Uncertainty relations with the variance and the quantum Fisher information

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Outline

How can we improve uncertainty relations?

- There are many approaches to improve uncertainty relations.

- We show a method that replaces the variance with the quantum Fisher information in some well known uncertainty relations.

- We use convex/concave roofs over the decompostions of the density matrix.

Outline

Quantum metrology

- Fundamental task in metrology

$$\varrho \quad \Rightarrow \quad U(\theta) = \exp(-iA\theta) \quad \Rightarrow \quad \varrho_\theta$$
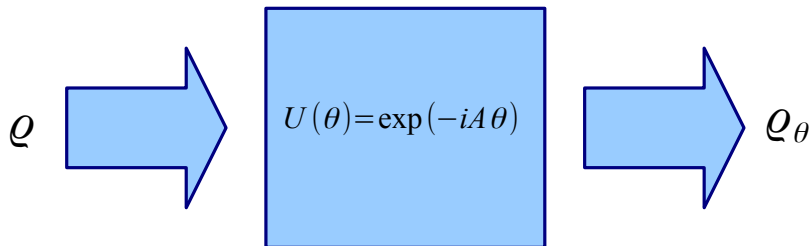
- We have to estimate θ in the dynamics

$$U = \exp(-iA\theta).$$

The quantum Fisher information

- Cramér-Rao bound on the precision of parameter estimation

$$(\Delta\theta)^2 \geq \frac{1}{mF_Q[\varrho, A]},$$

where $F_Q[\varrho, A]$ is the quantum Fisher information, and m is the number of independent repetitions.

- The quantum Fisher information is

$$F_Q[\varrho, A] = 2 \sum_{k,l} \frac{(\lambda_k - \lambda_l)^2}{\lambda_k + \lambda_l} |\langle k|A|l \rangle|^2,$$

where $\varrho = \sum_k \lambda_k |k\rangle\langle k|$.

Most important characteristics used for estimation

The quantum Fisher information is the convex roof of the variance

$$F_Q[\varrho, A] = 4 \min_{\{p_k, |\psi_k\rangle\}} \sum_k p_k (\Delta A)^2_{\psi_k},$$

where

$$\varrho = \sum_k p_k |\psi_k\rangle\langle\psi_k|.$$

[GT, D. Petz, Phys. Rev. A 87, 032324 (2013); S. Yu, arXiv1302.5311 (2013); GT, I. Apellaniz, J. Phys. A: Math. Theor. 47, 424006 (2014)]

- Convex roof over purifications.
  [R. Demkowicz-Dobrzański, J. Kołodyński, M. Guţă, Nature Comm. 2012.]

Similar relation for the variance

The quantum Fisher information is the convex roof of the variance

$$(\Delta A)^2{}_\varrho = \max_{\{p_k, |\psi_k\rangle\}} \sum_k p_k (\Delta A)^2{}_{\psi_k},$$

where

$$\varrho = \sum_k p_k |\psi_k\rangle\langle\psi_k|.$$

[GT, D. Petz, Phys. Rev. A 87, 032324 (2013);
GT, I. Apellaniz, J. Phys. A: Math. Theor. 47, 424006 (2014)]

A single relation for the QFI and the variance

The previous statements can be concisely reformulated as follows. For any decomposition $\{p_k, |\psi_k\rangle\}$ of the density matrix ϱ we have

$$\frac{1}{4} F_Q[\varrho, A] \leq \sum_k p_k (\Delta A)^2{}_{\psi_k} \leq (\Delta A)^2{}_\varrho,$$

where the upper and the lower bounds are both tight.

- Note that

$$F_Q[\varrho, A] \leq 4 (\Delta A)^2{}_\varrho,$$

  where we have an equality for pure states.

- The QFI appears as a "pair" of variance.

The quantum Fisher information vs. entanglement

- For separable states of N spin-1/2 particles

$$F_Q[\varrho, J_l] \leq N, \qquad l = x, y, z, \qquad J_l = \sum_{n=1}^{N} j_l^{(n)}.$$

[Pezze, Smerzi, Phys. Rev. Lett. 102, 100401 (2009); Hyllus, Gühne, Smerzi, Phys. Rev. A 82, 012337 (2010)]

