50.034 - Introduction to Probability and Statistics

Week 6 – Cohort Class

January–May Term, 2019

SINGAPORE UNIVERSITY OF
TECHNOLOGY AND DESIGN

CC Week 6 Assigned Readings: (Revise: 6.1, 6.2, 6.3, 6.4)

Outline of Cohort Class

- Recap: Joint/marginal/conditional distributions
- Recap: Conditional expectation
- Recap: Moments and moment generating functions
- **Mini-quiz 2**

Exercises on the following topics:

- Markov's inequality
- Chebyshev's inequality
- Central limit theorem

Recall: Joint distributions

The joint distribution of **any** R.V.'s X and Y is the collection of all probabilities of the form $\Pr\left((X, Y) \in C\right)$, for all sets $C \subseteq \mathbb{R}^2$.

Ways to describe the joint distribution of X and Y:

- joint probability mass function (only for discrete R.V.'s)

$$p(x, y) = \Pr(X = x \text{ and } Y = y) = \Pr\left((X, Y) = (x, y)\right).$$

- joint probability density function (only for continuous R.V.'s)

$$\Pr\left((X, Y) \in A\right) = \iint_A f(x, y)\, dx\, dy, \qquad \text{where } A \subseteq \mathbb{R}^2.$$

- joint cumulative distribution function (for *any* R.V.'s)

$$F(x, y) = \Pr(X \le x, Y \le y), \qquad \text{for } -\infty < x, y < \infty.$$

  - $F(a, b) = \sum_{x \le a} \sum_{y \le b} p(x, y)$. (discrete R.V.'s case)
  - $F(a, b) = \int_{-\infty}^{b} \int_{-\infty}^{a} f(x, y)\, dx\, dy$. (continuous R.V.'s case)

Recall: Marginal pmf/pdf/cdf

If X and Y are **discrete** R.V.'s with joint pmf $p(x, y)$, then:

- The marginal pmf of X is $p_X(x) = \sum_{y \in D_Y} p(x, y)$;
- The marginal pmf of Y is $p_Y(x) = \sum_{x \in D_X} p(x, y)$;

where D_X and D_Y are the sets of possible values for X and Y.

If X and Y are **continuous** R.V.'s with joint pdf $f(x, y)$, then:

- The marginal pdf of X is $f_X(x) = \int_{-\infty}^{\infty} f(x, y)\, dy$.
- The marginal pdf of X is $f_Y(y) = \int_{-\infty}^{\infty} f(x, y)\, dx$.

If X and Y are **arbitrary** R.V.'s with joint cdf $F(x, y)$, then:

- The marginal cdf of X is $F_X(x) = \lim_{y \to \infty} F(x, y)$.
- The marginal cdf of Y is $F_Y(y) = \lim_{x \to \infty} F(x, y)$.

Recall: Conditional distribution/pmf/pdf

Let $C' \subseteq \mathbb{R}$, and let X and Y be **arbitrary** R.V.'s. The conditional distribution of X given $Y \in C'$ is the collection of all conditional probabilities of the form $\Pr(X \in C | Y \in C')$ for all sets $C \subseteq \mathbb{R}$.

If X and Y are **discrete** R.V.'s with joint pmf $p(x, y)$, and if $y \in \mathbb{R}$ such that $p_Y(y) > 0$, then the conditional pmf of X given $Y = y$, is the function $p_{X|Y}(x|y) = \frac{p(x,y)}{p_Y(y)}$.

If X and Y are **continuous** R.V.'s with joint pdf $f(x, y)$, and if $y \in \mathbb{R}$ such that $f_Y(y) > 0$, then the conditional pdf of X given $Y = y$, is the function $f_{X|Y}(x|y) = \frac{f(x,y)}{f_Y(y)}$.

Recall: Conditional expectation

Recall: Conditional expectation $=$ $\mathbf{E}$[conditional distribution].

Let X, Y be R.V.'s, and let $C' \subseteq \mathbb{R}$ such that $\Pr(Y \in C') > 0$.
Definition: The conditional expectation of X given $Y \in C'$ is:

$$\mathbf{E}[X|Y \in C'] = \mathbf{E}\begin{bmatrix} \text{conditional distribution} \\ \text{of } X \text{ given } Y \in C'. \end{bmatrix}$$

- If C' is fixed, then $\mathbf{E}[X|Y \in C']$ is a fixed value.

