
\end{aligned}$$

$$= b_n - g_\infty \sum_{k=n+1}^{\infty} f_k + \sum_{k=1}^{n} f_k(g_{n-k} - g_\infty)$$

$$= \underbrace{b_n - g_\infty \mathbb{P}(X_1 > n)}_{=:\tilde{b}_n} + \underbrace{\sum_{k=0}^{n} f_k(g_{n-k} - g_\infty)}_{=:\tilde{g}_{n-k}}.$$

We define $\hat{g}_n := g_n - g_\infty$ and therefore get the rewritten renewal equation

$$\tilde{g}_n = \tilde{b}_n + \sum_{k=0}^{n} f_k \tilde{g}_{n-k}.$$

Since all requirements are met, we can apply Theorem (4.18) and obtain

$$\begin{aligned}\lim_{n\to\infty} \gamma^n \tilde{g}_n &= \frac{\sum_{k=0}^{\infty} \gamma^k \tilde{b}_k}{\sum_{k=0}^{\infty} k\gamma^k f_k} = \frac{\sum_{k=0}^{\infty} \gamma^k b_k - g_\infty \sum_{k=0}^{\infty} \gamma^k \mathbb{P}(X_1 > k)}{\gamma \sum_{k=0}^{\infty} \gamma^k \mathbb{P}(X_1 > k)} \\ &= \frac{1}{\gamma}\left\{ \frac{\sum_{k=0}^{\infty} \gamma^k b_k}{\sum_{k=0}^{\infty} \gamma^k \mathbb{P}(X_1 > k)} - g_\infty \right\} = \frac{1}{\gamma}\left\{ \frac{\sum_{k=0}^{\infty} \gamma^k b_k}{\sum_{k=0}^{\infty} \gamma^k \mathbb{P}(X_1 > k)} - \frac{b_\infty}{\alpha} \right\},\end{aligned}$$

using that

$$\sum_{k=0}^{\infty} k\gamma^k f_k = \sum_{k=1}^{\infty}\sum_{j=1}^{k} \gamma^k f_k = \sum_{j=1}^{\infty}\sum_{k=j}^{\infty} \gamma^k f_k = \sum_{j=1}^{\infty} \gamma^j \mathbb{P}(X_1 \geq j) = \sum_{j=0}^{\infty} \gamma^{j+1} \mathbb{P}(X_1 \geq j+1)$$

which makes use of the assumption that $\sum_{k=0}^{\infty} \gamma^k f_k = 1$.

**4.20 Example:** (the Lotka-Volterra model)

This example is taken from Bremaud (2017, p. 448). Let $g_n$ describe the average number of daugthers born at time $n \in \mathbb{N}_0$ and $f_k$ the average number of daughters born in the $k$-th year of the mother's life. Obviously, $(f_k)_{k\in\mathbb{N}_0}$ is no probability distribution in general since neither $f_k$ has to be in $[0, 1]$ nor does they have to satisfy $\sum_{k=0}^{\infty} f_k = 1$. We will refrain from doing so for the moment and define $\alpha(i)$ as the number of women if age $i$ at time 0. Furthermore, let $b_n$ be the average number of daughters born at time $n$ born by mothers which themselves were born at or before time 0 and $r_n$ the average number of daughters born at time $n$ by mothers which in contrast were born at time $t \in (0, n]$. Therefore, it is $g_n = b_n + r_n$. In addition to that, we can write $g_n$ as the by now well-knwon renewal equation

$$g_n = b_n + \sum_{k=0}^{n} f_k g_{n-k},$$

with $b_n = \sum_{i=0}^{\infty} \alpha(i) f_{n+i}$. In this context, the renewal equation is called Lotka-Volterra equation (Bremaud, 2017, p. 448). To differentiate between two cases denote

$$\varrho := \sum_{k=1}^{\infty} f_k.$$

Assume that $0 < \varrho < \infty$ and additionally $\varrho \neq 1$. Even though, $(f_k)_{k\in\mathbb{N}_0}$ is not inevitably a probability distribution we assume it to be aperiodic. In the case that $\varrho < 1$ there exists a $\gamma$ such that $\sum_{k=1}^{\infty} \gamma^k f_k = 1$. This case is referred to as exponential extinction and Theorem (4.18) is applicable.

On the other hand, in the case that $\varrho > 1$ there exists a $\gamma < 1$ such that $\sum_{k=1}^{\infty} \gamma^k f_k = 1$. This corresponds to exponential explosion (Bremaud, 2017, p. 448). In this case, the renewal equation is called *excessive* and Theorem (4.18) can be adapted accordingly.

This example demonstrates that Theorem (4.18) can also be applied in a non-probability context by manipulating the the sequence $(f_k)_{k\in\mathbb{N}_0}$ accordingly.

