Entanglement theory

Géza Tóth

[1]Theoretical Physics, University of the Basque Country UPV/EHU, Bilbao, Spain
[2]EHU Quantum Center, University of the Basque Country UPV/EHU
[3]IKERBASQUE, Basque Foundation for Science, Bilbao, Spain
[4]Donostia International Physics Center DIPC, San Sebastian, Spain
[5]Wigner Research Centre for Physics, Budapest, Hungary

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Outline

Entanglement - Pure states

- Q: What is entanglement for pure states?
- A: bipartite state can be a product state $|\Psi_A\rangle \otimes |\Psi_B\rangle$, or an entangled state.
- For instance, $|00\rangle$ and $|11\rangle$ are product states.
- $(|00\rangle + |11\rangle)/\sqrt{2}$ is an entangled state.
- We can always decide whether a pure state is entangled.

Entanglement - Mixed states

Definition

A quantum state is called separable if it can be written as a convex sum of product states as [Werner, 1989]

$$\varrho = \sum_k p_k \varrho_1^{(k)} \otimes \varrho_2^{(k)},$$

where p_k form a probability distribution ($p_k > 0, \sum_k p_k = 1$), and $\varrho_n^{(k)}$ are single-qudit density matrices.

A state that is not separable is called entangled.

- We cannot always decide whether the state is entangled.

Multipartite entanglement - $N > 2$ parties

Definition

A state is (fully) separable if it can be written as
$\sum_k p_k \varrho_1^{(k)} \otimes \varrho_2^{(k)} \otimes ... \otimes \varrho_N^{(k)}$.

Definition

A pure multi-qubit quantum state is called biseparable if it can be written as the tensor product of two multi-qubit states

$$|\Psi\rangle = |\Psi_1\rangle \otimes |\Psi_2\rangle.$$

Here $|\Psi\rangle$ is an N-qubit state.

A mixed state is called biseparable, if it can be obtained by mixing pure biseparable states.

Definition

If a state is not biseparable then it is called genuine multi-partite entangled.

k-particle entanglement

- Similarly, one can define N-qubit states with k-particle entanglement.
- N-particle entanglement $\equiv$ genuine multipartite entanglement.

k-producibility/k-entanglement

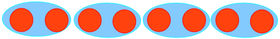
A pure state is *k-producible* if it can be written as

$$|\Phi\rangle = |\Phi_1\rangle \otimes |\Phi_2\rangle \otimes |\Phi_3\rangle \otimes |\Phi_4\rangle....$$

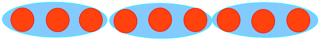
where $|\Phi_l\rangle$ are states of at most k qubits.

A mixed state is k-producible, if it is a mixture of k-producible pure states.
[e.g., Gühne, GT, NJP 2005.]

- If a state is not k-producible, then it is at least $(k+1)$-particle entangled.

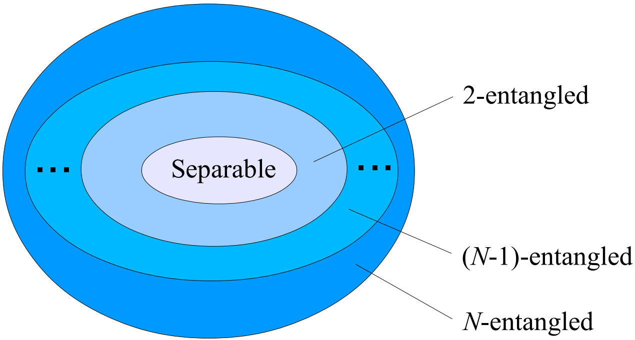


2-entangled



3-entangled

k-producibility/k-entanglement II



$(\lvert 00\rangle + \lvert 11\rangle) \otimes (\lvert 00\rangle + \lvert 11\rangle) \otimes (\lvert 00\rangle + \lvert 11\rangle)$	2-entangled
$(\lvert 000\rangle + \lvert 111\rangle) \otimes (\lvert 000\rangle + \lvert 111\rangle)$	3-entangled
$(\lvert 0000\rangle + \lvert 1111\rangle) \otimes (\lvert 0\rangle + \lvert 1\rangle)$	4-entangled

Examples

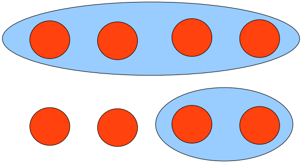
Examples

Two entangled states of four qubits:

$$|\mathrm{GHZ}_4\rangle = \frac{1}{\sqrt{2}}(|0000\rangle + |1111\rangle),$$

$$|\Psi_B\rangle = \frac{1}{\sqrt{2}}(|0000\rangle + |0011\rangle) = \frac{1}{\sqrt{2}}|00\rangle \otimes (|00\rangle + |11\rangle).$$

- The first state is genuine multipartite entangled, the second state is biseparable.



Outline

Separable states form a convex set

- If $\varrho_{\rm sep1}$ and $\varrho_{\rm sep2}$ are separable states then

$$p\varrho_{\rm sep1} + (1-p)\varrho_{\rm sep2}$$

  is also a separable state for $0 \le p \le 1$.

- Here,

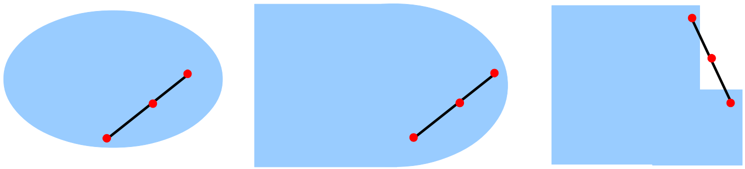
$$\varrho_{{\rm sep}m} = \sum_k p_k \varrho_{m,1}^{(k)} \otimes \varrho_{m,2}^{(k)}$$

  for $m = 1, 2$.

- Convexity of a reasonable requirement for a set of quantum states, otherwise strange phenomena can occur.

