# Computable Measure of Nonclassicality for Light

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We propose the entanglement potential (EP) as a measure of nonclassicality for quantum states of a single-mode electromagnetic field. It is the amount of two-mode entanglement that can be generated from the field using linear optics, auxiliary classical states, and ideal photodetectors. The EP detects nonclassicality, has a direct physical interpretation, and can be computed efficiently. These three properties together make it stand out from previously proposed nonclassicality measures. We derive closed expressions for the EP of important classes of states and analyze as an example of the degradation of nonclassicality in lossy channels.

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Quantum mechanics, and, in particular, quantum electrodynamics, enjoys impeccable internal consistency and shows unmatched agreement with experimental observations. Something remaining unresolved is determining its borderline to the classical world, where quantum rules are not observed. A first step towards understanding the quantum to classical transition is to find a description of classical states within quantum theory. One can then ask how much *nonclassicality* any given state possesses.

Coherent states [1] are generally accepted to be the most classical of the quantum states of a harmonic oscillator. They reflect the wavelike nature of fields, are generated by classical currents, and are pointer states in realistic environments. Adopting this notion, we call a state *classical* if it is a coherent state, or a mixture thereof (cf. [2]). All other quantum states are *nonclassical*.

With recent progress in quantum optics an increasing wealth of light fields are created deviating more and more from coherent states [3]. Such nonclassical states are distinguished by, e.g., sub-Poissonian photon statistics, squeezing, photon number oscillations, or negative values of the Wigner function. Most of these signatures can be quantified defining degrees of nonclassicality.

As a central problem, however, none of the above properties detects nonclassicality infallibly [4]: e.g., the state $|\alpha\rangle + |-\alpha\rangle$, with $\alpha \gg 1$ has Poissonian photon statistics, negligible squeezing, and yet is highly nonclassical. It is therefore desirable to have a general measure of nonclassicality. This could quantify a resource for a variety of applications that are brought about by the quantum features of light.

A clear-cut "universal" approach to quantifying nonclassicality has been formulated by Hillery [5]. He defined the trace distance of a state $\sigma$ from the set of classical states as a measure of its nonclassicality. This gives 0 for coherent-state mixtures and nonzero for any other state. However, its computation involves minimization over an infinite number of variables. Consequently Hillery's nonclassical distance has not yet been evaluated exactly for any nonclassical state (for alternative "distance-based" measures and approximations see [6–8]). Another universal measure, proposed by Lee [9], is the amount of Gaussian smoothing required to transform the Glauber $P$ function into a positive distribution and quantifies the noise necessary to wash out nonclassicality. This measure, however, lacks continuity: it lies between 0 and $1/2$ for Gaussian states, but is 1 for any non-Gaussian pure state, no matter how close it is to a Gaussian [10].

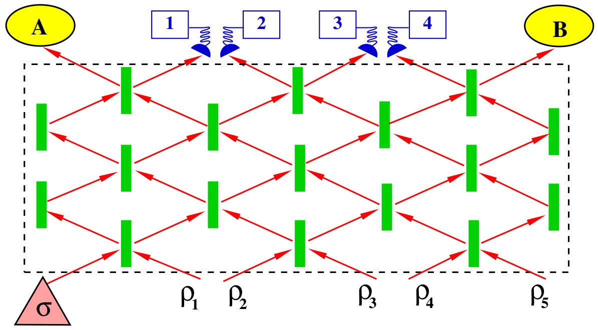
In this Letter we propose a universal measure of nonclassicality, the entanglement potential (EP), that can be computed efficiently for single-mode states of light. It is based on the observation that coherent states are the only pure states that produce uncorrelated outputs when split by a linear optical device [11,12]. We define the EP of a state $\sigma$ as the amount of two-mode entanglement (quantum correlations) that can be produced from $\sigma$ and auxiliary classical states with linear optics and photodetectors. A precise definition follows below.