## 4.3 Renewal Reward Theorem

Now we want to introduce another process, the so-called renewal reward process. To motivate this process, consider an insurance company that wants to describe the damage history of a client up to time $n \in \mathbb{N}_0$. We need a renewal counting process $(N(n))_{n\in\mathbb{N}_0}$ to get to know the number of cases of damage that have occurred up to time $n$. A renewal counting process is implied by a renewal process which we will again assume to be discrete. For example, the insurance company always records the claims at the end of the month. At each renewal time a claim is submitted. The amount of this damage can be understood as a random variable as well, which has its own arbitrary distribution. This leads to another sequence of random variables which we denote by $(Z_i)_{i\in\mathbb{N}}$. The total amount of the damage up to time $n$ would therefore be the sum of the first $N(n)$ damages. We call the implied sequence the renewal reward process. A cleaner representation provides the following definition:

**4.21 Definition:**

Given is a renewal process $(S_n)_{n\in\mathbb{N}_0}$ based on the i.i.d. sequence $(X_i)_{i\in\mathbb{N}}$, $X_i \in \mathbb{N}_0$ with renewal distribution $(f_n)_{n\in\mathbb{N}_0}$ and an i.i.d. sequence of random variables $(Z_i)_{i\in\mathbb{N}}$ with distribution $(g_n)_{n\in\mathbb{N}_0}$ and $(X_i, Z_i)_{i\in\mathbb{N}}$. The process $(\hat{Z}(n))_{n\in\mathbb{N}}$ defined by $\hat{Z}(n) := \sum_{i=1}^{N(n)} Z_i$ is called *renewal reward process.*

A naturally arising question is what average amount of damage the insurance company has to expect asymptotically from its clients. An answer to this question is provided be the following theorem (Bremaud, 2017, pp. 448-449).

**4.22 Theorem:**

Given is the situation from Definition (4.21) with $\mathbb{E}(X_1) < \infty$ and $\mathbb{E}|Z_1| < \infty$. Then applies:

1. $\lim_{n\to\infty} \frac{N(n)}{n} = \frac{1}{\mathbb{E}(X_1)}$
2. $\lim_{n\to\infty} \frac{\hat{Z}(n)}{n} = \frac{\mathbb{E}(Z_1)}{\mathbb{E}(X_1)}$

# 5 Delayed Renewal Processes

The task is again to observe the lifetime of a light bulb. But in contrast to Chapter 2, this time we start our observation at a random time, or in other terms, we missed the beginning. The light bulb may have already been on for a while before we began our observations, rather than being switched on for the first time at that moment. We cannot even say whether this is the first light bulb in the overall picture of the process but it is simply the first light bulb we see and we define it accordingly as the first. Since we started our observation some time later the lifetime of the first bulb will obviously be shorter in expectation than those of the succeeding ones. In general, even though the light bulbs are all the same and in fact have the same lifetime probability distribution from our perspective of the observer, the first light bulb will have a different distribution. For the following definition see Barbu and Limnios (2008, pp. 31-32).

For this chapter we assume $X_i > 0$ for $i = 1, 2, ...$ as we use the associated properties for a result which

## 5.1 General Delayed Renewal Processes

### 5.1 Definition:

Let $(X_i)_{i\in\mathbb{N}}$ be stochastically independent with $X_1$ having the distribution $b = (b_n)_{n\in\mathbb{N}_0}$ and $X_i$, $i \geq 2$, the distribution $f = (f_n)_{n\in\mathbb{N}_0}$. $f$ and $b$ are discrete probability distributions as defined in previous chapters. While $f$ is called *common distribution*, $b$ is referred to as the *initial distribution* (or *delayed distribution*). Then, $(S_n^D)_{n\in\mathbb{N}_0}$ defined as

$$S_n^D := \begin{cases} 0, & n = 0 \\ \sum_{i=1}^n X_i, & n \in \mathbb{N} \end{cases}$$

is called *delayed renewal process*. The corresponding renewal counting process is denoted as $(N^D(n))_{n\in\mathbb{N}_0}$ with $N^D(n) := \sum_{k=1}^\infty \mathbb{1}_{\{S_k \leq n\}}$.

Note that in the event of $f = b$ we have the ordinary renewal process as known from previous chapters.

We define the *associated renewal process* as $(S_n - S_1)_{n\geq 2}$, which is an ordinary renewal process. The probability that a renewal will occur at instant $n$ is given by $v_n := \sum_{k=0}^\infty \mathbb{P}(S_k = n)$. The same probability but for the associated renewal process is defined as $u_n := \sum_{k=2}^\infty \mathbb{P}(S_k - S_1 = n)$. The respective generating functions for $v$, $u$ and the initial distribution $b$ are given by $v(z) = \sum_{n=0}^\infty v_n z^n$, $u(z) = \sum_{n=0}^\infty u_n z^n$ and $b(z) = \sum_{n=0}^\infty b_n z^n$ (Barbu and Limnios, 2008, pp. 31-32).

A delayed renewal process is considered to be aperiodic or recurrent if the associated renewal process fulfills the corresponding properties.

Next, we want to develop a recursion formula for $(v_n)_{n\in\mathbb{N}_0}$ in order to derive a closed form for the generating function $v(z)$ as we did in Example (4.10) (Barbu and Limnios, 2008, pp. 32-33). We obtain

$$\begin{aligned}
v_n &= \sum_{r=0}^{\infty} \mathbb{P}(S_r = n) = \sum_{r=1}^{\infty}\sum_{k=0}^{\infty} \mathbb{P}(S_r = n \mid S_1 = k)\mathbb{P}(S_1 = k) \\
&= \sum_{k=0}^{n} \mathbb{P}(S_1 = k) \sum_{r=1}^{\infty} \mathbb{P}(S_r - S_1 = n - k) \\
&= \mathbb{P}(S_1 = n) \sum_{r=1}^{\infty} \mathbb{P}(S_r - S_1 = 0) + \sum_{k=0}^{n-1} \mathbb{P}(S_1 = k) \sum_{r=2}^{n-k} \mathbb{P}(S_r - S_1 = n - k) \\
&\overset{X_i \geq 0}{=} \mathbb{P}(S_1 = n) + \sum_{k=0}^{n-1} \mathbb{P}(S_1 = k) \sum_{r=2}^{n-k} \mathbb{P}(S_r - S_1 = n - k) = b_n + \sum_{k=0}^{n-1} b_k u_{n-k}
\end{aligned}$$

