50.034 – Introduction to Probability and Statistics

January–May Term, 2019

Homework Set 1

Due by: Week 2 Cohort Class (7 Feb 2019 or 8 Feb 2019)

Question 1. Express each of the following events in terms of the events A, B and C, as well as the operations of complements, unions and intersections:

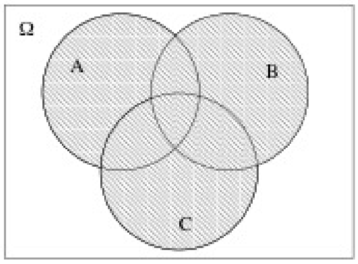
(i) at least one of the events A, B, C occurs;
(ii) all three events A, B, C occur;
(iii) exactly one of the events A, B, C occurs;
(iv) events A and B occur, but not C;
(v) either event A occurs or, if not, then B also does not occur.

In each case, draw the corresponding Venn diagrams.

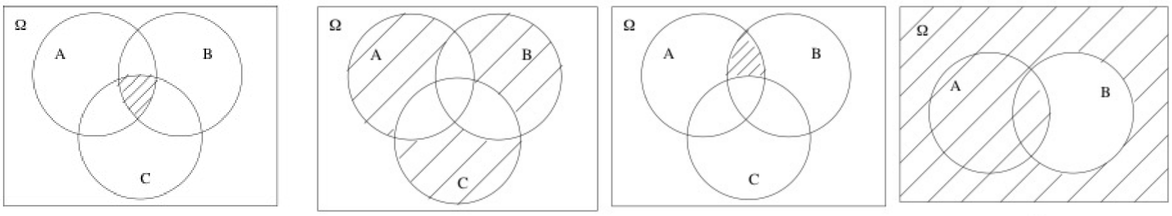
As an example, part (i) has been done for you:

(at least one of events A, B, C occurs) $= A \cup B \cup C$.

The corresponding Venn diagram is shown on the right.



Solution. The following Venn diagrams (from left to right) are for parts (ii)–(v):



(ii) $A \cap B \cap C$.
(iii) $(A \cap B^c \cap C^c) \cup (A^c \cap B \cap C^c) \cup (A^c \cap B^c \cap C)$.
(iv) $A \cap B \cap C^c$.
(v) $A \cup (A^c \cap B^c)$.

Question 2. This year's SUTD chess champion is to be selected by the following procedure. Brandon and Colin, the leading challengers, first play a two-game match. If one of them wins both games, he gets to play a two-game second round with Amanda, the current champion. Amanda retains her championship unless a second round is required and the challenger beats Amanda in both games. If Amanda wins the initial game of the second round, no more games are played. Furthermore, we know the following:

- Pr(Brandon will beat Colin in any particular game) = 0.6.
- Pr(Amanda will beat Brandon in any particular game) = 0.5.
- Pr(Amanda will beat Colin in any particular game) = 0.7.

Assume no tie games are possible and assume that the outcomes all games are independent of each other. Determine the following probabilities:

(i) Brandon will win the first round.
(ii) The second round will be required.
(iii) Amanda will retain her championship this year.

Solution. (i) Pr(Brandon wins 1st round) = Pr(Brandon beats Colin twice) = $(0.6)^2 = 0.36$.
(ii) Pr(2nd round required) = Pr(Brandon beats Colin twice)+Pr(Colin beats Brandon twice) $= (0.6)^2 + (1 - 0.6)^2 = 0.52$.
(iii) Pr(Amanda is champion) = 1 − Pr(Brandon is champion) − Pr(Colin is champion) $= 1 - (0.6)^2 \times (1 - 0.5)^2 - (0.4)^2 \times (1 - 0.7)^2 = 0.8956$.

Question 3. Peter has a peculiar pair of tetrahedron-shaped dice, each with four sides having the numbers $1, 2, 3, 4$ respectively. One die is red, and the other is blue. When he rolls both dice, the probability that the sum of the numbers on both dice equals k is proportional to k. For example, it is three times more likely to have a sum $k = 6$ than to have a sum $k = 2$, since 6 is three times of 2. All outcomes that result in a particular sum k are equally likely. For example, rolling a 1 on the red die and a 2 on the blue die, is equally likely as rolling a 2 on the red die and a 1 on the blue die.

(i) What is the probability that the sum is even?
(ii) What is the probability of Peter rolling a 3 on the red die and a 4 on the blue die?

