50.034 - Introduction to Probability and Statistics

Week 10 – Cohort Class

January–May Term, 2019

SINGAPORE UNIVERSITY OF TECHNOLOGY AND DESIGN

CC Week 10 Assigned Readings: (Revise: 8.2, 8.4, 8.5)

Outline of Cohort Class

Exercises on the following topics:

- χ^2 distribution
  - Includes practice for the use of χ^2 distribution table
- t-distribution
  - Includes practice for the use of t-distribution table
- Confidence intervals

Recall: Chi-squared distribution

The χ^2 distribution is a **special case of gamma distribution**.
Definition: A continuous R.V. X is called chi-squared (or χ^2) if its pdf is given by:

$$f(x) = \begin{cases} \frac{1}{2^{m/2}\Gamma(\frac{m}{2})} x^{(m/2)-1} e^{-0.5x}, & \text{if } x \geq 0; \\ 0, & \text{if } x < 0; \end{cases}$$

for some positive integer m, which is called the degree of freedom.

- We say that X is the χ^2 R.V. with m degrees of freedom, and we write $X \sim \chi^2(m)$.
- This distribution is **exactly the same** as the gamma distribution with parameters $\alpha = \frac{m}{2}$ and $\beta = \frac{1}{2}$.

Special Case: The following distributions are all exactly the same:

- The χ^2 distribution with two degrees of freedom.
- The gamma distribution with parameters $\alpha = 1$ and $\beta = \frac{1}{2}$.
- The exponential distribution with parameter $\frac{1}{2}$ (i.e. mean $= 2$).

Useful results involving χ^2 distribution

Theorem: If $Z \sim N(0, 1)$, then $Z^2 \sim \chi^2(1)$.

Theorem: Let $Y_1, \ldots, Y_n$ be **independent** R.V.'s, such that $Y_i \sim \chi^2(m_i)$ for each $1 \leq i \leq m$. Then the sum $Y_1 + \cdots + Y_n$ has the χ^2 distribution with $m_1 + \cdots + m_n$ degrees of freedom.

Corollary: Let $Z_1, \ldots, Z_n$ be iid **standard normal** R.V.'s. Then $(Z_1^2 + \cdots + Z_n^2) \sim \chi^2(n)$.

Corollary: Let $X_1, \ldots, X_n$ be iid **normal** R.V.'s with mean μ and variance σ^2. Then $\frac{1}{\sigma^2}\sum_{i=1}^n (X_i - \mu)^2 \sim \chi^2(n)$.

Theorem: Let $\{X_1, \ldots, X_n\}$ be a random sample of observable **normal** R.V.'s with variance σ^2, biased sample variance $\hat{\sigma}^2$, and sample mean $\hat{\mu}$. Then $\frac{1}{\sigma^2}\sum_{i=1}^n (X_i - \hat{\mu})^2 = \frac{n\hat{\sigma}^2}{\sigma^2} \sim \chi^2(n-1)$.

Exercise 1 (15 mins)

Suppose that a point (X, Y) on the xy-plane is randomly chosen, where X and Y are independent standard normal R.V.'s. Suppose we want to draw a circle on the xy-plane centered at the origin.

What is the smallest possible radius r for this circle, such that the random point (X, Y) is contained within the circle with probability at least 97.5%?

Hint: How is this question related to the χ^2 distribution?

	p								
.50	.60	.70	.75	.80	.90	.95	.975	.99	.995
.4549	.7083	1.074	1.323	1.642	2.706	3.841	5.024	6.635	7.879
1.386	1.833	2.408	2.773	3.219	4.605	5.991	7.378	9.210	10.60
2.366	2.946	3.665	4.108	4.642	6.251	7.815	9.348	11.34	12.84
3.357	4.045	4.878	5.385	5.989	7.779	9.488	11.14	13.28	14.86
4.351	5.132	6.064	6.626	7.289	9.236	11.07	12.83	15.09	16.75
5.348	6.211	7.231	7.841	8.558	10.64	12.59	14.45	16.81	18.55
6.346	7.283	8.383	9.037	9.803	12.02	14.07	16.01	18.48	20.28
7.344	8.351	9.524	10.22	11.03	13.36	15.51	17.53	20.09	21.95
8.343	9.414	10.66	11.39	12.24	14.68	16.92	19.02	21.67	23.59
9.342	10.47	11.78	12.55	13.44	15.99	18.31	20.48	23.21	25.19
10.34	11.53	12.90	13.70	14.63	17.27	19.68	21.92	24.72	26.76
11.34	12.58	14.01	14.85	15.81	18.55	21.03	23.34	26.22	28.30