Since it is $u_0 = 1$ by definition (see Chapter 2) the result is given by

$$v_n = \sum_{k=0}^{n} b_k u_{n-k}.$$

Analogous to Example (4.10) we can now multiply both side by $z^n$ and sum over $n$. In order to do so, we will need the fact, that $a(z)b(z) = (a * b)(z)$ which is quickly proved by

$$\begin{aligned}
(5.1) \qquad a(z)b(z) &= \sum_{m=0}^{\infty} a_m z^m \sum_{n=0}^{\infty} b_n z_n = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} a_m b_n z^{m+n} = \sum_{k=0}^{\infty} z^k \sum_{m,n \geq 0 \,:\, m+n=k} a_m * b_n \\
&= \sum_{k=0}^{\infty} z^k \sum_{l=0}^{k} a_{k-l} b_l = \sum_{k=0}^{\infty} (a * b)_k z^k = (a * b)(z).
\end{aligned}$$

Using this result we move on with

$$\begin{aligned}
(5.2) \qquad v(z) &:= \sum_{n=0}^{\infty} v_n z^n = \sum_{n=0}^{\infty}\sum_{k=0}^{n} b_k u_{n-k} z^n = \sum_{n=0}^{\infty} (b * u) z^n \\
&= (b * u)(z) \overset{\text{eq. (5.1)}}{=} b(z)u(z) \overset{\text{eq. (4.1)}}{=} \frac{b(z)}{1 - \tilde{f}(z)}.
\end{aligned}$$

In the last step we used the fact, that the associated renewal process is an ordinary one and therefore the results from Example (4.10) are applicable. The just derived results will come in handy for the proof of the following theorem since it is a simple application of the key renewal theorem (Theorem (4.14)) (Barbu and Limnios, 2008, p. 33).

**5.2 Theorem:** (renewal theorem for delayed renewal processes)

Consider a delayed recurrent renewal process $(S_n)_{n \in \mathbb{N}}$ with positive inter arrival times and an initial distribution $b = (b_n)_{n \in \mathbb{N}_0}$.

1. If the process is aperiodic, then

$$\lim_{n\to\infty} v_n = \frac{1}{\mu}\sum_{n=0}^{\infty} b_n.$$

2. If the chain is periodic of period $d > 1$, then for any positive integer l, $0 \le l < d$,

$$\lim_{n\to\infty} v_{l+nd} = \frac{d}{\mu}\sum_{n=0}^{\infty} b_{l+nd}.$$

Now, we consider the following example taken from Barbu and Limnios (2008, p. 32).

### 5.3 Example:

Consider i.i.d. Bernoulli trials with the characteristics {S=success, F=failure}. Generate a sequence of those Bernoulli trials and obtain for example

{S S S **S F S** F F **S F *S*** *F S* S S ...}

The bold letters indicate a pattern in the case that no overlapping is permitted and the italic letters are the one additionally occurring pattern in the case of overlapping. Note that at least three trials are required to obtain a pattern. Therefore, the first possible pattern is observable at the third trial. When we define the occurrence of a pattern as a renewal we receive a delayed renewal process.

Now consider the probability distribution $\mathbb{P}(S) = p$ and $\mathbb{P}(F) = q = 1-p$ with $0 < p < 1$. We are once again interested in the pattern 'SFS' and allow overlapping. Let the initital distribution be $(b_n)_{n\in\mathbb{N}_0}$, $(f_n)_{n\in\mathbb{N}_0}$ the common distribution of the waiting times $(X_n)_{n\ge 2}$, $\mu = \mathbb{E}(X_2)$ and $v_n$ the probability that a renewal occurs at time $n$.

First of all, we want to show that the successive occurrence times of the pattern form an aperiodic delayed recurrent renewal process.

Since the Bernoulli trials are independent, the waiting times between the occurrences are independent as well. Furthermore, the common distribution of waiting times $X_m$, $m \ge 2$, is given by

$$\begin{aligned}
f_0 &= \mathbb{P}(X_m = 0) = 0\\
f_1 &= \mathbb{P}(X_m = 1) = \mathbb{P}(\text{renewal at } S_{m-1}+1) = p^2q\\
f_2 &= \mathbb{P}(X_m = 2) = \mathbb{P}(\text{renewal at } S_{m-1}+2,\ \text{not a renewal at } S_{m-1}+1) = p^2q(1-p^2q)\\
f_n &= \mathbb{P}(X_m = n) = p^2q(1-p^2q)^{n-1}, \quad n \ge 2.
\end{aligned}$$

It can be seen that $(f_n)_{n\in\mathbb{N}_0}$ is independent of n and therefore a renewal process. The aperiodicity is easily checked by ensuring that the probability for a waiting time of 1 is not zero. If it were zero we would obtain because, of the discrete nature of the distribution, a period $d \ge 2$. Indeed we have $f_1 = p^2q$ and therefore the renewal process is aperiodic.