If we are given a specific value $Y = y$, then the conditional expectation of X given $Y = y$ is denoted by $\mathbf{E}[X|Y = y]$.

- Similarly, if y is fixed, then $\mathbf{E}[X|Y = y]$ is a fixed value.

Conditional expectation as a variable:

- We can think of $\mathbf{E}[X|Y \in C']$ as a function in terms of C'.
  - Different values of C' give different values for $\mathbf{E}[X|Y \in C']$.
- Similarly, we can think of $\mathbf{E}[X|Y = y]$ as a function of y.
  - Different values of y give different values for $\mathbf{E}[X|Y = y]$.

More generally, we can think of $\mathbf{E}[X|Y]$ as a **function of** Y.

- In other words, $\mathbf{E}[X|Y]$ is a random variable!

Moments and moment generating functions

Let X be a random variable, and let k be any positive integer.

- $\mathbf{E}[X^k]$ is called the k-th moment of X.
- $\mathbf{E}[(X-\mu)^k]$ is called the k-th central moment of X.
  - For this to make sense, $\mu = \mathbf{E}[X]$ should be finite.
- $\mathbf{E}[X] =$ first moment; $\mathrm{var}(X) =$ second central moment.

The moment generating function (mgf) of X is $\psi(t) = \mathbf{E}[e^{tX}]$.

- $\psi(t)$ (if it exists) depends only on the distribution of X.
- **Technical Note:** The domain of $\psi(t)$ is the set of all real values of t such that $\mathbf{E}[e^{tX}]$ exists.

Fact: If X and Y are R.V.'s with the same distribution, then X and Y must have the same mgf (if it exists).

- In other words, to check if X and Y are identically distributed, it suffices to check if their mgf's coincide (if they exist).

Exercise 1 (15 mins)

Let X be the exponential random variable with parameter $\lambda > 0$.

1. Find the moment generating function of X.
2. Using part 1., compute the third moment of X in terms of λ.

Hint: If X is a R.V. whose mgf $\psi(t)$ exists, then the derivatives of $\psi(t)$ at $t = 0$ generate the moments of X.

Exercise 1 - Solution

1. By the definition of an exponential R.V., the pdf of X is

$$f(x) = \begin{cases} \lambda e^{-\lambda x}, & \text{if } x \geq 0; \\ 0, & \text{otherwise.} \end{cases}$$

Recall: (Lecture 5) For any continuous R.V. with pdf $f(x)$, and for any function $h : \mathbb{R} \to \mathbb{R}$,

$$\mathbf{E}[h(X)] = \int_{-\infty}^{\infty} h(x) \cdot f(x)\, dx.$$

Thus,

$$\begin{aligned} \mathbf{E}[e^{tX}] &= \int_{-\infty}^{\infty} e^{tx} \cdot f(x)\, dx \\ &= \int_{0}^{\infty} e^{tx} \cdot \lambda e^{-\lambda x}\, dx \\ &= \lambda \int_{0}^{\infty} e^{(t-\lambda)x}\, dx \end{aligned}$$

Exercise 1 - Solution

1. (continued) If $t < \lambda$, then

$$\lambda \int_0^\infty e^{(t-\lambda)x}\,dx = \lambda\left[\tfrac{1}{t-\lambda}e^{(t-\lambda)x}\right]_{x=0}^{x=\infty}$$
$$= \lambda(0 - \tfrac{1}{t-\lambda})$$
$$= \frac{\lambda}{\lambda - t}.$$

Therefore, the moment generating function of X is

$$\psi(t) = \frac{\lambda}{\lambda - t}, \qquad \text{for } t < \lambda,$$

i.e. the domain of $\psi(t)$ is the open interval $(-\infty, \lambda)$.

Note: The exponential R.V. (with parameter λ) is an example of a R.V. whose mgf exists and has a domain that is not the whole of $\mathbb{R}$.