Exercise 1 - Solution

Note: (X, Y) is contained within the circle of radius r centered at the origin, if and only if $X^2 + Y^2 \leq r$.

- **Goal:** Find the smallest r such that $\Pr(X^2 + Y^2 \leq r) \geq 0.975$.

Both X and Y are standard normal R.V.'s, so $X^2 + Y^2$ has the χ^2-distribution with 2 degrees of freedom, i.e. $(X^2 + Y^2) \sim \chi^2(2)$.

From the χ^2 distribution table, $\Pr(X^2 + Y^2 \leq 7.378) = 0.975$.
Therefore the smallest possible for r is 7.378.

	p								
.50	.60	.70	.75	.80	.90	.95	.975	.99	.995
.4549	.7083	1.074	1.323	1.642	2.706	3.841	5.024	6.635	7.879
1.386	1.833	2.408	2.773	3.219	4.605	5.991	$\boxed{7.378}$	9.210	10.60
2.366	2.946	3.665	4.108	4.642	6.251	7.815	9.348	11.34	12.84
3.357	4.045	4.878	5.385	5.989	7.779	9.488	11.14	13.28	14.86
4.351	5.132	6.064	6.626	7.289	9.236	11.07	12.83	15.09	16.75
5.348	6.211	7.231	7.841	8.558	10.64	12.59	14.45	16.81	18.55
6.346	7.283	8.383	9.037	9.803	12.02	14.07	16.01	18.48	20.28
7.344	8.351	9.524	10.22	11.03	13.36	15.51	17.53	20.09	21.95
8.343	9.414	10.66	11.39	12.24	14.68	16.92	19.02	21.67	23.59
9.342	10.47	11.78	12.55	13.44	15.99	18.31	20.48	23.21	25.19
10.34	11.53	12.90	13.70	14.63	17.27	19.68	21.92	24.72	26.76
11.34	12.58	14.01	14.85	15.81	18.55	21.03	23.34	26.22	28.30

Exercise 2 (20 mins)

Let $\{X_1, \ldots, X_n\}$ be a random sample of n observable normal R.V.'s with variance σ^2 and biased sample variance $\hat{\sigma}^2$.
Find the smallest possible sample size n such that

$$\Pr\left(|\hat{\sigma}^2 - 1.275\sigma^2| \leq 0.265\sigma^2\right) \geq 0.2.$$

	p							
.50	.60	.70	.75	.80	.90	.95	.975	.9
.4549	.7083	1.074	1.323	1.642	2.706	3.841	5.024	6.6
1.386	1.833	2.408	2.773	3.219	4.605	5.991	7.378	9.2
2.366	2.946	3.665	4.108	4.642	6.251	7.815	9.348	11.3
3.357	4.045	4.878	5.385	5.989	7.779	9.488	11.14	13.2
4.351	5.132	6.064	6.626	7.289	9.236	11.07	12.83	15.0
5.348	6.211	7.231	7.841	8.558	10.64	12.59	14.45	16.8
6.346	7.283	8.383	9.037	9.803	12.02	14.07	16.01	18.4
7.344	8.351	9.524	10.22	11.03	13.36	15.51	17.53	20.0
8.343	9.414	10.66	11.39	12.24	14.68	16.92	19.02	21.6
9.342	10.47	11.78	12.55	13.44	15.99	18.31	20.48	23.2
10.34	11.53	12.90	13.70	14.63	17.27	19.68	21.92	24.7
11.34	12.58	14.01	14.85	15.81	18.55	21.03	23.34	26.2
12.34	13.64	15.12	15.98	16.98	19.81	22.36	24.74	27.6
13.34	14.69	16.22	17.12	18.15	21.06	23.68	26.12	29.1