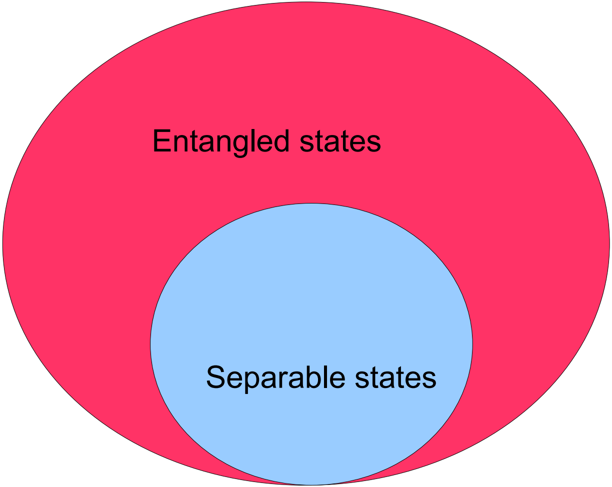
Separable states form a convex set II

- Convex sets and a non-convex set



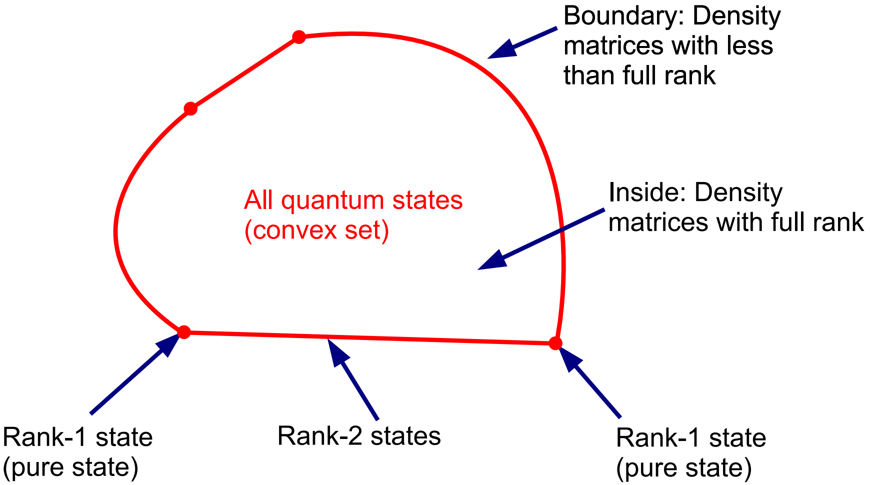
Separable states form a convex set III

- Set of separable states within the set of quantum states



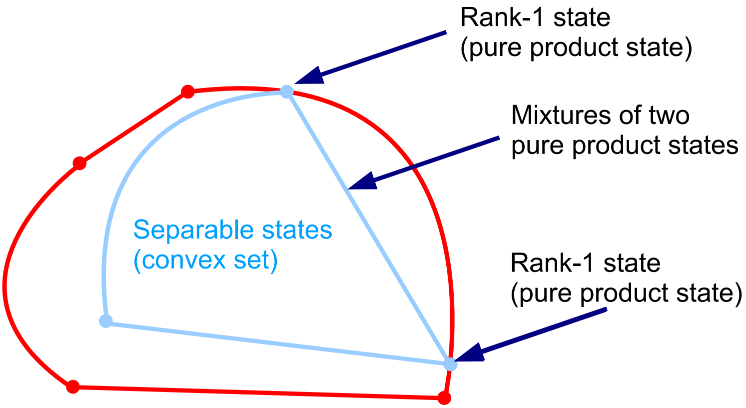
Separable states form a convex set III

- A more accurate picture



Separable states form a convex set IV

- Together with the set of separable states:



Outline

Entanglement witnesses

Definition

An entanglement witness $\mathcal{W}$ is an operator that is positive on all separable (biseparable) states.

Thus, $\mathrm{Tr}(\mathcal{W}\varrho) < 0$ signals entanglement (genuine multipartite entanglement).
[Horodecki 1996; Lewenstein, Kraus, Cirac, Horodecki 2000; Terhal 2000]

There are two main goals when searching for entanglement witnesses:

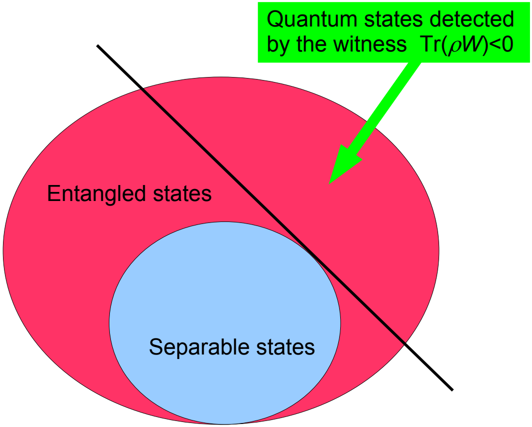
Large robustness to noise

Optimization

Few measurements

Convex sets for the entanglement witnesses

- Entanglement witnesses in the convex set picture



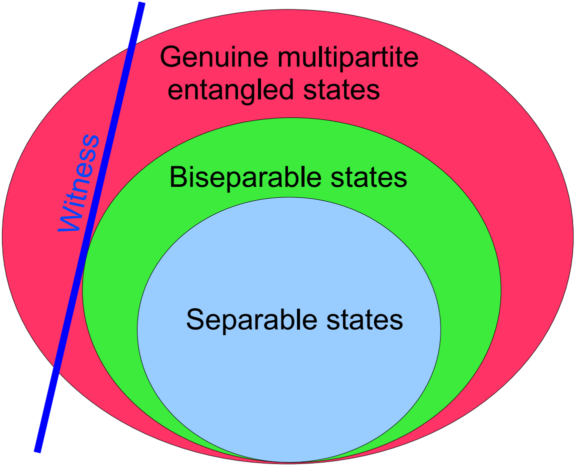
Convex sets for the entanglement witnesses II

- Entanglement witnesses can detect all entangled states since the set of separable states is convex.

- It is much more complicated to prove that a state is separable, since the set of entangled states is not convex.

Convex sets for the multipartite case

- The idea of convex sets also works for the multi-qubit case: A state is biseparable if it can be obtained by mixing pure biseparable states.



Outline

General scheme

- Witnesses are linear. Thus, the minimum of $\langle \mathcal{W} \rangle$ for separable states is the same as the minimum of $\langle \mathcal{W} \rangle$ for product states.

- A general scheme to get a witness is

$$\mathcal{W} = \mathcal{O} - \min_{\psi \text{ is of the form } \psi_1 \otimes \psi_2} \langle \mathcal{O} \rangle_\psi .$$

Witnesses based on correlations

Example

Witness with Heisenberg interaction

$$\mathcal{W}_{xyz} = \mathbb{1} \otimes \mathbb{1} + \sigma_x^{(1)} \otimes \sigma_x^{(2)} + \sigma_y^{(1)} \otimes \sigma_y^{(2)} + \sigma_z^{(1)} \otimes \sigma_z^{(2)}.$$

Proof. For product states of the form $|\Psi\rangle = |\Psi_1\rangle \otimes |\Psi_2\rangle$, we have

$$\langle \sigma_x \otimes \sigma_x \rangle + \langle \sigma_y \otimes \sigma_y \rangle + \langle \sigma_z \otimes \sigma_z \rangle = \sum_{l=x,y,z} \langle \sigma_l \rangle_{\Psi_1} \langle \sigma_l \rangle_{\Psi_2} \geq -1.$$

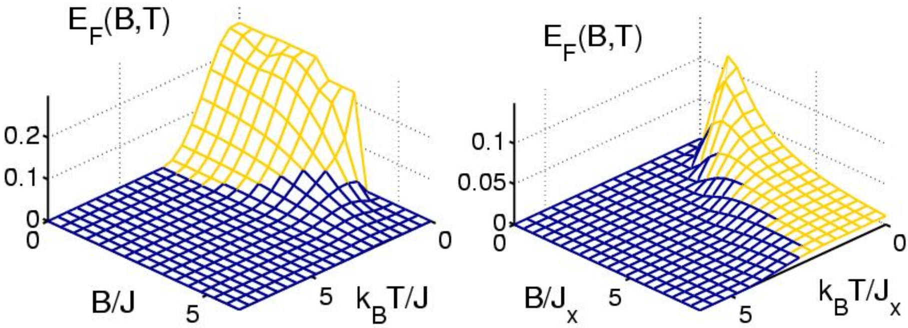
Here, we used the Cauchy-Schwarz inequality. Due to convexity, the inequality is also true for separable states.

