

Quantum metrology

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Outline

1 Motivation

- Why is quantum metrology interesting?

2 Simple examples of quantum metrology

- Classical case: Clock arm
- Quantum case: Single spin-1/2 particle
- Magnetometry with the fully polarized state
- Magnetometry with the spin-squeezed state
- Metrology with the GHZ state
- Dicke states
- Interferometry with squeezed photonic states

3 Entanglement theory

- Multipartite entanglement
- The spin-squeezing criterion

4 Quantum metrology using the quantum Fisher information

- Quantum Fisher information
- Quantum Fisher information in linear interferometers
- Noise and imperfections

Why is quantum metrology interesting?

- Recent technological development has made it possible to realize large coherent quantum systems, i.e., in cold gases, trapped cold ions or photons.
- Can such quantum systems outperform classical systems in something useful, i.e., metrology?

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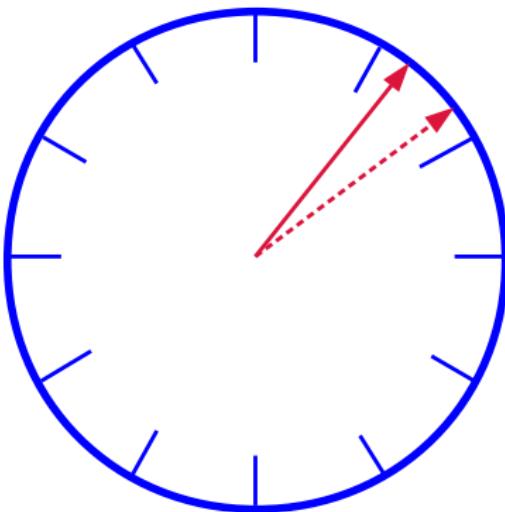
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Classical case: Estimating the angle of a clock arm

- Arbitrary precision ("in principle").



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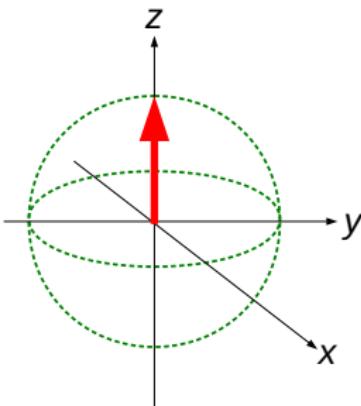
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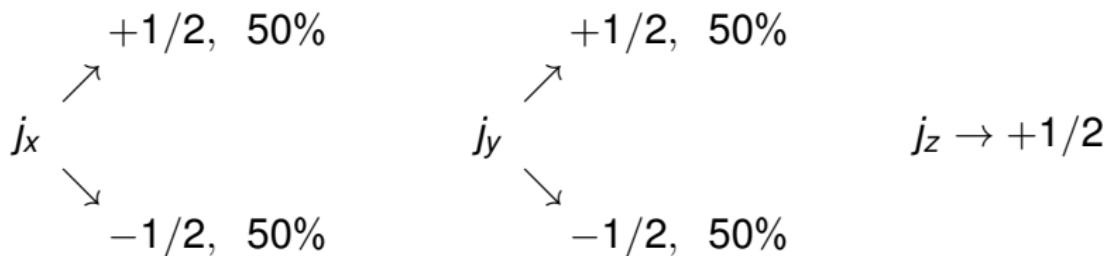
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Quantum case: A single spin-1/2 particle

- Spin-1/2 particle polarized in the z direction.



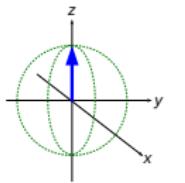
- We measure the spin components.



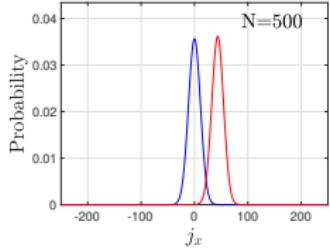
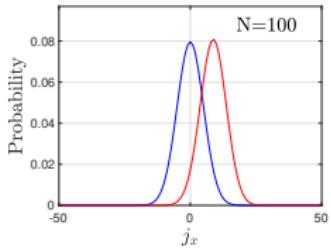
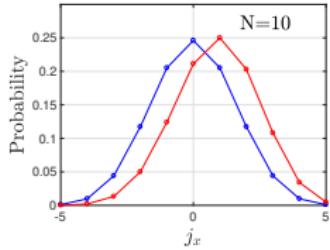
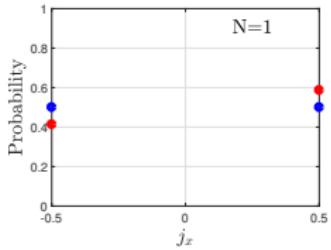
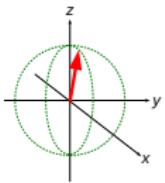
Quantum case: A single spin-1/2 particle II

- We cannot measure the three spin coordinates exactly j_x, j_y, j_z .
- In quantum physics, we can get only discrete outcomes in measurement. In this case, $+1/2$ and $-1/2$.
- A single spin-1/2 particle is not a good clock arm.

Several spin-1/2 particles



10° rotation
around y
⇒



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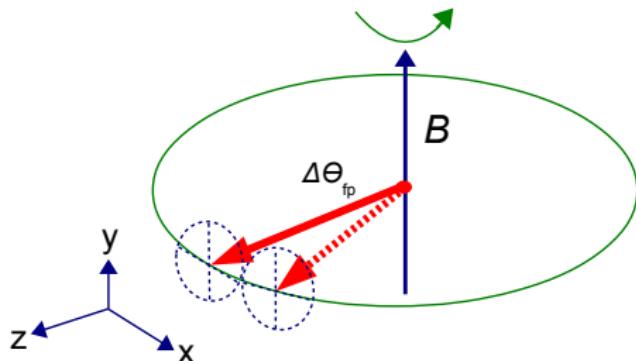
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Magnetometry with the fully polarized state

- N spin-1/2 particles, all fully polarized in the z direction.
- Magnetic field B points to the y direction.



- Note the uncertainty ellipses. $\Delta\theta_{fp}$ is the minimal angle difference we can measure.

Magnetometry with the fully polarized state II

- Collective angular momentum components

$$J_l := \sum_{n=1}^N j_l^{(n)}$$

for $l = x, y, z$, where $j_l^{(n)}$ are single particle operators.

- Dynamics

$$|\Psi\rangle = U_\theta |\Psi_0\rangle, \quad U_\theta = e^{-iJ_y\theta},$$

where $\hbar = 1$.

- Rotation around the y -axis.

