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Number operator-annihilation operator uncertainty as an alternative for the number-phase uncertainty relation

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Contents

1 Introduction

In this work we have considered the number operator-annihilation operator pair as a well behaved alternative of the number operator-phase operator canonical pair. Finding a Hermitian phase operator canonically conjugate to the number operator is not possible, and all the different approaches must make certain compromises. Along this paper, we discuss some of these different approaches and their properties, as well as their dissadvantages. In 1976, Lévy-Leblond suggested the use of non-Hermitian operators to describe some physical quantities. According to his ideas, we propose to use the harmonic oscillator's annihilation operator in the place of the phase operator. An uncertainty relation is derived for the number operator-annihilation operator pair in wich the bound on the product of uncertainties depends on the expectation value of the particle number. In order to derive this expression we must define what we understand by the uncertainty of a non-hermmitian operator, and we have followed Lévy-Leblond's definition to do so. The uncertainty relation that we have derived is

$$\left[(\Delta N)^2 + \frac{1}{4}\right]\left[(\Delta a)^2 + \frac{1}{2}\right] \geq \frac{\langle N\rangle}{4} + \frac{1}{8}. \tag{1}$$

In this formulation the bound is not a constant (as usual), but it is a quantity that can be easily controlled in many systems. We will also study the properties of the annihilation operator's eingenstates (coherent states), and also other families of states such as squeezed coherent states and photon added coherent states. Several setups for an experimental verification are also proposed, like trapped ions or a single mode electromagnetic field.

2 The quantum phase problem

Let us consider a quantum one-dimensional harmonic oscillator. The destruction and creation operators will be a and $a^\dagger$, respectively. The first attempt to introduce the phase operator was made by Dirac [4] in the context of the interaction between light and atoms. He proposed a contact transformation between the canonically conjugate variables a and $a^\dagger$, and N and ϕ defined by

$$a = N^{\frac{1}{2}} e^{i\phi}, \tag{2}$$

$$a^\dagger = N^{\frac{1}{2}} e^{-i\phi}. \tag{3}$$

As N and ϕ are canonically conjugate variables, we will try to quantize them as usual

$$[N, \phi] = i. \tag{4}$$

However, we will see that this commutation relation leads to a contradiction. For example, let us take matrix elements in the number state basis $|n'\rangle$, that is, the basis in wich $N|n'\rangle = n'|n'\rangle$. We get

$$\langle n|N\phi - \phi N|n'\rangle = i\langle n|n'\rangle \Rightarrow (n - n')\langle n|\phi|n'\rangle = i\delta_{nn'}. \tag{5}$$

For $n = n'$ we are led to an impossible equation $(0 = i)$. One source of problems is the fact that if we impose the equation $a = N^{\frac{1}{2}} e^{i\phi}$, then $e^{i\phi}$ must be equal to

$$e^{i\phi} = aN^{-\frac{1}{2}}. \tag{6}$$

But $N^{\frac{1}{2}}$ is not an invertible operator. As will be discussed later, another possible ansatz is

$$e^{i\phi} \equiv E = (N+1)^{-\frac{1}{2}} a = \sum_{n=0}^{\infty} |n\rangle\langle n+1|. \tag{7}$$

However, E is not unitary, thus ϕ cannot be Hermitian either.

Another source of problems is the fact that the number operator has a discrete and bounded spectrum. Thus, it is not possible to define a Hermitian operator conjugate to it (acting in the whole Hilbert space). This result is known as "Pauli Theorem". The Pauli Theorem was first stated by W. Pauli [41, 42, 43] before year 1933 in the context of the time operator in Quantum Mechanics. This theorem says that if A and B are two canonically conjugate Hermitian operators, then both spectra must be the whole real line. We will give a brief sketch of a proof following an article by Eric A. Galapagon [44]. Then we will discuss some formalisms that have been used to treat the pase quantum-mechanically.

2.1 Pauli Theorem

Pauli's argument goes as follows: Let A and B be two self-adjoint, canonically conjugate operators

$$[A, B] = i. \tag{8}$$

As A is self-adjoint, for all $\beta \in \mathbb{R}$, $U_\beta = e^{-i\beta A}$ is unitary. Now we can compute the commutator

$$[U_\beta, B] = \sum_{k=0}^{\infty} \frac{(-i\beta)^k}{k!}[A^k, B] = -\beta U_\beta. \tag{9}$$

Now, let φ_λ be an eingenvector of B with eingenvalue λ.Then, as a direct consequence of Eq.9, $U_\beta \varphi_\lambda$ is an eingenvector of B with eingenvalue $(\lambda + \beta)$

$$BU_\beta \varphi_\lambda = (\lambda + \beta)U_\beta \varphi_\lambda. \tag{10}$$

As β is an arbitrary real number, eingestates of B exist with arbitrary eingenvalue, so the spectrum of B is the whole real line. The same argument can be repeated to show that the spectum of A is also $\mathbb{R}$. Thus, as N's spectrum is not $\mathbb{R}$, there cannot exist a Hermitian operator conjugate to it. Finding a phase operator conjugate to N is equivalent to finding a time operator conjugate to the Hamiltonian, as for a harmonic oscilator we have $H = (N + \frac{1}{2})\hbar\omega$ and $\phi \approx \omega t$.

2.2 The Susskind and Glogower formalism

In this section, we will follow Carruthers and Nieto's notation to discuss Susskind and Glogower's formalism [5]. As we have mentioned before, they define E to be

$$E = (N+1)^{-\frac{1}{2}}a, \tag{11}$$
$$E^\dagger = a^\dagger (N+1)^{-\frac{1}{2}}. \tag{12}$$

E and $E^\dagger$ are the analogs of $e^{\pm i\phi}$. Similarly, one can construct the analogs of $\sin\phi$ and $\cos\phi$ as

$$S = \frac{1}{2i}(E - E^\dagger), \tag{13}$$
$$C = \frac{1}{2}(E + E^\dagger). \tag{14}$$

E is not unitary, but just "one-sided" unitary

$$EE^\dagger = Id, \tag{15}$$
$$E^\dagger E = Id - |0\rangle\langle 0|. \tag{16}$$

For these operators the following commutation relations hold

$$[C, N] = iS, \tag{17}$$
$$[S, N] = -iC, \tag{18}$$
$$[C, S] = \frac{i}{2}|0\rangle\langle 0|, \tag{19}$$
$$C^2 + S^2 = 1 - \frac{1}{2}|0\rangle\langle 0|. \tag{20}$$

In the last two equations there is a vacuum-state projector that makes C and S not to conmute and not to satisfy the usual trigonometric identity.

