# Quantum Wasserstein distance based on an optimization over separable states arXiv.2209.09925

G. Tóth[1,2,3,4,5] and J. Pitrik[5,6] 

[1]University of the Basque Country UPV/EHU, Bilbao, Spain
[2]EHU Quantum Center, University of the Basque Country UPV/EHU, Spain
[3]Donostia International Physics Center (DIPC), San Sebastián, Spain
[4]IKERBASQUE, Basque Foundation for Science, Bilbao, Spain
[5]Wigner Research Centre for Physics, Budapest, Hungary
[5]Alfréd Rényi Institute of Mathematics, Budapest, Hungary
[6]Department of Analysis,Budapest University of Technology and Economics, Budapest, Hungary

APS Spring meeting, 21 March 2023


F.Y.R.O.M. - The Former Yugoslav Republic of Macedonia
Greenland (DENMARK)
Jan Mayen (NORWAY)
Greenland Sea
Denmark Strait
Barents Sea
Norwegian Sea
Arctic Circle
Reykjavik
ICELAND
NORWAY
SWEDEN
FINLAND
RUSSIA
Hammerfest
Tromsø
Murmansk
Kiruna
White Sea
Arkhangel
Luleå
Oulu
Umeå
Trondheim
Gulf of Bothnia
Tampere
Lake Onega
Lake Ladoga
Tórshavn
Faroe Islands (DENMARK)
SHETLAND ISLANDS
ORKNEY ISLANDS
Bergen
Oslo
Gävle
Turku
Helsinki
St. Petersburg
Gulf of Finland
Stockholm
ALAND ISLANDS
Tallinn
ESTONIA
Rockall (U.K.)
HEBRIDES
Stavanger
North Atlantic Ocean
Aberdeen
Göteborg
Gotland
LATVIA
Riga
Glasgow
Edinburgh
North Sea
DENMARK
Copenhagen
Malmö
Baltic Sea
Öland
LITHUANIA
Vilnius
Vitsyebsk
Smolensk
Belfast
UNITED KINGDOM
Kaliningrad
RUSSIA
Minsk
Mahilyow
BELARUS
Dublin
IRELAND
Irish Sea
Isle of Man (U.K.)
Liverpool
Leeds
Manchester
Bornholm
Gdańsk
Hrodna
Homyel
Hamburg
Bremen
Berlin
Poznań
Warsaw
POLAND
Łódź
Brest
Birmingham
Cardiff
London
Amsterdam
NETH.
Rotterdam
Essen
Cologne
Leipzig
Wrocław
Kiev
Rivne
Celtic Sea
English Channel
Brussels
BEL.
Lille
Bonn
GERMANY
Prague
CZECH REPUBLIC
Brno
Kraków
L'viv
UKRAINE
Guernsey (U.K.)
Jersey (U.K.)
Frankfurt am Main
Luxembourg
LUX.
SLOVAKIA
Chernivtsi
Mykolaiv
Paris
Stuttgart
Bratislava
Chisinau
Strasbourg
Munich
Vienna
AUSTRIA
Budapest
Iaşi
MOLDOVA
Cluj-Napoca
Nantes
LIECH.
Vaduz
HUNGARY
ROMANIA
Zürich
Bern
SWITZ.
Geneva
FRANCE
Bay of Biscay
Lyon
Milan
Venice
SLOVENIA
Ljubljana
Zagreb
Bucharest
A Coruña
Bordeaux
Turin
CROATIA
BOSNIA AND HERZEGOVINA
Belgrade
Varna
Bilbao
Toulouse
Genoa
MONACO
SAN MARINO
Sarajevo
SERBIA AND MONTENEGRO
BULGARIA
Sofia
Andorra la Vella
ANDORRA
Marseille
Ligurian Sea
Florence
Porto
Zaragoza
Corsica
ITALY
Adriatic Sea
Skopje
F.Y.R.O.M.
Istanbul
PORTUGAL
Madrid
Barcelona
Rome
VATICAN CITY
Tirana
ALB.
Thessaloniki
Lisbon
SPAIN
Valencia
Balearic Sea
Naples
Tyrrhenian Sea
Aegean Sea
GREECE
Athens
Sardinia
BALEARIC ISLANDS
Sevilla
Mediterranean Sea
Cagliari
Palermo
Ionian Sea
Gibraltar (U.K.)
Málaga
Strait of Gibraltar
Sicily
Ceuta (SPAIN)
Alborán Sea
Melilla (SPAIN)
Oran
Algiers
Tunis
TUNISIA
Valletta
MALTA
Crete
Rabat
Casablanca
MOROCCO
ALGERIA
Scale 1: 19,500,000
Lambert Conformal Conic Projection, standard parallels 40°N and 56°N
300 Kilometers
300 Miles


