Number-phase uncertainty relations and bipartite entanglement detection in spin ensembles
Quantum 7, 914 (2023)

G. Vitagliano[1,2], M. Fadel[3,4], I. Apellaniz[2,5,6], M. Kleinmann[7,2], B. Lücke[8], C. Klempt[8,9], G. Tóth[2,5,10,11,12]

[1]IQOQI, Wien, Austria, [2]University of the Basque Country UPV/EHU, Bilbao, Spain
[3]Department of Physics, ETH Zürich, Zürich, Switzerland
[4]Department of Physics, University of Basel, Basel, Switzerland
[5]EHU Quantum Center, University of the Basque Country UPV/EHU, Spain
[6]University of Mondragon, Spain, [7]University of Siegen, Germany
[8]Institut für Quantenoptik, Leibniz Universität Hannover, Hannover, Germany
[9]Deutsches Zentrum für Luft- und Raumfahrt e.V. (DLR)
[10]Donostia International Physics Center (DIPC), San Sebastián, Spain
[11]IKERBASQUE, Basque Foundation for Science, Bilbao, Spain
[12]Wigner Research Centre for Physics, Budapest, Hungary

Outline

1. **Motivation**
   - Why entanglement is important?
2. **Spin squeezing and entanglement**
   - Entanglement
   - Collective measurements
   - The original spin-squeezing criterion
   - Generalized criteria for $j = \frac{1}{2}$
3. **Detecting multipartite entanglement of Dicke states**
   - Dicke state realized with a BEC of two-state atoms
4. **Detecting bipartite entanglement of Dicke states**
   - Bipartite entanglement from multipartite entanglement in BEC
   - Creating Dicke states in BEC
   - Entanglement detection in Dicke states
   - Our criteria work under realistic conditions

Why multipartite entanglement is important?

- Many experiments are aiming to create entangled states with many atoms.
- Full tomography is not possible, we still have to say something meaningful.
- Only collective quantities can be measured.
- Thus, entanglement detection seems to be a good idea.
- In many cases, we need to detect bipartite entanglement.

Outline

Entanglement

A state is (fully) separable if it can be written as

$$\sum_k p_k \varrho_1^{(k)} \otimes \varrho_2^{(k)} \otimes ... \otimes \varrho_N^{(k)}.$$

If a state is not separable then it is entangled.

Outline

Many-particle systems for j=1/2

- For spin-$\frac{1}{2}$ particles, we can measure the collective angular momentum operators:

$$J_l := \frac{1}{2}\sum_{k=1}^{N}\sigma_l^{(k)},$$

  where $l = x, y, z$ and $\sigma_l^{(k)}$ a Pauli spin matrices.

- We measure the expectation values $\langle J_l \rangle$.

- We can also measure the variances

$$(\Delta J_l)^2 := \langle J_l^2 \rangle - \langle J_l \rangle^2.$$

Outline

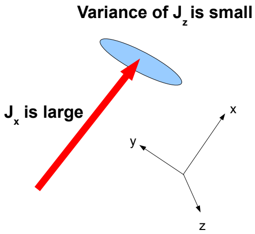
The standard spin-squeezing criterion

The spin squeezing criterion for entanglement detection is

$$\xi_s^2 = N \frac{(\Delta J_z)^2}{\langle J_x \rangle^2 + \langle J_y \rangle^2}.$$

[A. Sørensen, L.M. Duan, J.I. Cirac, P. Zoller, Nature 409, 63 (2001).]

- If $\xi_s^2 < 1$ then the state is entangled.
- States detected are like this:



- They are good for metrology!

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Generalized spin squeezing criteria for $j = \frac{1}{2}$

- Let us assume that for a system we know only

$$\vec{J} := (\langle J_{\mathrm{x}} \rangle, \langle J_{\mathrm{y}} \rangle, \langle J_{\mathrm{z}} \rangle),$$
$$\vec{K} := (\langle J_{\mathrm{x}}^2 \rangle, \langle J_{\mathrm{y}}^2 \rangle, \langle J_{\mathrm{z}}^2 \rangle).$$

- Then any state violating the following inequalities is entangled:

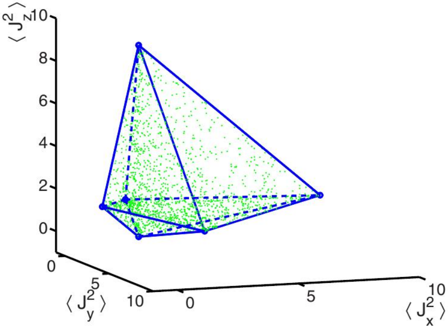
$$\langle J_x^2 \rangle + \langle J_y^2 \rangle + \langle J_z^2 \rangle \leq \frac{N(N+2)}{4},$$
$$(\Delta J_x)^2 + (\Delta J_y)^2 + (\Delta J_z)^2 \geq \frac{N}{2}, \qquad \text{(singlet states)}$$
$$\langle J_k^2 \rangle + \langle J_l^2 \rangle \leq (N-1)(\Delta J_m)^2 + \frac{N}{2}, \qquad \text{(Dicke states)}$$
$$(N-1)\left[(\Delta J_k)^2 + (\Delta J_l)^2\right] \geq \langle J_m^2 \rangle + \frac{N(N-2)}{4},$$

where k, l, m take all the possible permutations of x, y, z.

singlets: GT, Phys. Rev. A 69, 052327 (2004);
all Eqs.: GT, C. Knapp, O. Gühne, and H.J. Briegel, PRL 99, 250405 (2007);
spin-j: G. Vitagliano, P. Hyllus, I. L. Egusquiza, GT, PRL 107, 240502 (2011).

Generalized spin squeezing criteria for $j = \frac{1}{2}$ II

- Separable states are in the polytope



- We set $\langle J_l \rangle = 0$ for $l = x, y, z$.

Outline

Dicke states

- Dicke states: eigenstates of $\vec{J}^2 = J_x^2 + J_y^2 + J_z^2$ and J_z.
- Symmetric Dicke states of spin-1/2 particles, with $\langle J_z \rangle = \langle J_z^2 \rangle = 0$

$$|D_N\rangle = \binom{N}{\frac{N}{2}}^{-\frac{1}{2}} \sum_k \mathcal{P}_k \left(|0\rangle^{\otimes \frac{N}{2}} \otimes |1\rangle^{\otimes \frac{N}{2}} \right).$$

- Summing over all permutations.
- E.g., for four qubits they look like

$$|D_4\rangle = \frac{1}{\sqrt{6}} \left(|0011\rangle + |0101\rangle + |1001\rangle + |0110\rangle + |1010\rangle + |1100\rangle \right).$$

photons: N. Kiesel, C. Schmid, GT, E. Solano, H. Weinfurter, PRL 2007; Prevedel. *et al.,* PRL 2009; W. Wieczorek, R. Krischek, N. Kiesel, P. Michelberger, GT, H. Weinfurter, PRL 2009.

cold atoms: Lücke, Science 2011; Hamley *et al*, Nat. Phys. 2012.

Dicke states are useful because they ...

- ... possess strong multipartite entanglement, like GHZ states.

GT, JOSAB 2007.

- ... are optimal for quantum metrology, similarly to GHZ states.

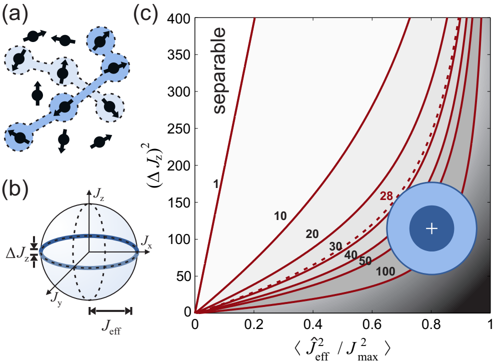
Hyllus *et al.*, PRA 2012; Lücke *et al.*, Science 2011;
GT, PRA 2012;
GT and Apellaniz, J. Phys. A, special issue for “50 year of Bell's theorem”, 2014.

- ... are macroscopically entangled, like GHZ states.

Fröwis, Dür, PRL 2011.

Multipartite entanglement

- Bose-Einstein condensate, 8000 particles. 28-particle entanglement is detected.



$$J^2_{\text{eff}} = J^2_x + J^2_y, \quad J_{\max} = \frac{N}{2}.$$

B. Lücke, J. Peise, G. Vitagliano, J. Arlt, L. Santos, GT, and C. Klempt, PRL 112, 155304 (2014).

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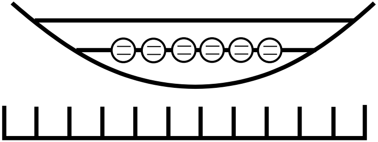
Bipartite entanglement from bosonic multipartite entanglement

- In the BEC, "all the particles are at the same place."
- In the usual formulation, entanglement is between spatially separated parties.
- Is multipartite entanglement within a BEC useful/real?
- Answer: yes!

