

Decoherence from spin environments

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We examine two exactly solvable models of decoherence—a central spin-system, (i) with and (ii) without a self-Hamiltonian, interacting with a collection of environment spins. In the absence of a self-Hamiltonian we show that in this model (introduced some time ago to illustrate environment-induced superselection) generic assumptions about the coupling strengths can lead to a universal (Gaussian) suppression of coherence between pointer states. On the other hand, we show that when the dynamics of the central spin is dominant a different regime emerges, which is characterized by a non-Gaussian decay and a dramatically different set of pointer states. We explore the regimes of validity of the Gaussian decay and discuss its relation to the spectral features of the environment and to the Loschmidt echo (or fidelity).

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I. INTRODUCTION

A central spin system \mathcal{S} interacting with an environment \mathcal{E} formed by N independent spins through the Hamiltonian

$$\mathcal{H}_{\mathcal{SE}} = \frac{1}{2} \sigma_z \otimes \sum_{k=1}^N g_k \sigma_z^{(k)} \quad (1)$$

may be the simplest solvable model of decoherence (we use the standard notation according to which $\sigma_i^{(k)}$ and σ_i , $i = x, y, z$ denote Pauli operators acting on the k th environmental spin and on the central system). This Hamiltonian was studied some time ago [1] as a simple model of decoherence. It was used to show that relatively straightforward assumptions about the dynamics can lead to the emergence of a preferred set of pointer states due to environment-induced superselection (einselection) [1,2]. Such models have gained additional importance in the past decade because of their relevance to quantum information processing [3].

The model described by Eq. (1) was particularly useful to illustrate the nature of decoherence in the context of a measurement. In such case the central spin is used as a simple (two state—one bit) approximation for a memory of classical apparatus. Then, it is natural to neglect the effect of the system's self-Hamiltonian. As a consequence, the eigenstates of the interaction Hamiltonian (1) emerge as preferred pointer states of the system (defined as the ones which are “least perturbed” by the interaction with the environment [2]). Thus the eigenstates of the σ_z operator (denoted here as $|0\rangle$ and $|1\rangle$, with eigenvalues $+1$ and -1 , respectively) are dynamically selected by the interaction with the environment. Indeed, these states are not perturbed by the interaction while other superpositions rapidly decay into their mixtures.

Neglecting the self-Hamiltonian of the system is not always a reasonable approximation. Studies of decoherence without such assumption have also been carried out using, mostly, the quantum Brownian motion as a paradigmatic example [4]. In such a case, pointer states do not coincide with the eigenstates of the interaction Hamiltonian but can range from coherent states for the QBM case [5] to eigenstates of

the system's Hamiltonian [6]. Their properties are determined by the interplay between the self-Hamiltonian and the interaction with the environment.

In this paper we will study a generalization of the above simple model described by the Hamiltonian

$$\mathcal{H}_T = \Delta \sigma_x + \frac{1}{2} \sigma_z \otimes \sum_{k=1}^N g_k \sigma_z^{(k)}. \quad (2)$$

This simple model includes both the effect of the evolution of the central system and its coupling with the spin environment.

The purpose of our study is twofold. First, in Sec. II we will analyze again the case where the central spin has no self-Hamiltonian ($\Delta=0$ above). Our goal is to show that—with a few additional natural and simple assumptions about the distribution of coupling strengths g_k —one can evaluate the exact time dependence of the reduced density matrix of the central spin. In fact, we will demonstrate that the off-diagonal components display a Gaussian (rather than exponential) decay. In this way we will exhibit a simple soluble example of a situation where the usual Markovian [7] assumptions about the evolution of a quantum open system are not satisfied at any time.

Then, in Sec. III we will consider the more complex case with nontrivial dynamics ($\Delta \neq 0$ above). We will show that, under the same natural assumptions made in Sec. II about the distribution of coupling strengths in the interaction Hamiltonian, the problem can also be solved exactly. The solution will enable us to study two very important features of the decoherence process. We will analyze the nature of pointer states and also the way in which the reduced density matrix of the central spin evolves in time. In this case, the decay of the off-diagonal component is not Gaussian but displays long-time algebraic tails which we obtain analytically. The most probable pointer states will be shown to range from eigenstates of σ_z in the small Δ limit to eigenstates of σ_x in the opposite limit of large Δ (result that can be expected based on the considerations presented in Ref. [6]). In Sec. IV we will summarize our results which, apart from their impli-

cations for decoherence, could also be relevant to quantum error correction [8] where precise knowledge of decoherence is essential to select an efficient strategy to defeat it.

