Quantum Wasserstein distance based on an optimization over separable states arXiv.2209.09925

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Crete
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Casablanca
MOROCCO
ALGERIA
TUNISIA
Valletta
MALTA
Scale 1: 19,500,000
Lambert Conformal Conic Projection, standard parallels 40°N and 56°N
300 Kilometers
300 Miles

Outline

1. **Motivation**
   - Connecting Wasserstein distance to entanglement theory?
2. **Background**
   - Quantum Wasserstein distance
   - Quantum Fisher information
3. **Wasserstein distance and separable states**
   - Quantum Wasserstein distance based on an optimization over separable states
   - Relation to entanglement conditions

Motivation

- Many distance measures are maximal for orthogonal states.
- Recently, the Wasserstein distance appeared, which is different and this makes it very useful.
- For the quantum case, surprisingly, the self-distance can be nonzero.
- Can we connect these to entanglement theory?

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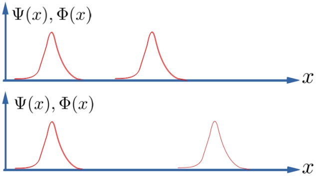
An important property of the Wasserstein distance

- Many distance measures are maximal for orthogonal states, e.g., for the following state-pairs.



- In the second example, the two states are further apart from each other, based on common sense.

- Wasserstein distance can recognize this since it is related to the "cost of moving sand from a distribution to the other one."

- It can be used for machine learning.

G. De Palma, M. Marvian, D. Trevisan, and S. Lloyd,
IEEE Transactions on Information Theory 67, 6627 (2021).

Quantum Wasserstein distance

- **Definition.**—The square of the distance between two quantum states described by the density matrices ϱ and σ is

$$
\begin{aligned}
D_{\mathrm{DPT}}(\varrho,\sigma)^2 = \frac{1}{2}\min_{\varrho_{12}} \sum_{n=1}^{N} \quad & \mathrm{Tr}[(H_n^T \otimes \mathbb{1} - \mathbb{1} \otimes H_n)^2 \varrho_{12}],\\
\text{s. t.} \quad & \varrho_{12} \in \mathcal{D},\\
& \mathrm{Tr}_2(\varrho_{12}) = \varrho^T,\\
& \mathrm{Tr}_1(\varrho_{12}) = \sigma,
\end{aligned}
$$

  where $\mathcal{D}$ is the set of density matrices, and H_n are Hermitian matrices.

- Note the relation to the representability problem.

  G. De Palma and D. Trevisan, Quantum optimal transport with quantum channels, Ann. Henri Poincaré 22, 3199 (2021).

Self-distance can be nonzero (unlike in the classical case)

- The self-distance of a state is

$$D_{\mathrm{DPT}}(\varrho,\varrho)^2 = \sum_{n=1}^{N} I_\varrho(H_n),$$

  where the Wigner-Yanase skew information is defined as

$$I_\varrho(H) = \mathrm{Tr}(H^2\varrho) - \mathrm{Tr}(H\sqrt{\varrho}H\sqrt{\varrho}).$$

- This connects connects Wasserstein distance and quantum metrolgy.

  G. De Palma and D. Trevisan, Quantum optimal transport with quantum channels, Ann. Henri Poincaré 22, 3199 (2021).

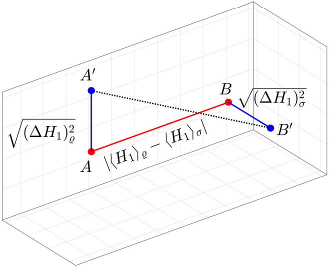
Wasserstein distance between a pure state $\varrho = |\Psi\rangle\langle\Psi|$ and a mixed state σ

- The distance is given as

$$
\begin{aligned}
&D_{\mathrm{DPT}}(\varrho,\sigma)^2 \\
&= \frac{1}{2}\sum_{n=1}^{N}\left[(\Delta H_n)^2{}_\varrho + (\Delta H_n)^2{}_\sigma + (\langle H_n\rangle_\varrho - \langle H_n\rangle_\sigma)^2\right],
\end{aligned}
$$

see the following figure, where $(\Delta H_n)^2$ is the variance.

Wasserstein distance between a pure state $\varrho = |\Psi\rangle\langle\Psi|$ and a mixed state σ II



- $N = 1$ with operator H_1.
- The quantum Wasserstein distance equals $1/2$ times the usual Euclidean distance between A' and B'.