The minimum for quantum states is –3. Such a minimum is obtained for the state

$$\frac{1}{\sqrt{2}} \left(|01\rangle - |10\rangle \right).$$

Witnesses based on correlations II

- This can be used in spin chains. If the energy is lower than the minimal energy of the classical model then the ground state is entangled.

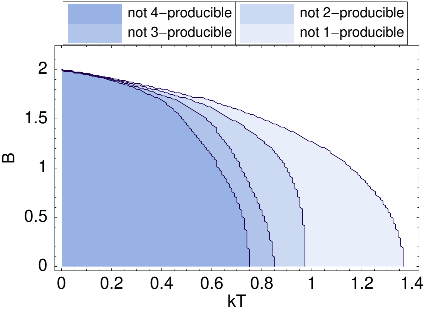


Heisenberg chain in an external field / Ising spin chain in a transverse field.

[G. Tóth, Phys. Rev. A 71, 010301(R) (2005); Č. Brukner and V. Vedral, e-print quant-ph/0406040; M. R. Dowling, A. C. Doherty, and S. D. Bartlett, Phys. Rev. A 70, 062113 2004.]

Witnesses for multipartite entanglement

- Can be used to obtain qualitative information on the thermal ground state.

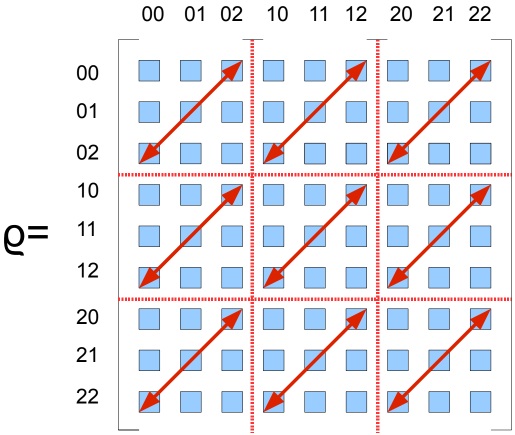


- XX-model in external field at finite temperature.

- not k-producible $\equiv$ at least $(k+1)$-particle entanglement

[O. Gühne, G. Tóth, New J. Phys. 7, 229 (2005); O. Gühne, G. Tóth, Phys. Rev. A 73, 052319 (2006).]

Partial transposition



$$(\varrho^{T2})_{kl,mn} = \varrho_{kn,ml}$$

PPT criterion-based witness I

- Let us take a bipartite separable state

$$\varrho_{\rm sep} = \sum_k p_k \varrho_1^{(k)} \otimes \varrho_2^{(k)}.$$

- After partial transposition on the second system, we get

$$\varrho_{\rm sep}^{T2} = \sum_k p_k \varrho_1^{(k)} \otimes (\varrho_2^{(k)})^T \geq 0.$$

- $\varrho_{\rm sep}^{T2}$ has a positive partial transpose (PPT).

- However, in general, there are states for which

$$\varrho^{T2} \ngeq 0.$$

They must be entangled.

[Peres,Horodecki,1997]

PPT criterion-based witness II

Witness for state $|\Psi\rangle$

We construct an entanglement witness that detects the state $|\Psi\rangle$ as entangled:

$$\mathcal{W} = |v\rangle\langle v|^{T1},$$

where $|v\rangle$ is the eigenvector of $|\Psi\rangle\langle\Psi|^{T1}$ with the smallest eigenvalue (which is negative).

Proof. For $|v\rangle$ we have $|\Psi\rangle\langle\Psi|^{T1}|v\rangle = \lambda|v\rangle$, where $\lambda < 0$. Then, we have

$$\mathrm{Tr}(\mathcal{W}|\Psi\rangle\langle\Psi|) = \mathrm{Tr}(|v\rangle\langle v||\Psi\rangle\langle\Psi|^{T1}) = \lambda < 0$$

using $Tr(A^{T1}B) = Tr(AB^{T1})$. Thus, the witness detects the state $|\Psi\rangle$ as entangled.

For every separable state

$$\mathrm{Tr}(\mathcal{W}\varrho_{\mathrm{sep}}) = \mathrm{Tr}(|v\rangle\langle v|\varrho_{\mathrm{sep}}^{T1}) > 0,$$

using $\varrho_{\mathrm{sep}}^{T1} \geq 0$ (previous slide).

Projector witness for bipartite systems

Projector witness for state $|\Psi\rangle$

A witness detecting entanglement in the vicinity of a pure state $|\Psi\rangle$ is

$$\mathcal{W}_{\Psi}^{(P)} := \lambda_{\Psi}^2 \mathbb{1} - |\Psi\rangle\langle\Psi|,$$

where λ_{Ψ}, is the maximum of the Schmidt coefficients for $|\Psi\rangle$.
[M. Bourennane, M. Eibl, C. Kurtsiefer, S. Gaertner, H. Weinfurter, O. Gühne, P. Hyllus, D. Bruß, M. Lewenstein, and A. Sanpera, Phys. Rev. Lett. 2004.]

Proof. Schmidt decomposition

$$|\Psi\rangle = \sum_k \lambda_k |k\rangle|k\rangle.$$

The maximum overlap with product states is $\max_k \lambda_k$. Hence,

$$\mathrm{Tr}(|\Psi\rangle\langle\Psi|\varrho_1 \otimes \varrho_2) \le \max_k \lambda_k^2.$$

Due to linearity,

$$\mathrm{Tr}(|\Psi\rangle\langle\Psi|\varrho_{\mathrm{sep}}) \le \max_k \lambda_k^2.$$

Projector witness for multipartite systems

- A witness detecting genuine multi-qubit entanglement in the vicinity of a pure state $|\Psi\rangle$ is

$$\mathcal{W}_{\Psi}^{(P)} := \lambda_{\Psi}^2 \mathbb{1} - |\Psi\rangle\langle\Psi|,$$

  where λ_Ψ is the maximum of the Schmidt coefficients for $|\Psi\rangle$, when all bipartitions are considered.

  [M. Bourennane, M. Eibl, C. Kurtsiefer, S. Gaertner, H. Weinfurter, O. Gühne, P. Hyllus, D. Bruß, M. Lewenstein, and A. Sanpera, Phys. Rev. Lett. 2004]

Projector witness: examples

- GHZ states (robustness to noise is $\frac{1}{2}$ for large N!)