Solution. First, we determine the probabilities of all possible outcomes. Since Pr(Sum $= k$) is proportional to k, there exists some constant p such that:

Red Die	Blue Die	Sum	Pr(Sum)
1	1	2	$2p$
1	2	3	$3p$
1	3	4	$4p$
1	4	5	$5p$
2	1	3	$3p$
2	2	4	$4p$
2	3	5	$5p$
2	4	6	$6p$
3	1	4	$4p$
3	2	5	$5p$
3	3	6	$6p$
3	4	7	$7p$
4	1	5	$5p$
4	2	6	$6p$
4	3	7	$7p$
4	4	8	$8p$
		Total:	$80p$

The sum of the probabilities of all possible outcomes must equal 1, hence $80p = 1$, i.e. $p = \frac{1}{80}$.

(i) Pr(sum being even) $= 2p + 4p + 4p + 6p + 4p + 6p + 6p + 8p = 40p = 0.5$.
(ii) Pr(red die $= 3$, blue die $= 4$) $= 7p = \frac{7}{80} = 0.0875$.

Question 4. There are 11 students in a party. Five of them are girls. In how many ways can these 11 students be arranged in a row if

(i) there are no restrictions?

(ii) the 5 girls must be together (forming a block)?

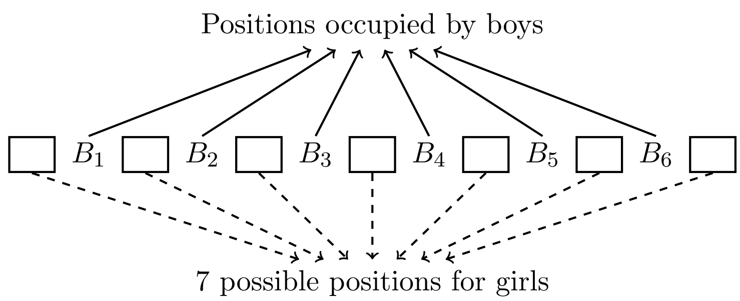
(iii) no 2 girls are adjacent?

Solution. (i) All permutations of the 11 students are allowed, so there are a total of $11! = 39916800$ ways to arrange these 11 students in a row with no restrictions.

(ii) If the 5 girls must be together and form a block, and the remaining 6 boys each form an individual block, then there are a total of 7 blocks. There are $5!$ ways to permute the girls within their block. Thus, the total number of possible ways is $7! \times 5! = 604800$.

(iii) We first arrange the boys, then consider arranging each girl one by one, either in front of the row (in front of the first boy), at the back of the row (behind the last boy), or in between two boys. First, there are $6!$ ways to arrange to arrange the boys. Next, for the five girls, there are:

- 7 possible positions for the first girl;
- 6 possible positions for the second girl;
- 5 possible positions for the third girl;
- 4 possible positions for the fourth girl;
- 3 possible positions for the fifth girl.



Therefore, the total number of possible ways is

$$6! \times 7 \times 6 \times 5 \times 4 \times 3 = 1814400.$$

Question 5. Of all the students in SUTD, 90% love at least one of the following three snacks: Hello Panda chocolate biscuits (Snack A), Tao Kae Noi crispy seaweed (Snack B), and Huang Fei Hong spicy peanuts (Snack C). 59% of the students love Snack A. 58% of the students love Snack B. 42% of the students love Snack C. Suppose we are given the following additional information:

- 32% of the students love both Snack A and Snack B.
- 29% of the students love both Snack A and Snack C.
- 34% of the students love both Snack B and Snack C.

Determine the probability that a randomly selected student loves Snack B and Snack C but does not love Snack A.

Solution. Let A, B and C be the events that a randomly selected students loves Snack A, Snack B, and Snack C respectively. We are given that

$$\Pr(A \cup B \cup C) = 0.9, \quad \Pr(A) = 0.59, \quad \Pr(B) = 0.58 \quad \Pr(C) = 0.42;$$
$$\Pr(A \cap B) = 0.32, \quad \Pr(A \cap C) = 0.29, \quad \Pr(B \cap C) = 0.34.$$

By the inclusion-exclusion principle,

$\Pr(A \cup B \cup C) = \Pr(A) + \Pr(B) + \Pr(C) - \Pr(A \cap B) - \Pr(A \cap C) - \Pr(B \cap C) + \Pr(A \cap B \cap C)$,

hence $\Pr(A \cap B \cap C) = 0.9 - 0.59 - 0.58 - 0.42 + 0.32 + 0.29 + 0.34 = 0.26$.

Therefore, the probability that a randomly selected student loves Snack B and Snack C but does not love Snack A is

$$\Pr(B \cap C \cap A^c) = \Pr(B \cap C) - \Pr(A \cap B \cap C) = 0.34 - 0.26 = 0.08.$$