# Outline

1. **Motivation**
   - Connecting Wasserstein distance to entanglement theory?

2. **Background**
   - Quantum Wasserstein distance
   - Quantum Fisher information

3. **Wasserstein distance and separable states**
   - Quantum Wasserstein distance based on an optimization separable states
   - Relation to entanglement conditions

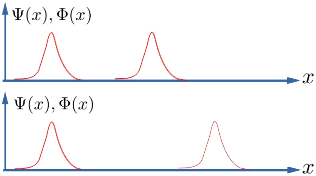
# Motivation

- Many distance measures are maximal for orthogonal states.
- Recently, the Wasserstein distance appeared, which is different and this makes it very useful.
- For the quantum case, surprisingly, the self-distance can be nonzero.
- Can we connect these to entanglement theory?

# Outline

# An important property of the Wasserstein distance

- The distance is often maximal for orthogonal states, e.g., for the following state-pairs.




- In the second example, the two states are further apart from each other, based on common sense.

- Wasserstein distance can recognize this since it is the "cost of moving sand from a distribution to the other one."

- It can be used for machine learning.

G. De Palma, M. Marvian, D. Trevisan, and S. Lloyd,
IEEE Transactions on Information Theory 67, 6627 (2021).

# Quantum Wasserstein distance

- **Definition.**—The square of the distance between two quantum states described by the density matrices $\varrho$ and $\sigma$ is

$$
\begin{aligned}
D_{\mathrm{DPT}}(\varrho,\sigma)^2 = \frac{1}{2}\min_{\varrho_{12}} \sum_{n=1}^{N} & \quad \mathrm{Tr}[(H_n^T \otimes \mathbb{1} - \mathbb{1} \otimes H_n)^2 \varrho_{12}], \\
\text{s. t.} & \quad \varrho_{12} \in \mathcal{D}, \\
& \quad \mathrm{Tr}_2(\varrho_{12}) = \varrho^T, \\
& \quad \mathrm{Tr}_1(\varrho_{12}) = \sigma,
\end{aligned}
$$

  where $\mathcal{D}$ is the set of density matrices.

- Note the relation to the representability problem.

  G. De Palma and D. Trevisan, Quantum optimal transport with quantum channels, Ann. Henri Poincaré 22, 3199 (2021).

# Self-distance can be nonzero (unlike in the classical case)

- The self-distance of a state is

$$D_{\mathrm{DPT}}(\varrho,\varrho)^2 = \sum_{n=1}^{N} I_\varrho(H_n),$$

  where the Wigner-Yanase skew information is defined as

$$I_\varrho(H) = \mathrm{Tr}(H^2\varrho) - \mathrm{Tr}(H\sqrt{\varrho}H\sqrt{\varrho}).$$

- This connects connects Wasserstein distance and quantum metrolgy.

  G. De Palma and D. Trevisan, Quantum optimal transport with quantum channels, Ann. Henri Poincaré 22, 3199 (2021).

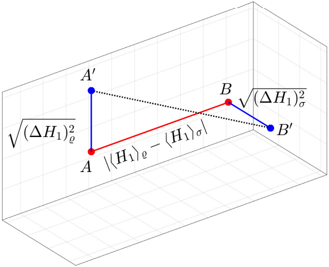
# Wasserstein distance between a pure state $\varrho = |\Psi\rangle\langle\Psi|$ and a mixed state $\sigma$

- The distance is given as

$$D_{\mathrm{DPT}}(\varrho,\sigma)^2 = \frac{1}{2}\sum_{n=1}^{N}\left[(\Delta H_n)^2{}_{\varrho} + (\Delta H_n)^2{}_{\sigma} + (\langle H_n\rangle_{\varrho} - \langle H_n\rangle_{\sigma})^2\right],$$

see the following figure.