Magnetometry with the fully polarized state III

- Let us assume, that we have an $M(\Theta)$ function.
- We know that there is an ΔM error in M .
- How much is the error $\Delta\theta$ in θ ?
- It is given by the classical **error propagation formula**:

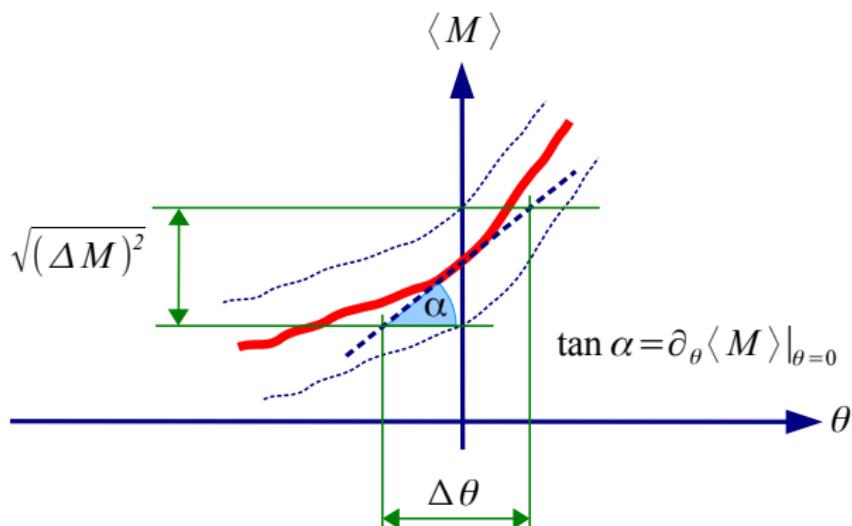
$$\Delta\theta \approx \frac{\Delta M}{dM/d\theta}.$$

- It tells us how the error in M "propagates" to θ .

Magnetometry with the fully polarized state IV

- Measure an operator M to get the estimate θ .
- To obtain the precision of estimation, we can use the **error propagation formula**

$$(\Delta\theta)_M^2 = \frac{(\Delta M)^2}{|\partial_\theta \langle M \rangle|^2} = \frac{(\Delta M)^2}{|\langle i[M, H] \rangle|^2}.$$



Magnetometry with the fully polarized state V

- In order to see the full picture, we need to consider ν measurements of M .
- We have to look for the average of the measured values

$$\bar{m} = \sum_{n=1}^{\nu} m_k.$$

- Then, if the measured probability distributions fulfill certain conditions, we can estimate the parameter with a precision

$$(\Delta\theta)^2 = \frac{1}{\nu} (\Delta\theta)_M^2 = \frac{1}{\nu} \frac{(\Delta M)^2}{|\partial_\theta \langle M \rangle|^2}.$$

[L. Pezze, A. Smerzi, M. K. Oberthaler, R. Schmied, and P. Treutlein,
"Quantum metrology with nonclassical states of atomic ensembles,"
Rev. Mod. Phys. 90, 035005 (2018).]

Magnetometry with the fully polarized state VI

- We consider the fully polarized states of N spin-1/2 particles

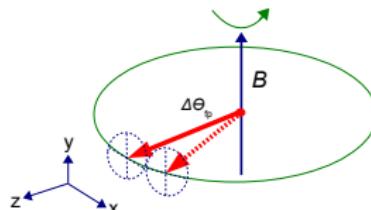
$$| +\frac{1}{2} \rangle^{\otimes N}.$$

- For this state,

$$\langle J_z \rangle = \frac{N}{2}, \quad \langle J_x \rangle = 0, \quad (\Delta J_x)^2 = \frac{N}{4}, \quad \langle J_z \rangle = \frac{N}{2} \cos \theta, \quad \langle J_x \rangle = \frac{N}{2} \sin \theta.$$

- We measure the operator

$$M = J_x.$$



- It is not like a classical clock arm, we have a nonzero uncertainty

$$(\Delta \theta)^2 = \frac{1}{\nu} \frac{(\Delta M)^2}{|\partial_\theta \langle M \rangle|^2} = \frac{1}{\nu} \frac{(\Delta J_x)^2}{\langle J_z \rangle^2} = \frac{1}{\nu} \frac{1}{N}.$$

Magnetometry with the fully polarized state VII

- Main result:

$$(\Delta\theta)^2 = \frac{1}{\nu N}.$$

- In some cold gas experiment, we can have $10^3 - 10^{12}$ particles.
- Later we will see that with a separable quantum state we cannot have a better precision.

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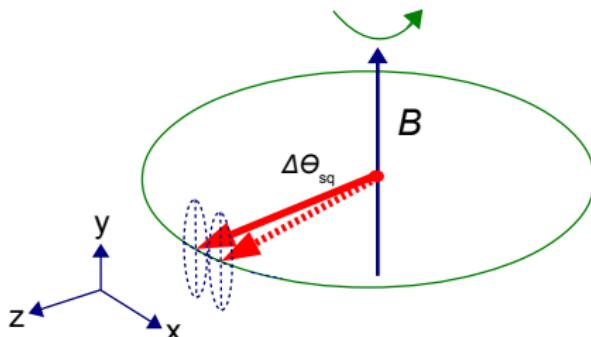
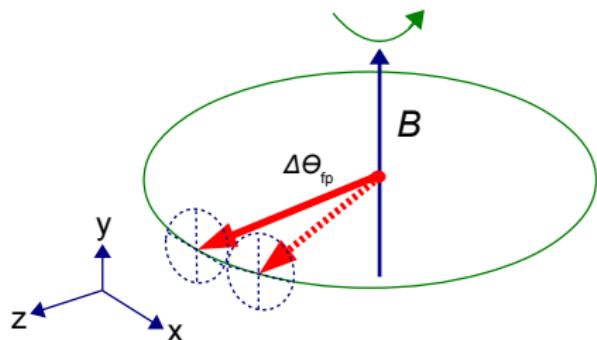
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Magnetometry with the spin-squeezed state

- We can increase the precision by spin squeezing



fully polarized state (fp)

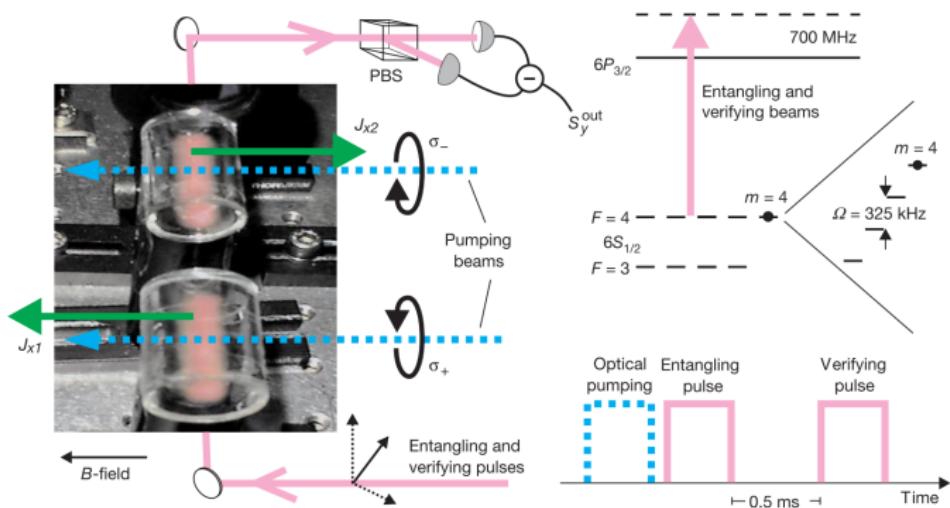
spin-squeezed state (sq)

$\Delta\theta_{fp}$ and $\Delta\theta_{sq}$ are the minimal angle difference we can measure.