II. STATIC SYSTEM—GAUSSIAN DECOHERENCE

Here we will consider the system described by Eq. (1). We begin by outlining how to solve this model exactly, and how to find the time dependence of the elements of the reduced density matrix of the system. Let us consider an initial state for the combined system environment of the form

$$|\Psi_{\mathcal{SE}}(0)\rangle = (a|0\rangle + b|1\rangle) \sum_{n=0}^{2^N-1} c_n |n\rangle. \quad (3)$$

Here $|n\rangle$ are the states of the computational basis of the environment that diagonalizes $\mathcal{H}_{\mathcal{SE}}$. The k th digit of the binary form of n , n_k , represents the state up or down in the Z axis of the k th spin of the environment. The main assumptions above are that the initial state is a product (no initial entanglement between the system and environment) and that the total state is pure. Both conditions can be easily relaxed, but choosing Eq. (3) simplifies the presentation. The state of \mathcal{SE} at an arbitrary time is given by

$$|\Psi_{\mathcal{SE}}(t)\rangle = a|0\rangle|\mathcal{E}_0(t)\rangle + b|1\rangle|\mathcal{E}_1(t)\rangle, \quad (4)$$

with

$$|\mathcal{E}_0(t)\rangle = \sum_{n=0}^{2^N-1} c_n e^{-iB_n t/2} |n\rangle = |\mathcal{E}_1(-t)\rangle, \quad (5)$$

and where

$$B_n = \sum_{k=1}^N (-1)^{n_k} g_k. \quad (6)$$

The reduced density matrix of the system is then

$$\begin{aligned} \rho_S &= \text{Tr}_{\mathcal{E}} |\Psi_{\mathcal{SE}}(t)\rangle \langle \Psi_{\mathcal{SE}}(t)| \\ &= |a|^2 |0\rangle \langle 0| + ab^* r(t) |0\rangle \langle 1| + a^* b r^*(t) |1\rangle \langle 0| \\ &\quad + |b|^2 |1\rangle \langle 1|, \end{aligned} \quad (7)$$

where the *decoherence factor* $r(t) = \langle \mathcal{E}_1(t) | \mathcal{E}_0(t) \rangle$ can be readily obtained:

$$r(t) = \sum_{n=0}^{2^N-1} |c_n|^2 e^{-iB_n t}. \quad (8)$$

It was shown in Ref. [1] (using some simplifications to be discussed below) that for $t > 0$, $r(t)$ decays rapidly to zero, so that the typical fluctuations of the off-diagonal terms of ρ_S will be small for large environments. Therefore the decoherence factor tends to zero

$$\langle |r(t)|^2 \rangle \xrightarrow{N \rightarrow \infty} 0,$$

leaving ρ_S approximately diagonal in a mixture of the pointer states $\{|0\rangle, |1\rangle\}$ which retain preexisting classical correlations.

We will show in this section that, for a fairly generic set of assumptions, the form of $r(t)$ can be further evaluated and that—quite universally—it turns out to be approximately Gaussian in time. To prove this we will only require that the couplings g_k of Eq. (1) are sufficiently concentrated near their average value so that their standard deviation $\langle (g_k - \langle g_k \rangle)^2 \rangle$ exists and is finite. When this condition is not fulfilled other sorts of time dependence become possible. In particular, $r(t)$ may be exponential when the distribution of couplings is, for example, Lorentzian.

To obtain our result we rewrite Eq. (8) as

$$r(t) = \int e^{-iBt} \eta(B) dB, \quad (9)$$

that is, the decoherence factor is the Fourier transform of a characteristic function

$$\eta(B) = \sum_{n=0}^{2^N-1} |c_n|^2 \delta(B - B_n). \quad (10)$$

Equation (10) is a particular case of the more general strength function or local density of states [9],

$$\eta(B) = \sum_{\lambda} |\langle \phi_{\lambda} | \Psi_{\mathcal{SE}} \rangle|^2 \delta(B - B_{\lambda}), \quad (11)$$

where $|\phi_{\lambda}\rangle$ are the eigenfunctions of the full Hamiltonian with eigenenergies B_{λ} .

The discussion of decoherence in our model is thus directly related to the study of the characteristic function of the distribution of coupling energies $\eta(B)$. Since the B_n 's are sums of g_k 's (that we assume are independent of each other), Eq. (9) makes $r(t)$ itself a product of characteristic functions of the distributions of the couplings g_k . Thus the distribution of B_n belongs to the class of the so-called *infinitely divisible distributions* [10,11]. The behavior of the decoherence factor $r(t)$ —characteristic function of an infinitely divisible distribution—depends only on the average and variance of the distributions of couplings weighted by the initial state of the environment [10,11].

Assuming that the variance of the couplings g_k is finite, we claim that for reasonable assumptions on the initial state of the environment (the coefficients c_n), and N sufficiently large, $\eta(B)$ has in general a Gaussian form. Therefore the decoherence factor decays as a Gaussian with time. We will show this behavior with some examples where an exact solution is possible, and discuss the regime of validity of the conjecture.