Entanglement is a genuine quantum feature that gives rise to the most striking "nonclassical" effects [13,14]. With the advent of quantum information technology (QIT) it has become a crucial resource for many applications. The leading role of quantum optics in the study of the foundations of quantum mechanics [15] and in the implementation of many QIT protocols has recently triggered a lot of interest in the characterization, generation, and distillation of entanglement between optical fields [16,17]. This provides EP with a direct physical meaning: single-mode nonclassicality as an entanglement resource. Moreover, it supplies a pool of results and methods from QIT for the study of nonclassicality.

At first glance, computation of EP seems prohibitively complicated. For any nonclassical state $\sigma$ one has to find the optimal linear optical transformation and auxiliary states to create the most two-mode entanglement. However, as we show below, the optimal linear optics entangler is the same for any state, and consists of a single beam splitter (BS) and an additional vacuum input.

A representation of the transformations that we allow in the definition of the EP is shown in Fig. 1 (top). A passive



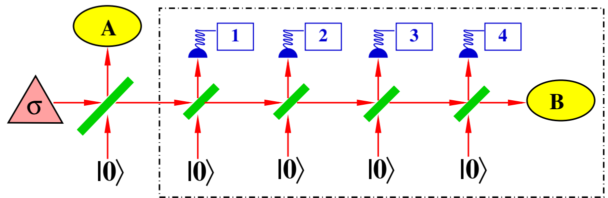




FIG. 1 (color online). (top) A linear optics "black box" (dashed line) creating entanglement between A and B from a nonclassical input state $\sigma$. Classical states enter via auxiliary input ports. Photon number measurements are made at the extra output ports. (bottom) An equivalent form of the black box for empty auxiliary ports. The dash-dotted box is local to B.

linear optical transformation can be modeled by a circuit of BS's (including phase shifters) [18]. This transforms input and auxiliary states according to a linear unitary map $a_i' = \sum_i U_{ij} a_j$ between input and output mode operators. At the output two modes are sent to A(lice) and B(ob), all others are measured by ideal photodetectors. Linear operations conditioned on a measurement outcome (e.g. [19]) are excluded, since they can increase the amount of existing nonclassicality.

The displacement of one input mode with $D(\alpha) = \exp(\alpha a^\dagger - \alpha^* a)$ amounts to (local) displacement of all output modes by amounts depending on the circuit of BS's. Mixing the displaced input modes translates to local mixing of the output modes with additional classical communication. As these operations cannot increase entanglement, vacuum ancillas are optimal for entangling the modes of Alice and Bob.

With vacuum ancillas, the circuit of BS's inside the box can be simplified to a standard form. As shown in Fig. 1 (bottom), this consists of a single BS splitting the input in two modes going to Alice and Bob, and a series of additional BS's further splitting the signal in Bob's side. Now all measurements can be carried out in Bob's auxiliary modes, and since local operations do not increase entanglement on average, Bob can expect no advantage from splitting off and measuring a part of his beam. The optimal entangling device is therefore a single BS. Although we currently lack of a general proof, all examples we checked analytically and numerically indicate that the transmissivity of the optimal BS is $1/2$ independent of the input state.

We denote by $U_{\rm BS}$ this 50:50 BS transformation, which induces the mapping $a = 2^{-1/2}(a_A + a_B)$ on the input mode annihilation operator.

Clearly, the EP is zero for classical states. Moreover, any decomposition of a given output two-mode mixed state in terms of pure states $\rho = \sum_i p_i |\Psi_i\rangle\langle\Psi_i|$ must be consistent term-by-term with a pure input $|\psi_i\rangle|0\rangle = U_{\rm BS}^{-1}|\Psi_i\rangle$: this is necessary for the corresponding input mixed state to have a vacuum auxiliary mode. Thus any separable output state must correspond to a convex combination of input states $|\alpha_i\rangle|0\rangle$, whereby all nonclassical input states, pure or mixed, will generate entanglement, and have a positive EP. This can be seen as an extension of the results of [12]. Since coherent displacement and phase shifting can be realized on a single BS with an additional strong coherent beam, the EP is invariant with respect to "classical" operations, as defined in [8].

