50.034 - Introduction to Probability and Statistics

Week 12 – Lecture 21

January–May Term, 2019

SINGAPORE UNIVERSITY OF TECHNOLOGY AND DESIGN

Lecture 21 Assigned Readings: 9.5, 9.6

Outline of Lecture

- Expectation of unbiased sample variance
- t-test
- One-sided versus two-sided t-test
- Significance level and p-values for t-test
- Non-central t-distribution
- Two-sample t-statistic
- Two-sample t-test

R.V.'s with unknown mean and unknown variance

Let $\{X_1, \ldots, X_n\}$ be a random sample with **unknown mean** μ and **unknown variance** σ^2.

Question: How do we use the observed values of $X_1, \ldots, X_n$ to get information about μ and σ^2?

- We know the sample mean $\overline{X}_n$ is approximately μ for large n.
  - (Lecture 11) By the law of large numbers, $\overline{X}_n \xrightarrow{\text{p}} \mu$.
- We also know that the unbiased sample variance s_n^2 is approximately σ^2 for large n.
  - (Lecture 18) The unbiased sample variance of $\{X_1, \ldots, X_n\}$ is

$$s_n^2 = s_n^2(X_1, \ldots, X_n) = \frac{1}{n-1} \sum_{i=1}^{n} (X_i - \overline{X}_n)^2.$$

  - **Note:** The unbiased sample variance is defined for **all** random samples, not just random samples consisting of normal R.V.'s.
  - (Lecture 18) $s_n^2 \xrightarrow{\text{p}} \sigma^2$.
- **Theorem:** $\mathbf{E}[s_n^2] = \sigma^2$ for **every** $n > 1$.
  - We shall see a proof on the next slide.

Expectation of unbiased sample variance

Theorem: Let $\{X_1, \ldots, X_n\}$ be **any** random sample with sample mean $\overline{X}_n$ and unbiased sample variance s_n^2. Let μ and σ^2 be the mean and variance of each X_i. Then $\mathbf{E}[s_n^2] = \sigma^2$ for all $n > 1$.

Proof: First, we shall compute $\mathbf{E}\Big[\sum_{i=1}^n (X_i - \overline{X}_n)^2\Big]$.

$$\mathbf{E}\Big[\sum_{i=1}^n (X_i - \overline{X}_n)^2\Big] = \mathbf{E}\Big[\sum_{i=1}^n \left(X_i^2 - 2X_i\overline{X}_n + \overline{X}_n^2\right)\Big]$$

$$= \mathbf{E}\left[\left(\sum_{i=1}^n \left[X_i^2 - \frac{2}{n}X_i(X_1 + \cdots + X_n)\right]\right) + \frac{1}{n}(X_1 + \cdots + X_n)^2\right].$$

Note that

$$\sum_{i=1}^n \left[X_i^2 - \frac{2}{n}X_i(X_1 + \cdots + X_n)\right] = \sum_{i=1}^n \left[\frac{n-2}{n}X_i^2 - \frac{2}{n}\sum_{\substack{1 \le j \le n \\ j \ne i}} X_i X_j\right].$$

$$(X_1 + \cdots + X_n)^2 = \sum_{i=1}^n \left[X_i^2 + \sum_{\substack{1 \le j \le n \\ k \ne j}} X_i X_j\right]$$

Expectation of unbiased sample variance (continued)

Thus, $\mathbf{E}\Big[\sum_{i=1}^n (X_i - \overline{X}_n)^2\Big]$ equals

$$\mathbf{E}\left[\sum_{i=1}^n \left(\tfrac{n-2}{n} X_i^2 - \tfrac{2}{n} \sum_{\substack{1 \le j \le n \\ j \ne i}} X_i X_j + \tfrac{1}{n} X_i^2 + \tfrac{1}{n} \sum_{\substack{1 \le j \le n \\ k \ne j}} X_i X_j\right)\right]$$

$$= \sum_{i=1}^n \left[\tfrac{n-1}{n} \mathbf{E}[X_i^2] - \tfrac{1}{n} \sum_{\substack{1 \le j \le n \\ j \ne i}} \mathbf{E}[X_i]\mathbf{E}[X_j]\right]$$