Let us consider first the simplest case where all couplings are equal, $g_k = g$, and all the spins of the environment have the same initial state,

$$|\Psi_{\mathcal{SE}}(0)\rangle = (a|0\rangle + b|1\rangle) \otimes (\alpha_k|0\rangle_k + \beta_k|1\rangle_k), \quad (12)$$

with $\alpha_k = \alpha$ and $\beta_k = \beta$ for all k . The decoherence factor then takes the simple form $r(t) = (|\alpha|^2 e^{igt} + |\beta|^2 e^{-igt})^N$. Expanding this expression we obtain

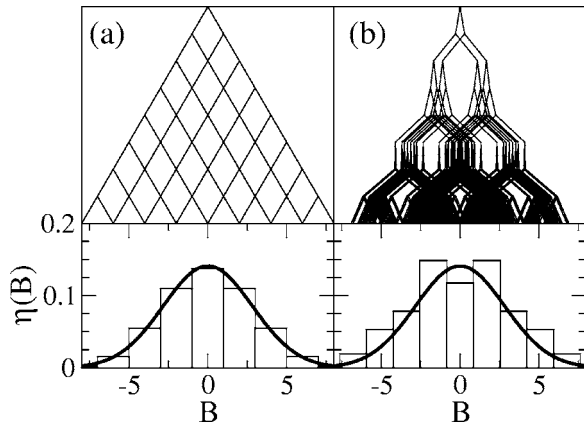


FIG. 1. The distribution of the energies obtains from the random walks with the steps given by the coupling size and in the direction ($+g_k$ or $-g_k$) biased by the probabilities $|\alpha_k|^2$ and $|\beta_k|^2$ as in Eq. (10) (although in these examples we set $|\alpha_k|^2 = 1/2$). (a) When all the couplings have the same size $g_k = g$ [Eq. (13)], a simple Newton's triangle leads to an approximate Gaussian for the distribution of energies. (b) When the couplings differ from step to step [Eq. (21)], the resulting distribution still has an approximately Gaussian envelope for large N .

$$r(t) = \sum_{k=0}^N \binom{N}{k} |\alpha|^{2k} |\beta|^{2(N-k)} \exp[-igt(2k - N)]. \quad (13)$$

As follows from the Laplace-de Moivre theorem [10], for sufficiently large N the coefficients of the binomial expansion of Eq. (13) can be approximated by a Gaussian,

$$\binom{N}{k} |\alpha|^{2k} |\beta|^{2(N-k)} \simeq \frac{e^{-(k - N|\alpha|^2)^2 / 2N|\alpha\beta|^2}}{\sqrt{2\pi N|\alpha\beta|^2}}. \quad (14)$$

Therefore for large N

$$\eta(B) \simeq \frac{\exp\left[-\frac{[B/g - N(|\alpha|^2 - |\beta|^2)]^2}{8N|\alpha\beta|^2}\right]}{\sqrt{8\pi N|\alpha\beta|^2}}. \quad (15)$$

This generic behavior can be interpreted as a result of the law of large numbers [10]: the energies B_n of the composite \mathcal{SE} system can be thought of as being the terminal points of an N -step random walk. The contribution of the k th spin of the environment to the random energy is $+g$ or $-g$ with probability $|\alpha|^2$ or $|\beta|^2$ respectively [Fig. 1(a)]. Therefore the set of all the resulting energies must have an (approximately) Gaussian distribution.

We can carry out the same argument in the more general case of Eq. (12) for different couplings and initial states for the spins of the environment. Here,

$$r(t) = \prod_{k=1}^N (|\alpha_k|^2 e^{ig_k t} + |\beta_k|^2 e^{-ig_k t}). \quad (16)$$

The “random walk” picture that yielded the distribution of the couplings remains valid [see Fig. 1(b)]. However, now the individual steps in the random walk are no longer all equal. Rather, they are given by the set $\{g_k\}$ [see Eq. (1)] with

each step g_k taken just once in a given walk. There are 2^N such distinct random walks W_n , one for every state $|n\rangle$ of the environment. Each walk contributes to $\eta(B)$ with the weight given by the product of the relevant $|\alpha_k|^2$ and $|\beta_k|^2$, or right ($k \in W_n^+$) and left ($k \in W_n^-$) “steps,” respectively. The weight of the n th walk is then given by

$$|c_n|^2 = \left(\prod_{k \in W_n^+} |\alpha_k|^2 \right) \left(\prod_{k \in W_n^-} |\beta_k|^2 \right). \quad (17)$$

The terminal points B_n of the random walks may or may not be degenerate: As seen in Fig. 1, in the degenerate case, the whole collection of 2^N random walks “collapses” into $N+1$ terminal energies. More typically, in the nondegenerate case (also displayed in Fig. 1), there are 2^N different terminal energies B_n . In any case, the “envelope” of the distribution $\eta(B)$ will be Gaussian, as we shall argue below.