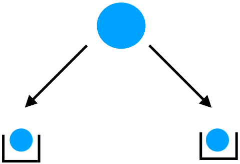
Bipartite entanglement from bosonic multipartite entanglement II

- Dilute cloud argument



See, e.g., P. Hyllus, L. Pezzé, A Smerzi and GT, PRA 86, 012337 (2012)

$|n_0 = 1\rangle |n_1 = 1\rangle$

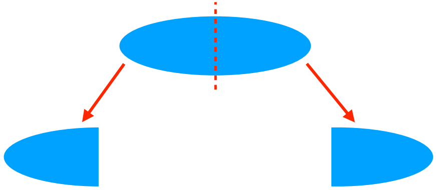


$\frac{1}{\sqrt{2}}(|01\rangle + |10\rangle)$

Bipartite entanglement from bosonic multipartite entanglement III

- After splitting it into two, we have bipartite entanglement if we had before multipartite entanglement.

- The splitting does not generate entanglement, if we consider projecting to a fixed particle number.



N. Killoran, M. Cramer, and M. B. Plenio, PRL 112, 150501 (2014).

Outline

Experiment in the group of Carsten Klempt at the University of Hannover

- Rubidium BEC, spin-1 atoms.
- Initially all atoms in the spin state $|j_z = 0\rangle$.
- Dynamics
$$H = a_0^2 a_{+1}^\dagger a_{-1}^\dagger + (a_0^\dagger)^2 a_{+1} a_{-1}.$$
Tunneling from mode 0 to the mode $+1$ and -1.
- Two-particle example:
$$|j_z = 0\rangle|j_z = 0\rangle \to \frac{1}{\sqrt{2}}(|j_z = +1\rangle|j_z = -1\rangle + |j_z = -1\rangle|j_z = +1\rangle)$$
$$= \text{Dicke state of 2 particles.}$$

Experiment in the group of Carsten Klempt at the University of Hannover II

- After some time, we have a state

$$|n_0, n_{-1}, n_{+1}\rangle = |N - 2n, n, n\rangle.$$

- That is, $N - 2n$ particles remained in the $|j_z = 0\rangle$ state, while $2n$ particles form a symmetric Dicke state given as

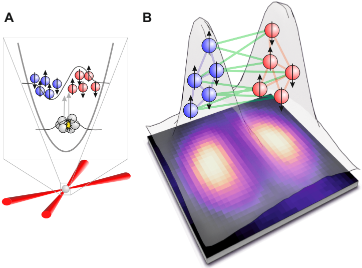
$$|D_N\rangle = \binom{N}{\frac{N}{2}}^{-\frac{1}{2}} \sum_k \mathcal{P}_k \left(|0\rangle^{\otimes \frac{N}{2}} \otimes |1\rangle^{\otimes \frac{N}{2}} \right),$$

  where we use $|0\rangle$ and $|1\rangle$ instead of $|j_z = -1\rangle$ and $|j_z = +1\rangle$.

- Half of the atoms in state $|0\rangle$, half of the atoms in state $|1\rangle$ + symmerization.

Experiment in the group of Carsten Klempt at the University of Hannover III

- Important: first excited spatial mode of the trap was used, not the ground state mode.
- It has two "bumps" rather than one, hence they had a split Dicke state.



[K. Lange, J. Peise, B. Lücke, I. Kruse, G. Vitagliano, I. Apellaniz, M. Kleinmann, G. Tóth, and C. Klempt, Entanglement between two spatially separated atomic modes, Science 360, 416 (2018).]

Correlations for Dicke states

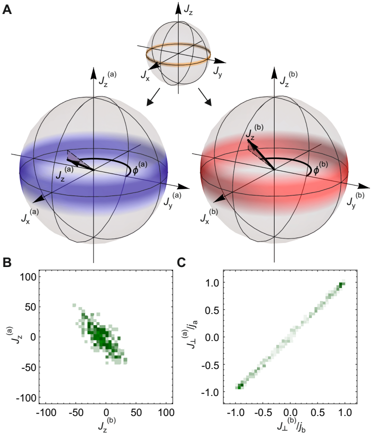
- For the Dicke state

$$\begin{aligned} (\Delta(J_x^a - J_x^b))^2 &\approx 0, \\ (\Delta(J_y^a - J_y^b))^2 &\approx 0, \\ (\Delta J_z)^2 = (\Delta(J_z^a + J_z^b))^2 &= 0. \end{aligned}$$

- Measurement results on well "*b*" can be predicted from measurements on "*a*"

$$\begin{aligned} J_x^b &\approx J_x^a, && \text{(correlation)} \\ J_y^b &\approx J_y^a, && \text{(correlation)} \\ J_z^b &= -J_z^a. && \text{(anti-correlation)} \end{aligned}$$

Correlations for Dicke states - experimental results



Here, $J_{\perp}^{(n)} = \cos\alpha J_{\mathrm{x}}^{(n)} + \sin\alpha J_{\mathrm{y}}^{(n)}$.

