



Entanglement and permutational symmetry

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- 1 Motivation
- 2 Entanglement criteria for bipartite systems
- 3 Symmetric bound entangled states—Bipartite case
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Motivation

- Symmetry is a central concept in quantum mechanics. Typically, the presence of some symmetry simplifies our calculations in physics.
- A particular type of symmetry, permutational symmetry appears in many systems studied in quantum optics.
- The separability problem is proven to be a very hard one. Thus, it is interesting to ask how permutational symmetry can simplify the separability problem.



Entanglement criteria for bipartite systems

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Two types of symmetries

Consider two d -dimensional quantum systems. We will examine two types of permutational symmetries, denoting the corresponding sets by \mathcal{I} and \mathcal{S} :

- 1 We call a state **permutationally invariant** (or just invariant, $\varrho \in \mathcal{I}$) if ϱ is invariant under exchanging the particles. That is, $F\varrho F = \varrho$, where the flip operator is $F = \sum_{ij} |ij\rangle\langle ji|$. The reduced state of two randomly chosen particles of a larger ensemble has this symmetry.

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- 2 We call a state **symmetric** ($\varrho \in \mathcal{S}$) if it acts on the symmetric subspace only. This is the state space of two d -state bosons.

Clearly, we have $\mathcal{S} \subset \mathcal{I}$.

Expectation value matrix

Definition

Expectation value matrix of a bipartite quantum state is

$$\eta_{kl}(\varrho) := \langle M_k \otimes M_l \rangle_{\varrho},$$

where M_k 's are local orthogonal observables for both parties, satisfying

$$\text{Tr}(M_k M_l) = \delta_{kl}.$$

- We can decompose the density matrix as

$$\varrho = \sum_{kl} \eta_{kl} M_k \otimes M_l.$$

Equivalence of several entanglement conditions for symmetric states

Observation 1. Let $\varrho \in \mathcal{S}$ be a symmetric state. Then the following separability criteria are equivalent:

- 1 ϱ has a positive partial transpose (PPT), $\varrho^{T_A} \geq 0$.
- 2 ϱ satisfies the Computable Cross Norm-Realignment (CCNR) criterion, $\|R(\varrho)\|_1 \leq 1$, where $R(\varrho)$ is the realignment map and $\|\dots\|_1$ is the trace norm.
- 3 $\eta \geq 0$, or, equivalently $\langle A \otimes A \rangle \geq 0$ for all observables A .
- 4 The correlation matrix, defined via the matrix elements as

$$C_{kl} := \langle M_k \otimes M_l \rangle - \langle M_k \otimes \mathbb{1} \rangle \langle \mathbb{1} \otimes M_l \rangle$$

is positive semidefinite: $C \geq 0$. [A.R. Usha Devi *et al.*, Phys. Rev. Lett. **98**, 060501 (2007).]

- 5 The state satisfies several variants of the Covariance Matrix Criterion (CMC). Latter are strictly stronger than the CCNR criterion for non-symmetric states.

Proof of Observation 1: Schmidt decomposition

Proof.

- For invariant states, η is a real symmetric matrix. It can be diagonalized by an orthogonal matrix O . The resulting diagonal matrix $\{\Lambda_k\}$ is the correlation matrix corresponding to the observables $M'_k = \sum O_{kl} M_l$.

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- Hence, any invariant state can be written as

$$\varrho = \sum_k \Lambda_k M'_k \otimes M'_k,$$

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- It can be shown that $-1 \leq \sum_k \Lambda_k \leq 1$ for invariant states and $\sum_k \Lambda_k = 1$ for symmetric states.

Proof of Observation 1: Equivalence of CCNR and

$$\eta \geq 0$$

- The Computable Cross Norm-Realignment (CCNR) can be formulated as follows: If

$$\sum_k |\Lambda_k| > 1$$

in the Schmidt decomposition, then the quantum state is entangled.

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- For symmetric states we have $\sum_k \Lambda_k = 1$, and $\sum_k |\Lambda_k| > 1$ is equivalent to

$$\Lambda_k < 0$$

for some k . Then $\langle M'_k \otimes M'_k \rangle < 0$ and η has a negative eigenvalue.

Proof of Observation 1: CCNR–PPT equivalence

Let us take an alternative definition of the CCNR criterion.

- The CCNR criterion states that if ϱ is separable, then $\|R(\varrho)\|_1 \leq 1$ where the realigned density matrix is $R(\varrho_{ij,kl}) = \varrho_{ik,jl}$. This just means that if

$$\|(\varrho F)^{T_A}\|_1 > 1$$

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[M.M. Wolf, Ph.D. Thesis, TU Braunschweig, 2003.]

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- Since for symmetric states

$$\varrho F = \varrho,$$

this condition is equivalent to $\|\varrho^{T_A}\|_1 > 1$. This is just the PPT criterion, since we have $\text{Tr}(\varrho^{T_A}) = 1$.