Let us compute the characteristic function $\eta(B)$. If we denote x_k the random variable that takes the value $+g_k$ or $-g_k$ with probability $|\alpha_k|^2$ or $|\beta_k|^2$, respectively, then its mean value a_k and its variance b_k are

$$a_k = (|\alpha_k|^2 - |\beta_k|^2)g_k,$$

$$b_k^2 = g_k^2 - a_k^2 = 4|\alpha_k|^2 |\beta_k|^2 g_k^2. \quad (18)$$

The behavior of the sums of N random variables x_k (and thus of their characteristic function) depends on whether the so-called Lindeberg condition holds [10]. It is expressed in terms of the cumulative variances $s_N^2 = \sum b_k^2$, and it is satisfied when the probability of the large individual steps is small; e.g.,

$$P(\max_{1 \leq k \leq N} |g_k - a_k| \geq \tau s_N) \xrightarrow{N \rightarrow \infty} 0, \quad (19)$$

for any positive constant τ . In effect, Lindeberg condition demands that s_N be finite: when it is met, the resulting distribution of energies $B = \sum x_k$ is Gaussian

$$\eta\left(\frac{B - \bar{B}_N}{s_N} < x\right) \xrightarrow{N \rightarrow \infty} \int_{-\infty}^x e^{-y^2/2} dy, \quad (20)$$

where $\bar{B}_N = \sum a_k$. This implies

$$\eta(B) \simeq \frac{1}{\sqrt{2\pi s_N^2}} \exp\left(-\frac{(B - \bar{B}_N)^2}{2s_N^2}\right), \quad (21)$$

an expression in excellent agreement with numerical results already for modest values of N . After applying the Fourier transform of Eq. (9), this distribution of energies yields a corresponding approximately Gaussian time dependence of $r(t)$ [Fig. 2],

$$r(t) \simeq e^{i\bar{B}_N t} e^{-s_N^2 t^2/2}, \quad (22)$$

valid only for times $t < 2\pi/\langle g \rangle$, at which a periodic revival occurs. Moreover, at least for short times of interest for, say, quantum error correction, $r(t)$ is approximately Gaussian already for relatively small values of N . This conclusion holds whenever the initial distribution of the couplings has a finite variance. Note that, in particular, we did not have to assume

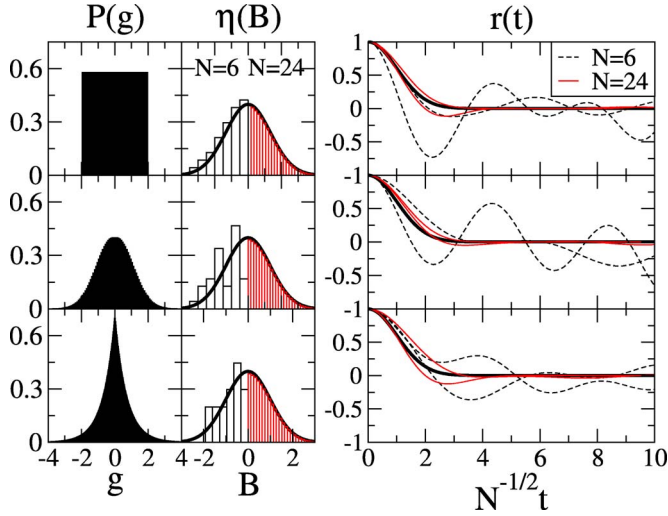


FIG. 2. (Color online) Assumed distribution of the couplings g_k , and resulting distribution of the eigenenergies B_n (left panels) for $N=6$ and $N=24$. In the case of $|\alpha_k|^2=1/2$ this distribution is in effect the “strength function” (local density of states). The corresponding decoherence factors $r(t)$ for different initial conditions with $N=6$ (dashed lines), $N=24$ (thin solid lines) and the average (bold line) is shown on the right. Note the rapid convergence to a Gaussian behavior.

“randomness” of the couplings g_k [see, e.g., Eqs. (13)–(15)].

A random initial state for the environment (not necessarily a product state) instead of Eq. (12) gives basically the same result. In this case typically $c_n \approx 2^{-N/2} e^{i\phi_n}$, with ϕ_n a random phase between 0 and 2π . From Eq. (10),

$$\eta(B) \approx \frac{1}{2^N} \sum_{k=0}^N \binom{N}{k} \delta[B - g(2k - N)], \quad (23)$$

and, as above, the Gaussian limit for large N applies.

It is also interesting to investigate cases when Lindeberg condition is not met. Here, the possible limit distributions are given by the stable (or Lévy) laws [11]. One interesting case is a Lorentzian distribution of couplings, which yields an exponential decay of the decoherence factor (see Fig. 3). Such a distribution could be obtained for instance by considering dipolar interaction between spins randomly placed in a sample. The long-range nature of the interaction gives rise to the Lorentzian distribution and therefore to the exponential decay that can be deduced by statistical arguments [12].

Relation to the Loschmidt echo

The Fourier transform of the strength function $\eta(B)$ is also related to the Loschmidt echo [13] (or fidelity) in the so-called Fermi golden rule regime. The fact that the purity and the fidelity have closely related decay rates has been recently shown [14] for the case of a bath composed of non-interacting harmonic oscillators. In this sense our results could be interpreted as an extension of the discussion of Ref. [14] to spin environments.