Now, let us look for the eingenvectors of the E operator

$$E|e^{i\phi}\rangle = e^{i\phi}|e^{i\phi}\rangle. \tag{21}$$

A set of solutions is given by

$$|e^{i\phi}\rangle = \sum_{n=0}^{\infty} e^{in\phi}|n\rangle. \tag{22}$$

These states form an overcomplete set, and they yield a resolution of the identity

$$\frac{1}{2\pi}\int_{-\pi}^{+\pi} \mathrm{d}\phi|e^{i\phi}\rangle\langle e^{i\phi}| = \mathbb{I}. \tag{23}$$

In Ref.[6], another family of eingenstates can be found

$$|\xi\rangle = (1-|\xi|^2)^{\frac{1}{2}}\sum_{n=0}^{\infty} \xi^n|n\rangle, \qquad |\xi| < 1. \tag{24}$$

Finally, it is important to write down the uncertainty relations satisfied by N, S and C. They are

$$\Delta N\Delta C \geq \frac{1}{2}\langle S\rangle, \tag{25}$$

$$\Delta N\Delta S \geq \frac{1}{2}\langle C\rangle, \tag{26}$$

$$\Delta C\Delta S \geq \frac{1}{4}\langle|0\rangle\langle 0|\rangle. \tag{27}$$

$$\tag{28}$$

2.3 The Pegg-Barnett formalism

Some investigators have considered the phase operator problem in spaces of finite dimension. Here we will talk about a formalism developed by Pegg and Barnett [7], wich makes use of a finite-dimensional space and then investigates the limiting form of the theory when the dimension of the space tends to infinity. In this formalism we consider the finite-dimensional vector space Ψ_s spanned by the $(s+1)$ vectors $\{|0\rangle, |1\rangle, |2\rangle, ..., |s\rangle\}$ (it is a $(s+1)$-dimensional space). In this space we can construct all states and operators as functions of s. Then we need a prescription for taking the limit $s \to \infty$. The limit can be carried in two ways: we can apply the limit $s \to \infty$ to the operators and states before calculating moments, or we can calculate expectation values as functions of s and then take the limit $s \to \infty$. For operators like N, a and $a^\dagger$ the two procedures give the same result. However, for the phase operator (wich we will construct soon) the two limiting procedures give different results, and we must use the second one.

Now we will use these ideas to construct a phase operator in the finite-dimensional space Ψ_s. First we will give a definition: Two operators A and B acting on Ψ_s are said to be conjugate if the following two properties are satisfied:

- The eigenstates of A are equally weighted superpositions of eigenstates of B and the eigenstates of B are equally weighted superpositions of eigenstates of A.
- The operator B is the generator of shifts in the eigenvalue of any eigenstate of A and the operator A is the generator of shifts in the eigenvalue of any eigenstate of B.

The eingenstates of the number operator are the states $|n\rangle$. An operator conjugate to N will have eigenstates $|\Theta\rangle$ like

$$|\theta\rangle = (s+1)^{-\frac{1}{2}} \sum_{n=0}^{s} \exp(in\theta)|n\rangle, \tag{29}$$

which is an equally weighted superposition of the number states.The number operator generates shifts in the value θ according to

$$\exp(iN\phi)|\theta\rangle = |\theta+\phi\rangle. \tag{30}$$

The states $|\theta\rangle$ form an overcomplete set and are not mutually orthogonal.Their overlap is

$$\langle\theta|\theta'\rangle = (s+1)^{-1}\frac{1-\exp[i(s+1)(\theta'-\theta)]}{1-\exp[i(\theta'-\theta)]}. \tag{31}$$

However, it is possible to construct a complete orthonormal set of $s+1$ states spanning Ψ_s by choosing θ to be

$$\theta_m = \theta_0 + \frac{2\pi m}{s+1}, \qquad (m = 0, 1, 2, ..., s), \tag{32}$$

where θ_0 is arbitrary. We can express $|n\rangle$ as a superposition of $|\theta_m\rangle$'s as

$$|n\rangle = (s+1)^{-\frac{1}{2}} \sum_{m=0}^{s} \exp(-in\theta_m) \left|\theta_m\right\rangle. \tag{33}$$

Now, the Hermitian phase operator is defined like this

$$\hat{\phi}_\theta = \sum_{m=0}^{s} \theta_m \left|\theta_m\right\rangle \left\langle\theta_m\right|. \tag{34}$$

It is easy to check that

$$\hat{\phi}_\theta \left|\theta_m\right\rangle = \theta_m \left|\theta_m\right\rangle. \tag{35}$$

This is, $|\theta_m\rangle$ are eigenstates of the phase operator with eigenvalues θ_m. They are called "phase states". The matrix elements of the phase operator in the number state basis are found to be

$$\langle n| \hat{\phi}_\theta |n\rangle = \theta_0 + \frac{\pi s}{s+1}, \qquad \text{(diagonal elements)} \tag{36}$$

$$\langle n'| \hat{\phi}_\theta |n\rangle = \frac{2\pi}{s+1}\frac{\exp[i(n'-n)\theta_0]}{\exp[2\pi i(n'-n)/(s+1)]-1}. \quad \text{(off-diagonal elements)} \tag{37}$$

