L05.01
Hashing I

50.004 Introduction to Algorithms

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CLRS Ch 11.1-11.3

Slides by A.Binder and based on Dr. Simon LUI

Today topics

- Hash table, hash function
  - Insert, search, delete in practice very efficient
  - Insert, delete: O(1)
  - search: O(1+α) α = load factor of the hash table
  - Hash table collisions
- Hash functions for non-numbers
- Good versus bad hash functions
- Runtime Analysis
- Table Doubling tricks for making search O(1) “on average” (not worst case complexity!!)

Hash tables

- A data structure that supports 'dictionary operations' search(key), insert(key), delete(key) very fast most of the time
- Heaps: insert O(log n), search O(n)
- Binary search trees: insert,search → O(height), worst case O(n)
- AVL trees insert, search → O(log n),
- Hash tables:
  - Insert O(1) worst case,
  - Search: O(n) worst case performance BUT
    - O(1+ α) average case performance
  - Delete: O(1) worst case performance*

* If you have the pointer to the object

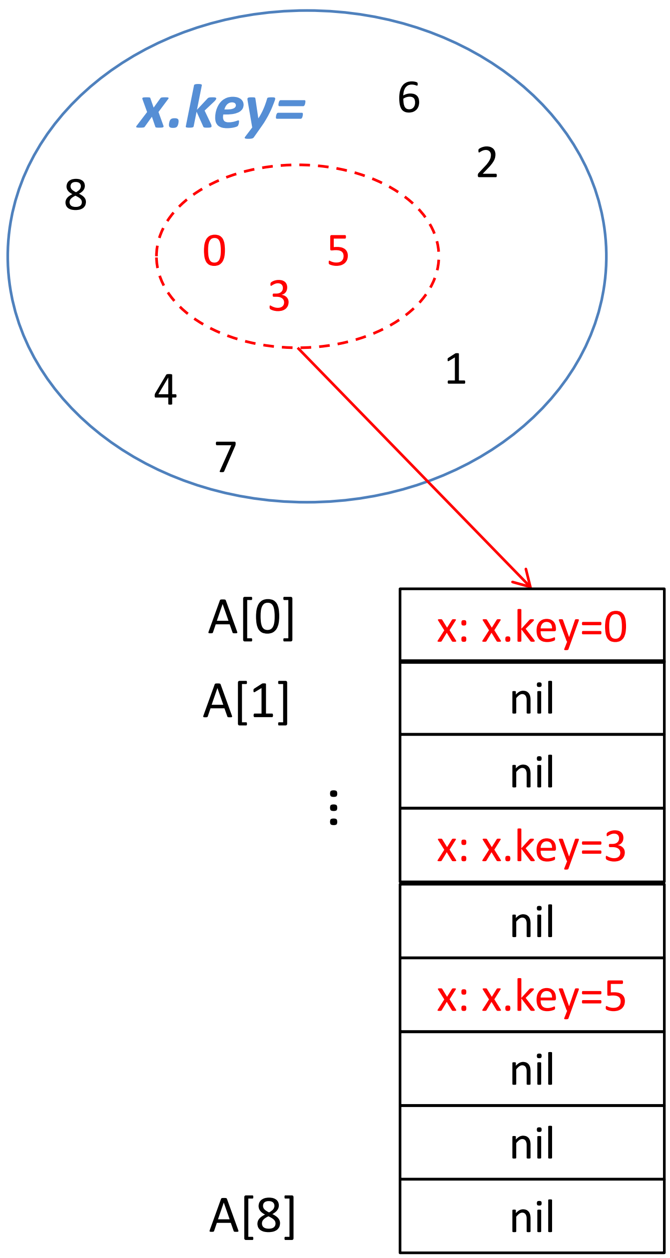
an array

- Suppose keys {0,…,l-1}, object x has key x.key
- Use an array A of length (l)
  - initialize with NULL pointers (python: list with empty entries)
- Insert(x):

  A[x.key]=x
- Delete(x)

  A[x.key]=NULL
- Search(x):

  Return True or False?
  (A[x.key]!=NULL)
- All O(1) operations



Array(ii)

- Use an array A of length (l), initialize with NULL pointers (or python: list with empty entries)
- Insert(x):

  A[x.key]=x

- When this is not practicable?

