Quantum Wasserstein distance based on an optimization over separable states arXiv.2209.09925

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F.Y.R.O.M. - The Former Yugoslav Republic of Macedonia
Greenland (DENMARK)
Jan Mayen (NORWAY)
Greenland Sea
Denmark Strait
Barents Sea
Norwegian Sea
Arctic Circle
Reykjavík
ICELAND
NORWAY
SWEDEN
FINLAND
RUSSIA
Tórshavn
Faroe Islands (DENMARK)
SHETLAND ISLANDS
ORKNEY ISLANDS
HEBRIDES
Rockall (U.K.)
North Atlantic Ocean
North Sea
UNITED KINGDOM
IRELAND
Dublin
London
Celtic Sea
English Channel
Bay of Biscay
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SLOVAKIA
DENMARK
LATVIA
LITHUANIA
ESTONIA
MOLDOVA
CROATIA
Mediterranean Sea
ALGERIA
MOROCCO
TUNISIA
Scale 1: 19,500,000
Lambert Conformal Conic Projection, standard parallels 40°N and 56°N
300 Kilometers
300 Miles

Outline

1. **Motivation**
   - Connecting Wasserstein distance to entanglement theory?
2. **Background**
   - Quantum Wasserstein distance
   - Quantum Fisher information
3. **Wasserstein distance and separable states**
   - Quantum Wasserstein distance based on an optimization separable states
   - Relation to entanglement conditions

Motivation

- Many distance measures are maximal for orthogonal states.
- Recently, the Wasserstein distance appeared, which is different and this makes it very useful.
- For the quantum case, surprisingly, the self-distance can be nonzero.
- Can we connect these to entanglement theory?

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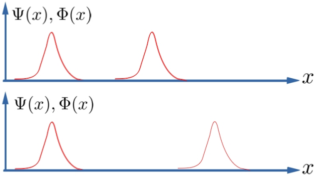
An important property of the Wasserstein distance

- The distance is often maximal for orthogonal states, e.g., for the following state-pairs.



- In the second example, the two states are further apart from each other, based on common sense.

- Wasserstein distance can recognize this since it is the "cost of moving sand from a distribution to the other one."

- It can be used for machine learning.

Quantum Wasserstein distance

- **Definition.**—The square of the distance between two quantum states described by the density matrices ϱ and σ is

$$
\begin{aligned}
D_{\mathrm{DPT}}(\varrho,\sigma)^2 = \frac{1}{2}\min_{\varrho_{12}} \sum_{n=1}^{N} \quad & \mathrm{Tr}[(H_n^T \otimes \mathbb{1} - \mathbb{1} \otimes H_n)^2 \varrho_{12}], \\
\text{s. t.} \quad & \varrho_{12} \in \mathcal{D}, \\
& \mathrm{Tr}_2(\varrho_{12}) = \varrho^T, \\
& \mathrm{Tr}_1(\varrho_{12}) = \sigma,
\end{aligned}
$$

  where $\mathcal{D}$ is the set of density matrices.

- Note the relation to the representability problem.

  G. De Palma and D. Trevisan, Quantum optimal transport with quantum channels, Ann. Henri Poincaré 22, 3199 (2021).

Self-distance can be nonzero (unlike in the classical case)

- It has been shown that for the self-distance of a state

$$D_{\mathrm{DPT}}(\varrho,\varrho)^2 = \sum_{n=1}^{N} I_\varrho(H_n)$$

holds, where the Wigner-Yanase skew information is defined as

$$I_\varrho(H) = \mathrm{Tr}(H^2\varrho) - \mathrm{Tr}(H\sqrt{\varrho}H\sqrt{\varrho}).$$

- This connects connects Wasserstein distance and quantum metrolgy.