Acín *et al.*, PRL 2001

$$\mathcal{W}^{(P)}_{\mathrm{GHZ}} := \frac{1}{2}\mathbb{1} - |\mathrm{GHZ}_N\rangle\langle\mathrm{GHZ}_N|$$

- Cluster states (obtained in Ising dynamics) Toth, Guhne, PRL 2005

$$\mathcal{W}^{(P)}_{\mathrm{CL}} := \frac{1}{2}\mathbb{1} - |\mathrm{CL}_N\rangle\langle\mathrm{CL}_N|.$$

- Symmetric Dicke state with $\langle J_z\rangle = 0$. Toth, JOSAB 2007

$$\mathcal{W}^{(P)}_{\mathrm{D(N,N/2)}} := \frac{1}{2}\frac{N}{N-1}\mathbb{1} - |\mathrm{D}_N^{(N/2)}\rangle\langle\mathrm{D}_N^{(N/2)}|.$$

- W-state (e.g., $|1000\rangle + |0100\rangle + |0010\rangle + |0001\rangle$)

$$\mathcal{W}^{(P)}_{\mathrm{W}} := \frac{N-1}{N}\mathbb{1} - |\mathrm{D}_N^{(1)}\rangle\langle\mathrm{D}_N^{(1)}|.$$

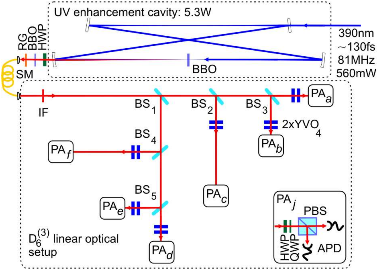
Outline

Experiment with photons

- A photon can have a horizontal (H) and a vertical (V) polarization.
- H/V can take the role of 0 and 1.
- Problem: photons do not interact with each other.

Photons II

$H^3 V^3 \longrightarrow$

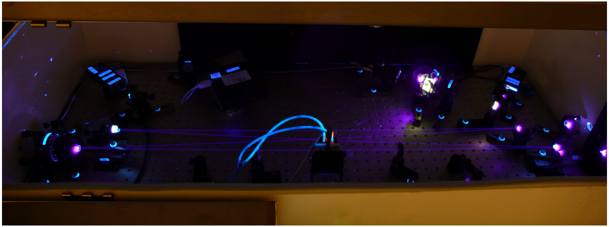


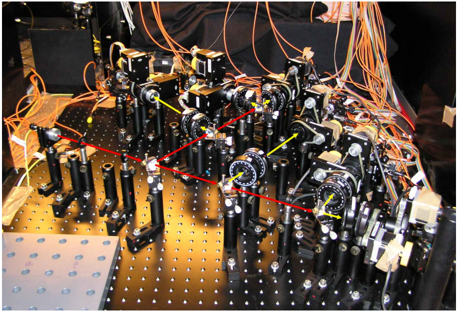
MPQ, Munich. Experiments with 6 photons.

W. Wieczorek, R. Krischek, N. Kiesel, P. Michelberger, G. Tóth, and H. Weinfurter, PRL 2009; another experiment: R. Prevedel, G. Cronenberg, M. S. Tame, M. Paternostro, P. Walther, M. S. Kim, and A. Zeilinger, PRL 2009.

$$|D_6^{(3)}\rangle = \frac{1}{\sqrt{20}} (|111000\rangle + |110100\rangle + ... + |000111\rangle).$$

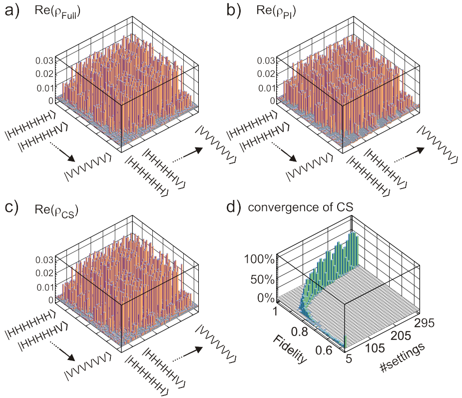
Photons III





Photons IV

6-qubit Quantum state tomography



[C. Schwemmer, G. Tóth, A. Niggebaum, T. Moroder, D. Gross, O. Gühne, and H. Weinfurter, Phys. Rev. Lett 113, 040503 (2014).]

Photons VI

- Entanglement witnesses can be used for entanglement detection:

> the state $|D_6^{(3)}\rangle$ leads to a reduction to only 21 measurement settings [15,16]. We have determined $\boxed{F_{D_6^{(3)}} = 0.654 \pm 0.024}$ with a measurement time of 31.5 h. This allows the application of the generic entanglement witness [10] $\boxed{\langle \mathcal{W}_g \rangle = 0.6 - F_{D_6^{(3)}} = -0.054 \pm 0.024}$ and thus proves genuine six-qubit entanglement of the observed state with a significance of 2 standard deviations (Fig. 4).

- For genuine multipartite entanglement, a fidelity of 0.6 is needed.

Outline

Variance-based criteria

For a bipartite system, for both parties

$$(\Delta A_k)^2 + (\Delta B_k)^2 \geq L_k.$$

For product states of the form $|\Psi\rangle = |\Psi_1\rangle \otimes |\Psi_2\rangle$, we have

$$(\Delta(A_1 + A_2))^2 = \langle (A_1 + A_2)^2 \rangle - \langle A_1 + A_2 \rangle^2 = (\Delta A_1)^2_{\Psi_1} + (\Delta A_2)^2_{\Psi_2}$$

since for product states

$$\langle A_1 A_2 \rangle - \langle A_1 \rangle \langle A_2 \rangle = 0.$$

Hence,

$$(\Delta(A_1 + A_2))^2 + (\Delta(B_1 + B_2))^2 \geq L_1 + L_2.$$

This is also true for separable states due to the convexity of separable states.

[See Gühne, Phys. Rev. Lett. (2004) for an exhaustive study.]

Variance-based criteria II

Example: we have

$$(\Delta x)^2(\Delta p)^2 \geq \frac{1}{4}.$$

Hence,

$$(\Delta x)^2 + (\Delta p)^2 \geq 1.$$

Then, for two-mode separable states

$$(\Delta(x_1 + x_2))^2 + (\Delta(p_1 - p_2))^2 \geq 2.$$

Any state violating this is entangled.

Note that

$$[x_1 + x_2, p_1 - p_2] = 0,$$

thus for entangle states the LHS can be zero!

[Generalization: L.M. Duan, G. Giedke, J.I. Cirac, P. Zoller, Phys. Rev. Lett (2000); R. Simon, Phys. Rev. Lett (2000).]

Outline

Many-particle systems

- For spin-$\frac{1}{2}$ particles, we can measure the collective angular momentum operators:

$$J_l := \frac{1}{2}\sum_{k=1}^{N}\sigma_l^{(k)},$$

where $l = x, y, z$ and $\sigma_l^{(k)}$ a Pauli spin matrices.

- We can also measure the

$$(\Delta J_l)^2 := \langle J_l^2\rangle - \langle J_l\rangle^2$$

variances.

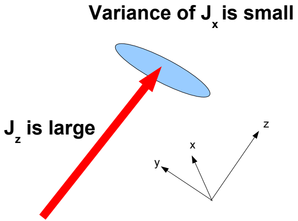
Spin squeezing

Definition

Uncertainty relation for the spin coordinates

$$(\Delta J_x)^2(\Delta J_y)^2 \geq \frac{1}{4}|\langle J_z\rangle|^2.$$

If $(\Delta J_x)^2$ is smaller than the standard quantum limit $\frac{1}{2}|\langle J_z\rangle|$ then the state is called spin squeezed (mean spin in the z direction!).
[M. Kitagawa and M. Ueda, Phys. Rev. A 47, 5138 (1993)]



Spin squeezing II

Definition

Spin squeezing criterion for the detection of quantum entanglement

$$\frac{(\Delta J_x)^2}{\langle J_y\rangle^2 + \langle J_z\rangle^2} \geq \frac{1}{N}.$$

If a quantum state violates this criterion then it is entangled.