The connection with fidelity is more easily seen if we write a generalized version of the Hamiltonian (1),

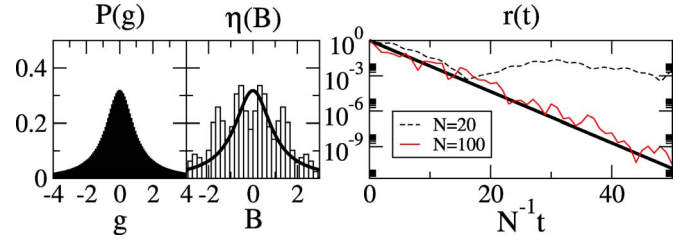


FIG. 3. (Color online) The same as Fig. 2 but for a Lorentzian distribution of the couplings g_k . In this case $r(t)$ decays exponentially. The histogram and the dashed line in $r(t)$ correspond to $N=20$, the straight thin line is a particular case for $N=100$, and the thick line is the average. We note that the convergence is slower than in the Gaussian case of Fig. 2, because realizations of g_k are more likely to have one or two dominant couplings. Therefore, although the average shows a clear exponential decay, fluctuations are noticeable even for large N .

$$\mathcal{H}_{SE} = \frac{1}{2} (|0\rangle\langle 0| \otimes \mathcal{H}_E^0 + |1\rangle\langle 1| \otimes \mathcal{H}_E^1). \quad (24)$$

The decoherence factor is then the overlap of the initial state of the environment $|\Psi_E(0)\rangle$ evolved with two different Hamiltonians,

$$r(t) = \langle \Psi_E(0) | e^{i\mathcal{H}_E^0 t/2} e^{-i\mathcal{H}_E^1 t/2} | \Psi_E(0) \rangle, \quad (25)$$

which clearly has the form of the amplitude of the Loschmidt echo for the environment with the two states of the system as the perturbation. In the model of Eq. (1), $\mathcal{H}_E^0 = -\mathcal{H}_E^1$ and thus

$$r(t) = \langle \Psi_E(0) | e^{-i\mathcal{H}_E^1 t} | \Psi_E(0) \rangle. \quad (26)$$

This expression is the survival probability of the initial state of the environment under the action of the Hamiltonian \mathcal{H}_E^1 , which is known to be the Fourier transform of the strength function [15]. This connection provides another way to understand Eq. (9).

III. DECOHERENCE AND DYNAMICS

In this section we will study the more general Hamiltonian of Eq. (2), that is, we will include a self-Hamiltonian to the central system. The results of the previous section will be contained in the limit of $\Delta=0$, however, we will see that for any finite Δ the behavior of the decoherence factor will be nontrivially different from what we obtained in the previous section.

Despite its more complex appearance, the model given by Eq. (2) is still exactly solvable [16]. Since the states $|n\rangle\langle n|$ of the environment commute with the Hamiltonian, we can write the evolution operator for the combined system environment as

$$U(t) = \prod_{n=0}^{2^N-1} U_{B_n}(t) \otimes |n\rangle\langle n|, \quad (27)$$

with

$$U_{B_n}(t) = I \cos(\Omega_n t) - i \frac{(\sigma_z B_n + \sigma_x \Delta)}{\Omega_n} \sin(\Omega_n t), \quad (28)$$

and $\Omega_n^2 = \Delta^2 + B_n^2$. The physical interpretation of this result is that for every state of the environment $|n\rangle$ the effective dynamics of the system is given by a magnetic field $\vec{\Omega}_n = (\Delta, 0, B_n)$ in the XZ plane. Seen from this perspective, the decoherence is produced by the dispersion of the fields B_n .

The reduced density matrix of the system at an arbitrary time t is

$$\rho(t) = \sum_{n=0}^{2^N-1} |c_n|^2 U_{B_n}(t) \rho(0) U_{B_n}^\dagger(t), \quad (29)$$

or, transforming the notation and using Eq. (10),

$$\rho(t) = \int U_B(t) \rho(0) U_B^\dagger(t) \eta(B) dB. \quad (30)$$

For simplicity, we will work with the polarization vector $\vec{\rho}$, such that $\rho = (I + \vec{\rho} \cdot \vec{\sigma})/2$. Thus

$$\vec{\rho}(t) = \int \vec{\rho}(t, B) \eta(B) dB \quad (31)$$

For an arbitrary time t , we find

$$\begin{aligned} p_x(t, B) &= p_x(0) \frac{\Delta^2 + B^2 \cos(2\Omega_B t)}{\Omega_B^2} - p_y(0) \frac{B}{\Omega_B} \sin(2\Omega_B t) \\ &\quad + p_z(0) \frac{2\Delta B}{\Omega_B^2} \sin^2(\Omega_B t), \end{aligned} \quad (32a)$$

$$p_y(t, B) = p_y(0) \cos(2\Omega_B t) + \frac{\sin(2\Omega_B t)}{\Omega_B} [p_x(0) B - \Delta p_z(0)], \quad (32b)$$

$$\begin{aligned} p_z(t, B) &= p_z(0) \frac{B^2 + \Delta^2 \cos(2\Omega_B t)}{\Omega_B^2} + p_x(0) \frac{2\Delta B}{\Omega_B^2} \sin^2(\Omega_B t) \\ &\quad + p_y(0) \frac{\Delta}{\Omega_B} \sin(2\Omega_B t). \end{aligned} \quad (32c)$$

According to the results of the previous section, in general for large N we can assume a Gaussian shape for $\eta(B)$. By using a Gaussian centered around zero,

$$\eta(B) = \frac{1}{\sqrt{2\pi s_N^2}} \exp(-B^2/2s_N^2), \quad (33)$$

Eqs. (32) simplify because the odd terms in B do not contribute to the final result.

