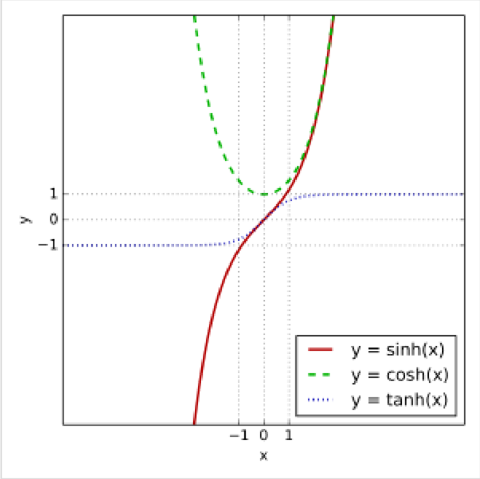
Hyperbolic functions



In mathematics, **hyperbolic functions** are analogues of the ordinary trigonometric functions, but defined using the hyperbola rather than the circle. Just as the points $(\cos t, \sin t)$ form a circle with a unit radius, the points $(\cosh t, \sinh t)$ form the right half of the unit hyperbola. Also, similarly to how the derivatives of $\sin(t)$ and $\cos(t)$ are $\cos(t)$ and $-\sin(t)$ respectively, the derivatives of $\sinh(t)$ and $\cosh(t)$ are $\cosh(t)$ and $+\sinh(t)$ respectively.

Hyperbolic functions occur in the calculations of angles and distances in hyperbolic geometry. They also occur in the solutions of many linear differential equations (such as the equation defining a catenary), cubic equations, and Laplace's equation in Cartesian coordinates. Laplace's equations are important in many areas of physics, including electromagnetic theory, heat transfer, fluid dynamics, and special relativity.

The basic hyperbolic functions are:[1]

- **hyperbolic sine** "sinh" (/ˈsɪŋ, ˈsɪntʃ, ˈʃaɪn/),[2]
- **hyperbolic cosine** "cosh" (/ˈkɒʃ, ˈkoʊʃ/),[3]

from which are derived:[4]

- **hyperbolic tangent** "tanh" (/ˈtæŋ, ˈtæntʃ, ˈθæn/),[5]
- **hyperbolic cosecant** "csch" or "cosech" (/ˈkoʊsɛtʃ, ˈkoʊʃɛk/[3])
- **hyperbolic secant** "sech" (/ˈsɛtʃ, ˈʃɛk/),[6]
- **hyperbolic cotangent** "coth" (/ˈkɒθ, ˈkoʊθ/),[7][8]

corresponding to the derived trigonometric functions.

The inverse hyperbolic functions are:

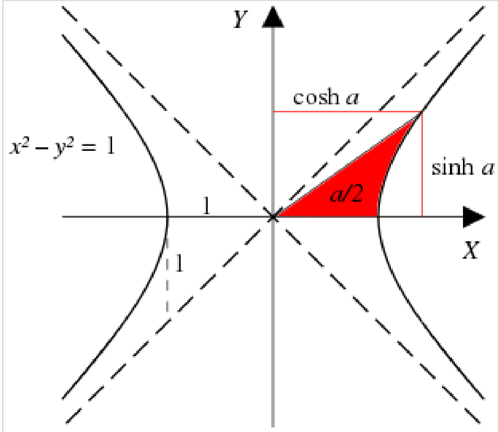
- **area hyperbolic sine** "arsinh" (also denoted "$\sinh^{-1}$", "asinh" or sometimes "arcsinh")[9][10][11]
- **area hyperbolic cosine** "arcosh" (also denoted "$\cosh^{-1}$", "acosh" or sometimes "arccosh")
- **area hyperbolic tangent** "artanh" (also denoted "$\tanh^{-1}$", "atanh" or sometimes "arctanh")
- **area hyperbolic cosecant** "arcsch" (also denoted "arcosech", "csch^{-1}", "$\operatorname{cosech}^{-1}$","acsch", "acosech", or sometimes "arccosh" or "arccosech")
- **area hyperbolic secant** "arsech" (also denoted "sech^{-1}", "asech" or sometimes "arcsech")
- **area hyperbolic cotangent** "arcoth" (also denoted "$\coth^{-1}$", "acoth" or sometimes "arccoth")

The hyperbolic functions take a real argument called a hyperbolic angle. The size of a hyperbolic angle is twice the area of its hyperbolic sector. The hyperbolic functions may be defined in terms of the legs of a right triangle covering this sector.

