

Detecting metrologically useful multiparticle entanglement of Dicke states

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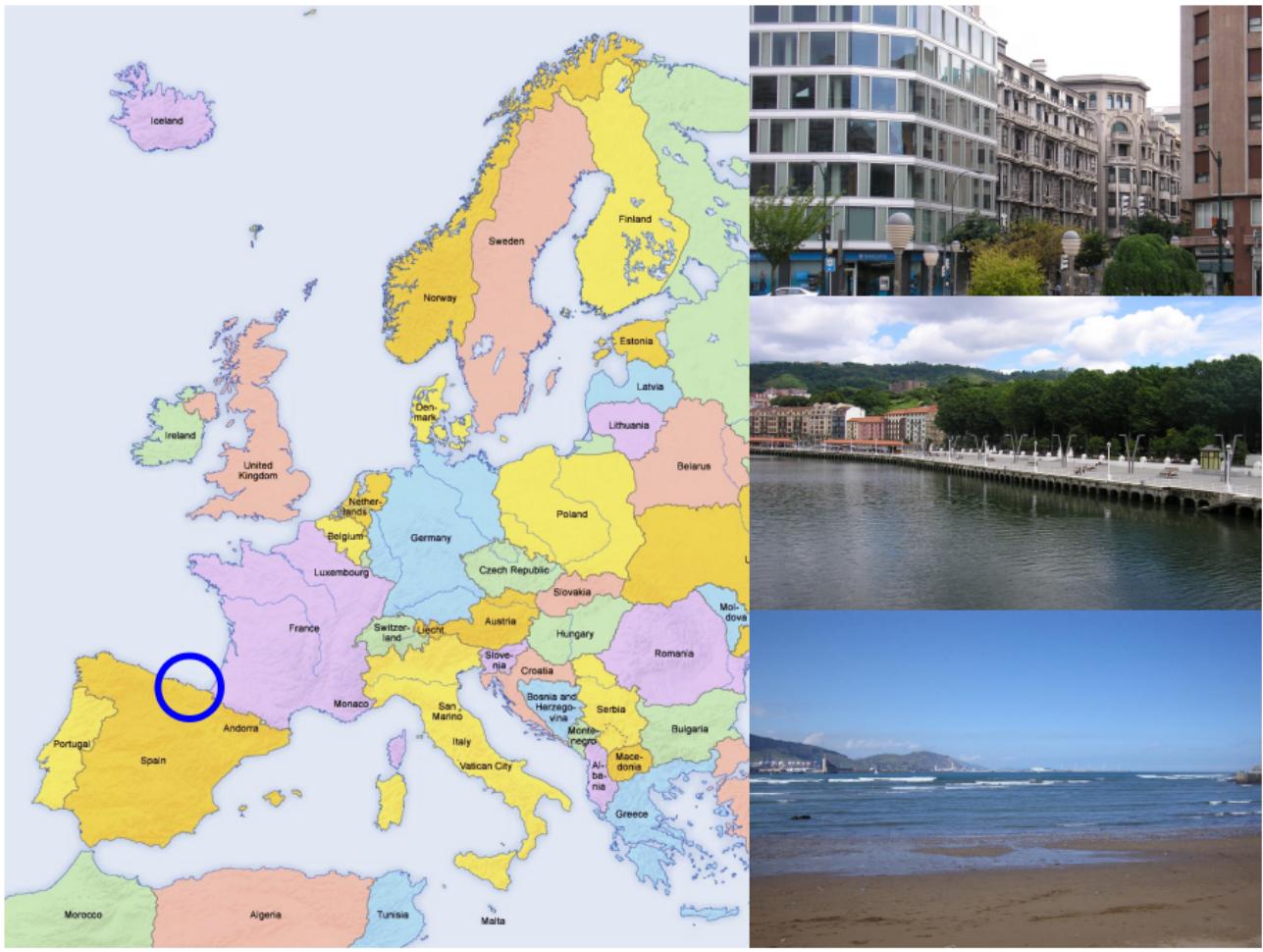
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Why multipartite entanglement is important?