Using these assumptions, we were not able to obtain a solution of the integral in Eq. (31) for arbitrary values of s_N and Δ . However, we can solve the two limiting cases $s_N \gg \Delta$ and $s_N \ll \Delta$, which turn out to give nontrivial results.

Let us consider first the case where $s_N \gg \Delta$, that is, where the central spin dynamics is so slow that its behavior should approach that obtained in the previous section. Indeed, for short times ($t \ll \Delta^{-1}$), using a Taylor expansion of Eqs. (32) around $\Delta=0$ one finds

$$\begin{aligned} p_x(t) &= p_x(0) e^{-2t^2 s_N^2}, \\ p_y(t) &= p_y(0) e^{-2t^2 s_N^2} - p_z(0) \frac{\Delta}{s_N} \sqrt{\frac{\pi}{2}} \text{Erf}(\sqrt{2} s_N t), \\ p_z(t) &= p_z(0) + p_y(0) \frac{\Delta}{s_N} \sqrt{\frac{\pi}{2}} \text{Erf}(\sqrt{2} s_N t), \end{aligned} \quad (34)$$

where $\text{Erf}(x)$ is the error function. To obtain the long-time behavior, we need to perform the integrals on B by stationary phase approximation. In the limit $t \gg \Delta^{-1}$ we find

$$\begin{aligned} p_x(t) &\approx p_x(0) \left[\gamma \left(\frac{\Delta}{\sqrt{2} s_N} \right) + \frac{1}{\sqrt{8\Delta s_N^2 t^3}} \cos \left(2\Delta t + \frac{3\pi}{4} \right) \right], \\ p_y(t) &\approx \sqrt{\frac{\Delta}{2s_N^2 t}} \left[p_y(0) \cos \left(2\Delta t + \frac{\pi}{4} \right) - p_z(0) \sin \left(2\Delta t + \frac{\pi}{4} \right) \right], \\ p_z(t) &\approx p_z(0) \left[1 - \gamma \left(\frac{\Delta}{\sqrt{2} s_N} \right) + \sqrt{\frac{\Delta}{2s_N^2 t}} \cos \left(2\Delta t + \frac{\pi}{4} \right) \right] \\ &\quad + p_y(0) \sqrt{\frac{\Delta}{2s_N^2 t}} \sin \left(2\Delta t + \frac{\pi}{4} \right), \end{aligned} \quad (35)$$

with $\gamma(x) = \sqrt{\pi} x e^{x^2} [1 - \text{Erf}(x)]$. In this limit, $\gamma(\Delta/\sqrt{2} s_N) \ll 1$.

Note that for any $\Delta \neq 0$ the X component of the polarization does not decay to zero, indicating that the decoherence process is not completely effective in this direction [the Y component does go to zero for large times due to the symmetry of Hamiltonian (2)]. Also, note that even a small self-Hamiltonian of the system always ends up turning a fast (Gaussian) decay into a slow (power law) one.

In the opposite limit of strong self-dynamics of the system, $s_N \ll \Delta$, we can obtain an expression valid for times larger than Δ^{-1} by expanding $\Omega_B \approx \Delta + B^2/2\Delta$. After some algebra,

$$\begin{aligned} p_x(t) &= p_x(0) \left[\gamma \left(\frac{\Delta}{\sqrt{2} s_N} \right) + \frac{s_N^2}{\Delta^2} \frac{\cos \left(2\Delta t + \frac{3}{2} \arctan \frac{2s_N^2 t}{\Delta} \right)}{\left(1 + \frac{4s_N^4 t^2}{\Delta^2} \right)^{3/4}} \right], \\ p_y(t) &= \frac{1}{\left(1 + \frac{4s_N^4 t^2}{\Delta^2} \right)^{1/4}} \left[p_y(0) \cos \left(2\Delta t + \frac{1}{2} \arctan \frac{2s_N^2 t}{\Delta} \right) \right. \\ &\quad \left. - p_z(0) \sin \left(2\Delta t + \frac{1}{2} \arctan \frac{2s_N^2 t}{\Delta} \right) \right], \end{aligned}$$

