

# Activating hidden metrological usefulness

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## Introduction

- It has been realized that entanglement can be a useful resource in very general metrological tasks. Even bound entangled states can be more useful than separable states [1,2]. Such states are called "useful" in short. However, there are highly entangled states that are not useful for metrology [3].
- In the spirit of Ref. [4], we show that some bipartite entangled quantum states that are not useful in linear interferometers become useful if several copies are considered or ancillas are added [5].
- To support our claims, we present a general method to find the *local* Hamiltonian for which a given bipartite quantum state provides the largest gain compared to separable states. Note that this task is different, and in a sense more complex, than maximizing the quantum Fisher information [5].

## Quantum Fisher information

- A basic metrological task in a *linear* interferometer is estimating the small angle  $\theta$  for a unitary dynamics  $U_\theta = \exp(-i\mathcal{H}\theta)$ , where the Hamiltonian is the sum of *local* terms. For bipartite systems it is

$$\mathcal{H} = \mathcal{H}_1 + \mathcal{H}_2, \quad (1)$$

where  $\mathcal{H}_i$  are single-subsystem operators.

- Cramér-Rao bound:

$$(\Delta\theta)^2 \geq \frac{1}{m\mathcal{F}_Q[\rho, \mathcal{H}]}, \quad (2)$$

where  $m$  is the number of independent repetitions, and the quantum Fisher information is defined by the formula

$$\mathcal{F}_Q[\rho, \mathcal{H}] = 2 \sum_{k,l} \frac{(\lambda_k - \lambda_l)^2}{\lambda_k + \lambda_l} |\langle k | \mathcal{H} | l \rangle|^2. \quad (3)$$

Here,  $\lambda_k$  and  $|k\rangle$  are the eigenvalues and eigenvectors, respectively, of the density matrix  $\rho$ , which is used as a probe state for estimating  $\theta$ .

## Metrological gain

- We define the metrological gain compared to separable states, for a given Hamiltonian, by [5]

$$g_{\mathcal{H}}(\rho) = \mathcal{F}_Q[\rho, \mathcal{H}] / \mathcal{F}_Q^{(\text{sep})}(\mathcal{H}), \quad (4)$$

where the separable limit for local Hamiltonians is

$$\mathcal{F}_Q^{(\text{sep})}(\mathcal{H}) = \sum_{n=1,2} [\sigma_{\max}(\mathcal{H}_n) - \sigma_{\min}(\mathcal{H}_n)]^2. \quad (5)$$

- We are interested in the quantity [5]

$$g(\rho) = \max_{\text{local } \mathcal{H}} g_{\mathcal{H}}(\rho), \quad (6)$$

where a local Hamiltonian is just the sum of single system Hamiltonians as in Eq. (1).

- The maximization task looks challenging since we have to maximize a fraction, where both the numerator and the denominator depend on the Hamiltonian.

## Ancilla

- For the  $3 \times 3$ -case, we consider the maximally entangled state mixed with noise

$$\rho_{AB}^{(p)} = (1-p)|\Psi^{(\text{me})}\rangle\langle\Psi^{(\text{me})}| + p\mathbb{1}/d^2, \quad (7)$$

which is useful if  $p < 0.3655$ .

- If a pure ancilla qubit is added [5]

$$\rho^{(\text{anc})} = |0\rangle\langle 0|_a \otimes \rho_{AB}^{(p)}. \quad \text{[Diagram: } a \text{ (ancilla), } A \text{ (system), } B \text{ (system)]}$$

then the state is useful if  $p < 0.3752$ .