- Full tomography is not possible, we still have to say something meaningful.
- Claiming “entanglement” is not sufficient for many particles.
- Many experiments are aiming to create entangled states with many atoms.
- Only collective quantities can be measured.

Outline

1 Introduction and motivation

2 Spin squeezing and entanglement

- Entanglement
- Collective measurements
- The original spin-squeezing criterion
- Generalized criteria for $j = \frac{1}{2}$

3 Spin squeezing for Dicke states

- Entanglement detection close to Dicke states
- Detection of multipartite entanglement close to Dicke states
- Experimental results

4 Detecting metrologically useful entanglement

- Basics of quantum metrology
- Metrology with measuring $\langle J_z^2 \rangle$
- Metrology with measuring any operator

Entanglement

A state is **(fully) separable** if it can be written as

$$\sum_k p_k \varrho_k^{(1)} \otimes \varrho_k^{(2)} \otimes \dots \otimes \varrho_k^{(N)}.$$

If a state is not separable then it is **entangled** (Werner, 1989).

- Separable states remain separable under local operations. (“Local operations and classical communication”)
- Separable states can be created without real quantum interaction. They are the “boring” states.

k -producibility/ k -entanglement

A pure state is **k -producible** if it can be written as

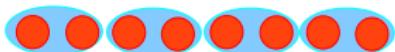
$$|\Phi\rangle = |\Phi_1\rangle \otimes |\Phi_2\rangle \otimes |\Phi_3\rangle \otimes |\Phi_4\rangle \dots$$

where $|\Phi_i\rangle$ are states of at most k qubits.

A mixed state is k -producible, if it is a mixture of k -producible pure states.

[e.g., O. Guhne and GT, New J. Phys 2005.]

- If a state is not k -producible, then it is at least **$(k + 1)$ -particle entangled**.



two-producible



three-producible

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Many-particle systems for $j=1/2$

- For spin- $\frac{1}{2}$ particles, we can measure the **collective angular momentum operators**:

$$J_l := \frac{1}{2} \sum_{k=1}^N \sigma_l^{(k)},$$

where $l = x, y, z$ and $\sigma_l^{(k)}$ a Pauli spin matrices.

- We can also measure the **variances**

$$(\Delta J_l)^2 := \langle J_l^2 \rangle - \langle J_l \rangle^2.$$

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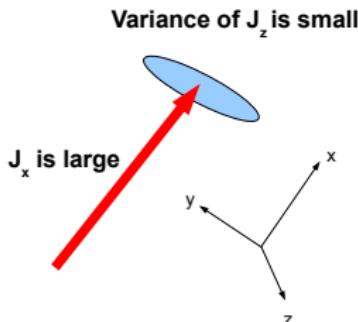
The standard spin-squeezing criterion

The spin squeezing criteria for entanglement detection is

$$\xi_s^2 = N \frac{(\Delta J_z)^2}{\langle J_x \rangle^2 + \langle J_y \rangle^2}.$$

[A. Sørensen, L.M. Duan, J.I. Cirac, P. Zoller, Nature 409, 63 (2001).]

- If $\xi_s^2 < 1$ then the state is entangled.
- States detected are like this:



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Generalized spin squeezing criteria for $j = \frac{1}{2}$

- Let us assume that for a system we know only

$$\vec{J} := (\langle J_x \rangle, \langle J_y \rangle, \langle J_z \rangle),$$
$$\vec{K} := (\langle J_x^2 \rangle, \langle J_y^2 \rangle, \langle J_z^2 \rangle).$$

