

General characteristics of multi-partite quantum systems

(Lecture of the Quantum Information class of the Master in Quantum Science and Technology)

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21 January, 2026

- Geometry of quantum states
 - A single qubit
 - A single qudit (qunit): d -dimensional systems

Bloch vector

- For a single qubit, **the density matrix has three real parameters**. It can be written as

$$\varrho = \frac{1}{2} \left(\mathbb{1} + \sum_{l=x,y,z} v_l \sigma_l \right), \quad (1)$$

where σ_l are the Pauli spin matrices.

- Using $\text{Tr}(\sigma_k \sigma_l) = 2\delta_{kl}$, we can write

$$\text{Tr}(\varrho^2) = \frac{1}{2} + \frac{1}{2} \sum_{l=x,y,z} v_l^2. \quad (2)$$

That is, the Bloch vector has a maximal length for pure states.

Bloch vector II

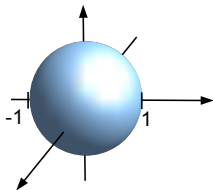
- From $\text{Tr}(\rho^2) \leq 1$, the condition for being physical is Eq. (??), which is equivalent to

$$\sum_{l=x,y,z} |v_l|^2 \leq 1. \quad (3)$$

The three-element vector is called the Bloch vector.

Bloch vector III

- Let us identify the points in (v_x, v_y, v_z) corresponding to physical states. They are in a ball.
- The pure states are on the surface.
- Mixed states are inside the Ball. This is because $\text{Tr}(\rho^2)$ is directly related to the length of the Bloch vector.
- The $|0\rangle$ and $|1\rangle$ correspond to the North and South Pole.
- $|0\rangle + \exp(-i\phi)|1\rangle$ correspond to points on the equator.



Set of physical quantum states for a single qubit. The axes correspond to v_l for $l = x, y, z$. Pure states correspond to points on the surface, mixed states correspond to internal points.

A single qudit (qunit): d -dimensional systems

- For higher dimensional systems the picture is much more complicated. Let us consider qudits with dimension d .
- Similarly to the case before, **a $d \times d$ Hermitian matrix with a unit trace has $d^2 - 1$ degrees of freedom.**
- Hence, we can write a density matrix as a linear combination of $d^2 - 1$ SU(d) generators as

$$\varrho = \frac{1}{d} \mathbb{1} + \frac{1}{2} \sum_{l=1}^{d^2-1} v_l g_l. \quad (4)$$

Here,

$$\text{Tr}(g_k g_l) = 2\delta_{kl}. \quad (5)$$

(Like for the Pauli matrices. Thus, we have something like the generalized Pauli matrices. $d = 3$: Gell-Mann matrices.)

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$$\varrho = \frac{1}{d} \mathbb{1} + \frac{1}{2} \sum_{l=1}^{d^2-1} v_l g_l. \quad (6)$$

Here,

$$\text{Tr}(g_k g_l) = 2\delta_{kl}. \quad (7)$$

- Like for the Pauli matrices. Thus, we have something like the generalized Pauli matrices. $d = 3$: for instance, Gell-Mann matrices.

A single qudit (qunit): d -dimensional systems II

- Gell-Mann matrices:

$$\begin{aligned}\lambda_1 &= \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} & \lambda_2 &= \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} & \lambda_3 &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\ \lambda_4 &= \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} & \lambda_5 &= \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix} \\ \lambda_6 &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} & \lambda_7 &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix} & \lambda_8 &= \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}.\end{aligned}$$

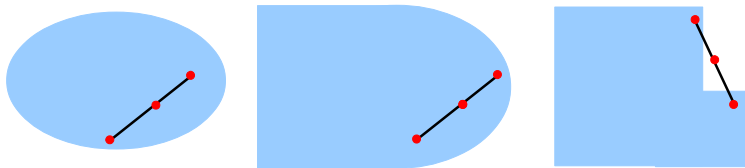
There are other possibilities: J. Lawrence, quant-ph/0403095.

A single qudit (qunit): d -dimensional systems III

- Let us again look at the points $(v_1, v_2, \dots, v_{d^2-1})$ corresponding to physical states.
- First note that the set is convex. This is because mixing two physical states ϱ_1 and ϱ_2 , we always get a physical state

$$\varrho = p\varrho_1 + (1 - p)\varrho_2. \quad (8)$$

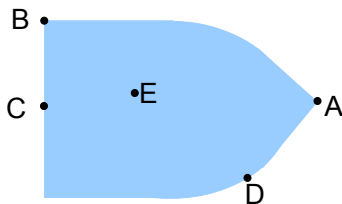
A single qudit (qunit): d -dimensional systems IV



Two convex objects and one that is not convex.

