Gradient magnetometry with various types of spin ensembles

Single atomic ensembles, chain of spins, two different ensembles and Bose-Einstein condensates

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Outline

1. Multiparametric Quantum Metrology
   - Cramér-Rao precision bound and quantum Fisher information
   - Multiparametric qFI matrix and simultaneous estimation

2. System setup and precision bounds of the gradient parameter estimation for various states
   - Gradient magnetometry and basic setup of the system
   - Precision bounds for various systems and different spin states
   - Saturability of the bounds

3. Conclusions

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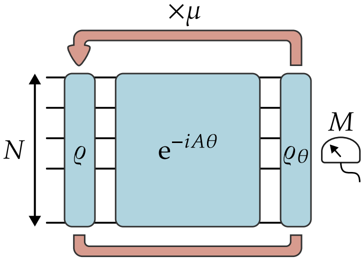
Quantum Metrology

Basic estimation task of the unknown parameter θ for quantum systems:



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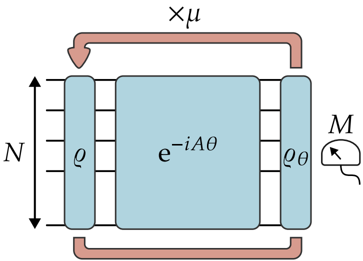

- The *Cramér-Rao (CR) bound* provides an upper bound for the precision

$$(\Delta\theta)^{-2} \leqslant \mu\mathcal{F}(\varrho, N, A(\theta), M, \ldots),$$

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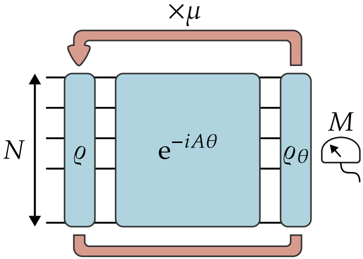

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  where $\mathcal{F}$ is the Fisher information.
- **Hint:** for separable states the best achievable precision scales as $(\Delta\theta)^{-2} \sim \mu N$.

- The Fisher information can be bounded from above with the quantum Fisher information (qFI).

Quantum Fisher information

For unitary transformations of the type $\varrho_\theta = \mathrm{e}^{-iA\theta}\, \varrho\, \mathrm{e}^{+iA\theta}$ where A is a Hermitian operator, the qFI computed on the eigenbasis of the state, $\varrho = \sum p_\lambda |\lambda\rangle\langle\lambda|$, is written as

$$\mathcal{F}_{\mathrm{Q}}[\varrho, A] = 2 \sum_{\lambda \neq \mu} \frac{(p_\lambda - p_\mu)^2}{p_\lambda + p_\mu} |\langle\lambda|A|\mu\rangle|^2,$$

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Properties of the qFI for a single parameter estimation problem

1. It is independent of the measurement. An optimal measurement exists though, which saturates the CR bound.

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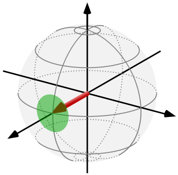

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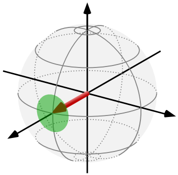

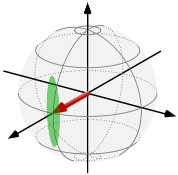

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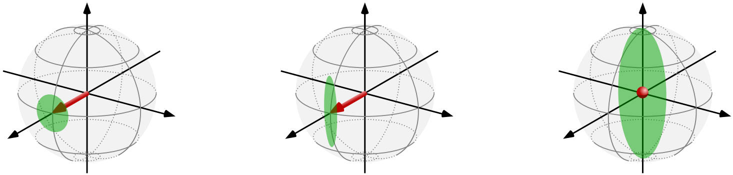

Entanglement

1. Separable states can achieve at most the so called Shot-noise limit (SNL),

$$\mathcal{F}_{\mathrm{Q}}[\varrho_{\mathrm{sep}}, H] \sim N.$$

2. An ultimate limit is obtained maximizing the qFI among all pure states

$$\max_{|\Psi\rangle} \mathcal{F}_{\mathrm{Q}}[|\Psi\rangle, H] = N^2,$$

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Entanglement criteria based on qFI

- Due to its tight relation with the variance, qFI has been used to improve some entanglement conditions.

[G Tóth (2022), PRR **4** 013075]

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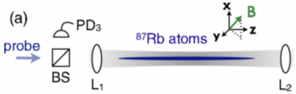
- States insensitive to a global rotation of the system have been prepared in an elongated trap.



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Multiparametric Quantum Metrology

Motivation

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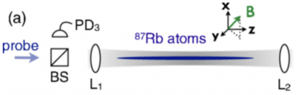

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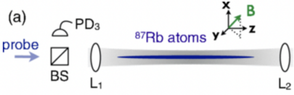

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$$\boldsymbol{B} = (0, 0, B_0) + (0, 0, xB_1) + \mathcal{O}(x^2).$$

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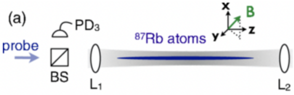

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In general, one cannot avoid a global rotation of the state.