To obtain a specific measure of nonclassicality an entanglement measure has to be chosen. The value of the EP of a state depends on this measure, and different choices may give rise to different orderings between states. Here we consider two alternatives.

A computable measure of nonclassicality for pure as well as for mixed single-mode states is obtained by taking the logarithmic negativity $E_{\mathcal{N}}$ [20] leading to the following definition for the entanglement potential,

$$\mathrm{EP}\,(\sigma) \equiv E_{\mathcal{N}}(\rho_\sigma) = \log_2 ||\rho_\sigma^{T_A}||_1. \tag{1}$$

Here $\rho_\sigma = U_{\rm BS}(\sigma \otimes |0\rangle\langle 0|)U_{\rm BS}^\dagger$, $\rho^{T_A}$ denotes the partial transpose of $\rho$, and $||\cdot||_1$ is the trace norm. A nonzero value of $E_{\mathcal{N}}$ reveals nonseparability, however, the converse is not true in general [21]. The so-called bound entangled states are not detected by the partial transposition criterion. Although examples of such non-Gaussian states exist [22], it remains an open question whether bound entanglement can arise in our setup, i.e., whether or not EP detects all nonclassical states.

An entanglement measure that does detect all entangled states is the relative entropy of entanglement $E_{\rm RE}$ [23]. We will call the induced nonclassicality measure *entropic entanglement potential* (EEP), defined as

$$\mathrm{EEP}\,(\sigma) \equiv \min_{\rho\in\mathcal{D}} \mathrm{Tr}\rho_\sigma(\log_2\rho_\sigma - \log_2\rho), \tag{2}$$

where minimization is carried out over the set $\mathcal{D}$ of all two-mode separable states. EEP detects all nonclassical states, and can be calculated for important classes of states. For finite dimensional mixed states, it can be numerically computed by an iterative procedure [24]. For pure states, it reduces to the von Neumann entropy in one of the output arms. Moreover, EEP gives a lower bound to the nonclassical relative entropy distance.

In the remainder of this Letter we calculate the EP and EEP for a variety of nonclassical states.

The EP and the EEP of a Fock state $|n\rangle$ are

$$\mathrm{EP}(n) = -n + 2\log_2 \sum_{k=0}^{n} \sqrt{\binom{n}{k}}, \tag{3}$$

$$\mathrm{EEP}(n) = n - 2^{-n} \sum_{k=0}^{n} \binom{n}{k} \log_2 \binom{n}{k}. \tag{4}$$

In the large-$n$ limit, the two diverge logarithmically and differ only in a constant: $\mathrm{EP}(n) \approx \frac{1}{2}\log_2(2\pi n)$; $\mathrm{EEP}(n) \approx \mathrm{EP}(n) - (1 - 1/\ln 4)$. The entanglement potential can thus detect the nonclassicality of Fock states, and shows that it increases with the photon number.

For any finite superposition of coherent states $|\Psi\rangle = \sum_k c_k |\alpha_k\rangle$ both EP and EEP can be calculated exactly using the "metric tensor" of [25]. This yields complicated formulas, a notable exception being the "odd coherent state" $|\alpha\rangle - |-\alpha\rangle$, for which both EP and EEP are 1, independent of $\alpha$. If the coherent states constituting $|\Psi\rangle$ are truly distinct $|\alpha_i - \alpha_k| \gg 1$, the nonclassicality is determined by the probability amplitudes $c_k$, and the $\alpha_k$ bring only an exponentially small correction. Such "macroscopic" coherent-state superpositions (MCSS) are typical examples of nonclassical states. Neglecting the correction arising from the overlaps, we obtain for the EP and the EEP of a MCSS state:

$$\mathrm{EP}(\Psi) \approx 2\log_2 \sum_k |c_k|, \tag{5}$$

$$\mathrm{EEP}(\Psi) \approx -\sum_k |c_k|^2 \log_2 |c_k|^2. \tag{6}$$

The most nonclassical superposition of $N$ coherent states is that where the probability amplitudes are of equal magnitude: in that case $\mathrm{EP} = \mathrm{EEP} = \log_2 N$.

