

# Quantum metrology

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# Outline

## 1 Motivation

- Why is quantum metrology interesting?

## 2 Simple examples of quantum metrology

- Classical case: Clock arm
- Quantum case: Single spin-1/2 particle
- Magnetometry with the fully polarized state
- Magnetometry with the spin-squeezed state
- Metrology with the GHZ state
- Dicke states
- Interferometry with squeezed photonic states

## 3 Entanglement theory

- Multipartite entanglement
- The spin-squeezing criterion

## 4 Quantum metrology using the quantum Fisher information

- Quantum Fisher information
- Quantum Fisher information in linear interferometers
- Noise and imperfections

# Why is quantum metrology interesting?

- Recent technological development has made it possible to realize large coherent quantum systems, i.e., in cold gases, trapped cold ions or photons.
- Can such quantum systems outperform classical systems in something useful, i.e., metrology?

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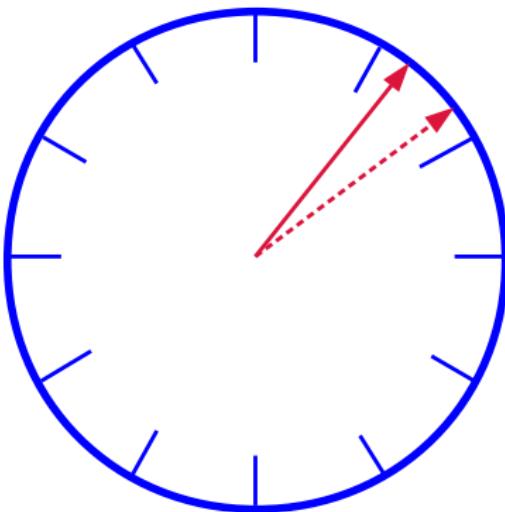
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# Classical case: Estimating the angle of a clock arm

- Arbitrary precision ("in principle").



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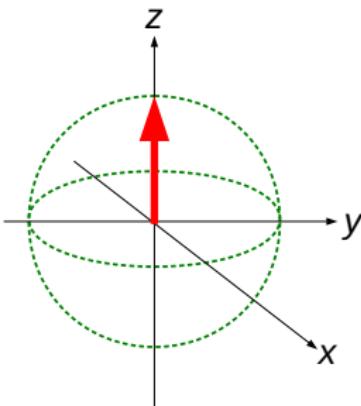
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# Quantum case: A single spin-1/2 particle

- Spin-1/2 particle polarized in the  $z$  direction.



- We measure the spin components.

$j_x$

+1/2, 50%

-1/2, 50%



$j_y$

+1/2, 50%

-1/2, 50%

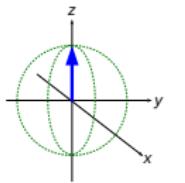


$j_z \rightarrow +1/2$

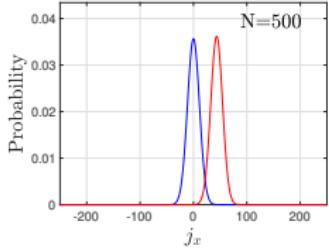
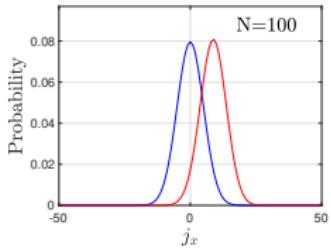
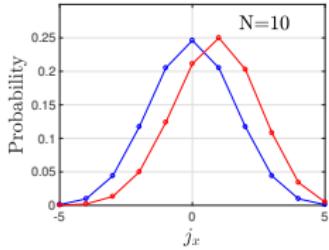
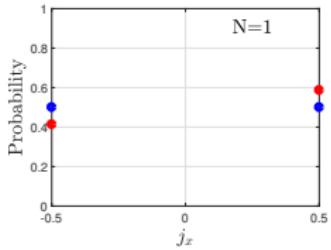
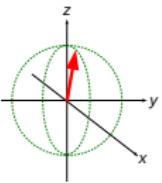
## Quantum case: A single spin-1/2 particle II

- We cannot measure the three spin coordinates exactly  $j_x, j_y, j_z$ .
- In quantum physics, we can get only discrete outcomes in measurement. In this case,  $+1/2$  and  $-1/2$ .
- A single spin-1/2 particle is not a good clock arm.

# Several spin-1/2 particles



10° rotation  
around  $y$   
⇒



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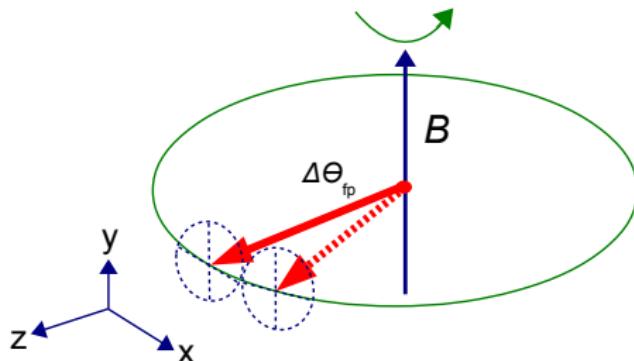
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# Magnetometry with the fully polarized state

- $N$  spin-1/2 particles, all fully polarized in the  $z$  direction.
- Magnetic field  $B$  points to the  $y$  direction.



- Note the uncertainty ellipses.  $\Delta\theta_{fp}$  is the minimal angle difference we can measure.

# Magnetometry with the fully polarized state II

- Collective angular momentum components

$$J_l := \sum_{n=1}^N j_l^{(n)}$$

for  $l = x, y, z$ , where  $j_l^{(n)}$  are single particle operators.

- Dynamics

$$|\Psi\rangle = U_\theta |\Psi_0\rangle, \quad U_\theta = e^{-iJ_y\theta},$$

where  $\hbar = 1$ .

- Rotation around the  $y$ -axis.

