

# Activating hidden metrological usefulness

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# Photos



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## 1 Motivation

- What are entangled states useful for?

## 2 Metrological gain and the optimal local Hamiltonian

- Metrological usefulness of a quantum state.
- Activation of metrological usefulness
- Optimal local Hamiltonian
- Bipartite pure entangled states

# What are entangled states useful for?

- Entanglement is needed for beating the shot-noise limit in quantum metrology.
- However, not all entangled states are more useful than separable states.
- Intriguing questions:
  - Can a quantum state become useful metrologically, if an ancilla or a second copy is added?
  - How to find the local Hamiltonian, for which a quantum state is the most useful compared to separable states?

## 1 Motivation

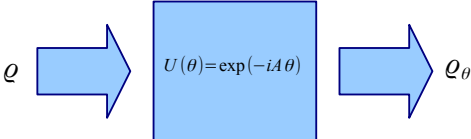
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# The quantum Fisher information

- Cramér-Rao bound on the precision of parameter estimation

$$(\Delta\theta)^2 \geq \frac{1}{mF_Q[\varrho, A]},$$


The diagram illustrates a quantum process. On the left, a quantum state  $\varrho$  is represented by a blue arrow pointing to the right. This arrow enters a light blue square box. Inside the box, the text  $U(\theta) = \exp(-iA\theta)$  is written. A second blue arrow points from the right side of the box to the final quantum state  $\varrho_\theta$ .

where where  $m$  is the number of independent repetitions and  $F_Q[\varrho, A]$  is the **quantum Fisher information**.

- The quantum Fisher information is

$$F_Q[\varrho, A] = 2 \sum_{k,l} \frac{(\lambda_k - \lambda_l)^2}{\lambda_k + \lambda_l} |\langle k|A|l \rangle|^2,$$

where  $\varrho = \sum_k \lambda_k |k\rangle \langle k|$ .

# The quantum Fisher information vs. entanglement

- For separable states of  $N$  qubits

$$F_Q[\varrho, J_I] \leq N, \quad I = x, y, z.$$

[Pezze, Smerzi, Phys. Rev. Lett. 102, 100401 (2009); Hyllus, Gühne, Smerzi, Phys. Rev. A 82, 012337 (2010)]

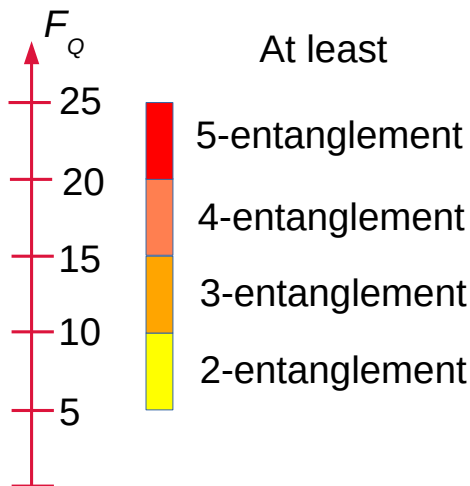
- For states with at most  $k$ -particle entanglement ( $k$  is divisor of  $N$ )

$$F_Q[\varrho, J_I] \leq kN.$$

[P. Hyllus *et al.*, Phys. Rev. A 85, 022321 (2012); GT, Phys. Rev. A 85, 022322 (2012)].

# The quantum Fisher information vs. entanglement

5 spin-1/2 particles





# Metrological usefulness

- Metrological gain for a given Hamiltonian

$$g_{\mathcal{H}}(\varrho) = \frac{\mathcal{F}_Q[\varrho, \mathcal{H}]}{\mathcal{F}_Q^{(\text{sep})}(\mathcal{H})},$$

where  $\mathcal{F}_Q^{(\text{sep})}(\mathcal{H})$  is the maximum of the QFI for separable states.

- Metrological gain optimized over all local Hamiltonians

$$g(\varrho) = \max_{\text{local } \mathcal{H}} g_{\mathcal{H}}(\varrho) = \max_{\text{local } \mathcal{H}} \frac{\mathcal{F}_Q[\varrho, \mathcal{H}]}{\mathcal{F}_Q^{(\text{sep})}(\mathcal{H})}.$$

- A state  $\varrho$  is useful if  $g(\varrho) > 1$ .
- The metrological gain is convex in the state.  
[G. Toth, T. Vertesi, P. Horodecki, R. Horodecki, PRL 2020.]
- We would like to determine  $g$ .