Exercise 1 - Solution

2. [compute the third moment of X]

First, we compute the first few derivatives of $\psi(t)$.

$$\psi'(t) = \frac{d}{dt}\left(\frac{\lambda}{\lambda - t}\right) = \frac{\lambda}{(\lambda - t)^2}.$$
$$\psi''(t) = \frac{d}{dt}\left(\frac{\lambda}{(\lambda - t)^2}\right) = \frac{2\lambda}{(\lambda - t)^3}.$$
$$\psi'''(t) = \frac{d}{dt}\left(\frac{2\lambda}{(\lambda - t)^3}\right) = \frac{6\lambda}{(\lambda - t)^4}.$$

Therefore, the third moment of X equals

$$\mathbf{E}[X^3] = \psi'''(0) = \frac{6\lambda}{(\lambda - 0)^4} = \frac{6}{\lambda^3}.$$

Mini-quiz 2 (15 mins)

Only writing materials are allowed. No calculators, notes, books, or cheat sheets are allowed. Don't worry, you won't need calculators.

If you are not present in class at the start of the quiz, you will not be given additional time to finish the quiz.

Remarks:

- There are no make-up mini-quizzes! If you arrive in class after the mini-quiz ends, or do not attend that cohort class, you will not have a chance to take the mini-quiz.
- To take into account unforeseen circumstances (e.g. mini-quiz missed due to illness), only the **best 3 of 4** mini-quiz scores will be counted towards your final grade.

Exercise 2 (20 mins)

The average length of an adult European rabbit is 40cm.

1. Give an upper bound on the probability that a randomly selected adult European rabbit is at least 50cm long.
2. Suppose that the standard deviation of the length distribution of adult European rabbits is 8cm. Find a lower bound on the probability that a randomly selected adult European rabbit is between 30cm and 50cm long.
3. Suppose we are given further that the lengths of adult European rabbits follow a normal distribution.
   - What is the actual probability that a randomly selected adult European rabbit is at least 50cm long?
   - Also, what is the actual probability that a randomly selected adult European rabbit is between 30cm and 50cm long?

x	$\Phi(x)$	x	$\Phi(x)$	x	$\Phi(x)$	x	$\Phi(x)$	x	$\Phi(x)$
0.00	0.5000	0.60	0.7257	1.20	0.8849	1.80	0.9641	2.40	0.9918
0.01	0.5040	0.61	0.7291	1.21	0.8869	1.81	0.9649	2.41	0.9920
0.02	0.5080	0.62	0.7324	1.22	0.8888	1.82	0.9656	2.42	0.9922
0.03	0.5120	0.63	0.7357	1.23	0.8907	1.83	0.9664	2.43	0.9925
0.04	0.5160	0.64	0.7389	1.24	0.8925	1.84	0.9671	2.44	0.9927
0.05	0.5199	0.65	0.7422	1.25	0.8944	1.85	0.9678	2.45	0.9929
0.06	0.5239	0.66	0.7454	1.26	0.8962	1.86	0.9686	2.46	0.9931

Exercise 2 - Solution

Let X be the length (in cm) of a randomly selected adult European rabbit.

1. We are given that $\mathbf{E}[X] = 40$. Thus, by Markov's inequality,

$$\Pr(X \geq 50) \leq \frac{\mathbf{E}[X]}{50} = 0.8.$$

2. We are given further that $\mathrm{var}(X) = 8^2 = 64$.
Thus, by Chebyshev's inequality,

$$\begin{aligned}\Pr(30 \leq X \leq 50) &= \Pr(|X - 40| \leq 10) = 1 - \Pr(|X - 40| > 10) \\ &\geq 1 - \frac{\mathrm{var}(X)}{10^2} = 1 - 0.64 = 0.36.\end{aligned}$$

[**Note:** Chebyshev's inequality gives $\Pr(|X - \mathbf{E}[X]| \geq 10) \leq \frac{\mathrm{var}(X)}{10^2}$.]

Exercise 2 - Solution

3. We are now given that X follows a normal distribution.
   - We already know that $\mathbf{E}[X] = 40$ and $\mathrm{var}(X) = 64$.
   - This means that $X \sim N(40, 64)$.

Let $Z = \frac{X-40}{8}$. Note that $Z \sim N(0,1)$.