# Wasserstein distance between a pure state $\varrho = |\Psi\rangle\langle\Psi|$ and a mixed state $\sigma$ II



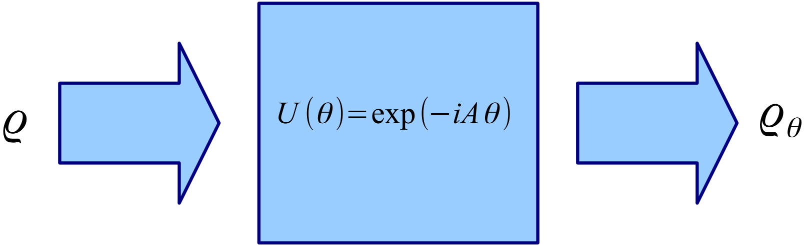

- $N = 1$ with operator $H_1$.
- The quantum Wasserstein distance equals $1/2$ times the usual Euclidean distance between $A'$ and $B'$.

GT and J. Pitrik, Quantum Wasserstein distance based on an optimization over separable states, arXiv:2209.09925.

# Outline

# Quantum metrology

- Fundamental task in metrology




- We have to estimate $\theta$ in the dynamics

$$U = \exp(-iA\theta).$$

# The quantum Fisher information

- Cramér-Rao bound on the precision of parameter estimation

$$(\Delta\theta)^2 \geq \frac{1}{mF_Q[\varrho, A]},$$

where $F_Q[\varrho, A]$ is the quantum Fisher information, and $m$ is the number of independent repetitions.

- The quantum Fisher information is

$$F_Q[\varrho, A] = 2\sum_{k,l} \frac{(\lambda_k - \lambda_l)^2}{\lambda_k + \lambda_l} |\langle k|A|l\rangle|^2,$$

where $\varrho = \sum_k \lambda_k |k\rangle\langle k|$.

# Formula based on convex roofs

The quantum Fisher information is the convex roof of the variance

$$F_Q[\varrho, A] = 4 \min_{\{p_k, |\psi_k\rangle\}} \sum_k p_k (\Delta A)^2_{\psi_k},$$

where

$$\varrho = \sum_k p_k |\psi_k\rangle\langle\psi_k|.$$

GT, D. Petz, Phys. Rev. A 87, 032324 (2013); S. Yu, arXiv1302.5311 (2013);
GT, I. Apellaniz, J. Phys. A: Math. Theor. 47, 424006 (2014).

# Similar relation for the variance

The quantum Fisher information is the convex roof of the variance

$$(\Delta A)^2{}_{\varrho} = \max_{\{p_k, |\psi_k\rangle\}} \sum_k p_k (\Delta A)^2{}_{\psi_k},$$

where

$$\varrho = \sum_k p_k |\psi_k\rangle\langle\psi_k|.$$

GT, D. Petz, Phys. Rev. A 87, 032324 (2013);

GT, I. Apellaniz, J. Phys. A: Math. Theor. 47, 424006 (2014).

# A single relation for the QFI and the variance

For any decomposition $\{p_k, |\psi_k\rangle\}$ of the density matrix $\varrho$ we have

$$\frac{1}{4}F_Q[\varrho, A] \le \sum_k p_k (\Delta A)^2{}_{\psi_k} \le (\Delta A)^2{}_{\varrho},$$

where the upper and the lower bounds are both tight.

- Note that

$$\frac{1}{4}F_Q[\varrho, A] \le (\Delta A)^2{}_{\varrho},$$

  where for pure states we have an equality.

- The QFI is strongly related to the variance.

