

Julia and Quasi-Julia Sets in Iterated Nonlinear Quantum Protocols

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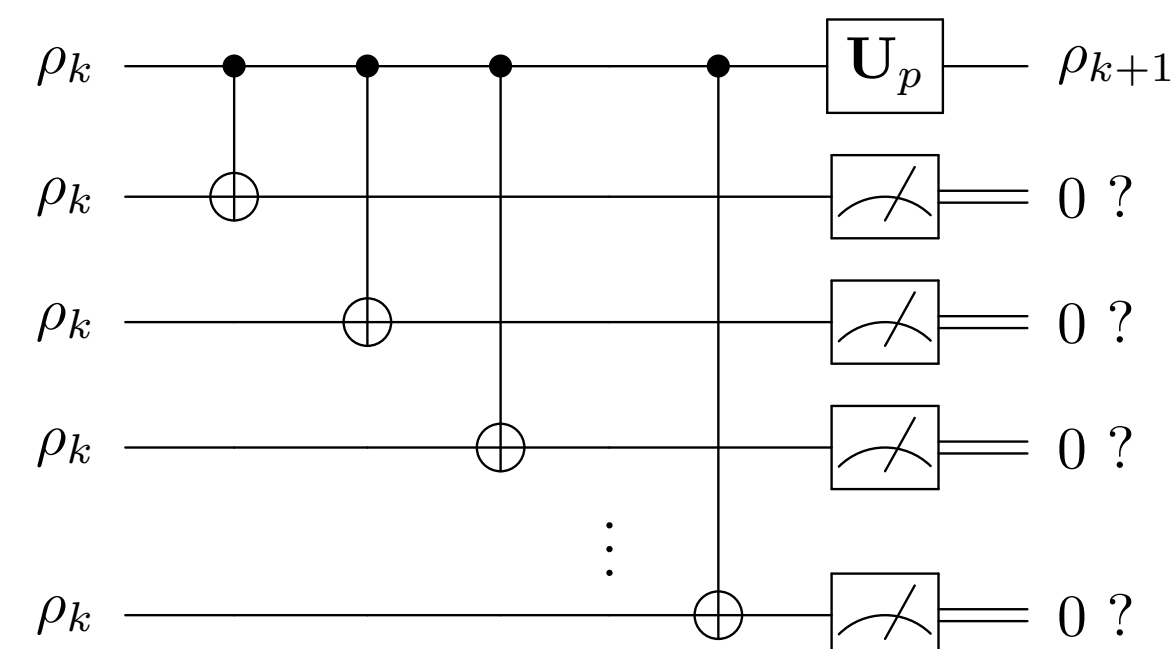


1. Abstract

Fractal structures often emerge in nonlinear dynamics. We study iterated, post-selection based quantum protocols built from CNOT networks and a single-qubit gate U_p , which induce nonlinear evolution. For pure states, the protocol yields degree- n rational maps on $\hat{\mathbb{C}}$ generating Julia sets; at lower purities, quasi-Julia sets appear as three-dimensional fractal boundaries in the Bloch sphere. Using box-counting and correlation dimensions, we quantify how fractal complexity depends on the parameter p and the order n , revealing discrete symmetries. We observe and explain a purity-driven phase transition at a critical purity P_c , linked to a mixed repelling fixed point on the inner boundary of the quasi-Julia set.

2. Nonlinear protocol

Quantum circuit realizing an n th-order nonlinear quantum transformation of the qubit state



n identical and independent qubits, all in the initial state ρ , are acted upon by $n-1$ two-qubit CNOT gates. Then, the $n-1$ target qubits are measured in the computational basis, and if all of the measurements result in “0”, a single-qubit gate is applied to the unmeasured qubit, whose state is transformed into ρ' .