- For states with at most k-particle entanglement (k is divisor of N)
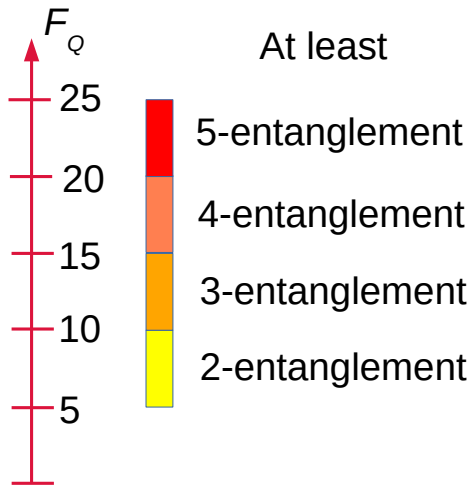
$$F_Q[\varrho, J_l] \leq kN.$$

[P. Hyllus *et al.*, Phys. Rev. A 85, 022321 (2012); GT, Phys. Rev. A 85, 022322 (2012)]. $\rightarrow$ Many experiments with cold gases and photons.

- In general

$$F_Q[\varrho, J_l] \leq N^2.$$

The quantum Fisher information vs. entanglement II

5 spin-1/2 particles

F_Q

At least

25 — 5-entanglement

20 — 4-entanglement

15 — 3-entanglement

10 — 2-entanglement

5

Outline

Robertson-Schrödinger inequality

The Robertson-Schrödinger inequality is defined as

$$(\Delta A)^2_\varrho (\Delta B)^2_\varrho \geq \tfrac{1}{4}|L_\varrho|^2,$$

where the lower bound is given by

$$L_\varrho = \sqrt{|\langle\{A,B\}\rangle_\varrho - 2\langle A\rangle_\varrho\langle B\rangle_\varrho|^2 + |\langle C\rangle_\varrho|^2},$$

$\{A,B\} = AB + BA$ is the anticommutator, and we used the definition

$$C = i[A,B].$$

Important: L_ϱ is neither convex nor concave in ϱ.

Heisenberg uncertainty

The Heisenberg inequality is defined as

$$(\Delta A)^2{}_\varrho (\Delta B)^2{}_\varrho \geq \tfrac{1}{4}|\langle C \rangle_\varrho|^2,$$

where we used the definition

$$C = i[A, B].$$

The two inequalities together

We have two inequalities

$$(\Delta A)^2_{\varrho}(\Delta B)^2_{\varrho} \geq \tfrac{1}{4}|L_{\varrho}|^2 \geq \tfrac{1}{4}|\langle C \rangle_{\varrho}|^2.$$

The Heisenberg uncertainty can be saturated only if

$$|L_{\varrho}| = |\langle C \rangle_{\varrho}|.$$

Outline

Robertson-Schrödinger inequality for ϱ_k

- Consider a decomposition to mixed states

$$\varrho = \sum_k p_k \varrho_k.$$

- For such a decomposition, for all ϱ_k the Robertson-Schrödinger inequality holds

$$(\Delta A)^2_{\varrho_k} (\Delta B)^2_{\varrho_k} \geq \tfrac{1}{4} |L_{\varrho_k}|^2.$$

- Let us consider the inequality

$$\left(\sum_k p_k a_k \right) \left(\sum_k p_k b_k \right) \geq \left(\sum_k p_k \sqrt{a_k b_k} \right)^2,$$

where $a_k, b_k \geq 0$.

Uncertainty with the variance and the QFI

- Hence, we arrive at

$$\left[\sum_k p_k (\Delta A)^2{}_{\varrho_k}\right]\left[\sum_k p_k (\Delta B)^2{}_{\varrho_k}\right] \geq \tfrac{1}{4}\left[\sum_k p_k L_{\varrho_k}\right]^2.$$

- We can choose the decomposition such that

$$\sum_k p_k (\Delta B)^2{}_{\varrho_k} = F_Q[\varrho, B]/4.$$

- Due to the concavity of the variance we also know that

$$\sum_k p_k (\Delta A)^2{}_{\varrho_k} \leq (\Delta A)^2.$$

- Hence, it follows that

$$(\Delta A)^2{}_{\varrho} F_Q[\varrho, B] \geq \left(\sum_k p_k L_{\varrho_k}\right)^2.$$

- In order to use the previous inequality, we need to know the decomposition $\{p_k, \varrho_k\}$ that minimizes it.

Uncertainty with the variance and the QFI II

- We can have a inequality where we do not need to know that decomposition

$$(\Delta A)^2{}_\varrho F_Q[\varrho, B] \geq \left(\min_{\{p_k, \varrho_k\}} \sum_k p_k L_{\varrho_k} \right)^2.$$

- On the right-hand side, the bound is defined based on a convex roof.

- It can be shown that we can move to pure state decompositions.

- We know that

$$L_{\psi_k} \geq |\langle C \rangle_{\psi_k}|$$

holds.