In complex analysis, the hyperbolic functions arise when applying the ordinary sine and cosine functions to an imaginary angle. The hyperbolic sine and the hyperbolic cosine are entire functions. As a result, the other hyperbolic functions are meromorphic in the whole complex plane.

By Lindemann–Weierstrass theorem, the hyperbolic functions have a transcendental value for every non-zero algebraic value of the argument.[12]

Hyperbolic functions were introduced in the 1760s independently by Vincenzo Riccati and Johann Heinrich Lambert.[13] Riccati used $Sc.$ and $Cc.$ (*sinus/cosinus circulare*) to refer to circular functions and $Sh.$ and $Ch.$ (*sinus/cosinus hyperbolico*) to refer to hyperbolic functions. Lambert adopted the names, but altered the abbreviations to those used today.[14] The abbreviations sh, ch, th, cth are also currently used, depending on personal preference.



A ray through the unit hyperbola $x^2 - y^2 = 1$ at the point $(\cosh a, \sinh a)$, where a is twice the area between the ray, the hyperbola, and the x-axis. For points on the hyperbola below the x-axis, the area is considered negative (see animated version with comparison with the trigonometric (circular) functions).

Notation

Definitions

There are various equivalent ways to define the hyperbolic functions.

Exponential definitions

In terms of the exponential function:[1][4]

- Hyperbolic sine: the odd part of the exponential function, that is,

$$\sinh x = \frac{e^x - e^{-x}}{2} = \frac{e^{2x} - 1}{2e^x} = \frac{1 - e^{-2x}}{2e^{-x}}.$$

- Hyperbolic cosine: the even part of the exponential function, that is,

$$\cosh x = \frac{e^x + e^{-x}}{2} = \frac{e^{2x} + 1}{2e^x} = \frac{1 + e^{-2x}}{2e^{-x}}.$$

- Hyperbolic tangent:

$$\tanh x = \frac{\sinh x}{\cosh x} = \frac{e^x - e^{-x}}{e^x + e^{-x}} = \frac{e^{2x} - 1}{e^{2x} + 1}.$$

- Hyperbolic cotangent: for $x \neq 0$,

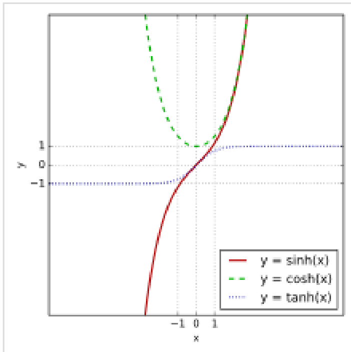
$$\coth x = \frac{\cosh x}{\sinh x} = \frac{e^x + e^{-x}}{e^x - e^{-x}} = \frac{e^{2x} + 1}{e^{2x} - 1}.$$

- Hyperbolic secant:

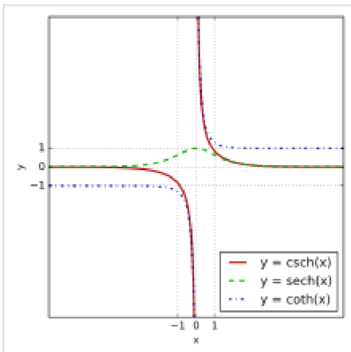
$$\operatorname{sech} x = \frac{1}{\cosh x} = \frac{2}{e^x + e^{-x}} = \frac{2e^x}{e^{2x} + 1}.$$

- Hyperbolic cosecant: for $x \neq 0$,

$$\operatorname{csch} x = \frac{1}{\sinh x} = \frac{2}{e^x - e^{-x}} = \frac{2e^x}{e^{2x} - 1}.$$



sinh, cosh and tanh



csch, sech and coth

Differential equation definitions

The hyperbolic functions may be defined as solutions of differential equations: The hyperbolic sine and cosine are the solution (s, c) of the system

$$\begin{aligned} c'(x) &= s(x), \\ s'(x) &= c(x), \end{aligned}$$

with the initial conditions $s(0) = 0, c(0) = 1$. The initial conditions make the solution unique; without them any pair of functions $(ae^x + be^{-x}, ae^x - be^{-x})$ would be a solution.

$\sinh(x)$ and $\cosh(x)$ are also the unique solution of the equation $f''(x) = f(x)$, such that $f(0) = 1$, $f'(0) = 0$ for the hyperbolic cosine, and $f(0) = 0$, $f'(0) = 1$ for the hyperbolic sine.