4. The invariant set of pure states

Pure quantum states, i.e. Bloch vectors satisfying $u^2 + v^2 + w^2 = 1$, form an invariant set of the nonlinear dynamics, since purity is preserved under every iteration:

$$u_k^2 + v_k^2 + w_k^2 = 1 \implies u_{k+1}^2 + v_{k+1}^2 + w_{k+1}^2 = 1.$$

Each pure state can be parameterized by a single complex variable $z \in \hat{\mathbb{C}}$ as

$$|\psi\rangle = \frac{1}{\sqrt{1+|z|^2}}(|0\rangle + z|1\rangle).$$

In this representation, one step of the nonlinear protocol corresponds to the rational map

$$f_p(z) = \frac{z^n - \bar{p}}{1 + pz^n}, \quad p \in \mathbb{C}.$$

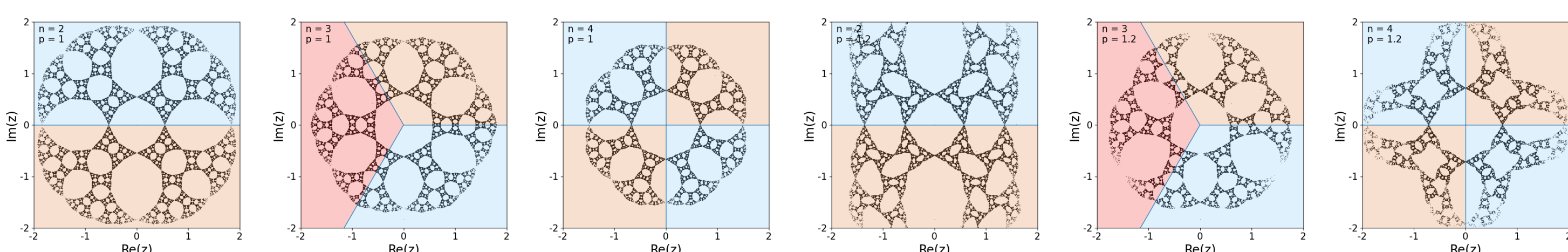
Thus, the pure-state dynamics reduce to the iteration

$$z_{k+1} = f_p(z_k),$$

defining a complex dynamical system on the Riemann sphere.

5. Julia set

The boundary between different convergence regions (including different parity regions) of the nonlinear map on the surface of the Bloch sphere—the Julia set—is a fractal.

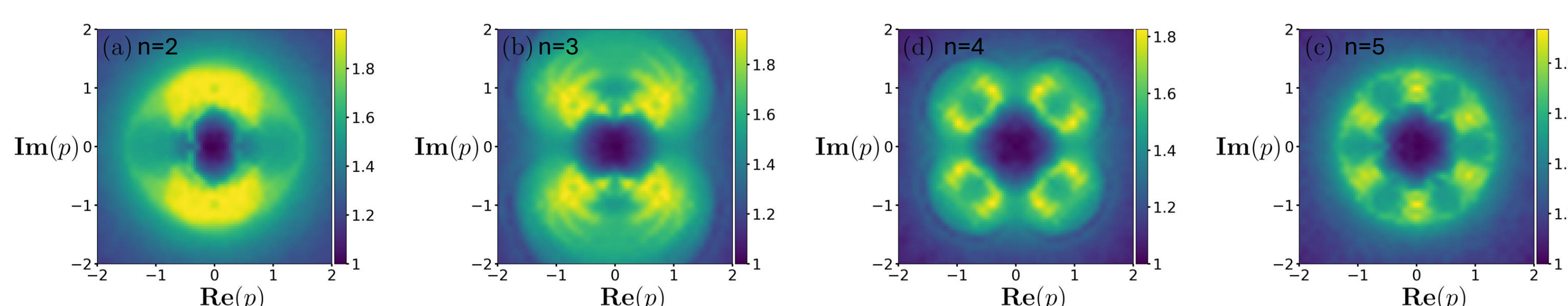


The Julia sets of the rational map have n -fold symmetry in the z -plane because of the z^n terms. In addition, the map is invariant (up to a change of coordinates) under the joint rotation

$$p \mapsto p e^{i\pi k/(n-1)}, \quad z \mapsto z e^{i\pi k/n}, \quad k \in \mathbb{Z},$$

which produces a discrete $2(n-1)$ -fold rotational symmetry in parameter space.