Exercise 2 - Solution

First Step: $\frac{n\hat{\sigma}^2}{\sigma^2} \sim \chi^2(n-1)$.

$$
\begin{aligned}
&\Pr\left(|\hat{\sigma}^2 - 1.275\sigma^2| \leq 0.265\sigma^2\right) \geq 0.2 \\
\iff &\Pr\left(|\tfrac{\hat{\sigma}^2}{\sigma^2} - 1.275| \leq 0.265\right) \geq 0.2 \\
\iff &\Pr\left(-0.265 \leq \tfrac{\hat{\sigma}^2}{\sigma^2} - 1.275 \leq 0.265\right) \geq 0.2 \\
\iff &\Pr\left(1.01 \leq \tfrac{\hat{\sigma}^2}{\sigma^2} \leq 1.54\right) \geq 0.2 \\
\iff &\Pr\left(1.01n \leq \tfrac{n\hat{\sigma}^2}{\sigma^2} \leq 1.54n\right) \geq 0.2
\end{aligned}
$$

Hence, what we need to do is to find a value for n (from the table of values of the χ^2 distribution) satisfying the following:

$$\Pr\left(\tfrac{n\hat{\sigma}^2}{\sigma^2} \leq 1.54n\right) - \Pr\left(\tfrac{n\hat{\sigma}^2}{\sigma^2} \leq 1.01n\right) \geq 0.2.$$

$$\Pr\left(\tfrac{(n-1)\hat{\sigma}^2}{\sigma^2} \leq 1.54(n-1)\right) - \Pr\left(\tfrac{(n-1)\hat{\sigma}^2}{\sigma^2} \leq 1.01(n-1)\right) < 0.2.$$

Exercise 2 - Solution

Looking through the values in the table, we find:

$$\Pr\left(\frac{6\hat{\sigma}^2}{\sigma^2} \leq 1.54 \times (6) = 9.24\right) - \Pr\left(\frac{6\hat{\sigma}^2}{\sigma^2} \leq 1.01 \times (6) = 6.06\right)$$

$$> \Pr\left(\frac{6\hat{\sigma}^2}{\sigma^2} \leq 9.236\right) - \Pr\left(\frac{6\hat{\sigma}^2}{\sigma^2} \leq 6.064\right) = 0.9 - 0.7 = 0.2.$$

$$\Pr\left(\frac{5\hat{\sigma}^2}{\sigma^2} \leq 1.54 \times (5) = 7.7\right) - \Pr\left(\frac{5\hat{\sigma}^2}{\sigma^2} \leq 1.01 \times (5) = 5.05\right)$$

$$< \Pr\left(\frac{5\hat{\sigma}^2}{\sigma^2} \leq 7.779\right) - \Pr\left(\frac{5\hat{\sigma}^2}{\sigma^2} \leq 4.878\right) = 0.9 - 0.7 = 0.2.$$

Therefore, the smallest sample size is $n = 6$.

				p				
.50	.60	.70	.75	.80	.90	.95	.975	.9
.4549	.7083	1.074	1.323	1.642	2.706	3.841	5.024	6.6
1.386	1.833	2.408	2.773	3.219	4.605	5.991	7.378	9.2
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3.357	4.045	$\boxed{4.878}$	5.385	5.989	$\boxed{7.779}$	9.488	11.14	13.2
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5.348	6.211	7.231	7.841	8.558	10.64	12.59	14.45	16.8
6.346	7.283	8.383	9.037	9.803	12.02	14.07	16.01	18.4

Confidence Intervals of Parameters

Let $0 < p < 1$, and let $\{X_1, \ldots, X_n\}$ be a random sample of observable R.V.'s that depend on some parameter θ.