Thus, the probability that a randomly selected adult European rabbit is at least 50cm long is:

$$\Pr(X \geq 50) = \Pr\left(\frac{X-40}{8} \geq \frac{50-40}{8}\right) = \Pr(Z \geq 1.25)$$
$$= 1 - \Phi(1.25) \approx 0.1056$$

[Note that 0.1056 is quite far below the upper bound 0.8 from part 1.]

x	$\Phi(x)$	x	$\Phi(x)$	x	$\Phi(x)$	x	$\Phi(x)$	x	$\Phi(x)$
0.00	0.5000	0.60	0.7257	1.20	0.8849	1.80	0.9641	2.40	0.9918
0.01	0.5040	0.61	0.7291	1.21	0.8869	1.81	0.9649	2.41	0.9920
0.02	0.5080	0.62	0.7324	1.22	0.8888	1.82	0.9656	2.42	0.9922
0.03	0.5120	0.63	0.7357	1.23	0.8907	1.83	0.9664	2.43	0.9925
0.04	0.5160	0.64	0.7389	1.24	0.8925	1.84	0.9671	2.44	0.9927
0.05	0.5199	0.65	0.7422	1.25	0.8944	1.85	0.9678	2.45	0.9929
0.06	0.5239	0.66	0.7454	1.26	0.8962	1.86	0.9686	2.46	0.9931
0.07	0.5279	0.67	0.7486	1.27	0.8980	1.87	0.9693	2.47	0.9932
0.08	0.5319	0.68	0.7517	1.28	0.8997	1.88	0.9699	2.48	0.9934
0.09	0.5359	0.69	0.7549	1.29	0.9015	1.89	0.9706	2.49	0.9936
0.10	0.5398	0.70	0.7580	1.30	0.9032	1.90	0.9713	2.50	0.9938

Exercise 2 - Solution

3. (continued)
The probability that a randomly selected adult European rabbit is between 30cm and 50cm long is:

$$\begin{aligned}
\Pr(30 \leq X \leq 50) &= \Pr\left(\tfrac{30-40}{8} \leq \tfrac{X-40}{8} \leq \tfrac{50-40}{8}\right) \\
&= \Pr(-1.25 \leq Z \leq 1.25) = \Phi(1.25) - \Phi(-1.25) \\
&= \Phi(1.25) - (1 - \Phi(1.25)) = 2 \cdot \Phi(1.25) - 1. \\
&\approx 0.7888
\end{aligned}$$

[Note that 0.7888 is quite far above the lower bound 0.36 from part 2.]

x	$\Phi(x)$	x	$\Phi(x)$	x	$\Phi(x)$	x	$\Phi(x)$	x	$\Phi(x)$
0.00	0.5000	0.60	0.7257	1.20	0.8849	1.80	0.9641	2.40	0.9918
0.01	0.5040	0.61	0.7291	1.21	0.8869	1.81	0.9649	2.41	0.9920
0.02	0.5080	0.62	0.7324	1.22	0.8888	1.82	0.9656	2.42	0.9922
0.03	0.5120	0.63	0.7357	1.23	0.8907	1.83	0.9664	2.43	0.9925
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0.06	0.5239	0.66	0.7454	1.26	0.8962	1.86	0.9686	2.46	0.9931
0.07	0.5279	0.67	0.7486	1.27	0.8980	1.87	0.9693	2.47	0.9932
0.08	0.5319	0.68	0.7517	1.28	0.8997	1.88	0.9699	2.48	0.9934
0.09	0.5359	0.69	0.7549	1.29	0.9015	1.89	0.9706	2.49	0.9936
0.10	0.5398	0.70	0.7580	1.30	0.9032	1.90	0.9713	2.50	0.9938
0.11	0.5438	0.71	0.7611	1.31	0.9049	1.91	0.9719	2.52	0.9941
0.12	0.5478	0.72	0.7642	1.32	0.9066	1.92	0.9726	2.54	0.9945

Exercise 3 (15 mins)

A surveyor is measuring the actual height of a building, which according to the construction plans is supposed to be 30 m.

He assumes that his instrument is properly calibrated, and that his measurement errors are independent, with mean 0 cm and standard deviation 5 cm. He plans to take n height measurements, and then compute the average of these measurements.

1. Estimate, using Chebyshev's inequality, how large n should be, if he wants to be at least 95% sure that the average of his n measurements is within 1 cm of the true height of the building.
2. Estimate, using the central limit theorem, how large n should be, if he wants to be at least 95% sure that the average of his n measurements is approximately within 1 cm of the true height of the building.