We can reach

$$(\Delta\theta)^2 < \frac{1}{\nu N}.$$

Spin squeezing in an ensemble of atoms via interaction with light

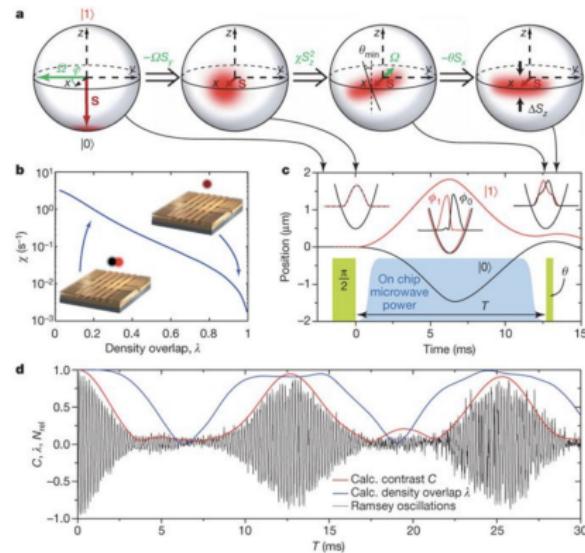


10^{12} atoms, room temperature.

Julsgaard, Kozhekin, Polzik, Nature 2001.

Spin squeezing in a Bose-Einstein Condensate via interaction between the particles

Figure 1: Spin squeezing and entanglement through controlled interactions on an atom chip.



M. F. Riedel, P. Böhi, Y. Li, T. W. Hänsch, A. Sinatra, and P. Treutlein,
Nature 464, 1170-1173 (2010).

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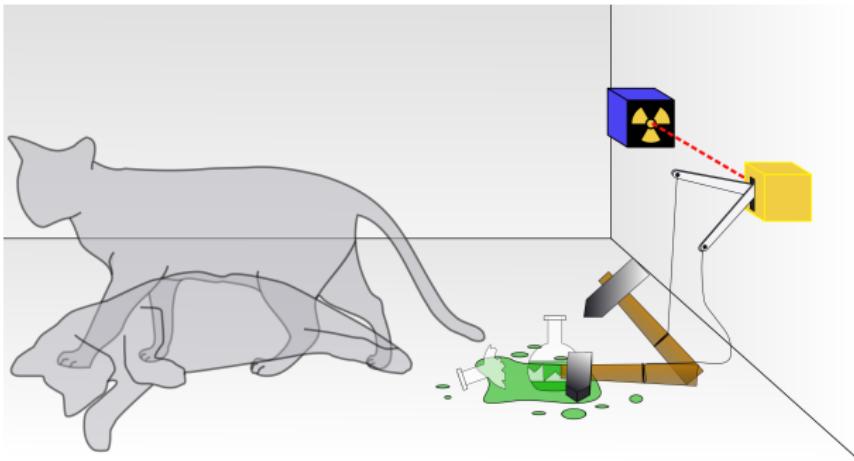
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GHZ state=Schrödinger cat state

- A superposition of two macroscopically distinct states



GHZ state

Greenberger-Horne-Zeilinger (GHZ) state

$$|GHZ_N\rangle = \frac{1}{\sqrt{2}}(|000\dots 00\rangle + |111\dots 11\rangle).$$

- Superposition of all atoms in state "0" and all atoms in state "1".

Metrology with the GHZ state

- Greenberger-Horne-Zeilinger (GHZ) state

$$|\text{GHZ}_N\rangle = \frac{1}{\sqrt{2}}(|000\dots 00\rangle + |111\dots 11\rangle),$$

- Unitary

$$|\Psi\rangle(\theta) = U_\theta |\text{GHZ}_N\rangle, \quad U_\theta = e^{-iJ_z\theta}.$$

- Dynamics

$$|\Psi\rangle(\theta) = \frac{1}{\sqrt{2}}(|000\dots 00\rangle + e^{-i\textcolor{red}{N}\theta}|111\dots 11\rangle),$$

Metrology with the GHZ state II

- We measure

$$M = \sigma_x^{\otimes N},$$

which is the parity in the x -basis.

- Expectation value and variance

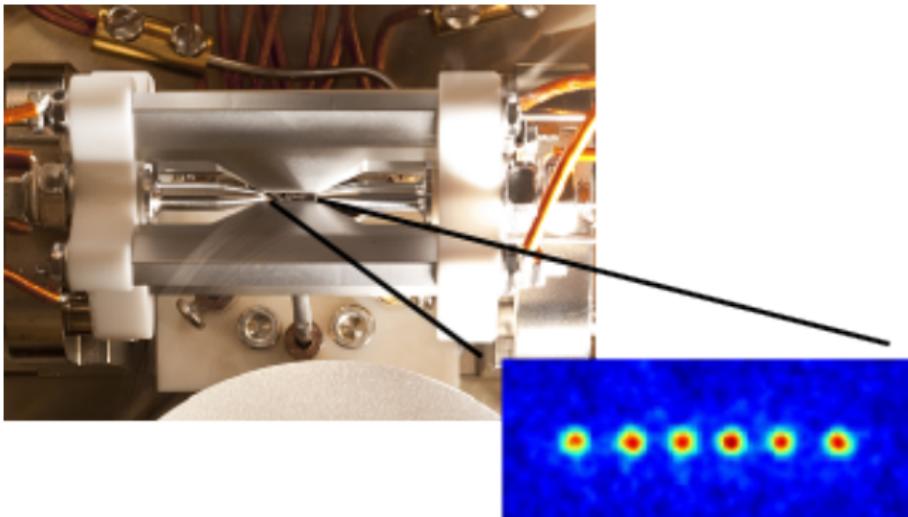
$$\langle M \rangle = \cos(\textcolor{red}{N}\theta), \quad (\Delta M)^2 = \sin^2(\textcolor{red}{N}\theta).$$

- For $\theta \approx 0$, the precision is

$$(\Delta\theta)^2 = \frac{1}{\nu} \frac{(\Delta M)^2}{|\partial_\theta \langle M \rangle|^2} = \frac{1}{\nu N^2}.$$

[e.g., photons: D. Bouwmeester, J. W. Pan, M. Daniell, H. Weinfurter and A. Zeilinger, Phys. Rev. Lett. 82, 1345 (1999);
ions: C. Sackett et al., Nature 404, 256 (2000).]