Next, we want to calculate the initial distribution. Since it is impossible for the pattern, consisting of three trials to occur in the first two trials, it is $b_0 = \mathbb{P}(X_1 = 0) = 0$ and

$b_1 = \mathbb{P}(X_1 = 1) = 0$. The probabilities for $n \geq 2$ are analog to the common distribution, i.e. $b_n = \mathbb{P}(X_1 = n) = p^2q(1-p^2q)^{n-2}$. Obviously the distributions of $X_1$ differs from $X_n$, $n \geq 2$. Thus, according to Definition (5.1) the renewal process is delayed. Furthermore, it is

$$\begin{aligned}\sum_{n=0}^{\infty} f_n &= 0 + p^2q + p^2q\sum_{n=2}^{\infty}(1-p^2q)n - 1 = p^2q + p^2q\sum_{n=0}^{\infty}(1-p^2q)n + 1 \\ &= p^2q + p^2q(1-p^2q)\sum_{n=0}^{\infty}(1-p^2q)n = p^2q + p^2q(1-p^2q)\frac{1}{1-(1-p^2q)} \\ &= p^2q + 1 - p^2q = 1\end{aligned}$$

and therefore a recurrent renewal process. Overall, the desired properties have now been demonstrated.

Since renewals still cannot arise in the first two trials, we get $v_0 = v_1 = 0$. On the other hand, for the probability for a renewal occurring at time $n$ for $n \geq 2$ we have

$$\begin{aligned}v_n &= \mathbb{P}(Z_n = 1) \\ &= \mathbb{P}(S \text{ at the } n\text{-th trial, } F \text{ at the } (n-1)\text{-th trial, } S \text{ at the } (n-2)\text{-th trial}) = p^2q.\end{aligned}$$

Since $v_n$ is independent of $n$ the limit is $\lim_{n\to\infty} v_n = p^2q$. We want to verify, that the result of Theorem (5.2) also holds true for this example. It is

$$\sum_{n=0}^{\infty} b_n = \sum_{n=2}^{\infty} p^2q(1-p^2q)^{n-2} = \sum_{n=0}^{\infty} p^2q(1-p^2q)^n = p^2q\frac{1}{1-(1-p^2q)} = 1.$$

Thus, with $\mu = 1/p^2q$ we obtain

$$\lim_{n\to\infty} v_n \overset{\text{Th. (5.2)}}{=} \frac{1}{\mu}\sum_{n=0}^{\infty} = \frac{1}{p^2q},$$

which is consistent with the limit we determined by foot before.

## 5.2 Stationary Renewal Process

Now we will take a look at a particular type of delayed renewal processes where the initial distribution has a certain property. This distribution is chosen so that $v_n$, which we have previously defined as the probability that a renewal will take place at time $n$, is independent of $n$. We refer to this process as *stationary renewal process* while the initial distribution is called *stationary distribution of the delayed renewal process* $(S_n)_{n\in\mathbb{N}}$ (Barbu and Limnios, 2008, pp. 34-35).

We want to derive this stationary distribution by considering a delayed renewal process $(S_n)_{n\in\mathbb{N}}$ with a recurrent initial distribution, $(b_n)_{n\in\mathbb{N}}$. Since our goal is to obtain a with respect to $n$ constant expression for $v_n$ we will denote it by $\Delta = v_n$ for $n \in \mathbb{N}_0$. Therefore, its generating function is given by (Barbu and Limnios, 2008, pp. 34-35)

$$v(z) = \sum_{n=0}^{\infty} \Delta z^n = \Delta \sum_{n=0}^{\infty} z^n = \frac{\Delta}{1-z}, \tag{5.3}$$

for $|z| < 1$. Using equation (5.2) we get

$$v(z) = b(z)u(z) \iff b(z) = \frac{v(z)}{u(z)} \overset{\text{eq. (5.3)}}{\iff} b(z) = \frac{\Delta}{1-z}(1 - \tilde{f}(z))$$

and therefore

$$b(z) = \sum_{n=0}^{\infty} b_n z^n = \Delta\left(\sum_{n=0}^{\infty} z^n\right)\left(1 - \sum_{n=1}^{\infty} f_n z^n\right).$$

With the help of a coefficient comparison, we now want to find expressions for $b_n$ for each $n \in \mathbb{N}_0$. We apply the Cauchy product on both series of the last equation,

$$\Delta\left(\sum_{n=0}^{\infty} z^n\right)\left(1 - \sum_{n=1}^{\infty} f_n z^n\right) = \Delta\left(\sum_{n=0}^{\infty} z^n - \sum_{n=0}^{\infty}\sum_{k=0}^{n} f_k z^n\right).$$

Thus, the coefficients of $z^n$ for each $n \in \mathbb{N}_0$ are given by

$$b_n = \begin{cases} \Delta, & n = 0 \\ \Delta\left(1 - \sum_{k=0}^{n} f_k\right) = \mathbb{P}(X_1 > n), & n \in \mathbb{N} \end{cases}. \tag{5.4}$$

