L02.01
sorting, master theorem

50.004 Introduction to Algorithm

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L02.01
sorting, master theorem

Pre- or post- readings:
Introduction to Algorithms CLRS book chapters:
Chapter 2 and Chapter 4

Objective

- Explain the sorting problem
  - Insertion sort is $O(n^2)$
  - Merge sort is $O(n\log n)$ (by divide and conquer)
- Analyze complexity of recursions
  - By expansion: the recursion tree method
  - By induction: the substitution method

Sorting problem

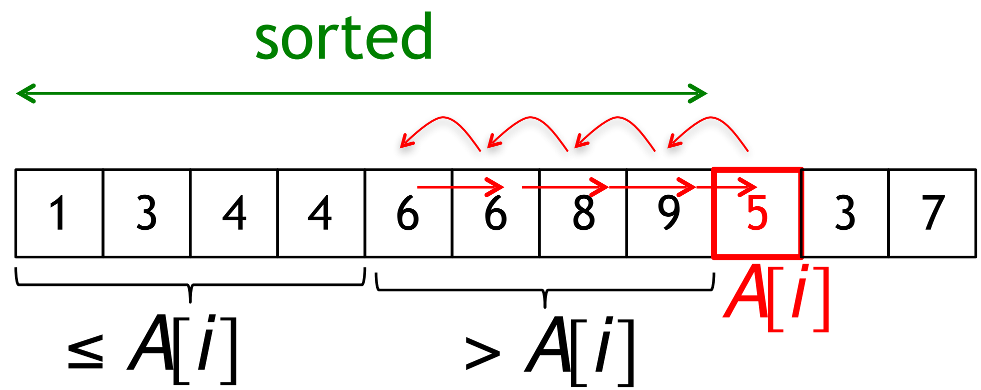
- Input: an array $A[0..n\text{-}1]$ of numbers
- Output: B. (a permutation of A) such that

$$B[0] \le B[1] \le \cdots \le B[n]$$

- in-place sort: if the sorted item occupy the same storage as the original one.
- Out-of-space sort: if the sorting algorithm used extra space to do the sorting

