Uncertainties with the variance and the quantum Fisher information

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Introduction

- Density matrices have an infinite number of convex decompositions. This is a feature of quantum mechanics not present in classical physics.
- So far, this fact is appreciated mostly in quantum information science. There, convex roofs over decompositions appear in the theory of entanglement measures.
- Recent findings show that such ideas can also be used in quantum metrology. It has turned out that the quantum Fisher information (QFI) is the convex roof of the variance, apart from a constant factor [1,2].
- Using the fact above, we will derive novel uncertainty relations with the QFI.

Quantum Fisher information

- A basic metrological task in a *linear* interferometer is estimating the small angle θ for a unitary dynamics
$$U_\theta = \exp(-i\mathcal{H}\theta). \quad (1)$$
- Cramér-Rao bound:
$$(\Delta\theta)^2 \geq \frac{1}{m\mathcal{F}_Q[\rho,\mathcal{H}]}, \quad (2)$$
where m is the number of indepedendent repetitions, and the quantum Fisher information is defined by the formula
$$\mathcal{F}_Q[\rho,\mathcal{H}] = 2\sum_{k,l}\frac{(\lambda_k-\lambda_l)^2}{\lambda_k+\lambda_l}|\langle k|\mathcal{H}|l\rangle|^2. \quad (3)$$
- Here, λ_k and $|k\rangle$ are the eigenvalues and eigenvectors, respectively, of the density matrix ρ, which is used as a probe state for estimating θ.

Error propagation formula

- Instead of the quantum Fisher information, let us consider the error propagation formula
$$(\Delta\theta)^2_M = \frac{(\Delta M)^2}{\langle i[M,\mathcal{H}]\rangle^2}, \quad (4)$$
which provides a bound on the quantum Fisher information
$$\mathcal{F}_Q[\rho,\mathcal{H}] \geq 1/(\Delta\theta)^2_M. \quad (5)$$
- The precision of the estimation is bounded as [3,4]
$$(\Delta\theta)^2 \geq \frac{1}{m}\min_A(\Delta\theta)^2_A, \quad (6)$$
where m is the number of independent repetitions.
- The minimum in Eq. (6) is taken if A is the symmetric logarithmic derivative.

QFI as convex roof

- Let us consider a density matrix of the form
$$\rho = \sum_k p_k|\psi_k\rangle\langle\psi_k|, \quad (7)$$
where $p_k > 0$ and $\sum_k p_k = 1$.
- The QFI is given as a convex roof [1,2]
$$F_Q[\rho,B] = 4\min_{\{p_k,|\psi_k\rangle\}}\sum_k p_k(\Delta B)^2_{\psi_k}, \quad (8)$$
where $\{p_k,|\psi_k\rangle\}$ refers to a decomposition of ρ.
- The variance is the concave roof of itself
$$(\Delta A)^2_\rho = \max_{\{p_k,|\psi_k\rangle\}}\sum_k p_k(\Delta A)^2_{\psi_k}. \quad (9)$$
- For any decomposition $\{p_k,|\psi_k\rangle\}$ we have
$$\frac{1}{4}F_Q[\rho,A] \leq \sum_k p_k(\Delta A)^2_{\psi_k} \leq (\Delta A)^2_\rho, \quad (10)$$
where the upper and the lower bounds are both tight.

RS uncertainty

- The Robertson-Schrödinger inequality is defined as
$$(\Delta A)^2_\rho(\Delta B)^2_\rho \geq \tfrac{1}{4}|L_\rho|^2, \quad (11)$$
where the lower bound is given by
$$L_\rho = \sqrt{|\langle\{A,B\}\rangle_\rho - 2\langle A\rangle_\rho\langle B\rangle_\rho|^2 + |\langle C\rangle_\rho|^2}, \quad (12)$$
$\{A,B\} = AB+BA$ is the anticommutator, and we used the definition $C = i[A,B]$.
- Let us consider the inequality
$$\left(\sum_k p_k a_k\right)\left(\sum_k p_k b_k\right) \geq \left(\sum_k p_k\sqrt{a_k b_k}\right)^2, \quad (13)$$
where $a_k, b_k \geq 0$.
- We get
$$\left[\sum_k p_k(\Delta A)^2_{\rho_k}\right]\left[\sum_k p_k(\Delta B)^2_{\rho_k}\right] \geq \tfrac{1}{4}\left[\sum_k p_k L_{\rho_k}\right]^2.$$

Convex roof/concave roof

- We have
$$(\Delta A)^2_\rho F_Q[\rho,B] \geq \left(\min_{\{p_k,|\psi_k\rangle\}}\sum_k p_k L_{\psi_k}\right)^2. \quad (14)$$
- A weaker but simpler relation is [5]
$$(\Delta A)^2_\rho F_Q[\rho,B] \geq |\langle C\rangle_\rho|^2. \quad (15)$$
- Another relation
$$(\Delta A)^2(\Delta B)^2 \geq \frac{1}{4}\left(\sum_k p_k L_{\rho_k}\right)^2. \quad (16)$$
- With a concave roof on the bound, we obtain
$$(\Delta A)^2_\rho(\Delta B)^2_\rho \geq \frac{1}{4}\left(\max_{\{p_k,\rho_k\}}\sum_k p_k L_{\rho_k}\right)^2. \quad (17)$$

Cramér-Rao bound

- Error propagation formula
$$(\Delta\theta)^2_A = \frac{(\Delta A)^2}{|\langle C\rangle|^2}. \quad (18)$$
- Based on the ideas above, we arrive at
$$(\Delta\theta)^2_A \geq \frac{1}{4\min_{\{p_k,|\psi_k\rangle\}}\left[\sum_k p_k(\Delta B)^2_{\psi_k}\right]}.$$
- Using (18) and (6), we get a lower bound on the precision of parameter estimation
$$(\Delta\theta)^2 \geq \frac{1}{m}\times\frac{1}{4\min_{\{p_k,|\psi_k\rangle\}}\left[\sum_k p_k(\Delta B)^2_{\psi_k}\right]}. \quad (19)$$

QFI+variance

- Let us start from the relation for pure states
$$(\Delta J_x)^2 + (\Delta J_y)^2 + (\Delta J_z)^2 \geq j, \quad (20)$$
where J_l are the spin components fulfilling
$$J_x^2 + J_y^2 + J_z^2 = j(j+1)\mathbb{1}. \quad (21)$$
- Equation (20) is valid for all decompositions of the type Eq. (7), hence
$$\sum_k p_k(\Delta J_x)^2_{\Psi_k} + \sum_k p_k(\Delta J_y)^2_{\Psi_k} + \sum_k p_k(\Delta J_z)^2_{\Psi_k} \geq j. \quad (22)$$
- We can choose a decomposition $\{p_k,|\Psi_k\rangle\}$ for which $\sum_k p_k(\Delta J_z)^2_{\Psi_k}$ is minimal.
- Hence,
$$(\Delta J_x)^2 + (\Delta J_y)^2 + \tfrac{1}{4}F_Q[\rho,J_z] \geq j. \quad (23)$$

Related bibliography

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