Gaussian states are displaced squeezed thermal states: $\rho(\alpha, r, \phi, \bar{n}) = D(\alpha)S(r, \phi)\rho_{\bar{n}} S(r, \phi)^\dagger D(\alpha)^\dagger$. Here $\rho_{\bar{n}}$ is a thermal state of average photon number $\bar{n}$, and $S(r, \phi) = \exp(\frac{1}{2} r[e^{i\phi}(a^\dagger)^2 - e^{-i\phi} a^2])$ is the squeezing operator. It is well known [26] that Gaussian states are classical for $r \leq r_c$, where the nonclassicality threshold is $r_c = \ln(2\bar{n} + 1)/2$. Using the results of Wolf *et al.* [17], it immediately follows that the EP of a general Gaussian state is given by the nonclassical part of the squeezing,

$$\mathrm{EP}[\rho(\alpha, r, \phi, \bar{n})] = \frac{r - r_c}{\ln 2}, \tag{7}$$

if $r > r_c$ and 0 otherwise. For Gaussian states EP detects nonclassicality (cf. [27]), and is a monotonous function of Lee's nonclassical depth [9]. The EEP of a pure Gaussian state is found by noting that for a BS, displacement in the input is mapped to (local) displacements at the output, and squeezing is mapped to two-mode squeezing followed by local squeezing in each mode. Hence, the EEP is the entropy in a single mode of the corresponding two-mode squeezed vacuum,

$$\mathrm{EEP}[\rho(\alpha, r, \phi, 0)] = \cosh^2(r/2)\log_2\cosh^2(r/2) - \sinh^2(r/2)\log_2\sinh^2(r/2). \tag{8}$$

For strong squeezing ($r \gg 1$) we again find approximate equality with EP up to a constant: $\mathrm{EEP}(\rho_{\alpha,r,\phi,0}) \approx r/\ln 2 - (2 - 1/\ln 2)$. For weak squeezing ($r \ll 1$), however, EEP increases more slowly with $r$: $\mathrm{EEP}(\rho_{\alpha,r,\phi,0}) \approx \log_2(re^{-1/2}/2)r^2/2$.

A quantitative measure allows us to investigate how much nonclassicality is lost in a physical process. As an example we study photon dissipation, the dominant decoherence process for states propagating in an optical fiber. The corresponding interaction picture master equation is $\dot{\rho} = \frac{1}{2}\gamma(2a\rho a^\dagger - a^\dagger a\rho - \rho a^\dagger a)$. For coherent states this results in an exponential damping of the amplitude, thus the Glauber $P$ function of any state changes as $P(\alpha) \rightarrow \xi^{-1}P(\xi^{-1/2}\alpha)$ with $\xi = \exp(-\gamma t)$.

For a coherent-state superposition undergoing photon loss the EP can be calculated exactly with the metric tensor [25]. For weak dissipation, decoherence dominates power loss, and if the state was a MCSS, the damped coherent states $|\xi^{1/2}\alpha_i\rangle$ are still approximately orthogonal. The whole state, however, becomes more and more entangled with the environment, resulting in a deterioration of its purity. This also reduces the nonclassicality:

$$\mathrm{EP}(t) \approx \log_2\Big(1 + 2\sum_{i<k} |\langle\alpha_i|\alpha_k\rangle|^{1-\xi(t)} |c_i||c_k|\Big). \tag{9}$$

Although the EP or EEP of MCSS states is independent of the phase-space distance of the constituent coherent states, this distance determines the decoherence behavior. For a two-component MCSS, at intermediate times $|\alpha_1 - \alpha_2|^{-2} < \gamma t \ll 1$ the EP is given by $\mathrm{EP}(t) \approx \exp(-t/T_D)2|c_1 c_2|/\ln 2$, with the well-known decoherence time scale $T_D = 2|\alpha_1 - \alpha_2|^{-2}\gamma^{-1}$.