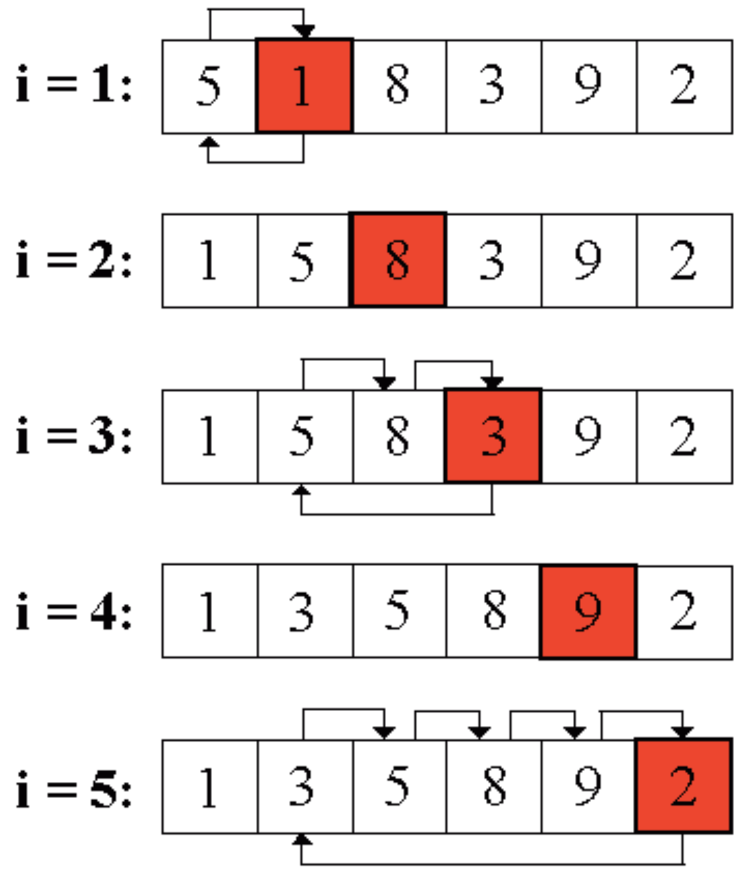
Insertion sort

Insertion sort



Principle: A[0:i-1] is sorted

Then, put A[i] in the right position

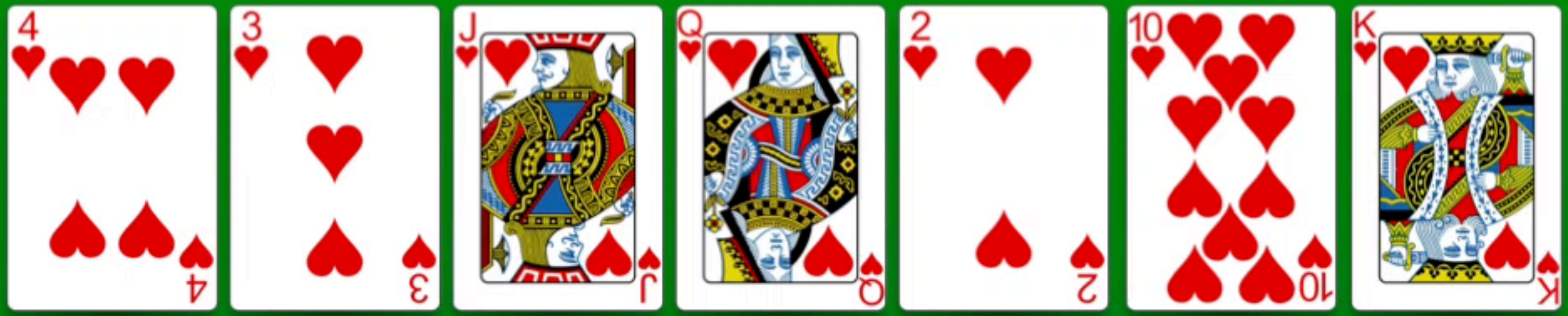


- For *i* in range(1,*n*):

  while A[i]<A[i-1]:

  swap A[i] with A[i-1]

  i -= 1



Insertion Sort

Another video example:

https://www.youtube.com/watch?v=mPEBjhl6oAU

Min 1:03

Complexity of insertion sort

- Worst-case running time $T(n)$ on an input of size n

- T(n) $= i$ comparisons and swaps at step i

$$= \sum_{i=1}^{n-1} i = \frac{n(n-1)}{2} = \Theta(n^2)$$

Divide and Conquer - revision

Divide and conquer solution

- Key idea:
  - **Divide** input into parts (smaller problems)
  - **Conquer** (solve) each part recursively
  - **Combine** results to obtain solution of original

$$
\begin{aligned}
T(n) = &\ \text{divide time} \\
&+ T(n_1) + T(n_2) + \ldots + T(n_k) \\
&+ \text{combine time}
\end{aligned}
$$

Lets construct the recursion!

9 3 4 220 1 3 10 5 8 7 2

Lets construct the recursion!

$$9|3|4|220|1|3|10|5|8|7|2$$

$\Theta(n)$

$$9\ 3\ 4\ 220\ 1 \qquad\qquad 3\ 10\ 5\ 8\ 7\ 2$$

Lets construct the recursion!

9 | 3 | 4 | 220 | 1 | 3 | 10 | 5 | 8 | 7 | 2

$\Theta(n)$ $2T(n/2)$

9 3 4 220 1 3 10 5 8 7 2

1 3 4 9 220 2 3 5 7 8 10

Lets construct the recursion!

$$9 \mid 3 \mid 4 \mid 220 \mid 1 \mid 3 \mid 10 \mid 5 \mid 8 \mid 7 \mid 2$$

$\Theta(n)$

9 3 4 220 1   3 10 5 8 7 2

$2T(n/2)$

1 3 4 9 220   2 3 5 7 8 10

$\Theta(n)$

1 2 3 3 4 5 7 8 9 10 220

Lets construct the recursion!