Multiparametric Quantum Metrology

Cramér-Rao matrix inequality

Consider the case on which the state is evolved by some set of unknown parameters in the following way

$$\varrho_{\boldsymbol{\theta}} = \mathrm{e}^{-i\sum_k A_k\theta_k}\,\varrho\,\mathrm{e}^{+i\sum_k A_k\theta_k}\,.$$

- In this case the CR bound is a matrix inequality for the covariance matrix

$$\mathrm{Cov}[\theta_i,\theta_j] \geqslant \frac{1}{\mu}(\mathcal{F}_{\mathrm{Q}}{}^{-1})_{i,j},$$

where $\mathrm{Cov}[\theta_i,\theta_j] = \langle\theta_i\theta_j\rangle - \langle\theta_i\rangle\langle\theta_j\rangle$.

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Symmetric logarithmic derivative

It is defined by the following condition $\mathcal{L}(\varrho, A_k)\,\varrho + \varrho\,\mathcal{L}(\varrho, A_k) = i2[\varrho, A_k]$. For simple cases it can be directly computed by

$$\mathcal{L}(\varrho, A_k) = 2\sum_{\lambda\neq\mu} \frac{p_\lambda - p_\mu}{p_\lambda + p_\mu} \langle\lambda|A_k|\mu\rangle|\lambda\rangle\langle\mu|.$$

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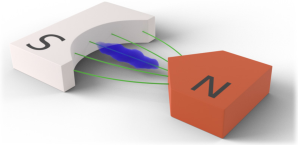

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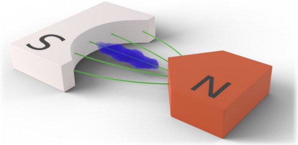

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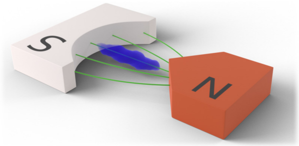

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- In this work we assume that the position state is either a Bose-Einstein condensate (BEC),

$$\varrho^{(\mathrm{x})} = (|\psi\rangle\langle\psi|)^{\otimes N},$$

  or an statistical mixture of point-like particles

$$\varrho^{(\mathrm{x})} = \int \frac{P(\boldsymbol{x})}{\langle \boldsymbol{x}|\boldsymbol{x}\rangle} |\boldsymbol{x}\rangle\langle\boldsymbol{x}|.$$

Gradient magnetometry and basic setup

- The atoms interact only with the magnetic field, $h^{(n)} = \gamma B_z{}^{(n)} \otimes j_z{}^{(n)}$, where $\gamma = g\mu_{\mathrm{B}}$. The collective Hamiltonian is

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- The two unknown parameters are B_0 and B_1 are encoded in b_0 and b_1 acting onto the state with the following unitary operator

$$U = \mathrm{e}^{-i(b_0 H_0 + b_1 H_1)},$$

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$$H_0 := J_z = \sum j_z{}^{(n)} \qquad \text{and} \qquad H_1 = \sum x^{(n)} \otimes j_z{}^{(n)}.$$

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In this work, we are interested on characterizing the precision bounds for $(\Delta b_1)^{-2}$ using the CR bound.

Gradient magnetometry and basic setup

Precision bounds for states **insensitive** to the homogeneous B_0

For states that commute with the homogeneous field, $[\varrho, J_z] = 0$, the precision bound is

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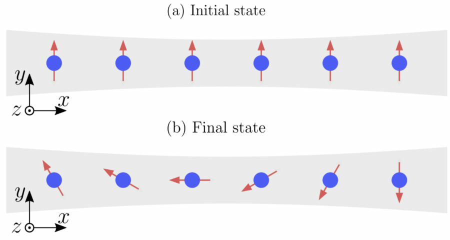

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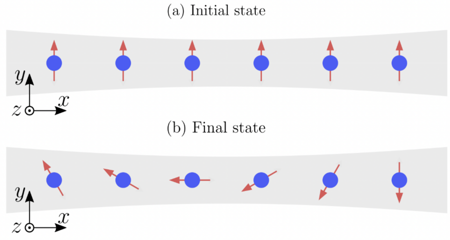

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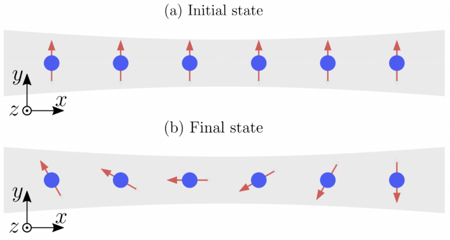

$$(\Delta b_1)^{-2} \leqslant \sum_{n,m} nma^2 \mathcal{F}_Q[|j\rangle_y^{\otimes N}, j_z{}^{(n)}, j_z{}^{(m)}] - \frac{\left(\sum_n na\mathcal{F}_Q[|j\rangle_y^{\otimes N}, j_z{}^{(n)}, J_z]\right)^2}{\mathcal{F}_Q[|j\rangle_y^{\otimes N}, J_z]}.$$

$$= 2a^2\frac{N^2-1}{12}Nj = 2\sigma^2 Nj$$

Double well of atoms

$$P(\boldsymbol{x}) = \prod_{n=1}^{N/2} \delta(x_n + a) \prod_{n=N/2+1}^{N} \delta(x_n - a)$$

