Universal Optimal Cloning of Arbitrary Quantum States: From Qubits to Quantum Registers

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We present the universal cloning transformation of states in arbitrary-dimensional Hilbert spaces. This unitary transformation attains the optimal fidelity of cloning as specified by Werner [Phys. Rev. A **58**, 1827 (1998)]. With this cloning transformation, pure as well as impure states can be optimally copied, and the quality of the copies does not depend on the state being copied. We discuss the properties of quantum clones. In particular, we show that in the limit of high dimension the fidelity of clones does not converge to zero but attains the limit $1/2$. We also show that our cloning transformation is most suitable for cloning of entanglement. [S0031-9007(98)07854-5]

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Suppose Alice has an *unknown* pure state of a quantum system, $|\Phi\rangle_a$, and that she would like to send information about this state to Bob. The two different ways she can proceed are what could be termed the classical method and the quantum method.

(1) Classical method.—Alice can perform an optimal measurement on her system (see Ref. [1] and references therein), and this allows her to estimate the state. The quality of this estimation is characterized by the mean fidelity $\bar{f}$ [1]. Taking into account the fact that Alice has only a single quantum object, the maximum value of the mean fidelity of the estimated state vector in an M-dimensional Hilbert space is [1]

$$\bar{f} = \frac{2}{M+1}\,. \tag{1}$$

After the measurement is performed Alice can communicate her result to Bob who can recreate the estimated state. Note that as soon as Alice performs the measurement the state $|\Phi\rangle_a$ is "lost," so that no further information can be gained.

(2) Quantum method.—Alice does not perform a measurement on her quantum system, but "swaps" the state $|\Phi\rangle_a$ with Bob. An *unknown* pure state of a quantum system can be swapped between two parties by a unitary transformation. To be specific, let us assume that Alice's quantum object is initially prepared in a pure quantum state $|\Phi\rangle_a$ given by

$$|\Phi\rangle_a = \sum_{i=1}^{M} \alpha_i |\Psi_i\rangle_a\,, \tag{2}$$

which lies in an M-dimensional Hilbert space $\mathcal{H}_a$ spanned by M orthonormal basis vectors $|\Psi_i\rangle_a$ ($i = 1,\ldots,M$). The complex amplitudes α_i are normalized to unity, i.e., $\sum|\alpha_i|^2 = 1$. Simultaneously Bob has the same quantum system but it is initially prepared in a specific (i.e., known) state $|0\rangle_b$ which is a vector in the M-dimensional Hilbert space $\mathcal{H}_b$. From the general rules of quantum mechanics it follows that there is a *unitary* transformation $\hat{S}$ acting on $\mathcal{H}_a \otimes \mathcal{H}_b$ which swaps Alice's and Bob's states, i.e.,

$$|\Phi\rangle_a|0\rangle_b \xrightarrow{\hat{S}} |0\rangle_a|\Phi\rangle_b\,. \tag{3}$$

In the case of qubits ($M = 2$) the swapping can be performed with the help of a simple quantum logic network composed of two C-NOT operations with the a (b) qubit being first the control (target) and then the target (control). Alternatively, one can utilize a nonunitary quantum teleportation protocol [2] to realize the swapping.

Comparing the two methods, we note three things. The first is that the quantum method transfers quantum information far better than the classical one. At the end of the quantum transfer, Bob actually has Alice's original state, while at the end of the classical procedure, he has only a pale imitation. The second point is that with the classical method, both Alice and Bob have information about the state. In fact, Alice can send the result of her measurement to as many people as she wishes, and each of them can make a very imperfect copy of her original quantum state. In the swapping scenario, however, only one person has the state $|\Phi\rangle$, Alice at the beginning of the procedure and Bob at the end. Third, it is worth noting that the quantum scenario requires shared entanglement (in the case of the teleportation) or the ability to perform nonlocal operations in the total Hilbert space of Alice and Bob (for the swapping). These requirements might be difficult to realize practically.