9 | 3 | 4 | 220 | 1 | 3 | 10 | 5 | 8 | 7 | 2

$\Theta(n)$

Read n numbers

9 3 4 220 1   3 10 5 8 7 2

$2T(n/2)$

1 3 4 9 220   2 3 5 7 8 10

2 x

Sort n/2 numbers

Merge n numbers

$\Theta(n)$

1 2 3 3 4 5 7 8 9 10 220

$$\Leftrightarrow T(n) = 2T(n/2) + \Theta(n)$$

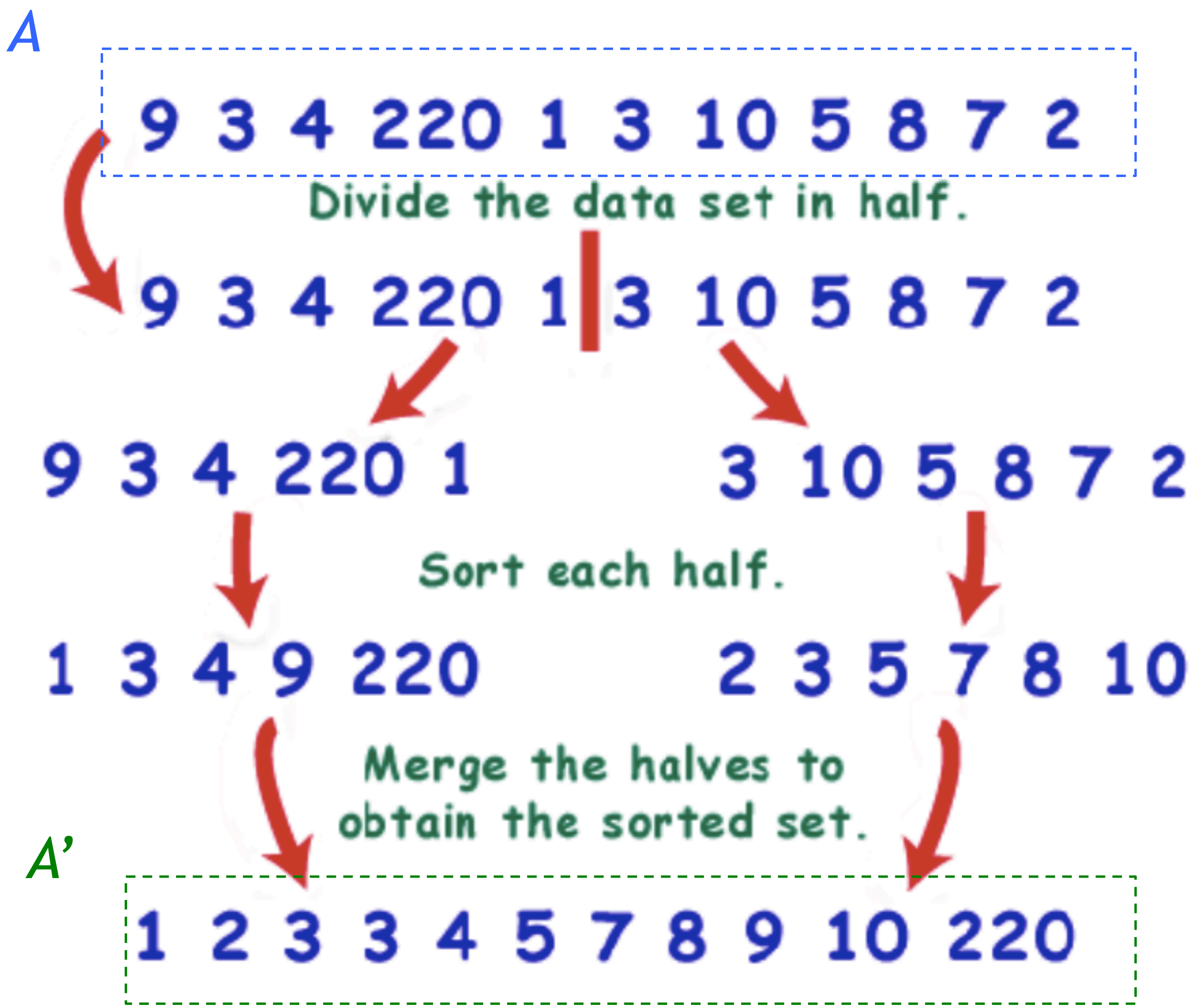
… Syntax for your reference

A = [2,4,6,8,10]

- A[] = [2,4,6,8,10]
- A[0] = [2]
- A[1] = [4]
- A[1:3] = [4,6,8]
- A[0:2] = [2,4,6]
- A[:2] = [2,4,6]
- A[1:] = [4,6,8,10]

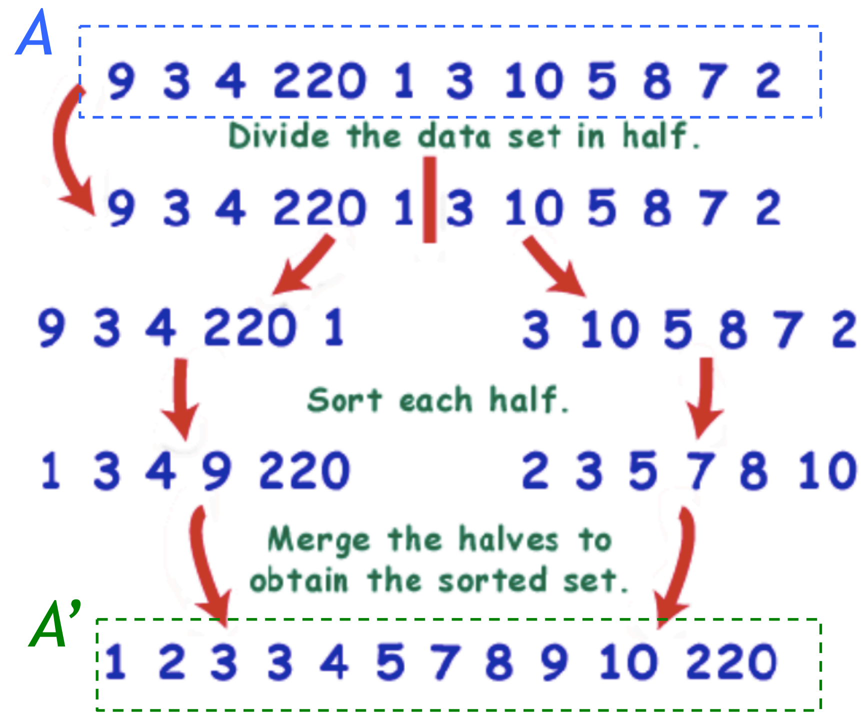
Merge sort

Merge sort



A
9 3 4 220 1 3 10 5 8 7 2
Divide the data set in half.
9 3 4 220 1 3 10 5 8 7 2
9 3 4 220 1
3 10 5 8 7 2
Sort each half.
1 3 4 9 220
2 3 5 7 8 10
Merge the halves to obtain the sorted set.
A'
1 2 3 3 4 5 7 8 9 10 220