## Magnetometry with the fully polarized state III

- Let us assume, that we have an  $M(\Theta)$  function.
- We know that there is an  $\Delta M$  error in  $M$ .
- How much is the error  $\Delta\theta$  in  $\theta$ ?
- It is given by the classical **error propagation formula**:

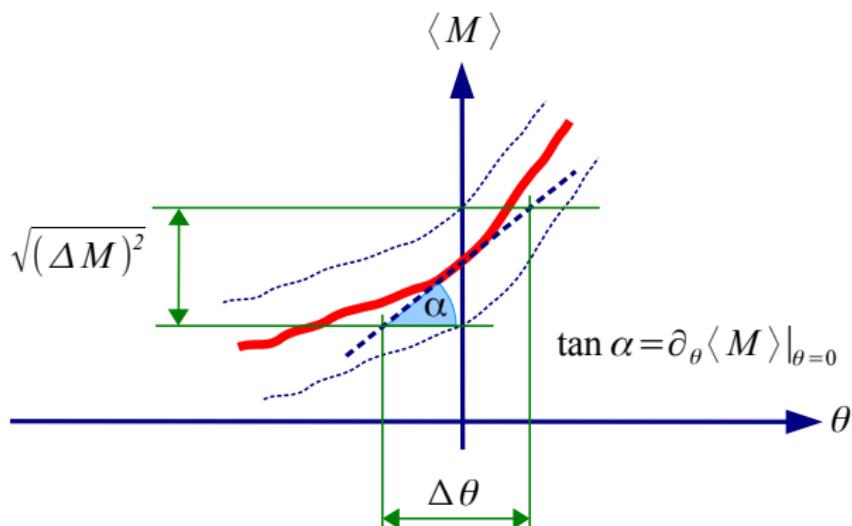
$$\Delta\theta \approx \frac{\Delta M}{dM/d\theta}.$$

- It tells us how the error in  $M$  "propagates" to  $\theta$ .

# Magnetometry with the fully polarized state IV

- Measure an operator  $M$  to get the estimate  $\theta$ .
- To obtain the precision of estimation, we can use the **error propagation formula**

$$(\Delta\theta)_M^2 = \frac{(\Delta M)^2}{|\partial_\theta \langle M \rangle|^2} = \frac{(\Delta M)^2}{|\langle i[M, H] \rangle|^2}.$$



# Magnetometry with the fully polarized state V

- In order to see the full picture, we need to consider  $\nu$  measurements of  $M$ .
- We have to look for the average of the measured values

$$\bar{m} = \sum_{n=1}^{\nu} m_k.$$

- Then, if the measured probability distributions fulfill certain conditions, we can estimate the parameter with a precision

$$(\Delta\theta)^2 = \frac{1}{\nu} (\Delta\theta)_M^2 = \frac{1}{\nu} \frac{(\Delta M)^2}{|\partial_\theta \langle M \rangle|^2}.$$

[ L. Pezze, A. Smerzi, M. K. Oberthaler, R. Schmied, and P. Treutlein,  
"Quantum metrology with nonclassical states of atomic ensembles,"  
Rev. Mod. Phys. 90, 035005 (2018). ]

# Magnetometry with the fully polarized state VI

- We consider the fully polarized states of  $N$  spin-1/2 particles

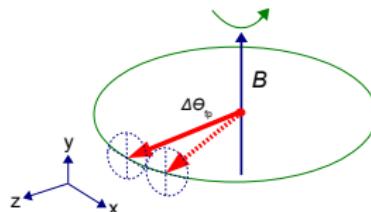
$$| +\frac{1}{2} \rangle^{\otimes N}.$$

- For this state,

$$\langle J_z \rangle = \frac{N}{2}, \quad \langle J_x \rangle = 0, \quad (\Delta J_x)^2 = \frac{N}{4}, \quad \langle J_z \rangle = \frac{N}{2} \cos \theta, \quad \langle J_x \rangle = \frac{N}{2} \sin \theta.$$

- We measure the operator

$$M = J_x.$$



- It is not like a classical clock arm, we have a nonzero uncertainty

$$(\Delta \theta)^2 = \frac{1}{\nu} \frac{(\Delta M)^2}{|\partial_\theta \langle M \rangle|^2} = \frac{1}{\nu} \frac{(\Delta J_x)^2}{\langle J_z \rangle^2} = \frac{1}{\nu} \frac{1}{N}.$$

# Magnetometry with the fully polarized state VII

- Main result:

$$(\Delta\theta)^2 = \frac{1}{\nu N}.$$

- In some cold gas experiment, we can have  $10^3 - 10^{12}$  particles.
- Later we will see that with a separable quantum state we cannot have a better precision.

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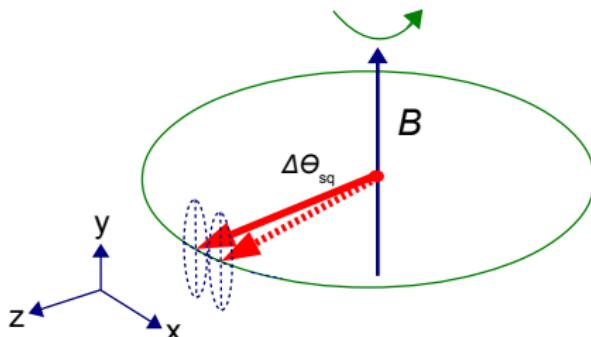
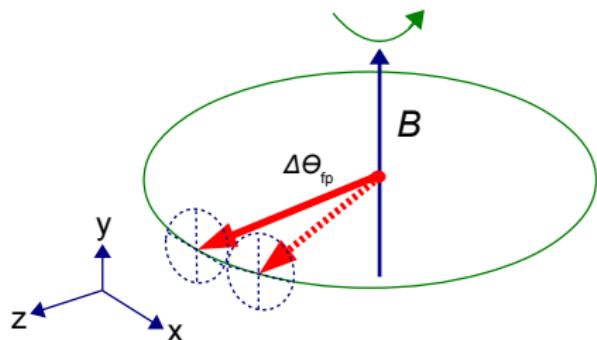
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# Magnetometry with the spin-squeezed state

- We can increase the precision by spin squeezing



fully polarized state (fp)

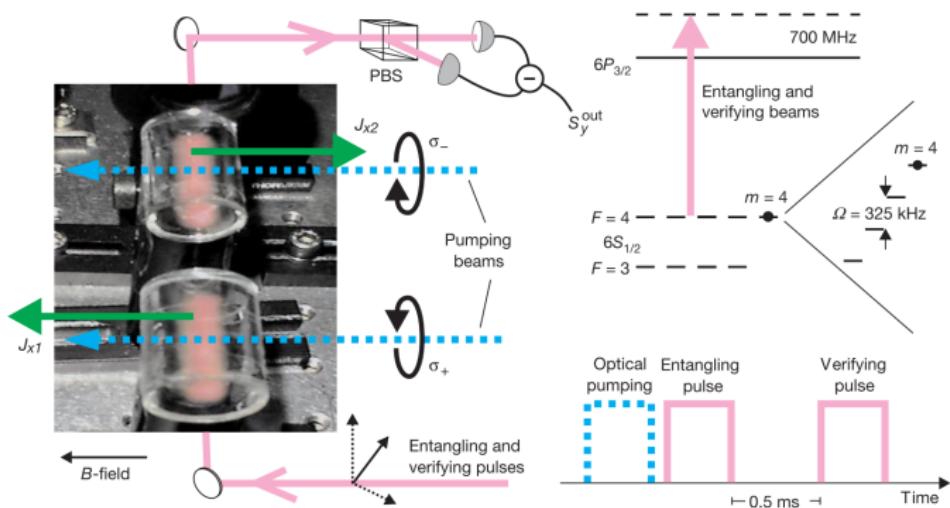
spin-squeezed state (sq)

$\Delta\theta_{fp}$  and  $\Delta\theta_{sq}$  are the minimal angle difference we can measure.