- Then any state violating the following inequalities is entangled:

$$\langle J_x^2 \rangle + \langle J_y^2 \rangle + \langle J_z^2 \rangle \leq \frac{N(N+2)}{4},$$

$$(\Delta J_x)^2 + (\Delta J_y)^2 + (\Delta J_z)^2 \geq \frac{N}{2}, \quad (\text{singlet})$$

$$\langle J_k^2 \rangle + \langle J_l^2 \rangle \leq (N-1)(\Delta J_m)^2 + \frac{N}{2}, \quad (\text{Dicke state})$$

$$(N-1)[(\Delta J_k)^2 + (\Delta J_l)^2] \geq \langle J_m^2 \rangle + \frac{N(N-2)}{4},$$

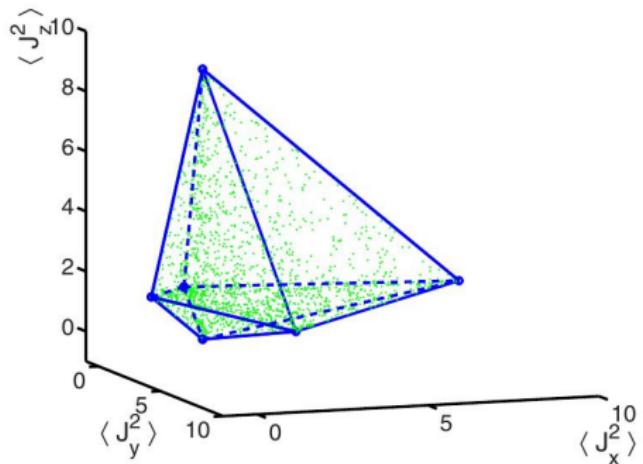
where k, l, m take all the possible permutations of x, y, z .

[GT, C. Knapp, O. Gühne, and H.J. Briegel, PRL 99, 250405 (2007)]

[Singlets: Behbood *et al.*, Phys. Rev. Lett. 2014; GT, Mitchell, New. J. Phys. 2010.]

Generalized spin squeezing criteria for $j = \frac{1}{2}$ II

- Separable states are in the polytope



- We set $\langle J_l \rangle = 0$ for $l = x, y, z$.

Spin squeezing criteria – Two-particle correlations

All quantities needed can be obtained with two-particle correlations

$$\langle J_I \rangle = N \langle j_I \otimes \mathbb{1} \rangle_{\varrho_{2p}}; \quad \langle J_I^2 \rangle = \frac{N}{4} + N(N-1) \langle j_I \otimes j_I \rangle_{\varrho_{2p}}.$$

- Here, the average 2-particle density matrix is defined as

$$\varrho_{2p} = \frac{1}{N(N-1)} \sum_{n \neq m} \varrho_{mn}.$$

- Still, we can detect states with a separable ϱ_{2p} .
- Still, as we will see, we can even detect multipartite entanglement!

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Dicke states

- Symmetric Dicke states with $\langle J_z \rangle = 0$ (simply “Dicke states” in the following) are defined as

$$|D_N\rangle = \left(\frac{N}{\frac{N}{2}}\right)^{-\frac{1}{2}} \sum_k \mathcal{P}_k \left(|0\rangle^{\otimes \frac{N}{2}} \otimes |1\rangle^{\otimes \frac{N}{2}} \right).$$

- E.g., for four qubits they look like

$$|D_4\rangle = \frac{1}{\sqrt{6}} (|0011\rangle + |0101\rangle + |1001\rangle + |0110\rangle + |1010\rangle + |1100\rangle).$$

[photons: Kiesel, Schmid, GT, Solano, Weinfurter, PRL 2007;
Prevedel, Cronenberg, Tame, Paternostro, Walther, Kim, Zeilinger, PRL 2007;
Wieczorek, Krischek, Kiesel, Michelberger, GT, and Weinfurter, PRL 2009]

[cold atoms: Lücke *et al.*, Science 2011; Hamley *et al.*, Science 2011; C. Gross *et al.*, Nature 2011]

Dicke states are useful because they ...

- ... possess strong multipartite entanglement, like GHZ states.

[GT, JOSAB 2007.]

- ... are optimal for quantum metrology, similarly to GHZ states.