Using this information, one can calculate $\langle\hat{\phi}_\theta\rangle$ in an arbitrary (in general, mixed) state of Ψ_s

$$\rho_s = \sum_{n=0}^{s}\sum_{n'=0}^{s} \rho_{nn'} \left|n\right\rangle \left\langle n'\right|, \tag{38}$$

$$\langle\hat{\phi}_\theta\rangle_s = Tr\{\rho_s\hat{\phi}_\theta\} = \sum_{n=0}^{s}\sum_{n'=0}^{s} \rho_{nn'} \langle n'| \hat{\phi}_\theta |n\rangle. \tag{39}$$

This is a function of s, and we must take the limit $s \to \infty$ in order to calculate the mean value in the actual Hilbert space. As an example we will calculate $(\Delta\hat{\phi}_\theta)^2$ in a number state: Firstly, we calculate $\langle\hat{\phi}_\theta\rangle$ as a function of the dimension, s

$$\langle\hat{\phi}_\theta\rangle_s = \langle n| \hat{\phi}_\theta |n\rangle_s = \theta_0 + \pi\frac{s^2+s}{s^2+2s+1}. \tag{40}$$

In the limit $s \to \infty$ this is

$$\lim_{s\to\infty} \langle \hat{\phi}_\theta \rangle_s = \theta_0 + \pi. \tag{41}$$

Similarly,

$$\lim_{s\to\infty} \langle n| \hat{\phi}_\theta^2 |n\rangle_s = \theta_0^2 + 2\pi\theta_0 + \frac{4}{3}\pi^2. \tag{42}$$

So we get

$$(\Delta\hat{\phi}_\theta)^2 = \frac{\pi^2}{3}. \tag{43}$$

It is interesting to note that a variance of $\pi^2/3$ is what would be expected in a classical state of random phase. For $e^{\pm i\phi}$, one can also use the usual trigonometric formulae

$$\cos\phi_\theta = \frac{1}{2}(e^i\phi + e^{-i\phi}), \tag{44}$$

$$\sin\phi_\theta = \frac{1}{2i}(e^i\phi - e^{-i\phi}). \tag{45}$$

One can readily show that

$$\cos^2\phi_\theta + \sin^2\phi_\theta = 1, \quad \text{(1 is the identity operator)} \tag{46}$$

$$[\cos\phi_\theta, \sin\phi_\theta] = 0, \tag{47}$$

$$\langle n| \cos^2\phi_\theta |n\rangle = \langle n| \sin^2\phi_\theta |n\rangle = \frac{1}{2}. \tag{48}$$

All of these expressions differ from the corresponding forms for the Susskind-Glogower operators. Thus, in the Pegg-Barnett theory cosine and sine operators satisfy the usual trigonometric identity for their squares, they commute, and their expectation values in the number states are consistent with these being random phase states, including the vacuum $(n = 0)$. All results so far are quite satisfactory. What then of the number-phase commutation relation, usually somewhat problematical in phase-operator theories? The diagonal and off-diagonal matrix elements of the commutator in the number state basis are

$$\langle n| [N, \hat{\phi}_\theta] |n\rangle = 0, \quad \text{(diagonal elements)} \tag{49}$$

$$\langle n'| [N, \hat{\phi}_\theta] |n\rangle = \frac{2\pi}{s+1} \frac{(n'-n)e^{i(n'-n)\theta_0}}{\exp[2\pi i(n'-n)/(s+1)] - 1}. \quad \text{(off-diagonal elements)} \tag{50}$$

Thus, for a general state of the form 38 one has

$$\langle [N, \hat{\phi}_\theta] \rangle = \lim_{s\to\infty} \langle [N, \hat{\phi}_\theta] \rangle_s = -i \sum_{n=0}^{\infty} \sum_{n'=0}^{\infty} \rho_{nn'}(1 - \delta_{nn'}) e^{i(n'-n)\theta_0}, \tag{51}$$

which differs from the expected formula $\langle [N, \hat{\phi}_\theta] \rangle = i$.

2.4 The Lévy-Leblond formalism

Lévy-Leblond (1976) [6] wrote about a different and very original solution: He proposed to use non-Hermitian operators to represent physical observables. Then he needs to define the variance for a non-Hermitian operator, since the usual definition ($(\Delta A)^2 = \langle \chi| A^2 |\chi\rangle$) is only valid for Hermitian ones. His idea is that now he can take the Susskind-Glogower's E operator and say that it is the phase operator, although it is not Hermitian. Literally, he writes: "Why be afraid of non-Hermitian operators?" ([6]) In Quantum Mechanics all observables are asked to be Hermitian, and the main argument is that Hermitian operators have real eingenvalues. If we choose Hermitians operators, we can say that their eingenvalues correspond to the measured quantities. But this argument is easy to refute saying that we could formulate Quantum Mechanics in such a way that measured quantities do not correespond to eingevalues, but to some real-valued function of the eingenvalues.

Another argument usually given to use Hermitian operators is that their eigenstates are orthonormal. This orthonormality has deep physical consecuences, because after a measurement the system is in an eigenstate of the operator measured, let's say, $|a\rangle$, and the probability amplitude for it to jump into another state with a different eigenvalue, $|a'\rangle$, is $\langle a|\, a'\rangle$, which is guarranteed to be 0 for Hermitian operators. If we use a non-Hermitian operator with non-orthogonal eigenstates, then repeated succesive measurements of that physical property will not necessarily give the same value. In usual Quantum Mechanics this occurs when we measure non-conmuting operators. But Lévy-Leblond says that there is not a reason to expect that several succesive measurements of the same property will give the same result, taking into account that we don't ask this property to the theory when we are measuring non-conmuting observables. Literally, he says: "the prejudice, then would be a purely epistemological one."

As we want to use non-Hermitian operators, we have to give a definition for the variance, as we said previously. Instead of using the usual definition,

$$(\Delta q)^2 \equiv \langle\chi|\, q^2\, |\chi\rangle - \langle\chi|\, q\, |\chi\rangle^2, \tag{52}$$

Lévy-Leblond proposes to use the generalization

$$(\Delta F)^2_{LL} \equiv \langle\chi|\, F^\dagger F\, |\chi\rangle - |\, \langle\chi|\, F\, |\chi\rangle\, |^2. \tag{53}$$

Note that (53) reduces to (52) for Hermitian operators. It can be rewritten as

$$(\Delta F)^2_{LL} = ||(F - \langle F\rangle I)\, |\chi\rangle\, ||^2. \tag{54}$$

The following properties are satisfied: (i) $(\Delta F)^2_{LL}$ is a nonnegative number, so a nonnegative ΔF always exists, and (ii) ΔF is zero if and only if $|\chi\rangle$ is an eigenstate of F.