# Formula based on an optimization in the two-copy space

- Two-copy formulation for the variance

$$(\Delta H)^2{}_{\Psi} = \mathrm{Tr}(\Omega|\Psi\rangle\langle\Psi|\otimes|\Psi\rangle\langle\Psi|),$$

where we define the operator

$$\Omega = H^2 \otimes \mathbb{1} - H \otimes H.$$

We can reformulate the convex roof as

$$\begin{aligned} \mathcal{F}_Q[\varrho, H] = \min_{\varrho_{12}} \quad & 4\mathrm{Tr}(\Omega\varrho_{12}), \\ \text{s. t.} \quad & \varrho_{12} \in \mathcal{S}', \\ & \mathrm{Tr}_2(\varrho_{12}) = \varrho. \end{aligned}$$

Here $\mathcal{S}'$ is the set of symmetric separable states.

GT, T. Moroder, and O. Gühne, Evaluating convex roof entanglement measures, Phys. Rev. Lett. 114, 160501 (2015); GT, D. Petz, Phys. Rev. A 87, 032324 (2013).

# Outline

# Quantum Wasserstein distance based on an optimization separable states

- **Definition**—We can also define

$$
\begin{aligned}
D_{\mathrm{DPT,sep}}(\varrho,\sigma)^2 = \frac{1}{2}\min_{\varrho_{12}} \sum_{n=1}^{N} \quad & \mathrm{Tr}[(H_n^T \otimes \mathbb{1} - \mathbb{1} \otimes H_n)^2 \varrho_{12}], \\
\text{s. t.} \quad & \varrho_{12} \in \mathcal{S}, \\
& \mathrm{Tr}_2(\varrho_{12}) = \varrho^T, \\
& \mathrm{Tr}_1(\varrho_{12}) = \sigma,
\end{aligned}
$$

where $S$ is the set of separable states.

GT and J. Pitrik, Quantum Wasserstein distance based on an optimization over separable states, arXiv:2209.09925.

# Self-distance

- The self-distance for $N = 1$ is

$$D_{\mathrm{DPT,sep}}(\varrho, \varrho)^2 = \frac{1}{4}\mathcal{F}_Q[\varrho, H_1].$$

- Note that

$$I_\varrho(A) \leq \frac{1}{4}F_Q[\varrho, A] \leq (\Delta A)^2{}_\varrho.$$

GT and J. Pitrik, Quantum Wasserstein distance based on an optimization over separable states, arXiv:2209.09925.

# Outline

# Entanglement of $\varrho_{12}$

- In general,
$$D_{\mathrm{DPT,sep}}(\varrho,\sigma) \geq D_{\mathrm{DPT}}(\varrho,\sigma).$$

- If the relation
$$D_{\mathrm{DPT,sep}}(\varrho,\sigma) > D_{\mathrm{DPT}}(\varrho,\sigma),$$
holds, then the optimal $\varrho_{12}$ for $D_{\mathrm{DPT}}(\varrho,\sigma)$ is entangled.

- Allowing an entangled $\varrho_{12}$ decreases the cost!

- Thus, an entangled $\varrho_{12}$ can be cheaper than a separable one.

# Bounds on the distance

- Let us choose a set of $H_n$ such that

$$\frac{1}{2}\sum_n \left\langle (H_n^T \otimes \mathbb{1} - \mathbb{1} \otimes H_n)^2 \right\rangle \geq \mathrm{const.}$$

  holds for separable states.

- E. g., $\{H_n\} = \{j_x, j_y, j_z\}$ and "const."$= j$.

- If the inequality

$$D_{\mathrm{DPT}}(\varrho, \sigma) < \mathrm{const.}$$

  holds, then the the optimal $\varrho_{12}$ for $D_{\mathrm{DPT}}(\varrho, \sigma)$ is entangled.

- Then, we will have a minimal distance

$$D_{\mathrm{DPT,sep}}(\varrho, \sigma) \geq \mathrm{const.}$$

# Summary

- For the quantum Wasserstein distance, the self-distance equals the quantum Fisher information if we restrict the optimization to separable states.

G. Tóth and J. Pitrik, arXiv:2209.09925.

THANK YOU FOR YOUR ATTENTION!
www.gtoth.eu