50.034 – Introduction to Probability and Statistics

January–May Term, 2019

Homework Set 4

Due by: Week 6 Monday's Lecture (4 Mar 2019)

Question 1. Let X and Y be continuous random variables such that (X, Y) must belong to the square in the xy-plane containing all points (x, y) that satisfy $0 \leq x \leq 1$ and $0 \leq y \leq 1$. Suppose that the joint cumulative distribution function of X and Y at every point (x, y) in this square is specified as follows:

$$F(x, y) = 0.1x^4y + 0.2x^3y^2 + 0.3x^2y^3 + 0.4xy^4.$$

Determine the following:

(i) The joint cumulative distribution function of X and Y (i.e. at every point in $\mathbb{R}^2$, not just in the square).

(ii) The joint probability density function of X and Y.

(iii) The cumulative distribution function of X.

Solution. (i) Consider the following cases:

Case 1: If $(x_0, y_0) \in \mathbb{R}^2$ satisfies $x_0 < 0$ or $y_0 < 0$, then $\{(x, y) \in \mathbb{R}^2 | x \leq x_0, y \leq y_0\}$ does not contain any point in the given square, hence $F(x_0, y_0) = 0$ in this case.

Case 2: If $(x_0, y_0) \in \mathbb{R}^2$ satisfies both $x_0 > 1$ and $y_0 > 1$, then $\{(x, y) \in \mathbb{R}^2 | x \leq x_0, y \leq y_0\}$ contains all points in the given square, hence $F(x_0, y_0) = 1$ in this case.

Case 3: If $(x_0, y_0) \in \mathbb{R}^2$ satisfies both $x_0 > 1$ and $0 \leq y_0 \leq 1$, then the intersection of $\{(x, y) \in \mathbb{R}^2 | x \leq x_0, y \leq y_0\}$ with the given square yields the subset:

$$\{(x, y) \in \mathbb{R}^2 | 0 \leq x \leq 1, 0 \leq y \leq y_0\},$$

hence $F(x_0, y_0) = F(1, y_0) = 0.1y + 0.2y^2 + 0.3y^3 + 0.4y^4$ in this case.

Case 4: If $(x_0, y_0) \in \mathbb{R}^2$ satisfies both $0 \leq x_0 \leq 1$ and $y_0 > 1$, then the intersection of $\{(x, y) \in \mathbb{R}^2 | x \leq x_0, y \leq y_0\}$ with the given square yields the subset:

$$\{(x, y) \in \mathbb{R}^2 | 0 \leq x \leq x_0, 0 \leq y \leq 1\},$$

hence $F(x_0, y_0) = F(x_0, 1) = 0.1x^4 + 0.2x^3 + 0.3x^2 + 0.4x$ in this case.

Every point $(x_0, y_0) \in \mathbb{R}^2$ either has been considered in one of the four cases above, or is contained in the given square. Therefore, by combining these four cases, together with the given value of $F(x, y)$ on the square, we get:

$$F(x, y) = \begin{cases} 0.1x^4y + 0.2x^3y^2 + 0.3x^2y^3 + 0.4xy^4, & \text{if } 0 \leq x \leq 1 \text{ and } 0 \leq y \leq 1; \\ 0.1y + 0.2y^2 + 0.3y^3 + 0.4y^4, & \text{if } x > 1 \text{ and } 0 \leq y \leq 1; \\ 0.1x^4 + 0.2x^3 + 0.3x^2 + 0.4x, & \text{if } 0 \leq x \leq 1 \text{ and } y > 1; \\ 1, & \text{if } x > 1 \text{ and } y > 1; \\ 0, & \text{otherwise.} \end{cases}$$

(ii) Let $f(x, y)$ denote the joint probability density function of X and Y. Since (X, Y) must belong to the given square, we know that $f(x, y) = 0$ whenever (x, y) is not in the square. Given any $(x_0, y_0) \in \mathbb{R}^2$, we know that

$$f(x_0, y_0) = \left. \frac{\partial^2 F(x, y)}{\partial x \partial y} \right|_{(x,y)=(x_0,y_0)},$$

provided this second-order partial derivative exists at $(x, y) = (x_0, y_0)$.
We check that

$$\begin{aligned}\frac{\partial^2}{\partial x \partial y}\left(0.1x^4y + 0.2x^3y^2 + 0.3x^2y^3 + 0.4xy^4\right) &= \frac{\partial}{\partial y}(0.4x^3y + 0.6x^2y^2 + 0.6xy^3 + 0.4y^4) \\ &= 0.4x^3 + 1.2x^2y + 1.8xy^2 + 1.6y^3.\end{aligned}$$