$$= \left(\sum_{i=1}^n \left[\tfrac{n-1}{n} (\mathbf{E}[X_i^2] - \mathbf{E}[X_i]^2) + \tfrac{n-1}{n} \mu^2\right]\right) - \tfrac{1}{n} (n(n-1))\mu^2$$

$$= \left(\sum_{i=1}^n \tfrac{n-1}{n} \mathrm{var}(X_i)\right) + (n-1)\mu^2 - (n-1)\mu^2$$

$$= (n-1)\sigma^2.$$

Therefore, $\mathbf{E}[s_n^2] = \mathbf{E}\Big[\frac{1}{n-1} \sum_{i=1}^n (X_i - \overline{X}_n)^2\Big] = \sigma^2$, for $n > 1$.

Recall: Main Theorem on t-distributions

Let $\{X_1, \ldots, X_n\}$ be a random sample of observable **normal** R.V.'s with mean μ and variance σ^2. Let $\overline{X}_n$ and s_n^2 be the sample mean and the **unbiased sample variance** respectively.

Most Important Theorem on t-distributions:

$\frac{\sqrt{n}(\overline{X}_n - \mu)}{s_n}$ has the t-distribution with $(n-1)$ degrees of freedom.

Useful Properties of the t-distribution:

Suppose Z has the t-distribution with m degrees of freedom.
Let $f(z)$ be the pdf of Z, and let $F(z)$ be the cdf of Z.

- The graph of $f(z)$ is symmetric about the point $z = 0$.
- $F(-z) = 1 - F(z)$ for all real numbers z.
- $F^{-1}(p) = -F^{-1}(1-p)$ for all real numbers $0 < p < 1$.

(Properties are similar to the properties of the **standard normal** distribution.)

Recall: Hypothesis Test

Model set-up: Let $X_1, \dots, X_n$ be observable R.V.'s with unknown parameter θ. Let Ω be the parameter space of θ.

- Goal: Perform hypothesis testing on the parameter θ.

1. Specify some **null hypothesis** $H_0 : \theta \in \Omega_0$.
   - $\Omega_0 \subseteq \Omega$ is a subset chosen based on your specific application.
   - You wish to test whether the "true" value of θ is not in Ω_0.
2. Specify some **test statistic** $T = T(X_1, \dots, X_n)$.
   - Your final decision will depend on the observed value of T.
3. Specify some **rejection region** $R \subseteq \mathbb{R}$.
   - This represents the region for where to reject H_0.
   - Note: R can be different from the complement of Ω_0.
4. Collect experimental evidence
   - Get observed values $X_1 = x_1, \dots, X_n = x_n$.
5. Final decision: To reject or not to reject?
   - "Reject H_0" if $T(x_1, \dots, x_n) \in R$.
   - "Do not reject H_0" if $T(x_1, \dots, x_n) \notin R$.

The entire test procedure is collectively called a hypothesis test.

t-test

Definition: A t-test is a hypothesis test $\mathcal{H}$ satisfying the following:

- The null hypothesis of $\mathcal{H}$ is $H_0 : \theta \in \Omega_0$, where θ is the **mean**.
- The test statistic of $\mathcal{H}$ would have the **t-distribution** on the condition that $\theta = \theta_0$ for some specific θ_0 in Ω_0.

Note: There are many different kinds of t-tests!

Three most important examples of t-tests:

Let $\{X_1, \ldots, X_n\}$ be a random sample of **normal** observable R.V.'s with **unknown mean** μ and **unknown variance** σ^2. Let $\overline{X}_n$, s_n^2 be the sample mean and the **unbiased sample variance** respectively.

Let μ_0 be some real constant, and define the R.V. $T = \frac{\sqrt{n}(\overline{X}_n - \mu_0)}{s_n}$.