Merge sort

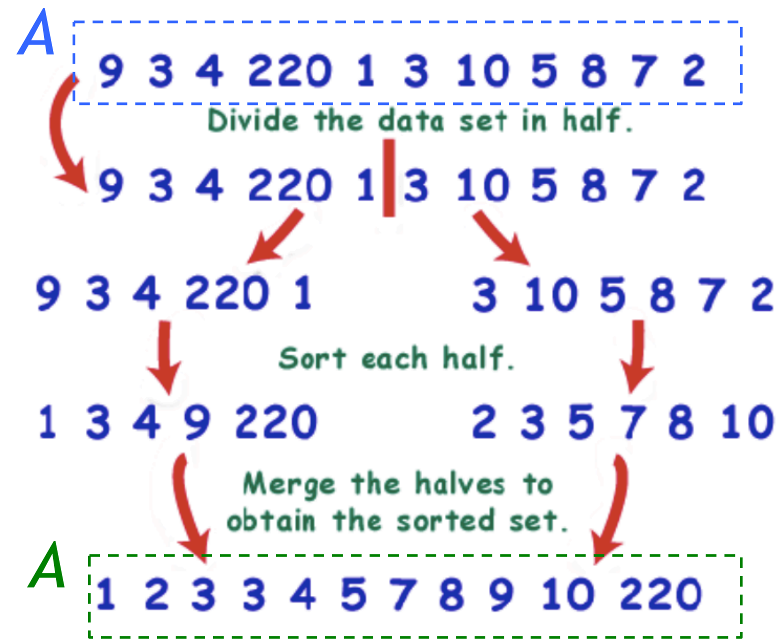


MergeSort(*A*):

- if *n*=1: done -> return
- recursively sort *A*[:*n*/2] -> *L*
- recursively sort *A*[*n*/2:] -> *R*
- merge *L* & *R* -> output *A'*

Merge sort

MergeSort(A):
- if n=1: done -> return
- recursively sort $A[:n/2]$ -> L
- recursively sort $A[n/2:]$ -> R
- merge L & R -> output A'



Time:
1. divide: $\Theta(n)$
2. recursion: $n_1=n_2=n/2$

   time = $T(n_1)+T(n_2)=2T(n/2)$
3. merge: $\Theta(n)$

total: $T(n)=2T(n/2)+\Theta(n)$

Merge sort

https://www.youtube.com/watch?v=mPEBjhl6oAU

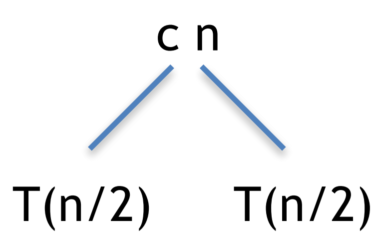
Min 1:53

Solve the recurrence formula

Solving recurrences by expansion (recursion tree)

- The recursion tree = sum of cost of nodes
- **<u>Example 1</u>**, Let's draw the tree of:

$$T(n) = 2T(n/2) + cn$$ c is a constant



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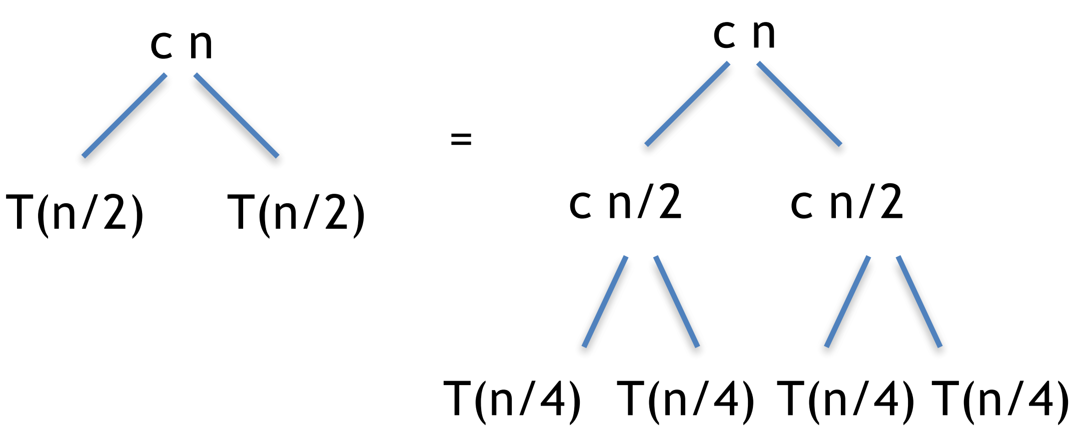