In the rest of his article Lévy-Leblond discusses the use of the operator $E = \sum_{n=0}^{\infty} |n\rangle\, \langle n+1|$ as the phase operator. It is Susskind and Glogower's phase operator, so he comes to the same results that were discussed in section B. He also comes to the additional relation

$$(\Delta N)^2[(\Delta\phi)^2 + \frac{1}{2}\langle P^{(0)}\rangle] \geq \frac{1}{4}[1 - (\Delta\phi)^2 - \langle P^{(0)}\rangle] \tag{55}$$

(ϕ stands for the E operator).

3 Number operator-annihilation operator uncertainty as an alternative of the number-phase uncertainty relation

Connected to Lévy-Leblond's ideas, we choose the annihilation operator to play the role of a phase operator. In the next sections we will contruct and study the properties of an uncertainty relation satisfied by the pair "number operator-annihilation operator". We will find interesting results for quantum optics, as well as for quantum information theory, because we will get a criterion for the detection of quantum entanglement.

A family of uncertainty relations with N and a has already appeared in Ref. [11], and has been used for the detection of quantum entanglement [13, 12, 14]. Such uncertainty relations made it possible to construct entanglement conditions with small experimental requirements. Remarkably, these conditions detect non-Gaussian entangled states that cannot be detected based on first and second moments of the quadrature components [15]. The uncertainty relation Eq. 1 can be seen as a single relation replacing the family of uncertainty relations described in Ref. [11]. For given $\langle N\rangle$, Eq. 1 identifies most of the values for the variances of N and a that are not allowed by quantum physics [16].

3.1 Variance of the annihilation operator

Before constructing uncertainty relations, we have to give a definition of the variance of a non-Hermitian operator. We will follow the definition given by Lévy-Leblond (Eq. 53)

$$(\Delta a)^2 = \langle a^\dagger a\rangle - \langle a^\dagger\rangle\langle a\rangle = \langle N\rangle - |\langle a\rangle|^2. \tag{56}$$

Let us discuss the meaning of $(\Delta a)^2$

- First of all, we notice that $(\Delta a)^2$ is zero only for coherent states, so it measures, in a sense, how close the quantum state is to a coherent state (Remember that a coherent state satisfies $a\,|\alpha\rangle = \alpha\,|\alpha\rangle\,, \quad \alpha \in \mathbb{C}$).

- We will now interpret the meaning of $(\Delta a)^2$ by relating it to the quadrature components: The quadrature components are defined as

$$x_\beta = \frac{ae^{+i\beta} + a^\dagger e^{-i\beta}}{\sqrt{2}}, \tag{57}$$

$$p_\beta = \frac{ae^{+i\beta} - a^\dagger e^{-i\beta}}{i\sqrt{2}}. \tag{58}$$

  A simple calculation yields

$$(\Delta a)^2 = \frac{(\Delta x_\beta)^2 + (\Delta p_\beta)^2}{2} - \frac{1}{2}. \tag{59}$$

  Here, β defines the two ortogonal quadrature components, x_β and p_β. When discussing the invariance properties of $(\Delta a)^2$, it is instructive to point out its connection to the correlation matrix, defined as

$$\Gamma_\beta = \begin{pmatrix} (\Delta x_\beta)^2 & \frac{1}{2}(\langle \Delta x_\beta \Delta p_\beta + \Delta p_\beta \Delta x_\beta \rangle) \\ \frac{1}{2}(\langle \Delta x_\beta \Delta p_\beta + \Delta p_\beta \Delta x_\beta \rangle) & (\Delta p_\beta)^2 \end{pmatrix}.$$

One can obtain $\Gamma_{\beta'}$ from Γ_β through orthogonal transformations. However, the trace of Γ, which equals $(\Delta x_\beta)^2 + (\Delta p_\beta)^2$, remains invariant under such transformations. Hence, $(\Delta x_\beta)^2 + (\Delta p_\beta)^2$ (and so $(\Delta a)^2$) is independent of the choice of β [21].

Thus, as it is independent from β, $(\Delta x_\beta)^2 + (\Delta p_\beta)^2$ seems to be a good measure of the uncertainty of the orthogonal quadrature components. An alternative measure could be $(\Delta x_\beta)^2 (\Delta p_\beta)^2$. However, it is not independent from β.

- In another context, $(\Delta a)^2$ can be expressed as

$$(\Delta a)^2 = \frac{1}{2\pi} \int_{\lambda=0}^{2\pi} (\Delta x_\lambda)^2 d\lambda - \frac{1}{2}. \tag{60}$$

  $(\Delta a)^2$ is connected to the average variance of the quadrature components x_β. That is, if β is choosen randomly between 0 and 2π, according to a uniform probability distribution, then $(\Delta a)^2 + \frac{1}{2}$ gives the expectation value of the quadrature variance $(\Delta x_\beta)^2$.

- Finally, let us see a connection of $(\Delta a)^2$ to important properties of the Wigner function of the quantum state. For the following discussion, as well as in the rest of this work, we will leave the β subscript, and will use x and p in the sense of x_0 and p_0, respectively. $(\Delta a)^2$ gives information on how sharp is the peak of the Wigner function $W(x,p)$ of the state since [22]

$$(\Delta x)^2 + (\Delta p)^2 = \int \left[(x - \langle x \rangle)^2 + (p - \langle p \rangle)^2\right] W(x,p) dx dp. \tag{61}$$

  For states with a non-negative Wigner function (i.e., squeezed coherent states), $2(\Delta a)^2 + 1$ is the sum of the width of the Wigner function in two orthogonal directions. The sharpest peak is obtained for coherent states for which Eq. (61) is the smallest.

