Entanglement measures
(How much is it entangled?)
(Lecture of the Quantum Information class of the Master in Quantum Science and Technology)

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UPV/EHU, Leioa
1 February, 2022

Outline

Entanglement measures

- After detecting entanglement, we have to ask how entangled the state is.

- It will turn out that entanglement is a resource.

Outline

General quantum operation

- The general quantum operation is defined as

$$\varrho' = \sum_k E_k \varrho E_k^\dagger$$

  with

$$\sum_k E_k^\dagger E_k = 1.$$

- E_k are Kraus operators.

- Generalized measurements, POVM (positive operator-valued measure).

- Special case: von Neumann measurements, when E_k are pairwise orthogonal projectors.

- Naimark's dilation theorem:
  general operation=
  von Neumann measurement on system+ancilla.

Outline

Local operations and classical communication (LOCC)

- LOCC are
  - local unitaries,
  - local von Neumann or POVM measurements,
  - local unitaries or measurements conditioned on measurement outcomes on the other party.

- Mathematical description of LOCC. Separable operations are a somewhat larger set, however, this set can easily be described.

$$\varrho' = \sum_k E_k^{(1)} \otimes E_k^{(2)} \varrho \left(E_k^{(1)} \otimes E_k^{(2)} \right)^\dagger$$

with

$$\sum_k \left(E_k^{(1)} \otimes E_k^{(2)} \right)^\dagger \left(E_k^{(1)} \otimes E_k^{(2)} \right) = 1.$$

Local operations and classical communication (LOCC) II

- Stochastic Local Operations and Classical Communication (SLOCC):
$$|\Psi\rangle' \leftarrow E_k^{(1)} \otimes E_k^{(2)} |\Psi\rangle$$
  It happens with some probability, not deterministic.

- LOCC cannot create entanglement. Separable states remain separable under LOCC.

- LOCC can create correlations.

Outline

Entropy of entanglement

- The von Neumann entropy is defined as

$$S(\varrho) = -\text{Tr}(\varrho \log_2 \varrho).$$

- It can be written with the eigenvalues of the density matrix as

$$S(\varrho) = -\sum_{k=1}^{d} \lambda_k \log_2 \lambda_k.$$

- For a pure state we have $\lambda_k = \{1, 0, 0, ..., 0\}$, and thus it is zero.
- Its maximal is for the completely mixed state for which $\lambda_k = \{\frac{1}{d}, \frac{1}{d}, \frac{1}{d}, ..., \frac{1}{d}\}$, and its value is $\log_2 d$.
- For a bipartite pure state, the entropy of entanglement is

$$E_E(|\Psi\rangle) = S(\text{Tr}_1(|\Psi\rangle\langle\Psi|)).$$

  That is, it is the von Neumann entropy of the reduced state is an entaglement measure.

Entropy of entanglement II

- Comments

    - It is one for two-qubit singlet states.

    - It is zero for product states.

    - It is invariant under $U_1 \otimes U_2$.