- If $\mathcal{H}$ is a hypothesis test with null hypothesis $H_0 : \mu \leq \mu_0$, test statistic T, and rejection region $[c, \infty)$, then $\mathcal{H}$ is a t-test.
- If $\mathcal{H}$ is a hypothesis test with null hypothesis $H_0 : \mu \geq \mu_0$, test statistic T, and rejection region $(-\infty, c]$, then $\mathcal{H}$ is a t-test.
- If $\mathcal{H}$ is a hypothesis test with null hypothesis $H_0 : \mu = \mu_0$, test statistic $|T|$, and rejection region $[c, \infty)$, then $\mathcal{H}$ is a t-test.

One-sided versus two-sided t-tests

Same assumptions as before:
Let $\{X_1, \ldots, X_n\}$ be a random sample of **normal** observable R.V.'s with **unknown mean** μ and **unknown variance** σ^2. Let $\overline{X}_n$, s_n^2 be the sample mean and the **unbiased sample variance** respectively. Let μ_0 be some real constant, and define the R.V. $T = \frac{\sqrt{n}(\overline{X}_n - \mu_0)}{s_n}$.

Definition:

- If $\mathcal{H}$ is a t-test with test statistic T and null hypothesis either $H_0 : \mu \leq \mu_0$ or $H_0 : \mu \geq \mu_0$ (with corresponding rejection region either $[c, \infty)$ or $(-\infty, c]$ respectively for some $c \in \mathbb{R}$), then we say that $\mathcal{H}$ is a one-sided t-test.
- If $\mathcal{H}$ is a t-test with test statistic $|T|$ and null hypothesis $H_0 : \mu = \mu_0$ (with corresponding rejection region $[c, \infty)$ for some $c \in \mathbb{R}$), then we say that $\mathcal{H}$ is a two-sided t-test.

Significance level of one-sided t-tests

Let $\{X_1, \ldots, X_n\}$ be a random sample of **normal** observable R.V.'s with **unknown mean** μ and **unknown variance** σ^2. Let $\overline{X}_n$, s_n^2 be the sample mean and the **unbiased sample variance** respectively. Let μ_0 and c_0 be fixed real numbers, and define $T = \frac{\sqrt{n}(\overline{X}_n - \mu_0)}{s_n}$.

Theorem: Suppose that c_0 is the $100(1-\alpha_0)$-percentile of the t-distribution with $n-1$ degrees of freedom.

- If $\mathcal{H}$ is a t-test with null hypothesis $H_0 : \mu \leq \mu_0$, test statistic T, and rejection region $[c, \infty)$, then $\mathcal{H}$ has significance level α_0 if and only if $c \geq c_0$.
- If $\mathcal{H}$ is a t-test with null hypothesis $H_0 : \mu \geq \mu_0$, test statistic T, and rejection region $(-\infty, c]$, then $\mathcal{H}$ has significance level α_0 if and only if $c \leq c_0$.

Intuition:

- If $H_0 : \mu \leq \mu_0$ is true, then any observed value $T = t$ should satisfy $t <$ "some small value".
- If $H_0 : \mu \geq \mu_0$ is true, then any observed value $T = t$ should satisfy $t >$ "some small value".

Significance level of two-sided t-tests

Let $\{X_1, \ldots, X_n\}$ be a random sample of **normal** observable R.V.'s with **unknown mean** μ and **unknown variance** σ^2. Let $\overline{X}_n$, s_n^2 be the sample mean and the **unbiased sample variance** respectively. Let μ_0 and c_0 be fixed real numbers, and define $T = \left|\frac{\sqrt{n}(\overline{X}_n - \mu_0)}{s_n}\right|$.

Theorem: Suppose that c_0 is the $100(1 - \frac{\alpha_0}{2})$-percentile of the t-distribution with $n-1$ degrees of freedom. If $\mathcal{H}$ is the t-test with null hypothesis $H_0 : \mu = \mu_0$, test statistic T, and rejection region $[c, \infty)$, then $\mathcal{H}$ has significance level α_0 if and only if $c \geq c_0$.

Intuition: If $H_0 : \mu = \mu_0$ is true, then any observed value $T = t$ should be approximately zero (remember that $T = \left|\frac{\sqrt{n}(\overline{X}_n - \mu_0)}{s_n}\right|$), so t should be less that "some small positive value".