Array(ii)

- When this is not practicable?
  - Set of possible objects/keys very large
    - Array of length l does not fit into memory!!
  - Don't know the values of keys in advance
  - Keys are not integers:
    - Filenames, student names, instances of an object class

Hash table + simple array

- When this is not practicable?
  - Set of possible objects/keys very large
  - Keys are not integers:
    - Filenames, student names,

- Solution: Hash function:
  - Let L be the space of all possible keys (e.g. L={all possible filenames})
  - Hash function (‘h’): a function that maps the space of all possible keys K onto {0,…,l-1}
  - h: K → {0,…,l-1},
- First idea (will refine it): Hash table is will be an array of size l,

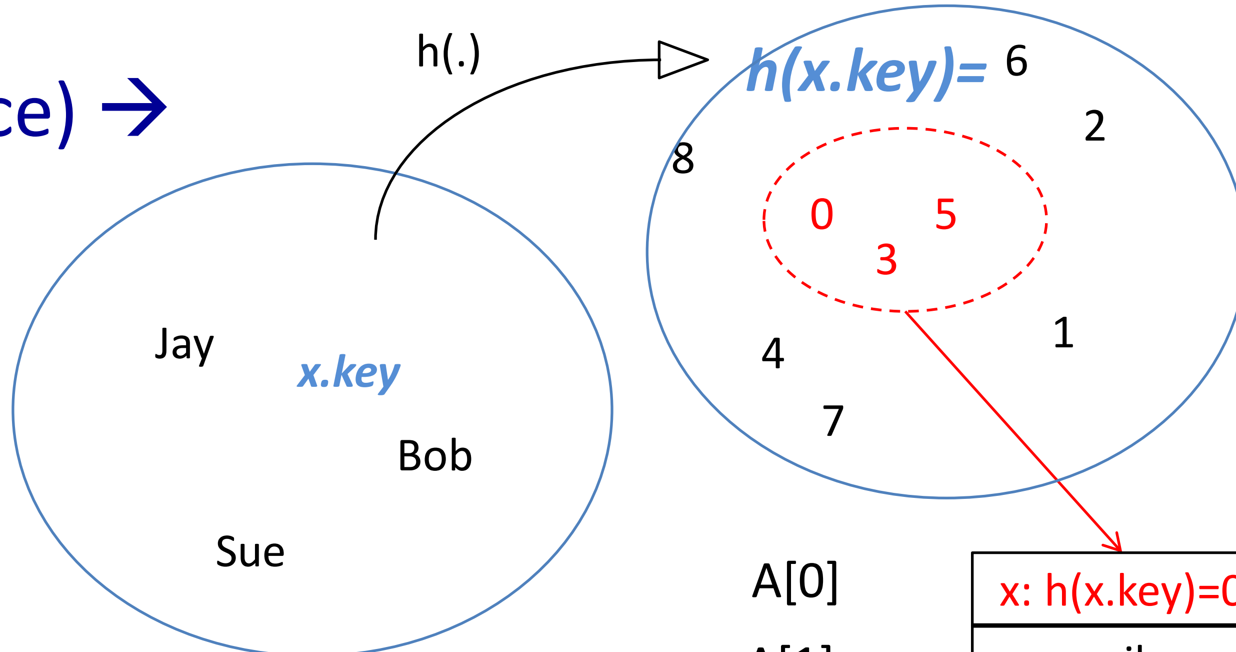
Hash function + simple Array

- h: K (key space) → {0,…,l-1},

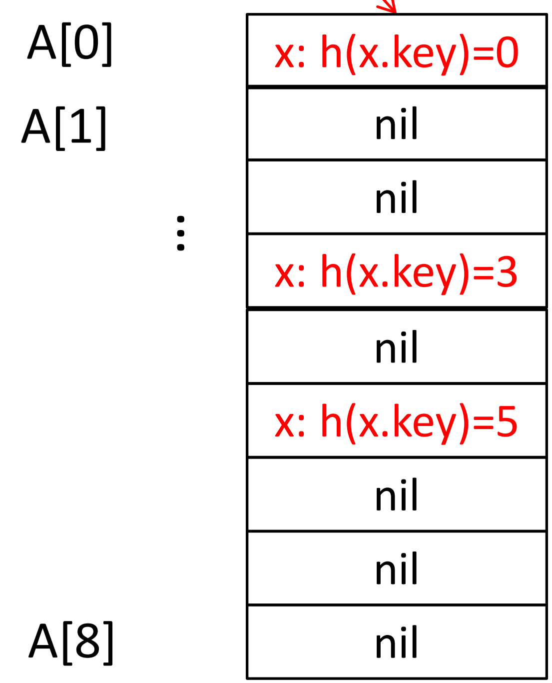
  h: Jay → 0,

  Bob →1,

  Sue → 2, …



- Insert(x):
  - Insert x into A[h(x.key)]
  - All happy?? collisions!!



Collisions

- Set of keys (e.g. K={all possible filenames}) is very large,
  - hash table is an array of length l, l typically much smaller than set of keys
  - If |K| > l, then there must be two keys in L such that h(key1)=h(key2)
  - Collision: two keys are hashed onto the same value: h(key1)=h(key2)