Formula (7) can be used to study the nonclassicality of Gaussian states in Gaussian channels. In particular, for a nonclassical Gaussian state ($r > r_c$) linearly coupled to a heat bath of mean photon number $n_T$ we find

$$\mathrm{EP}(t) = -\frac{1}{2}\log_2[e^{-\gamma t}e^{-2(r-r_c)} + (1 - e^{-\gamma t})(2n_T + 1)].$$

For $n_T > 0$ this formula is valid up to a finite time $t_c = \gamma^{-1}\ln[1 + (1 - e^{-2(r-r_c)})/(2n_T)]$: thereafter the state is classical [$\mathrm{EP}(t \geq t_c) = 0$]. We now concentrate on photon dissipation, i.e., $n_T = 0$, where the state remains nonclassical at any time. For weak squeezing, $0 < r - r_c \ll 1$, the above formula reduces to an exponential decay, $\mathrm{EP}(t) \approx (\ln 2)^{-1}(r - r_c)e^{-\gamma t}$. For strong squeezing, $r - r_c \gg 1$, we find a different decoherence behavior. Initially, EP decreases linearly with time,

$$\gamma t \ll e^{-2(r-r_c)} \quad : \quad \mathrm{EP}(t) \approx \frac{r - r_c}{\ln 2} - \frac{e^{2(r-r_c)}}{2\ln 2}\gamma t. \tag{10}$$

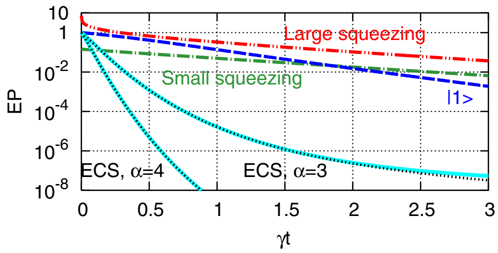



FIG. 2 (color online). The EP of Gaussian states with strong ($r - r_c = 5$, dash-dot-dotted) or weak ($r - r_c = 0.1$, dash-dotted) squeezing, the single-photon Fock state (dashed), and the even coherent states with $\alpha = 3$ and $\alpha = 4$. For these latter states, the exact values (dotted) and the approximation of Eq. (8) (solid lines) almost coincide.

The loss rate of EP is exponentially large in the initial squeezing. After a short time $\tau = \gamma^{-1}e^{-2(r-r_c)}$ the EP of the highly squeezed state follows a general curve:

$$\gamma t > e^{-2(r-r_c)} \quad : \quad \mathrm{EP}(t) \approx -\frac{1}{2}\log_2(1 - e^{-\gamma t}). \qquad (11)$$

The initial squeezing now adds only a minor correction, exponentially small in $r - r_c$, to the EP. On longer time scales, $\gamma t \gg 1$, the EP of strongly squeezed states also decreases exponentially, $\mathrm{EP}(t) = (2\ln 2)^{-1} \times (1 - e^{-2(r-r_c)})e^{-\gamma t}$. Here we explicitly included the small correction in $r$, which is the only memory of the initial parameters.

Figure 2 shows the time dependence of EP during photon loss for various states. Squeezed states retain EP better than other nonclassical states: although the weakly squeezed state ($r - r_c = 0.1$) initially has less EP than either the single-photon Fock state or the two examples of MCSS states, after $t > 2\gamma$ it is more valuable than these for entanglement generation. The most fragile states are the MCSS's, which lose EP on short time scales given by the $T_D$ mentioned above. Notice that (8) gives an excellent approximation of the EP for MCSS's.

The definition of EP in (1) has brought up the still open question of whether bound entanglement can be obtained from single-mode nonclassical light, auxiliary classical states, linear optics, and photodetectors. The study of the additivity properties of the entanglement potential—some examples show that it is superadditive—and the extension of the concept to multimode nonclassical fields are subjects of further research.

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