- Thus the null hypothesis H_0 should be rejected if t is at least "some small positive value".

Example 1

Let $\{X_1, \ldots, X_9\}$ be a random sample of normal observable R.V.'s with unknown mean μ and unknown variance σ^2.

Let $\mathcal{H} = \{\mathcal{H}_c\}_{c \in \mathbb{R}}$ be a collection of t-tests, where each $\mathcal{H}_c$ has the null hypothesis $H_0 : \mu \leq 1$, the test statistic $T = \frac{\sqrt{n}(\overline{X}_n - 1)}{s_n}$, and the rejection region $[c, \infty)$, where $\overline{X}_n$ denotes the sample mean, and s_n denotes the unbiased sample standard deviation.

1. Find the value of c that maximizes the power of $\mathcal{H}_c$ among all level 0.05 t-tests in $\mathcal{H}$.
2. Suppose we are given the observed value $T = 1.11$. What is the p-value of $\mathcal{H}$?

Example 1 - Solution

1. Notice that each $\mathcal{H}_c$ is a one-sided t-test. If $\mu = 1$, then T has the t-distribution with 8 degrees of freedom.
   - Note that $100(1 - 0.05) = 95$.
   - From the table of values for t-distributions, the 95th percentile of the t-distribution with 8 degrees of freedom is $c = 1.860$.
   - Thus, $\mathcal{H}_c$ has significance level 0.05 if and only if $c \geq 1.860$.

Table of the *t* Distribution

If X has a t distribution with m degrees of freedom, the table gives the value of x such that $\Pr(X \leq x) = p$.

m	$p = .55$	.60	.65	.70	.75	.80	.85	.90	.95	.975	.99	.995
1	.158	.325	.510	.727	1.000	1.376	1.963	3.078	6.314	12.706	31.821	63.657
2	.142	.289	.445	.617	.816	1.061	1.386	1.886	2.920	4.303	6.965	9.925
3	.137	.277	.424	.584	.765	.978	1.250	1.638	2.353	3.182	4.541	5.841
4	.134	.271	.414	.569	.741	.941	1.190	1.533	2.132	2.776	3.747	4.604
5	.132	.267	.408	.559	.727	.920	1.156	1.476	2.015	2.571	3.365	4.032
6	.131	.265	.404	.553	.718	.906	1.134	1.440	1.943	2.447	3.143	3.707
7	.130	.263	.402	.549	.711	.896	1.119	1.415	1.895	2.365	2.998	3.499
8	.130	.262	.399	.546	.706	.889	1.108	1.397	$\boxed{1.860}$	2.306	2.896	3.355
9	.129	.261	.398	.543	.703	.883	1.100	1.383	1.833	2.262	2.821	3.250
10	.129	.260	.397	.542	.700	.879	1.093	1.372	1.812	2.228	2.764	3.169

Example 1 - Solution (continued)

1. (continued) To maximize the power of $\mathcal{H}_c$ among all level 0.05 t-tests in $\mathcal{H}$, we need to find the smallest possible c satisfying $c \geq 1.860$, therefore $c = 1.860$.
2. Given $T = 1.11$, the t-test $\mathcal{H}_c$ rejects H_0 whenever $c \leq 1.11$.
   - From the table of values for t-distributions, 1.108 is the 85th percentile of the t-distribution with 8 degrees of freedom, which corresponds to the significance level $1 - 0.85 = 0.15$.
   - Therefore the p-value of $\mathcal{H}$ is ≈ 0.15.