Proof of Observation 1: Equivalence of $C \geq 0$ and $\eta \geq 0$

- Now we show that $C \geq 0 \Leftrightarrow \eta \geq 0$.
- The direction “ \Rightarrow ” is trivial, since for invariant states the matrix $\langle M_k \otimes \mathbb{1} \rangle \langle \mathbb{1} \otimes M_l \rangle$ is a projector and hence positive.
- The direction “ \Leftarrow ”: We make for a given state the special choice of observables $Q_k = M_k - \langle M_k \rangle$. Then, we just have $C(M_k) = \eta(Q_k)$.
- We know that $\eta(M_k) \geq 0 \Rightarrow \eta(Q_k) \geq 0$, even if Q_k are not pairwise orthogonal observables. Hence $C(M_k) \geq 0$ follows.

Proof of Observation 1: Covariance Matrix Criterion

- Variants of the Covariance Matrix Criterion:

$$\|C\|_1^2 \leq [1 - \text{Tr}(\varrho_A^2)][1 - \text{Tr}(\varrho_B^2)]$$

or

$$2 \sum |C_{ii}| \leq [1 - \text{Tr}(\varrho_A^2)] + [1 - \text{Tr}(\varrho_B^2)].$$

[O. Gühne *et al.*, PRL **99**, 130504 (2007); O. Gittsovich *et al.*, PRA **78**, 052319 (2008).]

- If ϱ is symmetric, the fact that C is positive semidefinite gives $\|C\|_1 = \text{Tr}(C) = \sum \Lambda_k - \sum_k \text{Tr}(\varrho_A M'_k)^2 = 1 - \text{Tr}(\varrho_A^2)$ and similarly, $\sum_i |C_{ii}| = \sum_i C_{ii} = 1 - \text{Tr}(\varrho_A^2)$.
- Hence, a state fulfilling $C \geq 0$ fulfills also both criteria. On the other hand, a state violating $C \geq 0$ must also violate these criteria, as they are strictly stronger than the CCNR criterion

Consequences

- Interesting result: For symmetric ϱ

$$\varrho^{T1} \geq 0 \iff \forall A : \langle A \otimes A \rangle \geq 0.$$

This relates the positivity of partial transposition to the sign of certain two-body correlations.

- Any symmetric state of the following form is PPT

$$\varrho_{\text{PPT}} = \sum_k p_k M_k \otimes M_k, \quad (1)$$

where p_k is a probability distribution, and M_k are pairwise orthogonal observables, i.e., $\text{Tr}(M_k^2) = 1$. Compare this to the definition of separability

$$\varrho_{\text{sep}} = \sum_k p_k \varrho_k \otimes \varrho_k, \quad (2)$$

where ϱ_k are observables, $\text{Tr}(\varrho_k) = 1$, $\varrho_k \geq 0$ and ϱ_k are pure, i.e., $\text{Tr}(\varrho_k^2) = 1$.

Consequences II

- Any symmetric state that can be written as

$$\varrho_{c+} = \sum_k c_k A_k \otimes A_k, \quad (3)$$

where $c_k > 0$, and A_k are some (not necessarily pairwise orthogonal) observables, is PPT. If ϱ_{c+} is permutationally invariant, then it does not violate the CCNR criterion.

- Multipartite case: A symmetric state of the form

$$\varrho_{\text{PPT}2:2} = \sum_k c_k A_k \otimes A_k \otimes A_k \otimes A_k \quad (4)$$

is PPT with respect to the $2 : 2$ partition. Example: Smolin state.

Are there symmetric bound entangled states?

- For symmetric states,

- 1 CCNR,
- 2 $\eta \geq 0$,
- 3 $C \geq 0$ and
- 4 CMC

are equivalent to the PPT criterion.

- It is then quite hard to find symmetric PPT entangled states.

Do symmetric bound entangled states exist at all?



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Symmetric bound entangled states

- Breuer presented, for even $d \geq 4$, a single parameter family of bound entangled states that are \mathcal{I} symmetric

$$\varrho_B = \lambda |\Psi_0^d\rangle\langle\Psi_0^d| + (1 - \lambda)\Pi_S^d.$$

[H.-P. Breuer, PRL **97**, 080501 (2006); see also K.G.H. Vollbrecht and M.M. Wolf, PRL **88**, 247901 (2002).]

- The state is PPT entangled for $0 \leq \lambda \leq 1/(d + 2)$. Here $|\Psi_0\rangle$ is the singlet state and Π_S is the normalized projector to the symmetric subspace.
- Idea to construct bound entangled states with an \mathcal{S} -symmetry: We embed a low dimensional entangled state into a higher dimensional Hilbert space, such that it becomes symmetric, while it remains entangled.

