

## 50.034 – Introduction to Probability and Statistics

January–May Term, 2019

### Homework Set 7

Due by: Week 11 Cohort Class (11 Apr 2019 or 12 Apr 2019)

**Note:** The tables of values for the  $\chi^2$  distribution and the  $t$  distribution can be found at the back of the course textbook.

**Question 1.** Let  $\{X_1, \dots, X_n\}$  be a random sample of  $n$  observable normal random variables with mean  $\mu$  and variance  $\sigma^2$ . Let  $\hat{\sigma}^2$  denote the sample variance of this random sample  $\{X_1, \dots, X_n\}$ . For each of the two inequalities given below, determine the smallest possible value for the sample size  $n$  such that the inequality is satisfied:

- (i)  $\Pr\left(\frac{\hat{\sigma}^2}{\sigma^2} \leq 1.2\right) \geq 0.8$ .
- (ii)  $\Pr\left(|\hat{\sigma}^2 - 0.77\sigma^2| \leq 0.37\sigma^2\right) \geq 0.725$ .

**Question 2.** Consider a statistical model consisting of observable exponential random variables  $X_1, \dots, X_n$  that are conditionally iid given the parameter  $\theta$ . Let  $\hat{\theta}$  be the maximum likelihood estimator of  $\theta$ . What is the conditional probability density function of the random variable  $n(\hat{\theta})^{-1}$  given  $\theta = 5$ ? (Hint: Theorem 5.7.7 of the course textbook may be useful.)

**Question 3.** An automated juice vending machine dispenses orange juice in cups. A total of 25 dispensed cups of orange juice are collected, and the amount of juice in each cup is measured. Consider a statistical model consisting of a random sample  $\{X_1, \dots, X_{25}\}$  of normal random variables with unknown mean  $\mu$  and unknown variance  $\sigma^2$ , where each  $X_i$  represents the amount of juice (in ml) in the  $i$ -th dispensed cup. Let  $\bar{X}_{25}$  and  $s_{25}^2$  be the sample mean and the unbiased sample variance respectively of  $\{X_1, \dots, X_{25}\}$ .

- (i) Find an exact 90% confidence interval for  $\mu$  in terms of  $\bar{X}_{25}$  and  $s_{25}^2$ .
- (ii) After measuring the amount of juice in these 25 dispensed cups, it was noticed that the 25 cups have a mean amount of 251 ml and a standard deviation of 4 ml. Using this information, find an observed value of the 90% confidence interval that you found in the previous part.

**Question 4.** At a Fuji apple farm in Fujisaki, Japan, 10 Fuji apples are randomly selected and weighed. Consider a statistical model consisting of a random sample  $\{X_1, \dots, X_{10}\}$  of normal random variables with unknown mean  $\mu$  and unknown variance  $\sigma^2$ , where each  $X_i$  represents the weight (in grams) of the  $i$ -th selected apple. Let  $\bar{X}_{10}$  and  $s_{10}^2$  be the sample mean and the unbiased sample variance respectively of  $\{X_1, \dots, X_{10}\}$ .

- (i) Find an exact 95% confidence interval for  $\mu$  in terms of  $\bar{X}_{10}$  and  $s_{10}^2$ .
- (ii) The measurements of all 10 weights (in grams) are indicated below:

148, 150, 155, 154, 152, 148, 155, 160, 152, 145.

Using these values, find an observed value of the 95% confidence interval for  $\mu$  that you found in the previous part.

**Question 5.** At a steel factory, the tensile strength of 20 randomly cut steel sample pieces are measured. Consider a statistical model consisting of a random sample  $\{X_1, \dots, X_{20}\}$  of normal random variables with unknown mean  $\mu$  and unknown variance  $\sigma^2$ , where each  $X_i$  represents the tensile strength (in MPa) of the  $i$ -th steel sample piece. Let  $\bar{X}_{20}$  and  $s_{20}^2$  be the sample mean and the unbiased sample variance respectively of  $\{X_1, \dots, X_{20}\}$ .

- (i) Find an exact 95% confidence interval for the variance  $\sigma^2$  in terms of  $\bar{X}_{20}$  and  $s_{20}^2$ .
- (ii) After the 20 measurements have been collected, it was noticed that the 20 steel sample pieces have a mean tensile strength of 355 MPa and a standard deviation of 25 MPa. Using this information, find an observed value of the 95% confidence interval that you found in the previous part.