Uncertainty relations with the variance and the quantum Fisher information — The Cramér-Rao bound as a convex roof

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Outline

1. **Motivation**
   - The Cramér-Rao bound

2. **The derivation**
   - The Cramér-Rao bound based on a convex roof

Proving the Cramér-Rao bound

- The Cramér-Rao bound is a fundamental relation in metrology.
- It is an expression with the quantum Fisher information (QFI), which is a complicated function of the state and the Hamiltonian.
- We will look for a simple proof of the Cramér-Rao bound based on fundamental uncertainty relations.
- We will exploit the fact that the QFI is the convex roof of the variance.

Outline

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Cramér-Rao bound

- Error propagation formula
$$(\Delta\theta)^2_A = \frac{(\Delta A)^2}{|\partial_\theta \langle A\rangle|^2} = \frac{(\Delta A)^2}{|\langle i[A,B]\rangle|^2}.$$
- If we measure A, then the precision of the estimation is bounded as
$$(\Delta\theta)^2 \geq \frac{1}{m}(\Delta\theta)^2{}_A,$$
where m is the number of independent repetitions.
- Let us consider a decomposition of the density matrix
$$\varrho = \sum_k p_k |\psi_k\rangle\langle\psi_k|.$$
- The Heisenberg uncertainty for the components ris
$$(\Delta A)^2{}_{\psi_k}(\Delta B)^2{}_{\psi_k} \geq \frac{1}{4}|\langle i[A,B]\rangle_{\psi_k}|^2.$$

Cramér-Rao bound II

- Let us consider the inequality
$$\left(\sum_k p_k a_k\right)\left(\sum_k p_k b_k\right) \geq \left(\sum_k p_k \sqrt{a_k b_k}\right)^2,$$
where $a_k, b_k \geq 0$.

- Hence, we arrive at
$$\left[\sum_k p_k (\Delta A)^2{}_{\psi_k}\right]\left[\sum_k p_k (\Delta B)^2{}_{\psi_k}\right] \geq \frac{1}{4}\left[\sum_k p_k |\langle i[A,B]\rangle_{\psi_k}|\right]^2.$$

- We can choose the decomposition such that
$$\sum_k p_k (\Delta B)^2{}_{\psi_k} = F_Q[\varrho, B]/4.$$

- Due to the concavity of the variance we also know that
$$\sum_k p_k (\Delta A)^2{}_{\psi_k} \leq (\Delta A)^2.$$

Cramér-Rao bound III

- Hence, it follows that

$$(\Delta A)^2{}_\varrho \left[4 \min_{p_k,\psi_k} \sum_k p_k (\Delta B)^2{}_{\psi_k}\right] \geq |\langle i[A,B]\rangle_{\psi_k}|^2.$$

- Then,

$$\frac{(\Delta A)^2{}_\varrho}{|\langle i[A,B]\rangle_{\psi_k}|^2} \geq \frac{1}{\left[4 \min_{p_k,\psi_k} \sum_k p_k (\Delta B)^2{}_{\psi_k}\right]}.$$

- Finally, for the precision of estimation, if we measure A and the Hamiltonian is B, we have

$$(\Delta\theta)^2 \geq \frac{1}{m}(\Delta\theta)^2{}_A = \frac{1}{m}\frac{(\Delta A)^2{}_\varrho}{|\langle i[A,B]\rangle_{\psi_k}|^2} \geq \frac{1}{m} \times \frac{1}{\underbrace{4 \min_{p_k,\psi_k} \sum_k p_k (\Delta B)^2{}_{\psi_k}}_{F_Q[\varrho,B], \text{ the QFI!}}}.$$

Summary

- We showed how to derive the Cramér-Rao bound with the convex roof of the variance.

See:

Géza Tóth and Florian Fröwis,

Uncertainty relations with the variance and the quantum Fisher information based on convex decompositions of density matrices,

Phys. Rev. Research 4, 013075 (2022).

THANK YOU FOR YOUR ATTENTION!

QUANTERA

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