Table of the t Distribution

If X has a t distribution with m degrees of freedom, the table gives the value of x such that $\Pr(X \leq x) = p$.

m	$p = .55$	.60	.65	.70	.75	.80	.85	.90	.95	.975	.99	.995
1	.158	.325	.510	.727	1.000	1.376	1.963	3.078	6.314	12.706	31.821	63.657
2	.142	.289	.445	.617	.816	1.061	1.386	1.886	2.920	4.303	6.965	9.925
3	.137	.277	.424	.584	.765	.978	1.250	1.638	2.353	3.182	4.541	5.841
4	.134	.271	.414	.569	.741	.941	1.190	1.533	2.132	2.776	3.747	4.604
5	.132	.267	.408	.559	.727	.920	1.156	1.476	2.015	2.571	3.365	4.032
6	.131	.265	.404	.553	.718	.906	1.134	1.440	1.943	2.447	3.143	3.707
7	.130	.263	.402	.549	.711	.896	1.119	1.415	1.895	2.365	2.998	3.499
8	.130	.262	.399	.546	.706	.889	$\boxed{1.108}$	1.397	1.860	2.306	2.896	3.355

Example 2

Let $\{X_1, \ldots, X_7\}$ be a random sample of normal observable R.V.'s with unknown mean μ and unknown variance σ^2.

Let $\mathcal{H} = \{\mathcal{H}_c\}_{c \in \mathbb{R}}$ be a collection of t-tests, where each $\mathcal{H}_c$ has the null hypothesis $H_0 : \mu = -4$, the test statistic $T = \left| \frac{\sqrt{n}(\overline{X}_n + 4)}{s_n} \right|$, and the rejection region $[c, \infty)$, where $\overline{X}_n$ denotes the sample mean, and s_n denotes the unbiased sample standard deviation.

1. Find the value of c that maximizes the power of $\mathcal{H}_c$ among all level 0.05 t-tests in $\mathcal{H}$.
2. Suppose we are given the observed value $T = 3.14$. What is the p-value of $\mathcal{H}$?

Example 2 - Solution

1. Notice that each $\mathcal{H}_c$ is a two-sided t-test. If $\mu = -4$, then the R.V. $\frac{\sqrt{n}(\overline{X}_n+4)}{s_n}$ has the t-distribution with 6 degrees of freedom.
   - Note that $100(1 - \frac{0.05}{2}) = 97.5$.
   - From the table of values for t-distributions, the 97.5th percentile of the t-distribution with 6 degrees of freedom is $c = 2.447$.
   - Thus, $\mathcal{H}_c$ has significance level 0.05 if and only if $c \geq 2.447$.

Table of the *t* Distribution

If X has a t distribution with m degrees of freedom, the table gives the value of x such that $\Pr(X \leq x) = p$.

m	$p = .55$	.60	.65	.70	.75	.80	.85	.90	.95	.975	.99	.995
1	.158	.325	.510	.727	1.000	1.376	1.963	3.078	6.314	12.706	31.821	63.657
2	.142	.289	.445	.617	.816	1.061	1.386	1.886	2.920	4.303	6.965	9.925
3	.137	.277	.424	.584	.765	.978	1.250	1.638	2.353	3.182	4.541	5.841
4	.134	.271	.414	.569	.741	.941	1.190	1.533	2.132	2.776	3.747	4.604
5	.132	.267	.408	.559	.727	.920	1.156	1.476	2.015	2.571	3.365	4.032
6	.131	.265	.404	.553	.718	.906	1.134	1.440	1.943	$\boxed{2.447}$	3.143	3.707
7	.130	.263	.402	.549	.711	.896	1.119	1.415	1.895	2.365	2.998	3.499
8	.130	.262	.399	.546	.706	.889	1.108	1.397	1.860	2.306	2.896	3.355
9	.129	.261	.398	.543	.703	.883	1.100	1.383	1.833	2.262	2.821	3.250
10	.129	.260	.397	.542	.700	.879	1.093	1.372	1.812	2.228	2.764	3.169

Example 2 - Solution (continued)

1. (continued) To maximize the power of $\mathcal{H}_c$ among all level 0.05 t-tests in $\mathcal{H}$, we need to find the smallest possible c satisfying $c \geq 2.447$, therefore $c = 2.447$.
2. Given $T = 3.14$, the t-test $\mathcal{H}_c$ rejects H_0 whenever $c \leq 3.14$.
   - From the table of values for t-distributions, 3.143 is the 99th percentile of the t-distribution with 6 degrees of freedom, which corresponds to the significance level $2(1 - 0.99) = 0.02$.
   - Therefore the p-value of $\mathcal{H}$ is ≈ 0.02.

