General characteristics of multi-partite quantum systems (Lecture of the Quantum Information class of the Master in Quantum Science and Technology)

Géza Tóth

Theoretical Physics, University of the Basque Country (UPV/EHU), Bilbao, Spain
Donostia International Physics Center (DIPC), San Sebastián, Spain
IKERBASQUE, Basque Foundation for Science, Bilbao, Spain
Wigner Research Centre for Physics, Budapest, Hungary

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Outline

1. **General characteristics of multi-partite quantum systems**
   - A. Classical bits
   - B. Quantum bit - pure states
   - C. Multi-qubit systems - pure states
   - D. Measurement
   - E. Mixed states and the density matrix
   - F. Geometry of quantum states
     - A single qubit

A single classical bit

- A classical bit can be either 0 or 1. Can we still use it to describe a real number between 0 and 1?
- For that, we need an ensemble of several classical bits

$$\{b_k\}_{k=1}^{M}, \tag{1}$$

where $b_k = 0$ or 1

- We can interpret the average value and the variance. That is,

$$\langle b \rangle = \frac{1}{M} \sum_k b_k, \tag{2}$$

and

$$(\Delta b)^2 = \frac{1}{M} \sum_k (b_k - \langle b \rangle)^2. \tag{3}$$

A single classical bit II

- This can also be given with probabilities:
- Let P_0 and P_1 be the probabilities of having a 0 or a 1.
- The expectation value and the variance are the function of P_0 and P_1. Since $P_0 + P_1 = 1$, we have a **single real degree of freedom** that describes the statistical properties of an ensemble of bits.
- Hence,

$$\langle b \rangle = P_1 \tag{4}$$

and

$$(\Delta b)^2 = P_0(0 - P_1)^2 + P_1(1 - P_1)^2. \tag{5}$$

Stochastic computing

- Stochastic computing uses random bits to calculate (John von Neumann, 1953).
- A random bit represents a real number between 0 and 1. Two random bits can easily be multiplied.

$$\langle b_1 b_2 \rangle = \langle b_1 \rangle \langle b_2 \rangle. \tag{6}$$

- We need many samples to get the average with small error.

Lectures on

PROBABILISTIC LOGICS AND THE SYNTHESIS OF RELIABLE

ORGANISMS FROM UNRELIABLE COMPONENTS

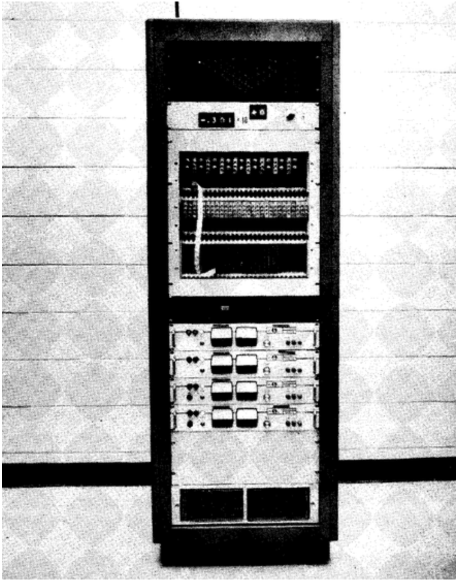
delivered by

PROFESSOR J. von NEUMANN

The Institute for Advanced Study
Princeton, N. J.

at the

Stochastic computing III



The RASCEL stochastic computer, circa 1969, Wikipedia.

Stochastic computing IV

Multiplication is possible with an AND gate.

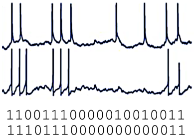


Figure 1.2: Similarity of biological signals and stochastic numbers; information is carried via pulses.

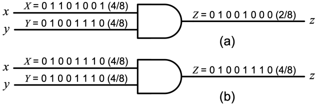


Figure 1.3: Stochastic multiplication: (a) accurate result with uncorrelated inputs; (b) inaccurate result due to correlated inputs.

A. Alaghi, The Logic of Random Pulses: Stochastic Computing, Ph.D. Thesis, University of Michigan, 2015.

Several classical bits

- N classical bits can be in one of the 2^N binary states. For example, for $N = 2$, these are 00, 01, 10 and 11.

- For $N = 2$, these are

$$P_{00}, P_{01}, P_{10}, P_{11}. \tag{7}$$

- The ensemble of the N-bit units can be described by the 2^N probabilities.