We can reach

$$(\Delta\theta)^2 < \frac{1}{\nu N}.$$

# Spin squeezing in an ensemble of atoms via interaction with light

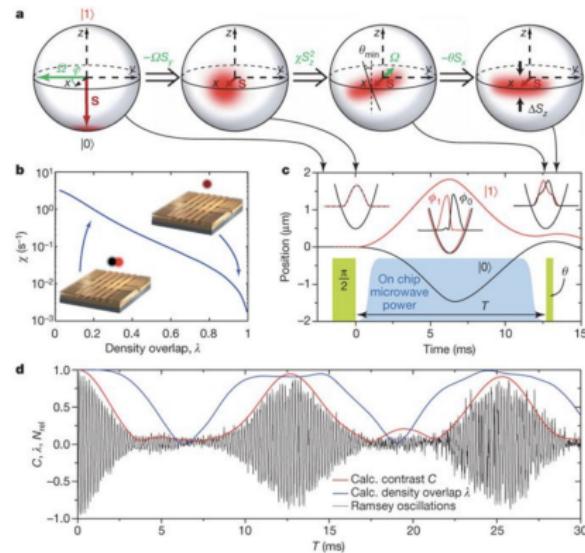


$10^{12}$  atoms, room temperature.

Julsgaard, Kozhekin, Polzik, Nature 2001.

# Spin squeezing in a Bose-Einstein Condensate via interaction between the particles

Figure 1: Spin squeezing and entanglement through controlled interactions on an atom chip.



M. F. Riedel, P. Böhi, Y. Li, T. W. Hänsch, A. Sinatra, and P. Treutlein,  
Nature 464, 1170-1173 (2010).

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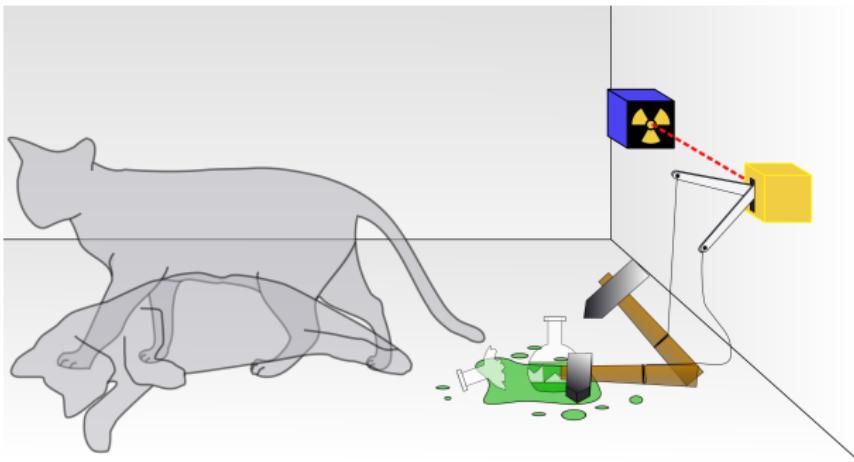
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# GHZ state=Schrödinger cat state

- A superposition of two macroscopically distinct states



# GHZ state

## Greenberger-Horne-Zeilinger (GHZ) state

$$|GHZ_N\rangle = \frac{1}{\sqrt{2}}(|000\dots 00\rangle + |111\dots 11\rangle).$$

- Superposition of all atoms in state "0" and all atoms in state "1".

# Metrology with the GHZ state

- Greenberger-Horne-Zeilinger (GHZ) state

$$|\text{GHZ}_N\rangle = \frac{1}{\sqrt{2}}(|000\dots 00\rangle + |111\dots 11\rangle),$$

- Unitary

$$|\Psi\rangle(\theta) = U_\theta |\text{GHZ}_N\rangle, \quad U_\theta = e^{-iJ_z\theta}.$$

- Dynamics

$$|\Psi\rangle(\theta) = \frac{1}{\sqrt{2}}(|000\dots 00\rangle + e^{-iN\theta}|111\dots 11\rangle),$$

# Metrology with the GHZ state II

- We measure

$$M = \sigma_x^{\otimes N},$$

which is the parity in the  $x$ -basis.

- Expectation value and variance

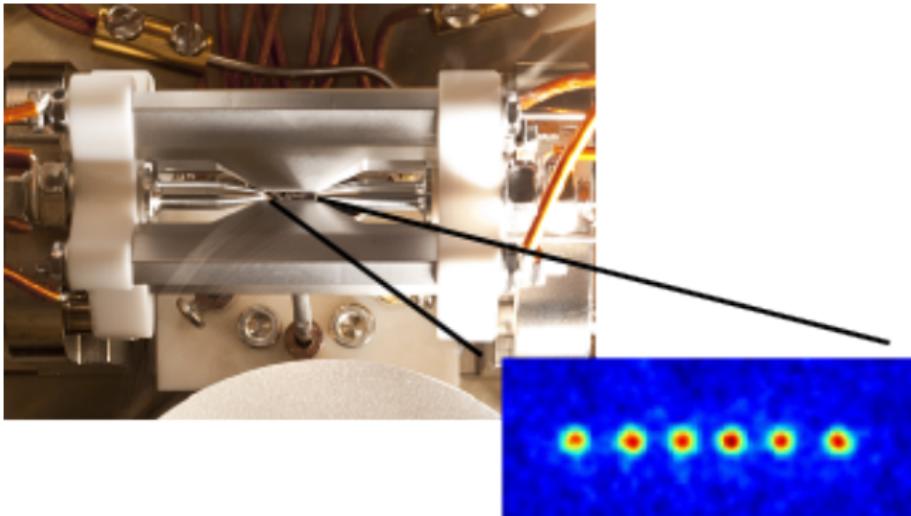
$$\langle M \rangle = \cos(\textcolor{red}{N}\theta), \quad (\Delta M)^2 = \sin^2(\textcolor{red}{N}\theta).$$