- If T_1 and T_2 are statistics of $\{X_1, \ldots, X_n\}$ such that $\Pr(T_1 < \theta < T_2) \geq p$ for all possible values of θ, then we say that the random open interval (T_1, T_2) is a 100p percent confidence interval for θ.
- We say that the confidence level is 100p percent.
- If T_1 and T_2 are statistics of $\{X_1, \ldots, X_n\}$ such that $\Pr(T_1 < \theta < T_2) = p$ for all possible values of θ, then the 100p percent confidence interval (T_1, T_2) is called exact.
- After the observed values $X_1 = x_1, \ldots, X_n = x_n$ are given, let $T_1 = t_1$ and $T_2 = t_2$ be the corresponding computed values. Then the open interval (t_1, t_2) is called the observed value of the confidence interval.

Important Note: A confidence interval is random! It is a **pair of R.V.'s** forming a random open interval.

- Different observed values for $X_1, \ldots, X_n$ give different actual open intervals.

Exercise 3 (5 mins)

Suppose you have 50 different observed values of a 90% confidence interval for some parameter θ, where θ is an unknown constant.

1. Among these 50 obtained open intervals, how many of them should we expect to not actually contain the "true" value of θ?
2. Suppose that based on actual observed values of the observable R.V.'s, you explicitly computed an open interval $(10.1, 17.2)$ that is an observed value of the 90% confidence interval. What can you conclude about the "true" value of θ?

(If you are not sure, discuss with your friends!)

Exercise 3 - Solution

1. A confidence interval (T_1, T_2) is a random open interval. Different observed values for the observable R.V.'s give different observed values for the confidence interval (T_1, T_2).

- By saying that (T_1, T_2) is a 90% confidence interval for θ, it means that 90% of all observed values (t_1, t_2) for (T_1, T_2) are open intervals that actually contain θ.

This means we should expect that $0.9 \times 50 = 45$ of the 50 open intervals found would actually contain the "true" value of θ. Therefore, we should expect roughly 5 "wrong" open intervals.

Exercise 3 - Solution

2. Given an explicit open interval $(10.1, 17.2)$, we would have no idea whether the interval is "correct" (i.e. contains the "true" value of θ), or "wrong" (i.e. doesn't contain the "true" value of θ).

- Since θ is a constant, either $\Pr(10.1 < \theta < 17.2) = 1$ or $\Pr(10.1 < \theta < 17.2) = 0$.
- We only know that if we observed lots of open intervals (observed values of the confidence interval) based on many different sets of observed values for the observable R.V.'s, then on average 90% of all these open intervals are "correct".

Note: This example shows the limitation of confidence intervals!

- The mean and relevance of confidence intervals is somewhat controversial in statistics!
- Yet, confidence intervals are quite commonly used (and wrongly interpreted) (e.g. in life sciences).

Recall: t-distribution

Let X be a continuous R.V.

Definition 1: We say X has a t-distribution if its pdf is given by:

$$f(x) = \frac{\Gamma\left(\frac{m+1}{2}\right)}{\sqrt{m\pi}\cdot\Gamma\left(\frac{m}{2}\right)}\left(1+\frac{x^2}{m}\right)^{-(m+1)/2} \quad \text{(for all } x\text{)}$$

for some positive integer m.

Definition 2: We say X has a t-distribution if

$$X = \frac{Z}{\sqrt{\frac{Y}{m}}},$$

for some positive integer m, where $Z \sim N(0,1)$, and $Y \sim \chi^2(m)$.

- Both definitions are equivalent. (See course textbook for a proof.)
- The positive integer m is called the degree of freedom.
- We say X has the t-distribution with m degrees of freedom.
  - We rarely say X is a t R.V., since this could be ambiguous.

Main Use: To model a normal R.V. with unknown mean and unknown variance after some transformation.

Unbiased sample variance

Let $\{X_1, \ldots, X_n\}$ be a random sample of observable R.V.'s.
Definition: The unbiased sample variance of $\{X_1, \ldots, X_n\}$ is

$$s_n^2(X_1, \ldots, X_n) = \frac{1}{n-1} \sum_{i=1}^{n} (X_i - \overline{X}_n)^2.$$

- Common notation for unbiased sample variance: s_n^2 or S_n^2.
- s_n^2 is also called the Bessel-corrected sample variance.
- $s_n = \sqrt{s_n^2}$ is called unbiased sample standard deviation.