Experiment in K. Lange *et al.*, Science 334, 773–776 (2011).

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Number-phase-like uncertainty

- We start from the sum of two Heisenberg uncertainty relations

$$(\Delta J_z)^2[(\Delta J_x)^2+(\Delta J_y)^2] \geq \frac{1}{4}(\langle J_x\rangle^2+\left\langle J_y\right\rangle^2).$$

  Then,

$$(\Delta J_z)^2[(\Delta J_x)^2+(\Delta J_y)^2]+\frac{1}{4}[(\Delta J_x)^2+(\Delta J_y)^2] \geq \frac{1}{4}(\left\langle J_x^2\right\rangle+\left\langle J_y^2\right\rangle).$$

- Simple algebra yields

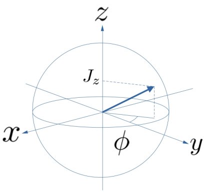
$$\left[(\Delta J_z)^2+\frac{1}{4}\right]\times\frac{(\Delta J_x)^2+(\Delta J_y)^2}{\langle J_x^2\rangle+\langle J_y^2\rangle} \geq \frac{1}{4}.$$

- Note that $\langle J_x^2\rangle$ appears, not $\langle J_x\rangle^2$.

Number-phase-like uncertainty II

- Uncertainty relation

$$\underbrace{\left[(\Delta J_{\mathrm{z}})^2 + \frac{1}{4}\right]}_{\sim \text{ fluctuation of } J_z} \times \underbrace{\frac{(\Delta J_{\mathrm{x}})^2 + (\Delta J_{\mathrm{y}})^2}{\langle J_{\mathrm{x}}^2 \rangle + \langle J_{\mathrm{y}}^2 \rangle}}_{\sim \text{ phase fluctuation}} \geq \frac{1}{4}.$$



Handwaving description:
J_z and ϕ cannot be defined both with high accuracy.

Normalized variables

- Let us introduce the normalized variables

$$\mathcal{J}_m^n = \frac{J_m^n}{\sqrt{j_n(j_n+1)}},$$

  where $m = x, y$ and $n = a, b$ (i.e., left well, right well), the total spin is

$$j_n = \frac{N_n}{2},$$

- Normalized variables $\rightarrow$ resistance to experimental imperfections.

Uncertainty with normalized variables

Our uncertainty relation is now

$$\left[(\Delta J_{\mathrm{z}})^2 + \frac{1}{4}\right]\left[(\Delta \mathcal{J}_{\mathrm{x}})^2 + (\Delta \mathcal{J}_{\mathrm{y}})^2\right] \geq \frac{1}{4}\left\langle \mathcal{J}_{\mathrm{x}}^2 + \mathcal{J}_{\mathrm{y}}^2 \right\rangle.$$

The two-well entanglement criterion

Main result I

For separable states,

$$\left[(\Delta J_z)^2 + \frac{1}{4}\right]\left[(\Delta \mathcal{J}_x^-)^2 + (\Delta \mathcal{J}_y^-)^2\right] \geq \frac{1}{16}\left\langle \mathcal{J}_x^2 + \mathcal{J}_y^2 \right\rangle^2$$

holds.

Here,

$$\begin{aligned} J_z &= J_z^a + J_z^b, \\ \mathcal{J}_m^- &= \mathcal{J}_m^a - \mathcal{J}_m^b \end{aligned}$$

for $m = x, y$.

The two-well EPR-Steering criterion

Main result II

For states with a hidden state model,

$$\left[(\Delta J_z)^2 + \frac{1}{4}\right]\left[(\Delta \mathcal{J}_x^-)^2 + (\Delta \mathcal{J}_y^-)^2\right] \geq \frac{1}{4}\left\langle (\mathcal{J}_x^a)^2 + (\mathcal{J}_y^a)^2 \right\rangle^2$$

holds.

Any state violating the inequality cannot be described by a hidden state model, i.e., the state is *steerable*.

