Alternative definition of hyperbolic cosine without relying on exponential function

Blue

Apr 17, 2014

1 Question

Ordinary trigonometric functions are defined independently of exponential function, and then shown to be related to it by Euler's formula.

Can one define hyperbolic cosine so that the formula

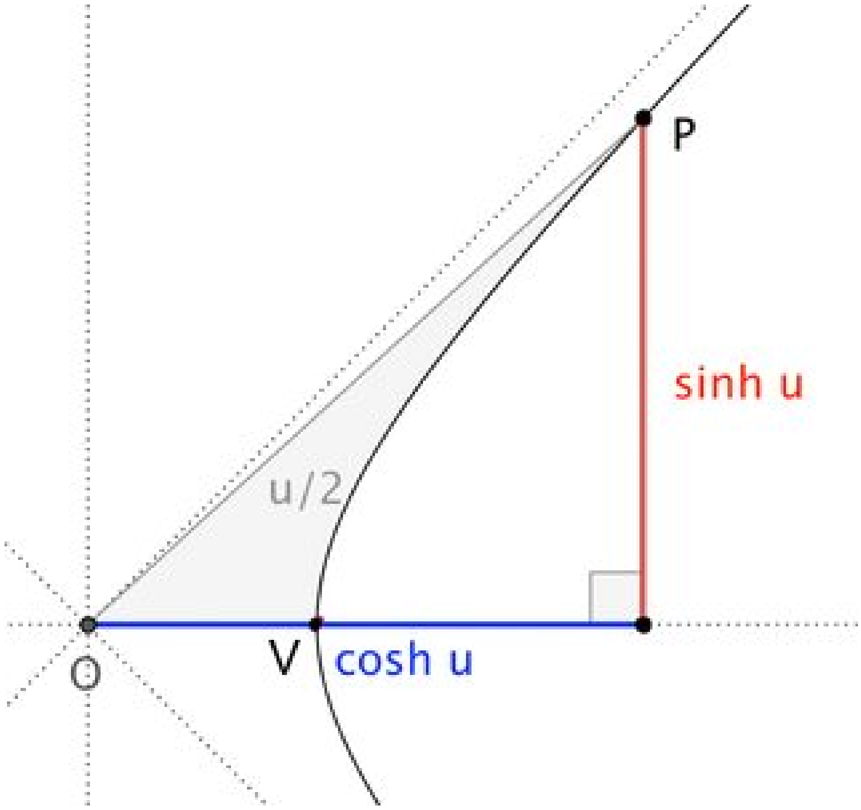
$$\cosh x = \frac{e^x + e^{-x}}{2}$$

becomes something to be *proven*?

2 Answer

One can define $\cosh u$ and $\sinh u$ geometrically as hyperbolic analogues of $\cos\theta$ and $\sin\theta$, taking $(\cosh u, \sinh u)$ to be points on the "unit hyperbola", $x^2 - y^2 = 1$. In that case, the relation between these values and exponentials does require proof. (I may have posted one on MSE at some point.)

