Uncertainty relations with the variance and the quantum Fisher information

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Benasque, 19 May 2022

Outline

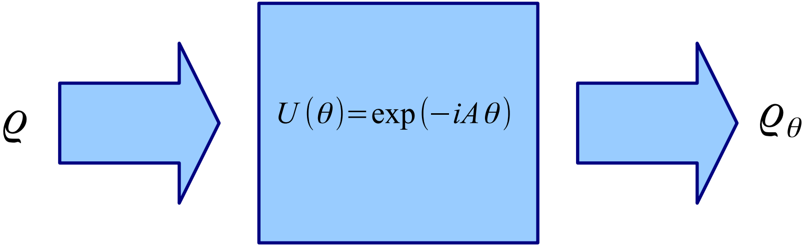
How can we improve uncertainty relations?

- There are many approaches to improve uncertainty relations.
- We show a method that replaces the variance with the quantum Fisher information in some well known uncertainty relations.
- We use convex/concave roofs over the decompostions of the density matrix.

Outline

Quantum metrology

- Fundamental task in metrology



- We have to estimate θ in the dynamics

$$U = \exp(-iA\theta).$$

The quantum Fisher information

- Cramér-Rao bound on the precision of parameter estimation

$$(\Delta\theta)^2 \geq \frac{1}{mF_Q[\varrho, A]},$$

where $F_Q[\varrho, A]$ is the quantum Fisher information, and m is the number of independent repetitions.

- The quantum Fisher information is

$$F_Q[\varrho, A] = 2\sum_{k,l} \frac{(\lambda_k - \lambda_l)^2}{\lambda_k + \lambda_l} |\langle k|A|l\rangle|^2,$$

where $\varrho = \sum_k \lambda_k |k\rangle\langle k|$.

Most important characteristics used for estimation

The quantum Fisher information is the convex roof of the variance

$$F_Q[\varrho, A] = 4 \min_{p_k, \Psi_k} \sum_k p_k (\Delta A)^2{}_k,$$

where

$$\varrho = \sum_k p_k |\Psi_k\rangle\langle\Psi_k|.$$

[GT, D. Petz, Phys. Rev. A 87, 032324 (2013); S. Yu, arXiv1302.5311 (2013); GT, I. Apellaniz, J. Phys. A: Math. Theor. 47, 424006 (2014)]

- Extended convexity for non-unitary dynamics.
  [S. Alipour, A. T. Rezakhani, Phys. Rev. A 91, 042104 (2015).]
- Convex roof over purifications.
  [R. Demkowicz-Dobrzański, J. Kołodyński, M. Guţă, Nature Comm. 2012.]

Similar relation for the variance

The quantum Fisher information is the convex roof of the variance

$$(\Delta A)^2{}_\varrho = \max_{p_k,\Psi_k} \sum_k p_k (\Delta A)^2{}_k,$$

where

$$\varrho = \sum_k p_k |\Psi_k\rangle\langle\Psi_k|.$$

[GT, D. Petz, Phys. Rev. A 87, 032324 (2013);
GT, I. Apellaniz, J. Phys. A: Math. Theor. 47, 424006 (2014)]

A single relation for the QFI and the variance

The previous statements can be concisely reformulated as follows. For any decomposition $\{p_k, |\psi_k\rangle\}$ of the density matrix ϱ we have

$$\frac{1}{4}F_Q[\varrho, A] \le \sum_k p_k (\Delta A)^2{}_{\psi_k} \le (\Delta A)^2{}_{\varrho},$$

where the upper and the lower bounds are both tight.

Outline

Robertson-Schrödinger inequality

The Robertson-Schrödinger inequality is defined as

$$(\Delta A)^2{}_\varrho(\Delta B)^2{}_\varrho \geq \frac{1}{4}|L_\varrho|^2,$$

where the lower bound is given by

$$L_\varrho = \sqrt{|\langle\{A, B\}\rangle_\varrho - 2\langle A\rangle_\varrho\langle B\rangle_\varrho|^2 + |\langle C\rangle_\varrho|^2},$$

$\{A, B\} = AB + BA$ is the anticommutator, and we used the definition

$$C = i[A, B].$$

Important: L_ϱ is neither convex nor concave in ϱ.

