50.034 – Introduction to Probability and Statistics

January–May Term, 2019

Homework Set 3

Due by: Week 4 Cohort Class (21 Feb 2019 or 22 Feb 2019)

Question 1. Let X be a discrete random variable with the following probability mass function

$$p_X(x) = \begin{cases} \frac{x^2}{a}, & \text{if } x \in \{-3, -2, -1, 0, 1, 2, 3\}; \\ 0, & \text{otherwise;} \end{cases}$$

where a is an unspecified constant.

(i) Find the values of a and $\mathbf{E}[X]$.
(ii) What is the probability mass function of the random variable $Y = (X - \mathbf{E}[X])^2$?
(iii) Using part (ii), calculate the variance of X.

Question 2. Consider the following game derived from a sequence of independent tosses of a fair coin. You start to play the game with exactly one dollar. You bet all your money (if you still have any) on each successive toss of the coin. If a head appears, you win twice of your bet. You will lose all of your money whenever you see a tail. Let X_n denote the amount of money (in dollars) that you have after the n-th coin toss.

(i) Determine the probability mass function of X_n.
(ii) Calculate the expectation $\mathbf{E}[X_n]$.
(iii) Calculate the variance $\text{var}(X_n)$.

Question 3. A particular circuit works if all ten of its component devices work. Each circuit is tested before leaving the factory. Each working circuit can be sold for k dollars, but each non-working circuit is worthless and must be thrown away. Each circuit can be built with either ordinary devices or ultra-reliable devices. An ordinary device has a failure probability of 0.1, while an ultra-reliable device has a failure probability of 0.05, independent of any other device. However, each ordinary device costs \$1, whereas an ultra-reliable device costs \$3. Should you build your circuit with ordinary devices or ultra-reliable devices in order to maximize your expected profit? Keep in mind that your answer will have to depend on the value of k.

Question 4. An airline sells 200 tickets for a certain flight on an airplane that has only 198 seats because, on average, 1 percent of those who purchase airline tickets do not appear for the departure of their flight. What is the probability that everyone who appears for the departure of this flight will have a seat? Give details for your solution, including any assumptions that you make.

Question 5. Bananas are slightly radioactive because they contain potassium, and potassium decays over time. Suppose that you have in storage a large crate of bananas, and radioactive particles from the bananas strike a target following a Poisson distribution with a rate of 120 strikes per hour.

(i) What is the probability that no radioactive particles from the bananas will strike the target in a one-minute period?

(ii) Given a radiation detector, what is the probability that the first radioactive particle strike on the target occurs within one minute?

Question 6. Let X be a continuous random variable with the cumulative distribution function

$$F(x) = \begin{cases} 0, & \text{if } x < 0; \\ \frac{1}{36}x^2, & \text{if } 0 \leq x \leq a; \\ \frac{x}{12} + \frac{2}{9}, & \text{if } a < x < b; \\ 1, & \text{if } x \geq b; \end{cases}$$

where a and b are unspecified constants. Find the values of a and b.