Since we assumed the initial distribution to be recurrent, we have

$$1 = \sum_{n=0}^{\infty} b_n \overset{\text{eq. (5.4)}}{=} \sum_{n=0}^{\infty} \Delta \mathbb{P}(X_1 > n) \iff \Delta = \frac{1}{\sum_{n=0}^{\infty} \mathbb{P}(X_1 > n)} = \frac{1}{\mu}.$$

Accordingly, the desired initial distribution is given by

$$b_n = \mathbb{P}(X_1 = n) = \frac{1}{\mu}\mathbb{P}(X_1 > n), \quad n \in \mathbb{N}_0. \tag{5.5}$$

Recall that the initial distribution should be chosen such that the generating function $v(z)$ is independent of $n$. We verify this property by first calculating $b(z)$ using the just obtained distribution, equation (5.5):

$$b(z) = \sum_{n=0}^{\infty} b_n z^n = \frac{1}{\mu}\sum_{n=0}^{\infty} \mathbb{P}(X_1 > n) z^n = \sum_{n=0}^{\infty}\sum_{k=n+1}^{\infty} \mathbb{P}(X_1 = k) z^n$$

$$= \frac{1}{\mu}\sum_{k=1}^{\infty} f_k \sum_{n=0}^{k-1} z^n = \frac{1}{\mu}\sum_{k=1}^{\infty} f_k \frac{1-z^k}{1-z} = \frac{1}{\mu}\frac{1}{1-z}\Bigg(\underbrace{\sum_{k=1}^{\infty} f_k}_{=1} - \underbrace{\sum_{k=1}^{\infty} f_k z^k}_{=\tilde{f}(z)}\Bigg)$$

$$= \frac{1}{\mu}\frac{1-\tilde{f}(z)}{1-z}.$$

Using equation (5.2) again provides

$$v(z) = b(z) \underbrace{u(z)}_{=1/(1-\tilde{f}(z))} = \frac{1}{\mu}\frac{1}{1-z} = \frac{1}{\mu}\sum_{n=0}^{\infty} z^n, \quad 0 \le z < 1$$

and therefore $v_n = 1/\mu$ for all $n \in \mathbb{N}_0$ which is consistent with our requirements. Barbu and Limnios (2008, p. 36) condensed this into the following result.

**5.4 Proposition:**

Let $(S_n)_{n\in\mathbb{N}}$ be a recurrent delayed renewal process with waiting times $(X_n)_{n\in\mathbb{N}_0}$ and $\mu := \mathbb{E}X_1 < \infty$.Then, $\mathbb{P}(X_1 = n) := 1/\mu\mathbb{P}(X_1 > n)$ is the unique choice for the initial distribution of the delayed renewal chain such that $v_n \equiv$ constant for all $n \in \mathbb{N}_0$. Moreover, this common constant is $1/\mu$. In this case $(S_n)_{n\in\mathbb{N}}$ is a stationary renewal process with the initial distribution being the stationary distribution.

In the next chapter we will examine among other things how the life time processes introduced at the end of Chapter 2 are related to the stationary distribution and obtain an intuitive explanation for the time invariant property from Proposition (5.4).

# 6 Lifetime Processes

Recall the three lifetime processes we defined in Chapter 2. They were given by $U(n) = n - S_{N(n)}$, $V(n) = S_{N(n)+1} - n$ and $L(n) = X_{N(n)+1}$. We want to determine their distribution as a function of $u_n$ in order to apply the renewal theorem in the following. For the backward recurrence time $U(n)$ we get (Mitov and Omey, 2014, pp. 61-62)

$$\begin{aligned}
\mathbb{P}(U(m) = n) &= \mathbb{P}(m - S_{N(m)} = n) = \mathbb{P}(S_{N(m)} = m - n) \\
&\overset{\text{LotP}}{=} \sum_{k=0}^{\infty} \mathbb{P}(S_{N(n)} = m - n,\, N(m) = k) = \sum_{k=0}^{\infty} \mathbb{P}(S_k = m - n,\, N(m) = k) \\
&\overset{\text{Feller}}{=} \sum_{k=0}^{\infty} \mathbb{P}(S_k = m - n,\, S_k \leq m < S_{k+1}) \\
&\overset{S_{N(m)}=m-n}{=} \sum_{k=0}^{\infty} \mathbb{P}(S_k = m - n,\, m - n \leq m < m - n + X_{k+1}) \\
&= \sum_{k=0}^{\infty} \mathbb{P}(S_k = m - n,\, 0 \leq m - (m - n) < X_{k+1}) \\
&= \sum_{k=0}^{\infty} \mathbb{P}(S_k = m - n,\, n < X_{k+1}) \overset{\text{s.i.}}{=} \sum_{k=0}^{\infty} \mathbb{P}(S_k = m - n)\mathbb{P}(n < X_{k+1}) \\
&= \mathbb{P}(X_1 > n) \sum_{k=0}^{\infty} \mathbb{P}(S_k = m - n) = u_{m-n}\mathbb{P}(X_1 > n).
\end{aligned}$$

The distribution of $V(n)$ is slightly more complex to obtain but we start similarly by conditioning on $N(m) + 1$ and then try to decompose the expression with the help of the conditional probability.