3.2 Uncertainty relation with the number and annihilation operators

Now that we have defined the variance of the annihilation operator and we have discussed its meanning, we are in the position to derive an uncertainty relation for it and the number operator. For deriving it we will start remembering the general Heisenberg uncertainty relation

$$(\Delta A)^2 (\Delta B)^2 \geq \frac{1}{4} |\langle [A, B] \rangle|^2. \tag{62}$$

Making use of it, and taking into account that $[N, p] = ix$ and $[N, x] = -ip$, we come to

$$\begin{aligned} (\Delta N)^2 (\Delta p)^2 &\geq \frac{1}{4} |\langle x \rangle|^2, \\ (\Delta N)^2 (\Delta x)^2 &\geq \frac{1}{4} |\langle p \rangle|^2. \end{aligned} \tag{63}$$

Now, using $|\langle x \rangle|^2 + |\langle p \rangle|^2 = 2|\langle a \rangle|^2$ and (59), we add these last two inequalities and we get

$$(\Delta N)^2 \Big[(\Delta a)^2 + \frac{1}{2} \Big] \geq \frac{1}{4} |\langle a \rangle|^2. \tag{64}$$

The bound in this uncertainty relation is not a constant, it depends on $\langle a \rangle$, which is zero for a wide class of states. It would be meaningful to find a similar relation with a constant bound or at least with a bound depending on a quantity that is easily measurable and controllable. We would like to construct a relation in wich the bound depends on $\langle N \rangle$ rather than on $\langle a \rangle$. For that, we add $[(\Delta a)^2 + \frac{1}{2}]/4$ to both sides of (64) and we get

$$\left[(\Delta N)^2 + \frac{1}{4}\right]\left[(\Delta a)^2 + \frac{1}{2}\right] \geq \frac{1}{4}\left[(\Delta a)^2 + \frac{1}{2} + |\langle a \rangle|^2\right] = \frac{1}{4}\left[\langle N \rangle + \frac{1}{2}\right]. \tag{65}$$

In the last step we have used $(\Delta a)^2 = \langle N \rangle - |\langle a \rangle|^2$. So we have shown that for any state of a quantum harmonic oscillator the following inequality is always fullfilled

$$\left[(\Delta N)^2 + \frac{1}{4}\right]\left[(\Delta a)^2 + \frac{1}{2}\right] \geq \frac{\langle N \rangle}{4} + \frac{1}{8}.$$

This is the result that we announced in the introduction (Eq. 1). The right hand side of Eq. 1 is minimal for the vacuum $|0\rangle$. In all other cases the right hand side is greater than $\frac{1}{8}$, thus the uncertainty finds some part of the $(\Delta a)^2$–$(\Delta N)^2$ plane inaccessible for quantum states.

Next, we will relate (1) to the uncertainty relation with N and E. For that, we determine the form of (1) for the case of large N and $(\Delta N)^2 \ll \langle N \rangle^2$. In this case $a \approx \sqrt{\langle N \rangle} E$ and we obtain

$$(\Delta N)^2 (\Delta E)^2 \gtrsim \frac{1}{4}\left[1 - (\Delta E)^2\right], \tag{66}$$

which is in accordance with the results of Ref. [6], (equation 63)

$$(\Delta N)^2 (\Delta E)^2 \geq \frac{1}{4}\left[1 - (\Delta E)^2 - \langle P^{(0)} \rangle\right], \tag{67}$$

where $P^{(0)} = |0\rangle \langle 0|$.

Finally, equation (63) can be improved by using the Robertson-Schrödinger inequalities [22]. First, they can be used to improve the two uncertainty relations in Eq. 63. Then, after steps similar to the previous ones, we obtain

$$\left[(\Delta N)^2 + \frac{1}{4}\right]\left[(\Delta a)^2 + \frac{1}{2}\right] \geq \frac{\langle N \rangle}{4} + \frac{1}{8} + \frac{1}{4}|\langle \{\Delta N, \Delta a\}_+ \rangle|^2, \tag{68}$$

where $\{A, B\}_+ = AB + BA$ is the anticommutator.

3.3 Tightness of the inequality

In what follows we will discuss the tightness of the inequality (1). We will investigate which quantum states are close to saturate it. We will see that the states saturating the left-hand side of it interpolate between coherent states and fock states.

Our inequality does not contain the highest possible lower bound. This is due to the fact that the two uncertainty relations (63) are saturated by different states. When we add them, the resulting inequality is not saturated by any of those states.

Let us see how close Fock states and coherent states are to saturating inequality (1). For Fock states we have

$$(\Delta a)^2_{Fock} = \langle N \rangle = n, \qquad (\Delta N)^2_{Fock} = 0,$$

since $\langle n| a |n\rangle \propto \langle n| n-1\rangle = 0$ For coherent states we have

$$(\Delta a)^2_{coh} = 0, \qquad (\Delta N)^2_{coh} = \langle N \rangle = |\alpha|^2.$$

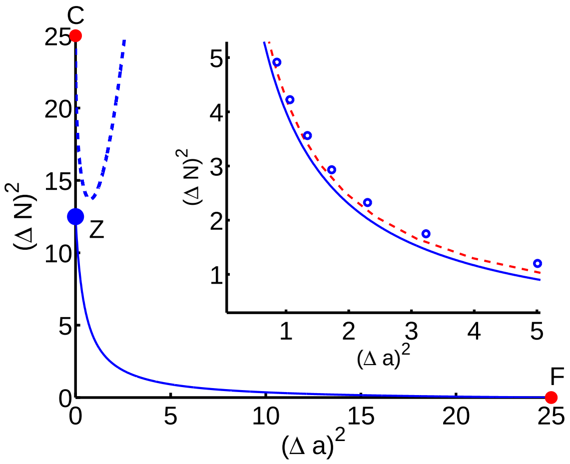

Figure 1: Numerical test of the inequality (1). F refers to the Fock state $|n = 25\rangle$, C refers to the coherent state $|\alpha = 5\rangle$, Z refers to the point that saturates inequality (1) for $(\Delta a)^2 = 0$. (solid) Boundary of the region defined by (1) for $\langle N\rangle = 25$. All points below this line correspond to aphysical $(\Delta a)^2$-$(\Delta N)^2$ values. (dashed) Points corresponding to squeezed coherent states. The equation of the curve is given in (80). Inset: (solid) Boundary of the region defined by (1). (dashed) Points corresponding to states with a Gaussian wave vector (72). (circles) States corresponding to photon added coherent states (86).

