

Uncertainty relations with the variance and the quantum Fisher information

— The Cramér-Rao bound as a convex roof

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Benasque, 19 May 2022

Outline

1 Motivation

- The Cramér-Rao bound

2 The derivation

- The Cramér-Rao bound based on a convex roof

Proving the Cramér-Rao bound

- The Cramér-Rao bound is a fundamental relation in metrology.
- It is an expression with the quantum Fisher information (QFI), which is a complicated function of the state and the Hamiltonian.
- We will look for a simple proof of the Cramér-Rao bound based on fundamental uncertainty relations.
- We will exploit the fact that the QFI is the convex roof of the variance.

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Cramér-Rao bound

- Error propagation formula

$$(\Delta\theta)_A^2 = \frac{(\Delta A)^2}{|\partial_\theta \langle A \rangle|^2} = \frac{(\Delta A)^2}{|\langle i[A, B] \rangle|^2}.$$

- The precision of the estimation is bounded as

$$(\Delta\theta)^2 \geq \frac{1}{m} \min_A (\Delta\theta)_A^2,$$

where m is the number of independent repetitions.

- Let us consider a decomposition of the density matrix

$$\varrho = \sum_k p_k |\psi_k\rangle \langle \psi_k|.$$

- The Heisenberg uncertainty for the components is

$$(\Delta A)_{\psi_k}^2 (\Delta B)_{\psi_k}^2 \geq \frac{1}{4} |\langle i[A, B] \rangle_{\psi_k}|^2.$$

Cramér-Rao bound II

- Let us consider the inequality

$$\left(\sum_k p_k a_k \right) \left(\sum_k p_k b_k \right) \geq \left(\sum_k p_k \sqrt{a_k b_k} \right)^2,$$

where $a_k, b_k \geq 0$.

- Hence, we arrive at

$$\left[\sum_k p_k (\Delta A)^2_{\psi_k} \right] \left[\sum_k p_k (\Delta B)^2_{\psi_k} \right] \geq \frac{1}{4} \left[\sum_k p_k |\langle i[A, B] \rangle_{\psi_k}| \right]^2.$$

- We can choose the decomposition such that

$$\sum_k p_k (\Delta B)^2_{\psi_k} = F_Q[\varrho, B]/4.$$

- Due to the concavity of the variance we also know that

$$\sum_k p_k (\Delta A)^2_{\psi_k} \leq (\Delta A)^2.$$

Cramér-Rao bound III

- Hence, it follows that

$$(\Delta A)^2_{\varrho} \left[4 \min_{p_k, \psi_k} \sum_k p_k (\Delta B)^2_{\psi_k} \right] \geq |\langle i[A, B] \rangle_{\varrho}|^2.$$

- Then,

$$\frac{(\Delta A)^2_{\varrho}}{|\langle i[A, B] \rangle_{\varrho}|^2} \geq \frac{1}{\left[4 \min_{p_k, \psi_k} \sum_k p_k (\Delta B)^2_{\psi_k} \right]}.$$

- Finally, for the precision of estimation, if we measure A and the Hamiltonian is B , we have

$$(\Delta \theta)^2 \geq \frac{1}{m} (\Delta \theta)^2_A = \frac{1}{m} \frac{(\Delta A)^2_{\varrho}}{|\langle i[A, B] \rangle_{\varrho}|^2} \geq \frac{1}{m} \times \underbrace{\frac{1}{4 \min_{p_k, \psi_k} \sum_k p_k (\Delta B)^2_{\psi_k}}}_{F_Q[\varrho, B], \text{ the QFI!}}.$$

Summary

- We showed how to derive the Cramér-Rao bound with the convex roof of the variance.

See:

Géza Tóth and Florian Fröwis,

Uncertainty relations with the variance and the quantum Fisher information based on convex decompositions of density matrices,

[Phys. Rev. Research 4, 013075 \(2022\)](#).

THANK YOU FOR YOUR ATTENTION!