$$\begin{aligned}
\mathbb{P}(V(m) = n) &= \mathbb{P}(S_{N(m)+1} - m = n) = \mathbb{P}(S_{N(m)+1} = n + m) \\
&= \sum_{k=1}^{\infty} \mathbb{P}(S_{N(m)+1} = n + m,\, N(m) + 1 = k) \\
&= \sum_{k=1}^{\infty} \mathbb{P}(S_k = n + m,\, N(m) = k - 1) \\
&\overset{\text{Feller}}{=} \sum_{k=1}^{\infty} \mathbb{P}(S_k = n + m,\, S_{k-1} \leq m < S_k) \\
&= \sum_{k=1}^{\infty} \mathbb{P}(S_k = m + n)\mathbb{P}(S_{k-1} \leq m < S_k \,|\, S_k = m + n)
\end{aligned}$$

$$= \sum_{k=1}^{\infty} \mathbb{P}(S_k = m+n)\mathbb{P}(S_{k-1} \leq m < S_k \,|\, S_{k-1} + X_k = m+n)$$

$$= \sum_{k=1}^{\infty} \mathbb{P}(S_{k-1} + X_k = m+n)\mathbb{P}(S_{k-1} \leq m < S_{k-1} + X_k \,|\, S_{k-1} + X_k = m+n)$$

$$= \sum_{k=1}^{\infty} \mathbb{P}(X_k = m+n-S_{k-1})\mathbb{P}(S_{k-1} \leq m < S_{k-1} + X_k \,|\, S_{k-1} + X_k = m+n)$$

$$= \sum_{k=1}^{\infty} \mathbb{P}(X_k = m+n-S_{k-1})\mathbb{P}(S_{k-1} \leq m)$$

$$= \sum_{k=1}^{\infty} \sum_{j=0}^{m} \mathbb{P}(X_k = m+n-j)\mathbb{P}(S_{k-1} = j)$$

$$= \sum_{j=0}^{m} \mathbb{P}(X_1 = m+n-j) \underbrace{\mathbb{P}(S_0 = j)}_{S_0 := 0 \implies 1 \Leftrightarrow j=0} + \sum_{k=2}^{\infty} \sum_{j=0}^{m} \mathbb{P}(X_k = m+n-j)\mathbb{P}(S_{k-1} = j)$$

$$= \mathbb{P}(X_1 = m+n) + \sum_{k=2}^{\infty} \sum_{j=1}^{m} \mathbb{P}(X_k = m+n-j)\mathbb{P}(S_{k-1} = j)$$

$$= \mathbb{P}(X_1 = m+n) + \sum_{j=1}^{m} \mathbb{P}(X_1 = m+n-j) \sum_{k=2}^{\infty} \mathbb{P}(S_{k-1} = j)$$

$$= \mathbb{P}(X_1 = m+n) + \sum_{j=1}^{m} \mathbb{P}(X_1 = m+n-j) \sum_{k=1}^{j} \mathbb{P}(S_k = j)$$

$$= \mathbb{P}(X_1 = m+n) + \sum_{j=1}^{m} \mathbb{P}(X_1 = m+n-j) \sum_{k=0}^{j} \mathbb{P}(S_k = j)$$

$$= \mathbb{P}(X_1 = m+n) + \sum_{j=1}^{m} \mathbb{P}(X_1 = m+n-j) u_j$$

$$= \sum_{j=0}^{m} \mathbb{P}(X_1 = m+n-j) u_j.$$

According to Mitov and Omey (2014, p. 62) we can put the distribution of the total lifetime in a similar form as we did with the backward recurrence time by

$$\mathbb{P}(L(m) = n) = \mathbb{P}(X_{N(m)+1} = n) = \sum_{k=0}^{\infty} \mathbb{P}(X_{k+1} = n,\, N(m) = k)$$

$$= \sum_{k=0}^{\infty} \mathbb{P}(X_{k+1} = n,\, S_k \leq m < S_{k+1})$$

$$
\begin{aligned}
&= \sum_{k=0}^{\infty} \mathbb{P}(X_{k+1} = n,\, S_k \le m < S_k + X_{k+1}) \\
&= \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} \mathbb{P}(X_{k+1} = n,\, S_k \le m < S_k + X_{k+1}, S_k = j) \\
&= \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} \mathbb{P}(X_{k+1} = n,\, j \le m < j + n, S_k = j) \\
&= \sum_{k=0}^{\infty} \sum_{j=m-n+1}^{m} \mathbb{P}(X_{k+1} = n)\mathbb{P}(S_k = j) \\
&= \mathbb{P}(X_1 = n) \sum_{j=m-n+1}^{m} \sum_{k=0}^{\infty} \mathbb{P}(S_k = j) \\
&= \mathbb{P}(X_1 = n) \sum_{j=m-n+1}^{m} u_j.
\end{aligned}
$$

Using the renewal theorem we can determine the limit of the distribution of the backward recurrence time and the total lifetime. We obtain

$$
\lim_{m\to\infty} \mathbb{P}(U(m) = n) = \lim_{m\to\infty} u_{m-n}\mathbb{P}(X_1 > n) \overset{\text{Th. (4.9)}}{=} \frac{1}{\mu}\mathbb{P}(X_1 > n)
$$

and

$$
\begin{aligned}
\lim_{m\to\infty} \mathbb{P}(L(m) = n) &= \lim_{m\to\infty} \mathbb{P}(X_1 = n) \sum_{j=m-n+1}^{m} u_j = \mathbb{P}(X_1 = n) \sum_{j=m-n+1}^{m} \lim_{m\to\infty} u_j \\
&\overset{\text{Th. (4.9)}}{=} \mathbb{P}(X_1 = n) \sum_{j=m-n+1}^{m} \frac{1}{\mu} = \frac{n}{\mu}\mathbb{P}(X_1 = n).
\end{aligned}
$$