Uncertainty with the variance and the QFI II

- Then, we can obtain the inequality

$$(\Delta A)^2_{\varrho} F_Q[\varrho, B] \geq \left(\min_{\{p_k, |\psi_k\rangle\}} \sum_k p_k |\langle C \rangle_{\psi_k}| \right)^2,$$

- Using

$$\sum_k p_k |\langle C \rangle_{\psi_k}| \geq \left| \sum_k p_k \langle C \rangle_{\psi_k} \right| \equiv |\langle C \rangle_{\varrho}|,$$

we arrive at the improved Heisenberg-Robertson uncertainty

$$(\Delta A)^2_{\varrho} F_Q[\varrho, B] \geq |\langle C \rangle_{\varrho}|^2.$$

Uncertainty with the variance and the QFI III

- The Heisenberg uncertainty

$$(\Delta A)^2_{\varrho}(\Delta B)^2_{\varrho} \geq \tfrac{1}{4}|\langle i[A, B]\rangle_{\varrho}|^2.$$

- The improved Heisenberg uncertinty

$$(\Delta A)^2_{\varrho} F_Q[\varrho, B] \geq |\langle i[A, B]\rangle_{\varrho}|^2.$$

- It has been derived originally with a different method in

  F. Fröwis, R. Schmied, and N. Gisin, Phys. Rev. A 92, 012102 (2015).

Conditions for saturation

- Conditions for saturating the relation with the simple bound

$$(\Delta A)^2{}_\varrho F_Q[\varrho, B] \geq \left(\min_{\{p_k, \varrho_k\}} \sum_k p_k L_{\varrho_k} \right)^2 \geq |\langle C \rangle_\varrho|^2.$$

- We have to have equality on the right-hand side.

- The optimal decomposotion can be made with pure components Ψ_k. Then, for all k, l we must have

$$\tfrac{1}{2} \langle \{A, B\} \rangle_{\psi_k} - \langle A \rangle_{\psi_k} \langle B \rangle_{\psi_k} = 0,$$

$$\begin{aligned}
(\Delta A)^2{}_{\psi_k} &= (\Delta A)^2{}_{\psi_l}, \\
(\Delta B)^2{}_{\psi_k} &= (\Delta B)^2{}_{\psi_l}, \\
|\langle C \rangle_{\psi_k}| &= |\langle C \rangle_\varrho|,
\end{aligned}$$

etc.

Outline

Uncertainty relation based on a concave roof

- For any decomposition $\{p_k, \varrho_k\}$ we have

$$(\Delta A)^2 (\Delta B)^2 \geq \frac{1}{4} \left(\sum_k p_k L_{\varrho_k} \right)^2,$$

where

$$L_\varrho = \sqrt{|\langle \{A, B\} \rangle_\varrho - 2\langle A \rangle_\varrho \langle B \rangle_\varrho|^2 + |\langle C \rangle_\varrho|^2}.$$

- We can even take a concave roof on the right-hand side

$$(\Delta A)^2{}_\varrho (\Delta B)^2{}_\varrho \geq \frac{1}{4} \left(\max_{\{p_k, \varrho_k\}} \sum_k p_k L_{\varrho_k} \right)^2.$$

- We prove that for qubits the above inequality is saturated for all states.

Any decomposition leads to a valid bound

- A simple inequality that is valid

$$(\Delta A)^2_{\varrho}(\Delta B)^2_{\varrho} \geq \frac{1}{4}\left(\sum_k \lambda_k L_{|k\rangle}\right)^2.$$

  if we have an eigendecompostion
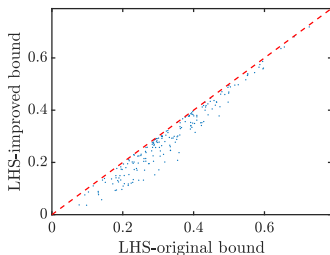
$$\varrho = \sum_k \lambda_k |k\rangle\langle k|.$$

- We can even look for concave roof numerically.
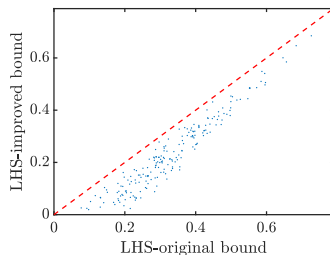
Numerical example

- For $d = 3$

$$(\Delta J_x)^2{}_\varrho (\Delta J_y)^2{}_\varrho \geq \frac{1}{4} \left(\max_{\{p_k, \varrho_k\}} \sum_k p_k L_{\varrho_k} \right)^2.$$

- Eigenvalues J_x and J_y are $-1, 0, +1$.