[Hyllus *et al.*, PRA 2012; Lücke *et al.*, Science 2011; GT, PRA 2012; GT and Apellaniz, JPHYSA, 2014.]

- ... are macroscopically entangled, like GHZ states.

[Fröwis, Dür, PRL 2011]

Spin Squeezing Inequality for Dicke states

- Let us rewrite the third inequality

$$\langle J_k^2 \rangle + \langle J_l^2 \rangle - \frac{N}{2} \leq (N-1)(\Delta J_m)^2.$$

- It detects states close to Dicke states since

$$\begin{aligned}\langle J_x^2 + J_y^2 \rangle &= \frac{N}{2} \left(\frac{N}{2} + 1 \right) = \text{max.,} \\ \langle J_z^2 \rangle &= 0.\end{aligned}$$

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Multipartite entanglement in spin squeezing

- We consider pure k -producible states of the form

$$|\Psi\rangle = \otimes_{I=1}^M |\psi_I\rangle,$$

where $|\psi_I\rangle$ is the state of at most k qubits.

Extreme spin squeezing

The spin-squeezing criterion for k -producible states is

$$(\Delta J_z)^2 \geq J_{\max} F_{\frac{k}{2}} \left(\frac{\sqrt{\langle J_x \rangle^2 + \langle J_y \rangle^2}}{J_{\max}} \right),$$

where $J_{\max} = \frac{N}{2}$ and we use the definition

$$F_j(X) := \frac{1}{j} \min_{\substack{\langle j_x \rangle \\ j}} (\Delta j_z)^2.$$

[Sørensen and Mølmer, Phys. Rev. Lett. 86, 4431 (2001);
experimental test: Gross, Zibold, Nicklas, Esteve, Oberthaler, Nature 464, 1165

Multipartite entanglement around Dicke states

- Measure the same quantities as before

$$(\Delta J_z)^2$$

and

$$\langle J_x^2 + J_y^2 \rangle.$$

- In contrast, for the original spin-squeezing criterion we measured $(\Delta J_z)^2$ and $\langle J_x \rangle^2 + \langle J_y \rangle^2$.
- Pioneering work: linear condition of Luming Duan, Phys. Rev. Lett. (2011). See also Zhang, Duan, New. J. Phys. (2014).

Multipartite entanglement - Our condition

- Sørensen-Mølmer condition for k -producible states

$$(\Delta J_z)^2 \geq J_{\max} F_{\frac{k}{2}} \left(\frac{\sqrt{\langle J_x \rangle^2 + \langle J_y \rangle^2}}{J_{\max}} \right).$$

- Combine it with

$$\langle J_x^2 + J_y^2 \rangle \leq J_{\max} \left(\frac{k}{2} + 1 \right) + \langle J_x \rangle^2 + \langle J_y \rangle^2,$$

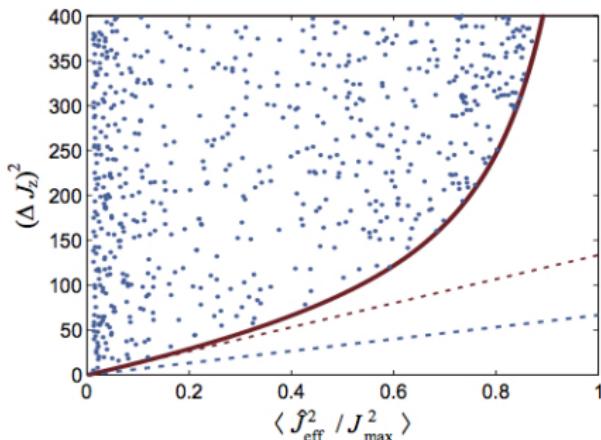
which is true for pure k -producible states. (Remember, $J_{\max} = \frac{N}{2}$.)

