# Activating hidden metrological usefulness Phys. Rev. Lett. 125, 020402 (2020) (open access) G. TÓTH<sup>1,2,3</sup>, T. VÉRTESI<sup>4</sup>, P. HORODECKI<sup>5,6</sup>, R. HORODECKI<sup>7,6</sup>

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#### Introduction

- It has been realized that entanglement can be a useful resource in very general metrological tasks. Even bound entangled states can be more useful than separable states [1,2]. Such states are called "useful" in short. However, there are highly entangled states that are not useful for metrology [3].
- ▶ In the spirit of Ref. [4], we show that some bipartite entangled quantum states that are not useful in linear interferometers become useful if several copies are considered or ancillas are added [5].
- ▶ To support our claims, we present a general method to find the local Hamiltonian for which a given bipartite quantum state provides the largest gain compared to separable states. Note that this task is different, and in a sense more complex, than maximizing the quantum Fisher information [5].

#### **Quantum Fisher information**

▶ A basic metrological task in a *linear* interferometer is estimating the small angle  $\theta$  for a unitary dynamics  $U_{\theta} = \exp(-i\mathcal{H}\theta)$ , where the Hamiltonian is the sum of local terms. For bipartite systems it is

$$\mathcal{H} = \mathcal{H}_1 + \mathcal{H}_2, \tag{1}$$

where  $\mathcal{H}_n$  are single-subsystem operators.

Cramér-Rao bound:

$$(\Delta \theta)^2 \ge \frac{1}{m \mathcal{F}_O[\rho, \mathcal{H}]},$$
 (2)

where m is the number of indepedendent repetitions, and the quantum Fisher information is defined by the formula

$$\mathcal{F}_{Q}[\rho,\mathcal{H}] = 2\sum_{k,l} \frac{(\lambda_{k} - \lambda_{l})^{2}}{\lambda_{k} + \lambda_{l}} |\langle k|\mathcal{H}|l\rangle|^{2}.$$
 (3)

Here,  $\lambda_k$  and  $|k\rangle$  are the eigenvalues and eigenvectors, respectively, of the density matrix p, which is used as a probe state for estimating  $\theta$ .

## Metrological gain

▶ We define the metrological gain compared to separable states, for a given Hamiltonian, by [5]

$$g_{\mathcal{H}}(\rho) = \mathcal{F}_{Q}[\rho, \mathcal{H}] / \mathcal{F}_{Q}^{(\text{sep})}(\mathcal{H}),$$
 (4)

where the separable limit for local Hamiltonians is

$$\mathcal{F}_{Q}^{(\text{sep})}(\mathcal{H}) = \sum_{n=1,2} [\sigma_{\text{max}}(\mathcal{H}_n) - \sigma_{\text{min}}(\mathcal{H}_n)]^2. \quad (5)$$

▶ We are interested in the quantity [5]

$$g(\mathbf{p}) = \max_{\text{local } \mathcal{H}} g_{\mathcal{H}}(\mathbf{p}), \tag{6}$$

where a local Hamiltonian is just the sum of single system Hamiltonians as in Eq. (1).

The maximization task looks challenging since we have to maximize a fraction, where both the numerator and the denominator depend on the Hamiltonian.

## Ancilla

For the  $3 \times 3$ -case, we consider the maximally entangled state mixed with noise

$$\rho_{AB}^{(p)} = (1-p)|\Psi^{(me)}\rangle\langle\Psi^{(me)}| + p\mathbb{1}/d^2,$$
(7)

which is useful if p < 0.3655.

▶ If a pure ancilla qubit is added [5]

$$\rho^{(\mathrm{anc})} = |0\rangle\langle 0|_a \otimes \rho_{AB}^{(p)}.$$

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then the state is useful if p < 0.3752.

The Hamiltonian is

$$\mathcal{H}^{(\mathrm{anc})} = 1.2C_{aA} \otimes \mathbb{1}_B + \mathbb{1}_{aA} \otimes D_B, \qquad (8)$$

$$C_{aA} = \frac{9}{20} (2\sigma_x + \sigma_z)_a \otimes |0\rangle\langle 0|_a$$
  
+  $\mathbb{1}_a \otimes (|2\rangle\langle 2|_a - |1\rangle\langle 1|_a),$ 

$$D = diag(+1, -1, +1). (9)$$

## Two copies

 $\blacktriangleright$  We consider now two copies of the noisy  $3 \times 3$  maximally entangled state [5]

$$\rho^{(tc)} = \rho_{AB}^{(p)} \otimes \rho_{A'B'}^{(p)}.$$



▶ Then, with the two-copy operator

$$\mathcal{H}^{(\text{tc})} = D_A \otimes D_{A'} \otimes \mathbb{1}_{BB'} + \mathbb{1}_{AA'} \otimes D_B \otimes D_{B'}, (10)$$

the state is useful if p < 0.4164.

- The Hamiltonians presented are not the optimal
- Let us look for the optimal Hamiltonians, for which  $g_{\mathcal{H}}$  is the largest.

# **Optimal Hamiltonian**

▶ Instead of the quantum Fisher information, let us consider the error propagation formula

$$(\Delta \theta)_M^2 = \frac{(\Delta M)^2}{\langle i[M, \mathcal{H}] \rangle^2},\tag{11}$$

which provides a bound on the quantum Fisher information

$$\mathcal{F}_Q[\rho, \mathcal{H}] \ge 1/(\Delta \theta)_M^2.$$
 (12)

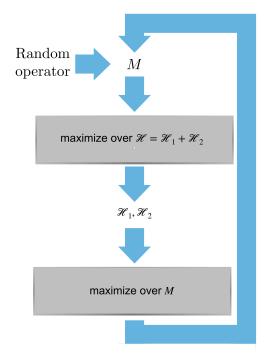
▶ We will minimize Eq. (11) using the idea [4]

$$\max_{\mathcal{H}} \mathcal{F}_{Q}[\rho, \mathcal{H}] = \max_{\mathcal{H}M} \frac{\left\langle i[M, \mathcal{H}] \right\rangle^{2}}{(\Delta M)^{2}}.$$
 (13)

Based on these, we realize a see-saw, optimizing alternatingly over  $\mathcal{H}$  and M.

	Analytic example	Numerics
Ancilla	0.3752	0.3941
Second copy	0.4164	0.4170

## **See-saw iteration**



#### Pure states

General case, pure state with a Schmidt decompo-

$$|\Psi\rangle = \sum_{k=1}^{s} \sigma_{k} |k\rangle_{a} |k\rangle_{B},$$

where s is the Schmidt number, and the real positive  $\sigma_k$  Schmidt coefficients are in a descending order.

▶ Direct calculation yields [4]

$$\mathcal{F}_{Q}[|\Psi\rangle,\mathcal{H}_{AB}] = 4(\Delta\mathcal{H}_{AB})_{\Psi}^{2}$$

$$= 8 \sum_{n=1,3,5,\dots,\tilde{s}-1} (\sigma_{n} + \sigma_{n+1})^{2},$$

which is larger than the separable bound,  $\mathcal{F}_{Q}^{(\mathrm{sep})}$  = 8, whenever the Schmidt rank is larger than 1. Here,  $\tilde{s}$  is the largest even number for which  $\tilde{s} \leq s$ . (For the Hamiltonian  $\mathcal{H}_{AB}$ , see Ref. [5].)

▶ In the limit of infinite copies, all entangled bipartite pure states are maximally useful [5].

# Related bibliography

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