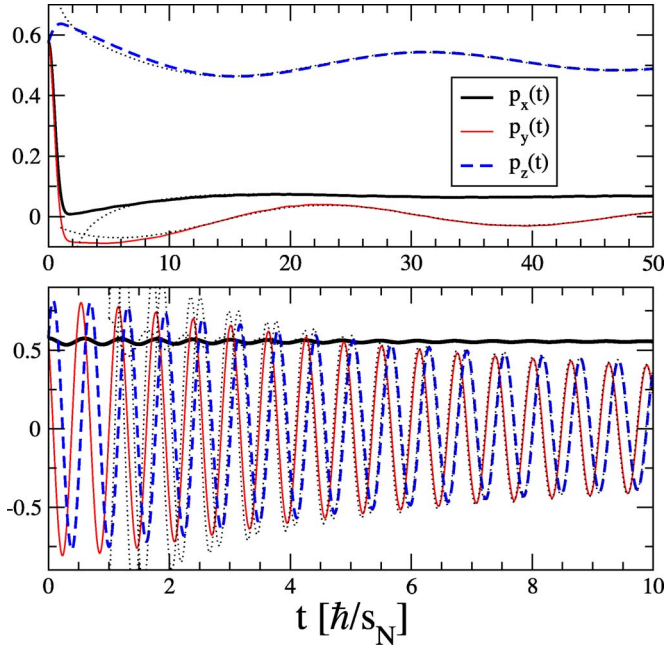


FIG. 4. (Color online) Behavior of the components of the polarization vector for long times for $\Delta/s_N=0.1$ (top panel) and $\Delta/s_N=5$ (bottom panel). Numerical results in solid lines and analytical predictions in dashed lines.

$$p_z(t) = p_z(0) \left[1 - \gamma \left(\frac{\Delta}{\sqrt{2}s_N} \right) + \frac{\cos \left(2\Delta t + \frac{1}{2} \arctan \frac{2s_N^2 t}{\Delta} \right)}{\left(1 + \frac{4s_N^4 t^2}{\Delta^2} \right)^{1/4}} \right] + p_y(0) \frac{\sin \left(2\Delta t + \frac{1}{2} \arctan \frac{2s_N^2 t}{\Delta} \right)}{\left(1 + \frac{4s_N^4 t^2}{\Delta^2} \right)^{1/4}}. \quad (36)$$

In the long-time limit ($t \gg \Delta/s_N^2$) these expressions are equal to Eqs. (35), only that now $\gamma(\Delta/\sqrt{2}s_N) \approx 1$. The results above for large and small Δ agree well with numerical simulations, as shown in Fig. 4.

Pointer basis

The above results allow us to draw some conclusions about the nature of the decoherence process and the pointer states which are dynamically selected by the environment. First, we can notice that for long times the polarization vector converges to a certain value (which, in general, depends on Δ and other parameters of the model). Second, we note that when $\Delta \neq 0$ the X component of the polarization vector does not decay to zero but is resilient to the interaction with the environment. This is the case even if the system interacts with the environment through the Z component of the spin.

The states which are dynamically selected by the environment are dramatically different in the two opposite regimes we examined above. For small values of Δ , the eigenstates of the Z component of the central spin are pointer states. They

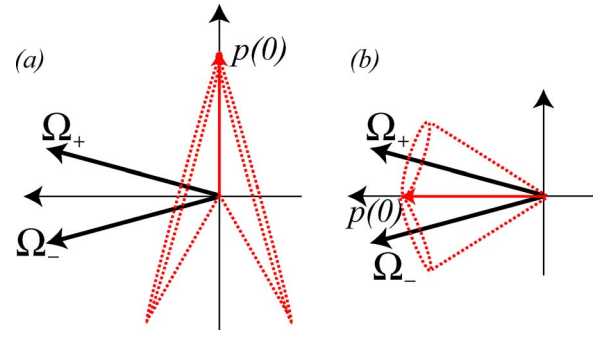


FIG. 5. (Color online) Schematic solution of the Hamiltonian (2) for two initial polarizations in the Z axis (left panel) and in the X axis (right panel). Supposing that B takes only two possible values, $\pm s_N$, the solution for each field B is the precession of the polarization around $\vec{\Omega}_{\pm} = (\Delta, 0, \pm s_N)$. The total polarization is the average of the two cones, which gives a small residue along the Z axis. The polarization in X is almost fully preserved.

are minimally perturbed by the interaction with the environment (in the previous section, where $\Delta=0$ was assumed, this emerged as an exact result since p_z is conserved). However, for large Δ (i.e., $\Delta \gg s_N$), the fact that $p_x(t \rightarrow \infty) \approx 1$ is a signature of the decoherence process selecting a completely different set of pointer states. In fact, in this case, the pointer states turn out to be eigenstates of the system Hamiltonian, which is proportional to σ_x . Thus this model enables us to examine these two very different situations: one where the interaction with the environment dominates ($\Delta \ll s_N$) and σ_z eigenstates are selected; the other where the self-Hamiltonian of the system dominates ($\Delta \gg s_N$) and σ_x eigenstates are selected.

The regime where the Hamiltonian of the system dominates (or, more precisely, where the environment is much slower than the system) was analyzed in a more general context before [6] and has a natural interpretation here: This behavior is simply the one corresponding to the strong decoupling regime observed in nuclear magnetic resonance [17]. There, the presence of a strong magnetic field in the X or Y axes effectively decouples the spectrally resolvable spins of a sample (whose interaction is Z dominant). The standard picture of this decoupling regime is that by rotating the polarization rapidly enough around X , any interaction in another axis is strongly suppressed and the spins effectively “decouple.”