Therefore,

$$f(x, y) = \begin{cases} 0.4x^3 + 1.2x^2y + 1.8xy^2 + 1.6y^3, & \text{if } 0 \le x \le 1 \text{ and } 0 \le y \le 1; \\ 0, & \text{otherwise.} \end{cases}$$

(iii) The cumulative distribution function of X is $F_X(x) = \lim_{y\to\infty} F(x, y)$. So from part (i),

$$F_X(x) = \begin{cases} 0.1x^4 + 0.2x^3 + 0.3x^2 + 0.4x, & \text{if } 0 \le x \le 1; \\ 1, & \text{if } x > 1; \\ 0, & \text{if } x < 0. \end{cases}$$

Question 2. Let X and Y be continuous random variables such that (X, Y) must belong to the square in the xy-plane containing all points (x, y) that satisfy $0 \le x \le 1$ and $0 \le y \le 1$. Suppose that the joint cumulative distribution function of X and Y at every point (x, y) in this square is specified as follows:

$$F(x, y) = 0.1x^4y + 0.5x^3y^2 + 0.3x^2y^3 + 0.1xy^4.$$

Determine the following:

(i) The joint probability density function of X and Y.
(ii) The marginal probability density function of X.
(iii) The marginal probability density function of Y.

Solution. (i) Let $f(x, y)$ denote the joint probability density function of X and Y. Since (X, Y) must belong to the given square, we know that $f(x, y) = 0$ whenever (x, y) is not in the square. Given any $(x_0, y_0) \in \mathbb{R}^2$, we know that

$$f(x_0, y_0) = \left.\frac{\partial^2 F(x, y)}{\partial x \partial y}\right|_{(x,y)=(x_0,y_0)},$$

provided this second-order partial derivative exists at $(x, y) = (x_0, y_0)$.
We check that

$$\begin{aligned}\frac{\partial^2}{\partial x \partial y}\left(0.1x^4y + 0.5x^3y^2 + 0.3x^2y^3 + 0.1xy^4\right) &= \frac{\partial}{\partial y}(0.4x^3y + 1.5x^2y^2 + 0.6xy^3 + 0.1y^4) \\ &= 0.4x^3 + 3x^2y + 1.8xy^2 + 0.4y^3.\end{aligned}$$

Therefore,

$$f(x, y) = \begin{cases} 0.4x^3 + 3x^2y + 1.8xy^2 + 0.4y^3, & \text{if } 0 \le x \le 1 \text{ and } 0 \le y \le 1; \\ 0, & \text{otherwise.} \end{cases}$$

(ii) We check that

$$\begin{aligned}\int_0^1 (0.4x^3 + 3x^2y + 1.8xy^2 + 0.4y^3)\,dy &= \Big[0.4x^3y + 1.5x^2y^2 + 0.6xy^3 + 0.1y^4\Big]_{y=0}^{y=1} \\ &= 0.4x^3 + 1.5x^2 + 0.6x + 0.1.\end{aligned}$$

Therefore, the marginal probability density function of X is

$$f_X(x) = \int_{-\infty}^{\infty} f(x,y)\,dy = \begin{cases} 0.4x^3 + 1.5x^2 + 0.6x + 0.1, & \text{if } 0 \le x \le 1; \\ 0, & \text{otherwise.} \end{cases}$$

(iii) We check that

$$\begin{aligned}\int_0^1 (0.4x^3 + 3x^2y + 1.8xy^2 + 0.4y^3)\,dx &= \Big[0.1x^4 + x^3y + 0.9x^2y^2 + 0.4xy^3\Big]_{x=0}^{x=1} \\ &= 0.1 + y + 0.9y^2 + 0.4y^3.\end{aligned}$$

Therefore, the marginal probability density function of Y is

$$f_Y(y) = \int_{-\infty}^{\infty} f(x,y)\,dx = \begin{cases} 0.1 + y + 0.9y^2 + 0.4y^3, & \text{if } 0 \le y \le 1; \\ 0, & \text{otherwise.} \end{cases}$$

Question 3. Let X and Y be continuous random variables with a joint probability density function given by:

$$f(x,y) = \begin{cases} k(1 + x^2 + x^2y - x^2y^2 - y^2 + y), & \text{if } 0 \le x \le 3 \text{ and } 0 \le y \le 1; \\ 0, & \text{otherwise;} \end{cases}$$

where k is some unspecified constant.

(i) Determine the value of k.
(ii) Are X and Y independent random variables? Justify your answer.