Hence, Fock states saturate the inequality. However, for coherent states (states with $(\Delta a)^2 = 0$) the particle number variance saturating the inequality is $(\Delta N)^2 = \frac{1}{2}\langle N\rangle$. This already shows that the lower bound in Eq. 1 cannot be optimal, as for $(\Delta a)^2 = 0$ there is not a quantum state with a smaller particle number variance than $\langle N\rangle$. States minimizing $(\Delta N)^2$ for $0 < (\Delta a)^2 < \langle N\rangle$, in a sense, interpolate between coherent states and Fock states.

In Fig.1, we plotted the $(\Delta a)^2 - (\Delta N)^2$ plane. From now on, as an example, we will concentrate on quantum states having $\langle N\rangle = 25$. The blue line corresponds to points that saturate the inequality (for $\langle N\rangle = 25$). Thus, no quantum state can have values of $[(\Delta a)^2, (\Delta N)^2]$ below this line (which is a hyperbola). Point F corresponds to the Fock state $|n = 25\rangle$ and point C corresponds to the coherent state $|\alpha = 5\rangle$. Point Z is the point where the hyperbolla reaches the $(\Delta N)^2$-axis ($(\Delta N)^2 = \frac{1}{2}\langle N\rangle$).

We will now examine the tightness numerically, through choosing appropriate trial states. Our search can be simplified noting that it is sufficient to search over wave vectors with non-negative real elements. To see this, let us consider a state of the form $|\Psi\rangle = \sum_n r_n e^{i\phi_n} |n\rangle$. For $\langle N^2\rangle$ we have

$$\langle N^2\rangle = \langle\Psi| N^2 |\Psi\rangle = \sum_{p,q} r_p r_q e^{i(\phi_p - \phi_q)} \langle q| N^2 |p\rangle = \sum_{p,q} r_p r_q e^{i(\phi_p - \phi_q)} p^2 \delta_{pq} = \sum_p r_p^2 p^2. \quad (69)$$

This is a number that does not depend on the ϕ's. Similarly, for $\langle N\rangle$

$$\langle N\rangle = \langle\Psi|\, N\, |\Psi\rangle = \sum_{p,q} r_p r_q e^{i(\phi_p-\phi_q)} \langle q|\, N\, |p\rangle = \sum_{p,q} r_p r_q e^{i(\phi_p-\phi_q)} p\delta_{pq} = \sum_p r_p p, \tag{70}$$

which doesn't depend on the ϕ's either. So $(\Delta N)^2$ doesn't depend on the ϕ's. Finally, for $|\langle a\rangle|$ we get

$$|\langle a\rangle| = |\sum_{p,q} r_p r_q e^{i(\phi_p-\phi_q)} \langle q|\, a\, |p\rangle\, | = |\sum_{p,q} r_p r_q e^{i(\phi_p-\phi_q)} \sqrt{p}\delta_{q,p-1}| = |\sum_p r_p r_{p-1} e^{i(\phi_p-\phi_{p-1})}\sqrt{p}|. \tag{71}$$

This number gets its maximmum (as a function of the ϕ_i's) when $\phi_i = \phi_j \quad \forall i \neq j$. On the other hand, if $|\langle a\rangle|$ is maximmum, $(\Delta a)^2 = \langle N\rangle - |\langle a\rangle|^2$ will be minimmum. Thus, if we are searching for the states having the smallest $[(\Delta N)^2 + \frac{1}{4}][(\Delta a)^2 + \frac{1}{2}]$, it is sufficient to search states with all $\phi_i = 0$.

We have choosen several families of trial states, all of them depending on two parameters, μ and ν. By fixing $\langle\Psi = (\mu,\nu)|\, N\, |\Psi = (\mu,\nu)\rangle = 25$ we get a constraint $f(\mu,\nu) = 0$, so we only have one free parameter, and by running it we will get a line in the $(\Delta a)^2 - (\Delta N)^2$ plane. We have done all these calculations numerically (with the help of MATLAB), not annalitically. We will discuss one by one all the families that we have tried.

3.3.1 Gaussian wave vector

We have called "Gaussian wave vector states" states of the form

$$|N_0, \Delta\rangle = \frac{1}{C}\sum_n \exp\left[-\frac{(n-N_0)^2}{4\Delta^2}\right], \tag{72}$$

where C is for normalization. They are not usual Gaussian states (in the sense that only the correlation matrix is needed to caracterize them), but we still call them "Gaussian". For these states $\langle N\rangle \approx N_0$ (wich we set equal to 25) and $(\Delta N)^2 \approx \Delta^2$. By changing the parameter Δ continuously from 0 to 5 we travel from a Fock state to a coherent state, always close to the line that saturates the inequality. It is represented in the inset of Fig. 1 as a dashed red line.

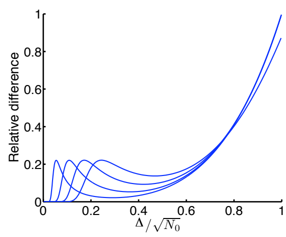

Figure 2: Numerical test of the inequality (1). The difference between the left-hand side and the right hand-side of (1) divided by the right hand side is shown for states (72) with a Gaussian wave vector for (from left to right) $N_0 = 100, 25, 10$ and 5.

In Fig.2 we have plotted the relative difference between the left hand side and the right hand side of Eq. 1 ((Left Hand Side - Right Hand Side)/Right Hand Side), as a function of $\Delta/\sqrt{N_0}$, for this family of states, for various values of N_0. The four curves correspond to $N_0 = 5, 10, 25, 100$ (from right to left, respectively). This graph shows that the relative difference doesn't grow much (in fact, it is smaller that 0.25) even if N_0 grows.