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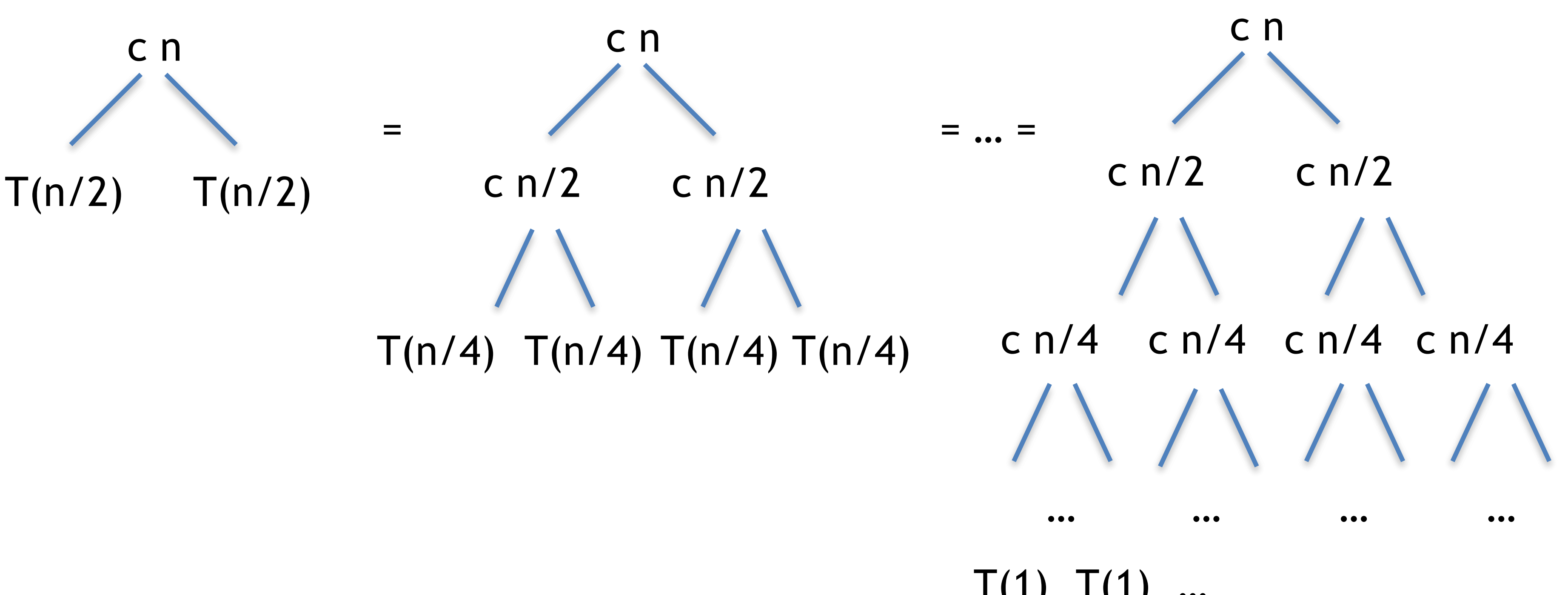

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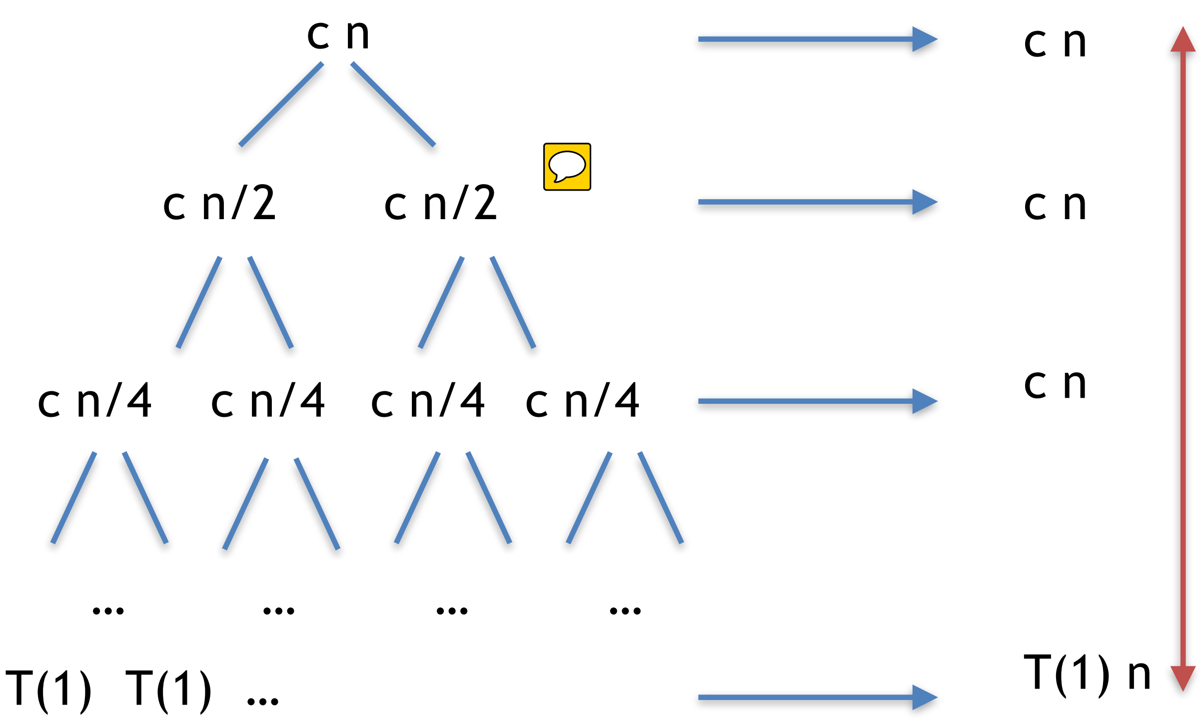


Hight of the tree:
log (n) levels

Sum each level, then sum all levels:

$$\boxed{\begin{aligned} T(n) &= nT(1) + cn\log n \\ &= \Theta(n\log n) \end{aligned}}$$

Solution by expansion (recursion tree)

- **Example 2, The recurrence tree of**

$$T(n) = T(n/2) + cn$$

c n

c n/2

c n/4

...