Condition for entanglement detection around Dicke states

$$(\Delta J_z)^2 \geq J_{\max} F_{\frac{k}{2}} \left(\frac{\sqrt{\langle J_x^2 + J_y^2 \rangle} - J_{\max} \left(\frac{k}{2} + 1 \right)}{J_{\max}} \right).$$

Due to convexity properties of the expression, this is also valid to mixed separable states.

Concrete example



- $N = 8000$ particles, and $J_{\text{eff}} = J_x^2 + J_y^2$.
- Red curve: boundary for 28-particle entanglement.
- Blue dashed line: linear condition given in [L.-M. Duan, Phys. Rev. Lett. 107, 180502 (2011).]
- Red dashed line: tangent of our curve.

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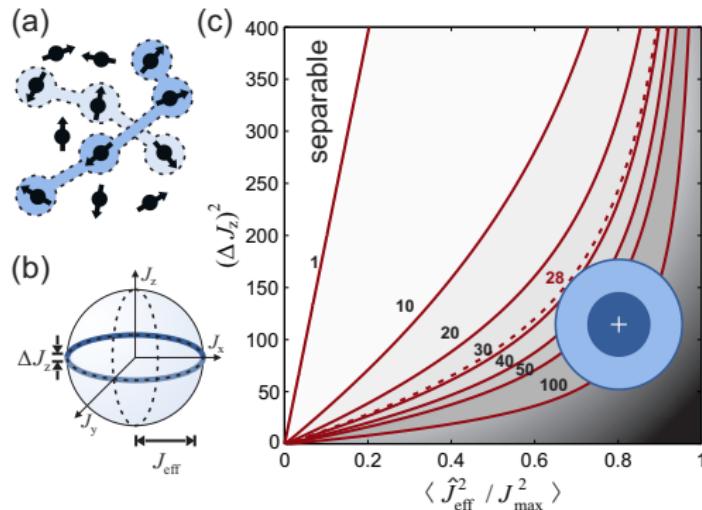
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Experimental results

- Bose-Einstein condensate, 8000 particles. 28-particle entanglement is detected.



Giuseppe Vitagliano



[Lücke *et al.*, Phys. Rev. Lett. 112, 155304 (2014).]