GT and J. Pitrik, Quantum Wasserstein distance based on an optimization over separable states, arXiv:2209.09925.

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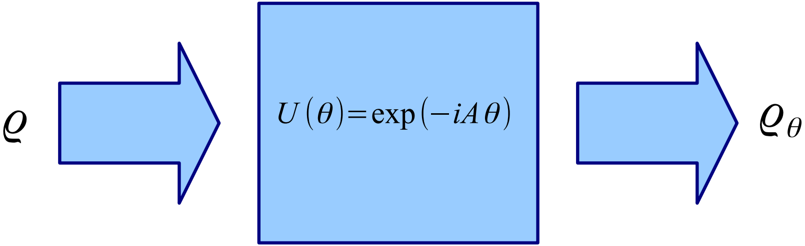
Quantum metrology

- Fundamental task in metrology



- We have to estimate θ in the dynamics

$$U = \exp(-iA\theta).$$

The quantum Fisher information

- Cramér-Rao bound on the precision of parameter estimation

$$(\Delta\theta)^2 \geq \frac{1}{mF_Q[\varrho, A]},$$

where $F_Q[\varrho, A]$ is the quantum Fisher information, and m is the number of independent repetitions.

- The quantum Fisher information is

$$F_Q[\varrho, A] = 2 \sum_{k,l} \frac{(\lambda_k - \lambda_l)^2}{\lambda_k + \lambda_l} |\langle k|A|l\rangle|^2,$$

where $\varrho = \sum_k \lambda_k |k\rangle\langle k|$.

Formula based on convex roofs

The quantum Fisher information is the convex roof of the variance times four

$$F_Q[\varrho, A] = 4 \min_{\{p_k, |\psi_k\rangle\}} \sum_k p_k (\Delta A)^2_{\psi_k},$$

where

$$\varrho = \sum_k p_k |\psi_k\rangle\langle\psi_k|.$$

GT, D. Petz, Phys. Rev. A 87, 032324 (2013); S. Yu, arXiv1302.5311 (2013);
GT, I. Apellaniz, J. Phys. A: Math. Theor. 47, 424006 (2014).

Formula based on concave roofs

The variance is the concave roof of itself

$$(\Delta A)^2{}_{\varrho} = \max_{\{p_k,|\psi_k\rangle\}} \sum_k p_k (\Delta A)^2{}_{\psi_k},$$

where

$$\varrho = \sum_k p_k |\psi_k\rangle\langle\psi_k|.$$

GT, D. Petz, Phys. Rev. A 87, 032324 (2013);

GT, I. Apellaniz, J. Phys. A: Math. Theor. 47, 424006 (2014).

A single relation for the QFI and the variance

For any decomposition $\{p_k, |\psi_k\rangle\}$ of the density matrix ϱ we have

$$\frac{1}{4}F_Q[\varrho, A] \le \sum_k p_k (\Delta A)^2_{\psi_k} \le (\Delta A)^2_{\varrho},$$

where the upper and the lower bounds are both tight.

- Note that
$$\frac{1}{4}F_Q[\varrho, A] \le (\Delta A)^2_{\varrho},$$
where for pure states we have an equality.

- The QFI is strongly related to the variance.

Formula based on an optimization in the two-copy space

- Two-copy formulation for the variance

$$(\Delta H)^2{}_\Psi = \mathrm{Tr}(\Omega|\Psi\rangle\langle\Psi|\otimes|\Psi\rangle\langle\Psi|),$$

where we define the operator

$$\Omega = H^2 \otimes \mathbb{1} - H \otimes H.$$

We can reformulate the convex roof as

$$\begin{aligned} F_Q[\varrho, H] = \min_{\varrho_{12}} \quad & 4\mathrm{Tr}(\Omega\varrho_{12}), \\ \text{s. t.} \quad & \varrho_{12} \in \mathcal{S}', \\ & \mathrm{Tr}_2(\varrho_{12}) = \varrho. \end{aligned}$$

Here $\mathcal{S}'$ is the set of symmetric separable states.

GT, T. Moroder, and O. Gühne, Evaluating convex roof entanglement measures, Phys. Rev. Lett. 114, 160501 (2015); GT, D. Petz, Phys. Rev. A 87, 032324 (2013).