At this point, one can ask whether it is possible to find a procedure which combines the desirable aspects of both of these methods. In particular, can one find a unitary transformation (unitary so that no quantum information is lost) which would result in both Alice and Bob having the state $|\Phi\rangle$ *simultaneously?* This unitary transformation, $\hat{U}$, would act in such a way that

$$|\Phi\rangle_a|0\rangle_b \xrightarrow{\hat{U}} |\Phi\rangle_a|\Phi\rangle_b\,, \tag{4}$$

for an arbitrary (unknown) input state $|\Phi\rangle$. Generalizing the proof of the Wootters-Zurek no-cloning theorem [3] it is easy to show that the linearity of quantum mechanics prohibits the existence of such a transformation (4).

This represents a major difference between quantum and classical information: it is possible to make perfect copies of classical information, but quantum information cannot be copied perfectly, i.e., quantum states cannot be perfectly cloned. Nevertheless, if the requirement that the copies be perfect is dropped, then it is possible to make quantum copies. This was first shown in Ref. [4], where a transformation which produces two copies of an arbitrary input qubit state ($M = 2$) was given. This transformation was shown to be optimal, in the sense that it maximizes the average fidelity between the input and output qubits, by Gisin and Massar [5] and by Bruss *et al.* [6]. Gisin and Massar have also been able to find copying transformations which produce k copies from l originals (where $k > l$) [5]. In addition, quantum logic networks for quantum copying machines of qubits have been developed [7], bounds have been placed on how good the copies can be [8,9], and asymmetric cloning has been proposed [10].

So far, all of the copying machines (transformations) which have been proposed copy qubits, which are two-level systems. Suppose instead that we would like to copy an entangled state of two or more qubits. One approach is to use the single-qubit cloners to individually copy each qubit. It is known that, in the case of two qubits, this will preserve some of the quantum correlations between the particles [11], but, as we shall see, it does not make a particularly good copy. The other alternative is to design a copy machine which copies higher-dimensional systems. That is what we shall do here.

We are particularly interested in how the quality of the copies scales with the dimensionality, M, of the system being cloned. What we find is that the fidelity of the copies decreases with M, as expected, but, somewhat surprisingly, does not go to zero as M goes to infinity.

Even though ideal cloning, i.e., the transformation (4), is prohibited by the laws of quantum mechanics for an *arbitrary* state (2), it is still possible to design quantum cloners which operate reasonably well. Here we note that for the swapping of quantum states it is enough to perform a unitary transformation on the Hilbert space $\mathcal{H}_a \otimes \mathcal{H}_b$. However, quantum cloning is best realized when the original and the copy quantum systems interact with an additional quantum system, the quantum cloner, and this is, in fact, what happens in the universal quantum cloning machine (UQCM) [4]. It can be specified by the following conditions.

(i) The state of the original system and its quantum copy at the output of the quantum cloner, described by density operators $\hat{\rho}_a^{(\mathrm{out})}$ and $\hat{\rho}_b^{(\mathrm{out})}$, respectively, are identical, i.e.,

$$\hat{\rho}_a^{(\mathrm{out})} = \hat{\rho}_b^{(\mathrm{out})}. \tag{5}$$

The reduced density operator $\hat{\rho}_a^{(\mathrm{out})}$ ($\hat{\rho}_b^{(\mathrm{out})}$) is obtained via tracing over the copier and the clone b (a) after the cloning is performed.