Table of useful values

$$\Phi(x) = \int_{-\infty}^{x} \frac{1}{(2\pi)^{1/2}} \exp\left(-\tfrac{1}{2}u^2\right) du$$

x	$\Phi(x)$	x	$\Phi(x)$	x	$\Phi(x)$	x	$\Phi(x)$	x	$\Phi(x)$
0.00	0.5000	0.60	0.7257	1.20	0.8849	1.80	0.9641	2.40	0.9918
0.01	0.5040	0.61	0.7291	1.21	0.8869	1.81	0.9649	2.41	0.9920
0.02	0.5080	0.62	0.7324	1.22	0.8888	1.82	0.9656	2.42	0.9922
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0.09	0.5359	0.69	0.7549	1.29	0.9015	1.89	0.9706	2.49	0.9936
0.10	0.5398	0.70	0.7580	1.30	0.9032	1.90	0.9713	2.50	0.9938
0.11	0.5438	0.71	0.7611	1.31	0.9049	1.91	0.9719	2.52	0.9941
0.12	0.5478	0.72	0.7642	1.32	0.9066	1.92	0.9726	2.54	0.9945
0.13	0.5517	0.73	0.7673	1.33	0.9082	1.93	0.9732	2.56	0.9948
0.14	0.5557	0.74	0.7704	1.34	0.9099	1.94	0.9738	2.58	0.9951
0.15	0.5596	0.75	0.7734	1.35	0.9115	1.95	0.9744	2.60	0.9953
0.16	0.5636	0.76	0.7764	1.36	0.9131	1.96	0.9750	2.62	0.9956
0.17	0.5675	0.77	0.7794	1.37	0.9147	1.97	0.9756	2.64	0.9959
0.18	0.5714	0.78	0.7823	1.38	0.9162	1.98	0.9761	2.66	0.9961
0.19	0.5753	0.79	0.7852	1.39	0.9177	1.99	0.9767	2.68	0.9963
0.20	0.5793	0.80	0.7881	1.40	0.9192	2.00	0.9773	2.70	0.9965
0.21	0.5832	0.81	0.7910	1.41	0.9207	2.01	0.9778	2.72	0.9967
0.22	0.5871	0.82	0.7939	1.42	0.9222	2.02	0.9783	2.74	0.9969
0.23	0.5910	0.83	0.7967	1.43	0.9236	2.03	0.9788	2.76	0.9971
0.24	0.5948	0.84	0.7995	1.44	0.9251	2.04	0.9793	2.78	0.9973
0.25	0.5987	0.85	0.8023	1.45	0.9265	2.05	0.9798	2.80	0.9974
0.26	0.6026	0.86	0.8051	1.46	0.9279	2.06	0.9803	2.82	0.9976
0.27	0.6064	0.87	0.8079	1.47	0.9292	2.07	0.9808	2.84	0.9977
0.28	0.6103	0.88	0.8106	1.48	0.9306	2.08	0.9812	2.86	0.9979
0.29	0.6141	0.89	0.8133	1.49	0.9319	2.09	0.9817	2.88	0.9980

Exercise 3 - Solution

1. Let X_i be the measurement error (in cm) of the i-th measurement. By assumption, $\{X_1, \ldots, X_n\}$ is a random sample, where each X_i has a mean of $\mu = 0$ and a variance of $\sigma^2 = 25$.

- Hence, $\mathbf{E}[\overline{X}_n] = 0$, and $\mathrm{var}(\overline{X}_n) = \frac{25}{n}$.

We want to determine the minimum possible value for n, so that the sample mean $\overline{X}_n$ satisfies $\Pr(|\overline{X}_n| \leq 1) \geq 0.95$, or equivalently, $\Pr(|\overline{X}_n| > 1) \leq 0.05$.

By Chebyshev's inequality,

$$\Pr(|\overline{X}_n - \mu| \geq 1) = \Pr(|\overline{X}_n| \geq 1) \leq \frac{\sigma^2}{n \cdot 1^2} = \frac{25}{n}.$$

Equating $\frac{25}{n} = 0.05$, we get $n = 500$.

Therefore, having $n = 500$ measurements would **guarantee** that $\Pr(|\overline{X}_n| \leq 1) \geq 0.95$.