Heisenberg uncertainty

The Heisenberg inequality is defined as

$$(\Delta A)^2{}_\varrho (\Delta B)^2{}_\varrho \geq \frac{1}{4} |\langle C \rangle_\varrho|^2,$$

where we used the definition

$$C = i[A, B].$$

The two inequalities together

We have two inequalities

$$(\Delta A)^2{}_\varrho(\Delta B)^2{}_\varrho \geq \frac{1}{4}|L_\varrho|^2 \geq \frac{1}{4}|\langle C\rangle_\varrho|^2.$$

The Heisenberg uncertainty can be saturated only if

$$|L_\varrho| = |\langle C\rangle_\varrho|.$$

Outline

Robertson-Schrödinger inequality for ϱ_k

- Consider a decomposition to mixed states

$$\varrho = \sum_k p_k \varrho_k.$$

- For such a decomposition, for all ϱ_k the Robertson-Schrödinger inequality holds

$$(\Delta A)^2{}_{\varrho_k} (\Delta B)^2{}_{\varrho_k} \geq \tfrac{1}{4} |L_{\varrho_k}|^2.$$

- Let us consider the inequality

$$\left(\sum_k p_k a_k \right) \left(\sum_k p_k b_k \right) \geq \left(\sum_k p_k \sqrt{a_k b_k} \right)^2,$$

where $a_k, b_k \geq 0$.

New inequality from decompositions

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Uncertainty with the variance and the QFI

- Hence, we arrive at

$$\left[\sum_k p_k (\Delta A)^2{}_{\varrho_k}\right]\left[\sum_k p_k (\Delta B)^2{}_{\varrho_k}\right] \geq \frac{1}{4}\left[\sum_k p_k L_{\varrho_k}\right]^2.$$

- We can choose the decomposition such that

$$\sum_k p_k (\Delta B)^2{}_{\varrho_k} = F_Q[\varrho, B]/4.$$

- Due to the concavity of the variance we also know that

$$\sum_k p_k (\Delta A)^2{}_{\varrho_k} \leq (\Delta A)^2.$$

- Hence, it follows that

$$(\Delta A)^2{}_{\varrho} F_Q[\varrho, B] \geq \left(\sum_k p_k L_{\varrho_k}\right)^2.$$

- In order to use the previous inequality, we need to know the decomposition $\{p_k, \varrho_k\}$ that minimizes it.

Uncertainty with the variance and the QFI II

- We can have a inequality where we do not need to know that decomposition

$$(\Delta A)^2_{\varrho} F_Q[\varrho, B] \geq \left(\min_{\{p_k, \varrho_k\}} \sum_k p_k L_{\varrho_k} \right)^2.$$

- On the right-hand side, the bound is defined based on a convex roof.

- It can be shown that we can move to pure state decompositions.

- We know that

$$L_{\psi_k} \geq |\langle C \rangle_{\psi_k}|$$

holds.

Uncertainty with the variance and the QFI II

- Then, we can obtain the inequality

$$(\Delta A)^2_\varrho F_Q[\varrho, B] \geq \left(\min_{\{p_k, |\psi_k\rangle\}} \sum_k p_k |\langle C \rangle_{\psi_k}| \right)^2,$$

- Using

$$\sum_k p_k |\langle C \rangle_{\psi_k}| \geq \left| \sum_k p_k \langle C \rangle_{\psi_k} \right| \equiv |\langle C \rangle_\varrho|,$$

we arrive at the improved Heisenberg-Robertson uncertainty

$$(\Delta A)^2_\varrho F_Q[\varrho, B] \geq |\langle C \rangle_\varrho|^2.$$

Uncertainty with the variance and the QFI III

- The Heisenberg uncertainty

$$(\Delta A)^2{}_\varrho (\Delta B)^2{}_\varrho \geq \frac{1}{4}|\langle i[A,B]\rangle_\varrho|^2.$$

- The improved Heisenberg uncertinty

$$(\Delta A)^2{}_\varrho F_Q[\varrho, B] \geq |\langle i[A,B]\rangle_\varrho|^2.$$

- It has been derived originally with a different method in

F. Fröwis, R. Schmied, and N. Gisin, Phys. Rev. A 92, 012102 (2015).

Conditions for saturation

- Conditions for saturating the relation with the simple bound

$$(\Delta A)^2{}_{\varrho} F_Q[\varrho, B] \geq \left(\min_{\{p_k,\varrho_k\}} \sum_k p_k L_{\varrho_k} \right)^2 \geq |\langle C \rangle_{\varrho}|^2.$$

- We have to have equality on the right-hand side.