(ii) If no *a priori* information about the *in* state of the original system is available, then it is reasonable to require that *all* pure states should be copied equally well. One way to implement this assumption is to design a quantum copier so that the distances between density operators of each system at the output $\hat{\rho}_j^{(\mathrm{out})}$ (where $j = a, b$) and the ideal density operator $\hat{\rho}_j^{(\mathrm{id})}$ which describes the *in* state of the original mode are input state independent. Quantitatively this means that if we employ the Bures distance [12,13]

$$d_B(\hat{\rho}_1, \hat{\rho}_2) = \sqrt{2}\,(1 - \mathrm{Tr}\sqrt{\hat{\rho}_1^{1/2}\hat{\rho}_2\hat{\rho}_1^{1/2}})^{1/2}, \tag{6}$$

as a measure of distance between two operators, then the quantum copier should be such that

$$d_B(\hat{\rho}_j^{(\mathrm{out})}; \hat{\rho}_j^{(\mathrm{id})}) = \mathrm{const}, \qquad j = a, b\,. \tag{7}$$

Here we note that other measures of the distance between two density operators (e.g., Hilbert-Schmidt norm) can be used to specify the universal cloning transformation. The final form of the transformation does not depend on the choice of the measure.

(iii) Finally, we would also like to require that the copies are as close as possible to the ideal output state, which is, of course, just the input state. This means that we want our quantum copying transformation to minimize the distance between the output state $\hat{\rho}_j^{(\mathrm{out})}$ of the copied qubit and the ideal state $\hat{\rho}_j^{(\mathrm{id})}$.

In looking for a universal cloning transformation which generates two imperfect copies from the original state, $|\Phi\rangle_a$, we note that the quality of the cloning will not depend on the particular state (in the given Hilbert space) which is going to be copied if and only if the output reduced density matrix is of the form

$$\hat{\rho}_j^{(\mathrm{out})} = s\hat{\rho}_j^{(\mathrm{id})} + \frac{1-s}{M}\,\hat{1}\,, \tag{8}$$

where $\hat{\rho}_j^{(\mathrm{id})} = |\Phi\rangle\langle\Phi|$ is the density operator describing the original state which is going to be cloned. This scaling form of Eq. (8) guarantees that the Bures distance (6) between the input and the output density operators is input state independent.

The quantum cloning machine we consider is itself an M-dimensional quantum system, and we shall let $|X_i\rangle_x$ ($i = 1, \ldots, M$) be an orthonormal basis of the cloning machine Hilbert space. This cloner is initially prepared in a particular state $|X\rangle_x$. The action of the cloning transformation can be specified by a unitary transformation acting on the basis vectors of the tensor product space of the original quantum system $|\Psi_i\rangle_a$, the copier, and an additional M-dimensional system which becomes the copy (which is initially prepared in a specific state $|0\rangle_b$). Let us consider the transformation of the basis vectors

$$|\Psi_i\rangle_a|0\rangle_b|X\rangle_x \longrightarrow c|\Psi_i\rangle_a|\Psi_i\rangle_b|X_i\rangle_x + d\sum_{j\neq i}^{M}(|\Psi_i\rangle_a|\Psi_j\rangle_b + |\Psi_j\rangle_a|\Psi_i\rangle_b)|X_j\rangle_x\,, \tag{9}$$

with real coefficients c and d. From the unitarity of the transformation (9) it follows that c and d satisfy the relation

$$c^2 + 2(M-1)d^2 = 1\,. \tag{10}$$

Using the transformation (9) we find that the particles a and b at the output of the cloner are in the same state (have the same reduced density matrixes), which is described by the density operator

$$\hat{\rho}_a^{(\mathrm{out})} = \hat{\rho}_b^{(\mathrm{out})} = \sum_{i=1}^{M}|\alpha_i|^2[c^2 + (M-2)d^2]|\Psi_i\rangle\langle\Psi_i| + \sum_{\substack{i,j=1\\ i\neq j}}^{M}\alpha_i\alpha_j^*[2cd + (M-2)d^2]|\Psi_i\rangle\langle\Psi_j| + d^2\hat{1}\,. \tag{11}$$

Now our task is to find the values for c and d such that the density operator in Eq. (11) takes the scaled form of Eq. (8). This directly guarantees the universality of the transformation (9).

