Quantum Wasserstein distance based on an optimization over separable states
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Outline

Motivation

- Many distance measures are maximal for orthogonal states.

- Recently, the Wasserstein distance appeared, which is different and this makes it very useful.

- For the quantum case, surprisingly, the self-distance can be nonzero.

- Can we connect these to entanglement theory?

Outline

An important property of the Wasserstein distance

- The distance is often maximal for orthogonal states, e.g., for the following state-pairs.
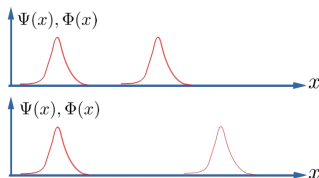
- In the second example the two states are further apart from each other, based on common sense.

- Wasserstein distance can recognize this since it is the "cost of moving sand from a distribution to the other one."

- It can be used for machine learning.

Quantum Wasserstein distance

- **Definition.**—The square of the distance between two quantum states described by the density matrices ϱ and σ is

$$
D_{\mathrm{DPT}}(\varrho, \sigma)^2 = \frac{1}{2} \min_{\varrho_{12}} \sum_{n=1}^{N} \quad \mathrm{Tr}[(H_n^T \otimes \mathbb{1} - \mathbb{1} \otimes H_n)^2 \varrho_{12}],
$$
$$
\mathrm{s.\,t.} \quad \varrho_{12} \in \mathcal{D},
$$
$$
\mathrm{Tr}_2(\varrho_{12}) = \varrho^T,
$$
$$
\mathrm{Tr}_1(\varrho_{12}) = \sigma,
$$

where $\mathcal{D}$ is the set of density matrices.

- Note the relation to the representability problem.

G. De Palma and D. Trevisan, Quantum optimal transport with quantum channels, Ann. Henri Poincaré 22, 3199 (2021).

Self-distance can be nonzero (unlike in the classical case)

- It has been shown that for the self-distance of a state

$$D_{\mathrm{DPT}}(\varrho, \varrho)^2 = \sum_{n=1}^{N} I_\varrho(H_n)$$

holds, where the Wigner-Yanase skew information is defined as

$$I_\varrho(H) = \mathrm{Tr}(H^2 \varrho) - \mathrm{Tr}(H \sqrt{\varrho} H \sqrt{\varrho}).$$

- This connects connects Wasserstein distance and quantum metrolgy.

G. De Palma and D. Trevisan, Quantum optimal transport with quantum channels, Ann. Henri Poincaré 22, 3199 (2021).
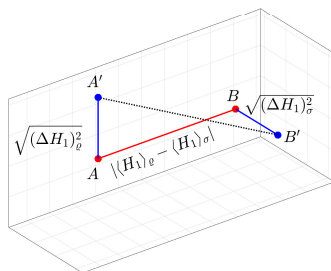
Wasserstein distance between a pure state $\varrho = |\Psi\rangle\langle\Psi|$ and a mixed state σ

- The distance is given as

$$D_{\mathrm{DPT}}(\varrho, \sigma)^2$$
$$= \frac{1}{2} \sum_{n=1}^{N} \left[(\Delta H_n)^2_{\varrho} + (\Delta H_n)^2_{\sigma} + (\langle H_n \rangle_{\varrho} - \langle H_n \rangle_{\sigma})^2 \right],$$

  see the following figure.

Wasserstein distance between a pure state $\varrho = |\Psi\rangle\langle\Psi|$ and a mixed state σ II

- $N = 1$ with operator H_1.

- The quantum Wasserstein distance equals 1/2 times the usual Euclidean distance between A' and B'.

Outline

Quantum metrology

- Fundamental task in metrology

$$\varrho \quad \Longrightarrow \quad \boxed{U(\theta) = \exp(-iA\theta)} \quad \Longrightarrow \quad \varrho_\theta$$
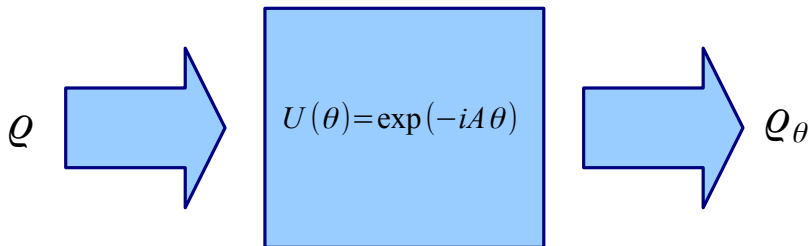
- We have to estimate θ in the dynamics

$$U = \exp(-iA\theta).$$

The quantum Fisher information

- Cramér-Rao bound on the precision of parameter estimation

$$(\Delta\theta)^2 \geq \frac{1}{mF_Q[\varrho, A]},$$

where $F_Q[\varrho, A]$ is the quantum Fisher information, and m is the number of independent repetitions.

- The quantum Fisher information is

$$F_Q[\varrho, A] = 2\sum_{k,l} \frac{(\lambda_k - \lambda_l)^2}{\lambda_k + \lambda_l} |\langle k|A|l\rangle|^2,$$

where $\varrho = \sum_k \lambda_k |k\rangle\langle k|$.

