

Activating hidden metrological usefulness

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Introduction

- It has been realized that entanglement can be a useful resource in very general metrological tasks. Even bound entangled states can be more useful than separable states [1,2]. Such states are called "useful" in short. However, there are highly entangled states that are not useful for metrology [3].
- In the spirit of Ref. [4], we show that some bipartite entangled quantum states that are not useful in linear interferometers become useful if several copies are considered or ancillas are added [5].
- To support our claims, we present a general method to find the *local* Hamiltonian for which a given bipartite quantum state provides the largest gain compared to separable states. Note that this task is different, and in a sense more complex, than maximizing the quantum Fisher information [5].

Quantum Fisher information

- A basic metrological task in a *linear* interferometer is estimating the small angle θ for a unitary dynamics $U_\theta = \exp(-i\mathcal{H}\theta)$, where the Hamiltonian is the sum of *local* terms. For bipartite systems it is

$$\mathcal{H} = \mathcal{H}_1 + \mathcal{H}_2, \quad (1)$$

where \mathcal{H}_i are single-subsystem operators.

- Cramér-Rao bound:

$$(\Delta\theta)^2 \geq \frac{1}{m\mathcal{F}_Q[\rho, \mathcal{H}]}, \quad (2)$$

where m is the number of independent repetitions, and the quantum Fisher information is defined by the formula

$$\mathcal{F}_Q[\rho, \mathcal{H}] = 2 \sum_{k,l} \frac{(\lambda_k - \lambda_l)^2}{\lambda_k + \lambda_l} |\langle k | \mathcal{H} | l \rangle|^2. \quad (3)$$

Here, λ_k and $|k\rangle$ are the eigenvalues and eigenvectors, respectively, of the density matrix ρ , which is used as a probe state for estimating θ .

Metrological gain

- We define the metrological gain compared to separable states, for a given Hamiltonian, by [5]

$$g_{\mathcal{H}}(\rho) = \mathcal{F}_Q[\rho, \mathcal{H}] / \mathcal{F}_Q^{(\text{sep})}(\mathcal{H}), \quad (4)$$

where the separable limit for local Hamiltonians is

$$\mathcal{F}_Q^{(\text{sep})}(\mathcal{H}) = \sum_{n=1,2} [\sigma_{\max}(\mathcal{H}_n) - \sigma_{\min}(\mathcal{H}_n)]^2. \quad (5)$$

- We are interested in the quantity [5]

$$g(\rho) = \max_{\text{local } \mathcal{H}} g_{\mathcal{H}}(\rho), \quad (6)$$

where a local Hamiltonian is just the sum of single system Hamiltonians as in Eq. (1).

- The maximization task looks challenging since we have to maximize a fraction, where both the numerator and the denominator depend on the Hamiltonian.

Ancilla

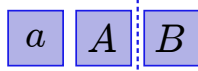
- For the 3×3 -case, we consider the maximally entangled state mixed with noise

$$\rho_{AB}^{(p)} = (1-p)|\Psi^{(\text{me})}\rangle\langle\Psi^{(\text{me})}| + p\mathbb{1}/d^2, \quad (7)$$

which is useful if $p < 0.3655$.

- If a pure ancilla qubit is added [5]

$$\rho^{(\text{anc})} = |0\rangle\langle 0|_a \otimes \rho_{AB}^{(p)}.$$



then the state is useful if $p < 0.3752$.