- For  $\theta \approx 0$ , the precision is

$$(\Delta\theta)^2 = \frac{1}{\nu} \frac{(\Delta M)^2}{|\partial_\theta \langle M \rangle|^2} = \frac{1}{\nu N^2}.$$

[ e.g., photons: D. Bouwmeester, J. W. Pan, M. Daniell, H. Weinfurter and A. Zeilinger, Phys. Rev. Lett. 82, 1345 (1999);  
ions: C. Sackett et al., Nature 404, 256 (2000). ]

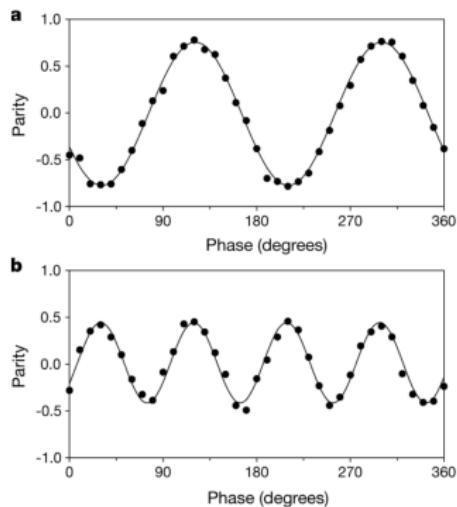
# Metrology with the GHZ state III



Quantum Computation with Trapped Ions, Innsbruck

# Metrology with the GHZ state IV

Figure 2: Determination of  $p_{(\uparrow\downarrow)}$ .



a, Interference signal for two ions; b, four ions. After the entanglement operation of Fig. 1, an analysis pulse with relative phase  $\varphi$  is applied on the single-ion  $|+\rangle \leftrightarrow |-\rangle$  transition. As  $\varphi$  is varied, the parity of the  $N$  ions oscillates as  $\cos N\varphi$ , and the amplitude of the oscillation is twice the magnitude of the density-matrix element  $p_{(\uparrow\downarrow)}$ . Each data point represents an average of 1,000 experiments, corresponding to a total integration time of roughly 10 s for each graph.

For four ions the curve oscillates faster than for two ions.

[ ions: C. Sackett et al., Nature 404, 256 (2000). ]

# Metrology with the GHZ state IV

- We reached the Heisenberg-limit

$$(\Delta\theta)^2 = \frac{1}{\nu N^2}.$$

- The fully polarized state reached only the shot-noise limit

$$(\Delta\theta)^2 = \frac{1}{\nu N}.$$

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# Dicke states

- Symmetric Dicke states with  $\langle J_z \rangle = 0$  (simply “Dicke states” in the following) are defined as

$$|D_N\rangle = \left(\frac{N}{\frac{N}{2}}\right)^{-\frac{1}{2}} \sum_k \mathcal{P}_k \left( |0\rangle^{\otimes \frac{N}{2}} \otimes |1\rangle^{\otimes \frac{N}{2}} \right).$$

- E.g., for four qubits they look like

$$|D_4\rangle = \frac{1}{\sqrt{6}} (|0011\rangle + |0101\rangle + |1001\rangle + |0110\rangle + |1010\rangle + |1100\rangle).$$

[photons: Kiesel, Schmid, GT, Solano, Weinfurter, PRL 2007;  
Prevedel, Cronenberg, Tame, Paternostro, Walther, Kim, Zeilinger, PRL 2007;  
Wieczorek, Krischek, Kiesel, Michelberger, GT, and Weinfurter, PRL 2009]

[cold atoms: Lücke *et al.*, Science 2011; Hamley *et al.*, Science 2011; C. Gross *et al.*, Nature 2011]

# Metrology with Dicke states. Clock arm = noise

- For our symmetric Dicke state

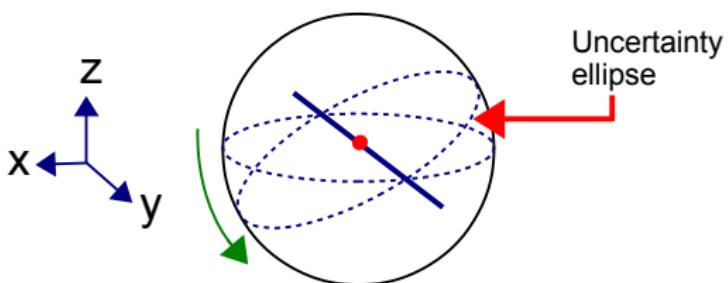
$$\langle J_l \rangle = 0, l = x, y, z, \quad \langle J_z^2 \rangle = 0, \quad \langle J_x^2 \rangle = \langle J_y^2 \rangle = \text{large}.$$

- Linear metrology

$$U = \exp(-iJ_y\theta).$$

- Measure  $\langle J_z^2 \rangle$  to estimate  $\theta$ .

(We cannot measure first moments, since they are zero.)



# Metrology with Dicke states

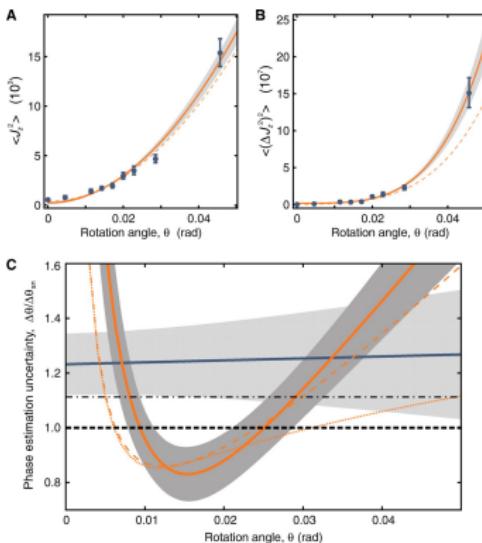
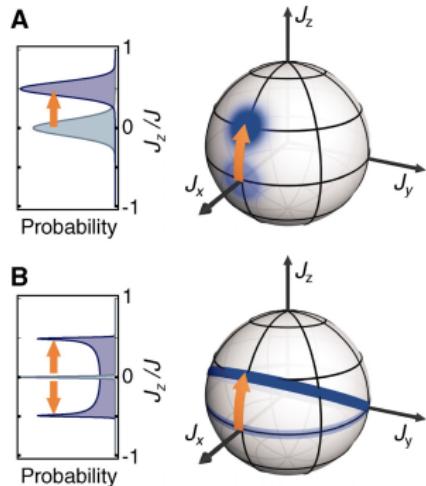
- Dicke states are more robust to noise than GHZ states. (Even if they loose a particle, they remain entangled).
- Dicke states can also reach the Heisenberg-scaling like GHZ states.