We estimate complexity via box-counting. For $p \approx 0$ the boundary degenerates toward a smooth curve with $D_B \approx 1$; as $|p|$ grows, the boundary becomes increasingly intricate and D_B approaches values characteristic of space-filling planar fractals (typically $1 < D_B < 2$). Across $n = 2, 3, 4, 5$, the peak-complexity regions align with the $2(n-1)$ symmetry sectors, while large $|p|$ tends to simplify the boundary again.



3. Time evolution

The nonlinear dynamics is expressed as an iterated function system (IFS):

$$\rho_{k+1} = U_p \mathbf{T}(\rho_k) U_p^\dagger.$$

The transformation \mathbf{T} encodes the effect of the n -qubit entangling operation, measurement, and post-selection:

$$\mathbf{T}(\rho_k) = \frac{\odot^n \rho_k}{\text{Tr}(\odot^n \rho_k)} = \frac{1}{\rho_{k,11}^n + \rho_{k,22}^n} \begin{pmatrix} \rho_{k,11}^n & \rho_{k,12}^n \\ \rho_{k,21}^n & \rho_{k,22}^n \end{pmatrix}.$$

The remaining qubit is then transformed by

$$U(p) = \frac{1}{\sqrt{1+|p|^2}} \begin{pmatrix} 1 & p \\ -\bar{p} & 1 \end{pmatrix}, \quad p \in \mathbb{C}.$$

Iterating this rule defines a nonlinear dynamical system on the Bloch sphere. The time evolution can be expressed as a system of difference equations for the Bloch coordinates:

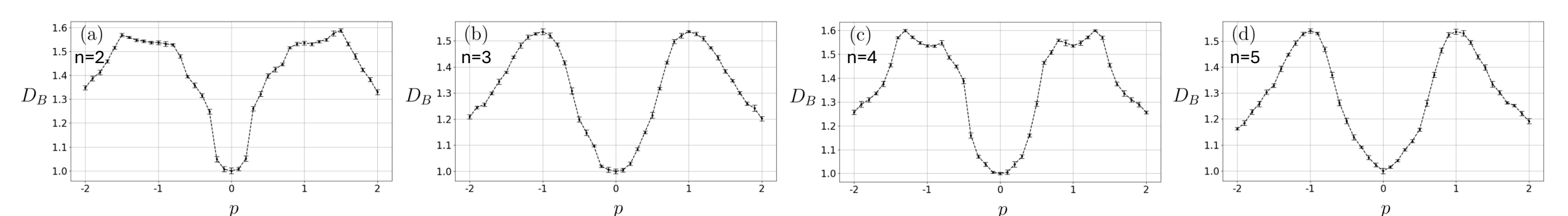
$$\begin{aligned} u_{k+1} &= U(u_k, v_k, w_k) = \rho_{k+1,12} + \rho_{k+1,21} \\ v_{k+1} &= V(u_k, v_k, w_k) = i(\rho_{k+1,12} - \rho_{k+1,21}) \\ w_{k+1} &= W(u_k, v_k, w_k) = \rho_{k+1,11} - \rho_{k+1,22} \end{aligned}$$

This framework enables analysis of fixed points, convergence properties, and stability.