- Since, again, the sum of all the probablities is 1, **we need** $2^N - 1$ **real degrees of freedom** to describe the statistical properties of such an ensemble.

Several classical bits II

- Let us consider some function of N bits $f(k)$, where k is now an N bit number.
- Then, the expectation value of f is

$$\langle f \rangle = \sum_{k=0}^{2^N-1} p_k f(k) = \vec{p}\vec{f}, \tag{8}$$

where k is an N-bit number, i.e., an integer between 0 and $2^N - 1$. We put the $f(k)$'s into a vector $\vec{f}$. We also put the p_k probabilities into $\vec{p}$.

Several classical bits III

- We can also write

$$\langle f^2 \rangle = \sum_k p_k [f(k)]^2 \tag{9}$$

Hence,

$$(\Delta f)^2 = \sum_k p_k [f(k)]^2 - \left(\sum_k p_k f(k) \right)^2 . \tag{10}$$

These were relevant, since in the quantum case, we will have similar expressions.

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Quantum bit - pure states

- A quantum bit (=two-state system, spin-$\frac{1}{2}$ particle) can be in a pure state
$$|q\rangle = \alpha|0\rangle + \beta|1\rangle, \tag{11}$$
where α and β are complex numbers, and the normalisation condition $|\alpha|^2 + |\beta|^2 = 1$.

- Note that the overall phase does not matter, thus a pure quantum bit is described by two degrees of freedom.

- The two complex coefficients have **4 real degrees of freedom.**

- However, due to the normalisation condition and the arbitrariness of the overall phase we are left with **two degrees of freedom**.)

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Multi-qubit systems - pure states

- What about a two-qubit system? What kind of states it can be in? One could think on qubit 1 in state

$$|q_1\rangle = \alpha_1|0\rangle + \beta_1|1\rangle, \tag{12}$$

  and qubit 2 in state

$$|q_2\rangle = \alpha_2|0\rangle + \beta_2|1\rangle. \tag{13}$$

- However, we all know that the general state of the two-qubit system can be given as

$$|q_{12}\rangle = \alpha_{00}|00\rangle + \alpha_{01}|01\rangle + \alpha_{10}|00\rangle + \alpha_{11}|01\rangle. \tag{14}$$

Multi-qubit systems - pure states II

- In general, for N qubits we need N complex numbers. Again the state has to be normalized and the overall phase does not matter, thus this means $2 \times 2^N - 2$ real degrees of freedom.
- We can place the coefficients in a vector, called state vector and write

$$|\Psi\rangle = \begin{pmatrix} \alpha_{00} \\ \alpha_{01} \\ \alpha_{10} \\ \alpha_{11} \end{pmatrix}. \tag{15}$$

- The properties of the state vector are: it is normalized

$$\langle\Psi|\Psi\rangle = 1. \tag{16}$$

Multi-qubit systems - pure states III

- An overall phase does not matter:

$$e^{-i\theta}|\Psi\rangle \tag{17}$$

  describes the same state for any θ.

- The expectation value of an operator for a pure state can be obtained as

$$\langle A\rangle = \langle\Psi|A|\Psi\rangle = \mathrm{Tr}(A|\Psi\rangle\langle\Psi|). \tag{18}$$

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Measurement

- The von Neumann measurement in the z basis results is eithet 0 or 1. If the state was $\alpha|0\rangle + \beta|1\rangle$, then we get a statistical mixture of 0 and 1, with the probabilities

$$P_0 = |\alpha|^2, \tag{19}$$

  and

$$P_1 = |\beta|^2. \tag{20}$$

  That is, from an ensemble of quantum bits we get an ensemble of classical bits.

- If we measure in the x basis, we get another classical ensemble.
- For a multi-qubit system, if we measure in the some basis (e.g., x, y or z), we get an ensemble of N-bit systems. However, for exach choice of basis we get a different classical ensemble.

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Mixed states and the density matrix

- So far we were talking about pure states.
- In reality, in an experiment we do not have a situation where a machine always produces the $|\Psi_1\rangle$ state.
- Sometimes it makes mistakes, and produces the $|\Psi_k\rangle$ states for $k = 2, 3, ...$ How to describe such a situation?

$|\Psi_1\rangle$	p_1
$|\Psi_2\rangle$	p_2
$|\Psi_3\rangle$	p_3
...	...