[Metrology with cold gases: B. Lücke, M Scherer, J. Kruse, L. Pezze, F. Deuretzbacher, P. Hyllus, O. Topic, J. Peise, W. Ertmer, J. Arlt, L. Santos, A. Smerzi, C. Klemp, Science 2011.]

[Metrology with photons: R. Krischek, C. Schwemmer, W. Wieczorek, H. Weinfurter, P. Hyllus, L. Pezze, A. Smerzi, PRL 2011.]

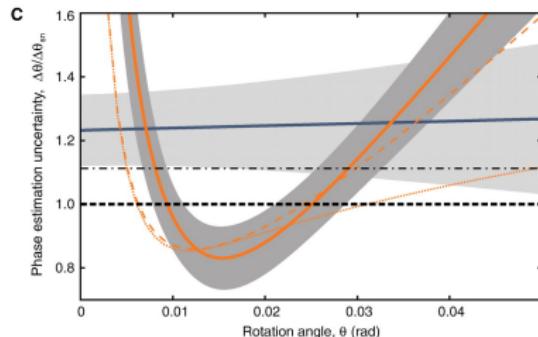
# Metrology with Dicke states II

Experiment with cold gas of 8000 atoms.



[Lücke M. Scherer, Kruse, Pezzé, Deuretzbacher, Hyllus, Topic, Peise, Ertmer, Arlt, Santos, Smerzi, Klempt, Science 2011.]

# Metrology with Dicke states III



- $\Delta\theta$  is the precision of estimating  $\theta$ .
- $\Delta\theta_{sn}$  means the "shot-noise" uncertainty. This is the smallest uncertainty that could be achieved by separable states.
- Black dashed line = the level corresponding to  $\Delta\theta/\Delta\theta_{sn} = 1$ .
- Orange solid line = precision of estimating the angle  $\theta$  in the experiment.
- The orange solid line is below the black dashed line around  $\theta \approx 0.015$ . Hence, the uncertainty  $\Delta\theta$  is smaller than that could be achieved by any separable state, and hence the state of the system is entangled.

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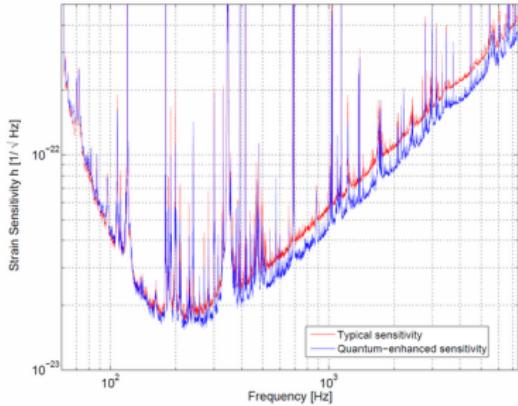
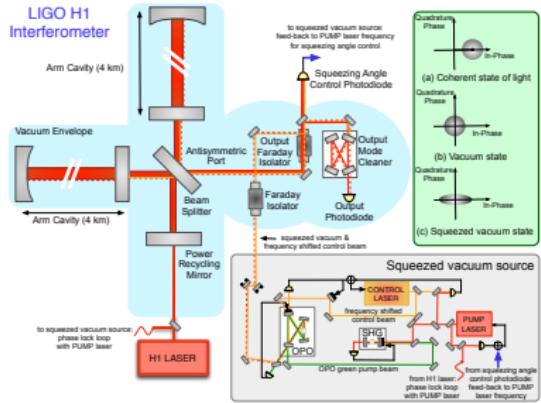
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# LIGO gravitational wave detector

The performance was enhanced with squeezed light.



The role of clock arm is played by the squeezed coherent state.

[ J. Aasi et al., Nature Photonics 2013. ]

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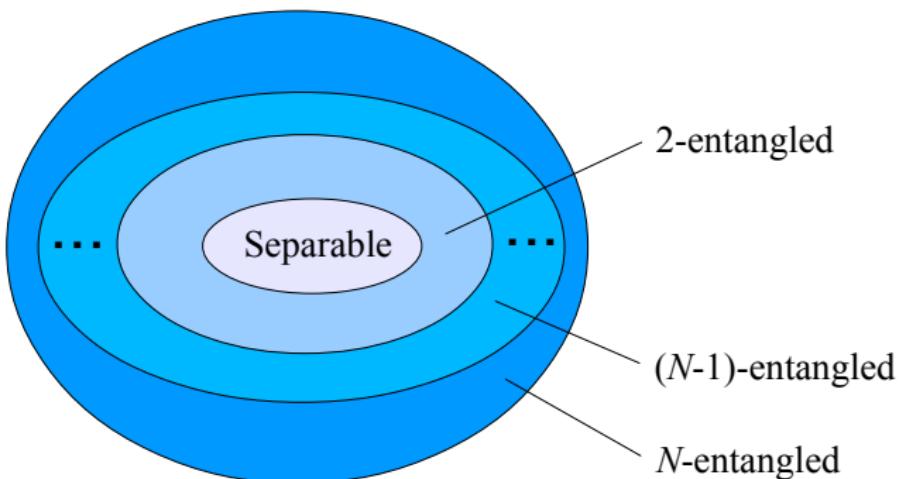
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# $k$ -producibility/ $k$ -entanglement



$$(|00\rangle + |11\rangle) \otimes (|00\rangle + |11\rangle) \otimes (|00\rangle + |11\rangle) \quad 2\text{-entangled}$$

$$(|000\rangle + |111\rangle) \otimes (|000\rangle + |111\rangle) \quad 3\text{-entangled}$$

$$(|0000\rangle + |1111\rangle) \otimes (|0\rangle + |1\rangle) \quad 4\text{-entangled}$$

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# Multipartite entanglement in spin squeezing

- We consider pure  $k$ -producible states of the form

$$|\Psi\rangle = \otimes_{I=1}^M |\psi_I\rangle,$$

where  $|\psi_I\rangle$  is the state of at most  $k$  qubits.