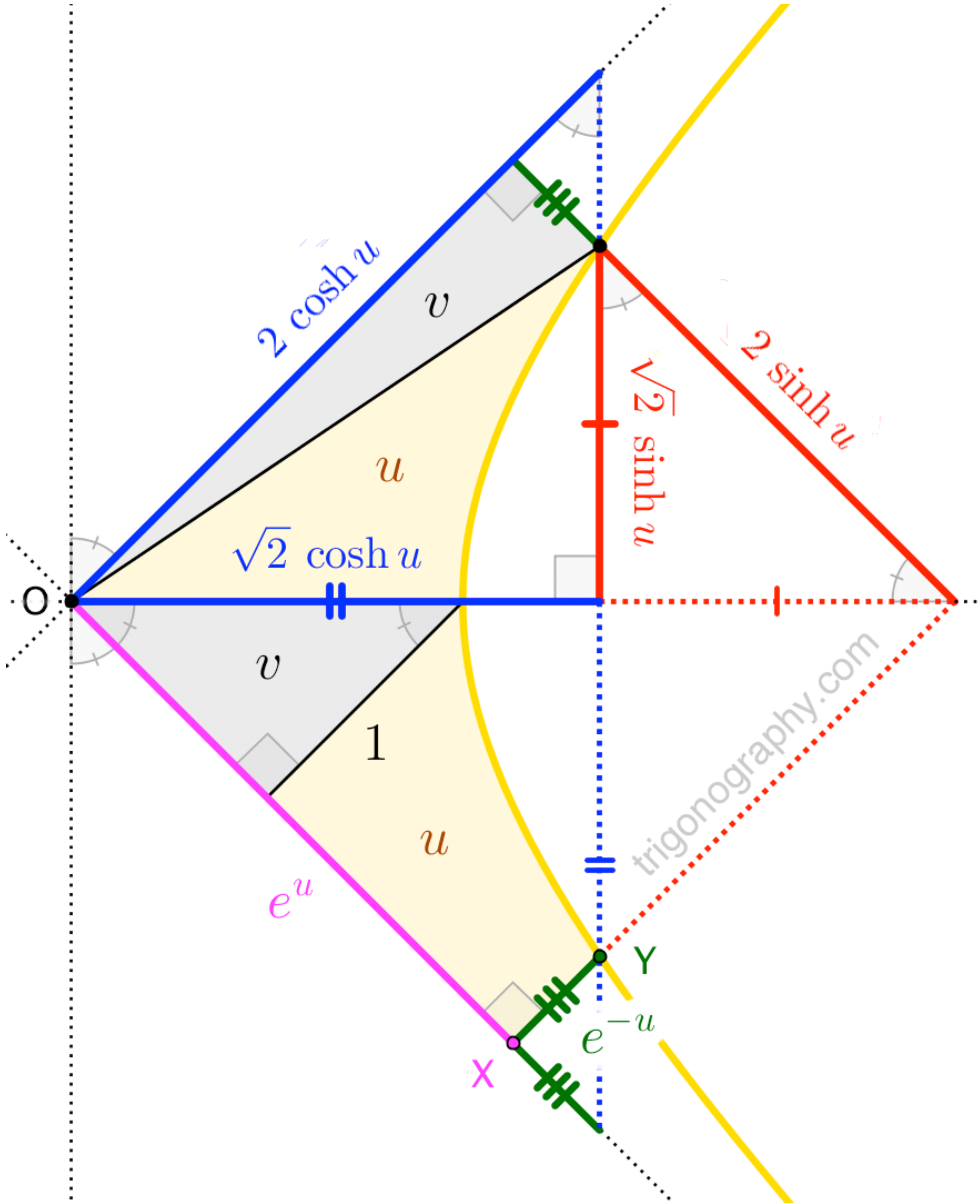
The more-geometrically-minded of us take $\cosh u$ and $\sinh u$ to be defined via the "unit hyperbola", $x^2 - y^2 = 1$, in a manner directly analogous to $\cos\theta$ and $\sin\theta$. Specifically, given P a point on the hyperbola with vertex V, and defining u as *twice*(?!) the area of the hyperbolic sector OVP, then $\cosh u$ and $\sinh u$ are, respectively the x- and y-coordinates of P.



Just as in circular trig, we can assign measures u (in "hyperbolic radians") to angles --- from the flat angle (when $u = 0$) to half a right angle (when $u = \infty$) --- and associate those measures with the lengths of the corresponding cosh and sinh segments. And, just as in circular trig (prior to the advent of imaginary numbers), we might be forgiven for suspecting that the correspondences $u \longleftrightarrow \cosh u$ and $u \longleftrightarrow \sinh u$ are "non-arithmetical", which is to say: that no *arithmetical formula* converts angle measures to their associated trig values.

However, it turns out that the correspondences are not non-arithmetical; to find the appropriate arithmetical conversion formula, all we need is a bit of calculus ...

