50.034 – Introduction to Probability and Statistics

January–May Term, 2019

Homework Set 3

Due by: Week 4 Cohort Class (21 Feb 2019 or 22 Feb 2019)

Question 1. Let X be a discrete random variable with the following probability mass function

$$p_X(x) = \begin{cases} \frac{x^2}{a}, & \text{if } x \in \{-3,-2,-1,0,1,2,3\}; \\ 0, & \text{otherwise;} \end{cases}$$

where a is an unspecified constant.

(i) Find the values of a and $\mathbf{E}[X]$.
(ii) What is the probability mass function of the random variable $Y = (X - \mathbf{E}[X])^2$?
(iii) Using part (ii), calculate the variance of X.

Solution. (i) Since $p_X(x)$ is a probability mass function, it must satisfy

$$1 = \sum_{x=-3}^{3} p_X(x) = \sum_{x=-3}^{3} \frac{x^2}{a} = \frac{1}{a}(9+4+1+0+1+4+9) = \frac{28}{a},$$

therefore $a = 28$. Consequently,

$$\mathbf{E}[X] = \sum_{x=-3}^{3} x \cdot p(x) = \sum_{x=-3}^{3} \frac{x^3}{28} = \frac{1}{28}(-27-8-1+0+1+8+27) = 0.$$

(ii) The following table shows the value of Y for each given value of X, together with its corresponding probability:

x	-3	-2	-1	0	1	2	3
$p(x)$	$\frac{9}{28}$	$\frac{1}{7}$	$\frac{1}{28}$	0	$\frac{1}{28}$	$\frac{1}{7}$	$\frac{9}{28}$
$Y \mid X = x$	9	4	1	0	1	4	9

We see that Y can take only three possible values with non-zero probability, namely 1, 4 and 9. Furthermore, for each such value, there are two corresponding values for X. For example, if $p_Y(y)$ denotes the probability mass function of Y, then

$$p_Y(9) = \Pr(Y = 9) = \Pr(X = -3) + \Pr(X = 3) = p_X(-3) + p_X(3).$$

Thus, the probability mass function of Y is

$$p_Y(y) = \begin{cases} \frac{1}{14}, & \text{if } y = 1; \\ \frac{2}{7}, & \text{if } y = 4; \\ \frac{9}{14}, & \text{if } y = 9; \\ 0, & \text{otherwise.} \end{cases}$$

(iii) By definition, $\text{var}(X) = \mathbf{E}(X - \mathbf{E}[X])^2$, hence from part (ii), we have

$$\text{var}(X) = \mathbf{E}[Y] = 1 \cdot p_Y(1) + 4 \cdot p_Y(4) + 9 \cdot p_Y(9) = 1(\tfrac{1}{14}) + 4(\tfrac{2}{7}) + 9(\tfrac{9}{14}) = 7.$$

Question 2. Consider the following game derived from a sequence of independent tosses of a fair coin. You start to play the game with exactly one dollar. You bet all your money (if you still have any) on each successive toss of the coin. If a head appears, you win twice of your bet. You will lose all of your money whenever you see a tail. Let X_n denote the amount of money (in dollars) that you have after the n-th coin toss.

(i) Determine the probability mass function of X_n.
(ii) Calculate the expectation $\mathbf{E}[X_n]$.
(iii) Calculate the variance $\mathrm{var}(X_n)$.

Solution. (i) After the n-th coin toss, the total amount of money is either 2^n dollars or 0 dollars, depending on whether there are n consecutive heads or not respectively. The probability of getting n consecutive heads is $(\frac{1}{2})^n = \frac{1}{2^n}$. Hence, the probability mass function of X_n is

$$p_{X_n}(x) = \begin{cases} \frac{1}{2^n}, & \text{if } x = 2^n; \\ 1 - \frac{1}{2^n}, & \text{if } x = 0; \\ 0, & \text{otherwise.} \end{cases}$$

(ii) Using the probability mass function of X_n from part (i), we get

$$\mathbf{E}[X_n] = 2^n \cdot \frac{1}{2^n} + 0 \cdot (1 - \frac{1}{2^n}) = 1.$$

(iii) Note that $\mathbf{E}[X_n^2] = (2^n)^2 \cdot \frac{1}{2^n} + 0^2 \cdot (1 - \frac{1}{2^n}) = 2^n$. Thus,

$$\mathrm{var}(X_n) = \mathbf{E}[X_n^2] - (\mathbf{E}[X_n])^2 = 2^n - 1^2 = 2^n - 1.$$

Question 3. A particular circuit works if all ten of its component devices work. Each circuit is tested before leaving the factory. Each working circuit can be sold for k dollars, but each non-working circuit is worthless and must be thrown away. Each circuit can be built with either ordinary devices or ultra-reliable devices. An ordinary device has a failure probability of 0.1, while an ultra-reliable device has a failure probability of 0.05, independent of any other device. However, each ordinary device costs \$1, whereas an ultra-reliable device costs \$3. Should you build your circuit with ordinary devices or ultra-reliable devices in order to maximize your expected profit? Keep in mind that your answer will have to depend on the value of k.