In comparison, the biased sample variance of $\{X_1, \ldots, X_n\}$ (which we saw in Lecture 16) is

$$\hat{\sigma}^2(X_1, \ldots, X_n) = \frac{1}{n} \sum_{i=1}^{n} (X_i - \overline{X}_n)^2.$$

- $\hat{\sigma} = \sqrt{\hat{\sigma}^2}$ is called biased sample standard deviation.
- Both $\hat{\sigma}^2$ and s_n^2 are commonly used, so be careful to state whether you mean biased or unbiased sample variance!

Note: $s_n^2 = \frac{n}{n-1}\hat{\sigma}^2$. (They are different by a factor of $\frac{n}{n-1}$.)

Main Theorem on t-distributions

Let $\{X_1, \ldots, X_n\}$ be a random sample of observable **normal** R.V.'s with mean μ and variance σ^2. Let $\overline{X}_n$ and s_n^2 be the sample mean and the **unbiased sample variance** respectively.

Important Theorem: The random variable

$$\frac{\sqrt{n}(\overline{X}_n - \mu)}{s_n} = \frac{\sqrt{n}(\overline{X}_n - \mu)}{\frac{1}{n-1}\sum_{i=1}^{n}(X_i - \overline{X}_n)^2}$$

has the t-distribution with $(n-1)$ degrees of freedom.

- In comparison, $\frac{\sqrt{n}(\overline{X}_n - \mu)}{\sigma}$ has the standard normal distribution.

Connection to Confidence Intervals:

Let $0 < p < 1$ be a fixed real number.

- If σ^2 is known, then the standard normal distribution can be used to find a $100p$-percent confidence interval for μ.
- If σ^2 is unknown, then the t-distribution can be used to find a $100p$-percent confidence interval for μ.

Exercise 4 (15 mins)

Let $\{X_1, \ldots, X_9\}$ be a random sample of 9 observable normal R.V.'s with unknown mean μ and unknown variance σ^2. Let $\overline{X}_9$ and s_9^2 be the sample mean and the unbiased sample variance respectively of $\{X_1, \ldots, X_9\}$.

Find an exact 90% confidence interval for μ in terms of $\overline{X}_9$ and s_9^2.

m	$p = .55$	.60	.65	.70	.75	.80	.85	.90	.95	.975	.99
1	.158	.325	.510	.727	1.000	1.376	1.963	3.078	6.314	12.706	31.821
2	.142	.289	.445	.617	.816	1.061	1.386	1.886	2.920	4.303	6.965
3	.137	.277	.424	.584	.765	.978	1.250	1.638	2.353	3.182	4.541
4	.134	.271	.414	.569	.741	.941	1.190	1.533	2.132	2.776	3.747
5	.132	.267	.408	.559	.727	.920	1.156	1.476	2.015	2.571	3.365
6	.131	.265	.404	.553	.718	.906	1.134	1.440	1.943	2.447	3.143
7	.130	.263	.402	.549	.711	.896	1.119	1.415	1.895	2.365	2.998
8	.130	.262	.399	.546	.706	.889	1.108	1.397	1.860	2.306	2.896
9	.129	.261	.398	.543	.703	.883	1.100	1.383	1.833	2.262	2.821
10	.129	.260	.397	.542	.700	.879	1.093	1.372	1.812	2.228	2.764
11	.129	.260	.396	.540	.697	.876	1.088	1.363	1.796	2.201	2.718
12	.128	.259	.395	.539	.695	.873	1.083	1.356	1.782	2.179	2.681
13	.128	.259	.394	.538	.694	.870	1.079	1.350	1.771	2.160	2.650
14	.128	.258	.393	.537	.692	.868	1.076	1.345	1.761	2.145	2.624

Exercise 4 - Solution

Note that $Z = \frac{\sqrt{9}(\overline{X}_9 - \mu)}{s_9} = \frac{3(\overline{X}_9 - \mu)}{s_9}$ has the t-distribution with 8 degrees of freedom.