Complex trigonometric definitions

Hyperbolic functions may also be deduced from trigonometric functions with complex arguments:

- Hyperbolic sine:[1]

  $$\sinh x = -i \sin(ix).$$

- Hyperbolic cosine:[1]

  $$\cosh x = \cos(ix).$$

- Hyperbolic tangent:

  $$\tanh x = -i \tan(ix).$$

- Hyperbolic cotangent:

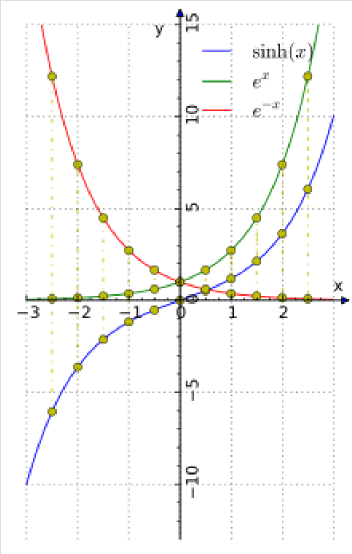
  $$\coth x = i \cot(ix).$$

- Hyperbolic secant:

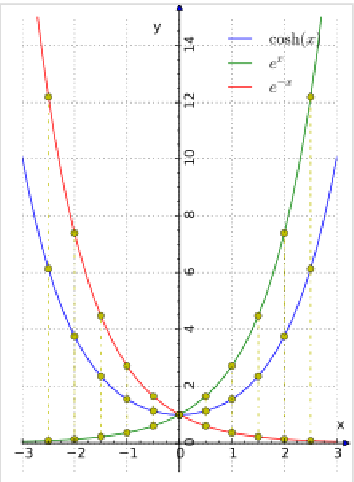
  $$\operatorname{sech} x = \sec(ix).$$

- Hyperbolic cosecant:

  $$\operatorname{csch} x = i \csc(ix).$$



sinh *x* is half the difference of e^x and e^{-x}



cosh *x* is the average of e^x and e^{-x}

where i is the imaginary unit with $i^2 = -1$.

The above definitions are related to the exponential definitions via Euler's formula (See § Hyperbolic functions for complex numbers below).

Characterizing properties

Hyperbolic cosine

It can be shown that the area under the curve of the hyperbolic cosine (over a finite interval) is always equal to the arc length corresponding to that interval:[15]

$$\text{area} = \int_a^b \cosh x \, dx = \int_a^b \sqrt{1 + \left(\frac{d}{dx} \cosh x\right)^2} \, dx = \text{arc length.}$$

Hyperbolic tangent

The hyperbolic tangent is the (unique) solution to the differential equation $f' = 1 - f^2$, with $f(0) = 0$.[16][17]

Useful relations

The hyperbolic functions satisfy many identities, all of them similar in form to the trigonometric identities. In fact, **Osborn's rule**[18] states that one can convert any trigonometric identity for θ, 2θ, 3θ or θ and φ into a hyperbolic identity, by expanding it completely in terms of integral powers of sines and cosines, changing sine to sinh and cosine to cosh, and switching the sign of every term containing a product of two sinhs.

Odd and even functions:

$$\begin{aligned} \sinh(-x) &= -\sinh x \\ \cosh(-x) &= \cosh x \end{aligned}$$

Hence:

$$\begin{aligned}\tanh(-x) &= -\tanh x\\ \coth(-x) &= -\coth x\\ \operatorname{sech}(-x) &= \operatorname{sech} x\\ \operatorname{csch}(-x) &= -\operatorname{csch} x\end{aligned}$$

Thus, $\cosh x$ and $\operatorname{sech} x$ are even functions; the others are odd functions.

$$\begin{aligned}\operatorname{arsech} x &= \operatorname{arcosh}\left(\frac{1}{x}\right)\\ \operatorname{arcsch} x &= \operatorname{arsinh}\left(\frac{1}{x}\right)\\ \operatorname{arcoth} x &= \operatorname{artanh}\left(\frac{1}{x}\right)\end{aligned}$$

Hyperbolic sine and cosine satisfy:

$$\begin{aligned}\cosh x + \sinh x &= e^x\\ \cosh x - \sinh x &= e^{-x}\\ \cosh^2 x - \sinh^2 x &= 1\end{aligned}$$

the last of which is similar to the Pythagorean trigonometric identity.

