Entanglement measures (How much is it entangled?) (Lecture of the Quantum Information class of the Master in Quantum Science and Technology)

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Outline

1. **Entanglement measures (How much is it entangled?)**
   - Motivation
   - A. General quantum operation
   - B. Local operations and classical communication (LOCC)
   - C. Entanglement of formation
   - D. Concurrence

Entanglement measures

- After detecting entanglement, we have to ask how entangled the state is.

- It will turn out that entanglement is a resource.

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General quantum operation

- The general quantum operation is defined as

$$\varrho' = \sum_k E_k \varrho E_k^\dagger$$

with

$$\sum_k E_k^\dagger E_k = 1.$$

- E_k are Kraus operators.
- Generalized measurements, POVM (positive operator-valued measure).
- Special case: von Neumann measurements, when E_k are pairwise orthogonal projectors.
- Naimark's dilation theorem:
  general operation=
  von Neumann measurement on system+ancilla.

Outline

Local operations and classical communication (LOCC)

- LOCC are
  - local unitaries,
  - local von Neumann or POVM measurements,
  - local unitaries or measurements conditioned on measurement outcomes on the other party.

- Mathematical description of LOCC. Separable operations are a somewhat larger set, however, this set can easily be described.

$$\varrho' = \sum_k E_k^{(1)} \otimes E_k^{(2)} \varrho \left(E_k^{(1)} \otimes E_k^{(2)} \right)^\dagger$$

with

$$\sum_k \left(E_k^{(1)} \otimes E_k^{(2)} \right)^\dagger \left(E_k^{(1)} \otimes E_k^{(2)} \right) = 1.$$

Local operations and classical communication (LOCC) II

- Stochastic Local Operations and Classical Communication (SLOCC):

  $$|\Psi\rangle' \leftarrow E_k^{(1)} \otimes E_k^{(2)} |\Psi\rangle$$

  It happens with some probability, not deterministic.

- LOCC cannot create entanglement. Separable states remain separable under LOCC.

- LOCC can create correlations.

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Entropy of entanglement

- The von Neumann entropy is defined as

$$S(\varrho) = -\mathrm{Tr}(\varrho \log_2 \varrho).$$

- It can be written with the eigenvalues of the density matrix as

$$S(\varrho) = -\sum_{k=1}^{d} \lambda_k \log_2 \lambda_k.$$

- For a pure state we have $\lambda_k = \{1, 0, 0, ..., 0\}$, and thus it is zero.
- Its maximal is for the completely mixed state for which $\lambda_k = \{\frac{1}{d}, \frac{1}{d}, \frac{1}{d}, ..., \frac{1}{d}\}$, and its value is $\log_2 d$.
- For a bipartite pure state, the entropy of entanglement is

$$E_E(|\Psi\rangle) = \mathrm{S}(\mathrm{Tr}_1(|\Psi\rangle\langle\Psi|)).$$

That is, it is the von Neumann entropy of the reduced state is an entaglement measure.

Entropy of entanglement II

- Comments
  - It is one for two-qubit singlet states.
  - It is zero for product states.
  - It is invariant under $U_1 \otimes U_2$.

Entanglement of formation

- For mixed states, the entanglement of formation is the convex roof of the von Neumann entropy of the reduced state.

$$E_F = \min_{|\Psi_k\rangle, p_k} \sum_k p_k E_E(|\Psi_k\rangle),$$

- The optimization is over all decompositions of the state of the type

$$\varrho = \sum_k p_k |\Psi_k\rangle\langle\Psi_k|.$$

- E_F tells us, in the asymptotic limit, how many singlets we need to create the state.
- Is it easy to compute? No. For 2×2 systems, there is an explicit formula with the concurrence. For larger systems, there is not a general method.

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Entanglement of formation

- For two qubits, E_F can be calculated explicitly (Wootters, 1997).
- Special case: for pure states the concurrence is

$$C(|\Psi\rangle) = |\langle\Psi|\tilde{\Psi}\rangle| = 2|a_{11}a_{22} - a_{12}a_{21}|,$$

where

$$|\Psi\rangle = \begin{pmatrix} a_{11} \\ a_{12} \\ a_{21} \\ a_{22} \end{pmatrix}.$$

- It is related to the linear entropy of the reduced state.

$$C = \sqrt{2(1 - \mathrm{Tr}(\rho_{\mathrm{red}}^2)}, \quad (1)$$

where

$$\rho_{\mathrm{red}} = \mathrm{Tr}_2(|\Psi\rangle\langle\Psi|). \quad (2)$$

Entanglement of formation II

- Now we have to compute E_F from C.
- We also nee that

$$\epsilon(c) = H_2\left(\frac{1+\sqrt{1-c^2}}{2}\right).$$

  Here

$$H_2 = -x\log_2 x - (1-x)\log_2(1-x).$$

- Then, E_F can be obtained as

$$E_F(\varrho) = \epsilon(C(\varrho)).$$

Entanglement of formation III

- For mixed states, the concurrence is defined as

$$C(\varrho) = \max(0, \lambda_1 - \lambda_2 - \lambda_3 - \lambda_4),$$

where λ_k's are, in a decreasing order, the eigenvalues of

$$R = \sqrt{\sqrt{\varrho}\tilde{\varrho}\sqrt{\varrho}},$$

and

$$\tilde{\varrho} = (\sigma_y \otimes \sigma_y)\varrho^*(\sigma_y \otimes \sigma_y).$$