- Application: Quantum metrology, magnetometry. Used many times in experiments.
- Violation of the spin-squeezing entanglement criterion $\Rightarrow$ violation of the spin Kitagawa-Udea criterion

[A. Sørensen *et al.*, Nature **409**, 63 (2001)]

Spin squeezing - Atomic ensembles

- Atoms interact with light, then the light is measured.
- Room temperature atoms: Hald, Sørensen, Schori, Polzik, PRL 1999
- Cold atoms: Sewell, Koschorreck, Napolitano, Dubost, Behbood, Mitchell, PRL 2012
- Cold atoms in a cavity: Leroux, Schleier-Smith, Vuletić, PRL 2011; Hosten, Engelsen, Krishnakumar, Kasevich, "Measurement noise 100 times lower than the quantum-projection limit using entangled atoms", Nature 2016.

Bose-Einstein condensate people

Netflix movie "Spectral"

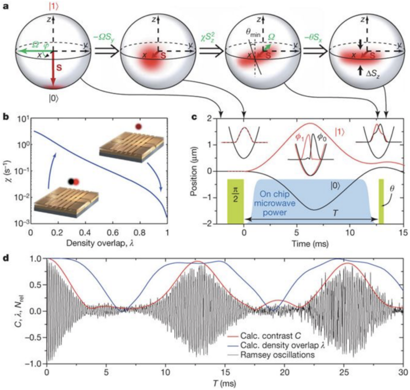
Filmed in Budapest

Spin squeezing - BEC

- Atoms interact with each other.
- Estéve, Gross, Weller, Giovanazzi, Oberthaler, "Squeezing and entanglement in a Bose-Einstein condensate", Nature 2008.
- Riedel, Böhi, Li, Hänsch, Sinatra, Treutlein, "Atom-chip-based generation of entanglement for quantum metrology". Nature 2010.

Spin squeezing - BEC

Figure 1: Spin squeezing and entanglement through controlled interactions on an atom chip.



M. F. Riedel, P. Böhi, Y. Li, T. W. Hänsch, A. Sinatra, and P. Treutlein, Nature 464, 1170-1173 (2010).

Spin squeezing - Varying particle number

- The experiment is repeated many times. Each time we find a somewhat different particle number.
- Postselecting for a given particle number is not feasible.
- Density matrix:
$$\varrho = \sum_N p_N \varrho_N,$$
where p_N are probabilities and ϱ_N are states.
- ϱ is entangled iff at least one of the ϱ_N is entangled.
[Hyllus, Pezze, Smerzi, PRL 2010.]

Outline

Complete set of the generalized spin squeezing criteria

- Let us assume that for a system we know only

$$\vec{J} := (\langle J_x\rangle, \langle J_y\rangle, \langle J_z\rangle),$$

$$\vec{K} := (\langle J_x^2\rangle, \langle J_y^2\rangle, \langle J_z^2\rangle).$$

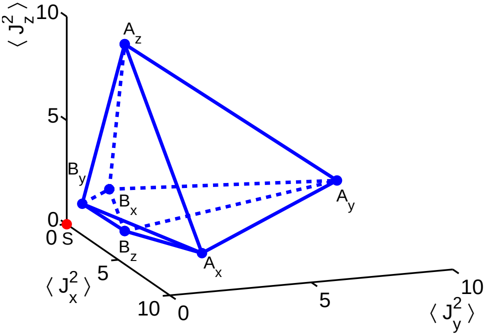
- Then any state violating the following inequalities is entangled

$$\begin{aligned}
\langle J_x^2\rangle + \langle J_y^2\rangle + \langle J_z^2\rangle &\le N(N+2)/4, && \text{(always true)}\\
(\Delta J_x)^2 + (\Delta J_y)^2 + (\Delta J_z)^2 &\ge N/2, && \text{(singlet)}\\
\langle J_k^2\rangle + \langle J_l^2\rangle - N/2 &\le (N-1)(\Delta J_m)^2, && \text{(Dicke state)}\\
(N-1)\left[(\Delta J_k)^2 + (\Delta J_l)^2\right] &\ge \langle J_m^2\rangle + N(N-2)/4, && \text{(planar sq. state)}
\end{aligned}$$

where k, l, m takes all the possible permutations of x, y, z.
[GT, C. Knapp, O. Gühne, and H.J. Briegel, Phys. Rev. Lett. 2007]

The polytope

- The previous inequalities, *for fixed* $\langle J_{x/y/z} \rangle$, describe a polytope in the $\langle J^2_{x/y/z} \rangle$ space.
- Separable states correspond to points inside the polytope. Note: Convexity comes up again!



Spin squeezing criteria – Two-particle correlations

All quantities needed can be obtained with two-particle correlations

$$\langle J_l \rangle = N \langle j_l \otimes \mathbb{1} \rangle_{\varrho_{\rm 2p}}; \quad \langle J_l^2 \rangle = \frac{N}{4} + N(N-1) \langle j_l \otimes j_l \rangle_{\varrho_{\rm 2p}}.$$

- Here, the average 2-particle density matrix is defined as

$$\varrho_{\rm 2p} = \frac{1}{N(N-1)} \sum_{n \neq m} \varrho_{mn}.$$

- Still, we can detect states with a separable $\varrho_{\rm 2p}$.
- Still, as we will see, we can even detect multipartite entanglement!

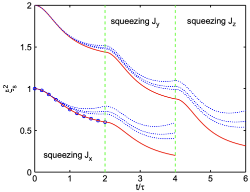
Singlet state

- Singlet states are ground states of antiferromagnetic Hamiltonians.
- The permutationally invariant singlet is
$$\varrho_{\rm singlet} \propto \lim_{T\rightarrow 0} e^{-\frac{J_x^2+J_y^2+J_z^2}{T}}.$$
- For such a state, for large N we have
$$\varrho_{\rm 2p} \approx \frac{\mathbb{1}}{4},$$
still it is detected as entangled by our criterion!
- Such a state has been created in cold atoms.

Singlet state experiments

- Creating singlets by squeezing the uncertainties of spin components → incoherent process, "creating entanglement with decoherence".

G. Tóth and M. W. Mitchell, Generation of macroscopic singlet states in atomic ensembles, New J. Phys. 12 053007 (2010).



- Entanglement in singlet states in $> 10^6$ cold Rb atoms.

N. Behbood, F. Martin Ciurana, G. Colangelo, M. Napolitano, GT, R. J. Sewell, and M. W. Mitchell, Generation of Macroscopic Singlet States in a Cold Atomic Ensemble, Phys. Rev. Lett. 113, 093601 (2014).

- Entanglement in singlet states in $> 10^{13}$ hot alkali atoms.

J. Kong, R. Jiménez-Martínez, C. Troullinou, V. G. Lucivero, GT, and Morgan W. Mitchell, Measurement-induced, spatially-extended entanglement in a hot, strongly-interacting atomic system,

Nat. Commun. 11, 2415 (2020).