Edit. (Two years later!) Check the edit history for an inelegant argument that I now streamline with the help of this trigonograph, in which lengths from the unit hyperbola have been scaled by $\sqrt{2}$ (and, thus, areas by 2):



Because the hyperbola is rectangular, we have that $|OX| \cdot |XY|$ is a constant (here, 1), which guarantees that the regions labeled v have the same area (namely, 1/2), and therefore that the regions labeled u have the same area (namely, u). Now, the bit of calculus I promised, to evaluate u as the area under the reciprocal curve:

$$u = \int_1^{|OX|} \frac{1}{t} dt = \ln |OX| \quad \rightarrow \quad |OX| = e^u \quad \rightarrow \quad |XY| = \frac{1}{e^u}$$

With that, we clearly have

$$2\,\sinh u \;=\; e^u - e^{-u} \qquad\qquad 2\,\cosh u \;=\; e^u + e^{-u}$$

as desired. Easy-peasy!

End of edit.

That hyperbolic radians are defined via doubling the area of a hyperbolic sector may seem at odds with the common definition of circular radians in terms of arc-length, but it's hard to argue with success, given the elegance of the formulas above. Even so, the hyperbolic twice-the-sector-area definition can be seen as directly analogous to the circular case, since circular radians are also definable as "twice-the-sector-area": In the unit circle, the sector with angle measure $\pi/2$ radians has area $\pi/4$ (it's a quarter-circle), the sector with angle measure π radians has area $\pi/2$ (it's a half-circle), and the "sector" with angle measure 2π radians has area π (it's the full circle); in these, and all other, cases, the angle measure is twice the sector area.

Would I be correct in assuming that, like with the circular trig functions, if z gives the arc length from the vertex to the point (x, y) on the hyperbola $x^2 - y^2 = r^2$, with a sign of positive or negative according to whether y is positive or negative, then $\cosh z$ could also be defined as the ratio x/r, and $\sinh z$ as y/r? And then in the unit hyperbola, these ratios simply reduce to coordinates and the arc length becomes half the sector area? This would be an even nicer analogy to circular trigonometry.

You *can* define cosh and sinh based on an arc-length parameter (your z); however, hyperbolic arc-length cannot be expressed in terms of elementary functions. (Lengths of curves are almost-always trickier to calculate than the areas they bound; circles (& lines) are the primary exceptions.) The length of arc $V'P'$ involves $\int \sqrt{1+x^4}/x^2 dx$, which is quite non-trivial, so hyperbolic trig values would effectively be "non-arithmetical" functions of an arc-length-based angle measure. It's *certainly* not the case that arc-length is twice the sector area.

I realise that this is quite an old post, but I don't quite see how the hyperbola being rectangular implies $|\overline{OX}| \cdot |\overline{XY}|$ is a constant. From what I understand, rectangular hyperbolas are hyperbolas where the asymptotes are perpendicular, but I'm not quite sure how the result follows from this.

"I don't quite see how the hyperbola being rectangular implies $|\overline{OX}| \cdot |\overline{XY}|$ is a constant." The least-inspired way to show that is to take the rectangular hyperbola eqn $x^2 - y^2 = a^2$ and do a coordinate transformation that rotates by 45° (so that the asymptotes align with the coordinate axes); the result has the form $xy = k^2$. The product $|\overline{OX}| \cdot |\overline{XY}|$ corresponds to $x \cdot y$ in that relation. ... I believe there's a nice geometric proof of the relation using fundamental hyperbola properties, but I can't recall it at the moment.

Using matrices, I found that the equation $x^2 - y^2 = \left(\sqrt{2}\right)^2$ is equivalent to $xy = 1$ if we rotate the graph 45° anticlockwise. (This makes sense because in both cases the closest point to the origin is $\sqrt{2}$ units away.) I love this answer because it explains how the hyperbolic functions are connected to the exponential: a common way to define the exponential function is by first defining the logarithm as $\log x = \int_1^x \frac{1}{t}\, dt$. And the graph of $y = 1/t$ is a hyperbola!