Comparing Eqs. (8) and (11) we find that c and d must satisfy the equation

$$c^2 = 2cd\,. \tag{12}$$

Taking into account the normalization condition in Eq. (10), we find that

$$c^2 = \frac{2}{(M+1)}\,, \qquad d^2 = \frac{1}{2(M+1)}\,, \tag{13}$$

from which it follows that the scaling factor s is

$$s = c^2 + (M-2)d^2 = \frac{(M+2)}{2(M+1)}\,. \tag{14}$$

If $M = 2$, the transformation (9) then reduces to the cloning transformation for qubits introduced in Ref. [4]. For this case the optimality of the cloning transformation (i.e., that $s = 2/3$ is the maximum possible value of the scaling factor) has been proven by Gisin and Massar [5] (see also later work by Bruss *et al.* [6]). We have numerically tested the optimality of the cloner described by the unitary transformation (9). Werner [14] has recently analyzed general limits on the fidelity of universal cloning. His results independently confirm that the transformation (9) is optimal.

We note that the scaling factor, which describes the quality of the copy, is a decreasing function of M. This is not surprising, because a quantum state in a large-dimensional space contains more quantum information than one in a small-dimensional space (e.g., a state in a 4-dimensional space contains information about two qubits while a state in a 2-dimensional space describes only a single qubit), so that, as M increases, one is trying to copy more and more quantum information. On the other hand, it is interesting to note that in the limit $M \to \infty$, i.e., in the case where the Hilbert space of the given quantum system is infinite dimensional (e.g., a quantum-mechanical harmonic oscillator), the cloning can still be performed efficiently with the scaling factor equal to $1/2$.

In order to confirm that the quality of the copies which the cloning transformation (9) produces is input state independent (i.e., all states are cloned equally well), we evaluate the Bures distance (6) between the density operators describing the output of the cloner and the ideal clone. In our particular case, we find that the distance between $\hat{\rho}_a^{(\mathrm{out})}$ and $\hat{\rho}_a^{(\mathrm{id})}$ depends only on the dimension of the Hilbert space M, but not on the state which is cloned, i.e.,

$$d_B(\hat{\rho}_a^{(\mathrm{out})}, \hat{\rho}_a^{(\mathrm{id})}) = \sqrt{2}\left(1 - \sqrt{\frac{M+3}{2(M+1)}}\right)^{1/2}. \tag{15}$$

The Bures distance in Eq. (15) is maximal when states in an infinite-dimensional Hilbert space are cloned, and in that case we find $\lim_{M\to\infty} d_B(\hat{\rho}_{a_k}^{(\mathrm{out})}, \hat{\rho}_{a_k}^{(\mathrm{id})}) = \sqrt{2-\sqrt{2}}$. This means that, even for an infinite-dimensional system, reasonable cloning can be performed, which is reflected in the fact that the corresponding scaling factor s is equal to $1/2$.

Using the transformation in Eq. (9), we can also find the state of the copy machine after the cloning has been performed,

$$\hat{\rho}_x^{(\mathrm{out})} = 2d^2(\hat{\rho}_x^{(\mathrm{id})})^T + 2d^2\hat{1}\,, \tag{16}$$

i.e., the cloner is left in a state proportional to the transposed state of the original quantum system. The von Neumann entropy of the copier at the output reflects the degree of entanglement between the copies and the copier. As expected, this entropy does not depend on the state to be copied and is just a function of the dimension of the Hilbert space, i.e., $S = \ln(M+1) - (2\ln 2)/(M+1)$. This is again an increasing function of M which reflects the fact that the copies and the copier become increasingly correlated as M increases.