- The Hamiltonian is

$$\mathcal{H}^{(\text{anc})} = 1.2C_{aA} \otimes \mathbb{1}_B + \mathbb{1}_{aA} \otimes D_B, \quad (8)$$

where

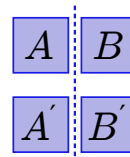
$$C_{aA} = \frac{9}{20} (2\sigma_x + \sigma_z)_a \otimes |0\rangle\langle 0|_A + \mathbb{1}_a \otimes (|2\rangle\langle 2|_A - |1\rangle\langle 1|_A),$$

$$D = \text{diag}(+1, -1, +1). \quad (9)$$

Two copies

- We consider now two copies of the noisy 3×3 maximally entangled state [5]

$$\rho^{(\text{tc})} = \rho_{AB}^{(p)} \otimes \rho_{A'B'}^{(p)}.$$



- Then, with the two-copy operator

$$\mathcal{H}^{(\text{tc})} = D_A \otimes D_{A'} \otimes \mathbb{1}_{BB'} + \mathbb{1}_{AA'} \otimes D_B \otimes D_{B'}, \quad (10)$$

the state is useful if $p < 0.4164$.

- The Hamiltonians presented are not the optimal ones.

- Let us look for the optimal Hamiltonians, for which $g_{\mathcal{H}}$ is the largest.

Optimal Hamiltonian

- Instead of the quantum Fisher information, let us consider the error propagation formula

$$(\Delta\theta)_M^2 = \frac{(\Delta M)^2}{\langle i[M, \mathcal{H}] \rangle^2}, \quad (11)$$

which provides a bound on the quantum Fisher information

$$\mathcal{F}_Q[\rho, \mathcal{H}] \geq 1/(\Delta\theta)_M^2. \quad (12)$$

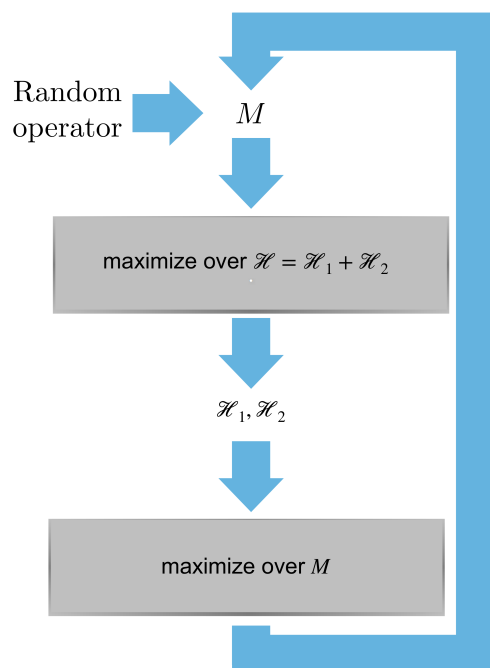
- We will minimize Eq. (11) using the idea [4]

$$\max_{\mathcal{H}} \mathcal{F}_Q[\rho, \mathcal{H}] = \max_{\mathcal{H}, M} \frac{\langle i[M, \mathcal{H}] \rangle^2}{(\Delta M)^2}. \quad (13)$$

Based on these, we realize a see-saw, optimizing alternately over \mathcal{H} and M .

	Analytic example	Numerics
Ancilla	0.3752	0.3941
Second copy	0.4164	0.4170

See-saw iteration



Pure states

- General case, pure state with a Schmidt decomposition

$$|\Psi\rangle = \sum_{k=1}^s \sigma_k |k\rangle_a |k\rangle_b,$$

where s is the Schmidt number, and the real positive σ_k Schmidt coefficients are in a descending order.

- Direct calculation yields [4]

$$\begin{aligned} \mathcal{F}_Q[|\Psi\rangle, \mathcal{H}_{AB}] &= 4(\Delta\mathcal{H}_{AB})_{\Psi}^2 \\ &= 8 \sum_{n=1,3,5,\dots,\tilde{s}-1} (\sigma_n + \sigma_{n+1})^2, \end{aligned}$$

which is larger than the separable bound, $\mathcal{F}_Q^{(\text{sep})} = 8$, whenever the Schmidt rank is larger than 1. Here, \tilde{s} is the largest even number for which $\tilde{s} \leq s$. (For the Hamiltonian \mathcal{H}_{AB} , see Ref. [5].)

- In the limit of infinite copies, all entangled bipartite pure states are maximally useful [5].

Related bibliography

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