K={0,…,100}

$$h(k) = k \bmod l = k \bmod 9$$

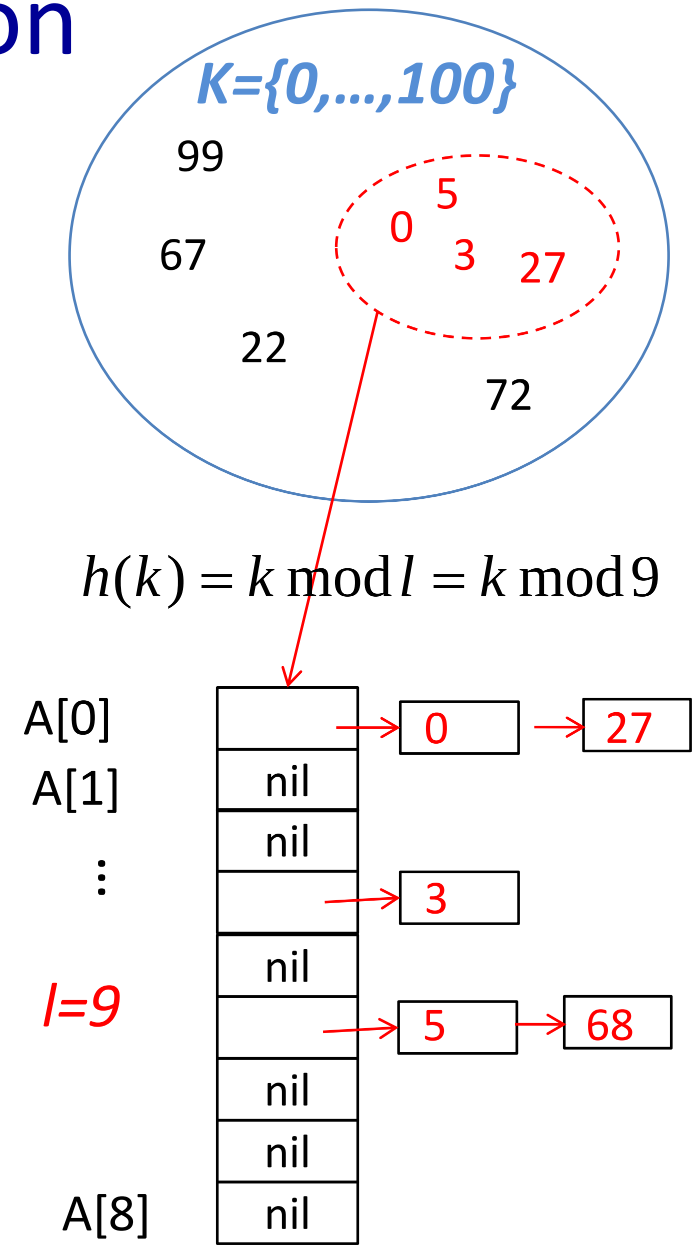
h(1)=h(37)=1

A[0]	nil
A[1]	x: x.key =1, x: x.key =37
⋮	nil
	nil
	nil
	nil
A[6]	nil

Hash function + Hash table with chaining -- formal definition

- Hash function:
  – K = space of all possible keys e.g. K={0,…,100}
  – Hash function ('h') maps L onto {0,…,l-1}, where l is the size of the has table
  – h: K → {0,…,l-1},

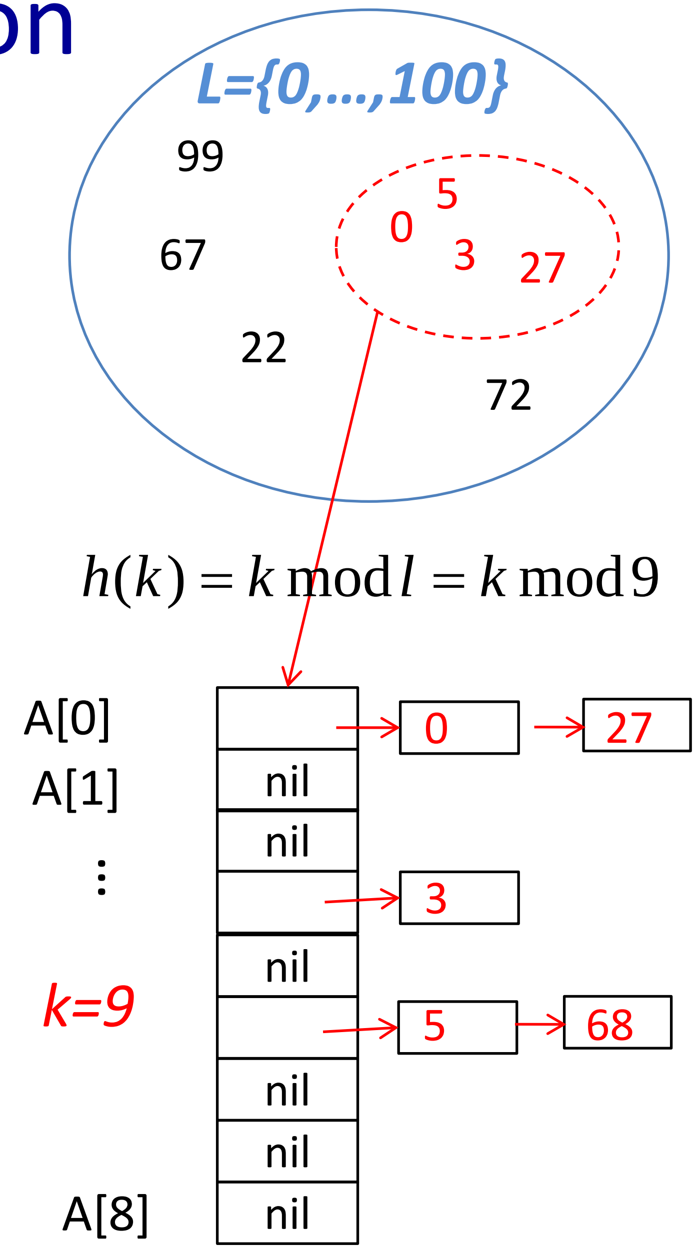
- Hash table is an array of size l,
  – each element in the table is a [doubly]-linked list* ("chaining")
  – Allows to insert multiple elements for the same hash value



