

50.034 – Introduction to Probability and Statistics

January–May Term, 2019

Homework Set 4

Due by: Week 6 Monday's Lecture (4 Mar 2019)

Question 1. Let X and Y be continuous random variables such that (X, Y) must belong to the square in the xy -plane containing all points (x, y) that satisfy $0 \leq x \leq 1$ and $0 \leq y \leq 1$. Suppose that the joint cumulative distribution function of X and Y at every point (x, y) in this square is specified as follows:

$$F(x, y) = 0.1x^4y + 0.2x^3y^2 + 0.3x^2y^3 + 0.4xy^4.$$

Determine the following:

- (i) The joint cumulative distribution function of X and Y (i.e. at every point in \mathbb{R}^2 , not just in the square).
- (ii) The joint probability density function of X and Y .
- (iii) The cumulative distribution function of X .

Question 2. Let X and Y be continuous random variables such that (X, Y) must belong to the square in the xy -plane containing all points (x, y) that satisfy $0 \leq x \leq 1$ and $0 \leq y \leq 1$. Suppose that the joint cumulative distribution function of X and Y at every point (x, y) in this square is specified as follows:

$$F(x, y) = 0.1x^4y + 0.5x^3y^2 + 0.3x^2y^3 + 0.1xy^4.$$

Determine the following:

- (i) The joint probability density function of X and Y .
- (ii) The marginal probability density function of X .
- (iii) The marginal probability density function of Y .

Question 3. Let X and Y be continuous random variables with a joint probability density function given by:

$$f(x, y) = \begin{cases} k(1 + x^2 + x^2y - x^2y^2 - y^2 + y), & \text{if } 0 \leq x \leq 3 \text{ and } 0 \leq y \leq 1; \\ 0, & \text{otherwise;} \end{cases}$$

where k is some unspecified constant.

- (i) Determine the value of k .
- (ii) Are X and Y independent random variables? Justify your answer.

Question 4. Let X and Y be continuous random variables with a joint probability density function given by:

$$f(x, y) = \begin{cases} c(5x^4 + 6x^2y + 8xy^3), & \text{if } 0 \leq x \leq 2 \text{ and } 0 \leq y \leq 2; \\ 0, & \text{otherwise;} \end{cases}$$

where c is some unspecified constant.

- (i) Determine the value of c .
- (ii) Determine the conditional probability density function of X given $Y = y$.
- (iii) Determine the conditional probability $\Pr(X \geq 1|Y = 1)$.

Question 5. Let X and Y be continuous random variables, such that the marginal probability density function of Y is

$$f_Y(y) = \begin{cases} \frac{2}{3}y^3 + \frac{1}{3}y + \frac{2}{3}, & \text{if } 0 \leq y \leq 1; \\ 0, & \text{otherwise;} \end{cases}$$

and such that the conditional probability density function of X given $Y = y$ is

$$f_{X|Y}(x|y) = \begin{cases} \frac{k(x^3 + xy + 2y^3)}{2y^3 + y + 2}, & \text{if } 0 \leq x \leq 2; \\ 0, & \text{otherwise;} \end{cases}$$

where k is an unspecified constant.

- (i) What is the value of k ?
- (ii) What is the conditional probability density function of Y given $X = x$?