We also note that the linearity of quantum mechanics implies not only that ideal cloning of the form given in Eq. (4) *does not* exist but also that there is no universal cloning transformation which would result in a separable output of the form

$$\hat{\rho}_{ab}^{(\mathrm{out})} = s\hat{\rho}_a^{(\mathrm{out})} \otimes \hat{\rho}_b^{(\mathrm{out})} + \frac{1-s}{M^2}\hat{1}_a \otimes \hat{1}_b\,, \tag{17}$$

such that the reduced density operators $\hat{\rho}_a^{(\mathrm{out})}$ and $\hat{\rho}_b^{(\mathrm{out})}$ have the scaled form (8). In other words, the cloning transformation which satisfies the conditions (i)–(iii) produces two clones which are entangled. From this one may adopt the following interpretation of cloning: After the cloning, the information is distributed in such a way that some of it is in the copies, some is in the entanglement between the copies, some is in the copy machine, and some is in the entanglement between the copies and the copy machine. The information in the entanglement and in the copy machine is effectively lost if we just look at the copies, and this is why the copies are not perfect.

Until now we have considered only the cloning of pure states. Nevertheless, the cloning transformation (9) can be applied successfully for the universal cloning of arbitrary *impure* states. To be specific, let us assume the most general density operator $\hat{\rho}_a^{(\mathrm{in})} = \sum_{i,j} A_{ij} |\Psi_i\rangle\langle\Psi_j|$. It can be directly shown that, with the cloning transformation described above, one obtains the two clones at the output with the reduced density operators given in the scaled form (8). The scaling factor is the same as for pure states. This proves once again the universality of the cloning transformation—arbitrary unknown states (pure or impure) are universally cloned with the same fidelity which does not depend on the input state.

Finally, we compare two methods of cloning quantum registers. In particular, we shall consider cloning an entangled state of two qubits. We assume that the two qubits are prepared in the state

$$|\Phi\rangle_{a_0 b_0} = \alpha|00\rangle_{a_0 b_0} + \beta|11\rangle_{a_0 b_0}, \tag{18}$$

where, for simplicity, we have taken α and β to be real, and $\alpha^2 + \beta^2 = 1$. First, we shall consider the case in which each of the two qubits a_0 and b_0 is copied *locally* by two independent quantum copiers [4]. Each of these two copiers is described by the transformation (9) with $M = 2$. Second, we shall consider a *nonlocal* cloning of the two-qubit state (18) when this system is cloned via the unitary transformation (9) with $M = 4$, i.e., the cloner in this case acts nonlocally on the two qubits. Our chief task will be to analyze how inseparability is cloned in these two scenarios, but we shall also examine the quality of the copies which are produced. From the Peres-Horodecki theorem [15–17], it follows that the state (18) is inseparable for all values of α^2 such that $0 < \alpha^2 < 1$.

Comparing the results of the local and nonlocal cloning we find that in the first case the two-qubit clones are inseparable if

$$\frac{1}{2} - \frac{\sqrt{39}}{16} \le \alpha^2 \le \frac{1}{2} + \frac{\sqrt{39}}{16}, \tag{19}$$

while in the case of *nonlocal* cloning the two-qubit clones are inseparable for a much wider range of the parameter α, i.e.,

$$\frac{1}{2} - \frac{\sqrt{2}}{3} \le \alpha^2 \le \frac{1}{2} + \frac{\sqrt{2}}{3}. \tag{20}$$

We conclude that quantum inseparability can be copied better (i.e., for a much larger range of the parameter α) by using a nonlocal copier than when two local copiers are used. Obviously, there is a price to be paid—nonlocal cloners are likely to be more difficult to implement in practice than local cloners.

The quality of the copies which are produced by the two different methods are also different. Local copying has the disadvantage that the quality of the copies it produces depends on the state being copied, so that, in general, the copies are not in the scaled form which appears in Eq. (8). However, in the special case $\alpha = \beta = 1/\sqrt{2}$ they are, and the scaling factor is $4/9$. This can be compared to the input-state-independent value of $s = 3/5$ for the nonlocal copier. Thus, we see that the nonlocal copier produces better copies.

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