Solution. Let X_{ord} be the random variable representing the profit in dollars (sales minus cost) of a circuit built with ordinary devices, and let X_{ultra} be the random variable representing the profit in dollars of a circuit built with ultra-reliable devices. Note that

$$\mathbf{E}[X_{\text{ord}}] = k(1 - 0.1)^{10} - 10 = k \cdot 0.9^{10} - 10;$$
$$\mathbf{E}[X_{\text{ultra}}] = k(1 - 0.05)^{10} - 30 = k \cdot 0.95^{10} - 30.$$

Now, $\mathbf{E}[X_{\text{ultra}}] \geq \mathbf{E}[X_{\text{ord}}] \iff k \cdot 0.95^{10} - 30 \geq k \cdot 0.9^{10} - 10 \iff k \geq \frac{20}{(0.95^{10} - 0.9^{10})} \approx 79.98$, therefore we conclude that to maximize the expected profit, we should use ultra-reliable devices if $k \geq 80$, and use ordinary devices if $k < 80$.

Question 4. An airline sells 200 tickets for a certain flight on an airplane that has only 198 seats because, on average, 1 percent of those who purchase airline tickets do not appear for the departure of their flight. What is the probability that everyone who appears for the departure of this flight will have a seat? Give details for your solution, including any assumptions that you make.

Solution. Let X be the random variable representing the number of people who do not appear for their flight. Note that everyone who appears for the departure of this flight will have a seat if and only if $X \geq 2$. We can assume that the X is a binomial random variable with parameters $n = 200$ and $p = 0.01$. Since $n > 500$ and $np = 2 < 5$, we can approximate the binomial distribution by a Poisson distribution with parameter $\lambda = 200(0.01) = 2$. Therefore, the probability is $\Pr(X \geq 2) = 1 - \Pr(X \leq 1) \approx 1 - (\frac{2^0 e^{-2}}{0!} + \frac{2^1 e^{-2}}{1!}) = 1 - 3e^{-2} \approx 0.5940$.

Question 5. Bananas are slightly radioactive because they contain potassium, and potassium decays over time. Suppose that you have in storage a large crate of bananas, and radioactive particles from the bananas strike a target following a Poisson distribution with a rate of 120 strikes per hour.

(i) What is the probability that no radioactive particles from the bananas will strike the target in a one-minute period?

(ii) Given a radiation detector, what is the probability that the first radioactive particle strike on the target occurs within one minute?

Solution. (i) Let X be the number of radioactive particles that strike the target in a one-minute period. Since the number of radioactive particle strikes per hour follows a Poisson distribution with parameter 120, we infer that X is a Poisson random variable with parameter $\frac{120}{60} = 2$. Hence, the probability mass function of X is

$$p_X(x) = \begin{cases} \frac{2^x e^{-2}}{x!}, & \text{if } x = 0, 1, 2, \ldots; \\ 0, & \text{otherwise.} \end{cases}$$

Consequently, the probability that no radioactive particles from the bananas will strike the target in a one-minute period is $\Pr(X = 0) = e^{-2} \approx 0.1353$.

(ii) Let Y be the waiting time in minutes until the first radioactive particle strikes the target. Y can be modeled as an exponential random variable with parameter 2 (i.e. the occurrence rate is 2). The probability density function of Y is

$$f_Y(y) = \begin{cases} 2e^{-2y}, & \text{if } y \geq 0; \\ 0, & \text{if } y < 0. \end{cases}$$

Therefore, the probability that the first radioactive particle strike on the target occurs within one minute is

$$\Pr(Y \leq 1) = \int_0^1 2e^{-2y}\, dy = \Big[- e^{-2y} \Big]_{y=0}^{y=1} = 1 - e^{-2} \approx 0.8647.$$

Question 6. Let X be a continuous random variable with the cumulative distribution function

$$F(x) = \begin{cases} 0, & \text{if } x < 0; \\ \frac{1}{36}x^2, & \text{if } 0 \leq x \leq a; \\ \frac{x}{12} + \frac{2}{9}, & \text{if } a < x < b; \\ 1, & \text{if } x \geq b; \end{cases}$$

where a and b are unspecified constants. Find the values of a and b.

Solution. Note that the cumulative distribution function of a continuous random variable is always a continuous function. Hence, $\lim_{x \to b^-} F(x) = F(b)$, or equivalently, $\frac{b}{12} + \frac{2}{9} = 1$, which implies that $b = \frac{28}{3}$.

Similarly, $\lim_{x \to a^+} F(x) = F(a)$, or equivalently, $\frac{a}{12} + \frac{2}{9} = \frac{1}{36}a^2$, which implies that $a^2 - 3a - 8 = 0$. Solving for a, we get $a = \frac{3 \pm \sqrt{41}}{2} \approx 4.702$ or -1.702. Since we are given that $a \geq 0$, we must then have $a = \frac{3+\sqrt{41}}{2} \approx 4.702$.