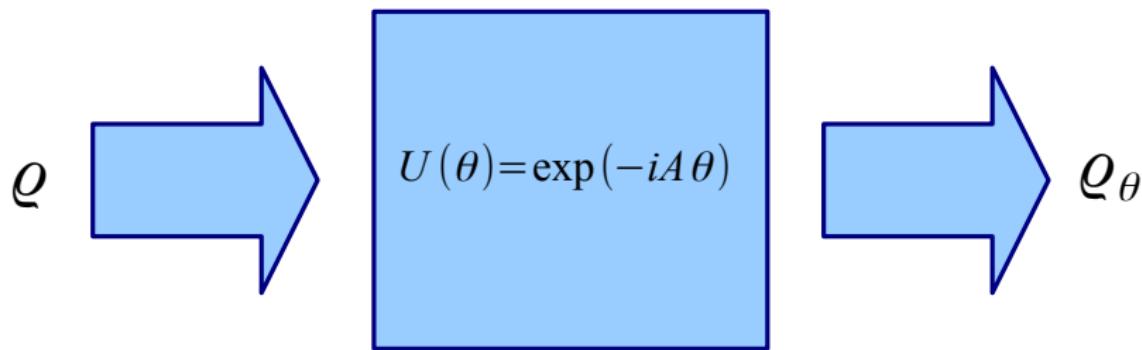


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Quantum metrology

- Fundamental task in metrology



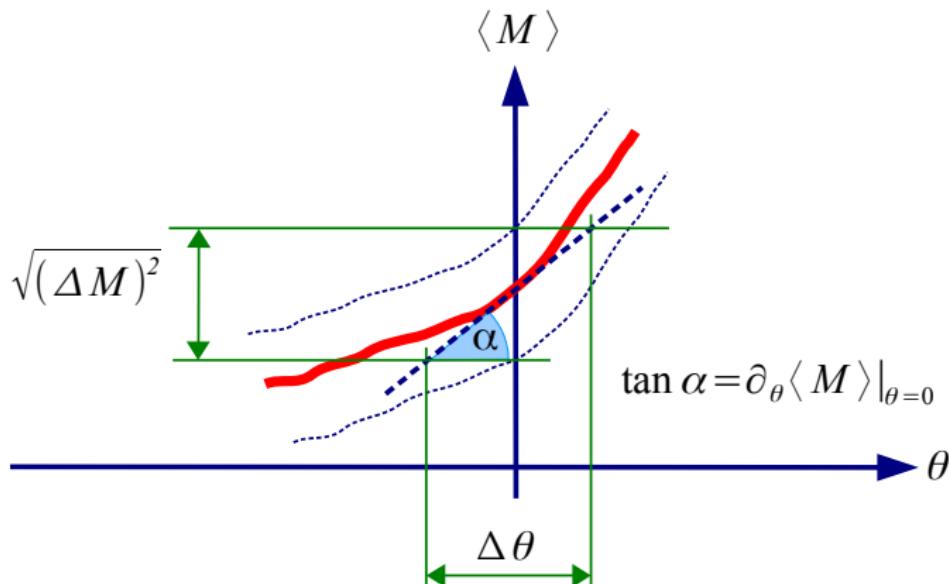
- We have to estimate θ in the dynamics

$$U = \exp(-iA\theta).$$

Precision of parameter estimation

- Measure an operator M to get the estimate θ . The precision is

$$(\Delta\theta)^2 = \frac{(\Delta M)^2}{|\partial_\theta \langle M \rangle|^2}.$$



The quantum Fisher information

- Cramér-Rao bound on the precision of parameter estimation

$$(\Delta\theta)^2 \geq \frac{1}{F_Q[\varrho, A]}, \quad (\Delta\theta)^{-2} \leq F_Q[\varrho, A].$$

where $F_Q[\varrho, A]$ is the **quantum Fisher information**.

- The quantum Fisher information is

$$F_Q[\varrho, A] = 2 \sum_{k,l} \frac{(\lambda_k - \lambda_l)^2}{\lambda_k + \lambda_l} |\langle k | A | l \rangle|^2,$$

where $\varrho = \sum_k \lambda_k |k\rangle\langle k|$.

The quantum Fisher information vs. entanglement

- For separable states

$$F_Q[\varrho, J_I] \leq N.$$

[Pezze, Smerzi, Phys. Rev. Lett. 102, 100401 (2009); Hyllus, Guhne, Smerzi, Phys. Rev. A 82, 012337 (2010)]

- For states with at most k -particle entanglement (k is divisor of N)

$$F_Q[\varrho, J_I] \leq kN.$$

[Hyllus *et al.*, Phys. Rev. A 85, 022321 (2012); GT, Phys. Rev. A 85, 022322 (2012)].

- Macroscopic superpositions (e.g, GHZ states, Dicke states)

$$F_Q[\varrho, J_I] \propto N^2$$

[Frowis, Dur, New J. Phys. 14 093039 (2012).]

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Metrology with Dicke states

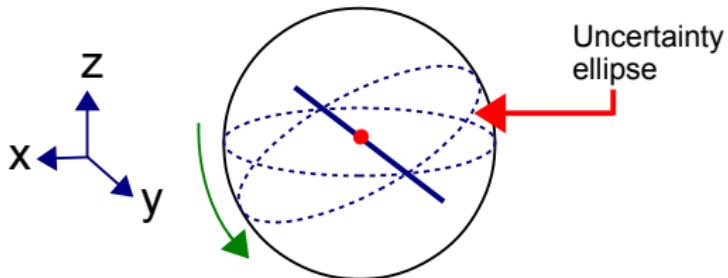
- For Dicke state

$$\langle J_l \rangle = 0, l = x, y, z, \quad \langle J_z^2 \rangle = 0, \quad \langle J_x^2 \rangle = \langle J_y^2 \rangle = \text{large}.$$

- Linear metrology

$$U = \exp(-iJ_y\theta).$$

- Measure $\langle J_z^2 \rangle$ to estimate θ . (We cannot measure first moments, since they are zero.)