$$h(k) = k \bmod l = k \bmod 9$$

Hash function + Hash table with chaining -- formal definition

- Idea: map a very large set of keys to a small number of hash-values
- Efficient insert, delete in worst case O(1)
- Efficient search:
  – Worst case O(n),
  – average case O(1+alpha)
  – average case O(1)* under some assumptions



Hash functions

- Chained-hash-Insert(A,x) O(1) – compute h(x.key) + 1 insert
  - Insert element x at the head of linked list A[h(x.key)]

- Chained-hash-search(A, k) O(???)
  - Search for an element with key k in list A[h(k)]

Worst case performance is what and why?

Hash functions

- Chained-hash-Insert(A,x) O(1) – compute h(x.key) + 1 insert
  – Insert element x at the head of linked list A[h(x.key)]

- Chained-hash-search(A, k) O(???) – will analyze this later
  – Search for an element with key r in list A[h(r)]

- Chained-hash-Delete(A,x): O(1) [O(???) if using the key]

  We assume here that we have a pointer to x, that the memory address of element x

  – Go to x in A[h(x.key)] using the address
  – Link predecessor and successor of x in this doubly linked-list (use link to previous and next element of x in the list!)
  – Delete x from the list A[h(x.key)] using the pointer

Further questions …

- How to hash filenames and non-numerical entries?
- What is a good hash function?
- Runtimes ? Load factors?

Hashing for non-numerical input

- Latin?

‘pt’ → (112,116)$\xrightarrow{radix\ 128} 112 \cdot 128^1 + 116 \cdot 128^0$

- Written Chinese? Tamil? Jawi? Traditional Chinese? Old Javanese ?

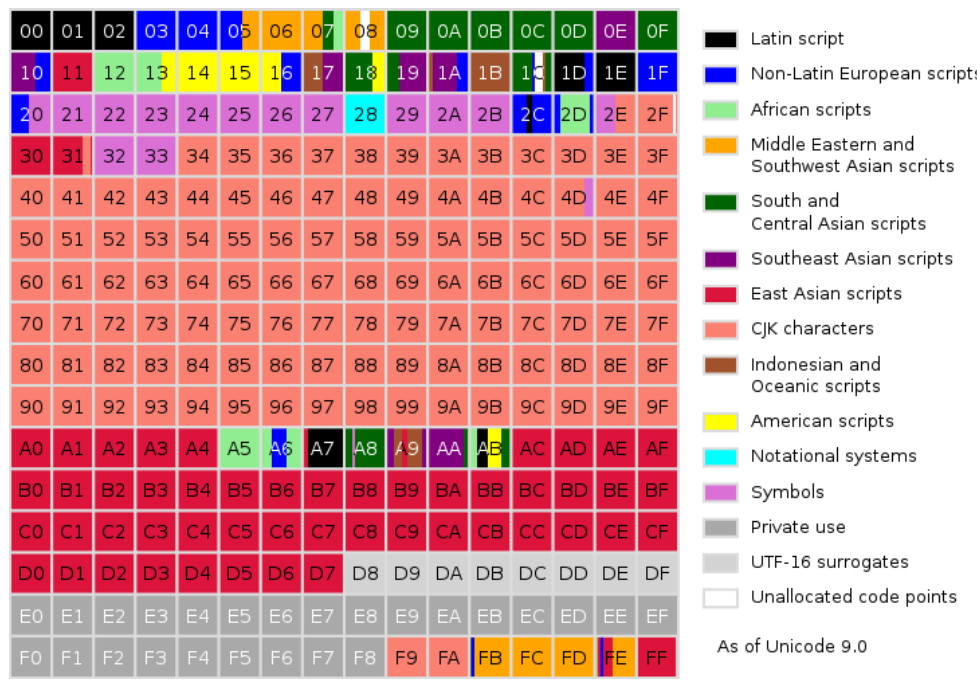
Hashing for non-numerical input

Written Chinese? Tamil? Jawi?

Old Javanese ? Traditional Chinese? Sanskrit?

