Activating hidden metrological usefulness

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Introduction

- It has been realized that entanglement can be a useful resource in very general metrological tasks. Even bound entangled states can be more useful than separable states [1]. However, there are highly entangled states that are not useful for metrology [2].
- In the spirit of Ref. [3], we show that some bipartite entangled quantum states that are not useful in linear interferometers become useful if several copies are considered or ancillas are added [4].
- To support our claims, we present a general method to find the *local* Hamiltonian for which a given bipartite quantum state provides the largest gain compared to separable states. Note that this task is different, and in a sense more complex, than maximizing the quantum Fisher information [4].

Quantum Fisher information

- A basic metrological task in a *linear* interferometer is estimating the small angle θ for a unitary dynamics $U_\theta = \exp(-i\mathcal{H}\theta)$, where the Hamiltonian is the sum of *local* terms. For bipartite systems it is

$$\mathcal{H} = \mathcal{H}_1 + \mathcal{H}_2, \qquad (1)$$

where $\mathcal{H}_n$ are single-subsystem operators.

- Cramér-Rao bound:

$$(\Delta\theta)^2 \geq \frac{1}{m\mathcal{F}_Q[\rho,\mathcal{H}]}, \qquad (2)$$

where m is the number of indepedendent repetitions, and the quantum Fisher information is defined by the formula

$$\mathcal{F}_Q[\rho,\mathcal{H}] = 2\sum_{k,l}\frac{(\lambda_k-\lambda_l)^2}{\lambda_k+\lambda_l}|\langle k|\mathcal{H}|l\rangle|^2. \qquad (3)$$

Here, λ_k and $|k\rangle$ are the eigenvalues and eigenvectors, respectively, of the density matrix ρ, which is used as a probe state for estimating θ.

Metrological gain

- We define the metrological gain compared to separable states, for a given Hamiltonian, by [4]

$$g_{\mathcal{H}}(\rho) = \mathcal{F}_Q[\rho,\mathcal{H}]/\mathcal{F}_Q^{(\mathrm{sep})}(\mathcal{H}), \qquad (4)$$

where the separable limit for local Hamiltonians is

$$\mathcal{F}_Q^{(\mathrm{sep})}(\mathcal{H}) = \sum_{n=1,2}[\sigma_{\max}(\mathcal{H}_n) - \sigma_{\min}(\mathcal{H}_n)]^2. \qquad (5)$$

- We are interested in the quantity [4]

$$g(\rho) = \max_{\mathrm{local}\,\mathcal{H}} g_{\mathcal{H}}(\rho), \qquad (6)$$

where a local Hamiltonian is just the sum of single system Hamiltonians as in Eq. (1).

- The maximization task looks challenging since we have to maximize a fraction, where both the numerator and the denominator depend on the Hamiltonian.

Ancilla

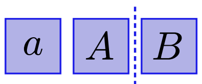
- For the 3×3-case, we consider the maximally entangled state mixed with noise

$$\rho_{AB}^{(p)} = (1-p)|\Psi^{(\mathrm{me})}\rangle\langle\Psi^{(\mathrm{me})}| + p\mathbb{1}/d^2, \qquad (7)$$

which is useful if $p < 0.3655$.

- If a pure ancilla qubit is added [4]

$$\rho^{(\mathrm{anc})} = |0\rangle\langle 0|_a \otimes \rho_{AB}^{(p)}.$$



then the state is useful if $p < 0.3752$.