G. De Palma and D. Trevisan, Quantum optimal transport with quantum channels, Ann. Henri Poincaré 22, 3199 (2021).

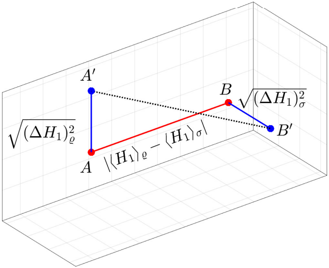
Wasserstein distance between a pure state $\varrho = |\Psi\rangle\langle\Psi|$ and a mixed state σ

- The distance is given as

$$\begin{aligned} &D_{\mathrm{DPT}}(\varrho, \sigma)^2 \\ &= \frac{1}{2} \sum_{n=1}^{N} \left[(\Delta H_n)^2{}_{\varrho} + (\Delta H_n)^2{}_{\sigma} + (\langle H_n \rangle_{\varrho} - \langle H_n \rangle_{\sigma})^2 \right], \end{aligned}$$

see the following figure.

Wasserstein distance between a pure state $\varrho = |\Psi\rangle\langle\Psi|$ and a mixed state σ II



- $N = 1$ with operator H_1.
- The quantum Wasserstein distance equals $1/2$ times the usual Euclidean distance between A' and B'.

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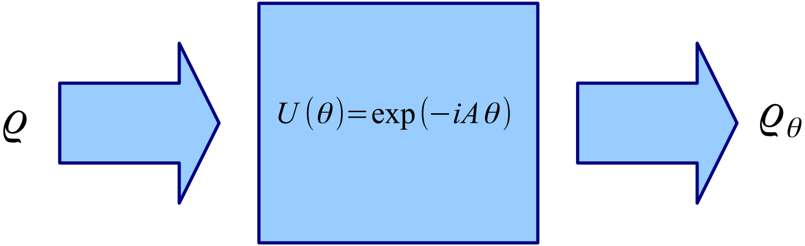
Quantum metrology

- Fundamental task in metrology



- We have to estimate θ in the dynamics

$$U = \exp(-iA\theta).$$

The quantum Fisher information

- Cramér-Rao bound on the precision of parameter estimation

$$(\Delta\theta)^2 \geq \frac{1}{mF_Q[\varrho, A]},$$

where $F_Q[\varrho, A]$ is the quantum Fisher information, and m is the number of independent repetitions.

- The quantum Fisher information is

$$F_Q[\varrho, A] = 2 \sum_{k,l} \frac{(\lambda_k - \lambda_l)^2}{\lambda_k + \lambda_l} |\langle k|A|l\rangle|^2,$$

where $\varrho = \sum_k \lambda_k |k\rangle\langle k|$.

Formula based on convex roofs

The quantum Fisher information is the convex roof of the variance

$$F_Q[\varrho, A] = 4 \min_{\{p_k, |\psi_k\rangle\}} \sum_k p_k (\Delta A)^2{}_{\psi_k},$$

where

$$\varrho = \sum_k p_k |\psi_k\rangle\langle\psi_k|.$$

GT, D. Petz, Phys. Rev. A 87, 032324 (2013); S. Yu, arXiv1302.5311 (2013);
GT, I. Apellaniz, J. Phys. A: Math. Theor. 47, 424006 (2014).

Similar relation for the variance

The quantum Fisher information is the convex roof of the variance

$$(\Delta A)^2{}_{\varrho} = \max_{\{p_k, |\psi_k\rangle\}} \sum_k p_k (\Delta A)^2{}_{\psi_k},$$

where

$$\varrho = \sum_k p_k |\psi_k\rangle\langle\psi_k|.$$

GT, D. Petz, Phys. Rev. A 87, 032324 (2013);

GT, I. Apellaniz, J. Phys. A: Math. Theor. 47, 424006 (2014).

A single relation for the QFI and the variance

For any decomposition $\{p_k, |\psi_k\rangle\}$ of the density matrix ϱ we have

$$\frac{1}{4}F_Q[\varrho, A] \le \sum_k p_k (\Delta A)^2{}_{\psi_k} \le (\Delta A)^2{}_{\varrho},$$

where the upper and the lower bounds are both tight.

- Note that

$$\frac{1}{4}F_Q[\varrho, A] \le (\Delta A)^2{}_{\varrho},$$

  where for pure states we have an equality.