Well, that is usually simply taken to be the definition, but given that

$$\cos x = \cosh ix$$

you may be asking for a proof that

$$\cos x = \frac{e^{ix} + e^{-ix}}{2}$$

From Taylor's theorem, we know that

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

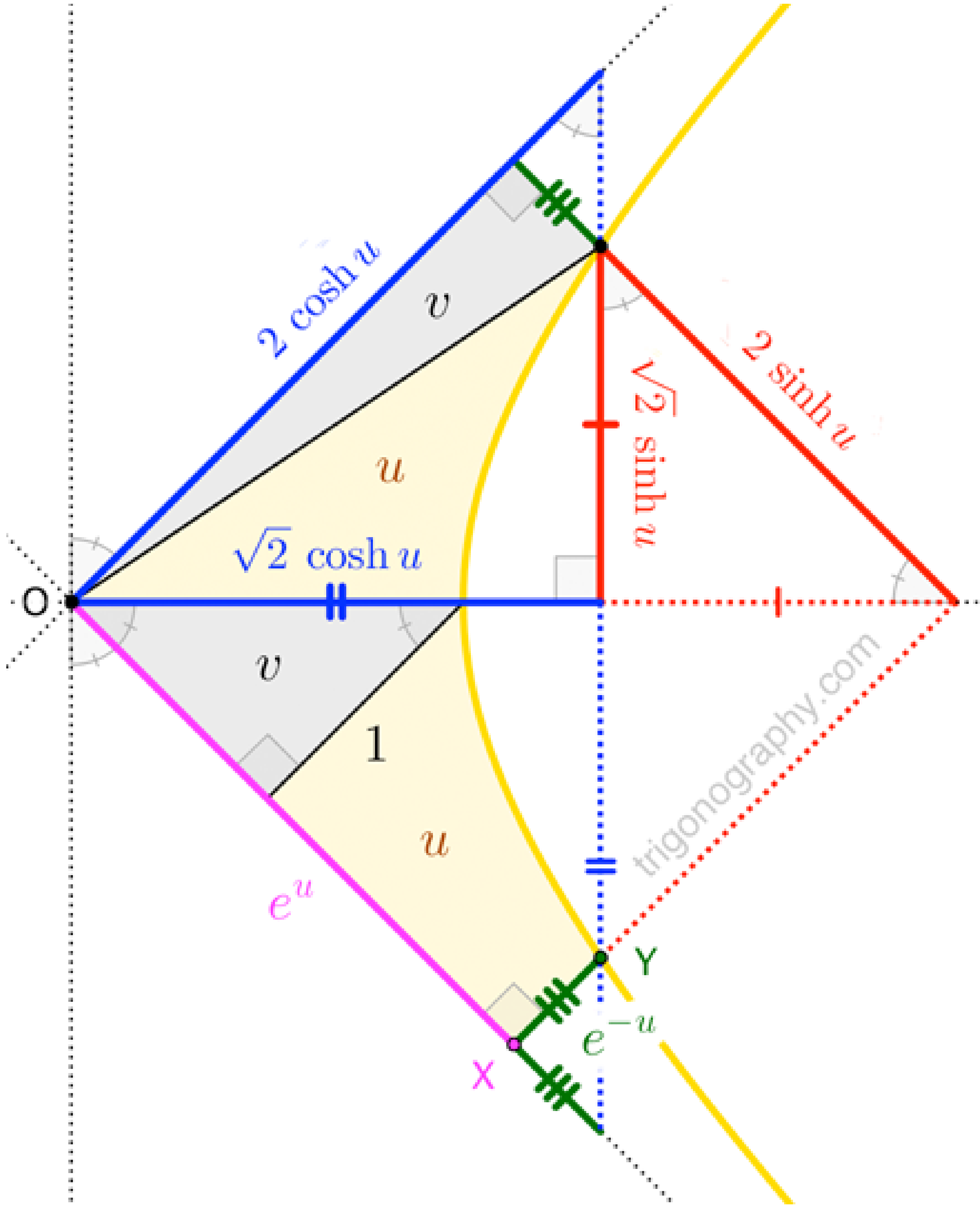
So

$$e^{ix} = \sum_{n=0}^{\infty} \frac{(ix)^n}{n!} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} + i \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} = \cos x + i \sin x$$

Using $e^{ix} = \cos x + i \sin x$, express $e^{ix} + e^{-ix}$ in terms of $\cos x$, noting that the cosine function is even and the sine function is odd.