T(1)

T(n)
= cn + cn/2 + cn/4 + …. + 1
= cn (1 + ½ + ¼ + …. + 1/2$^{\log n}$)
= cn (2)
= Θ(n)

Master Theorem

Can we generalize our observations?

There are three cases

- Each level's value is the same
  - Total value = Θ(number of levels x sum of each level)
- Each level's value is decreasing
  - total value = Θ(root node's value)
- Each level's value is increasing
  - total value = Θ(number of leaves)
    - Since each level node is Θ(1)

What is master theorem

Consider $T(n) = aT(n/b) + f(n)$

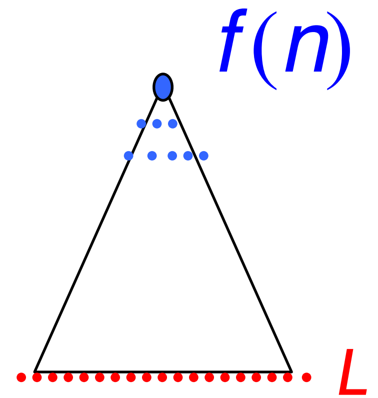
- The master Theorem can find the complexity of T(n), according to f(n)

The Master Theorem

Consider $T(n) = aT(n/b) + f(n)$

$\Rightarrow \; h = \#\text{ of levels} = \log_b n = \Theta(\log n)$

$\Rightarrow \; L = \#\text{ of leaves} = a^h = a^{\log_b n} = n^{\log_b a}$



1)if $f(n) = O(L^{1-\varepsilon}) = O(n^{\log_b a-\varepsilon})$

$\Rightarrow T(n) = \Theta(L) = \Theta(n^{\log_b a})$

i.e. value of f(n) is geometrically increasing down the tree

The Master Theorem

Consider $T(n) = aT(n/b) + f(n)$

$\Rightarrow h = \#\text{ of levels} = \log_b n = \Theta(\log n)$

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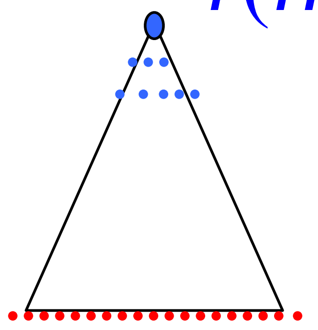
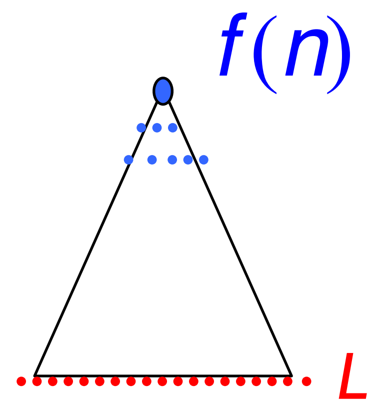