A single qudit (qunit): d -dimensional systems V

- On the next figure we will show the set of quantum states.
- The coordinate axis could be the v_l , for example.
- Inside the set there are the density matrices with full rank.
- On the boundary there are the states with less than full rank, such as for example rank-1 states, which are pure states.



Set of physical quantum states. Note that the set is convex.
A,B,D: rank-1 states. C: rank-2 state. E: full rank states.

A single qudit (qunit): d -dimensional systems VI

- **Observation.** The following inequality is true

$$\lambda_{\min}(A + B) \geq \lambda_{\min}(A) + \lambda_{\min}(B). \quad (9)$$

Proof. Let us consider that for a Hermitian matrix X we have

$$\lambda_{\min}(X) = \min_{\psi} \langle \psi | X | \psi \rangle. \quad (10)$$

Then, for A and B Hermitian matrices we have

$$\begin{aligned} \lambda_{\min}(A + B) &= \min_{\psi} \langle \psi | A + B | \psi \rangle \geq \min_{\psi} \langle \psi | A | \psi \rangle + \min_{\psi} \langle \psi | B | \psi \rangle \\ &= \lambda_{\min}(A) + \lambda_{\min}(B). \end{aligned} \quad (11)$$

□

We can prove similarly that

$$\lambda_{\max}(A + B) \leq \lambda_{\max}(A) + \lambda_{\max}(B). \quad (12)$$

Full rank states

- Using this, we can say the following.

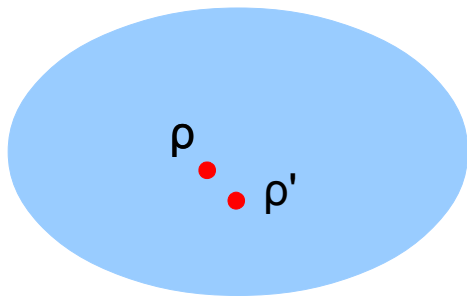
- **Observation.** Full-rank states are inside the set.

Proof. If the state is full rank, it means that for some small ϵ

$$\varrho' = \varrho + \epsilon H \tag{13}$$

is also physical, where H is a trace 0 Hermitian matrix. Why is that? See also the next figure.

Full rank states II



We take an internal state ϱ and consider the states ϱ' in its neighborhood.

Full rank states III

- It is physical since
 - 1 Trace is 1.
 - 2 Hermitian.
 - 3 Full rank means that

$$\lambda_{\min}(\varrho) > 0, \quad \lambda_{\max}(\varrho) < 1. \quad (14)$$

Eigenvalues are nonzero for small ϵ . This is because

$$\lambda_{\max}(\varrho) + \lambda_{\max}(\epsilon H) \geq \lambda_k(\varrho') \geq \lambda_{\min}(\varrho) + \lambda_{\min}(\epsilon H). \quad (15)$$

Here we have

$$\lambda_{\min}(\epsilon H) = \begin{cases} +\epsilon \lambda_{\min}(H), & \text{if } \epsilon \geq 0, \\ -|\epsilon| \lambda_{\max}(H), & \text{if } \epsilon < 0. \end{cases} \quad (16)$$

Similar statement holds for $\lambda_{\max}(\epsilon H)$. \square

Non-full-rank states

- **Observation.** Non-full-rank states are on the surface of the set.
- *Proof.* If the state is not full rank, then it has zero eigenvalues. Thus, there is an H such that ϱ' is aphysical for any $\epsilon > 0$ or any $\epsilon < 0$.
- To be more explicit, let us write

$$\varrho = UDU^\dagger, \quad (17)$$

such that D contains the eigenvalues. Here,

$$D = \text{diag}(\lambda_1, \lambda_2, \lambda_3, \dots, \lambda_d), \quad (18)$$

and the eigenvectors are

$$U = [|\Psi_1\rangle, |\Psi_2\rangle, |\Psi_3\rangle, \dots, |\Psi_d\rangle]. \quad (19)$$

Non-full-rank states II

- Assume that $\lambda_d = 0$. Then,

$$\varrho' = \varrho + \epsilon(|\Psi_d\rangle\langle\Psi_d| - \mathbb{1}/d) \quad (20)$$

has a negative eigenvalue for any $\epsilon < 0$. The Identity is needed to make the expression zero-trace.

- This is because the eigenvalues of this matrix are

$$D' = \text{diag}(\lambda_1 - \epsilon/d, \lambda_2 - \epsilon/d, \lambda_3 - \epsilon/d, \dots, \lambda_d + \epsilon(1 - 1/d)). \quad (21)$$

□