3 Exponential Forms of Hyperbolic Sine and Cosine

18 July, 2016



$$2\sinh u \;=\; e^{u} - e^{-u}$$

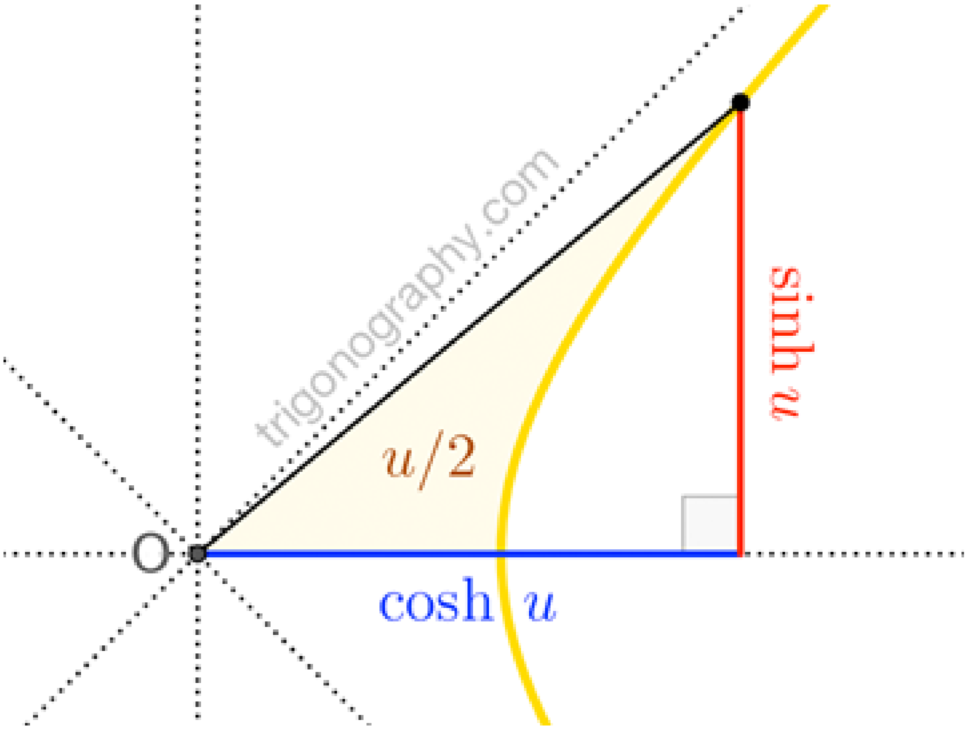
$$2\cosh u \;=\; e^{u} + e^{-u}$$

$$|\overline{OX}|\cdot|\overline{XY}| \;\equiv\; 1 \quad\rightarrow\quad u = \int_{1}^{|\overline{OX}|}\frac{1}{t}dt \;=\; \ln|\overline{OX}|$$

$$\rightarrow\quad |\overline{OX}| = e^{u} \quad\text{and}\quad |\overline{XY}| = e^{-u}$$

Some background. Geometrically, we define $\sinh u$ and $\cosh u$ by direct analogy with $\sin\theta$ and $\cos\theta$: as certain perpendicular segments associated with an arc of the "unit hyperbola", $x^2 - y^2 = 1$.

While θ is usually interpreted as the length of a circular arc, we note that it is also twice the area of the corresponding circular sector. The hyperbolic parameter u is interpreted via area, as well; today's trigonograph shows why:



Conveniently scaling lengths in the unit hyperbola figure by $\sqrt{2}$ —and, thus, scaling areas by 2 —we see that $\sinh u$ and $\cosh u$ are directly computable from u via the exponential function!

(In fact, we can say the same of $\sin\theta$ and $\cos\theta$, but we need complex exponentials for that.)