## Extreme spin squeezing

The spin-squeezing criterion for  $k$ -producible states is

$$(\Delta J_x)^2 \geq J_{\max} F_{\frac{k}{2}} \left( \frac{\sqrt{\langle J_y \rangle^2 + \langle J_z \rangle^2}}{J_{\max}} \right),$$

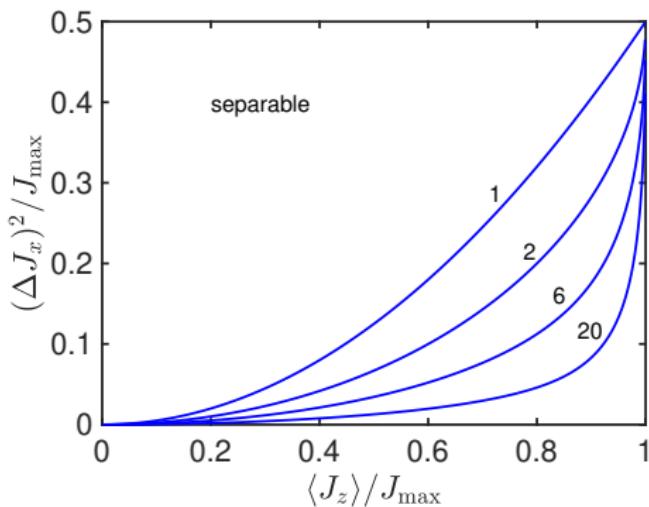
where  $J_{\max} = \frac{N}{2}$  and we use the definition

$$F_j(Z) := \frac{1}{j} \min_{\frac{\langle j_z \rangle}{j} = Z} (\Delta j_x)^2.$$

[Sørensen and Mølmer, Phys. Rev. Lett. 86, 4431 (2001); experimental test: Gross, Zibold, Nicklas, Esteve, Oberthaler, Nature 464, 1165 (2010).]

# Multipartite entanglement in spin squeezing

- Larger and larger multipartite entanglement is needed to larger and larger squeezing ("extreme spin squeezing").



- $N = 100$  spin-1/2 particles,  $J_{\max} = N/2$ .

[Sørensen and Mølmer, Phys. Rev. Lett. 86, 4431 (2001); experimental test: Gross, Zibold, Nicklas, Esteve, Oberthaler, Nature 464, 1165 (2010).]

# Our experience so far

- We find that more spin squeezing/better precision needs more entanglement.
- Question: Is this general?
- Answer: Yes.

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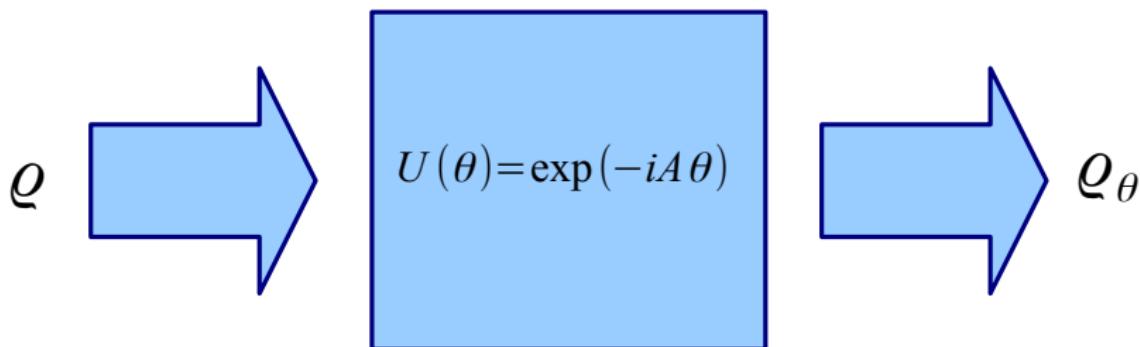
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# Quantum metrology

- Fundamental task in metrology



- We have to estimate  $\theta$  in the dynamics

$$U = \exp(-iA\theta).$$

# The quantum Fisher information

## Cramér-Rao bound on the precision of parameter estimation

For the variance of the parameter estimation

$$(\Delta\theta)^2 \geq \frac{1}{\nu F_Q[\varrho, A]}$$

holds, where  $\nu$  is the number of repetitions and  $F_Q[\varrho, A]$  is the **quantum Fisher information**.

- The bound includes any estimation strategy, even POVM's.
- The quantum Fisher information is

$$F_Q[\varrho, A] = 2 \sum_{k,l} \frac{(\lambda_k - \lambda_l)^2}{\lambda_k + \lambda_l} |\langle k | A | l \rangle|^2,$$

where  $\varrho = \sum_k \lambda_k |k\rangle\langle k|$ .

# Convexity of the quantum Fisher information

- For pure states, it equals four times the variance,

$$F_Q[|\Psi\rangle, A] = 4(\Delta A)^2_{\Psi}.$$

- For mixed states, it is convex

$$F_Q[\varrho, A] \leq \sum_k p_k F_Q[|\Psi_k\rangle, A],$$

where

$$\varrho = \sum_k p_k |\Psi_k\rangle\langle\Psi_k|.$$

## Quantum Fisher information - Some basic facts

- The larger the quantum Fisher information, the larger the achievable precision.
- For the totally mixed state it is zero for any  $A$

$$F_Q[\varrho_{\text{cm}}, A] = 0,$$

where  $\varrho_{\text{cm}} = \mathbb{1}/d$  is the completely mixed state and  $d$  is the dimension.

- This is logical: the completely mixed states does not change under any Hamiltonian.
- For any state  $\varrho$  that commutes with  $A$ , i.e.,  $\varrho A - A\varrho = 0$  we have

$$F_Q[\varrho, A] = 0.$$

## Best operator to measure

- The error propagation formula is bounded from above as

$$(\Delta\theta)_M^2 = \frac{(\Delta M)^2}{|\partial_\theta \langle M \rangle|^2} \geq \frac{1}{F_Q[\varrho, H]}.$$

- The inequality is saturated for the following operator, i.e., the **symmetric logarithmic derivative**

$$M = 2i \sum_{k,l} \frac{\lambda_k - \lambda_l}{\lambda_k + \lambda_l} |k\rangle\langle l| \langle k|A|l\rangle,$$

where  $\varrho = \sum_k \lambda_k |k\rangle\langle k|$ .

