50.034 – Introduction to Probability and Statistics

January–May Term, 2019

Homework Set 6

Due by: Week 10 Cohort Class (4 Apr 2019 or 5 Apr 2019)

Question 1. Phone calls are made to a customer service centre following a Poisson distribution with θ calls per minute. A data set consisting of 100 randomly selected one-minute periods yields an average of 1.8 calls. If the prior distribution of θ is an exponential distribution with mean 2, then what is the posterior probability density function of θ?

Question 2. A new device has been invented. To test for its reliability, 100 prototypes of the device are made. Consider a statistical model consisting of observable exponential random variables $X_1, \dots, X_{100}$ that are conditionally iid given the parameter θ. Each X_i represents the time to failure (in hours) of the i-th selected prototype. Suppose that θ is a continuous random variable with the following prior probability density function

$$\xi(\theta) = \begin{cases} 4e^{-4\theta}, & \text{if } \theta \geq 0; \\ 0, & \text{if } \theta < 0. \end{cases}$$

If we are given that the sample mean of $\{X_1, \dots, X_{100}\}$ is 4.23, then what is the posterior probability density function of θ?

Question 3. A new movie titled "Revengers: Finite War" has been released. To determine the approval rating θ ($0 \leq \theta \leq 1$), consider a statistical model where $X_1, \dots, X_{10}$ are observable Bernoulli random variables that are conditionally iid given the parameter θ. Assume that the prior distribution of θ is the uniform distribution on the interval $[0, 1]$. The random variables $X_1, \dots, X_{10}$ represent the approval ratings of 10 randomly selected individuals, where $X_i = 1$ if the i-th selected individual liked the movie, and $X_i = 0$ otherwise. We are given the following:

$$X_1 = 1, X_2 = 1, X_3 = 1, X_4 = 0, X_5 = 1, X_6 = 1, X_7 = 1, X_8 = 1, X_9 = 0, X_{10} = 1;$$

that is, exactly 8 of these 10 individuals liked the movie.

(i) What is the posterior probability density function of θ?

(ii) What is the posterior probability that θ is strictly larger than 0.85?

Question 4. Suppose that the lengths of the raccoons in North America have a normal distribution for which the mean θ is unknown, and the standard deviation is known to be 15 cm. Suppose that the prior distribution of θ is a normal distribution for which the mean is 50 cm, and the standard deviation is 10 cm. Given that 50 raccoons in North America were selected at random, and given that their average length was found to be 54.5 cm, what is the Bayes estimate of θ with respect to the squared error loss function?

Question 5. Let $X_1, \dots, X_n$ be continuous random variables that are conditionally iid given the parameter θ. Suppose that the conditional probability density function of each X_i is given as follows:

$$f_{X_i}(x_i|\theta) = \begin{cases} 2\theta x^{2\theta-1}, & \text{if } 0 < x_i < 1; \\ 0, & \text{otherwise.} \end{cases}$$

(i) What is the likelihood function of θ?

(ii) What is the log-likelihood function of θ?

(iii) What is the maximum likelihood estimate of θ?