Table of the *t* Distribution

If X has a t distribution with m degrees of freedom, the table gives the value of x such that $\Pr(X \leq x) = p$.

m	$p = .55$	.60	.65	.70	.75	.80	.85	.90	.95	.975	.99	.995
1	.158	.325	.510	.727	1.000	1.376	1.963	3.078	6.314	12.706	31.821	63.657
2	.142	.289	.445	.617	.816	1.061	1.386	1.886	2.920	4.303	6.965	9.925
3	.137	.277	.424	.584	.765	.978	1.250	1.638	2.353	3.182	4.541	5.841
4	.134	.271	.414	.569	.741	.941	1.190	1.533	2.132	2.776	3.747	4.604
5	.132	.267	.408	.559	.727	.920	1.156	1.476	2.015	2.571	3.365	4.032
6	.131	.265	.404	.553	.718	.906	1.134	1.440	1.943	2.447	$\boxed{3.143}$	3.707
7	.130	.263	.402	.549	.711	.896	1.119	1.415	1.895	2.365	2.998	3.499
8	.130	.262	.399	.546	.706	.889	1.108	1.397	1.860	2.306	2.896	3.355

p-values of t-tests

Let $\{X_1, \ldots, X_n\}$ be a random sample of **normal** observable R.V.'s with **unknown mean** μ and **unknown variance** σ^2. Let $\overline{X}_n$, s_n^2 be the sample mean and the **unbiased sample variance** respectively. Let μ_0 be a fixed real number, and define $T = \frac{\sqrt{n}(\overline{X}_n - \mu_0)}{s_n}$.

Theorem:

- (**One-sided t-test**) Let $\mathcal{H}$ be a t-test with null hypothesis $H_0 : \mu \leq \mu_0$ or $H_0 : \mu \geq \mu_0$, and test statistic T. Given the observed value $T = t$, suppose that t is the $100(1 - \alpha_0)$-th percentile of the t-distribution with $n - 1$ degrees of freedom. Then the p-value of $\mathcal{H}$ is α_0.
- (**Two-sided t-test**) Let $\mathcal{H}$ be a t-test with null hypothesis $H_0 : \mu = \mu_0$ and test statistic $|T|$. Given the observed value $T = t$, suppose that $|t|$ is the $100(1 - \frac{\alpha_0}{2})$-th percentile of the t-distribution with $n - 1$ degrees of freedom. Then the p-value of $\mathcal{H}$ is α_0.

A closer look at the test statistic of the t-test

Let $\{X_1, \ldots, X_n\}$ be a random sample of **normal** R.V.'s with mean μ and variance σ^2. Let $\overline{X}_n$ and s_n^2 be the sample mean and the **unbiased sample variance** respectively. Let $\mu_0 \in \mathbb{R}$ be fixed.

Consider a t-test with test statistic $T = \frac{\sqrt{n}(\overline{X}_n - \mu_0)}{s_n}$.

- **What we know:** If $\mu = \mu_0$, then T has the t-distribution with $(n-1)$ degrees of freedom.
- **Question:** What if $\mu \neq \mu_0$? What then can we say about the distribution of T?

Key observation: We can rewrite $\frac{\sqrt{n}(\overline{X}_n - \mu_0)}{s_n}$ as $\frac{\frac{\sqrt{n}(\overline{X}_n - \mu_0)}{\sigma}}{\frac{s_n}{\sigma}}$.

- The numerator $\frac{\sqrt{n}(\overline{X}_n - \mu_0)}{\sigma}$ has the normal distribution with mean $\frac{\sqrt{n}(\mu - \mu_0)}{\sigma}$ and variance 1.
- If $\hat{\sigma}_n^2$ is the biased sample variance of $\{X_1, \ldots, X_n\}$, then $\frac{n\hat{\sigma}_n{}^2}{\sigma^2} = \frac{(n-1)s_n^2}{\sigma^2} \sim \chi^2(n-1)$.
- Thus, the denominator $\frac{s_n}{\sigma}$ equals $\sqrt{\frac{Y}{n-1}}$, where $Y \sim \chi^2(n-1)$.

