50.034 – Introduction to Probability and Statistics

January–May Term, 2019

Homework Set 5

Due by: Week 6 Cohort Class (7 Mar 2019 or 8 Mar 2019)

Reminder: There is Mini-quiz 2 in Week 6 during your cohort class.

Question 1. Suppose an investor comes to you, wishing to invest $\$85,000$ completely in a portfolio consisting of some combination of Stock A or Stock B, where Stock A costs \$100 per share, and Stock B costs \$10 per share. Assume that the number of shares for each stock must be a whole number. (So if the portfolio consists of a shares of Stock A and b shares of Stock B, then $100a + 10b = 85000$ for some non-negative integers a and b.)

Let X and Y be random variables representing the rates of return (in dollars) per share of Stock A and Stock B respectively, for a period of one year. From your research on these two stocks, you believe that X and Y have variances 16 and 4 respectively, while their correlation is $\rho(X, Y) = -0.5$. What should the investment portfolio be to minimize the variance of the overall return after one year?

Question 2. A new electronics shop has opened. As part of its promotional advertising, the owner has organized an instant lucky draw via a digital "spin-the-wheel", where every customer who enters the lucky draw wins a \$2 cash voucher with probability 0.8, wins a \$5 cash voucher with probability 0.15, wins a \$10 cash voucher with probability 0.049, and wins a \$1000 cash voucher with probability 0.001. Customers can only enter this lucky draw if they are the first 500 customers, and as long as nobody has won a \$1000 cash voucher yet. Once someone wins a \$1000 cash voucher, the lucky draw ends immediately after that. If none of the first 500 customers has won a \$1000 cash voucher, then the lucky draw ends. What is the expected total amount (in dollars) of cash vouchers given out for the lucky draw?

(Hint: What if you let customers spin the wheel "for fun", even after the lucky draw has ended?)

Question 3. Let X and Y be independent random variables, and suppose they have moment generating functions given by:

$$\psi_X(t) = e^{t^2+t}, \quad \text{for } -\infty < t < \infty;$$
$$\psi_Y(t) = e^{t^2-t}, \quad \text{for } -\infty < t < \infty;$$

respectively.

(i) What is the moment generating function of $2X + 3Y$?
(ii) What is the mean of $2X + 3Y$?
(iii) What is the variance of $2X + 3Y$?

Question 4. An automated juice vending machine dispenses orange juice, whose amount per cup follows a normal distribution with a mean of 250 ml and a standard deviation of 27 ml.

(i) What should the minimum cup size be (in ml), so that the orange juice dispensed would overflow at most 1% of the time?
(ii) If the cup size is fixed at 300 ml, what is the probability that the orange juice dispensed for one cup would overflow?

Question 5. A new animal species has recently been discovered in the heart of the Amazon rainforest. Let X be the random variable representing the weight (in grams) of this animal, and let Y be the random variable representing the length (in cm) of this animal. Suppose that X and Y have the bivariate normal distribution with means $\mu_X = 230$, $\mu_Y = 6.3$, variances $\sigma_X^2 = 3600$, $\sigma_Y^2 = 0.49$, and correlation $\rho = 0.65$.

(i) Suppose that a randomly selected animal of this species has length 7.0 cm. What is the probability that it has weight at least 280 g?

(ii) Suppose that a second randomly selected animal of this species has weight 200 g. What is its expected length?

(Hint: Did you read Section 5.10 of the course textbook?)