Unicode https://en.wikipedia.org/wiki/Unicode

- Most languages encoded in 6 planes, each with 65535 entries (unicode supports 17 planes)
- Shown: plane 0

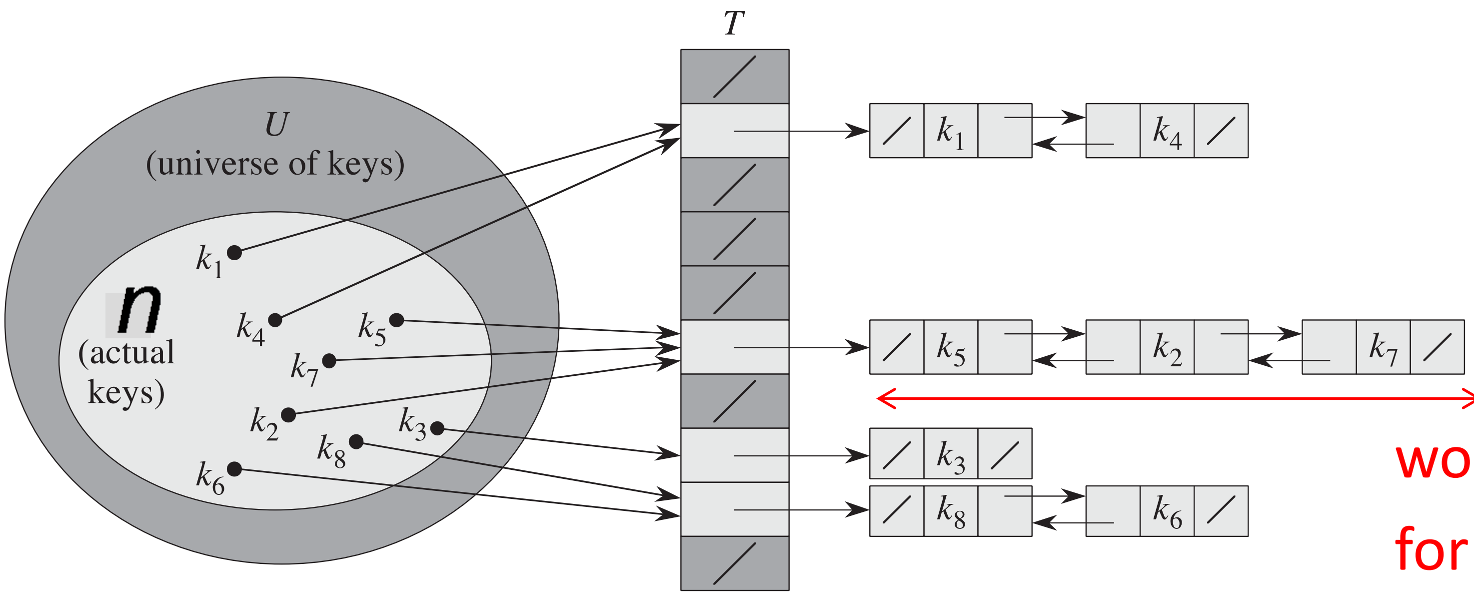


What is a good hash function?

- Scenario: N keys are coming in, unknown order

- Worst case: ??

Resolving collisions: chaining



worse case

for given *h*()

= $\Theta(n)$

insert(k) -> $O(1)$

delete(k) -> $O(1)$

search(k) -> $\Theta(n)$

What is a good hash function?

- Scenario: N keys are coming in, unknown order
- Worst case: hashes all keys into the same slot.
  - search: Have to go through one linked list with n elements
- Best case:
  - How n keys should be distributed onto l lists for fast search??

What is a good hash function?

- Scenario: N keys are coming in, unknown order
- Worst case: hashes all keys into the same slot.
  – search: Have to go through one linked list with n elements
- Best case: they are distributed equally likely into all slots
  – All slots are filled with approximately the same number of keys search is faster in worst case … n/l elements
- Bad case: a few slots have high load, most of table is empty “clustering of keys”

What is a good hash function?

- Scenario: N keys are coming in, unknown order

- Worst case: hashes all keys into the same slot.
  - search: Have to go through one linked list with n elements

- Best case: they are distributed equally likely into all slots
  - All slots are filled with approximately the same number of keys search is faster in worst case … n/l elements

- Simple uniform hashing assumption: any given element is equally likely to hash into any of the l slots, independently of where any other element has hashed to.
  - Getting close to it – design of a hash function

Why it is hard to achieve the simple uniform hashing assumption?

- Distribution of incoming keys unknown
- Can have correlations, not independent
  - E.g. programmers use variable names that follow rules of their spoken languages. nobody writes code with random variable names.
  - Infections in english end all on "itis"
  - Other examples?

- Example of a hash function that satisfies U.H.A.
- K= random real numbers 0 <=r<1, uniformly distributed, have l slots.
- h(k) = floor (k l)

- Think about: being close to simple uniform hashing assumption depends on what factors ?