# Maximally entangled state

- Difficult to obtain  $g(\varrho)$  and the optimal local Hamiltonian for any  $\varrho$ .
- As a first step, we consider the  $d \times d$  maximally entangled state

$$|\Psi^{(\text{me})}\rangle = \frac{1}{\sqrt{d}} \sum_{k=1}^d |k\rangle|k\rangle.$$

- The optimal Hamiltonian is

$$\mathcal{H}^{(\text{me})} = D \otimes \mathbb{1} + \mathbb{1} \otimes D,$$

where

$$D = \text{diag}(+1, -1, +1, -1, \dots).$$

# Maximally entangled state II

The  $3 \times 3$  noisy quantum state

$$\varrho_{AB}^{(p)} = (1 - p)|\psi^{(\text{me})}\rangle\langle\psi^{(\text{me})}| + p\mathbb{1}/d^2,$$

is useful if

$$p < \frac{25 - \sqrt{177}}{32} \approx 0.3655,$$

while for larger  $p$ 's it is not useful.

- Note that it is entangled if

$$p < \frac{2}{3}.$$

## 1 Motivation

- What are entangled states useful for?

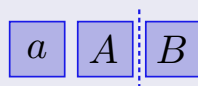
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# Activation by an ancilla qubit

If a pure ancilla qubit is added

$$\varrho^{(\text{anc})} = |0\rangle\langle 0|_a \otimes \varrho_{AB}^{(p)}.$$



then the state is useful if

$$p < 0.3752.$$

(For a single copy, the limit was  $p < 0.3655$ .)

- The Hamiltonian is

$$\mathcal{H}^{(\text{anc})} = 1.2 C_{aA} \otimes \mathbb{1}_B + \mathbb{1}_{aA} \otimes D_B,$$

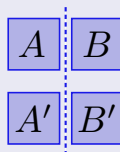
where

$$C_{aA} = \frac{9}{20} (2\sigma_x + \sigma_z)_a \otimes |0\rangle\langle 0|_A + \mathbb{1}_a \otimes (|2\rangle\langle 2|_A - |1\rangle\langle 1|_A).$$

# Activation by a second copy

If a second copy is added

$$\varrho^{(\text{tc})} = \varrho_{AB}^{(p)} \otimes \varrho_{A'B'}^{(p)}.$$



then the state is useful if

$$p < 0.4164.$$

(For a single copy, the limit was  $p < 0.3655$ .)

- The Hamiltonian is

$$\mathcal{H}^{(\text{tc})} = D_A \otimes D_{A'} \otimes \mathbb{1}_{BB'} + \mathbb{1}_{AA'} \otimes D_B \otimes D_{B'}.$$

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# Method for maximizing $g$

- Metrological gain optimized over all local Hamiltonians

$$g(\varrho) = \max_{\text{local } \mathcal{H}} \frac{\mathcal{F}_Q[\varrho, \mathcal{H}]}{\mathcal{F}_Q^{(\text{sep})}(\mathcal{H})} \quad \begin{array}{l} \leftarrow \text{metrological performance of } \varrho \\ \leftarrow \text{best metrological performance of} \\ \text{separable states} \end{array}$$

- It is a fundamental quantity metrology!
- Difficult to compute, since  $\mathcal{H}$  is in both the numerator and the denominator!
- We reduce the problem to maximize  $\mathcal{F}_Q$  over a set of local Hamiltonians.



# Method for finding the optimal local Hamiltonian I

- Direct maximization of  $\mathcal{F}_Q[\varrho, \mathcal{H}]$  over  $\mathcal{H}$  is difficult: it is convex in  $\mathcal{H}$ .
- Let us consider the error propagation formula

$$(\Delta\theta)^2_M = \frac{(\Delta M)^2}{\langle i[M, \mathcal{H}] \rangle^2},$$

which provides a bound on the quantum Fisher information

$$\mathcal{F}_Q[\varrho, \mathcal{H}] \geq 1/(\Delta\theta)^2_M.$$

[M. Hotta and M. Ozawa, Phys. Rev. A 2004; B. M. Escher, arXiv:1212.2533; F. Fröwis, R. Schmied, and N. Gisin, Phys. Rev. A 2015. For a summary, see, e.g., the Supplemental Material of Tóth, Vértesi, Horodecki, Horodecki, PRL 2020.]