One also has

$$\begin{aligned}\operatorname{sech}^2 x &= 1 - \tanh^2 x\\ \operatorname{csch}^2 x &= \coth^2 x - 1\end{aligned}$$

for the other functions.

Sums of arguments

$$\begin{aligned}\sinh(x+y) &= \sinh x \cosh y + \cosh x \sinh y\\ \cosh(x+y) &= \cosh x \cosh y + \sinh x \sinh y\\ \tanh(x+y) &= \frac{\tanh x + \tanh y}{1 + \tanh x \tanh y}\end{aligned}$$

particularly

$$\begin{aligned}\cosh(2x) &= \sinh^2 x + \cosh^2 x = 2\sinh^2 x + 1 = 2\cosh^2 x - 1\\ \sinh(2x) &= 2\sinh x \cosh x\\ \tanh(2x) &= \frac{2\tanh x}{1 + \tanh^2 x}\end{aligned}$$

Also:

$$\begin{aligned}\sinh x + \sinh y &= 2\sinh\left(\frac{x+y}{2}\right)\cosh\left(\frac{x-y}{2}\right)\\ \cosh x + \cosh y &= 2\cosh\left(\frac{x+y}{2}\right)\cosh\left(\frac{x-y}{2}\right)\end{aligned}$$

Subtraction formulas

$$\begin{aligned}\sinh(x-y) &= \sinh x \cosh y - \cosh x \sinh y \\ \cosh(x-y) &= \cosh x \cosh y - \sinh x \sinh y \\ \tanh(x-y) &= \frac{\tanh x - \tanh y}{1 - \tanh x \tanh y}\end{aligned}$$

Also:[19]

$$\begin{aligned}\sinh x - \sinh y &= 2\cosh\left(\frac{x+y}{2}\right)\sinh\left(\frac{x-y}{2}\right) \\ \cosh x - \cosh y &= 2\sinh\left(\frac{x+y}{2}\right)\sinh\left(\frac{x-y}{2}\right)\end{aligned}$$

Half argument formulas

$$\begin{aligned}\sinh\left(\frac{x}{2}\right) &= \frac{\sinh x}{\sqrt{2(\cosh x + 1)}} &&= \operatorname{sgn} x \sqrt{\frac{\cosh x - 1}{2}} \\ \cosh\left(\frac{x}{2}\right) &= \sqrt{\frac{\cosh x + 1}{2}} \\ \tanh\left(\frac{x}{2}\right) &= \frac{\sinh x}{\cosh x + 1} &&= \operatorname{sgn} x \sqrt{\frac{\cosh x - 1}{\cosh x + 1}} = \frac{e^x - 1}{e^x + 1}\end{aligned}$$

where sgn is the sign function.

If $x \neq 0$, then[20]

$$\tanh\left(\frac{x}{2}\right) = \frac{\cosh x - 1}{\sinh x} = \coth x - \operatorname{csch} x$$

Square formulas

$$\begin{aligned}\sinh^2 x &= \tfrac{1}{2}(\cosh 2x - 1) \\ \cosh^2 x &= \tfrac{1}{2}(\cosh 2x + 1)\end{aligned}$$

Inequalities

The following inequality is useful in statistics: $\cosh(t) \leq e^{t^2/2}$ [21]

It can be proved by comparing term by term the Taylor series of the two functions.

Inverse functions as logarithms

$$
\begin{aligned}
\operatorname{arsinh}(x) &= \ln\left(x + \sqrt{x^2 + 1}\right) \\
\operatorname{arcosh}(x) &= \ln\left(x + \sqrt{x^2 - 1}\right) && x \geq 1 \\
\operatorname{artanh}(x) &= \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right) && |x| < 1 \\
\operatorname{arcoth}(x) &= \frac{1}{2}\ln\left(\frac{x+1}{x-1}\right) && |x| > 1 \\
\operatorname{arsech}(x) &= \ln\left(\frac{1}{x} + \sqrt{\frac{1}{x^2} - 1}\right) = \ln\left(\frac{1 + \sqrt{1 - x^2}}{x}\right) && 0 < x \leq 1 \\
\operatorname{arcsch}(x) &= \ln\left(\frac{1}{x} + \sqrt{\frac{1}{x^2} + 1}\right) && x \neq 0
\end{aligned}
$$