- The Hamiltonian is

$$\mathcal{H}^{(\mathrm{anc})} = 1.2C_{aA}\otimes\mathbb{1}_B + \mathbb{1}_{aA}\otimes D_B,$$

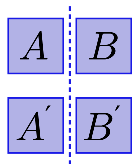
where

$$\begin{aligned} C_{aA} &= \frac{9}{20}(2\sigma_x+\sigma_z)_a\otimes|0\rangle\langle 0|_a \\ &+ \mathbb{1}_a\otimes(|2\rangle\langle 2|_a - |1\rangle\langle 1|_a). \end{aligned}$$

Two copies

- We consider now two copies of the noisy 3×3 maximally entangled state [4]

$$\rho^{(\mathrm{tc})} = \rho_{AB}^{(p)}\otimes\rho_{A'B'}^{(p)}.$$



- Then, with the two-copy operator

$$\mathcal{H}^{(\mathrm{tc})} = D_a\otimes D_{A'}\otimes\mathbb{1}_{BB'} + \mathbb{1}_{AA'}\otimes D_B\otimes D_{B'}, \qquad (8)$$

where

$$D = \mathrm{diag}(+1,-1,+1,-1,\ldots), \qquad (9)$$

the state is useful if $p < 0.4164$.

Optimal Hamiltonian

- Instead of the quantum Fisher information, let us consider the error propagation formula

$$(\Delta\theta)_M^2 = \frac{(\Delta M)^2}{\langle i[M,\mathcal{H}]\rangle^2}, \qquad (10)$$

which provides a bound on the quantum Fisher information

$$\mathcal{F}_Q[\rho,\mathcal{H}] \geq 1/(\Delta\theta)_M^2. \qquad (11)$$

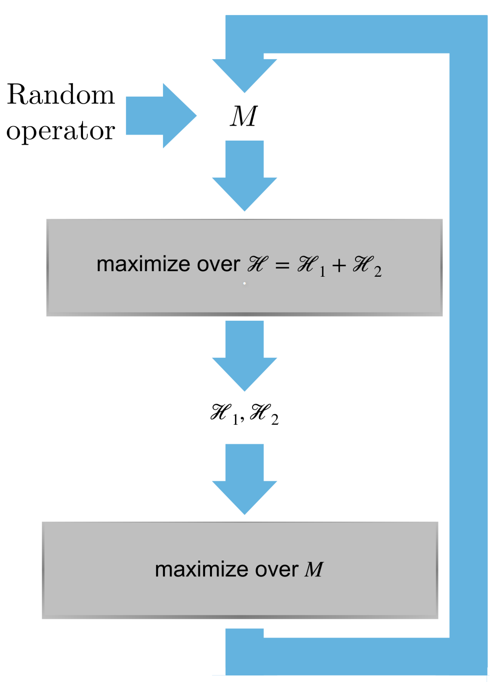
- We will minimize Eq. (10) using the idea [4]

$$\max_{\mathcal{H}}\mathcal{F}_Q[\rho,\mathcal{H}] = \max_{\mathcal{H},M}\frac{\langle i[M,\mathcal{H}]\rangle^2}{(\Delta M)^2}. \qquad (12)$$

Based on these ideas, we realize a see-saw, optmize alternatingly over $\mathcal{H}$ and M.

	Analytic example	Numerics
Ancilla	0.3752	0.3941
Second copy	0.4164	0.4170

See-saw iteration



Pure states

- General case, pure state with a Schmidt decomposition

$$|\Psi\rangle = \sum_{k=1}^{s}\sigma_k|k\rangle_a|k\rangle_B,$$

where s is the Schmidt number, and the real positive σ_k Schmidt coefficients are in a descending order.

- Direct calculation yields [4]

$$\begin{aligned} \mathcal{F}_Q[|\Psi\rangle,\mathcal{H}_{AB}] &= 4(\Delta\mathcal{H}_{AB})_\Psi^2 \\ &= 8\sum_{n=1,3,5,\ldots,\tilde{s}-1}(\sigma_n+\sigma_{n+1})^2, \end{aligned}$$

which is larger than the separable bound, $\mathcal{F}_Q^{(\mathrm{sep})} = 8$, whenever the Schmidt rank is larger than 1. Here, $\tilde{s}$ is the largest even number for which $\tilde{s}\leq s$. (For the Hamiltonian $\mathcal{H}_{AB}$, see Ref. [4].)

- In the limit of infinite copies, all entangled bipartite pure states are maximally useful [4].

Related bibliography

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