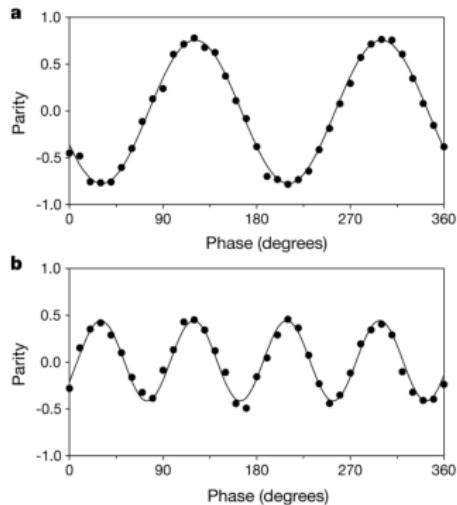
Metrology with the GHZ state III



Quantum Computation with Trapped Ions, Innsbruck

Metrology with the GHZ state IV

Figure 2: Determination of $p_{(\uparrow\downarrow)}$.



a, Interference signal for two ions; b, four ions. After the entanglement operation of Fig. 1, an analysis pulse with relative phase φ is applied on the single-ion $|+\rangle \leftrightarrow |-\rangle$ transition. As φ is varied, the parity of the N ions oscillates as $\cos N\varphi$, and the amplitude of the oscillation is twice the magnitude of the density-matrix element $p_{(\uparrow\downarrow)}$. Each data point represents an average of 1,000 experiments, corresponding to a total integration time of roughly 10 s for each graph.

For four ions the curve oscillates faster than for two ions.
ions: C. Sackett et al., Nature 404, 256 (2000).

Metrology with the GHZ state IV

- We reached the Heisenberg-limit

$$(\Delta\theta)^2 = \frac{1}{\nu N^2}.$$

- The fully polarized state reached only the shot-noise limit

$$(\Delta\theta)^2 = \frac{1}{\nu N}.$$

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Dicke states

- Symmetric Dicke states with $\langle J_z \rangle = 0$ (simply “Dicke states” in the following) are defined as

$$|D_N\rangle = \left(\frac{N}{\frac{N}{2}}\right)^{-\frac{1}{2}} \sum_k \mathcal{P}_k \left(|0\rangle^{\otimes \frac{N}{2}} \otimes |1\rangle^{\otimes \frac{N}{2}} \right).$$

- E.g., for four qubits they look like

$$|D_4\rangle = \frac{1}{\sqrt{6}} (|0011\rangle + |0101\rangle + |1001\rangle + |0110\rangle + |1010\rangle + |1100\rangle).$$

[photons: Kiesel, Schmid, GT, Solano, Weinfurter, PRL 2007;
Prevedel, Cronenberg, Tame, Paternostro, Walther, Kim, Zeilinger, PRL 2007;
Wieczorek, Krischek, Kiesel, Michelberger, GT, and Weinfurter, PRL 2009]
cold atoms: Lücke *et al.*, Science 2011; Hamley *et al.*, Science 2011; C. Gross *et al.*,
Nature 2011

Metrology with Dicke states. Clock arm = noise

- For our symmetric Dicke state

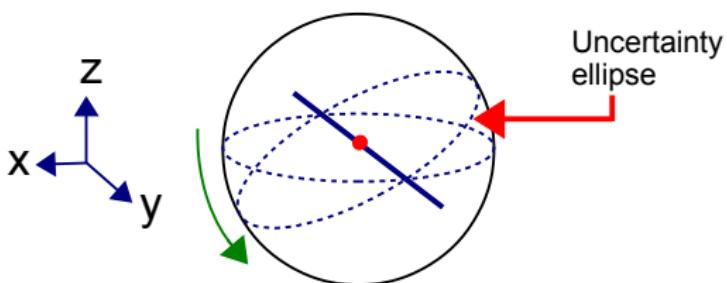
$$\langle J_l \rangle = 0, l = x, y, z, \quad \langle J_z^2 \rangle = 0, \quad \langle J_x^2 \rangle = \langle J_y^2 \rangle = \text{large}.$$

- Linear metrology

$$U = \exp(-iJ_y\theta).$$

- Measure $\langle J_z^2 \rangle$ to estimate θ .

(We cannot measure first moments, since they are zero.)



Metrology with Dicke states

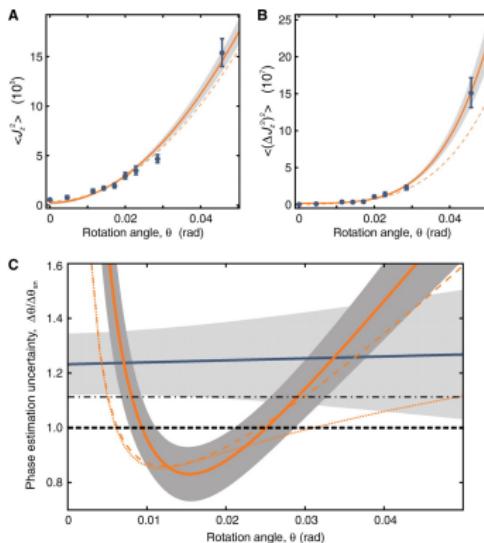
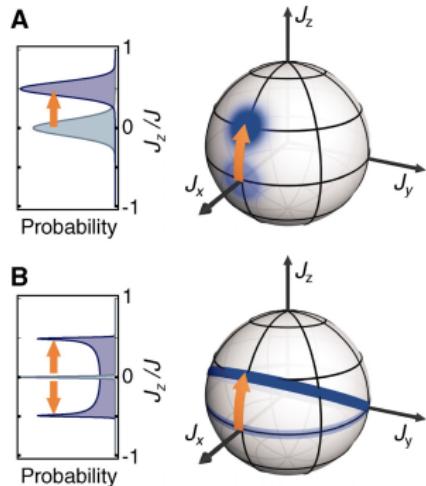
- Dicke states are more **robust to noise** than GHZ states. (Even if they loose a particle, they remain entangled).
- Dicke states can also reach the **Heisenberg-scaling** like GHZ states.

Metrology with cold gases: B. Lücke, M Scherer, J. Kruse, L. Pezze, F. Deuretzbacher, P. Hyllus, O. Topic, J. Peise, W. Ertmer, J. Arlt, L. Santos, A. Smerzi, C. Klempert, Science 2011.

Metrology with photons: R. Krischek, C. Schwemmer, W. Wieczorek, H. Weinfurter, P. Hyllus, L. Pezze, A. Smerzi, PRL 2011.

Metrology with Dicke states II

Experiment with cold gas of 8000 atoms.



Lücke M. Scherer, Kruse, Pezzé, Deuretzbacher, Hyllus, Topic, Peise, Ertmer, Arlt, Santos, Smerzi, Klempt, Science 2011.

