50.034 – Introduction to Probability and Statistics

January–May Term, 2019

Homework Set 2

Due by: Week 3 Cohort Class (14 Feb 2019 or 15 Feb 2019)

Question 1. An old magnetic tape storing information in binary form has been corrupted. Suppose you are trying to save as much information as possible from this magnetic tape. Due to the damage on the tape, you know that there will be errors in the reading. The probability that you correctly detect a 0 is 0.9, while the probability that you correctly detect a 1 is 0.85. Given that each digit is either a 0 or a 1 with equal probability, and given that your reading is a 1, what is the probability that this is a correct reading?

Solution. Define the following events:

- A_0 = "Selected digit is actually 0".
- A_1 = "Selected digit is actually 1".
- B = "Selected digit is read as 1".

Since each digit is either a 0 or a 1 with equal probability, we have $\Pr(A_0) = \Pr(A_1) = 0.5$. Note that events A_0 and A_1 are mutually exclusive and exhaustive. Thus by Bayes' theorem,

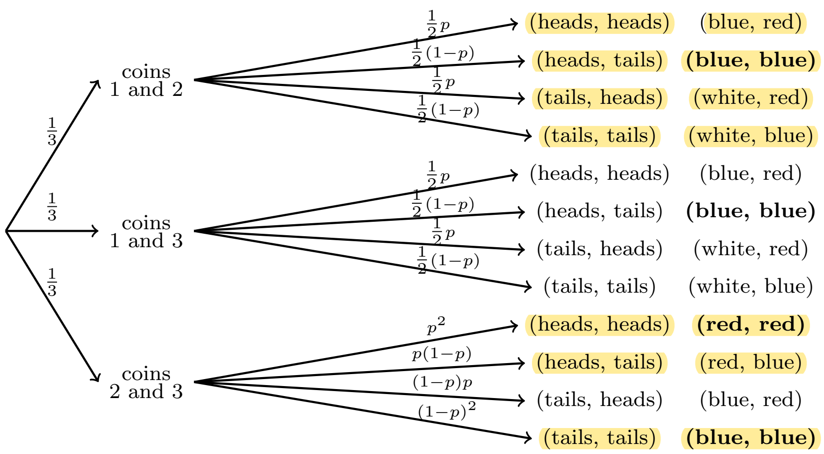
$$\Pr(A_1|B) = \frac{\Pr(B|A_1)\Pr(A_1)}{\Pr(B|A_0)\Pr(A_0) + \Pr(B|A_1)\Pr(A_1)} = \frac{(0.85)(0.5)}{(1-0.9)(0.5) + (0.85)(0.5)}$$
$$\approx 0.8947.$$

Question 2. You are given three coins. The first coin is a fair coin painted blue on the heads side and white on the tails side. The other two coins are biased, such that the probability of getting heads is p. Both biased coins are painted blue on the tails side and red on the heads side. You conduct the following experiment: You select two of the three coins at random, then you toss both selected coins.

(i) Describe the outcomes in the sample space of this experiment.

(ii) It was experimentally determined that the probability that the sides that land face up have the same color is $\frac{29}{96}$. What are the possible values of p?

Solution. (i) There are two stages to this experiment: the selection of two coins, and the tossing of the two selected coins. There are three different ways that two coins can be selected: the 1st and 2nd, the 1st and 3rd, and the 2nd and 3rd. Each of these pairs are equally likely to be selected. For each pair, the tosses have four possible outcomes: (heads, heads), (heads, tails), (tails, heads), (tails, tails). Hence, the sample space can be described as follows:



(ii) In the previous diagram, the outcomes corresponding to pairs of faces with the same color are in bold. Thus, the probability that the sides that land face up have the same color is

$$\begin{aligned}\Pr(\text{same color}) &= \Pr((\text{blue, blue})) + \Pr((\text{red, red})) = \tfrac{1}{3}\Big[\tfrac{1}{2}(1-p) + \tfrac{1}{2}(1-p) + p^2 + (1-p)^2\Big] \\ &= \tfrac{1}{3}\big(2p^2 - 3p + 2\big).\end{aligned}$$

We are given that $\Pr(\text{same color}) = \frac{29}{96}$, hence $\frac{1}{3}\big(2p^2 - 3p + 2\big) = \frac{29}{96}$, or equivalently,

$$64p^2 - 96p + 35 = (8p-5)(8p-7) = 0.$$

Therefore, the possible values of p are $\frac{5}{8}$ and $\frac{7}{8}$.