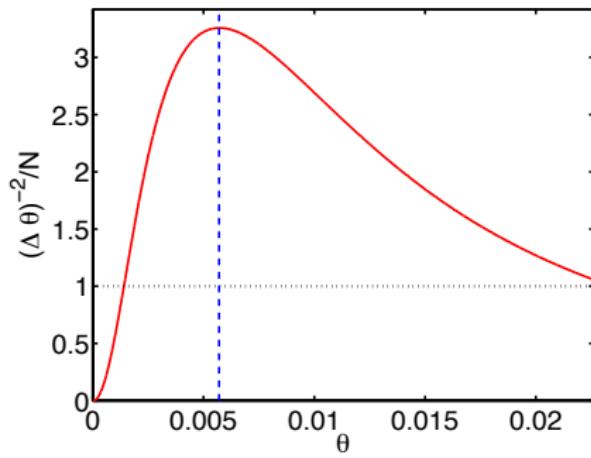


Metrology with Dicke states II

We measure $\langle J_z^2 \rangle$ to estimate θ . The precision is given by the error-propagation formula

$$(\Delta\theta)^2 = \frac{(\Delta J_z^2)^2}{|\partial_\theta \langle J_z^2 \rangle|^2}.$$

- Precision as a function of θ for some noisy Dicke state
[remember: $(\Delta\theta)^{-2} \leq F_Q[\varrho, J_y]$.]



Formula for maximal precision

Parameter value for the maximum

$$\tan^2 \theta_{\text{opt}} = \sqrt{\frac{(\Delta J_z^2)^2}{(\Delta J_x^2)^2}}.$$

Consistency check: for the noiseless Dicke state we have $(\Delta J_z^2)^2 = 0$, hence $\theta_{\text{opt}} = 0$.

Iagoba Apellaniz



Formula for maximal precision II

Maximal precision with a closed formula

$$(\Delta\theta)_{\text{opt}}^2 = \frac{2\sqrt{(\Delta J_z^2)^2(\Delta J_x^2)^2 + 4\langle J_x^2 \rangle - 3\langle J_y^2 \rangle - 2\langle J_z^2 \rangle(1 + \langle J_x^2 \rangle)} + 6\langle J_z J_x^2 J_z \rangle}{4(\langle J_x^2 \rangle - \langle J_z^2 \rangle)^2}.$$

- Given in terms of collective observables, like spin-squeezing criteria.

[Apellaniz, Lücke, Peise, Klempert, GT, New J. Phys. 17, 083027 (2015).]

Formula for maximal precision III

- Some things are difficult to measure, they can be bounded

$$\langle J_z J_x^2 J_z \rangle = \frac{\langle J_z (J_x^2 + J_y^2) J_z \rangle}{2} = \frac{\langle J_z (J_x^2 + J_y^2 + J_z^2) J_z \rangle - \langle J_z^4 \rangle}{2} \leq \frac{N(N+2)}{8} \langle J_z^2 \rangle - \frac{1}{2} \langle J_z^4 \rangle.$$

- Equality holds for symmetric states.

[Apellaniz, Lücke, Peise, Klempert, GT, New J. Phys. 17, 083027 (2015).]

“Witnessing” the quantum Fisher information

- Important concept: we detect metrological usefulness **without carrying out the metrological task.**
- Advantages:
 - The experiment can be simpler, we do not need dynamics.
 - Possibly, we could obtain $F_Q \propto N^2$ scaling for larger N , since we are not affected by the noisy dynamics.

[Demkowicz-Dobrzański, Kołodyński, Guća, Nat. Commun. 3, 1063 (2012).]

- Relevant for all methods estimating F_Q .