The contribution of the position of the particles:

$$\int x_n P(\boldsymbol{x})\, \mathrm{d}\boldsymbol{x} = \begin{cases} -a \\ +a \end{cases} \quad \text{and} \quad \int x_n x_m P(\boldsymbol{x})\, \mathrm{d}\boldsymbol{x} = \begin{cases} +a^2 \\ -a^2 \end{cases}$$

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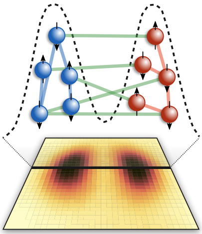

[K Langle *et al.* (2018), Sci. **360** 6387]

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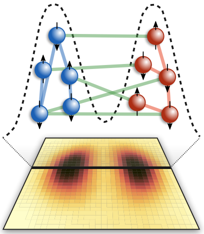

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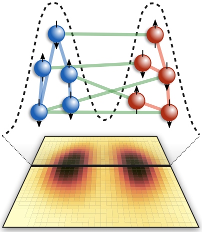

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This bound is quite similar to the Heisenberg limit obtained by the GHZ state under a global rotation. The double well setup can also be modeled by $H_0 = J_z$ and $H_1 = J_z^{(\mathrm{L})} - J_z^{(\mathrm{R})}$. One can easily connect both pictures applying a π rotation on one subensemble along the x-direction.

Double well of atoms

Product spin states

For states of the type $|\psi\rangle^{(\mathrm{L})} \otimes |\psi\rangle^{(\mathrm{R})}$, we have that

$$\mathcal{F}_{\mathrm{Q}}[|\psi\rangle^{(\mathrm{L})} \otimes |\psi\rangle^{(\mathrm{R})}, j_z{}^{(n)}, j_z{}^{(m)}] = \begin{cases} \mathcal{F}_{\mathrm{Q}}[|\psi\rangle, j_z{}^{(n)}, j_z{}^{(m)}] & \text{if } n \text{ and } m \text{ same well} \\ 0 & \text{otherwise} \end{cases}$$

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Hence, the precision bounds can be simply computed for $N/2$ particles from one of the wells as

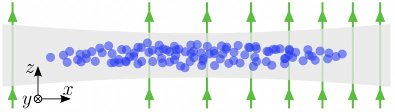
$$(\Delta b_1)^{-2} \leqslant 2\sigma^2 \mathcal{F}_{\mathrm{Q}}[|\psi\rangle, J_z{}^{(N/2)}]$$

$\lvert\psi\rangle^{(N/2)}$	$\mathcal{F}_{\mathrm{Q}}[\lvert\psi\rangle, J_z]$	$(\Delta b_1)^{-2} \leqslant$
$\lvert\mathrm{GHZ}\rangle^{(N/2)}$	$(N/2)^2$	$\sigma^2 N^2/2$
$\lvert j\rangle_y{}^{(N/2)}$	Nj	$2\sigma^2 Nj$
$\lvert\mathrm{D}\rangle_x{}^{(N/2)}$	$N(N+4)/8$	$\sigma^2 N(N+4)/4$

Single ensemble of point-like atoms

Permutationally invariant PDF

$$P(\boldsymbol{x}) = \frac{1}{N!} \sum_{k \in S_N} \mathcal{P}_k[P(\boldsymbol{x})]$$



[N Behbood *et al.* (2014), PRL **113** 093601]

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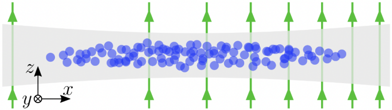

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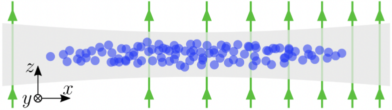

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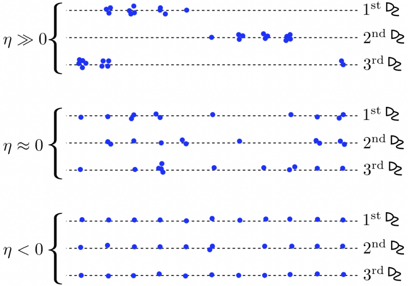

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Precision bounds for various spin states

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More in PRA ***97****, 053603 (2018)*

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Because of their spatial properties, Bose-Einstein condensates (BECs) are another interesting platform for gradient metrology.

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The precision bounds for states sensitive and insensitive to the homogeneous fields is

$$(\Delta b_1)^{-2} \leqslant \mathcal{F}_Q[\varrho^{(s)}, H_1] = 4\sigma^2 \operatorname{tr}\left[\sum_n (j_z{}^{(n)})^2 \varrho^{(s)}\right].$$

Even if BECs can be used for gradient magnetometry, they cannot overcome the SNL.

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Outline

Saturability of the bounds

As it was mentioned in the introduction, the following condition is necessary

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For all PI states

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Thank you for your attention!