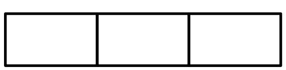
Good or bad hash function?

- $H(k) = k \ mod \ 2^p$
  - Returns p lowest bits, when k is written in binary representation

Case 1: K= uniformly distributed integers {0, 2^{2p}-1}

Case 2: K= odd numbers, uniformly distributed

Case 3: K={ total size in byte of an array of floats, array lengths are uniformly distributed, each single entry has 4 byte length }



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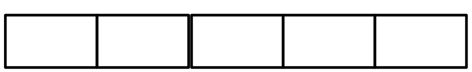

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Why it is hard to achieve the simple uniform hashing assumption?

- Knowledge of distribution of keys helps to design better hash functions.

- array Bytelength example: p bits starting at bit #2 is better
  - lowest p bits have bad reputation in C.S. ;-)

p=10 lowest bits

- For array lengths in big data :D

- Now some hash functions
- Mostly heuristics … hacks that work well in practice

Hash function example: The division method

$h(k) = k \ mod \ l$ **Example:** l = 19 → h(93) = 17

Choice of l: $l = 2^p$ (powers of 2) is risky.

l too close to a power of 2 can have surprising effects.
Example: encode strings in base 2^p

pt → (112,116) → $112 \cdot 2^1 + 116 \cdot 2^0$

If $l = 2^p$-1, then all permutations of a string result in the same hash ! (proof on request …)

A good choice of l: A prime not close to 2^p for any integer p
Results in funny hash table sizes

Hash function example: The multiplication method

$$h(k) = \lfloor l\,(\,(kA) mod\ 1)\rfloor = \lfloor l\ (\ kA\ -\ \lfloor kA\rfloor\,)\ \rfloor$$

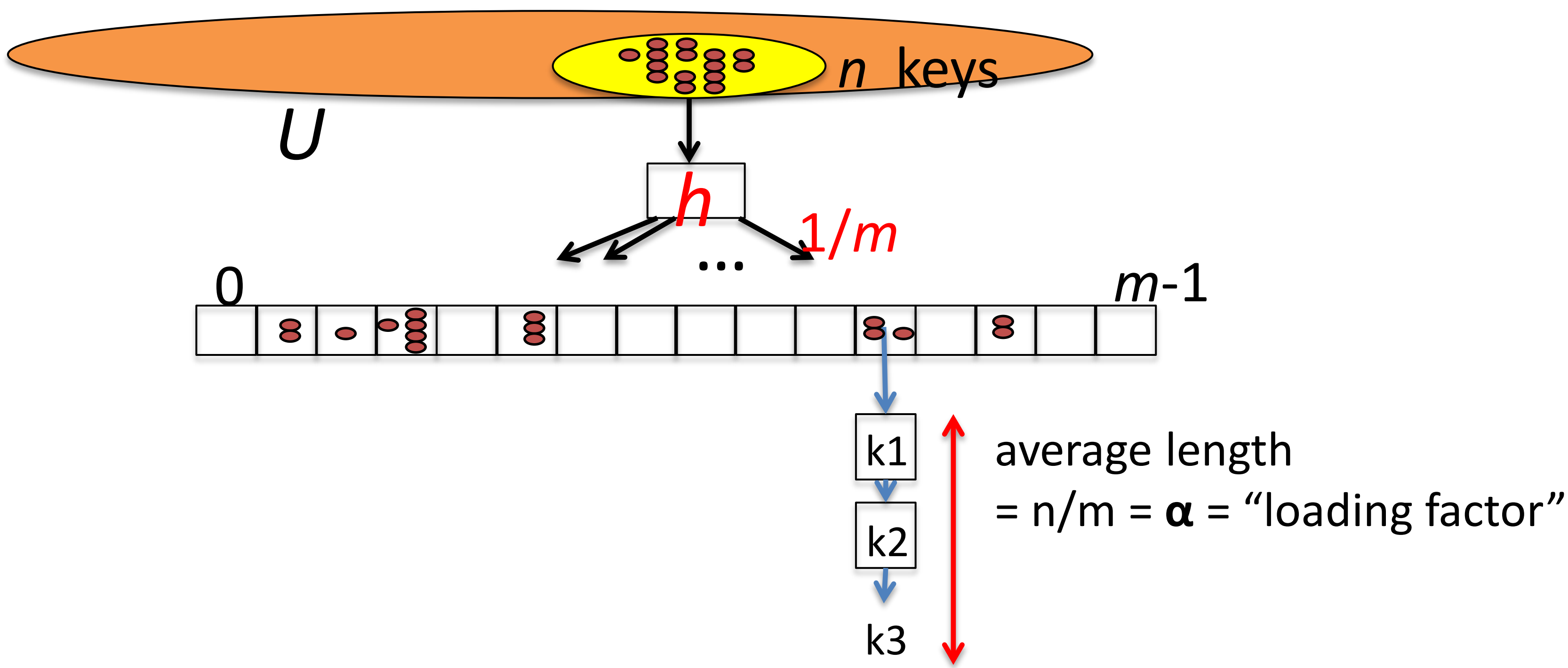
- **Method:**
  - Choose constant A in the range $0 < A < 1$.
  - Multiply key k by A.
  - Extract the fractional part of kA.
  - Multiply the fractional part by m.
  - Take the floor of the result.
- In short, the hash function is $h(k) = \lfloor m\ (kA \bmod 1)\rfloor$, where $kA \bmod 1 = kA - \lfloor kA \rfloor$ = fractional part of kA.

