50.034 – Introduction to Probability and Statistics
January–May Term, 2019

Homework Set 8
Due by: Week 12 Cohort Class (17 Apr 2019 or 18 Apr 2019)
NOTE: **Make-up for Friday's cohort class** (originally 19th April)
will be on **17th April** (Wednesday), 2–4pm, CC14 (2.507).
Reminder: There is **Mini-quiz 4** in Week 12 during your cohort class.

Note: The tables of values for the standard normal distribution, the χ^2 distribution, and the t-distribution, can be found at the back of the course textbook.

Question 1. Let $\{X_1, \ldots, X_{10}\}$ be a random sample of observable normal random variables with unknown mean μ and known variance 16. Let $\overline{X}_{10}$ denote the sample mean of $\{X_1, \ldots, X_{10}\}$. Let $\mathcal{H} = \{\mathcal{H}_c\}_{c \in \mathbb{R}}$ be a collection of hypothesis tests, where each $\mathcal{H}_c$ represents a hypothesis test with null hypothesis $H_0 : \mu = 2$, test statistic $T = |\overline{X}_{10} - 2|$, and rejection region $R = [c, \infty)$.
  (i) Find the value of $c \in \mathbb{R}$ such that $\mathcal{H}_c$ is a level 0.05 test that has the highest power among all level 0.05 tests in $\mathcal{H}$.
  (ii) Determine the p-value of $\mathcal{H}$, given the following observed values:

$$X_1 = 1, X_2 = 2, X_3 = 1, X_4 = 2, X_5 = 1, X_6 = 2, X_7 = 1, X_8 = 2, X_9 = 1, X_{10} = 2.$$

Question 2. Let $\{X_1, \ldots, X_{20}\}$ be a random sample of observable normal random variables with unknown mean μ and unknown variance σ^2. Let $\overline{X}_{20}$ and s_{20}^2 denote the sample mean and unbiased sample variance respectively of $\{X_1, \ldots, X_{20}\}$. Let $\mathcal{H} = \{\mathcal{H}_c\}_{c \in \mathbb{R}}$ be a collection of hypothesis tests, where each $\mathcal{H}_c$ represents a hypothesis test with null hypothesis $H_0 : \mu = 0$, test statistic $T = \left| \frac{\overline{X}_{10}}{s_{20}} \right|$, and rejection region $R = [c, \infty)$.
  (i) Find the value of $c \in \mathbb{R}$ such that $\mathcal{H}_c$ is a level 0.05 test that has the highest power among all level 0.05 tests in $\mathcal{H}$.
  (ii) Determine the p-value of $\mathcal{H}$, given the following observed values:

$$X_1 = -1, X_2 = 0, X_3 = 1, X_4 = 0, X_5 = -1, X_6 = 0, X_7 = 1, X_8 = 0, X_9 = -1, X_{10} = 0.$$

Question 3. Let $\{X_1, \ldots, X_{20}\}$ be a random sample of 20 observable normal random variables with unknown mean μ and unknown variance σ^2. Let $\overline{X}_{20}$ and $\hat{\sigma}_{20}^2$ denote the sample mean and biased sample variance respectively of $\{X_1, \ldots, X_{20}\}$. Consider a hypothesis test $\mathcal{H}$ with null hypothesis $H_0 : \sigma^2 = 25$. Let $T = \hat{\sigma}_{20}^2$ be the test statistic, and let $R = [c, \infty)$ be the rejection region of $\mathcal{H}$, where c is some constant to be determined. Find the value of c that maximizes the power of $\mathcal{H}$ at significance level 0.1.

Question 4. Let $\{X_1, \ldots, X_{20}\}$ be a random sample of observable Poisson random variables with unknown mean θ. Suppose we are given that the parameter space of θ contains only two possible values 1 and 2. Find the most powerful hypothesis test with significance level 0.05, such that its null hypothesis is $H_0 : \theta = 1$. Please give a complete description of this most powerful hypothesis test $\mathcal{H}$, including the test statistic and rejection region of $\mathcal{H}$.

Question 5. Let $\{X_1, \ldots, X_{15}\}$ be a random sample of observable Bernoulli random variables with unknown parameter θ. Assume that the parameter space of θ is the interval $[0, 1]$. Find a uniformly most powerful hypothesis test with significance level 0.1, such that its null hypothesis is $H_0 : \theta \geq 0.4$. Please give a complete description of this uniformly most powerful hypothesis test $\mathcal{H}$, including the test statistic and rejection region of $\mathcal{H}$.