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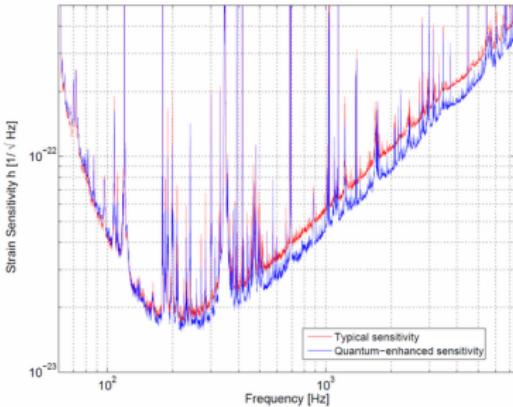
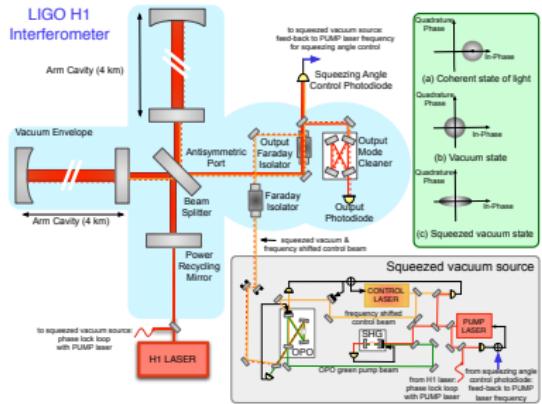
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LIGO gravitational wave detector

The performance was enhanced with squeezed light.



The role of clock arm is played by the squeezed coherent state.

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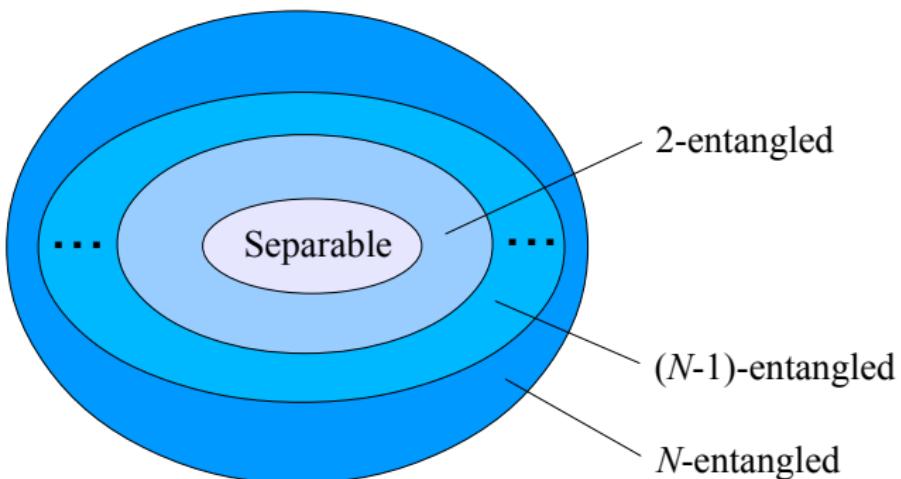
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k -producibility/ k -entanglement



$$(|00\rangle + |11\rangle) \otimes (|00\rangle + |11\rangle) \otimes (|00\rangle + |11\rangle) \quad 2\text{-entangled}$$

$$(|000\rangle + |111\rangle) \otimes (|000\rangle + |111\rangle) \quad 3\text{-entangled}$$

$$(|0000\rangle + |1111\rangle) \otimes (|0\rangle + |1\rangle) \quad 4\text{-entangled}$$

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Multipartite entanglement in spin squeezing

- We consider pure k -producible states of the form

$$|\Psi\rangle = \otimes_{I=1}^M |\psi_I\rangle,$$

where $|\psi_I\rangle$ is the state of at most k qubits.

Extreme spin squeezing

The spin-squeezing criterion for k -producible states is

$$(\Delta J_x)^2 \geq J_{\max} F_{\frac{k}{2}} \left(\frac{\sqrt{\langle J_y \rangle^2 + \langle J_z \rangle^2}}{J_{\max}} \right),$$

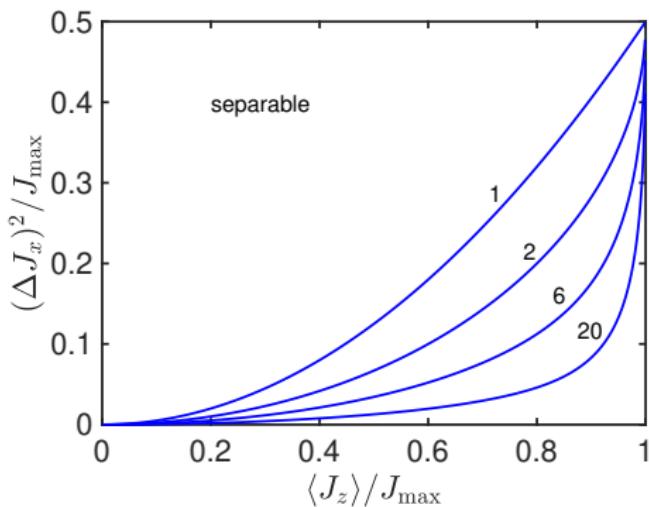
where $J_{\max} = \frac{N}{2}$ and we use the definition

$$F_j(Z) := \frac{1}{j} \min_{\frac{\langle j_z \rangle}{j} = Z} (\Delta j_x)^2.$$

Sørensen and Mølmer, Phys. Rev. Lett. 86, 4431 (2001); experimental test:
Gross, Zibold, Nicklas, Esteve, Oberthaler, Nature 464, 1165 (2010).

Multipartite entanglement in spin squeezing

- Larger and larger multipartite entanglement is needed to larger and larger squeezing ("extreme spin squeezing").



- $N = 100$ spin-1/2 particles, $J_{\max} = N/2$.

Sørensen and Mølmer, Phys. Rev. Lett. 86, 4431 (2001); experimental test: Gross, Zibold, Nicklas, Esteve, Oberthaler, Nature 464, 1165 (2010).

Our experience so far

- We find that more spin squeezing/better precision needs more entanglement.
- Question: Is this general?
- Answer: Yes.