[Pezze, Smerzi, PRL 2009; Zhang, Duan, NJP 2014; Augusiak, Kołodyński, Streletsov, Bera, A. Acín, Lewenstein, arXiv:1506.08837]

Experimental test of our formula

- Trying the bound for the experimental data for $N = 7900$ particles

$$\begin{aligned}\langle J_z^2 \rangle &= 112 \pm 31, & \langle J_z^4 \rangle &= 40 \times 10^3 \pm 22 \times 10^3, \\ \langle J_x^2 \rangle &= 6 \times 10^6 \pm 0.6 \times 10^6, & \langle J_x^4 \rangle &= 6.2 \times 10^{13} \pm 0.8 \times 10^{13}.\end{aligned}$$

- Hence, we obtain

$$\frac{(\Delta\theta)_{\text{opt}}^{-2}}{N} \geq 3.7 \pm 1.5.$$

- Remember, for states for at most k -particle entanglement we have

$$(\Delta\theta)^{-2} \leq F_Q[\varrho, J_l] \leq kN.$$

- Thus, four-particle entanglement is detected for this particular measurement.

Interpretation of the results

k -particle multipartite entanglement is easier to get than *metrologically useful* k -particle multipartite entanglement.

- k -particle entangled state:

$$(k \text{ particles}) \otimes [(k - 1) \text{ particles}] \otimes [(k - 1) \text{ particles}] \otimes \dots$$

- Metrologically useful k -particle entangled state:

$$(k \text{ particles}) \otimes |\text{GHZ}_{k-1}\rangle \otimes |\text{GHZ}_{k-1}\rangle \otimes \dots,$$

where the state of $(k$ particles) is better metrologically than $|\text{GHZ}_{k-1}\rangle$.

- Mixed states: we need **detectable entanglement**.

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Most important characteristics used for estimation

The quantum Fisher information is the convex roof of the variance

$$F_Q[\varrho, A] = 4 \min_{p_k, \Psi_k} \sum_k p_k (\Delta A)_k^2,$$

where

$$\varrho = \sum_k p_k |\Psi_k\rangle\langle\Psi_k|.$$

[GT, Petz, Phys. Rev. A 87, 032324 (2013); Yu, arXiv1302.5311 (2013);
GT, I. Apellaniz, J. Phys. A: Math. Theor. 47, 424006 (2014)]

- Thus, it is similar to entanglement measures that are also defined by convex roofs.

Let us put it in context

Due to convexity of F_Q and the concavity of the variance

$$\frac{1}{4}F_Q[\varrho, A] \leq \sum_k p_k (\Delta A)_k^2 \leq (\Delta A)^2_{\varrho}.$$

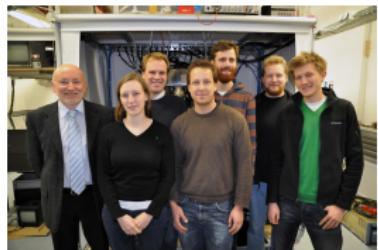
Both inequalities are tight.

We can also interpret this based on splitting the variance to “quantum” and “classical” parts

$$(\Delta A)^2 = \sum_k p_k [(\Delta A)_k^2 + (\langle A \rangle - \langle A \rangle_k)^2].$$

The quantum Fisher information is the minimal “quantum part” of the variance. [GT, Petz, Phys. Rev. A 87, 032324 (2013)]

Project participants



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Siegen

Summary

- Detection of multipartite entanglement and metrological usefulness close to Dicke states, by measuring collective quantities only. ↗ **Talk of lagoba Apellaniz**

Vitagliano, Apellaniz, Egusquiza, GT, PRA (2014).

Lücke, Peise, Vitagliano, Arlt, Santos, GT, Klempt, PRL 112, 155304 (2014), we also physics.aps.org;

Apellaniz, Lücke, Peise, Klempt, GT, New J. Phys. 17, 083027 (2015);

Apellaniz, Kleinmann, Gühne, GT, arxiv: arXiv:1511.05203.

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