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Quantum Wasserstein distance based on an optimization over separable states

- **Definition**—We can also define

$$
\begin{aligned}
D_{\mathrm{DPT,sep}}(\varrho,\sigma)^2 = \frac{1}{2}\min_{\varrho_{12}} \sum_{n=1}^{N} \quad & \mathrm{Tr}[(H_n^T \otimes \mathbb{1} - \mathbb{1} \otimes H_n)^2 \varrho_{12}], \\
\text{s. t.} \quad & \varrho_{12} \in \mathcal{S}, \\
& \mathrm{Tr}_2(\varrho_{12}) = \varrho^T, \\
& \mathrm{Tr}_1(\varrho_{12}) = \sigma,
\end{aligned}
$$

where S is the set of separable states.

GT and J. Pitrik, Quantum Wasserstein distance based on an optimization over separable states, arXiv:2209.09925.

Quantum Wasserstein distance based on an optimization over separable states II

- For two-qubits, it is computable numerically with semidefinite programming.
- For systems of larger dimensions, one can obtain a very good lower bound based on an optimization over states with a positive partial transpose (PPT).
- Even better lower bounds can be obtained.

P. Horodecki, Phys. Lett. A 232, 333 (1997);
A. Peres, Phys. Rev. Lett. 77, 1413 (1996);
A. C. Doherty, P. A. Parrilo, and F. M. Spedalieri, Phys. Rev. A 69, 022308 (2004).

Self-distance

- The self-distance for $N=1$ is

$$D_{\mathrm{DPT,sep}}(\varrho,\varrho)^2 = \frac{1}{4} F_Q[\varrho, H_1].$$

- Note that

$$I_\varrho(A) \le \frac{1}{4} F_Q[\varrho, A] \le (\Delta A)^2{}_\varrho.$$

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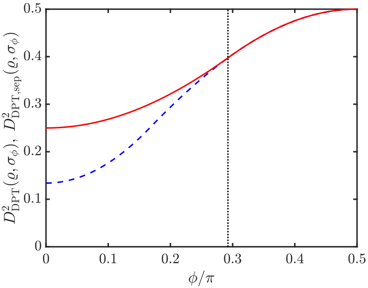
Entanglement of ϱ_{12}

- In general,
$$D_{\mathrm{DPT,sep}}(\varrho,\sigma) \geq D_{\mathrm{DPT}}(\varrho,\sigma).$$
- If the relation
$$D_{\mathrm{DPT,sep}}(\varrho,\sigma) > D_{\mathrm{DPT}}(\varrho,\sigma)$$
holds, then the optimal ϱ_{12} for $D_{\mathrm{DPT}}(\varrho,\sigma)$ is entangled.
- Allowing an entangled ϱ_{12} decreases the cost!
- Thus, an entangled ϱ_{12} can be cheaper than a separable one.

Comparison of the two types of Wasserstein distance

- Let us consider the distance between two single-qubit mixed states
$$\varrho = \frac{1}{2}|1\rangle\langle 1|_x + \frac{1}{2}\cdot\frac{\mathbb{1}}{2},$$
and
$$\sigma_\phi = e^{-i\frac{\sigma_y}{2}\phi}\varrho e^{+i\frac{\sigma_y}{2}\phi}.$$



Bounds on the distance

- Entanglement condition: Let us choose a set of H_n such that

$$\frac{1}{2}\sum_n \left\langle (H_n^T \otimes \mathbb{1} - \mathbb{1} \otimes H_n)^2 \right\rangle \geq \text{const.}$$

  holds for separable states.

- E. g., $\{H_n\} = \{j_x, j_y, j_z\}$ and "const."$= j$.

- If the inequality

$$D_{\rm DPT}(\varrho, \sigma) < \text{const.}$$

  holds, then all optimal ϱ_{12} states for $D_{\rm DPT}(\varrho, \sigma)$ are entangled.

- Then, we will have a minimal distance

$$D_{\rm DPT,sep}(\varrho, \sigma) \geq \text{const.}$$

Summary

- For the quantum Wasserstein distance, we restrict the optimization to separable states.

- Then, the self-distance equals the quantum Fisher information over four.

- We found a fundamental connection from quantum optimal transport to quantum entanglement theory and quantum metrology.

G. Tóth and J. Pitrik, arXiv:2209.09925.

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