$$2)\ \text{if } f(n) = \Theta(L) = \Theta(n^{\log_b a})$$

$$\Rightarrow T(n) = \Theta(L \log n) = \Theta(n^{\log_b a} \log n)$$

i.e value of each level are (roughly) equal

The Master Theorem



Consider $T(n) = aT(n/b) + f(n)$

$\Rightarrow h = \#$ of levels $= \log_b n = \Theta(\log n)$

$\Rightarrow L = \#$ of leaves $= a^h = a^{\log_b n} = n^{\log_b a}$

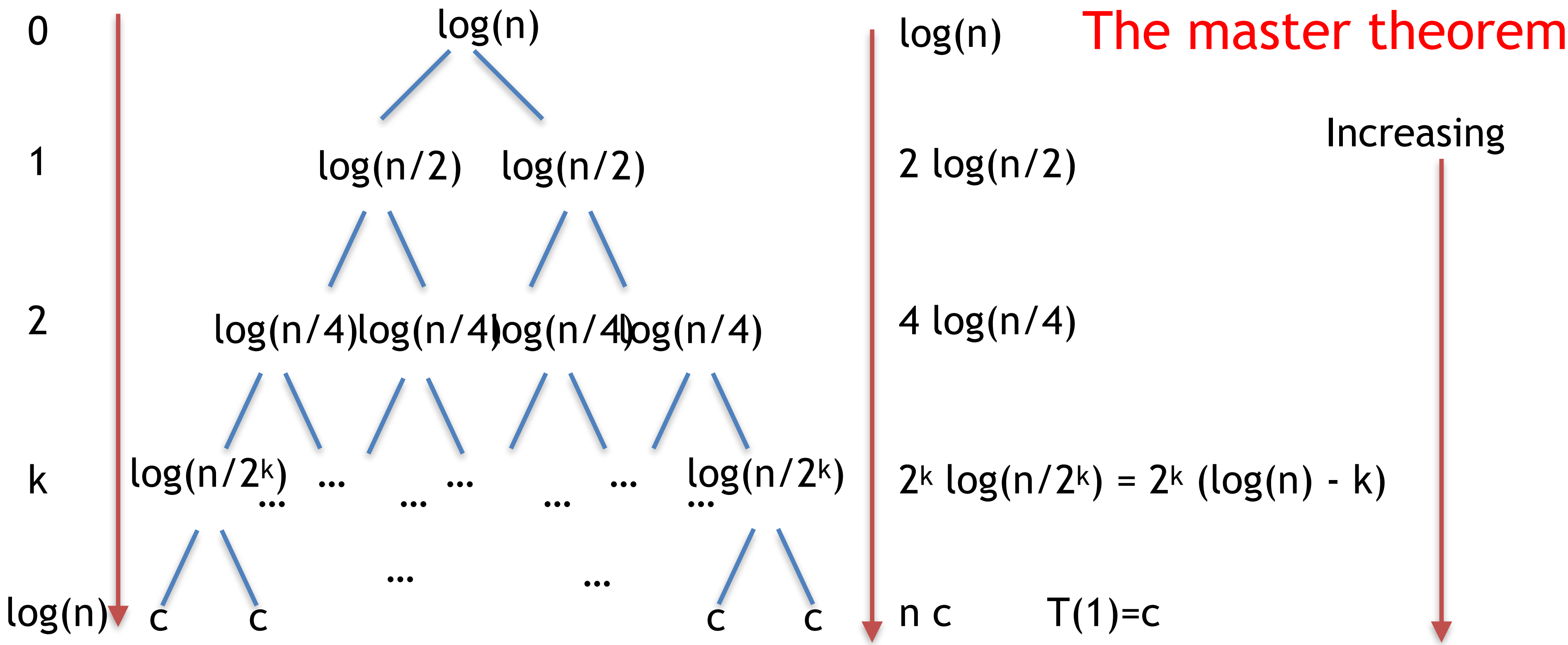
$\boxed{\begin{array}{l} \text{3) if } f(n) = \Omega(L^{1+\varepsilon}) = \Omega(n^{\log_b a+\varepsilon}) \\ \Rightarrow T(n) = \Theta(f(n)) \end{array}}$

i.e. the value of f(n)
Is geometrically decreasing
down the tree

Examples

$$T(n) = 2T(n/2) + \log n$$

$$\boxed{\begin{aligned} &1) \text{ if } f(n) = O(L^{1-\varepsilon}) = O(n^{\log_b a - \varepsilon}) \\ &\Rightarrow T(n) = \Theta(L) = \Theta(n^{\log_b a}) \end{aligned}}$$



f(n) = log n, a = 2, b = 2

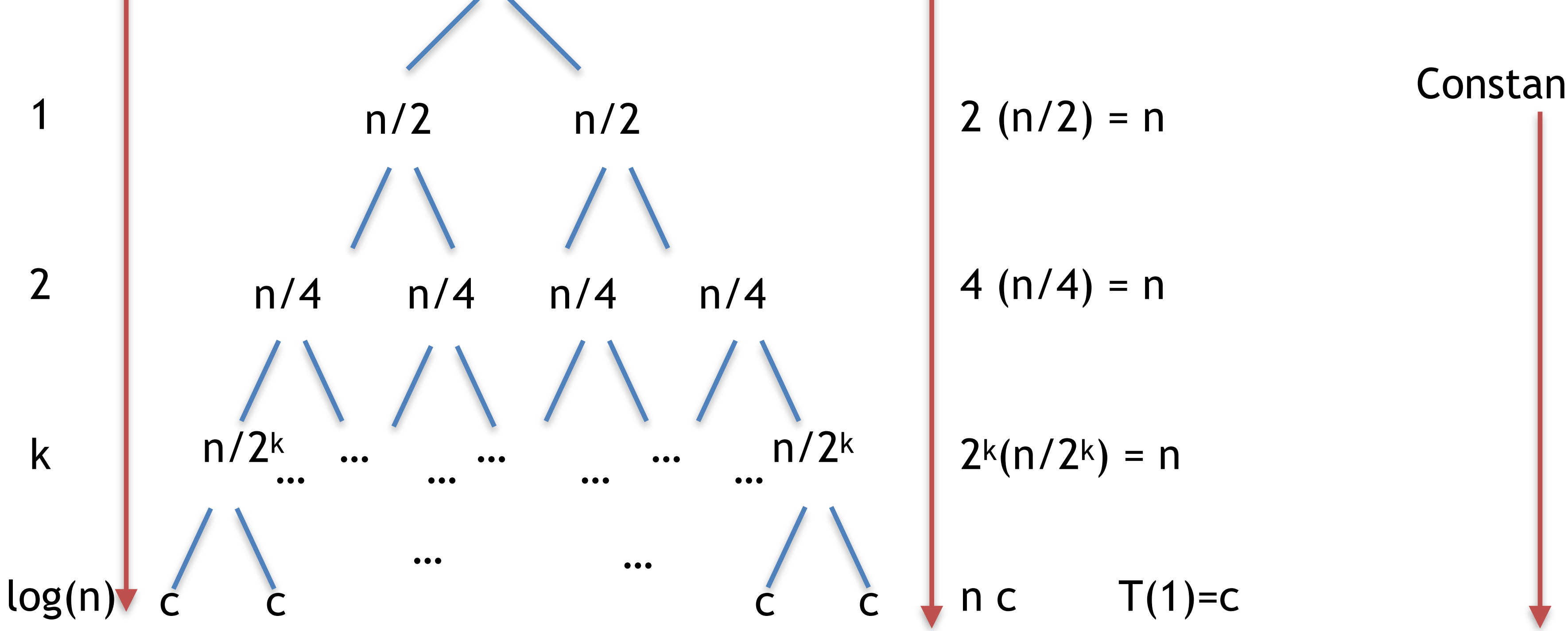
The values of each level are geometrically increasing down the tree. So, $T(n) = \Theta(n)$