- The optimal decomposotion can be made with pure components Ψ_k. Then, for all k, l we must have

$$\tfrac{1}{2}\langle\{A, B\}\rangle_{\psi_k} - \langle A\rangle_{\psi_k}\langle B\rangle_{\psi_k} = 0,$$

$$\begin{aligned} (\Delta A)^2{}_{\psi_k} &= (\Delta A)^2{}_{\psi_l}, \\ (\Delta B)^2{}_{\psi_k} &= (\Delta B)^2{}_{\psi_l}, \end{aligned}$$

$$|\langle C\rangle_{\psi_k}| = |\langle C\rangle_{\varrho}|,$$

etc.

Outline

Cramér-Rao bound

- Error propagation formula

$$(\Delta\theta)^2_A = \frac{(\Delta A)^2}{|\partial_\theta \langle A \rangle|^2} = \frac{(\Delta A)^2}{|\langle C \rangle|^2}.$$

- If we measure A, then the precision of the estimation is bounded as

$$(\Delta\theta)^2 \geq \frac{1}{m} {(\Delta\theta)^2}_A,$$

where m is the number of independent repetitions.

- Cramér-Rao bound based on a convex roof

$$(\Delta\theta)^2 \geq \frac{1}{m} \times \frac{1}{\underbrace{4 \min_{\{p_k, |\psi_k\rangle\}} \left[\sum_k p_k {(\Delta B)^2}_{\psi_k} \right]}_{F_Q[\varrho, B]}}.$$

Outline

Uncertainty relation based on a concave roof

- For any decomposition $\{p_k, \varrho_k\}$ we have

$$(\Delta A)^2(\Delta B)^2 \geq \frac{1}{4}\left(\sum_k p_k L_{\varrho_k}\right)^2,$$

where

$$L_\varrho = \sqrt{|\langle\{A,B\}\rangle_\varrho - 2\langle A\rangle_\varrho\langle B\rangle_\varrho|^2 + |\langle C\rangle_\varrho|^2}.$$

- We can even take a concave roof on the right-hand side

$$(\Delta A)^2{}_\varrho(\Delta B)^2{}_\varrho \geq \frac{1}{4}\left(\max_{\{p_k,\varrho_k\}}\sum_k p_k L_{\varrho_k}\right)^2.$$

- We prove that for qubits the above inequality is saturated for all states.

Any decomposition leads to a valid bound

- A simple inequality that is valid

$$(\Delta A)^2{}_\varrho (\Delta B)^2{}_\varrho \geq \frac{1}{4}\left(\sum_k \lambda_k L_{|k\rangle}\right)^2.$$

  if we have an eigendecompostion

$$\varrho = \sum_k \lambda_k |k\rangle\langle k|.$$

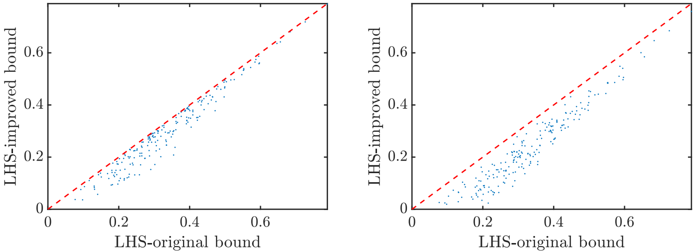
- We can even look for concave roof numerically.

Numerical example

- For $d = 3$

$$(\Delta J_x)^2{}_{\varrho}(\Delta J_y)^2{}_{\varrho} \geq \frac{1}{4}\left(\max_{\{p_k,\varrho_k\}} \sum_k p_k L_{\varrho_k}\right)^2.$$

- Eigenvalues J_x and J_y are $-1, 0, +1$.