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Problem 1: Varying particle number

- The experiment is repeated many times. Each time we find a somewhat different particle number.

- Postselecting for a given particle number is not feasible.

- Hence, the density matrix is of the form

$$\varrho = \sum_{j_a, j_b} Q_{j_a, j_b} \varrho_{j_a, j_b},$$

  where ϱ_{j_a, j_b} are states with $2j_a$ and $2j_b$ particles in the two wells, Q_{j_a, j_b} are probabilities.

- ϱ is entangled iff at least one of the ϱ_{j_a, j_b} is entangled.

Problem 1: Varying particle number II

- Splitting noise: Even if we have a constant total particle number, the ensemble will not be evenly split.

- Probability distribution for having $N/2 + x$ particles

$$p_x = 2^{-N}\binom{N}{N/2 + x}.$$

- Variance

$$\mathrm{var}(N_\mathrm{a}) = \mathrm{var}(x) = \langle x^2 \rangle = \frac{N}{4.}$$

- Collective variance

$$[\Delta(J_l^a - J_l^b)]^2 \approx \sum_{x=-N/2}^{N/2} p_x \left(\frac{N}{8} + \frac{1}{2}x^2\right) = \frac{N}{8} + \frac{1}{2}\mathrm{var}(x) = \frac{N}{4}.$$

Twice as large due to the unequal splitting.

Problem 1: Varying particle number III

- $N/2 : N/2$ splitting:

$$[\Delta(J_l^a - J_l^b)]^2 = \frac{N}{8}.$$

- Real splitting with partition noise:

$$[\Delta(J_l^a - J_l^b)]^2 \approx \frac{N}{4}.$$

Problem 1: Varying particle number IV

- Solution: Let use the normalized quantity mentioned before

$$\mathcal{J}_l^- = \frac{1}{\sqrt{j_a(j_a+1)}} J_l^a - \frac{1}{\sqrt{j_b(j_b+1)}} J_l^b$$

  for $l = x, y$.

- For the variance of $\mathcal{J}_l$ we obtain

$$(\Delta \mathcal{J}_l^-)^2 \approx \frac{N}{N^2/2 + 4N - 2x^2}.$$

  After splitting $|x| \lesssim \sqrt{N/4}$.

- We have

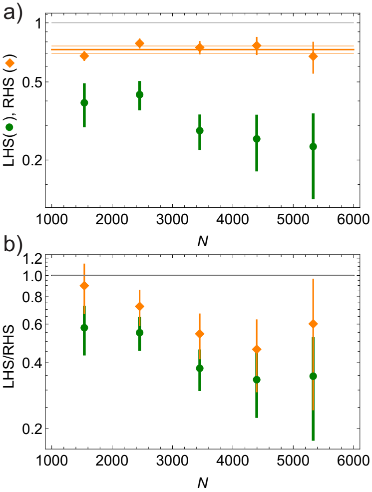
$$(\Delta \mathcal{J}_l^-)^2 \approx \frac{2}{N}.$$

  $(\Delta \mathcal{J}_l)^2$ is not sensitive to the fluctuation of x if N is large.

Problem 2: States are not always symmertic in a BEC of two-state atoms

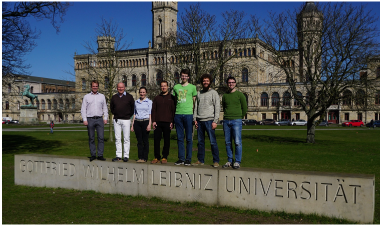
- Ideally, the BEC is in a single spatial mode.
- The state of an ensemble of the two-state atoms must be symmetric.
- In practice, the BEC is not in a single spatial mode, so there is no perfect symmetry.
- Our criterion must hadle this.

Violation of the criterion: entanglement is detected II



LHS/RHS for (top) our present work, and (bottom) for Science 2018.

Collaborators on entanglement conditions for double-well Dicke states



C. Klempt, I. Kruse, J. Peise,
K. Lange, B. Lücke

Hannover

G. Vitagliano

IQOQI, Wien

I. Apellaniz

Bilbao (G.T.)

M. Fadel

ETH Zürich, Basel

M. Kleinmann

U. of Siegen

Summary

- Detection of bipartite entanglement and EPR steering close to Dicke states. It works also for split spin-squeezed states.
- Non-symmetric states within the wells and a varying particle number can also be handled.

G. Vitagliano, I. Apellaniz, M. Fadel, M. Kleinmann,
B. Lücke, C. Klempt, and G. Tóth,
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