Non-central t-distribution

Recall: If X is a continuous R.V., then X has a t-distribution if there is some positive integer m such that

$$X = \frac{Z}{\sqrt{\frac{Y}{m}}},$$

where $Z \sim N(0,1)$, and $Y \sim \chi^2(m)$.

- This is one of the two equivalent definitions of t-distribution.

Definition: A R.V. X is said to have a non-central t-distribution if there exist a positive integer m and a real number ψ such that

$$X = \frac{Z}{\sqrt{\frac{Y}{m}}},$$

where $Z \sim N(\psi, 1)$, and $Y \sim \chi^2(m)$.

- m is the degree of freedom. ψ is the non-centrality parameter.
- We say that X has the t-distribution with m degrees of freedom and non-centrality parameter ψ.

The distribution of the test statistic of the t-test

Let $\{X_1, \ldots, X_n\}$ be a random sample of **normal** R.V.'s with mean μ and variance σ^2. Let $\overline{X}_n$ and s_n^2 be the sample mean and the **unbiased sample variance** respectively. Let $\mu_0 \in \mathbb{R}$ be fixed.

Consider a t-test with test statistic $T = \frac{\sqrt{n}(\overline{X}_n - \mu_0)}{s_n}$.

Theorem: T has the non-central t-distribution with $n-1$ degrees of freedom and non-centrality parameter $\psi = \frac{\sqrt{n}(\mu - \mu_0)}{\sigma}$.

Remark: There are many statistical software that can "compute" the cdf of a non-cental t-distribution.

- In this course, we shall not be computing non-central t-distributions, but it is good to know that the distribution of the test statistic T can actually be computed numerically for any value of μ_0.

Two-sample t-statistic

Note: t-tests also make sense on two random samples.

- Let $\{X_1, \ldots, X_n\}$ be a random sample of **normal** observable R.V.'s with **unknown mean** μ_X and **unknown variance** σ^2.
  - Let $\overline{X}_n$, s_X^2 be the sample mean and the **unbiased sample variance** respectively.
- Let $\{Y_1, \ldots, Y_m\}$ be a random sample of **normal** observable R.V.'s with **unknown mean** μ_Y and **unknown variance** σ^2.
  - Let $\overline{X}_n$, s_X^2 be the sample mean and the **unbiased sample variance** respectively.
- Here, we assume every X_i and Y_j have the same variance σ^2.

Definition: The two-sample t-statistic of $\{X_1, \ldots, X_n\}$ and $\{Y_1, \ldots, Y_m\}$ is the R.V.

$$T = \frac{\sqrt{n+m-2}(\overline{X}_n - \overline{Y}_m)}{\sqrt{\frac{1}{n} + \frac{1}{m}}\sqrt{(n-1)s_X^2 + (m-1)s_Y^2}}.$$

Theorem: If $\mu_X = \mu_Y$, then the **two-sample t-statistic** has the t-distribution with $m + n - 2$ degrees of freedom.

Two-sample t-test

Definition: A two-sample t-test is a t-test that uses the two-sample t-statistic (or its absolute value) as the test statistic.

Three most important examples of two-sample t-tests:
Let $\{X_1, \ldots, X_n\}$ and $\{Y_1, \ldots, Y_m\}$ be two random samples of **normal** observable R.V.'s, where each X_i has **unknown mean** μ_X, each Y_j has **unknown mean** μ_Y, and all of the X_i's and Y_j's have a **common unknown variance** σ^2. Let $c \in \mathbb{R}$, and let T be the **two-sample t-statistic** of $\{X_1, \ldots, X_n\}$ and $\{Y_1, \ldots, Y_m\}$.

- The t-test with null hypothesis $H_0 : \mu_X \leq \mu_Y$, test statistic T, and rejection region $[c, \infty)$ is a two-sample t-test.
- The t-test with null hypothesis $H_0 : \mu_X \geq \mu_Y$, test statistic T, and rejection region $(-\infty, c]$ is a two-sample t-test.
- The t-test with null hypothesis $H_0 : \mu_X = \mu_Y$, test statistic $|T|$, and rejection region $[c, \infty)$ is a two-sample t-test.

