

# 50.034 – Introduction to Probability and Statistics

January–May Term, 2019

## Homework Set 8

Due by: Week 12 Cohort Class (17 Apr 2019 or 18 Apr 2019)

**NOTE: Make-up for Friday's cohort class** (originally 19th April)

will be on **17th April** (Wednesday), 2–4pm, CC14 (2.507).

**Reminder:** There is **Mini-quiz 4** in Week 12 during your cohort class.

**Note:** The tables of values for the standard normal distribution, the  $\chi^2$  distribution, and the  $t$ -distribution, can be found at the back of the course textbook.

**Question 1.** Let  $\{X_1, \dots, X_{10}\}$  be a random sample of observable normal random variables with unknown mean  $\mu$  and known variance 16. Let  $\bar{X}_{10}$  denote the sample mean of  $\{X_1, \dots, X_{10}\}$ . Let  $\mathcal{H} = \{\mathcal{H}_c\}_{c \in \mathbb{R}}$  be a collection of hypothesis tests, where each  $\mathcal{H}_c$  represents a hypothesis test with null hypothesis  $H_0 : \mu = 2$ , test statistic  $T = |\bar{X}_{10} - 2|$ , and rejection region  $R = [c, \infty)$ .

- (i) Find the value of  $c \in \mathbb{R}$  such that  $\mathcal{H}_c$  is a level 0.05 test that has the highest power among all level 0.05 tests in  $\mathcal{H}$ .
- (ii) Determine the  $p$ -value of  $\mathcal{H}$ , given the following observed values:

$$X_1 = 1, X_2 = 2, X_3 = 1, X_4 = 2, X_5 = 1, X_6 = 2, X_7 = 1, X_8 = 2, X_9 = 1, X_{10} = 2.$$

**Question 2.** Let  $\{X_1, \dots, X_{20}\}$  be a random sample of observable normal random variables with unknown mean  $\mu$  and unknown variance  $\sigma^2$ . Let  $\bar{X}_{20}$  and  $s_{20}^2$  denote the sample mean and unbiased sample variance respectively of  $\{X_1, \dots, X_{20}\}$ . Let  $\mathcal{H} = \{\mathcal{H}_c\}_{c \in \mathbb{R}}$  be a collection of hypothesis tests, where each  $\mathcal{H}_c$  represents a hypothesis test with null hypothesis  $H_0 : \mu = 0$ , test statistic  $T = \left| \frac{\bar{X}_{20}}{s_{20}} \right|$ , and rejection region  $R = [c, \infty)$ .

- (i) Find the value of  $c \in \mathbb{R}$  such that  $\mathcal{H}_c$  is a level 0.05 test that has the highest power among all level 0.05 tests in  $\mathcal{H}$ .
- (ii) Determine the  $p$ -value of  $\mathcal{H}$ , given the following observed values:

$$\begin{array}{cccccccccccc} X_1 = -1, & X_2 = 0, & X_3 = 1, & X_4 = 0, & X_5 = -1, & X_6 = 0, & X_7 = 1, & X_8 = 0, & X_9 = -1, & X_{10} = 0, \\ X_{11} = 1, & X_{12} = 0, & X_{13} = -1, & X_{14} = 0, & X_{15} = 1, & X_{16} = 0, & X_{17} = -1, & X_{18} = 0, & X_{19} = 1, & X_{20} = 0. \end{array}$$

**Question 3.** Let  $\{X_1, \dots, X_{20}\}$  be a random sample of 20 observable normal random variables with unknown mean  $\mu$  and unknown variance  $\sigma^2$ . Let  $\bar{X}_{20}$  and  $\hat{\sigma}_{20}^2$  denote the sample mean and biased sample variance respectively of  $\{X_1, \dots, X_{20}\}$ . Consider a hypothesis test  $\mathcal{H}$  with null hypothesis  $H_0 : \sigma^2 = 25$ . Let  $T = \hat{\sigma}_{20}^2$  be the test statistic, and let  $R = [c, \infty)$  be the rejection region of  $\mathcal{H}$ , where  $c$  is some constant to be determined. Find the value of  $c$  that maximizes the power of  $\mathcal{H}$  at significance level 0.1.

**Question 4.** Let  $\{X_1, \dots, X_{20}\}$  be a random sample of observable Poisson random variables with unknown mean  $\theta$ . Suppose we are given that the parameter space of  $\theta$  contains only two possible values 1 and 2. Find the most powerful hypothesis test with significance level 0.05, such that its null hypothesis is  $H_0 : \theta = 1$ . Please give a complete description of this most powerful hypothesis test  $\mathcal{H}$ , including the test statistic and rejection region of  $\mathcal{H}$ .

**Question 5.** Let  $\{X_1, \dots, X_{15}\}$  be a random sample of observable Bernoulli random variables with unknown parameter  $\theta$ . Assume that the parameter space of  $\theta$  is the interval  $[0, 1]$ . Find a uniformly most powerful hypothesis test with significance level 0.1, such that its null hypothesis is  $H_0 : \theta \geq 0.4$ . Please give a complete description of this uniformly most powerful hypothesis test  $\mathcal{H}$ , including the test statistic and rejection region of  $\mathcal{H}$ .