Examples

$$T(n) = 2T(n/2) + n$$

$$2) \text{ if } f(n) = \Theta(L) = \Theta(n^{\log_b a})$$
$$\Rightarrow T(n) = \Theta(L \log n) = \Theta(n^{\log_b a} \log n)$$

The master theorem



f(n) = n, a = 2, b = 2
The values of each level are geometrically equal
So, $T(n) = \Theta(n \log n)$

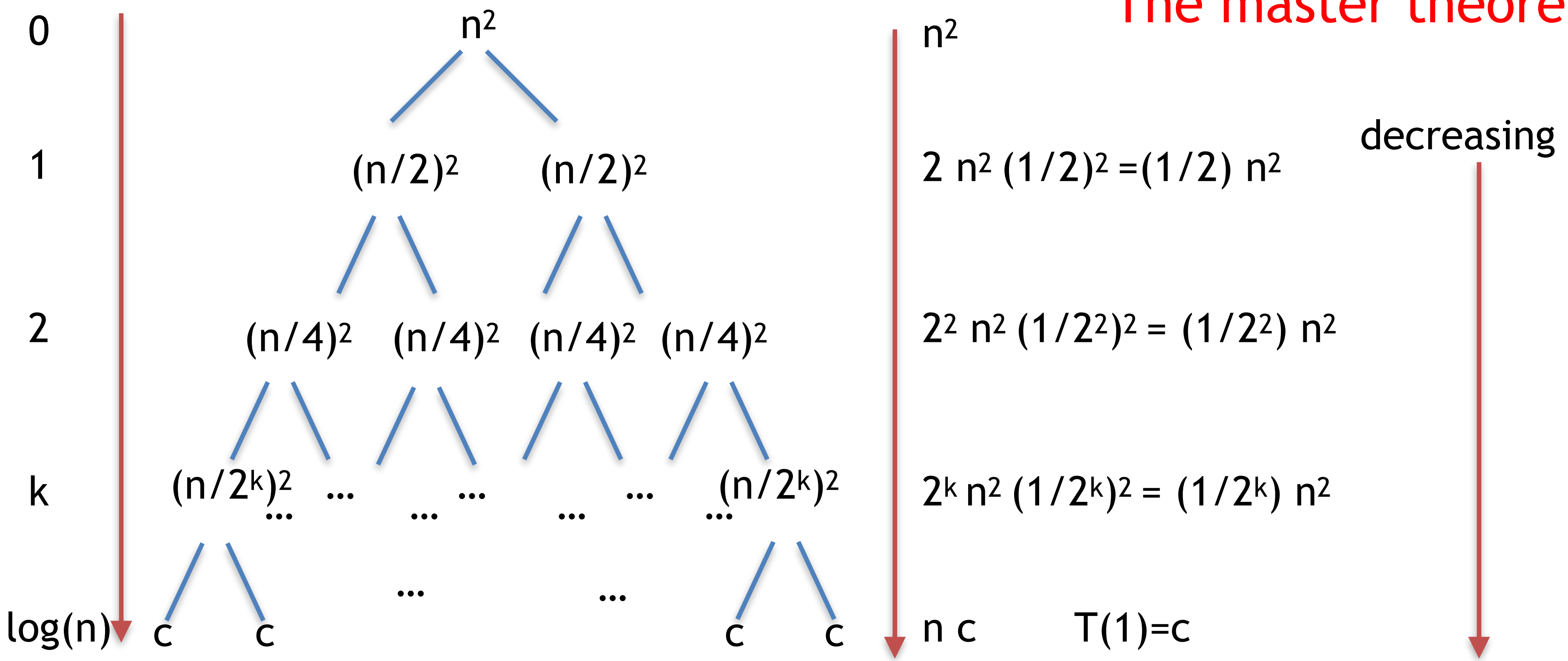
Examples

$$T(n) = 2T(n/2) + n^2$$

3) if $f(n) = \Omega(L^{1+\varepsilon}) = \Omega(n^{\log_b a+\varepsilon})$

$\Rightarrow T(n) = \Theta(f(n))$

The master theorem



$f(n) = n^2$, $a = 2$, $b = 2$

The values of each level are geometrically decreasing down the tree

So, $T(n) = \Theta(n^2)$