Using the eigendecomposition　　Numerical search

Outline

Uncertainty relations with a variance and the QFI

- Similar ideas work even for a sum of two variances. For example, for a continuous variable system

$$(\Delta x)^2 + (\Delta p)^2 \geq 1$$

holds, where x and p are the position and momentum operators.

- Hence, for any decompositions of the density matrix it follows that

$$\sum_k p_k (\Delta x)^2{}_{\psi_k} + \sum_k p_k (\Delta p)^2{}_{\psi_k} \geq 1.$$

- For p we choose the decomposition that leads to the minimal value for the average variance, i.e., the QFI over four.

- Then, since $\sum_k p_k (\Delta x)^2{}_{\psi_k} \leq (\Delta x)^2$ holds, it follows that

$$(\Delta x)^2 + \tfrac{1}{4} F_Q[\varrho, p] \geq 1.$$

Uncertainty relations with two variances and the QFI

- Let us start from the relation for pure states

$$(\Delta J_x)^2 + (\Delta J_y)^2 + (\Delta J_z)^2 \geq j,$$

where J_l are the spin components fulfilling

$$J_x^2 + J_y^2 + J_z^2 = j(j+1)\mathbb{1}.$$

- Based on similar ideas we arriving at

$$(\Delta J_x)^2 + (\Delta J_y)^2 + \frac{1}{4}F_Q[\varrho, J_z] \geq j.$$

- See parallel publication in

S.-H. Chiew and M. Gessner, Phys. Rev. Research 4, 013076 (2022).

Uncertainty relations with two variances and the QFI II

- For a spin-j particle, the following inequality bounds from below the metrological usefulness of the state

$$F_Q[\varrho, J_z] \geq 4j - 4(\Delta J_x)^2 - 4(\Delta J_y)^2 =: B_{FQ}.$$

- For instance, this can be used in BEC of two-state atoms to bound $F_Q[\varrho, J_z]$ with the variances.

Outline

Simple observation to prove more relations

- Let us consider a relation

$$\underbrace{(\Delta A)^2{}_{\varrho}}_{\text{variance}} \geq \underbrace{g(\varrho)}_{\text{convex in } \varrho},$$

  which is true for pure states.

- If $g(\varrho)$ is convex in density matrices, then

$$\frac{1}{4} F_Q[\varrho, A] \geq g(\varrho)$$

  holds for mixed states.

- *Proof.* $\frac{1}{4} F_Q[\varrho, A]$ is given as a convex roof of the variance.

- It is the largest convex function that equals $(\Delta A)^2{}_{\varrho}$ for all pure states. ■

Repeating the proof for the two variances and the QFI

- We rewrite the relation with three variances as

$$\underbrace{(\Delta J_x)^2}_{\text{variance}} \geq \underbrace{j - (\Delta J_y)^2 - (\Delta J_z)^2}_{\text{convex in } \varrho},$$

- The right-hand side is convex in ϱ and the left-hand side is a variance.
- Hence,

$$\frac{1}{4} F_Q[\varrho, J_z] \geq j - (\Delta J_x)^2 - (\Delta J_y)^2.$$

Extreme spin squeezing

- For a particle with spin-j

$$\underbrace{(\Delta J_x)^2}_{\text{variance}} \geq \underbrace{jF_j(\langle J_z\rangle/j)}_{\text{convex in } \varrho}$$

holds, where $F_j(X)$ is a convex function defined as

$$F_j(X) = \min_{\varrho:\langle J_z\rangle = Xj} \frac{(\Delta J_x)^2}{j}.$$

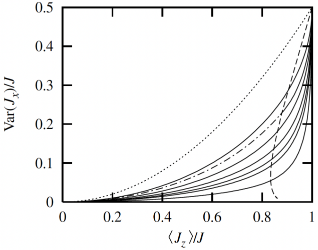


FIG. 1. Maximal squeezing for different values of J. The curves starting at the origin represent the minimum obtainable variance as a function of the mean spin. Starting from above, the curves represent $J = 1/2$, 1, 3/2, 2, 3, 4, 5, and 10. The dotted curve for $J = 1/2$ is the limit identified in Ref. [11]. The solid

Application for extreme spin squeezing

- The metrological usefulness of a state is bounded with the spin-length as
$$F_Q[\varrho, J_x] \geq 4jF_j(\langle J_z \rangle / j).$$