Formula based on convex roofs

The quantum Fisher information is the convex roof of the variance

$$F_Q[\varrho, A] = 4 \min_{\{p_k, |\psi_k\rangle\}} \sum_k p_k (\Delta A)^2{}_{\psi_k},$$

where

$$\varrho = \sum_k p_k |\psi_k\rangle\langle\psi_k|.$$

GT, D. Petz, Phys. Rev. A 87, 032324 (2013); S. Yu, arXiv1302.5311 (2013);
GT, I. Apellaniz, J. Phys. A: Math. Theor. 47, 424006 (2014).

Similar relation for the variance

The quantum Fisher information is the convex roof of the variance

$$(\Delta A)^2{}_{\varrho} = \max_{\{p_k, |\psi_k\rangle\}} \sum_k p_k (\Delta A)^2{}_{\psi_k},$$

where

$$\varrho = \sum_k p_k |\psi_k\rangle\langle\psi_k|.$$

GT, D. Petz, Phys. Rev. A 87, 032324 (2013);
GT, I. Apellaniz, J. Phys. A: Math. Theor. 47, 424006 (2014).

A single relation for the QFI and the variance

For any decomposition $\{p_k, |\psi_k\rangle\}$ of the density matrix ϱ we have

$$\frac{1}{4}F_Q[\varrho, A] \le \sum_k p_k (\Delta A)^2_{\psi_k} \le (\Delta A)^2_\varrho,$$

where the upper and the lower bounds are both tight.

- Note that

$$\frac{1}{4}F_Q[\varrho, A] \le (\Delta A)^2_\varrho,$$

  where for pure states we have an equality.

- The QFI appears as a "pair" of variance.

Formula based on an optimization in the two-copy space

- We can use a two-copy formulation for the variance

$$(\Delta H)^2{}_\Psi = \mathrm{Tr}(\Omega|\Psi\rangle\langle\Psi| \otimes |\Psi\rangle\langle\Psi|),$$

where we define the operator

$$\Omega = H^2 \otimes \mathbb{1} - H \otimes H.$$

We can reformulate the convex roof as

$$\mathcal{F}_Q[\varrho, H] = \min_{\varrho_{12}} \quad 4\mathrm{Tr}(\Omega\varrho_{12}),$$
$$\text{s.t.} \quad \varrho_{12} \in \mathcal{S}',$$
$$\mathrm{Tr}_2(\varrho_{12}) = \varrho.$$

Here $\mathcal{S}'$ is the set of symmetric separable states.

GT, T. Moroder, and O. Gühne, Evaluating convex roof entanglement measures, Phys. Rev. Lett. 114, 160501 (2015); GT, D. Petz, Phys. Rev. A 87, 032324 (2013).

Outline

Quantum Wasserstein distance based on an optimization over separable states

- **Definition**—We can also define

$$D_{\mathrm{DPT,sep}}(\varrho, \sigma)^2 = \frac{1}{2} \min_{\varrho_{12}} \sum_{n=1}^{N} \quad \mathrm{Tr}[(H_n^T \otimes \mathbb{1} - \mathbb{1} \otimes H_n)^2 \varrho_{12}],$$

$$\text{s. t.} \quad \varrho_{12} \in \mathcal{S},$$
$$\mathrm{Tr}_2(\varrho_{12}) = \varrho^T,$$
$$\mathrm{Tr}_1(\varrho_{12}) = \sigma,$$

where S is the set of separable states.

GT and J. Pitrik, Quantum Wasserstein distance based on an optimization over separable states, arXiv:2209.09925.

Self-distance

- The self-distance for $N = 1$ is

$$D_{\mathrm{DPT,sep}}(\varrho, \varrho)^2 = \frac{1}{4} \mathcal{F}_Q[\varrho, H_1].$$

- Note that

$$I_\varrho(A) \leq \frac{1}{4} F_Q[\varrho, A] \leq (\Delta A)^2{}_\varrho.$$

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Outline

Entanglement of ϱ_{12}

- In general,
$$D_{\mathrm{DPT,sep}}(\varrho, \sigma) \geq D_{\mathrm{DPT}}(\varrho, \sigma).$$

- If the relation
$$D_{\mathrm{DPT,sep}}(\varrho, \sigma) > D_{\mathrm{DPT}}(\varrho, \sigma),$$
holds, then the optimal ϱ_{12} for $D_{\mathrm{DPT}}(\varrho, \sigma)$ is entangled.

- Allowing an entangled ϱ_{12} decreases the cost!

- Thus, an entangled ϱ_{12} can be cheaper than a separable one.

Bounds on the distance

- Let us choose a set of H_n such that

$$\frac{1}{2} \sum_n \left\langle (H_n^T \otimes \mathbb{1} - \mathbb{1} \otimes H_n)^2 \right\rangle \geq \text{const.}$$

holds for separable states. (The, we have an entanglement criterion. E. g., $\{H_n\} = \{j_x, j_y, j_z\}$ and "const."$= j$.)

- If the inequality

$$D_{\text{DPT}}(\varrho, \sigma) < \text{const.}$$

holds, then the the optimal ϱ_{12} for $D_{\text{DPT}}(\varrho, \sigma)$ is entangled.

- Then, we will have a minimal distance

$$D_{\text{DPT,sep}}(\varrho, \sigma) \geq \text{const.}$$

Summary

- For the Wasserstein distance, the self-distance equals the quantum Fisher information if we restrict the optimization to separable states.

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