- The Hamiltonian is

$$\mathcal{H}^{(\text{anc})} = 1.2C_{aA} \otimes \mathbb{1}_B + \mathbb{1}_{aA} \otimes D_B,$$

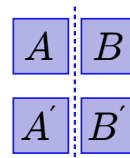
where

$$C_{aA} = \frac{9}{20} (2\sigma_x + \sigma_z)_a \otimes |0\rangle\langle 0|_A + \mathbb{1}_a \otimes (|2\rangle\langle 2|_A - |1\rangle\langle 1|_A).$$

## Two copies

- We consider now two copies of the noisy  $3 \times 3$  maximally entangled state [5]

$$\rho^{(\text{tc})} = \rho_{AB}^{(p)} \otimes \rho_{A'B'}^{(p)}.$$



- Then, with the two-copy operator

$$\mathcal{H}^{(\text{tc})} = D_a \otimes D_{A'} \otimes \mathbb{1}_{BB'} + \mathbb{1}_{AA'} \otimes D_B \otimes D_{B'}, \quad (8)$$

where

$$D = \text{diag}(+1, -1, +1, -1, \dots), \quad (9)$$

the state is useful if  $p < 0.4164$ .

## Optimal Hamiltonian

- Instead of the quantum Fisher information, let us consider the error propagation formula

$$(\Delta\theta)_M^2 = \frac{(\Delta M)^2}{\langle i[M, \mathcal{H}] \rangle^2}, \quad (10)$$

which provides a bound on the quantum Fisher information

$$\mathcal{F}_Q[\rho, \mathcal{H}] \geq 1/(\Delta\theta)_M^2. \quad (11)$$

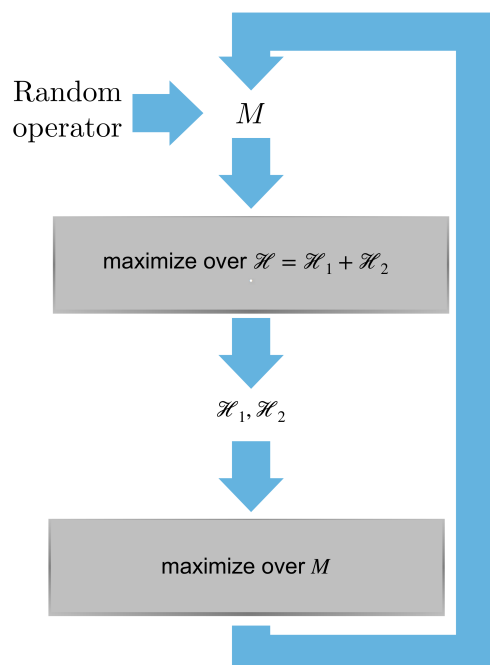
- We will minimize Eq. (10) using the idea [4]

$$\max_{\mathcal{H}} \mathcal{F}_Q[\rho, \mathcal{H}] = \max_{\mathcal{H}, M} \frac{\langle i[M, \mathcal{H}] \rangle^2}{(\Delta M)^2}. \quad (12)$$

Based on these, we realize a see-saw, optimizing alternately over  $\mathcal{H}$  and  $M$ .

	Analytic example	Numerics
Ancilla	0.3752	0.3941
Second copy	0.4164	0.4170

## See-saw iteration



## Pure states

- General case, pure state with a Schmidt decomposition

$$|\Psi\rangle = \sum_{k=1}^s \sigma_k |k\rangle_A |k\rangle_B,$$

where  $s$  is the Schmidt number, and the real positive  $\sigma_k$  Schmidt coefficients are in a descending order.

- Direct calculation yields [4]

$$\begin{aligned} \mathcal{F}_Q[|\Psi\rangle, \mathcal{H}_{AB}] &= 4(\Delta\mathcal{H}_{AB})_\Psi^2 \\ &= 8 \sum_{n=1,3,5,\dots,\tilde{s}-1} (\sigma_n + \sigma_{n+1})^2, \end{aligned}$$

which is larger than the separable bound,  $\mathcal{F}_Q^{(\text{sep})} = 8$ , whenever the Schmidt rank is larger than 1. Here,  $\tilde{s}$  is the largest even number for which  $\tilde{s} \leq s$ . (For the Hamiltonian  $\mathcal{H}_{AB}$ , see Ref. [5].)

- In the limit of infinite copies, all entangled bipartite pure states are maximally useful [5].

## Related bibliography

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