Advantage: value of m is not critical (can be 2^p)
Disadvantage: slower than division method

Average Run time analysis

- Chained-hash-search(A, k) O(???)
  - Search for an element with key k in list A[h(k)]

Under the simple uniform hashing assumption



Average Run time analysis

- Let α=N/l be the load factor of a hash table

- Under the simple uniform hashing assumption, the average runtime of chained-hash-search is O(1+α)
  - This is an average runtime result,
  - Depends on the design of the hash!!

Average Run time analysis

- Case 1: Search for a key that is not in the table
- Case 2: Search for a key that is in the table

- Case 1: search in A[h(k)]
  - A[h(k)] has on average $\alpha = N/l$ elements
  - Must go through these elements until element x is found
  - O(1) from computing h(k) and looking up in A[.]

  - So Case 1: $O(1 + \alpha)$,

Out of curriculum part

- Case 1: Search for a key that is not in the table
- Case 2: Search for a key that is in the table

- Case 2: O(1 + alpha),
- Read CLRS if you want to know more …

Two keys collide if: $1\{h(k_i) == \ h(k_l)\}$

elements inserted before are those that were inserted after element i, that's why sum i+1…n

$$s = \frac{1}{n}\sum_{i=1}^{n} E[1 + \sum_{l=1+i}^{n} 1\{h(k_i) == \ h(k_l)\}\ ]$$

Code here

- Under the simple uniform hashing assumption, the average runtime of chained-hash-search is $O(1+\alpha)$

- Under the simple uniform hashing assumption, the average runtime of chained-hash-search is $O(1+\alpha)$

  that is $O(1)$ if α is upper bounded (independent of N).
- "α being upper bounded" Requires what? (as one adds more and more entries into the hash table)

why resize a hash table?

- Under the simple uniform hashing assumption, the average runtime of chained-hash-search is $O(1+\alpha)$

  that is $O(1)$ if α is upper bounded (independent of N).
- "α being upper bounded" Requires what? (as one adds more and more entries into the hash table)
- Hash table needs to be resized "from time to time" when load factor α becomes too large
- We will see: for search complexity $O(1)$ we need to learn about two tricks
  - 1. the right strategy of growing a table
  - 2. only as an average over many insert steps
- $O(1)$ can only be attained on average over a large number of inserts ("amortized analysis")

- GrowTable(A, l')
  - Make Table A' of size l'
  - Build new hash function h'
  - Rehash:

    For item in A.items():

    A'.insert(item) %using h' as new hash inside

why we do not reuse/extend the existing hash?

- Complexity ?

In What way resize a hash table?

- Complexity ? $O(n+l+l')$
  - l for visiting each table entry in old table
  - N for processing all objects stored in there
  - l' for making of new table

- How much make the table bigger?
- $l'=l+1$?

In What way resize a table ?

- Resize Complexity O(n+l+l')

- How much make the table bigger?
- resize by 1
  - l'=l+1 whenever n=m, start at m=1
  - Sum over resizes in l: $1+2+3+\cdots+n \;=\frac{n(n+1)}{2}=$ $O(n^2)$ Quadratic, slow

- resize: adding a constant to the size
  - l'=l+c whenever N/l = t ???

In What way resize a table ?

- Resize Complexity O(n+l+l’)

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- resize by 1
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  - Sum over resizes in l: $1+2+3+\cdots+n = \frac{n(n+1)}{2} = O(n^2)$ Quadratic, slow

- resize: adding a constant to the size
  - l’=l+c whenever N/l = t load factor hits a threshold t
    → Same problem as above!!!

  - suppose N/l= t, next resize time: N’ such that N’/(l+c)=t (= next resize)
  - N’-N= (l+c)t-lt = ct → resize after every ct steps, N’= {1,2,3,4,5} ct
  - resize complexity is θ(n+l+l’) =θ(N’) = some constant · {1,2,3,4,5} · ct
  - Last resize time: $\left\lfloor \frac{N_{final}}{ct} \right\rfloor$

  Overall resize complexity is:

  $$s = O(ct + 2ct + 3ct + \ldots + \left\lfloor \frac{n}{ct} \right\rfloor ct) = O(\frac{\left\lfloor \frac{n}{ct} \right\rfloor \left(\left\lfloor \frac{n}{ct} \right\rfloor + 1 \right) ct}{2}) = O(n^2)$$

In What way resize a table ?