- *Proof.* For the components of the angular momentum
$$(\Delta J_x)^2 \geq jF_j(\langle J_z \rangle / j)$$
holds. Then, it follows
$$\frac{1}{4}F_Q[\varrho, J_x] \geq jF_j(\langle J_z \rangle / j).$$
$\blacksquare$

Application for extreme spin squeezing II

- It is clear that if $\langle J_z \rangle > 0$ then $F_j(\langle J_z \rangle / j) > 0$. Hence, it follows that $F_Q[\varrho, J_x] > 0$.
- Thus, if the z-component of the angular momentum has a non-zero expectation value then the state can be used for metrology with the Hamiltonian J_x.

Outline

CV systems

- Bosonic mode, canonical x and p operators. For coherent states, $|\alpha\rangle$

$$(\Delta x)^2 = (\Delta p)^2 = \frac{1}{2}$$

holds. For mixtures of coherent states

$$\varrho_{\rm mc} = \sum_k p_k |\alpha_k\rangle\langle\alpha_k|$$

due to the concavity of the variance and the convexity of the QFI

$$(\Delta x)^2, (\Delta p)^2 \geq \frac{1}{2}, \quad F_Q[x, \varrho], F_Q[p, \varrho] \leq 2.$$

CV systems II

- For a mixture of products of coherent states $\alpha_k^{(l)}$ of the form

$$\varrho_{\text{sepc}} = \sum_k p_k |\alpha_k^{(1)}\rangle\langle\alpha_k^{(1)}| \otimes |\alpha_k^{(2)}\rangle\langle\alpha_k^{(2)}|$$

  for the the collective quantites

$$[\Delta(x_1 \pm x_2)]^2 \geq 1; \quad [\Delta(p_1 \pm p_2)]^2 \geq 1.$$

  Moreover,

$$F_Q[\varrho, p_1 \pm p_2] \leq 4; \quad F_Q[\varrho, x_1 \pm x_2] \leq 4.$$

  For such states the multi-variable Glauber-Sudarshan P function is non-negative.

CV systems III

- Consider entanglement detection in two-mode systems with uncertainty relations.

- A well-known entanglement criterion is

$$[\Delta(x_1 + x_2)]^2 + [\Delta(p_1 - p_2)]^2 \geq 2.$$

If a quantum state violates it, then it is entangled.

L.-M. Duan, G. Giedke, J. I. Cirac, and P. Zoller, Phys. Rev. Lett. 84, 2722 (2000);
R. Simon, Phys. Rev. Lett. 84, 2726 (2000).

CV systems IV

- For a two-mode state, the following uncertainty relation holds

$$[\Delta(x_1+x_2)]^2+[\Delta(p_1-p_2)]^2 \geq \\ 4/F_Q[\varrho,p_1+p_2]+4/F_Q[\varrho,x_1-x_2].$$

  As a consequence, states violating the entanglement condition are metrologically more useful than states that are the mixtures of products of coherent states.

- *Proof.* We start from the relations

$$\begin{aligned} [\Delta(x_1+x_2)]^2 F_Q[\varrho,p_1+p_2] &\geq 4, \\ [\Delta(p_1-p_2)]^2 F_Q[\varrho,x_1-x_2] &\geq 4. \end{aligned}$$

- Then, in both inequalities we divide by the term containing the QFI. Finally, we sum the two resulting inequalities.

CV systems V

- The violation of the entanglement criterion given implies the violation of one of the inequalities for the QFI.
- Thus, violation of the uncertainty relation-based entanglement condition also means that the state has larger metrological usefulness than mixtures of products of coherent states. ■
- We did not prove that violating the entanglement condition leads to larger metrological usefulness than that of separable states in general.
- Even for pure product states $F_Q[\varrho, x_1 \pm x_2]$ or $F_Q[\varrho, p_1 \pm p_2]$ can be arbitrarily large for two bosonic modes.

Summary

- We showed how to derive new uncertainty relations with the variance and the quantum Fisher information based on simple convexity arguments.

See:

Géza Tóth and Florian Fröwis,

Quantum states with a positive partial transpose
are useful for metrology,

Phys. Rev. Research 4, 013075 (2022).

THANK YOU FOR YOUR ATTENTION!

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