Using the eigendecomposition   Numerical search

Outline

Uncertainty relations with a variance and the QFI

- Similar ideas work even for a sum of two variances. For example, for a continuous variable system

$$(\Delta x)^2 + (\Delta p)^2 \geq 1$$

holds, where x and p are the position and momentum operators.

- Hence, for any decompositions of the density matrix it follows that

$$\sum_k p_k (\Delta x)^2_{\psi_k} + \sum_k p_k (\Delta p)^2_{\psi_k} \geq 1.$$

- For p we choose the decomposition that leads to the minimal value for the average variance, i.e., the QFI over four.

- Then, since $\sum_k p_k (\Delta x)^2_{\psi_k} \leq (\Delta x)^2$ holds, it follows that

$$(\Delta x)^2 + \tfrac{1}{4} F_Q[\varrho, p] \geq 1.$$

Uncertainty relations with two variances and the QFI

- Let us start from the relation for pure states

$$(\Delta J_x)^2 + (\Delta J_y)^2 + (\Delta J_z)^2 \geq j,$$

where J_l are the spin components fulfilling

$$J_x^2 + J_y^2 + J_z^2 = j(j+1)\mathbb{1}.$$

- Based on similar ideas we arriving at

$$(\Delta J_x)^2 + (\Delta J_y)^2 + \tfrac{1}{4}F_Q[\varrho, J_z] \geq j.$$

- See parallel publication in

S.-H. Chiew and M. Gessner, Phys. Rev. Research 4, 013076 (2022).

Outline

Simple observation to prove further relations

- Let us consider a relation

$$\underbrace{(\Delta A)^2_\varrho}_{\text{variance}} \geq \underbrace{g(\varrho)}_{\text{convex in } \varrho} \ ,$$

  which is true for pure states.

- If $g(\varrho)$ is convex in density matrices, then

$$\frac{1}{4} F_Q[\varrho, A] \geq g(\varrho)$$

  holds for mixed states.

- *Proof.* $\frac{1}{4} F_Q[\varrho, A]$ is given as a convex roof of the variance.

- It is the largest convex function that equals $(\Delta A)^2_\varrho$ for all pure states. ∎

Repeating the proof for the two variances and the QFI

- We rewrite the relation with three variances as

$$\underbrace{(\Delta J_x)^2}_{\text{variance}} \geq \underbrace{j - (\Delta J_y)^2 - (\Delta J_z)^2}_{\text{convex in } \varrho},$$

- The right-hand side is convex in ϱ and the left-hand side is a variance.

- Hence,

$$\frac{1}{4} F_Q[\varrho, J_z] \geq j - (\Delta J_x)^2 - (\Delta J_y)^2.$$

Outline

CV systems (for spin systems, the derivation is similar, but longer)

- Consider entanglement detection in two-mode systems with uncertainty relations.

- A well-known entanglement criterion is

$$[\Delta(x_1 + x_2)]^2 + [\Delta(p_1 - p_2)]^2 \geq 2.$$

If a quantum state violates it, then it is entangled.

L.-M. Duan, G. Giedke, J. I. Cirac, and P. Zoller, Phys. Rev. Lett. 84, 2722 (2000);
R. Simon, Phys. Rev. Lett. 84, 2726 (2000).

CV systems II

- For a two-mode state, the following uncertainty relation holds

$$[\Delta(x_1 + x_2)]^2 + [\Delta(p_1 - p_2)]^2 \geq$$
$$4/F_Q[\varrho, p_1 + p_2] + 4/F_Q[\varrho, x_1 - x_2].$$

  As a consequence, we know something about metrology with states violating the entanglement condition (details are in the paper).

- *Proof.* We start from the relations

$$[\Delta(x_1 + x_2)]^2 F_Q[\varrho, p_1 + p_2] \geq 4,$$
$$[\Delta(p_1 - p_2)]^2 F_Q[\varrho, x_1 - x_2] \geq 4.$$

- Then, in both inequalities we divide by the term containing the QFI. Finally, we sum the two resulting inequalities. ∎

Summary

- We showed how to derive new uncertainty relations with the variance and the quantum Fisher information based on simple convexity arguments.

See:

Géza Tóth and Florian Fröwis,

Uncertainty relations with the variance and the quantum Fisher information based on convex decompositions of density matrices,

Phys. Rev. Research 4, 013075 (2022).

THANK YOU FOR YOUR ATTENTION!

QUANTERA   Alexander von Humboldt
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