We can therefore formulate the following theorem (Mitov and Omey, 2014, p. 62):

**6.1 Theorem:**

Assume that $0 < \mu < \infty$. Then

1. $\lim_{m\to\infty} \mathbb{P}(U(m) = n) = \frac{1}{\mu}\mathbb{P}(X_1 > n)$
2. $\lim_{m\to\infty} \mathbb{P}(V(m) = n) = \frac{1}{\mu}\mathbb{P}(X_1 \ge n)$
3. $\lim_{m\to\infty} \mathbb{P}(L(m) = n) = \frac{n}{\mu}\mathbb{P}(X_1 = n)$.

Note that we have already proved the second assertion in Example (4.16).

**6.2 Example:** (Bernoulli distribution part 5)
Consider a renewal process like in Example (2.10)). We want to determine the distributions of the three lifetime processes, we calculated earlier. We begin with the backward recurrence time:

$$\mathbb{P}(U(m) = n) = u_{m-n}\mathbb{P}(X_1 > n) \overset{\text{eq. (2.4)}}{=} \frac{1}{p}\mathbb{P}(X_1 > n).$$

For the forward recurrence time we get

$$\begin{aligned}\mathbb{P}(V(m) = n) &= \sum_{j=0}^{m} \mathbb{P}(X_1 = m + n - j)u_j \overset{\text{eq. (2.4)}}{=} \frac{1}{p}\sum_{j=0}^{m} \mathbb{P}(X_1 = m + n - j) \\ &= \frac{1}{p}\Big(\mathbb{P}(X_1 = m + n) + ... + \mathbb{P}(X_1 = n)\Big) \xrightarrow{n\to\infty} \frac{1}{p}\sum_{k=0}^{\infty} \mathbb{P}(X_1 = n + k) \\ &= \frac{1}{p}\mathbb{P}(X_1 \geq n).\end{aligned}$$

The last distribution is for the total lifetime and easily derived by

$$\mathbb{P}(L(m) = n) = \mathbb{P}(X_1 = n) \sum_{j=m-n+1}^{m} \frac{1}{p} = \mathbb{P}(X_1 = n)\frac{n}{p} = \frac{n}{\mu}\mathbb{P}(X_1 = n).$$

Comparing those results with Theorem (6.1) we find that these are consistent.

Next we want to derive a renewal equation for the tail of the distribution of the forward recurrence time $V(m)$ and solve it with the help of generating functions as we already did in previous chapters to find a closed form for the said distribution (Mitow and Omey, 2014, pp. 63-64). By conditioning on the $X_1 = i$ we get

$$\mathbb{P}(V(m) > n) = \sum_{i=1}^{\infty} \mathbb{P}(V(m) > \,|\, X_1 = i)\mathbb{P}(X_1 = i) = \sum_{i=1}^{\infty} \mathbb{P}(V(m) > \,|\, X_1 = i)f_i.$$

We would like to learn more about the conditional probability and divide it into three cases. First, assume that $i > m + n$. Since $S_{N(m)+1}$ consists of at least $X_1$ no matter the value of $N(m)$ it is

$$\begin{aligned}\mathbb{P}(S_{N(m)+1} - m > n \,|\, X_1 > m + n) &= \mathbb{P}(\sum_{k=1}^{N(m)+1} \underbrace{X_k}_{>0} > m + n \,|\, X_1 > m + n) \\ &\geq \mathbb{P}(X_1 > m + n \,|\, X_1 > m + n) = 1.\end{aligned}$$

For the second case consider $m < i \leq m+n$. Since $N(m) = \sup\{l \in \mathbb{N}_0 \,|\, S_l \leq m\}$ by definition, it is $N(m) = 0$ since $X_1 > m$. Therefore,

$$\begin{aligned}
\mathbb{P}(V(m) > n \,|\, m < X_1 \leq m+n) &= \mathbb{P}(S_{N(m)+1} > m+n \,|\, m < X_1 \leq m+n) \\
&= \mathbb{P}(\sum_{k=1}^{N(m)+1} X_k > m+n \,|\, m < X_1 \leq m+n) \\
&= \mathbb{P}(X_1 > m+n \,|\, m < X_1 \leq m+n) = 0,
\end{aligned}$$

because $X_1 > m+n$ given $X_1 \leq m+n$ is obviously a contradiction. The third and final case considers the event of $i \leq m$:

$$\begin{aligned}
\mathbb{P}(V(m) > n \,|\, X_1 = i) &= \mathbb{P}(S_{N(m)+1} > m+n \,|\, X_1 = i) \\
&= \mathbb{P}(X_1 + \sum_{k=2}^{N(m)+1} X_k > m+n \,|\, X_1 = i) \\
&= \mathbb{P}(i + \sum_{k=2}^{N(m)+1} X_k > m+n) = \mathbb{P}(\sum_{k=1}^{N(m-i)+1} X_k > m+n) \\
&= \mathbb{P}(S_{N(m-i)+1} > m+n) = \mathbb{P}(V(m-i) > n).
\end{aligned}$$