Exercise 3 - Solution

2. As before, let X_i be the measurement error (in cm) of the i-th measurement. We want to determine the minimum possible value for n, so that the sample mean $\overline{X}_n$ satisfies $\Pr(|\overline{X}_n| \leq 1) \geq 0.95$.

- Recall: $\mathbf{E}[\overline{X}_n] = 0$, and $\mathrm{var}(\overline{X}_n) = \frac{25}{n}$.

By the central limit theorem, $\overline{X}_n$ is approximately normal with mean 0 and variance $\frac{25}{n}$. Hence, if we define

$$Z = \frac{\overline{X}_n}{\frac{5}{\sqrt{n}}} = \frac{\overline{X}_n \cdot \sqrt{n}}{5},$$

then $Z \sim N(0, 1)$ approximately.

$$\begin{aligned}
\Pr(|\overline{X}_n| \leq 1) &= \Pr\left(\frac{-\sqrt{n}}{5} \leq \frac{\overline{X}_n \cdot \sqrt{n}}{5} \leq \frac{\sqrt{n}}{5}\right) \\
&= \Pr(-0.2\sqrt{n} \leq Z \leq 0.2\sqrt{n}) \\
&\approx \Phi(0.2\sqrt{n}) - (1 - \Phi(0.2\sqrt{n})) \\
&= 2 \cdot \Phi(0.2\sqrt{n}) - 1,
\end{aligned}$$

where $\Phi(z)$ denotes the standard normal cdf.

Exercise 3 - Solution

2. (continued) So far: $\Pr(|\overline{X}_n| \le 1) \approx 2 \cdot \Phi(0.2\sqrt{n}) - 1$ (by CLT). Thus, it suffices to find a value of n such that

$$2 \cdot \Phi(0.2\sqrt{n}) - 1 \ge 0.95,$$

or equivalently, $\Phi(0.2\sqrt{n}) \ge 0.975$. From the table, the closest value we can find for z satisfying $\Phi(z) = 0.975$ is $z = 1.96$. Therefore $0.2\sqrt{n} \ge 1.96$, which implies $n \ge \frac{1}{(0.2)^2} \times 1.96^2 \approx 96$. In other words, 96 measurements would be sufficient.

x	$\Phi(x)$	x	$\Phi(x)$	x	$\Phi(x)$	x	$\Phi(x)$	x	$\Phi(x)$
0.00	0.5000	0.60	0.7257	1.20	0.8849	1.80	0.9641	2.40	0.9918
0.01	0.5040	0.61	0.7291	1.21	0.8869	1.81	0.9649	2.41	0.9920
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0.09	0.5359	0.69	0.7549	1.29	0.9015	1.89	0.9706	2.49	0.9936
0.10	0.5398	0.70	0.7580	1.30	0.9032	1.90	0.9713	2.50	0.9938
0.11	0.5438	0.71	0.7611	1.31	0.9049	1.91	0.9719	2.52	0.9941
0.12	0.5478	0.72	0.7642	1.32	0.9066	1.92	0.9726	2.54	0.9945
0.13	0.5517	0.73	0.7673	1.33	0.9082	1.93	0.9732	2.56	0.9948
0.14	0.5557	0.74	0.7704	1.34	0.9099	1.94	0.9738	2.58	0.9951
0.15	0.5596	0.75	0.7734	1.35	0.9115	1.95	0.9744	2.60	0.9953
0.16	0.5636	0.76	0.7764	1.36	0.9131	$\boxed{1.96}$	$\boxed{0.9750}$	2.62	0.9956
0.17	0.5675	0.77	0.7794	1.37	0.9147	1.97	0.9756	2.64	0.9959

Summary

- Recap: Joint/marginal/conditional distributions
- Recap: Conditional expectation
- Recap: Moments and moment generating functions
- **Mini-quiz 2**

Exercises on the following topics:

- Markov's inequality
- Chebyshev's inequality
- Central limit theorem

Reminders:

- The **mid-term exam** will be held in Week 8 (Wednesday, 2–4pm), at the **Multi-purpose hall**.
  - Tested on all materials from Lectures 1–12 and Cohort classes weeks 1–6. (This is Week 6.)
  - Lecture 14 (Week 8 Tuesday) will be a review lecture.
- Class participation assignment is also due in Week 8 during Cohort Class (both report and presentation).