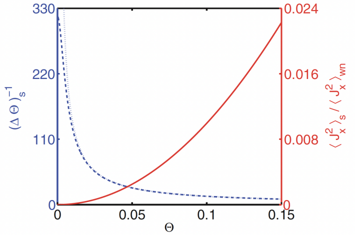
Applications of singlets

- Singlets are invariant under $U^{\otimes N}$, i. e., under homogeneous fields.
- Gradient field destroys the singlet $\rightarrow$ Gradient metrology.

I. Apellaniz, I. Urizar-Lanz, Z. Zimborás, P. Hyllus, and GT, Precision bounds for gradient magnetometry with atomic ensembles, Phys. Rev. A 97, 053603 (2018).

S. Altenburg, M. Oszmaniec, S. Wölk, O. Gühne, Estimation of gradients in quantum metrology, Phys. Rev. A 96, 042319 (2017).

I. Urizar-Lanz, P. Hyllus, I. L. Egusquiza, M. W. Mitchell, and GT, Macroscopic singlet states for gradient magnetometry, Phys. Rev. A 88, 013626 (2013).



Entanglement conditions based on the two-body density matrix

- Spin squeezing conditions with collective variables based on the two-body density matrix.
- All detected states have an entangled two-qubit density matrix, violating the PPT criterion.

Spin Squeezing Inequalities and Entanglement of *N* Qubit States,
J. K. Korbicz, J. I. Cirac, and M. Lewenstein,
Phys. Rev. Lett. 95, 120502 (2005).

Outline

Multipartite entanglement in spin squeezing

- We consider pure k-producible states of the form

$$|\Psi\rangle = \otimes_{n=1}^{M} |\psi^{(n)}\rangle,$$

where $|\psi^{(n)}\rangle$ is the state of at most k qubits.

The spin-squeezing criterion for k-producible states is

$$(\Delta J_z)^2 \geqslant J_{\max} F_{\frac{k}{2}} \left(\frac{\sqrt{\langle J_x \rangle^2 + \langle J_y \rangle^2}}{J_{\max}} \right),$$

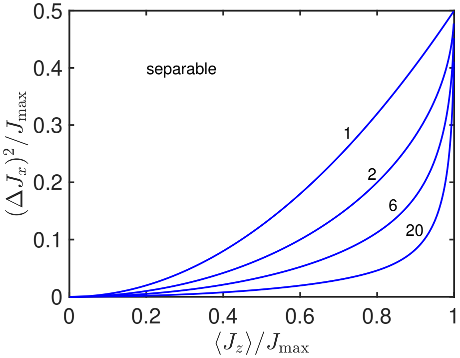
where $J_{\max} = \frac{N}{2}$ and we use the definition

$$F_j(X) := \frac{1}{j} \min_{\frac{\langle j_x \rangle}{j} = X} (\Delta j_z)^2.$$

[Sørensen and Mølmer, Phys. Rev. Lett. 86, 4431 (2001);
experimental test: C. Gross *et al.*, Nature 464, 1165 (2010).]

Multipartite entanglement in spin squeezing

- Larger and larger multipartite entanglement is needed to larger and larger squeezing ("extreme spin squeezing").



- $N = 100$ spin-1/2 particles, $J_{\max} = N/2$.

[Sørensen and Mølmer, Phys. Rev. Lett. 86, 4431 (2001); experimental test: Gross, Zibold, Nicklas, Esteve, Oberthaler, Nature 464, 1165 (2010).]

Multipartite entanglement around Dicke states

- Measure the quantities

$$(\Delta J_z)^2$$

  and

$$\langle J_x^2 + J_y^2 \rangle.$$

- In contrast, for the original spin-squeezing criterion we measured $(\Delta J_z)^2$ and $\langle J_x \rangle^2 + \langle J_y \rangle^2$.

- Pioneering work: linear condition of Luming Duan, Phys. Rev. Lett. (2011). See also Zhang, Duan, NJP (2014).

Multipartite entanglement - Dicke states

- Sørensen-Mølmer condition for k-producible states

$$(\Delta J_z)^2 \geqslant J_{\max} F_{\frac{k}{2}} \left(\frac{\sqrt{\langle J_x \rangle^2 + \langle J_y \rangle^2}}{J_{\max}} \right).$$

- Combine it with

$$\langle J_x^2 + J_y^2 \rangle \leqslant J_{\max}(\tfrac{k}{2} + 1) + \langle J_x \rangle^2 + \langle J_y \rangle^2,$$

which is true for pure k-producible states. (Remember, $J_{\max} = \frac{N}{2}$.)

Condition for entanglement detection around Dicke states

$$(\Delta J_z)^2 \geqslant J_{\max} F_{\frac{k}{2}} \left(\frac{\sqrt{\langle J_x^2 + J_y^2 \rangle - J_{\max}(\frac{k}{2} + 1)}}{J_{\max}} \right).$$

Due to convexity properties of the expression, this is also valid to mixed separable states.

Outline

Dicke states II

PRL **112,** 155304 (2014) PHYSICAL REVIEW LETTERS week ending 18 APRIL 2014

Detecting Multiparticle Entanglement of Dicke States

Bernd Lücke,[1] Jan Peise,[1] Giuseppe Vitagliano,[2] Jan Arlt,[3] Luis Santos,[4] Géza Tóth,[2,5,6] and Carsten Klempt[1]
[1]*Institut für Quantenoptik, Leibniz Universität Hannover, Welfengarten 1, D-30167 Hannover, Germany*
[2]*Department of Theoretical Physics, University of the Basque Country UPV/EHU, P.O. Box 644, E-48080 Bilbao, Spain*
[3]*QUANTOP, Institut for Fysik og Astronomi, Aarhus Universitet, 8000 Århus C, Denmark*
[4]*Institut für Theoretische Physik, Leibniz Universität Hannover, Appelstraße 2, D-30167 Hannover, Germany*
[5]*IKERBASQUE, Basque Foundation for Science, E-48011 Bilbao, Spain*
[6]*Wigner Research Centre for Physics, Hungarian Academy of Sciences, P.O. Box 49, H-1525 Budapest, Hungary*
(Received 27 February 2014; published 17 April 2014)

Recent experiments demonstrate the production of many thousands of neutral atoms entangled in their spin degrees of freedom. We present a criterion for estimating the amount of entanglement based on a measurement of the global spin. It outperforms previous criteria and applies to a wider class of entangled states, including Dicke states. Experimentally, we produce a Dicke-like state using spin dynamics in a Bose-Einstein condensate. Our criterion proves that it contains at least genuine 28-particle entanglement. We infer a generalized squeezing parameter of $-11.4(5)$ dB.

DOI: 10.1103/PhysRevLett.112.155304 PACS numbers: 67.85.−d, 03.67.Bg, 03.67.Mn, 03.75.Mn

Entanglement, one of the most intriguing features of quantum mechanics, is nowadays a key ingredient for many applications in quantum information science [1,2], quantum simulation [3,4], and quantum-enhanced metrology [5]. Entangled states with a large number of particles cannot be characterized via full state tomography [6], which is routinely used in the case of photons [7,8], trapped ions [9], or superconducting circuits [10,11]. A reconstruction of the full density matrix is hindered and finally prevented by the exponential increase of the required number of measurements. Furthermore, it is technically impossible to address all individual particles or even fundamentally forbidden if the particles occupy the same quantum state. Therefore, the entanglement of many-particle states is best characterized by measuring the expectation values and variances of the components of the collective spin $\mathbf{J} = (J_x, J_y, J_z)^T = \sum_i \mathbf{s}_i$, the sum of all individual spins $\mathbf{s}_i$ in the ensemble.