Derivatives

$$
\begin{aligned}
\frac{d}{dx}\sinh x &= \cosh x \\
\frac{d}{dx}\cosh x &= \sinh x \\
\frac{d}{dx}\tanh x &= 1 - \tanh^2 x = \operatorname{sech}^2 x = \frac{1}{\cosh^2 x} \\
\frac{d}{dx}\coth x &= 1 - \coth^2 x = -\operatorname{csch}^2 x = -\frac{1}{\sinh^2 x} && x \neq 0 \\
\frac{d}{dx}\operatorname{sech} x &= -\tanh x \operatorname{sech} x \\
\frac{d}{dx}\operatorname{csch} x &= -\coth x \operatorname{csch} x && x \neq 0
\end{aligned}
$$

$$
\begin{aligned}
\frac{d}{dx}\operatorname{arsinh} x &= \frac{1}{\sqrt{x^2+1}} \\
\frac{d}{dx}\operatorname{arcosh} x &= \frac{1}{\sqrt{x^2-1}} && 1 < x \\
\frac{d}{dx}\operatorname{artanh} x &= \frac{1}{1-x^2} && |x| < 1 \\
\frac{d}{dx}\operatorname{arcoth} x &= \frac{1}{1-x^2} && 1 < |x| \\
\frac{d}{dx}\operatorname{arsech} x &= -\frac{1}{x\sqrt{1-x^2}} && 0 < x < 1 \\
\frac{d}{dx}\operatorname{arcsch} x &= -\frac{1}{|x|\sqrt{1+x^2}} && x \neq 0
\end{aligned}
$$

Second derivatives

Each of the functions $\sinh$ and $\cosh$ is equal to its second derivative, that is:

$$
\frac{d^2}{dx^2}\sinh x = \sinh x
$$

$$
\frac{d^2}{dx^2}\cosh x = \cosh x\,.
$$

All functions with this property are linear combinations of $\sinh$ and $\cosh$, in particular the exponential functions e^{x} and e^{-x}.[22]

Standard integrals

$$\begin{aligned}
\int \sinh(ax)\,dx &= a^{-1}\cosh(ax) + C \\
\int \cosh(ax)\,dx &= a^{-1}\sinh(ax) + C \\
\int \tanh(ax)\,dx &= a^{-1}\ln(\cosh(ax)) + C \\
\int \coth(ax)\,dx &= a^{-1}\ln\left|\sinh(ax)\right| + C \\
\int \operatorname{sech}(ax)\,dx &= a^{-1}\arctan(\sinh(ax)) + C \\
\int \operatorname{csch}(ax)\,dx &= a^{-1}\ln\left|\tanh\left(\frac{ax}{2}\right)\right| + C = a^{-1}\ln\left|\coth(ax) - \operatorname{csch}(ax)\right| + C = -a^{-1}\operatorname{arcoth}(\cosh(ax)) + C
\end{aligned}$$

The following integrals can be proved using hyperbolic substitution:

$$\begin{aligned}
\int \frac{1}{\sqrt{a^2+u^2}}\,du &= \operatorname{arsinh}\left(\frac{u}{a}\right) + C \\
\int \frac{1}{\sqrt{u^2-a^2}}\,du &= \operatorname{sgn} u \operatorname{arcosh}\left|\frac{u}{a}\right| + C \\
\int \frac{1}{a^2-u^2}\,du &= a^{-1}\operatorname{artanh}\left(\frac{u}{a}\right) + C \qquad u^2 < a^2 \\
\int \frac{1}{a^2-u^2}\,du &= a^{-1}\operatorname{arcoth}\left(\frac{u}{a}\right) + C \qquad u^2 > a^2 \\
\int \frac{1}{u\sqrt{a^2-u^2}}\,du &= -a^{-1}\operatorname{arsech}\left|\frac{u}{a}\right| + C \\
\int \frac{1}{u\sqrt{a^2+u^2}}\,du &= -a^{-1}\operatorname{arcsch}\left|\frac{u}{a}\right| + C
\end{aligned}$$

where *C* is the constant of integration.

Taylor series expressions

It is possible to express explicitly the Taylor series at zero (or the Laurent series, if the function is not defined at zero) of the above functions.

$$\sinh x = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \cdots = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!}$$

This series is convergent for every complex value of x. Since the function $\sinh x$ is odd, only odd exponents for x occur in its Taylor series.

$$\cosh x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \cdots = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!}$$

This series is convergent for every complex value of x. Since the function $\cosh x$ is even, only even exponents for x occur in its Taylor series.