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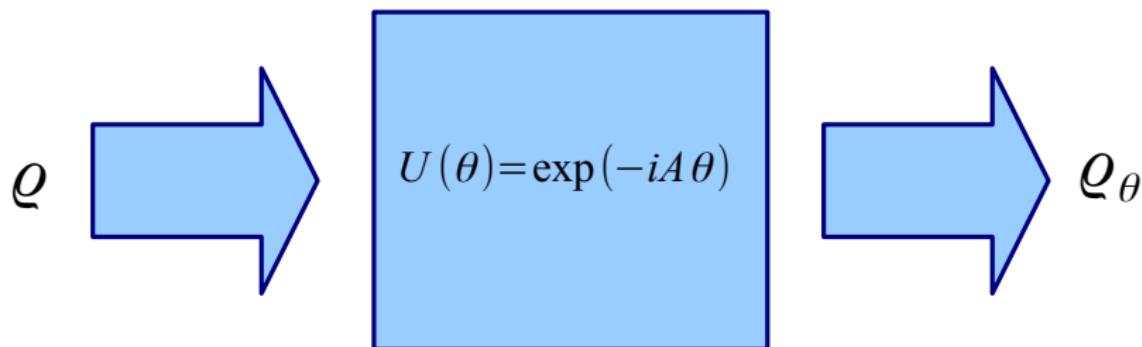
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Quantum metrology

- Fundamental task in metrology



- We have to estimate θ in the dynamics

$$U = \exp(-iA\theta).$$

The quantum Fisher information

Cramér-Rao bound on the precision of parameter estimation

For the variance of the parameter estimation

$$(\Delta\theta)^2 \geq \frac{1}{\nu F_Q[\varrho, A]}$$

holds, where ν is the number of repetitions and $F_Q[\varrho, A]$ is the **quantum Fisher information**.

- The bound includes any estimation strategy, even POVM's.
- The quantum Fisher information is

$$F_Q[\varrho, A] = 2 \sum_{k,l} \frac{(\lambda_k - \lambda_l)^2}{\lambda_k + \lambda_l} |\langle k | A | l \rangle|^2,$$

where $\varrho = \sum_k \lambda_k |k\rangle\langle k|$.

Convexity of the quantum Fisher information

- For pure states, it equals four times the variance,

$$F_Q[|\Psi\rangle, A] = 4(\Delta A)^2_{\Psi}.$$

- For mixed states, it is convex

$$F_Q[\varrho, A] \leq \sum_k p_k F_Q[|\Psi_k\rangle, A],$$

where

$$\varrho = \sum_k p_k |\Psi_k\rangle\langle\Psi_k|.$$

Quantum Fisher information - Some basic facts

- The larger the quantum Fisher information, the larger the achievable precision.
- For the totally mixed state it is zero for any A

$$F_Q[\varrho_{\text{cm}}, A] = 0,$$

where $\varrho_{\text{cm}} = \mathbb{1}/d$ is the completely mixed state and d is the dimension.

- This is logical: the completely mixed states does not change under any Hamiltonian.
- For any state ϱ that commutes with A , i.e., $\varrho A - A\varrho = 0$ we have

$$F_Q[\varrho, A] = 0.$$

Best operator to measure

- The error propagation formula is bounded from above as

$$(\Delta\theta)_M^2 = \frac{(\Delta M)^2}{|\partial_\theta \langle M \rangle|^2} \geq \frac{1}{F_Q[\varrho, H]}.$$

- The inequality is saturated for the following operator, i.e., the **symmetric logarithmic derivative**

$$L = 2i \sum_{k,l} \frac{\lambda_k - \lambda_l}{\lambda_k + \lambda_l} |k\rangle\langle l| \langle k|H|l\rangle,$$

where $\varrho = \sum_k \lambda_k |k\rangle\langle k|$.

- The QFI can be obtained as

$$F_Q[\varrho, H] = \text{Tr}(L^2 \varrho).$$

QFI as a convex roof

The quantum Fisher information is the convex roof of the variance times four

$$F_Q[\varrho, A] = 4 \min_{\{p_k, \Psi_k\}} \sum_k p_k (\Delta A)^2_{\Psi_k},$$

where

$$\varrho = \sum_k p_k |\Psi_k\rangle\langle\Psi_k|.$$

GT, D. Petz, Phys. Rev. A 87, 032324 (2013); S. Yu, arXiv1302.5311 (2013);
GT, I. Apellaniz, J. Phys. A: Math. Theor. 47, 424006 (2014)

- It is like entanglement measures, e. g., Entanglement of Formation.
- Convex roof over purifications.

R. Demkowicz-Dobrzański, J. Kołodyński, M. Guță, Nature Comm. 2012.

The variance as a concave roof

The variance is the concave roof of itself

$$(\Delta A)^2_{\varrho} = \sup_{\{p_k, \Psi_k\}} \sum_k p_k (\Delta A)^2_{\Psi_k},$$

where

$$\varrho = \sum_k p_k |\Psi_k\rangle\langle\Psi_k|.$$

GT, D. Petz, Phys. Rev. A 87, 032324 (2013).

Summary of statements

- Decompose ϱ as

$$\varrho = \sum_k p_k |\Psi_k\rangle\langle\Psi_k|.$$

- Then,

$$\frac{1}{4} F_Q[\varrho, A] \leq \sum_k p_k (\Delta A)_{\Psi_k}^2 \leq (\Delta A)_{\varrho}^2$$

holds.

- Both inequalities can be saturated by some decomposition.

Uncertainty relation with the QFI

- We use that for pure states

$$(\Delta A)^2_{\varrho} (\Delta B)^2_{\varrho} \geq \frac{1}{4} |\langle C \rangle_{\varrho}|^2$$

holds with $C = i[A, B]$.

- From the Cauchy-Schwarz inequality

$$\left[\sum_k p_k (\Delta A)^2_{\Psi_k} \right] \left[\sum_k p_k (\Delta B)^2_{\Psi_k} \right] \geq \frac{1}{4} \left[\sum_k p_k |\langle C \rangle_{\Psi_k}|^2 \right]^2.$$

- With a convex roof, we obtain a relation tighter than the Heisenberg uncertainty

$$(\Delta A)^2_{\varrho} F_Q[\varrho, B] \geq |\langle C \rangle_{\varrho}|^2. \quad \text{Note: } 4(\Delta B)^2_{\varrho} \geq F_Q[\varrho, B].$$

Presented in Fröwis, Schmied, Gisin PRA 2018;
derived in this way GT, Fröwis, PR Research 2022.

Uncertainty relation with the QFI II

- For pure states

$$(\Delta J_x)^2_{\varrho} + (\Delta J_y)^2_{\varrho} + (\Delta J_z)^2_{\varrho} \geq j.$$

- Then, we obtain

$$\sum_k p_k (\Delta J_x)^2_{\psi_k} + \sum_k p_k (\Delta J_y)^2_{\psi_k} + \sum_k p_k (\Delta J_z)^2_{\psi_k} \geq j.$$

- With a convex roof, we obtain a relation stronger than the Heisenberg uncertainty

$$(\Delta J_x)^2_{\varrho} + (\Delta J_y)^2_{\varrho} + \frac{F_Q[\varrho, J_z]}{4} \geq j. \quad \text{Note: } (\Delta J_z)^2_{\varrho} \geq \frac{F_Q[\varrho, J_z]}{4}.$$

QFI as an optimization over separable states

- The QFI can be obtained as an optimization over separable states,

$$\begin{aligned} F_Q[\varrho, H] = \frac{1}{8} \min_{\varrho_{12}} \quad & \text{Tr}[(H \otimes \mathbb{1} - \mathbb{1} \otimes H)^2 \varrho_{12}], \\ \text{s. t.} \quad & \varrho_{12} \in \mathcal{S}, \\ & \text{Tr}_2(\varrho_{12}) = \varrho, \\ & \text{Tr}_1(\varrho_{12}) = \varrho, \end{aligned}$$

where \mathcal{S} is the set of separable states.