Note: The first two two-sample t-tests are called one-sided, while the third two-sample t-test is called two-sided.

Significance level of two-sample t-test

Let $\{X_1, \ldots, X_n\}$ and $\{Y_1, \ldots, Y_m\}$ be two random samples of **normal** observable R.V.'s, where each X_i has **unknown mean** μ_X, each Y_j has **unknown mean** μ_Y, and all of the X_i's and Y_j's have a **common unknown variance** σ^2. Let $c_0 \in \mathbb{R}$, and let T be the **two-sample t-statistic** of $\{X_1, \ldots, X_n\}$ and $\{Y_1, \ldots, Y_m\}$.

Theorem: Suppose that c_0 is the $100(1-\alpha_0)$-percentile of the t-distribution with $n+m-2$ degrees of freedom.

- If $\mathcal{H}$ is a t-test with null hypothesis $H_0 : \mu_X \leq \mu_Y$, test statistic T, and rejection region $[c, \infty)$, then $\mathcal{H}$ has significance level α_0 if and only if $c \geq c_0$.
- If $\mathcal{H}$ is a t-test with null hypothesis $H_0 : \mu_X \geq \mu_Y$, test statistic T, and rejection region $(-\infty, c]$, then $\mathcal{H}$ has significance level α_0 if and only if $c \leq c_0$.

Theorem: Suppose that c_0 is the $100(1-\frac{\alpha_0}{2})$-percentile of the t-distribution with $n+m-2$ degrees of freedom. If $\mathcal{H}$ is the t-test with null hypothesis $H_0 : \mu_X = \mu_Y$, test statistic $|T|$, and rejection region $[c, \infty)$, then $\mathcal{H}$ has significance level α_0 if and only if $c \geq c_0$.

p-values of two-sample t-tests

Let $\{X_1, \ldots, X_n\}$ and $\{Y_1, \ldots, Y_m\}$ be two random samples of **normal** observable R.V.'s, where each X_i has **unknown mean** μ_X, each Y_j has **unknown mean** μ_Y, and all of the X_i's and Y_j's have a **common unknown variance** σ^2. Let $c_0 \in \mathbb{R}$, and let T be the **two-sample t-statistic** of $\{X_1, \ldots, X_n\}$ and $\{Y_1, \ldots, Y_m\}$.

Theorem:

- (**One-sided two-sample t-test**) Let $\mathcal{H}$ be a t-test with null hypothesis $H_0 : \mu_X \leq \mu_Y$ or $H_0 : \mu_X \geq \mu_Y$, and test statistic T. Given the observed value $T = t$, suppose that t is the $100(1-\alpha_0)$-th percentile of the t-distribution with $n+m-2$ degrees of freedom. Then the p-value of $\mathcal{H}$ is α_0.
- (**Two-sided two-sample t-test**) Let $\mathcal{H}$ be a t-test with null hypothesis $H_0 : \mu_X = \mu_Y$ and test statistic $|T|$. Given the observed value $T = t$, suppose that $|t|$ is the $100(1-\frac{\alpha_0}{2})$-th percentile of the t-distribution with $n+m-2$ degrees of freedom. Then the p-value of $\mathcal{H}$ is α_0.

Summary

- Expectation of unbiased sample variance
- t-test
- One-sided versus two-sided t-test
- Significance level and p-values for t-test
- Non-central t-distribution
- Two-sample t-statistic
- Two-sample t-test

Reminders:

There is **mini-quiz 4** (15mins) this week during Cohort Class.

- Final mini-quiz! Tested on all materials from Lectures 15–20 and Cohort classes weeks 9–11. Today's lecture is Lecture 21.

Make-up class for **this week**'s Friday's Cohort Class

- Originally on 19th April (Good Friday).
- Make-up: On 17th April (Wednesday), 2–4pm, CC14 (2.507).
    - So your mini-quiz 4 will be on Wednesday!
- This Thursday's cohort classes are on as usual.