Mixed states and the density matrix

- What is the expectation value of an operator in such a system? We can write it as

$$\langle A \rangle = \sum_k p_k \langle \Psi_k | A | \Psi_k \rangle = \mathrm{Tr}\left(A \sum_k p_k |\Psi_k\rangle\langle\Psi_k| \right). \tag{21}$$

- This can be rewritten as

$$\langle A \rangle = \mathrm{Tr}(\varrho A), \tag{22}$$

where

$$\varrho = \sum_k p_k |\Psi_k\rangle\langle\Psi_k| \tag{23}$$

is the density matrix (Neumann, Landau).

- Note that if ϱ is diagonal, we obtain

$$\langle A \rangle = \mathrm{Tr}(\varrho A) = \sum_k \varrho_{kk} A_{kk}. \tag{24}$$

That is, A is written in the eigenbasis of ϱ. This is the scalar product of two vectors as in $\langle f \rangle = \vec{p}\vec{f}$ [given in Eq. (8)].

Mixed states and the density matrix II

- The density matrix describes the state completely. Now we see, why the overall phase does not matter:

$$e^{-i\theta}|\Psi_k\rangle\langle\Psi_k|e^{+i\theta} = |\Psi_k\rangle\langle\Psi_k|. \tag{25}$$

- The properties of the density matrix are

$$\begin{aligned} \varrho &= \varrho^\dagger, \\ \varrho &\geq 0, \\ \mathrm{Tr}(\varrho) &= 1. \end{aligned} \tag{26}$$

- **A $2^N \times 2^N$ density matrix has $4^N - 1$ real parameters.**

- For $N = 1$, this means 3 real parameters, corresponding to the three coordinates of the Bloch vector. For r $N = 2$, this means 8 real parameters.

Mixed states and the density matrix III

- We can also say that
$$\mathrm{Tr}(\varrho^2) \leq 1. \tag{27}$$
It is one only for pure (rank-1) states.

- The density matrix can be decomposed into the sum of pure states in many ways. The decomposition
$$\varrho = \sum_k p_k |\Psi_k\rangle\langle\Psi_k| \tag{28}$$
is not unique, i.e., it is not necessarily an eigendecomposition. This has a large importance for entanglement theory.

Mixed states and the density matrix IV

Summary:

	N bits	N qubits
Number of DOF	$2^N - 1$	$4^N - 1$
Description	$\vec{p}$	ϱ
Expectation value	$\vec{f}\vec{p}$	$\mathrm{Tr}(A\varrho)$
Normalization	$\sum_k p_k = 1$	$\mathrm{Tr}(\varrho) = 1$

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Bloch vector

- For a single qubit, **the density matrix has three real parameters.** It can be written as

$$\varrho = \frac{1}{2}\left(\mathbb{1} + \sum_{l=x,y,z} v_l \sigma_l\right), \tag{29}$$

  where σ_l are the Pauli spin matrices.

- Using $\mathrm{Tr}(\sigma_k \sigma_l) = 2\delta_{kl}$, we can write

$$\mathrm{Tr}(\varrho^2) = \frac{1}{2} + \frac{1}{2}\sum_{l=x,y,z} v_l^2. \tag{30}$$

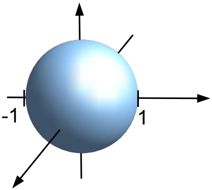
  That is, the Bloch vector has a maximal length for pure states.

Bloch vector II

- From $\mathrm{Tr}(\varrho^2) \leq 1$, the condition for being physical is Eq. (26), which is equvalent to
$$\sum_{l=x,y,z} |v_l|^2 \leq 1. \tag{31}$$
The three-element vector is called the Bloch vector.

Bloch vector III

- Let us identify the points in (v_x, v_y, v_z) corresponding to physical states. They are in a ball.
- The pure states are on the surface.
- Mixed states are inside the Ball. This is because $\mathrm{Tr}(\varrho^2)$ is directly related to the length of the Bloch vector.
- The $|0\rangle$ and $|1\rangle$ correspond to the North and South Pole.
- $|0\rangle + \exp(-i\phi)|1\rangle$ correspond to points on the equator.



Set of physical quantum states for a single qubit. The axes correspond to v_l for $l = x, y, z$. Pure states correspond to points on the surface, mixed states correspond to internal points.