Question 3. Let A and B be events contained in a sample space Ω, such that $A \subseteq B$. Can A and B be independent events? Justify your answer with as much details as possible.

Solution. By definition, A and B are independent if and only if $\Pr(A \cap B) = \Pr(A)\Pr(B)$. Since $A \subseteq B$, we know that $\Pr(A \cap B) = \Pr(A)$. Therefore, A and B are independent if and only if either $\Pr(B) = 1$ or $\Pr(A) = 0$. This could happen, for example if $B = \Omega$, or if $A = \varnothing$.

Question 4. Let X be a continuous random variable with the following probability density function

$$f(x) = \begin{cases} kx^3 \ln x, & \text{if } 2 \le x \le 4; \\ 0, & \text{otherwise;} \end{cases}$$

where k is an unspecified constant.

(i) Find the value of k.
(ii) Find the value of $\Pr(X \ge 1)$.

Solution. (i) Since $f(x)$ is a probability density function, it must satisfy

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_2^4 kx^3 \ln x\,dx = 1.$$

Using integration by parts, we have

$$\int kx^3 \ln x\,dx = \big[k\tfrac{x^4}{4}\ln x\big] - \int k\tfrac{x^4}{4}\cdot\tfrac{1}{x}\,dx = k\tfrac{x^4}{4}\ln x - \big[k\tfrac{x^4}{16}\big] + \text{constant},$$

hence

$$\begin{aligned}1 &= \int_2^4 kx^3 \ln x\,dx = k\Big[\tfrac{x^4}{4}\ln x - \tfrac{x^4}{16}\Big]_{x=2}^{x=4} = k\big[(64\ln 4 - 16) - (4\ln 2 - 1)\big] \\ &= k(124\ln 2 - 15),\end{aligned}$$

therefore $k = \frac{1}{124\ln 2 - 15} \approx 0.01409$.

(ii) Since $f(x) = 0$ for all $x < 2$ and $x > 4$, it follows that

$$\Pr(X \ge 1) = \int_1^{\infty} f(x)\,dx = \int_2^4 \tfrac{1}{124\ln 2 - 15} x^3 \ln x\,dx = 1.$$

Question 5. Let X be a continuous random variable with the following probability density function

$$f(x) = \begin{cases} \frac{k}{x^2+2x+2}, & \text{if } -2 \le x \le 0; \\ 0, & \text{otherwise;} \end{cases}$$

where k is an unspecified constant.

(i) Find the value of k.
(ii) Find the value of $\Pr(X \le -\frac{1}{\sqrt{3}})$.

Solution. (i) Since $f(x)$ is a probability density function, it must satisfy

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_{-2}^{0} \frac{k}{x^2+2x+2}\,dx = 1.$$

Let $u = x+1$, and note that $\frac{dx}{du} = \frac{d}{du}(u-1) = 1$. Thus,

$$\begin{aligned} \int \frac{k}{x^2+2x+2}\,dx &= \int \frac{k}{u^2+1}\frac{dx}{du}\,du = \int \frac{k}{u^2+1}\,du = k\tan^{-1}(u) + \text{constant} \\ &= k\tan^{-1}(x+1) + \text{constant} \end{aligned}$$

hence

$$1 = \int_{-2}^{0} \frac{k}{x^2+2x+2}\,dx = k\Big[\tan^{-1}(x+1)\Big]_{x=-2}^{x=0} = k\big[\tan^{-1}(1) - \tan^{-1}(-1)\big] = k\cdot\tfrac{\pi}{2},$$

therefore $k = \frac{2}{\pi} \approx 0.6366$.

(ii) Using $k = \frac{2}{\pi}$ from the previous part,

$$\begin{aligned} \Pr(X \le -\tfrac{1}{\sqrt{3}}) &= \int_{-\infty}^{-\frac{1}{\sqrt{3}}} f(x)\,dx = \int_{-2}^{-\frac{1}{\sqrt{3}}} \frac{2}{\pi}\cdot\frac{1}{x^2+2x+2}\,dx \\ &= \Big[\tfrac{2}{\pi}\tan^{-1}(x+1)\Big]_{x=-2}^{x=-\frac{1}{\sqrt{3}}} = \tfrac{2}{\pi}\big(-\tfrac{\pi}{6} - (-\tfrac{\pi}{4})\big) \\ &= \frac{1}{6}. \end{aligned}$$