- The QFI appears as a "pair" of variance.

Formula based on an optimization in the two-copy space

- We can use a two-copy formulation for the variance

$$(\Delta H)^2{}_{\Psi} = \mathrm{Tr}(\Omega|\Psi\rangle\langle\Psi| \otimes |\Psi\rangle\langle\Psi|),$$

where we define the operator

$$\Omega = H^2 \otimes \mathbb{1} - H \otimes H.$$

We can reformulate the convex roof as

$$\begin{aligned} \mathcal{F}_Q[\varrho, H] = \min_{\varrho_{12}} \quad & 4\mathrm{Tr}(\Omega\varrho_{12}), \\ \text{s. t.} \quad & \varrho_{12} \in \mathcal{S}', \\ & \mathrm{Tr}_2(\varrho_{12}) = \varrho. \end{aligned}$$

Here $\mathcal{S}'$ is the set of symmetric separable states.

GT, T. Moroder, and O. Gühne, Evaluating convex roof entanglement measures, Phys. Rev. Lett. 114, 160501 (2015); GT, D. Petz, Phys. Rev. A 87, 032324 (2013).

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Quantum Wasserstein distance based on an optimization separable states

- **Definition**—We can also define

$$
\begin{aligned}
D_{\mathrm{DPT,sep}}(\varrho,\sigma)^2 = \frac{1}{2}\min_{\varrho_{12}} \sum_{n=1}^{N} \quad & \mathrm{Tr}[(H_n^T \otimes \mathbb{1} - \mathbb{1} \otimes H_n)^2 \varrho_{12}], \\
\text{s. t.} \quad & \varrho_{12} \in \mathcal{S}, \\
& \mathrm{Tr}_2(\varrho_{12}) = \varrho^T, \\
& \mathrm{Tr}_1(\varrho_{12}) = \sigma,
\end{aligned}
$$

where S is the set of separable states.

GT and J. Pitrik, Quantum Wasserstein distance based on an optimization over separable states, arXiv:2209.09925.

Self-distance

- The self-distance for $N = 1$ is

$$D_{\mathrm{DPT,sep}}(\varrho,\varrho)^2 = \frac{1}{4}\mathcal{F}_Q[\varrho, H_1].$$

- Note that

$$I_\varrho(A) \leq \frac{1}{4}F_Q[\varrho, A] \leq (\Delta A)^2{}_\varrho.$$

GT and J. Pitrik, Quantum Wasserstein distance based on an optimization over separable states, arXiv:2209.09925.

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Entanglement of ϱ_{12}

- In general,

$$D_{\mathrm{DPT,sep}}(\varrho,\sigma) \geq D_{\mathrm{DPT}}(\varrho,\sigma).$$

- If the relation

$$D_{\mathrm{DPT,sep}}(\varrho,\sigma) > D_{\mathrm{DPT}}(\varrho,\sigma),$$

  holds, then the optimal ϱ_{12} for $D_{\mathrm{DPT}}(\varrho,\sigma)$ is entangled.

- Allowing an entangled ϱ_{12} decreases the cost!

- Thus, an entangled ϱ_{12} can be cheaper than a separable one.

Bounds on the distance

- Let us choose a set of H_n such that

$$\frac{1}{2}\sum_n \left\langle (H_n^T \otimes \mathbb{1} - \mathbb{1} \otimes H_n)^2 \right\rangle \geq \text{const.}$$

  holds for separable states. (The, we have an entanglement criterion. E. g., $\{H_n\} = \{j_x, j_y, j_z\}$ and "const."$= j$.)

- If the inequality

$$D_{\rm DPT}(\varrho, \sigma) < \text{const.}$$

  holds, then the the optimal ϱ_{12} for $D_{\rm DPT}(\varrho, \sigma)$ is entangled.

- Then, we will have a minimal distance

$$D_{\rm DPT,sep}(\varrho, \sigma) \geq \text{const.}$$

Summary

- For the Wasserstein distance, the self-distance equals the quantum Fisher information if we restrict the optimization to separable states.

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