An 8×8 symmetric bound entangled states

- We consider the symmetric state



$$\hat{\rho} = \lambda \Pi_a^{d_2} \otimes |\Psi_0^d\rangle\langle\Psi_0^d| + (1 - \lambda) \Pi_s^{d_2} \otimes \Pi_s^d.$$

Here, $\Pi_a^{d_2}$ and $\Pi_s^{d_2}$ are normalized projectors to the two-qudit symmetric/antisymmetric subspace with dimension d_2 . Thus, $\hat{\rho}$ is symmetric.

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Here, $\Pi_a^{d_2}$ and $\Pi_s^{d_2}$ are normalized projectors to the two-qudit symmetric/antisymmetric subspace with dimension d_2 . Thus, $\hat{\varrho}$ is symmetric.

- If the original system is of dimension $d \times d$ then the system of $\hat{\varrho}$ is of dimension $dd_2 \times dd_2$. Since ϱ_B is the reduced state of $\hat{\varrho}$, if the first is entangled, then the second is also entangled.
- For $d_2 = 2$ and $d = 4$, numerical calculation shows that $\hat{\varrho}$ is PPT for $\lambda < 0.062$.

This provides an example of an \mathcal{S} symmetric bound entangled state of size 8×8 .



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Symmetric bound entangled state via numerics–

Basic idea

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- It is known that such a state is either separable with respect to all bipartitions or it is entangled with respect to all bipartitions.

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- Thus any state that is PPT with respect to the $\frac{N}{2} : \frac{N}{2}$ partition while NPT with respect to some other partition is bound entangled with respect to the $\frac{N}{2} : \frac{N}{2}$ partition.



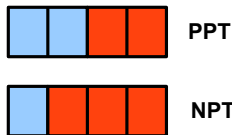
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- Since the state is symmetric, it can straightforwardly be mapped to a $(\frac{N}{2} + 1) \times (\frac{N}{2} + 1)$ bipartite symmetric state.

Symmetric bound entangled state via numerics II

- To obtain such a multiqubit state, one has to first generate an initial random state ρ that is PPT with respect to the $\frac{N}{2} : \frac{N}{2}$ partition.

Symmetric bound entangled state via numerics II

- To obtain such a multiqubit state, one has to first generate an initial random state ϱ that is PPT with respect to the $\frac{N}{2} : \frac{N}{2}$ partition.
- Then, we compute the minimum nonzero eigenvalue of the partial transpose of ϱ with respect to all other partitions

$$\lambda_{\min}(\varrho) := \min_k \min_l \lambda_l(\varrho^{T_{l_k}}).$$

If $\lambda_{\min}(\varrho) < 0$ then the state is bound entangled with respect to the $\frac{N}{2} : \frac{N}{2}$ partition. If it is non-negative then we start an optimization process for decreasing this quantity.

Symmetric bound entangled state via numerics II

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- We generate another random density matrix $\Delta\varrho$, and check the properties of

$$\varrho' = (1 - \varepsilon)\varrho + \varepsilon\Delta\varrho, \tag{5}$$

where $0 < \varepsilon < 1$ is a small constant. If ϱ' is still PPT with respect to the $\frac{N}{2} : \frac{N}{2}$ partition and $\lambda_{\min}(\varrho') < \lambda_{\min}(\varrho)$ then we use ϱ' as our new random initial state ϱ .

3×3 symmetric bound entangled state

- Repeating this procedure, we obtained a four-qubit symmetric state this way

$$\rho_{BE4} = \begin{pmatrix} 0.22 & 0 & 0 & -0.059 & 0 \\ 0 & 0.176 & 0 & 0 & 0 \\ 0 & 0 & 0.167 & 0 & 0 \\ -0.059 & 0 & 0 & 0.254 & 0 \\ 0 & 0 & 0 & 0 & 0.183 \end{pmatrix}.$$

The basis states are $|0\rangle := |0000\rangle$, $|1\rangle := \text{sym}(|1000\rangle)$, $|2\rangle := \text{sym}(|1100\rangle)$, ...

- The state is bound entangled with respect to the $2 : 2$ partition. This corresponds to a 3×3 bipartite symmetric bound entangled state, demonstrating the simplest possible symmetric bound entangled state.

Five- and six-qubit fully PPT entangled states

- Our method can be straightforwardly generalized to create multipartite bound entangled states of the symmetric subspace, such that *all* bipartitions are PPT (“fully PPT states”).
- We found such a state for five and six qubits.
- Note that these states are **both fully PPT and genuine multipartite entangled**. It is further interesting to relate this to the Peres conjecture, stating that fully PPT states cannot violate a Bell inequality.



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Conclusions

- In summary, we have discussed entanglement in symmetric systems.
- We showed that for states that are in the symmetric subspace several relevant entanglement conditions, especially the PPT criterion, the CCNR criterion, and the criterion based on covariance matrices coincide.
- We proved the existence of symmetric bound entangled states, in particular, a 3×3 , five-qubit and six-qubit symmetric PPT entangled states.
- See G. Tóth and O. Gühne, PRL 102, 170503 (2009).

*** THANK YOU ***