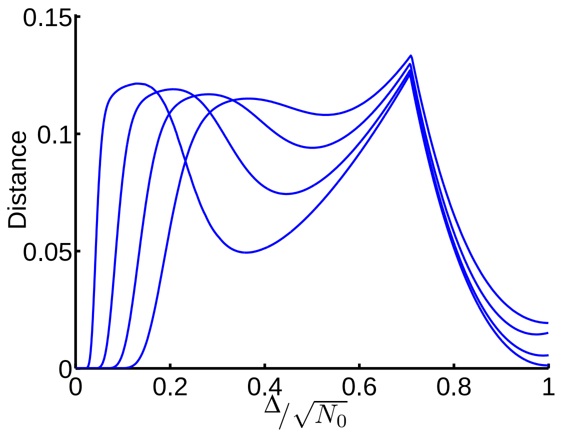

Figure 3: Distance of the points corresponding to states with a Gaussian wave vector (72) from the curve corresponding to states that saturate (1) for (from left to right) $N_0 = 100, 25, 10$ and 5. Δ and N_0 are the parameters of the state (72).

In Fig.3 we plot the distance on the $(\Delta a)^2$–$(\Delta N)^2$ plane from Gaussian states to the curve as a function of $\Delta/\sqrt{N_0}$ for various values of N_0. As before, the curves correspond to $N_0 = 5, 10, 25, 100$ (from right to left, respectively). By distance, we mean the following: given a value of Δ, compute $(\Delta a)^2$ and $(\Delta N)^2$ for that state (Eq. 72, remember that we have set $N_0 = 25$). Then compute the "euclidean distance" in the graph from the state to the curve (1), this is, the distance you would measure with a rod. This one is the distance that we have plotted. We have done this numerically. It is interesting to note that Gaussian states remain close to the "minimmum uncertainty curve" (closer than 0.15) even for large values of N_0. This means that for large $\langle N\rangle$ Gaussian states are practically the inteligent states of the uncertainty relation (1). (Intelligent states for an uncertainty relation are those that saturate it, this is, for which the equality holds. Minimmum Uncertainty States (MUS) are those Intelligent states that have the smallest possible value for the mean values that appear (if they appear)).

The states Eq.72 are a subset of the number-phase intelligent states called $|g'\rangle$ presented in Ref. [9]. There, the coefficients of $|n\rangle$ have a Gaussian dependence on n, just as in Eq.72, however, the phase of the coefficients is not zero but has a linear dependence on n. A state vector with a phase with a linear dependence on n has the same value for $(\Delta a)^2, (\Delta N)^2$ and $\langle N\rangle$ as a state with zero phase. Thus, all the $|g'\rangle$ states presented in Ref. [9] are very close to saturate Eq.1. Hence, our inequality makes it possible to define number-phase intelligent states and amplitude squeezing [10] without a reference to a phase operator.

There are other states known to be number-phase intelligent states [24, 25]. We now consider the state presented in Ref. [26]. They are defined as the superposition of coherent states on a circle

$$|\alpha_0, u\rangle \propto \int_{-\infty}^{+\infty} \exp(-\frac{1}{2}u^2\phi^2 - i\delta\phi)\,|\alpha_0 e^{i\phi}\rangle\, d\phi, \tag{73}$$

where $\delta = \alpha_0^2$. The overlap with Fock states is

$$\begin{aligned} \langle\alpha_0, u|n\rangle &\propto \frac{\alpha_0^n}{\sqrt{n!}} \exp\left[-\frac{(n-\delta)^2}{2u^2}\right] \\ &= \langle\alpha_0|n\rangle \times \exp\left[-\frac{(n-\delta)^2}{2u^2}\right]. \end{aligned} \tag{74}$$

The second expression stresses the fact that we have the overlap of a coherent state $|\alpha_0\rangle$ and the Fock state $|n\rangle$, multiplied by a Gaussian centered around α_0^2, that is, the expectation value of the particle number for $|\alpha_0\rangle$. Thus, $|\alpha_0, u\rangle$ has an almost Gaussian wave vector for large N in the number basis. Hence, these states give similar results numerically, as Eq.72 does for our uncertainty relation Eq.1.

3.3.2 Squeezed coherent states

Another interesting family of states is the family of squeezed coherent states. They are defined by

$$|\alpha, \zeta\rangle = D(\alpha)S(\zeta)\,|0\rangle\,, \tag{75}$$

where D is the displacement operator and S is the squeezing operator

$$\begin{aligned} D(\alpha) &= \exp\,(\alpha a^\dagger - \alpha^* a), \\ S(\zeta) &= \exp\Big(-\frac{\zeta}{2}a^{\dagger 2} + \frac{\zeta^*}{2}a^2\Big). \end{aligned} \tag{76}$$

Next, we will use that

$$\begin{aligned} D^\dagger(\alpha) a D(\alpha) &= a + \alpha, \\ S^\dagger(\zeta) a S(\zeta) &= a\cosh(s) - a^\dagger e^{i\vartheta}\sinh(s), \end{aligned} \tag{77}$$

where $\zeta = se^{i\vartheta}$. Hence, with $\alpha = |\alpha|e^{i\theta}$, we obtain

$$\begin{aligned} \langle N\rangle_{|\alpha,\zeta\rangle} &= \sinh^2(s) + |\alpha|^2, \\ (\Delta a)^2_{|\alpha,\zeta\rangle} &= \sinh^2 s, \\ (\Delta N)^2_{|\alpha,\zeta\rangle} &= |\alpha|^2\left[\cosh(2s) - \sinh(2s)\cos(2\theta - \vartheta)\right] \\ &+ 2\sinh^2(s)\left[1 + \sinh^2(s)\right]. \end{aligned} \tag{78}$$