## QFI as a convex roof

The quantum Fisher information is the convex roof of the variance times four

$$F_Q[\varrho, A] = 4 \min_{\{p_k, \Psi_k\}} \sum_k p_k (\Delta A)^2_{\Psi_k},$$

where

$$\varrho = \sum_k p_k |\Psi_k\rangle\langle\Psi_k|.$$

[GT, D. Petz, Phys. Rev. A 87, 032324 (2013); S. Yu, arXiv1302.5311 (2013);  
GT, I. Apellaniz, J. Phys. A: Math. Theor. 47, 424006 (2014)]

- It is like entanglement measures, e. g., Entanglement of Formation.
- Convex roof over purifications.

[R. Demkowicz-Dobrzański, J. Kołodyński, M. Guća, Nature Comm. 2012.]

# The variance as a concave roof

- We have a family of generalized variances, which are concave in the state and all equal to the variance for pure states.
- There is a largest of such functions defined by the concave roof.

The usual variance is

$$(\Delta A)^2_{\varrho} = \sup_{\{p_k, \Psi_k\}} \sum_k p_k (\Delta A)^2_{\Psi_k},$$

where

$$\varrho = \sum_k p_k |\Psi_k\rangle\langle\Psi_k|.$$

GT, D. Petz, Phys. Rev. A 87, 032324 (2013).

# Summary of statements

- Decompose  $\varrho$  as

$$\varrho = \sum_k p_k |\Psi_k\rangle\langle\Psi_k|.$$

- Then,

$$\frac{1}{4} F_Q[\varrho, A] \leq \sum_k p_k (\Delta A)_{\Psi_k}^2 \leq (\Delta A)_{\varrho}^2$$

holds.

- Both inequalities can be saturated by some decomposition.

# Quantum Fisher information and the fidelity

The quantum Fisher information appears in the Taylor expansion of  $F_B$

$$F_B(\varrho, \varrho_\theta) = 1 - \frac{\theta^2 F_Q[\varrho, A]}{4} + \mathcal{O}(\theta^3),$$

where

$$\varrho_\theta = \exp(-iA\theta)\varrho \exp(+iA\theta).$$

- Bures fidelity

$$F_B(\varrho_1, \varrho_2) = \text{Tr} \left( \sqrt{\sqrt{\varrho_1} \varrho_2 \sqrt{\varrho_1}} \right)^2.$$

- Clearly,

$$0 \leq F_B(\varrho_1, \varrho_2) \leq 1.$$

The fidelity is 1 only if  $\varrho_1 = \varrho_2$ .

# Outline

## 1 Motivation

- Why is quantum metrology interesting?

## 2 Simple examples of quantum metrology

- Classical case: Clock arm
- Quantum case: Single spin-1/2 particle
- Magnetometry with the fully polarized state
- Magnetometry with the spin-squeezed state
- Metrology with the GHZ state
- Dicke states
- Interferometry with squeezed photonic states

## 3 Entanglement theory

- Multipartite entanglement
- The spin-squeezing criterion

## 4 Quantum metrology using the quantum Fisher information

- Quantum Fisher information
- Quantum Fisher information in linear interferometers
- Noise and imperfections

# Magnetometry with a linear interferometer

- The Hamiltonian  $A$  is defined as

$$A = J_I = \sum_{n=1}^N j_I^{(n)}, \quad I \in \{x, y, z\}.$$

There are no interaction terms.

- The dynamics rotates all spins in the same way.

# Quantum Fisher information for separable states

- Let us consider a pure product state of  $N$  qubits

$$|\Psi\rangle_{\text{prod}} = |\Psi^{(1)}\rangle \otimes |\Psi^{(2)}\rangle \otimes \dots \otimes |\Psi^{(N)}\rangle.$$

- Since this is a pure state, we have  $F_Q[\varrho, J_I] = 4(\Delta J_I)^2|_{|\Psi\rangle_{\text{prod}}}.$
- Then, for the product state we have

$$(\Delta J_I)^2|_{|\Psi\rangle_{\text{prod}}} = \sum_{n=1}^N (\Delta j_I^{(n)})^2|_{|\Psi^{(n)}\rangle} \leq N \times \frac{1}{4},$$

where we used that for qubits  $(\Delta j_I^{(n)})^2 \leq 1/4$ .

- Since the quantum Fisher information is convex in the state, the bound is also valid for a mixture of product states, i.e., separable states

$$F_Q[\varrho, J_I] \leq N.$$

# The quantum Fisher information vs. entanglement

- For separable states of  $N$  spin-1/2 particles (qubits)

$$F_Q[\varrho, J_l] \leq N, \quad l = x, y, z.$$

[Pezze, Smerzi, Phys. Rev. Lett. 102, 100401 (2009); Hyllus, Ghne, Smerzi, Phys. Rev. A 82, 012337 (2010)]

- For states with at most  $k$ -qubit entanglement ( $k$  is divisor of  $N$ )

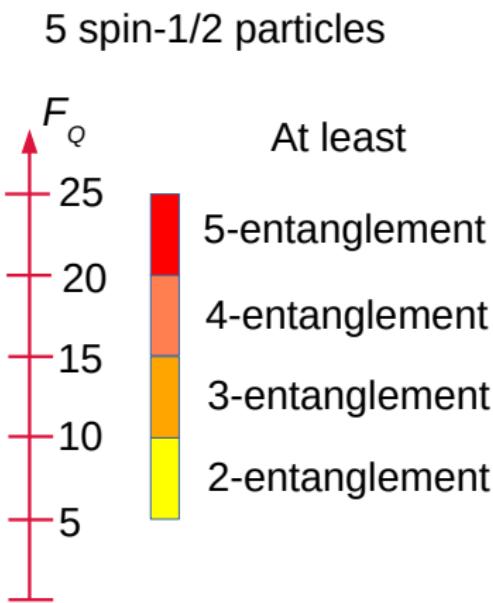
$$F_Q[\varrho, J_l] \leq kN.$$

[P. Hyllus *et al.*, Phys. Rev. A 85, 022321 (2012); GT, Phys. Rev. A 85, 022322 (2012)].

- Bound for all quantum states of  $N$  qubits

$$F_Q[\varrho, J_l] \leq N^2.$$

# The quantum Fisher information vs. entanglement



(Using the  $F_Q[\varrho, J_l] \leq kN$ . Note that there is a slightly better bound.)

# Let us use the Cramér-Rao bound

- For separable states

$$(\Delta\theta)^2 \geq \frac{1}{\nu N}, \quad l = x, y, z.$$

[Pezze, Smerzi, Phys. Rev. Lett. 102, 100401 (2009); Hyllus, Gühne, Smerzi, Phys. Rev. A 82, 012337 (2010)]

- For states with at most  $k$ -particle entanglement ( $k$  is divisor of  $N$ )

$$(\Delta\theta)^2 \geq \frac{1}{\nu k N}.$$

[P. Hyllus *et al.*, Phys. Rev. A 85, 022321 (2012); GT, Phys. Rev. A 85, 022322 (2012)].