GT and J. Pitrik, Quantum 7, 914 (2023);

GT, Moroder, Gühne, PRL 2015;

GT, Petz, PRA 2012.

Quantum Fisher information and the fidelity

The quantum Fisher information appears in the Taylor expansion of F_B

$$F_B(\varrho, \varrho_\theta) = 1 - \frac{\theta^2 F_Q[\varrho, A]}{4} + \mathcal{O}(\theta^3),$$

where

$$\varrho_\theta = \exp(-iA\theta)\varrho\exp(+iA\theta).$$

- Bures fidelity

$$F_B(\varrho_1, \varrho_2) = \text{Tr} \left(\sqrt{\sqrt{\varrho_1}\varrho_2\sqrt{\varrho_1}} \right)^2.$$

- Clearly,

$$0 \leq F_B(\varrho_1, \varrho_2) \leq 1.$$

The fidelity is 1 only if $\varrho_1 = \varrho_2$.

Outline

1 Motivation

- Why is quantum metrology interesting?

2 Simple examples of quantum metrology

- Classical case: Clock arm
- Quantum case: Single spin-1/2 particle
- Magnetometry with the fully polarized state
- Magnetometry with the spin-squeezed state
- Metrology with the GHZ state
- Dicke states
- Interferometry with squeezed photonic states

3 Entanglement theory

- Multipartite entanglement
- The spin-squeezing criterion

4 Quantum metrology using the quantum Fisher information

- Quantum Fisher information
- Quantum Fisher information in linear interferometers
- Noise and imperfections

Magnetometry with a linear interferometer

- The Hamiltonian A is defined as

$$A = J_I = \sum_{n=1}^N j_I^{(n)}, \quad I \in \{x, y, z\}.$$

There are no interaction terms.

- The dynamics rotates all spins in the same way.

Quantum Fisher information for separable states

- Let us consider a pure product state of N qubits

$$|\Psi\rangle_{\text{prod}} = |\Psi^{(1)}\rangle \otimes |\Psi^{(2)}\rangle \otimes \dots \otimes |\Psi^{(N)}\rangle.$$

- Since this is a pure state, we have $F_Q[\varrho, J_I] = 4(\Delta J_I)^2|_{|\Psi\rangle_{\text{prod}}}.$
- Then, for the product state we have

$$(\Delta J_I)^2|_{|\Psi\rangle_{\text{prod}}} = \sum_{n=1}^N (\Delta j_I^{(n)})^2|_{|\Psi^{(n)}\rangle} \leq N \times \frac{1}{4},$$

where we used that for qubits $(\Delta j_I^{(n)})^2 \leq 1/4$.

- Since the quantum Fisher information is convex in the state, the bound is also valid for a mixture of product states, i.e., separable states

$$F_Q[\varrho, J_I] \leq N.$$

The quantum Fisher information vs. entanglement

- For separable states of N spin-1/2 particles (qubits)

$$F_Q[\varrho, J_l] \leq N, \quad l = x, y, z.$$

Pezze, Smerzi, Phys. Rev. Lett. 102, 100401 (2009);

Hyllus, Gühne, Smerzi, Phys. Rev. A 82, 012337 (2010)

- For states with at most k -qubit entanglement (k is divisor of N)

$$F_Q[\varrho, J_l] \leq kN.$$

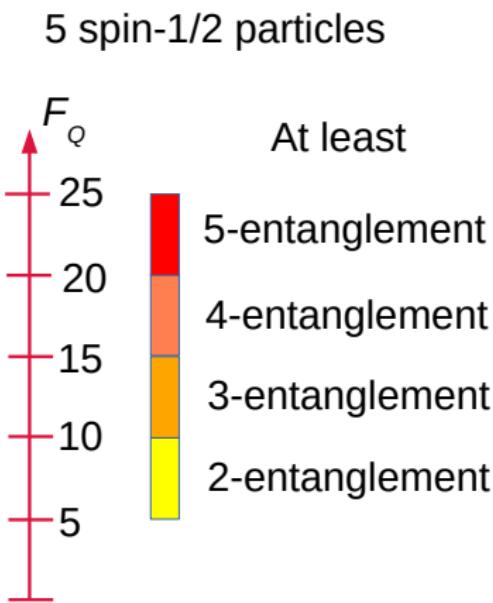
P. Hyllus *et al.*, Phys. Rev. A 85, 022321 (2012);

GT, Phys. Rev. A 85, 022322 (2012).

- Bound for all quantum states of N qubits

$$F_Q[\varrho, J_l] \leq N^2.$$

The quantum Fisher information vs. entanglement



(Using the $F_Q[\varrho, J_l] \leq kN$. Note that there is a slightly better bound.)

Let us use the Cramér-Rao bound

- For separable states

$$(\Delta\theta)^2 \geq \frac{1}{\nu N}, \quad l = x, y, z.$$

Pezze, Smerzi, Phys. Rev. Lett. 102, 100401 (2009);

Hyllus, Gühne, Smerzi, Phys. Rev. A 82, 012337 (2010)

- For states with at most k -particle entanglement (k is divisor of N)

$$(\Delta\theta)^2 \geq \frac{1}{\nu k N}.$$

P. Hyllus *et al.*, Phys. Rev. A 85, 022321 (2012);

GT, Phys. Rev. A 85, 022322 (2012).

- Bound for all quantum states

$$(\Delta\theta)^2 \geq \frac{1}{\nu N^2}.$$

Spin squeezing and QFI

Spin squeezing parameter

$$\xi_s^2 = N \frac{(\Delta J_x)^2}{\langle J_y \rangle^2 + \langle J_z \rangle^2}.$$

If $\xi_s^2 < 1$ then the state is entangled.

Sørensen, Duan, Cirac, Zoller, Nature 2001.

Based on QFI

$$\chi^2 = \frac{N}{F[\varrho, J_z]}.$$

If $\chi^2 < 1$ then the state is entangled.

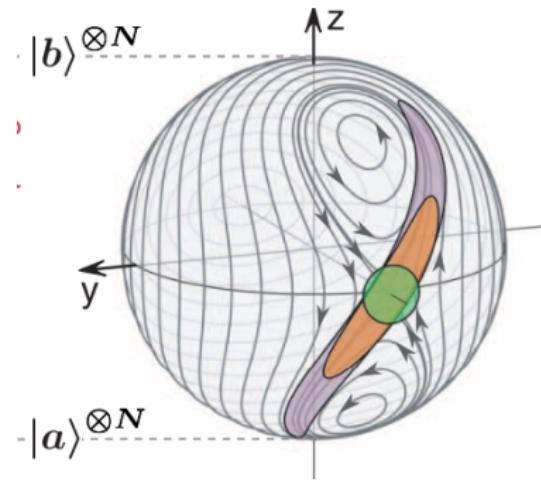
It can be proven that

$$\xi_s^2 \geq \chi^2.$$

Thus, χ^2 detects more states than ξ_s .