Overall, we can conclude that

$$\mathbb{P}(V(m) > \,|\, X_1 = i) = \begin{cases} 1, & \text{if } i > m+n \\ 0, & \text{if } m < i \leq m+n \\ \mathbb{P}(V(m-i) > n), & \text{if } i \leq m \end{cases}. \tag{6.1}$$

This leads to the following renewal equation:

$$\begin{aligned}
\mathbb{P}(V(m) > n) &= \sum_{i=1}^{\infty} \mathbb{P}(V(m) > n \,|\, X_1 = i) f_i \\
&= \sum_{i=m+n+1}^{\infty} \underbrace{\mathbb{P}(V(m) > n \,|\, X_1 = i)}_{=1} f_i + \sum_{i=m+1}^{m+n} \underbrace{\mathbb{P}(V(m) > n \,|\, X_1 = i)}_{=0} f_i \\
&\quad + \sum_{i=1}^{m} \mathbb{P}(V(m) > n \,|\, X_1 = i) f_i \\
&= \sum_{i=m+n+1}^{\infty} f_i + \sum_{i=0}^{m} \mathbb{P}(V(m-i) > n) f_i
\end{aligned}$$

$$= \mathbb{P}(X_1 > m+n) + \sum_{i=0}^{m} \mathbb{P}(V(m-i) > n) f_i.$$

Define two generating functions as $g_n(z) := \sum_{m=0}^{\infty} \mathbb{P}(V(m) > n) z^m$ and $b(z) := \sum_{m=0}^{\infty} \mathbb{P}(X_1 > m+n) z^m$. When we multiply both sides of the renewal equation by $z^m$ and sum over $m$ as we already did in previous chapters, we obtain

$$\sum_{m=0}^{\infty} \mathbb{P}(V(m) > n) z^m = \sum_{m=0}^{\infty} \mathbb{P}(X_1 > m+n) z^m + \sum_{m=0}^{\infty} \sum_{i=0}^{m} \mathbb{P}(V(m-i) > n) f_i z^m,$$

which becomes

$$g_n(z) = b(z) + g_n(z) \tilde{f}(z) \tag{6.2}$$

with equation (5.1). We can rearrange equation (6.2) and obtain

$$g_n(z)(1 - \tilde{f}(z)) = b(z) \quad \Leftrightarrow \quad g_n(z) = \frac{b(z)}{1 - \tilde{f}(z)} \overset{\text{eq. (4.1)}}{=} b(z) u(z). \tag{6.3}$$

If we write out the generating functions, we get

$$\sum_{m=0}^{\infty} \mathbb{P}(V(m) > n) z^m \overset{\text{eq. (6.3)}}{=} \sum_{m=0}^{\infty} \sum_{i=0}^{m} \mathbb{P}(X_1 > m+n-i) u_i z^m,$$

which leads to the fact, that the tail distribution of $V(m)$ is given by

$$\mathbb{P}(V(m) > n) = \sum_{i=0}^{m} \mathbb{P}(X_1 > m+n-i) u_i.$$

Now we will once again determine the distribution of the forward recurrence time for geometrically distributed inter arrival times, using only the results from Chapters 2 and 4. This the Exercise 7.6.1 taken from Pinsky and Karlin (2011).

**6.3 Example:**

Consider a renewal process with geometrically distributed inter arrival times $X_1, X_2, ...$, i.e. $f_k = \mathbb{P}(X_1 = k) = p(1-p)^{k-1}$, for $k = 1, 2, ...$ and $p \in (0, 1)$. We aim to determine the distribution of the forward recurrence time $V(n) = S_{N(n)+1} - n$. In order to do, so we want to use Lemma (2.15). When we set $g_n(m) = \mathbb{P}(V(n) = m)$ we get with equation (4.6)

$$g_n(m) = f_{m+n} + \sum_{k=0}^{n} g_{n-k}(m) f_k.$$

Furthermore, in Example (2.16) we already determined the solution of the basic renewal equation with geometrically distributed inter arrival times. We got

$$h_n = \begin{cases} 1, & n = 0 \\ p, & n > 0 \end{cases}.$$

as solution of $h_n = \delta_n + \sum_{k=0}^{n} f_k h_{n-k}$. Therefore, we are able to apply Lemma (2.15):

$$\begin{aligned}
g_n(m) &= \sum_{k=0}^{n} b_{n-k} h_k \overset{\text{Ex. (2.16)}}{=} f_m + \sum_{k=1}^{n} f_{m+n-k} p \\
&= p(1-p)^{m-1} + \sum_{k=1}^{n} p^2 (1-p)^{m+n-k} = p(1-p)^{m-1} + \sum_{k=0}^{n-1} p^2 (1-p)^{m+n-k-1} \\
&= p(1-p)^{m-1} + p^2 (1-p)^{m+n-1} \sum_{k=0}^{n-1} (1-p)^{-k} \\
&= p(1-p)^{m-1} + p^2 (1-p)^{m+n-1} \frac{1-(1-p)^{-n}}{-1} \\
&= p(1-p)^{m-1} - p(1-p)^{m+n} - p(1-p)^{m}
\end{aligned}$$

and therefore

$$\mathbb{P}(V(n) = m) = p(1-p)^{m-1} - p(1-p)^{m+n} - p(1-p)^{m}$$

as the distribution of the the forward recurrence time.

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