There is an instructive physical picture to understand these results. Instead of using a continuous distribution for B , let us suppose that B can only take two values, $\eta(B) = [\delta(B - s_N) + \delta(B + s_N)]/2$, with $s_N \ll \Delta$. The classical solution for the evolution of the polarization vector is the precession of \vec{p} around $\vec{\Omega}_{\pm} = (\Delta, 0, \pm s_N)$, as shown in Fig. 5 for two possible initial conditions of \vec{p} . The polarization vector of the reduced system is the average of the two cones corresponding to the precession around $\vec{\Omega}_+$ and $\vec{\Omega}_-$. The presence of a small component Δ in the Z axis tilts the precession cones so that their average is almost 1 in the X direction and has a small residue on the Z axis (the Y component cancels due to the symmetry).

IV. CONCLUSIONS

We have studied a very simple model of decoherence due to spin environments. We showed that the decoherence factor will generically have a Gaussian decay when there is no self-Hamiltonian for the system. We note that similar behavior was observed for short times by studying the decoherence process in models where the largest energy scale is the system-environment interaction strength [18]. A model similar to Eq. (1) is used in the NMR setting [12] to compute corrections to the second and fourth moments of the decay of the polarization signal. Here the idea is to treat the interaction with surrounding spins as an effective local magnetic field that shifts the Larmor frequency inhomogeneously across the sample. The statistical treatment used in Ref. [12] contrasts with the exact solution presented in this work, even in the presence of a self-Hamiltonian of the central spin. Thus our model has applicability and relevance to a larger class of physical situations. There is a substantial body of work [16,19–21] on decoherence due to spin environments, stimulated in part by the interests of quantum computation. Our results are most relevant for quantum error correction and other strategies to fight decoherence in a quantum computer. For example, Gaussian time dependence of the decoherence factor would suggest a different (more frequent) error correction than the exponential dependence often assumed with little or no justification.

We also showed how by adding a self-Hamiltonian for the system one can dramatically change the main features of the decoherence process. Even for the case of slow dynamics of the system, we found that for long times the initial Gaussian behavior changes to a power law. On the other end, when the self-Hamiltonian is much stronger than the interaction with the environment, the whole process changes its nature. The decay is predominantly a power law. Moreover, the pointer states now correspond to eigenstates of the system rather than eigenstates of the system-environment interaction [6]. For illustrative purposes, our results are summarized schematically in Fig. 6 using the Bloch sphere representation.

Our results, though interesting, arise from a very simplified model. A logical step for future research is the inclusion of intrabath interactions. The entanglement thus created between spin baths will surely have an impact on the amount of decoherence in the system [22].

Possible experimental applications of our considerations are in nuclear magnetic resonance, and in any other situation

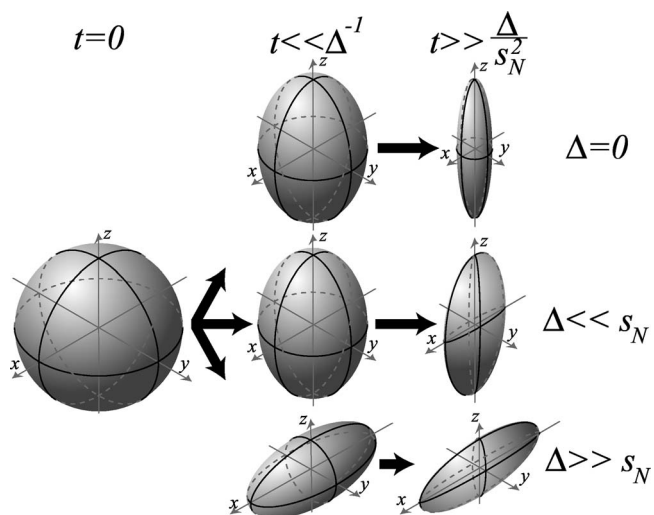


FIG. 6. Bloch sphere representation of our main results. Possible initial states at $t=0$ are represented on the left by a full sphere. Intermediate times (center) are similar for $\Delta=0$ and $\Delta \ll s_N$, where decoherence reduces the Bloch sphere equally fast (Gaussian) in the X and Y axes. For large Δ , Y and Z axes are decohered in a slower algebraic way. For long times, right panel, the $\Delta=0$ case is completely decohered to the Z axis, while the small Δ retains some polarization along the X axis. The large Δ case is almost the opposite, retaining almost all polarization in the X axis and a small residue in the Z axis.

where two-level systems interact with spin environments. Another area of impact of our results is in the characterization of the process that leads to redundancy in the environment of the classical information about the system [23]. The relation between the decoherence factor and the strength function might prove useful in the physical setting of strongly interacting fermions, where it has been shown that the strength function takes a Gaussian shape [24]. It is our hope that the simple analytic model described here will assist in gaining further insights into these fascinating problems.

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