Pezze, Smerzi, PRL 2009.

Non-Gaussian entangled states



H. Strobel, W. Muessel, D. Linnemann, T. Zibold, D. B. Hume, L. Pezzè, A. Smerzi,
M. K. Oberthaler, Fisher information and entanglement of non-Gaussian spin states,
Science 2014.

Multiparameter metrology

The Cramér-Rao bound can be generalized to this case as

$$C - \nu^{-1} F^{-1} \geq 0,$$

where the inequality means that the left-hand side is a positive semidefinite matrix, C is now the **covariance matrix** with elements

$$C_{mn} = \langle \theta^{(m)} \theta^{(n)} \rangle - \langle \theta^{(m)} \rangle \langle \theta^{(n)} \rangle,$$

and F is the **Fisher matrix**, and its elements are given as

$$F_{mn} \equiv F_Q[\varrho, H^{(m)}, H^{(n)}].$$

Here, the quantum Fisher information with two operators is given as

$$F_Q[\varrho, A, B] = 2 \sum_{k,l} \frac{(\lambda_k - \lambda_l)^2}{\lambda_k + \lambda_l} \langle k | A | l \rangle \langle l | B | k \rangle.$$

Simple examples

- Commuting generators for measuring the homogenous field and the gradient

$$\begin{aligned} H_1 &= \sum_n J_z^{(n)}, \\ H_2 &= \sum_n x_n J_z^{(n)}. \end{aligned}$$

I. Apellaniz, I. Urizar-Lanz, Z. Zimborás, P. Hyllus, and GT, PRA 2018.

I. Urizar-Lanz, P. Hyllus, I. L. Egusquiza, M. W. Mitchell, and GT, Macroscopic singlet states for gradient magnetometry, PRA 2013.

- Measuring rotation angles around three directions

$$F_Q[\varrho, J_x] + F_Q[\varrho, J_y] + F_Q[\varrho, J_z] \leq N(N+2).$$

Hyllus *et al.*, Phys. Rev. A 85, 022321 (2012); GT, Phys. Rev. A 85, 022322 (2012).

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Noisy metrology: Simple example

- A particle with a state ϱ_1 passes through a map that turns its internal state to the fully mixed state with some probability p as

$$\epsilon_p(\varrho_1) = (1 - p)\varrho_1 + p\frac{1}{2}.$$

- This map acts in parallel on all the N particles

$$\epsilon_p^{\otimes N}(\varrho) = \sum_{n=0}^N p_n \varrho_n,$$

where the state obtained after n particles decohered into the completely mixed state is

$$\varrho_n = \frac{1}{N!} \sum_k \Pi_k \left[\left(\frac{1}{2} \right)^{\otimes n} \otimes \text{Tr}_{1,2,\dots,n}(\varrho) \right] \Pi_k^\dagger.$$

The summation is over all permutations Π_k . The probabilities are

$$p_n = \binom{N}{n} p^n (1-p)^{(N-n)}.$$

Noisy metrology: Simple example II

- Rewriting it

$$\epsilon_p^{\otimes N}(\varrho) = \sum_{n=0}^N p_n \frac{1}{N!} \sum_k \Pi_k \left[\left(\frac{1}{2} \right)^{\otimes n} \otimes \text{Tr}_{1,2,\dots,n}(\varrho) \right] \Pi_k^\dagger = \sum_n p_n \varrho_n.$$

- For the noisy state

$$(\Delta J_x)^2 \geq \sum_n p_n (\Delta J_x)^2_{\varrho_n} \geq \sum_n p_n \frac{n}{4} = \frac{pN}{4}.$$

- Hence, for the precision shot-noise scaling follows

$$(\Delta \theta)^2 = \frac{(\Delta J_x)^2}{\langle J_z \rangle^2} \geq \frac{\frac{pN}{4}}{\frac{N^2}{4}} = \frac{p}{N}.$$

Noisy metrology: General treatment

- In the most general case, uncorrelated single particle noise leads to shot-noise scaling after some particle number.

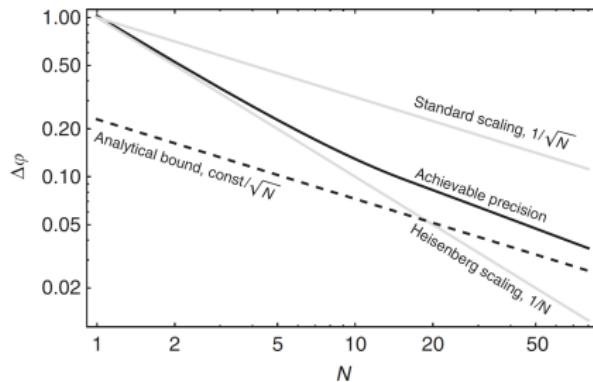


Figure from

R. Demkowicz-Dobrzański, J. Kołodyński, M. Guć, Nature Comm. 2012.

- Correlated noise is different.

Take home message

- Quantum physics makes it possible to obtain bounds for precision of the parameter estimation in realistic many-particle quantum systems.
- Shot-noise limit: Non-entangled states lead to $(\Delta\theta)^2 \geq \frac{1}{\nu N}$.
- Heisenberg limit: Fully entangled states can lead to $(\Delta\theta)^2 = \frac{1}{\nu N^2}$.
- At the end, noise plays a central role.

Reviews

- M. G. A. Paris, Quantum estimation for quantum technology, *Int. J. Quantum Inf.* 7, 125 (2009).
- V. Giovannetti, S. Lloyd, and L. Maccone, Advances in quantum metrology, *Nat. Photonics* 5, 222 (2011).
- C. Gross, Spin squeezing, entanglement and quantum metrology with Bose-Einstein condensates, *J. Phys. B: At., Mol. Opt. Phys.* 45, 103001 (2012).
- R. Demkowicz-Dobrzanski, M. Jarzyna, and J. Kolodynski, Chapter four-quantum limits in optical interferometry, *Prog. Opt.* 60, 345 (2015).
- L. Pezze, A. Smerzi, M. K. Oberthaler, R. Schmied, and P. Treutlein, Non-classical states of atomic ensembles: fundamentals and applications in quantum metrology, *Rev. Mod. Phys.* 90, 035005 (2018).

Summary

- We reviewed quantum metrology from a quantum information point of view.

See:

Géza Tóth and Iagoba Apellaniz,

Quantum metrology from a quantum information science perspective,

J. Phys. A: Math. Theor. 47, 424006 (2014),
special issue "50 years of Bell's theorem"
(open access).

Please see the slides at www.gtoth.eu.