- Complexity ? O(n+l+l')
- How much make the table bigger?

- resize by doubling
  - l'=2l whenever α =N/l = t   load factor hits a threshold t

$\frac{N}{l} = t$, next resize time: $\frac{N'}{l'} = \frac{N'}{2l} = t$

→ $N' - N = lt = N$ → $N' = 2N$,

With some initial $N_0 = l_0 \cdot\ t$, (l_0 is initial table size),

In What way resize a table ?

- resize by doubling
  - l'=2l whenever N/l = t
  - N/l=t, next resize time: N'=2N, starts at some initial sample size N_0
  - Resize when sample size reaches $N_0, 2N_0, 4N_0, \dots, 2^t N_0$
  - resize complexity is θ(n+l+l') =θ(n)

    with $n = N_0, 2N_0, 4N_0, \dots, 2^t N_0$
  - Let be last resize time when $N_{final} = 2^{t_f}$

  Overall resize Complexity is

$$s = O(N_0 + 2N_0 + 4N_0 + \cdots + 2^{t_f} N_0)$$

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    with $n = N_0, 2N_0, 4N_0, \ldots, 2^t N_0$
  - Let be last resize time when $N_{final} = 2^{t_f} N_0$

  overall resize Complexity is

  $s = O(N_0 + 2N_0 + 4N_0 + \cdots + 2^{t_f} N_0) = O(N_0\, 2^{t_f+1})$

  !!!!!!!!!!

  $= O(2N_0 2^{t_f}) = O(2N_{final})$

In What way resize a table ?

- resize by doubling
  - l'=2l whenever N/l = t
  - resize complexity is θ(n+l+l') =θ(n)= θ(l)

Overall resize Complexity is

$s = O(N_{final})$ NOT O(N^2)

- That's why table doubling!

Table doubling

- Do we get the nice $O(N_{final})$ when not doubling but

$l' = l \cdot r,\ r > 1$???

Table doubling

- Do we get the nice $O(N_{final})$ when not doubling but

$l' = l \cdot r,\ r > 1$??

Yes! proof relies on the geometric series

$$r^0 + r^1 + r^2 + \cdots = \sum_{k=0}^{t} r^k = \frac{r^{t+1} - 1}{r - 1}$$

Table doubling in general

- table doubling = resize a hash table by doubling (or exponentially by multiplying with a factor r>1)
- $l' = l \cdot r,\ r > 1$

Overall resize Complexity for table doubling is

$$O(N_{final})$$

Now we can show under what conditions Hash-search is O(1)

amortized analysis

- ***amortized analysis*** = average the time required to perform a sequence of operations over the number of operations
  - $N_{final} = 2^{t_f} N_0$ hash table inserts,
  - all necessary table doublings as we insert in total N_{final} many keys
- Amortized costs for insert + table growing is O(???)
  - Hash table inserts: N_{final} O(1)
  - Table doubling costs: O(N_{final})
  - Total costs: O(N_{final})+ N_{final} O(1) = $\mathrm{O}(N_{final})$
- Average cost for search (IF simple uniform hashing assumption HOLDS) is:
  - Hash table search: O(1+α)
    - α is bounded independent of N_{final} by table doubling strategy, so

Hash table search: O(1) !!!

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  - Table doubling costs: O(N_{final})
  - Total costs: O(N_{final})+ N_{final} O(1) = $\mathrm{O}(N_{final})$
  - Amortized (=average) cost over N_{final} insertions:

$$\frac{\mathrm{O}(N_{final})}{N_{final}} = O(1)$$

- Average cost for search (IF simple uniform hashing assumption HOLDS) is:
  - Hash table search: O(1+α)
    - α is bounded independent of N_{final} by table doubling strategy, so

Hash table search: O(1) !!!

Final Result:

- Under the simple uniform hashing assumption:

> The average search time of a chained hash table with table doubling strategy is O(1)

Why ?(because we do table doubling whenever the load factor α goes above a fixed threshold t (=independent of n), therefore $O(1+ \alpha) <= O(1+t)$ which is O(1) when considered as a function of the number of keys n)

- The amortized cost for the necessary insertions and table doublings (for keeping α bounded by a constant) are also O(1)

- This is a double average
  - $O(1+ \alpha)$ contains averaging of distribution of keys into hash slots (simple uniform hashing assumption)
  - Amortization Analysis of Table doubling contains averaging over a large number of sequential insertions

Conclusions…

- Chained hash:
  - an array s.t. each entry is a linked list
  - a hash function that maps keys onto array indices
  - Map n keys from a very large key space onto hash table of size m
- may have hash collisions
  - Hash operations: insert,delete as O(1) worst case
  - Search O(n) worst, O(1+α) average case (simple uniform hashing assumption)
  - When combined with table doubling: O(1) average case\
  - Amortized costs for table doubling+ insertions: also O(1)

- Good hash functions: close to simple uniform hashing assumption each key has equal chance to end up in any of the bins … ensures on average equal load across the whole table