Let $F(z)$ be the cdf of Z. Then for any real number $r > 0$,

$$\begin{aligned}
\Pr(\overline{X}_9 - r < \mu < \overline{X}_9 + r) &= \Pr(-r < \overline{X}_9 - \mu < r) \\
&= \Pr\left(-\frac{3r}{s_9} < \frac{3(\overline{X}_9 - \mu)}{s_9} < \frac{3r}{s_9}\right) \\
&= F\left(\frac{3r}{s_9}\right) - F\left(-\frac{3r}{s_9}\right) \\
&= F\left(\frac{3r}{s_9}\right) - \left(1 - F\left(\frac{3r}{s_9}\right)\right) \\
&= 2 \cdot F\left(\frac{3r}{s_9}\right) - 1.
\end{aligned}$$

Thus, $\Pr(\overline{X}_9 - r < \mu < \overline{X}_9 + r) = 0.9 \Leftrightarrow F\left(\frac{3r}{s_9}\right) = 0.95.$

Exercise 4 - Solution

What we want: $F\left(\frac{3r}{s_9}\right) = 0.95$ (for 8 degrees of freedom).
From the table, the closest value of z satisfying $F(z) = 0.95$ for 8 degrees of freedom is $z = 1.860$.
Hence $\frac{3r}{s_9} \approx 1.860$, i.e. $r \approx \frac{1.860}{3} s_9 = 0.62 s_9$.
Therefore, an exact 90% confidence interval for μ is

$$(\overline{X}_9 - 0.62 s_9, \overline{X}_9 + 0.62 s_9).$$

m	$p = .55$	.60	.65	.70	.75	.80	.85	.90	.95	.975	.99
1	.158	.325	.510	.727	1.000	1.376	1.963	3.078	6.314	12.706	31.821
2	.142	.289	.445	.617	.816	1.061	1.386	1.886	2.920	4.303	6.965
3	.137	.277	.424	.584	.765	.978	1.250	1.638	2.353	3.182	4.541
4	.134	.271	.414	.569	.741	.941	1.190	1.533	2.132	2.776	3.747
5	.132	.267	.408	.559	.727	.920	1.156	1.476	2.015	2.571	3.365
6	.131	.265	.404	.553	.718	.906	1.134	1.440	1.943	2.447	3.143
7	.130	.263	.402	.549	.711	.896	1.119	1.415	1.895	2.365	2.998
8	.130	.262	.399	.546	.706	.889	1.108	1.397	$\boxed{1.860}$	2.306	2.896
9	.129	.261	.398	.543	.703	.883	1.100	1.383	1.833	2.262	2.821
10	.129	.260	.397	.542	.700	.879	1.093	1.372	1.812	2.228	2.764
11	.129	.260	.396	.540	.697	.876	1.088	1.363	1.796	2.201	2.718
12	.128	.259	.395	.539	.695	.873	1.083	1.356	1.782	2.179	2.681

Exercise 5 (20 mins)

Let $\{X_1, \ldots, X_9\}$ be a random sample of 9 observable normal R.V.'s with unknown mean μ and unknown variance σ^2. Let $\overline{X}_9$ and $\hat{\sigma}_9^2$ be the sample mean and the biased sample variance respectively of $\{X_1, \ldots, X_9\}$.

Find an exact 90% confidence interval for the **variance** σ^2 in terms of $\overline{X}_9$ and $\hat{\sigma}_9^2$.

Hint: See next slide for the table of values for the $\boldsymbol{\chi^2}$ **distribution** (both halves of the table are shown).