- Bound for all quantum states

$$(\Delta\theta)^2 \geq \frac{1}{\nu N^2}.$$

# Spin squeezing and QFI

## Spin squeezing parameter

$$\xi_s^2 = N \frac{(\Delta J_x)^2}{\langle J_y \rangle^2 + \langle J_z \rangle^2}.$$

If  $\xi_s^2 < 1$  then the state is entangled.  
[Sørensen, Duan, Cirac, Zoller, Nature (2001).]

## Based on QFI

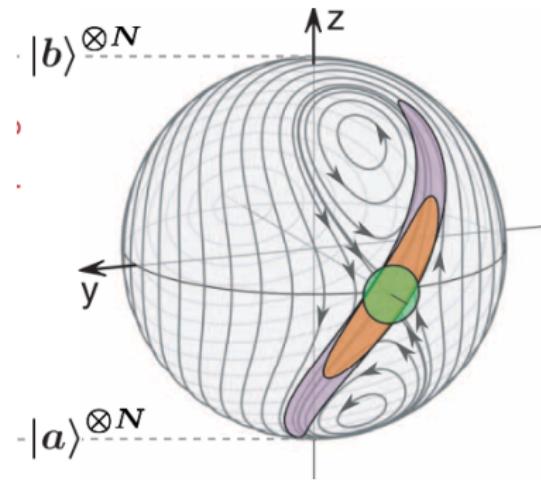
$$\chi^2 = \frac{N}{F[\varrho, J_z]}.$$

If  $\chi^2 < 1$  then the state is entangled.  
It can be proven that

$$\xi_s^2 \geq \chi^2.$$

Thus,  $\chi^2$  detects more states than  $\xi_s$ .  
[Pezze, Smerzi, PRL 2009.]

# Non-Gaussian entangled states



H. Strobel, W. Muessel, D. Linnemann, T. Zibold, D. B. Hume, L. Pezzè, A. Smerzi,  
M. K. Oberthaler, Fisher information and entanglement of non-Gaussian spin states,  
Science 2014.

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## Noisy metrology: Simple example

- A particle with a state  $\varrho_1$  passes through a map that turns its internal state to the fully mixed state with some probability  $p$  as

$$\epsilon_p(\varrho_1) = (1 - p)\varrho_1 + p\frac{1}{2}.$$

- This map acts in parallel on all the  $N$  particles

$$\epsilon_p^{\otimes N}(\varrho) = \sum_{n=0}^N p_n \varrho_n,$$

where the state obtained after  $n$  particles decohered into the completely mixed state is

$$\varrho_n = \frac{1}{N!} \sum_k \Pi_k \left[ \left( \frac{1}{2} \right)^{\otimes n} \otimes \text{Tr}_{1,2,\dots,n}(\varrho) \right] \Pi_k^\dagger.$$

The summation is over all permutations  $\Pi_k$ . The probabilities are

$$p_n = \binom{N}{n} p^n (1-p)^{(N-n)}.$$

## Noisy metrology: Simple example II

- Rewriting it

$$\epsilon_p^{\otimes N}(\varrho) = \sum_{n=0}^N p_n \frac{1}{N!} \sum_k \Pi_k \left[ \left( \frac{1}{2} \right)^{\otimes n} \otimes \text{Tr}_{1,2,\dots,n}(\varrho) \right] \Pi_k^\dagger = \sum_n p_n \varrho_n.$$

- For the noisy state

$$(\Delta J_x)^2 \geq \sum_n p_n (\Delta J_x)^2_{\varrho_n} \geq \sum_n p_n \frac{n}{4} = \frac{pN}{4}.$$

- Hence, for the precision shot-noise scaling follows

$$(\Delta \theta)^2 = \frac{(\Delta J_x)^2}{\langle J_z \rangle^2} \geq \frac{\frac{pN}{4}}{\frac{N^2}{4}} \propto \frac{1}{N}.$$

# Noisy metrology: General treatment

- In the most general case, uncorrelated single particle noise leads to shot-noise scaling after some particle number.

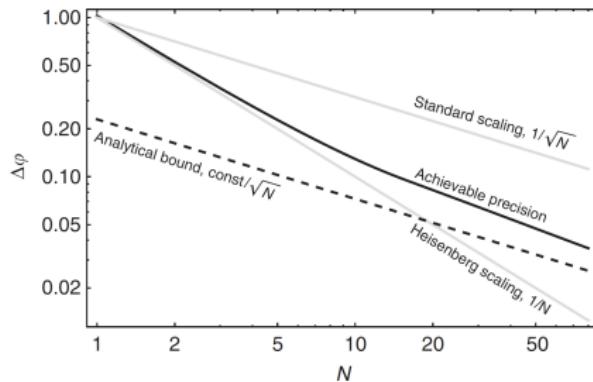


Figure from

[R. Demkowicz-Dobrzański, J. Kołodyński, M. Guća, Nature Comm. 2012.]

- Correlated noise is different.

## Take home message

- Quantum physics makes it possible to obtain bounds for precision of the parameter estimation in realistic many-particle quantum systems.
- Shot-noise limit: Non-entangled states lead to  $(\Delta\theta)^2 \geq \frac{1}{\nu N}$ .
- Heisenberg limit: Fully entangled states can lead to  $(\Delta\theta)^2 = \frac{1}{\nu N^2}$ .
- At the end, noise plays a central role.

## Reviews

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- V. Giovannetti, S. Lloyd, and L. Maccone, Advances in quantum metrology, *Nat. Photonics* 5, 222 (2011).
- C. Gross, Spin squeezing, entanglement and quantum metrology with Bose-Einstein condensates, *J. Phys. B: At., Mol. Opt. Phys.* 45, 103001 (2012).
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- L. Pezze, A. Smerzi, M. K. Oberthaler, R. Schmied, and P. Treutlein, Non-classical states of atomic ensembles: fundamentals and applications in quantum metrology, *Rev. Mod. Phys.* 90, 035005 (2018).

# Summary

- We reviewed quantum metrology from a quantum information point of view.

See:

Géza Tóth and Iagoba Apellaniz,

Quantum metrology from a quantum information science perspective,

J. Phys. A: Math. Theor. 47, 424006 (2014),  
special issue "50 years of Bell's theorem"  
(open access).

Please see the slides at [www.gtoth.eu](http://www.gtoth.eu).