The sum of the sinh and cosh series is the infinite series expression of the exponential function.

The following series are followed by a description of a subset of their domain of convergence, where the series is convergent and its sum equals the function.

$$\begin{aligned}
\tanh x &= x - \frac{x^3}{3} + \frac{2x^5}{15} - \frac{17x^7}{315} + \cdots = \sum_{n=1}^{\infty} \frac{2^{2n}(2^{2n}-1)B_{2n}x^{2n-1}}{(2n)!}, \qquad |x| < \frac{\pi}{2} \\
\coth x &= x^{-1} + \frac{x}{3} - \frac{x^3}{45} + \frac{2x^5}{945} + \cdots = \sum_{n=0}^{\infty} \frac{2^{2n}B_{2n}x^{2n-1}}{(2n)!}, \qquad 0 < |x| < \pi \\
\operatorname{sech} x &= 1 - \frac{x^2}{2} + \frac{5x^4}{24} - \frac{61x^6}{720} + \cdots = \sum_{n=0}^{\infty} \frac{E_{2n}x^{2n}}{(2n)!}, \qquad |x| < \frac{\pi}{2} \\
\operatorname{csch} x &= x^{-1} - \frac{x}{6} + \frac{7x^3}{360} - \frac{31x^5}{15120} + \cdots = \sum_{n=0}^{\infty} \frac{2(1-2^{2n-1})B_{2n}x^{2n-1}}{(2n)!}, \qquad 0 < |x| < \pi
\end{aligned}$$

where:

- B_n is the nth Bernoulli number
- E_n is the nth Euler number

Infinite products and continued fractions

The following expansions are valid in the whole complex plane:

$$\sinh x = x\prod_{n=1}^{\infty}\left(1+\frac{x^2}{n^2\pi^2}\right) = \cfrac{x}{1 - \cfrac{x^2}{2\cdot 3 + x^2 - \cfrac{2\cdot 3x^2}{4\cdot 5 + x^2 - \cfrac{4\cdot 5x^2}{6\cdot 7 + x^2 - \ddots}}}}$$

$$\cosh x = \prod_{n=1}^{\infty}\left(1+\frac{x^2}{(n-1/2)^2\pi^2}\right) = \cfrac{1}{1 - \cfrac{x^2}{1\cdot 2 + x^2 - \cfrac{1\cdot 2x^2}{3\cdot 4 + x^2 - \cfrac{3\cdot 4x^2}{5\cdot 6 + x^2 - \ddots}}}}$$

$$\tanh x = \cfrac{1}{\cfrac{1}{x} + \cfrac{1}{\cfrac{3}{x} + \cfrac{1}{\cfrac{5}{x} + \cfrac{1}{\cfrac{7}{x} + \ddots}}}}$$

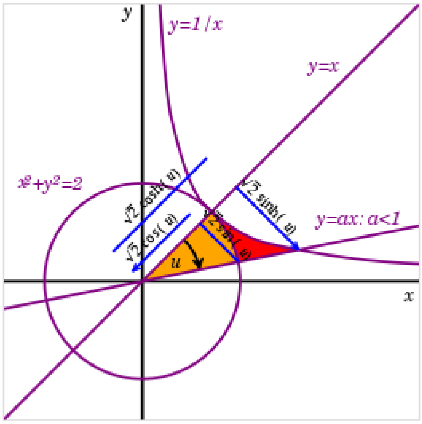
Comparison with circular functions

The hyperbolic functions represent an expansion of trigonometry beyond the circular functions. Both types depend on an argument, either circular angle or hyperbolic angle.

Since the area of a circular sector with radius r and angle u (in radians) is $r^2u/2$, it will be equal to u when $r = \sqrt{2}$. In the diagram, such a circle is tangent to the hyperbola $xy = 1$ at (1,1). The yellow sector depicts an area and angle magnitude. Similarly, the yellow and red regions together depict a hyperbolic sector with area corresponding to hyperbolic angle magnitude.

The legs of the two right triangles with hypotenuse on the ray defining the angles are of length $\sqrt{2}$ times the circular and hyperbolic functions.

The hyperbolic angle is an invariant measure with respect to the squeeze mapping, just as the circular angle is invariant under rotation.[23]



Circle and hyperbola tangent at (1,1) display geometry of circular functions in terms of circular sector area *u* and hyperbolic functions depending on hyperbolic sector area *u*.