Useful χ^2 distribution values for Exercise 5

m	p = .005	.01	.025	.05	.10	.20	.25	.30	.40
1	.0000	.0002	.0010	.0039	.0158	.0642	.1015	.1484	.2750
2	.0100	.0201	.0506	.1026	.2107	.4463	.5754	.7133	1.022
3	.0717	.1148	.2158	.3518	.5844	1.005	1.213	1.424	1.869
4	.2070	.2971	.4844	.7107	1.064	1.649	1.923	2.195	2.753
5	.4117	.5543	.8312	1.145	1.610	2.343	2.675	3.000	3.655
6	.6757	.8721	1.237	1.635	2.204	3.070	3.455	3.828	4.570
7	.9893	1.239	1.690	2.167	2.833	3.822	4.255	4.671	5.493
8	1.344	1.647	2.180	2.732	3.490	4.594	5.071	5.527	6.423
9	1.735	2.088	2.700	3.325	4.168	5.380	5.899	6.393	7.357
10	2.156	2.558	3.247	3.940	4.865	6.179	6.737	7.267	8.295

.50	p = .60	.70	.75	.80	.90	.95	.975	.9
.4549	.7083	1.074	1.323	1.642	2.706	3.841	5.024	6.6
1.386	1.833	2.408	2.773	3.219	4.605	5.991	7.378	9.2
2.366	2.946	3.665	4.108	4.642	6.251	7.815	9.348	11.3
3.357	4.045	4.878	5.385	5.989	7.779	9.488	11.14	13.2
4.351	5.132	6.064	6.626	7.289	9.236	11.07	12.83	15.0
5.348	6.211	7.231	7.841	8.558	10.64	12.59	14.45	16.8
6.346	7.283	8.383	9.037	9.803	12.02	14.07	16.01	18.4
7.344	8.351	9.524	10.22	11.03	13.36	15.51	17.53	20.0
8.343	9.414	10.66	11.39	12.24	14.68	16.92	19.02	21.6
9.342	10.47	11.78	12.55	13.44	15.99	18.31	20.48	23.2

Exercise 5 - Solution

Note that $Z = \frac{9\hat{\sigma}_9^2}{\sigma^2}$ has the χ^2-distribution with 8 degrees of freedom.

Let $F(z)$ be the cdf of Z. Then for any real numbers $c_1, c_2 > 0$,

$$\begin{aligned}
\Pr(c_1\hat{\sigma}_9^2 < \sigma^2 < c_2\hat{\sigma}_9^2) &= \Pr\left(\tfrac{1}{c_2\hat{\sigma}_9^2} < \tfrac{1}{\sigma^2} < \tfrac{1}{c_1\hat{\sigma}_9^2}\right) \\
&= \Pr\left(\tfrac{9}{c_2} < \tfrac{9\hat{\sigma}_9^2}{\sigma^2} < \tfrac{9}{c_1}\right) \\
&= F\left(\tfrac{9}{c_1}\right) - F\left(\tfrac{9}{c_2}\right)
\end{aligned}$$

Thus, we want to find some suitable values for c_1, c_2 such that

$$F\left(\tfrac{9}{c_1}\right) - F\left(\tfrac{9}{c_2}\right) = 0.9.$$

Note: There are infinitely many possible pairs of answers for c_1, c_2.

Exercise 5 - Solution

What we want: Values $c_1, c_2 > 0$ such that $F(\frac{9}{c_1}) - F(\frac{9}{c_2}) = 0.9$.

For example, since $0.95 - 0.05 = 0.9$ and since the χ^2 distribution table gives $F(15.51) = 0.95$ and $F(2.732) = 0.05$ (for 8 degrees of freedom), it means that we could set $\frac{9}{c_1} = 15.51$ and $\frac{9}{c_2} = 2.732$.

In other words, $c_1 = \frac{9}{15.51} \approx 0.5803$ and $c_2 = \frac{9}{2.732} \approx 3.2943$.

Therefore, $(0.5803\hat{\sigma}_9^2, 3.2943\hat{\sigma}_9^2)$ is an exact 90% confidence interval for σ^2.

- Note: There are many possible exact 90% confidence intervals for σ^2.
- e.g. $(0, \frac{9}{13.36}\hat{\sigma}_9^2)$ and $(\frac{9}{3.490}\hat{\sigma}_9^2, \infty)$ are two other possible exact 90% confidence intervals for σ^2.

Summary

Exercises on the following topics:

- χ^2 distribution
  - Includes practice for the use of χ^2 distribution table
- t-distribution
  - Includes practice for the use of t-distribution table
- Confidence intervals