# Method for finding the optimal Hamiltonian II

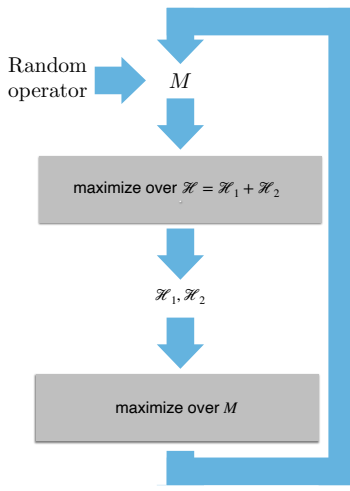
The maximum over local Hamiltonians can be obtained as

$$\max_{local \mathcal{H}} \mathcal{F}_Q[\varrho, \mathcal{H}] = \max_{local \mathcal{H}} \max_M \frac{\langle i[M, \mathcal{H}] \rangle_{\varrho}^2}{(\Delta M)^2}.$$

Similar idea for optimizing over the state, rather than over  $\mathcal{H}$ :

[K. Macieszczak, arXiv:1312.1356; K. Macieszczak, M. Fraas, and R. Demkowicz-Dobrzański, New J. Phys. 16, 113002 (2014); Tóth and Vértesi, Phys. Rev. Lett. (2018).]

# See-saw algorithm



The precision  
cannot get worse  
with the iteration!

Note that  $\mathcal{H}_1, \mathcal{H}_2$  fulfill

$$c_n \mathbb{1} \pm \mathcal{H}_n \geq 0.$$

# Numerical results

- We remember that the  $3 \times 3$  isotropic state is useful if

$$p < \frac{25 - \sqrt{177}}{32} \approx 0.3655.$$

- Then, we have the following results for activation.

	Analytic example	Numerics
Ancilla	0.3752	0.3941
Second copy	0.4164	0.4170

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# Single copy of pure states

All entangled bipartite pure states are metrologically useful.

- *Proof.*—For the two-qubit case, see  
P. Hyllus, O. Gühne, and A. Smerzi, *Phys. Rev. A* 82, 012337 (2010).
- General case, pure state with a Schmidt decomposition

$$|\psi\rangle = \sum_{k=1}^s \sigma_k |k\rangle_a |k\rangle_b,$$

where  $s$  is the Schmidt number, and the real positive  $\sigma_k$  Schmidt coefficients are in a descending order.

- We define

$$\mathcal{H}_A = \sum_{n=1,3,5,\dots,\tilde{s}-1} |+\rangle\langle+|_{A,n,n+1} - |-\rangle\langle-|_{A,n,n+1},$$

where  $\tilde{s}$  is the largest even number for which  $\tilde{s} \leq s$ , and

$$|\pm\rangle_{A,n,n+1} = (|n\rangle_a \pm |n+1\rangle_a) / \sqrt{2}.$$

# Single copy of pure states II

- We define  $\mathcal{H}_B$  in a similar manner.
- We also define the collective Hamiltonian

$$\mathcal{H}_{AB} = \mathcal{H}_A \otimes \mathbb{1} + \mathbb{1} \otimes \mathcal{H}_B.$$

Then, we have  $\langle \mathcal{H}_{AB} \rangle_\psi = 0$ .

- Direct calculation yields

$$\mathcal{F}_Q[|\Psi\rangle, \mathcal{H}_{AB}] = 4(\Delta \mathcal{H}_{AB})^2_\psi = 8 \sum_{n=1,3,5,\dots,\tilde{s}-1} (\sigma_n + \sigma_{n+1})^2,$$

which is larger than the separable bound,  $\mathcal{F}_Q^{(\text{sep})} = 8$ , whenever the Schmidt rank is larger than 1. □

# Infinite number of copies

In the infinite copy limit, all bipartite pure entangled states are maximally useful.

[ Supplemental Material of Tóth, Vértesi, Horodecki, Horodecki, PRL 2020. ]



# Summary

- Some entangled quantum states that are not useful metrologically, can still be made useful, if an ancilla or an additional copy is added.
- We have shown a general method to get the optimal local Hamiltonian for a quantum state.
- All bipartite pure entangled states are useful metrologically. If infinite number of copies are given, they are maximally useful.

See:

Géza Tóth, Tamás Vértesi, Paweł Horodecki, Ryszard Horodecki,

Activating hidden metrological usefulness,

[Phys. Rev. Lett. 125, 020402 \(2020\). \(open access\)](#)

THANK YOU FOR YOUR ATTENTION!