Solution. (i) Since $f(x,y)$ is a joint probability density function, it must satisfy

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dx\,dy = \int_0^1\int_0^3 k(1 + x^2 + x^2y - x^2y^2 - y^2 + y)\,dx\,dy = 1.$$

We check that

$$\begin{aligned}&\int_0^1\int_0^3 k(1 + x^2 + x^2y - x^2y^2 - y^2 + y)\,dx\,dy \\ &= k \cdot \int_0^1 \Big[x + \tfrac{1}{3}x^3 + \tfrac{1}{3}x^3y - \tfrac{1}{3}x^3y^2 - y^2x + xy\Big]_{x=0}^{x=3} dy \\ &= k \cdot \int_0^1 (12 + 12y - 12y^2)\,dy \\ &= k \cdot \Big[12y + 6y^2 - 4y^3\Big]_{y=0}^{y=1} \\ &= 14k,\end{aligned}$$

therefore $k = \frac{1}{14}$.

(ii) First, we compute the marginal probability density function of X:

$$
\begin{aligned}
f_X(x) &= \int_{-\infty}^{\infty} f(x,y)\,dy = \int_0^1 \tfrac{1}{14}(1+x^2+x^2y-x^2y^2-y^2+y)\,dy \\
&= \tfrac{1}{14}\Big[y+x^2y+\tfrac12 x^2y^2-\tfrac13 x^2y^3-\tfrac13 y^3+\tfrac12 y^2\Big]_{y=0}^{y=1} \\
&= \tfrac{1}{14}(1+x^2+\tfrac12 x^2-\tfrac13 x^2-\tfrac13+\tfrac12) \\
&= \tfrac{1}{12}(x^2+1)
\end{aligned}
$$

if $0 \le x \le 3$, and $f_X(x) = 0$ otherwise.

Next, we compute the marginal probability density function of Y:

$$
\begin{aligned}
f_Y(y) &= \int_{-\infty}^{\infty} f(x,y)\,dx = \int_0^3 \tfrac{1}{14}(1+x^2+x^2y-x^2y^2-y^2+y)\,dx \\
&= \tfrac{1}{14}\Big[x+\tfrac13 x^3+\tfrac13 x^3y-\tfrac13 x^3y^2-y^2x+xy\Big]_{x=0}^{x=3} \\
&= \tfrac{6}{7}(1+y-y^2)
\end{aligned}
$$

if $0 \le y \le 1$, and $f_Y(y) = 0$ otherwise.

Now, we check that $\frac{1}{12}(x^2+1)\cdot\frac{6}{7}(1+y-y^2) = \frac{1}{14}(1+x^2+x^2y-x^2y^2-y^2+y)$, hence $f(x,y) = f_X(x)f_Y(y)$, i.e. the joint probability density function of X and Y is the product of the marginal probability density functions of X and Y. Therefore, we conclude that X and Y are independent random variables.

Question 4. Let X and Y be continuous random variables with a joint probability density function given by:

$$
f(x,y) = \begin{cases} c(5x^4+6x^2y+8xy^3), & \text{if } 0 \le x \le 2 \text{ and } 0 \le y \le 2; \\ 0, & \text{otherwise;} \end{cases}
$$

where c is some unspecified constant.

(i) Determine the value of c.

(ii) Determine the conditional probability density function of X given $Y = y$.

(iii) Determine the conditional probability $\Pr(X \ge 1 | Y = 1)$.

Solution. (i) Since $f(x,y)$ is a joint probability density function, it must satisfy

$$
\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dx\,dy = \int_0^2\int_0^2 c(5x^4+6x^2y+8xy^3)\,dx\,dy = 1.
$$

We check that

$$
\begin{aligned}
&\int_0^2\int_0^2 c(5x^4+6x^2y+8xy^3)\,dx\,dy = c\cdot\int_0^2 \Big[x^5+2x^3y+4x^2y^3\Big]_{x=0}^{x=2} dy \\
&= c\cdot\int_0^2 (16y^3+16y+32)\,dy = c\cdot\Big[4y^4+8y^2+32y\Big]_{y=0}^{y=2} \\
&= 160c,
\end{aligned}
$$

therefore $c = \frac{1}{160}$.

(ii) We first calculate the marginal probability density function of Y:

$$f_Y(y) = \int_{-\infty}^{\infty} f(x,y)\,dx = \int_0^2 \tfrac{1}{160}(5x^4 + 6x^2y + 8xy^3)\,dx$$
$$= \tfrac{1}{160}\Big[x^5 + 2x^3y + 4x^2y^3\Big]_{x=0}^{x=2} = \tfrac{1}{10}(y^3 + y + 2)$$

if $0 \le y \le 2$, and $f_Y(y) = 0$ otherwise.
We check that

$$\frac{\tfrac{1}{160}(5x^4 + 6x^2y + 8xy^3)}{\tfrac{1}{10}(y^3 + y + 2)} = \frac{5x^4 + 6x^2y + 8xy^3}{16(y^3 + y + 2)}.$$