In particular, the spin-squeezing parameter $\xi^2 = N(\Delta J_z)^2/(\langle J_x\rangle^2 + \langle J_y\rangle^2)$ defines the class of spin-squeezed states for $\xi^2 < 1$. This inequality can be used to verify the presence of entanglement, since all spin-squeezed states are entangled [12]. Large clouds of entangled neutral atoms are typically prepared in such spin-squeezed states, as shown in thermal gas cells [13], at ultracold temperatures [14–16], and in Bose-Einstein

quantified by means of the so-called entanglement depth, defined as the number of particles in the largest nonseparable subset [see Fig. 1(a)]. There have been numerous experiments detecting multiparticle entanglement involving up to 14 qubits in systems, where the particles can be addressed individually [9,20–24]. Large ensembles of neutral atoms

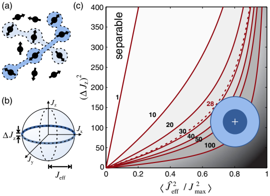


FIG. 1 (color online). Measurement of the entanglement depth for a total number of 8000 atoms. (a) The entanglement depth is given by the number of atoms in the largest nonseparable subset

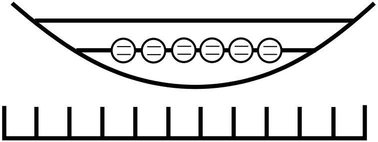
Outline

Bipartite entanglement from bosonic multipartite entanglement

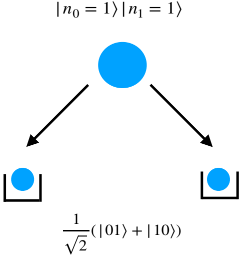
- In the BEC, "all the particles are at the same place."
- In the usual formulation, entanglement is between spatially separated parties.
- Is multipartite entanglement within a BEC useful/real?
- Answer: yes!

Bipartite entanglement from bosonic multipartite entanglement II

- Dilute cloud argument



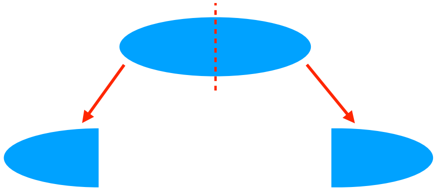
See, e.g., P. Hyllus, L. Pezzé, A Smerzi and GT, PRA 86, 012337 (2012)



Bipartite entanglement from bosonic multipartite entanglement III

- After splitting it into two, we have bipartite entanglement if we had before multipartite entanglement.

- The splitting does not generate entanglement, if we consider projecting to a fixed particle number.



N. Killoran, M. Cramer, and M. B. Plenio, PRL 112, 150501 (2014).

Outline

Experiment in the group of Carsten Klempt at the University of Hannover

- Rubidium BEC, spin-1 atoms.

- Initially all atoms in the spin state $|j_z = 0\rangle$.

- Dynamics

$$H = a_0^2 a_{+1}^\dagger a_{-1}^\dagger + (a_0^\dagger)^2 a_{+1} a_{-1}.$$

  Tunneling from mode 0 to the mode $+1$ and -1.

- Two-particle example:

$$|j_z = 0\rangle|j_z = 0\rangle \to \frac{1}{\sqrt{2}}(|j_z = +1\rangle|j_z = -1\rangle + |j_z = -1\rangle|j_z = +1\rangle)$$
$$= \text{Dicke state of 2 particles.}$$

Experiment in the group of Carsten Klempt at the University of Hannover II

- After some time, we have a state

$$|n_0, n_{-1}, n_{+1}\rangle = |N_{\mathrm{total}} - 2n, n, n\rangle.$$

- That is, $N_{\mathrm{total}} - 2n$ particles remained in the $|j_z = 0\rangle$ state, while $N = 2n$ particles form a symmetric Dicke state given as

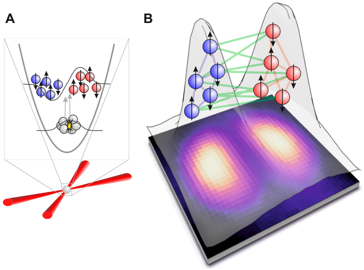
$$|D_N\rangle = \binom{N}{\frac{N}{2}}^{-\frac{1}{2}} \sum_k \mathcal{P}_k \left(|0\rangle^{\otimes \frac{N}{2}} \otimes |1\rangle^{\otimes \frac{N}{2}} \right),$$

  where we use $|0\rangle$ and $|1\rangle$ instead of $|j_z = -1\rangle$ and $|j_z = +1\rangle$.

- Half of the atoms in state $|0\rangle$, half of the atoms in state $|1\rangle$ + symmerization.

Experiment in the group of Carsten Klempt at the University of Hannover III

- Important: first excited spatial mode of the trap was used, not the ground state mode.
- It has two "bumps" rather than one, hence they had a split Dicke state.



K. Lange, J. Peise, B. Lücke, I. Kruse, G. Vitagliano, I. Apellaniz, M. Kleinmann, G. Tóth, and C. Klempt, Entanglement between two spatially separated atomic modes, Science 360, 416 (2018).

Correlations for Dicke states

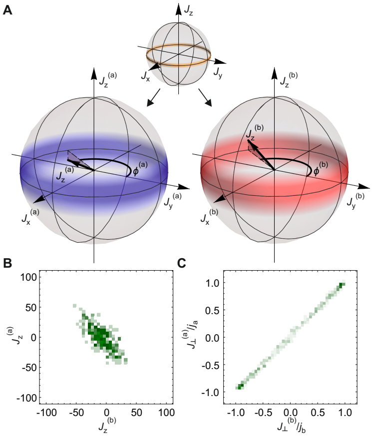
- For the Dicke state

$$\begin{aligned}(\Delta(J_x^a - J_x^b))^2 &\approx 0,\\ (\Delta(J_y^a - J_y^b))^2 &\approx 0,\\ (\Delta J_z)^2 = (\Delta(J_z^a + J_z^b))^2 &= 0.\end{aligned}$$

- Measurement results on well "*b*" can be predicted from measurements on "*a*"

$$\begin{aligned}J_x^b &\approx J_x^a, & &\text{(correlation)}\\ J_y^b &\approx J_y^a, & &\text{(correlation)}\\ J_z^b &= -J_z^a. & &\text{(anti-correlation)}\end{aligned}$$

Correlations for Dicke states - experimental results



Here, $J_\perp^{(n)} = \cos\alpha J_x^{(n)} + \sin\alpha J_y^{(n)}$.

Experiment in K. Lange *et al.*, Science 334, 773–776 (2011).