The Gudermannian function gives a direct relationship between the circular functions and the hyperbolic functions that does not involve complex numbers.

The graph of the function $a\ \cosh(x/a)$ is the catenary, the curve formed by a uniform flexible chain, hanging freely between two fixed points under uniform gravity.

Relationship to the exponential function

The decomposition of the exponential function in its even and odd parts gives the identities

$$e^x = \cosh x + \sinh x,$$

and

$$e^{-x} = \cosh x - \sinh x.$$

Combined with Euler's formula

$$e^{ix} = \cos x + i \sin x,$$

this gives

$$e^{x+iy} = (\cosh x + \sinh x)(\cos y + i \sin y)$$

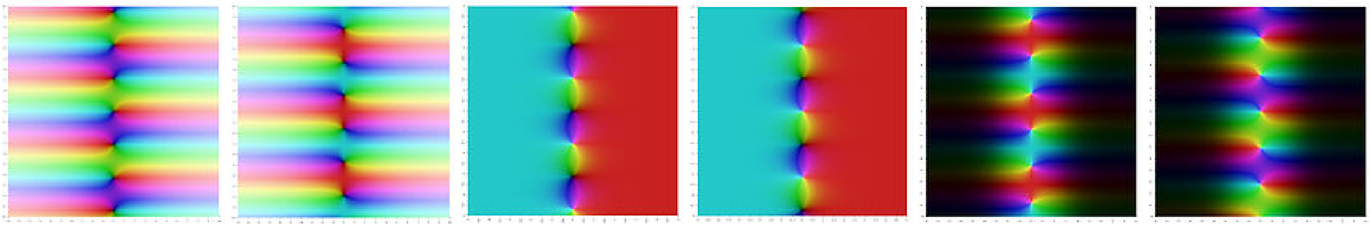
for the general complex exponential function.

Additionally,

$$e^x = \sqrt{\frac{1+\tanh x}{1-\tanh x}} = \frac{1+\tanh \frac{x}{2}}{1-\tanh \frac{x}{2}}$$

Hyperbolic functions for complex numbers

Hyperbolic functions in the complex plane



$\sinh(z)$ $\cosh(z)$ $\tanh(z)$ $\coth(z)$ $\operatorname{sech}(z)$ $\operatorname{csch}(z)$

Since the exponential function can be defined for any complex argument, we can also extend the definitions of the hyperbolic functions to complex arguments. The functions $\sinh z$ and $\cosh z$ are then holomorphic.

Relationships to ordinary trigonometric functions are given by Euler's formula for complex numbers:

$$\begin{aligned} e^{ix} &= \cos x + i \sin x \\ e^{-ix} &= \cos x - i \sin x \end{aligned}$$

so:

$$\begin{aligned}
\cosh(ix) &= \frac{1}{2}\left(e^{ix} + e^{-ix}\right) = \cos x \\
\sinh(ix) &= \frac{1}{2}\left(e^{ix} - e^{-ix}\right) = i \sin x \\
\cosh(x+iy) &= \cosh(x)\cos(y) + i \sinh(x)\sin(y) \\
\sinh(x+iy) &= \sinh(x)\cos(y) + i \cosh(x)\sin(y) \\
\tanh(ix) &= i \tan x \\
\cosh x &= \cos(ix) \\
\sinh x &= -i \sin(ix) \\
\tanh x &= -i \tan(ix)
\end{aligned}$$

Thus, hyperbolic functions are periodic with respect to the imaginary component, with period $2\pi i$ (πi for hyperbolic tangent and cotangent).

See also

- e (mathematical constant)
- Equal incircles theorem, based on sinh
- Hyperbolic growth
- Inverse hyperbolic functions
- List of integrals of hyperbolic functions
- Poinsot's spirals
- Sigmoid function
- Soboleva modified hyperbolic tangent
- Trigonometric functions

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External links

- "Hyperbolic functions" (https://www.encyclopediaofmath.org/index.php?title=Hyperbolic_functions), *Encyclopedia of Mathematics*, EMS Press, 2001 [1994]
- Hyperbolic functions (https://planetmath.org/hyperbolicfunctions) on PlanetMath
- GonioLab (https://web.archive.org/web/20071006172054/http://glab.trixon.se/): Visualization of the unit circle, trigonometric and hyperbolic functions (Java Web Start)
- Web-based calculator of hyperbolic functions (http://www.calctool.org/CALC/math/trigonometry/hyperbolic)