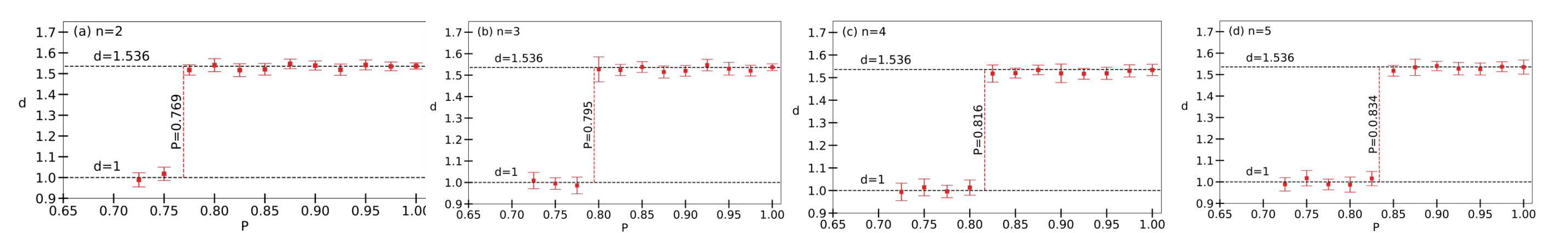
6. Quasi-Julia set

For mixed initial states, the Julia set generalizes into a quasi-Julia set, a three-dimensional fractal boundary embedded in the Bloch sphere. It separates basins of attraction leading either to purification or to mixed states, extending the self-similar structure of the pure-state Julia set into the mixed-state regime.

To characterize it, we study its sections on constant-purity surfaces, where the fractal dimension can be measured.



The dimension of these sections reveals a phase-transition-like phenomenon: above a critical purity P_c , the cross-sections retain fractal complexity with non-integer dimension, while below P_c , the boundary collapses to smooth curves with dimension ≈ 1 . The critical purity P_c thus marks the threshold where the quasi-Julia set disappears from constant-purity sections, explaining the abrupt loss of fractal structure. We find that P_c coincides with the purity of a mixed, repelling fixed point that sits on the inner boundary of the quasi-Julia set.

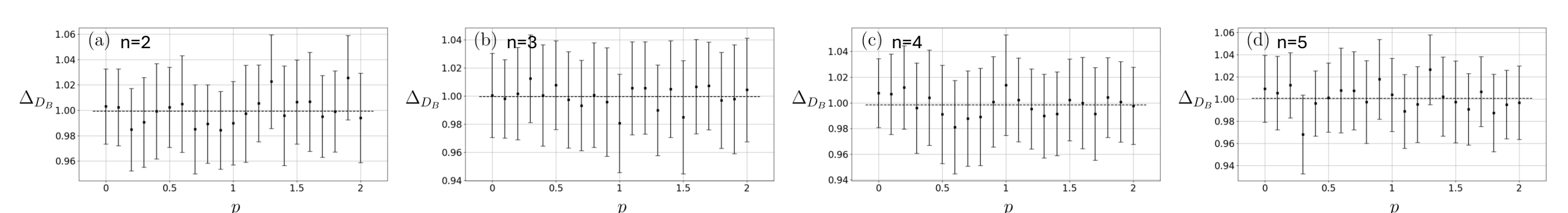


7. Fractal dimension of the quasi-Julia set

We directly measured the box-counting dimension of the quasi-Julia set embedded in the three-dimensional Bloch sphere ($D_B^{(3D)}$). Its difference to the dimension of the constant-purity cross-sections, including the pure-state Julia set at $P = 1$ ($D_B^{(2D)}$), follows the generic intersection rule

$$D_B^{(2D)} = D_B^{(3D)} + 2 - 3 = D_B^{(3D)} - 1,$$

so that $\Delta D_B \equiv D_B^{(3D)} - D_B^{(2D)} \approx 1$ for $P > P_c$, whereas for $P < P_c$ the section no longer intersects the quasi-Julia set.



10. References

- [1] A. Portik, O. Kálmán, I. Jex, and T. Kiss, Iterated n th-order nonlinear quantum dynamics with mixed initial states, *Physics Letters A* (2022).
- [2] M. Malachov, I. Jex, O. Kálmán, and T. Kiss, Phase transition in iterated quantum protocols for noisy inputs, *Chaos* (2019).

11. Acknowledgment

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