Thus, the conditional probability density function of X given $Y = y$ is

$$f_{X|Y}(x|y) = \frac{f(x,y)}{f_Y(y)} = \begin{cases} \dfrac{5x^4 + 6x^2y + 8xy^3}{16(y^3 + y + 2)}, & \text{if } 0 \le x \le 2; \\ 0, & \text{otherwise.} \end{cases}$$

(iii) Substituting $y = 1$ into $f_{X|Y}(x|y)$ from the previous part, we get

$$f_{X|Y}(x|1) = \begin{cases} \tfrac{1}{64}(5x^4 + 6x^2 + 8x), & \text{if } 0 \le x \le 2; \\ 0, & \text{otherwise;} \end{cases}$$

thus

$$\Pr(X \ge 1 | Y = 1) = \int_1^{\infty} f_{X|Y}(x|1)\,dx = \int_1^2 \tfrac{1}{64}(5x^4 + 6x^2 + 8x)\,dx$$
$$= \Big[\tfrac{1}{64}(x^5 + 2x^3 + 4x^2)\Big]_{x=1}^{x=2}$$
$$= \tfrac{57}{64}.$$

Question 5. Let X and Y be continuous random variables, such that the marginal probability density function of Y is

$$f_Y(y) = \begin{cases} \tfrac{2}{3}y^3 + \tfrac{1}{3}y + \tfrac{2}{3}, & \text{if } 0 \le y \le 1; \\ 0, & \text{otherwise;} \end{cases}$$

and such that the conditional probability density function of X given $Y = y$ is

$$f_{X|Y}(x|y) = \begin{cases} \dfrac{k(x^3 + xy + 2y^3)}{2y^3 + y + 2}, & \text{if } 0 \le x \le 2; \\ 0, & \text{otherwise;} \end{cases}$$

where k is an unspecified constant.

(i) What is the value of k?
(ii) What is the conditional probability density function of Y given $X = x$?

Solution. (i) A conditional probability density function is a legitimate probability density function, so we must have

$$\int_{-\infty}^{\infty} f_{X|Y}(x|y)\,dx = \int_0^2 \frac{k(x^3 + xy + 2y^3)}{2y^3 + y + 2}\,dx = 1.$$

We check that

$$\begin{aligned}
\int_0^2 \frac{k(x^3+xy+2y^3)}{2y^3+y+2}\,dx &= \frac{k}{2y^3+y+2}\cdot\int_0^2 (x^3+xy+2y^3)\,dx \\
&= \frac{k}{2y^3+y+2}\cdot\left[\tfrac{1}{4}x^4+\tfrac{1}{2}x^2y+2xy^3\right]_{x=0}^{x=2} \\
&= \frac{k}{2y^3+y+2}\cdot(4+2y+4y^3) = 2k,
\end{aligned}$$

therefore $k=\frac{1}{2}$.

(ii) By the law of total probability for continuous random variables, the marginal probability density function of X is

$$\begin{aligned}
f_X(x) &= \int_{-\infty}^{\infty} f_{X|Y}(x|y)f_Y(y)\,dy = \int_0^1 \frac{\frac{1}{2}(x^3+xy+2y^3)}{2y^3+y+2}\cdot(\tfrac{2}{3}y^3+\tfrac{1}{3}y+\tfrac{2}{3})\,dy \\
&= \int_0^1 \tfrac{1}{6}(x^3+xy+2y^3)\,dy = \left[\tfrac{1}{6}x^3y+\tfrac{1}{12}xy^2+\tfrac{1}{12}y^4\right]_{y=0}^{y=1} \\
&= \tfrac{1}{6}x^3+\tfrac{1}{12}x+\tfrac{1}{12}
\end{aligned}$$

if $0\le x\le 2$, and $f_X(x)=0$ otherwise.

Thus, by the Bayes' theorem for continuous random variables,

$$f_{Y|X}(y|x) = \frac{f_{X|Y}(x|y)f_Y(y)}{f_X(x)} = \frac{\frac{1}{6}(x^3+xy+2y^3)}{\frac{1}{6}x^3+\frac{1}{12}x+\frac{1}{12}} = \frac{2x^3+2xy+4y^3}{2x^3+x+1}$$

for each $0\le y\le 1$, and $f_{Y|X}(y|x)=0$ otherwise.

In other words, the conditional probability density function of Y given $X=x$ is

$$f_{Y|X}(y|x) = \begin{cases} \dfrac{2x^3+2xy+4y^3}{2x^3+x+1}, & \text{if } 0\le y\le 1; \\ 0, & \text{otherwise.} \end{cases}$$