Outline

Number-phase-like uncertainty

- We start from the sum of two Heisenberg uncertainty relations

$$(\Delta J_z)^2[(\Delta J_x)^2 + (\Delta J_y)^2] \geq \frac{1}{4}(\langle J_x \rangle^2 + \left\langle J_y \right\rangle^2).$$

Then,

$$(\Delta J_z)^2[(\Delta J_x)^2 + (\Delta J_y)^2] + \frac{1}{4}[(\Delta J_x)^2 + (\Delta J_y)^2] \geq \frac{1}{4}(\left\langle J_x^2 \right\rangle + \left\langle J_y^2 \right\rangle).$$

- Simple algebra yields

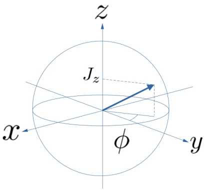
$$\left[(\Delta J_z)^2 + \frac{1}{4}\right] \times \frac{(\Delta J_x)^2 + (\Delta J_y)^2}{\langle J_x^2 \rangle + \langle J_y^2 \rangle} \geq \frac{1}{4}.$$

- Note that $\langle J_x^2 \rangle$ appears, not $\langle J_x \rangle^2$.

Number-phase-like uncertainty II

- Uncertainty relation

$$\underbrace{\left[(\Delta J_z)^2 + \frac{1}{4}\right]}_{\sim \text{ fluctuation of } J_z} \times \underbrace{\frac{(\Delta J_x)^2 + (\Delta J_y)^2}{\langle J_x^2 \rangle + \langle J_y^2 \rangle}}_{\sim \text{ phase fluctuation}} \geq \frac{1}{4}.$$



Handwaving description:
J_z and ϕ cannot be defined both with high accuracy.

The two-well entanglement criterion

Main result

For separable states,

$$\left[(\Delta J_z)^2 + \frac{1}{4}\right]\left[(\Delta \mathcal{J}_x^-)^2 + (\Delta \mathcal{J}_y^-)^2\right] \geq \frac{1}{16}\left\langle \mathcal{J}_x^2 + \mathcal{J}_y^2 \right\rangle^2$$

holds. $|D_N\rangle$: $\frac{1}{4}$ $\approx \frac{4}{N}$ ≈ 1 $\frac{LHS}{RHS} = \frac{1}{N}$

Here,

$$\begin{aligned} J_z &= J_z^a + J_z^b, \\ \mathcal{J}_x^- &= \mathcal{J}_x^a - \mathcal{J}_x^b, \\ \mathcal{J}_y^- &= \mathcal{J}_y^a - \mathcal{J}_y^b, \end{aligned}$$

where $\mathcal{J}_x$ and $\mathcal{J}_y$ denote normalized quantities. Similar criterion for EPR steering.

Outline

Problem 1: Varying particle number

- The experiment is repeated many times. Each time we find a somewhat different particle number.
- Postselecting for a given particle number is not feasible.
- Density matrix:

$$\varrho = \sum_{N_a,N_b} Q_{N_a,N_b} \varrho_{N_a,N_b},$$

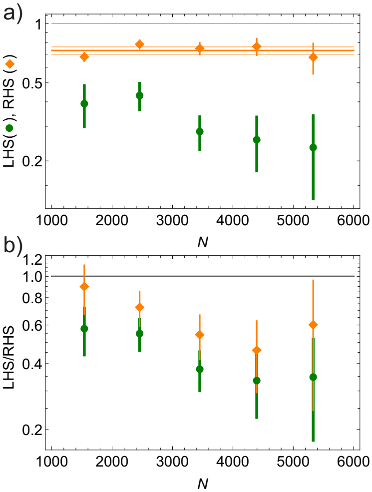
  where Q_{N_a,N_b} are probabilities and ϱ_{N_a,N_b} are states.

- ϱ is entangled iff at least one of the ϱ_{N_a,N_b} is entangled.
  [Hyllus, Pezze, Smerzi, PRL 2010.]

Problem 2: States are not always symmetric in a BEC of two-state atoms

- Ideally, the BEC is in a single spatial mode.
- Hence, the state of an ensemble of the two-state atoms must be symmetric.
- In practice, the BEC is not in a single spatial mode, hence the state is no perfectly symmetric.
- Our criterion must hadle this.

Violation of the criterion: entanglement is detected II



LHS/RHS for (top) Quantum 2024, and (bottom) for Science 2018.

The two-well entanglement criterion

Other experiments creating bipartite entanglement in BEC, published back-to-back:

Spatially separated parts of a spin-squeezed Bose-Einstein condensate. Two components condensate:

M. Fadel, T. Zibold, B. Décamps, and P. Treutlein,
Spatial entanglement patterns and Einstein-Podolsky-Rosen steering in Bose-Einstein condensates,
Science 360, 409 (2018).

Spatially separated parts of a spin-squeezed Bose-Einstein condensate. Spin-1 particles, initial state is 0:

P. Kunkel, M. Prüfer, H. Strobel, D. Linnemann, A. Frölian, T. Gasenzer, M. Gärttner, and M. K. Oberthaler,
Spatially distributed multipartite entanglement enables EPR steering of atomic clouds,
Science 360, 413 (2018).

High-dimensional entanglement

- Dimension = Schmidt number of the bipartite state.

$$|\Psi\rangle = \sum_{k=1}^{r} \lambda_k |k\rangle_A \otimes |k\rangle_B,$$

where r is the Schmidt number. $|_A\langle k|l\rangle_A| = \delta_{kl}$, and $|_B\langle k|l\rangle_B| = \delta_{kl}$.

- Can be extended to mixed states via comvex sets.

- There are Schmidt number witensses

A. Sanpera, D. Bruß, and M. Lewenstein,
Schmidt-number witnesses and bound entanglement,
Phys. Rev. A 63, 050301(R), (2001).

High-dimensional entanglement II

- Several bipartite correlations must be measured.

M. Krenn, M. Huber, R. Fickler, R. Lapkiewicz, S. Ramelow, A. Zeilinger Generation and confirmation of a (100x100)-dimensional entangled quantum system, PNAS 2014

O. Lib, S. Liu, R. Shekel, Q. He, M. Huber, Y. Bromberg, G. Vitagliano, Experimental certification of high-dimensional entanglement with randomized measurements, PRL2025

Entanglement quantification in atomic ensembles

- They consider broad families of entanglement criteria that are based on variances of arbitrary operators.
- Analytically lower bounds for the best separable approximation (BSA) and the generalized robustness (GR).

[M. Fadel, A. Usui, M. Huber, N. Friis, and G. Vitagliano, PRL 2021.]

Reviews

- R. Horodecki, P. Horodecki, M. Horodecki, and K. Horodecki, "Quantum entanglement", Rev. Mod. Phys. 81, 865–942 (2009).
- L.Amico, R. Fazio, A. Osterloh, and V. Vedral. "Entanglement in many-body systems", Rev. Mod. Phys. 80, 517–576 (2008).
- N. Friis, G. Vitagliano, M. Malik, and M. Huber. "Entanglement certification from theory to experiment", Nat. Rev. Phys. 1, 72–87 (2019).

Conclusions

- We discussed a method to construct entanglement conditions:
  - Entanglement witnesses, i.e., conditions linear in operator expectation values,
  - Nonlinear entanglement witnesses, spin squeezing.
- For the transparencies, see

  www.gtoth.eu

- See also

  O. Gühne and G. Tóth,

  Entanglement detection,

  Phys. Rep. 474, 1 (2009); arxiv:0811.2803.

THANK YOU FOR YOUR ATTENTION!