For given $|\alpha|$ and s, the variance $(\Delta N)^2$ in Eq. 78 is minimal if $\cos(2\theta - \vartheta) = 1$. This is fulfilled, for example, if $\theta = \vartheta = 0$, that is, both ζ and α are real and nonnegative. So we again have a two-parametric family of states, the two parameters being the non-negative real numbers α and ζ. For it, we have

$$\begin{aligned} (\Delta N)^2_{\min}(|\alpha|, s) &= |\alpha|^2\left[\cosh(2s) - \sinh(2s)\right] \\ &+ 2\sinh^2(s)\left[1 + \sinh^2(s)\right]. \end{aligned} \tag{79}$$

Putting together (78) and (79), we get the smallest possible $(\Delta N)^2$ for squeezed coherent states, for given $(\Delta a)^2$ and $\langle N\rangle$ as

$$\begin{aligned} (\Delta N)^2_{\min} &= \left[\langle N\rangle - (\Delta a)^2\right]\left[\sqrt{1 + (\Delta a)^2} - \sqrt{(\Delta a)^2}\right]^2 \\ &+ 2(\Delta a)^2[1 + (\Delta a)^2]. \end{aligned} \tag{80}$$

This curve is represented in Fig.1 by a dashed blue line.

Let us interpret this result. Since $(\Delta a)^2$ and $(\Delta N)^2$ are invariant under a rotation around the origin in the x–p plane, we can start from coherent states $|\alpha\rangle$ with a real and positive α. Then, the state we considered for the curve Eq. 80 corresponds to squeezing the x quadrature component. This type of squeezing is called "number-squeezing" in the literature (e.g., Ref. [28]) and it reduces the number variance for small amount of squeezing. Thus, for small squeezing Eq. 80 is not far from the bound given by Eq. 1. However, squeezing further, the number variance starts to grow. Thus, for large $(\Delta a)^2$ there are no squeezed coherent states giving an almost minimal particle number variance, and one has to look for non-Gaussian states for that.

Finally, note that from Eq. 75 and Eq. 77, it follows that the squeezed coherent states are the eigenvectors of operators of the type

$$\mu a + \nu a^\dagger, \tag{81}$$

where μ and ν are complex numbers [28, 27].

3.3.3 Displaced Fock states

Displaced Fock states are defined as

$$|\alpha, n\rangle = D(\alpha)\,|n\rangle\,. \tag{82}$$

With Eq. 77, we obtain

$$\begin{aligned}
&\langle N\rangle_{|\alpha,n\rangle} = n + |\alpha|^2,\\
&(\Delta N)^2_{|\alpha,n\rangle} = (2n+1)|\alpha|^2,\\
&(\Delta a)^2_{|\alpha,n\rangle} = n.
\end{aligned} \tag{83}$$

Hence, for displaced Fock states we get the equation

$$(\Delta N)^2 = \left[2(\Delta a)^2 + 1\right]\left[\langle N\rangle - (\Delta a)^2\right], \tag{84}$$

where $(\Delta a)^2$ must be a non-negative integer. It is fulfilled both by Fock states and coherent states. However, other points on the $(\Delta a)^2 - (\Delta N)^2$ plane fulfilling Eq. 84 are very far from saturating Eq. 1.

Finally, from Eq.82 and Eq.77, it follows that a squeezed coherent state $|\alpha, n\rangle$ satisfies the following eigenvalue equation

$$(a^\dagger a - \alpha a^\dagger - \alpha^* a)\,|\alpha, n\rangle = (n - |\alpha|^2)\,|\alpha, n\rangle\,. \tag{85}$$

3.3.4 Photon-added coherent states

Photon added coherent states are defined as [30]

$$|\alpha, m\rangle \propto (a^\dagger)^m\,|\alpha\rangle\,. \tag{86}$$

They are very close to saturating (1). In Fig.1 (inset) we have plotted points in the places where the photon added coherent states lie. As m is a discrete parameter (integer), they do not define a curve on the $(\Delta a)^2$–$(\Delta N)^2$ plane, but they lie on discrete points.

3.3.5 Eigenstates of $a^\dagger a$ + const. $\times a$

According to Heisenberg's method, states that minimize the uncertainty product $(\Delta X)^2(\Delta Y)^2$ for Hermitian X and Y with a constant commutator, are the eigenstates of $X + icY$, where c is some constant [31]. While in Eq. 1 we do not have the product of the uncertainties of two Hermitian observables, this still can give the idea of considering the states $|d,k\rangle$ defined through the eigenvalue equation

$$(a^\dagger a + da)\,|d,k\rangle = k\,|d,k\rangle\,, \tag{87}$$

where d and k are constants. We will study the case in which they are real numbers so that we have a two-parametric family of states.

These states interpolate between Fock and coherent states, as most of the families considered in this work. They reduce to the Fock state $|k\rangle$ when $d = 0$ and to a coherent state when d is very big ($d \to \infty$).

Writing $|d,k\rangle$ as $\sum_k c_n\,|n\rangle$, we obtain

$$\begin{aligned}
&(a^\dagger a + da)\Big(\sum_{j=0}^{\infty} c_j\,|j\rangle\Big) = k\Big(\sum_{j=0}^{\infty} c_j\,|j\rangle\Big),\\
&\sum_{j=0}^{\infty} c_j\Big(j\,|j\rangle + d\sqrt{j}\,|j-1\rangle\Big) = \sum_{j=0}^{\infty} kc_j\,|j\rangle\,,\\
&\sum_{j=0}^{\infty} c_j j\,|j\rangle + \sum_{j=1}^{\infty} dc_j\sqrt{j}\,|j-1\rangle = \sum_{j=0}^{\infty} kc_j\,|j\rangle\,,\\
&\sum_{j=0}^{\infty} c_j j\,|j\rangle + \sum_{j=0}^{\infty} dc_{j+1}\sqrt{j+1}\,|j\rangle = \sum_{j=0}^{\infty} kc_j\,|j\rangle\,,\\
&\Rightarrow \sum_{j=0}^{\infty}\Big(c_j j + dc_{j+1}\sqrt{j+1} - kc_j\Big)\,|j\rangle = 0.
\end{aligned} \tag{88}$$

As $\{|j\rangle\}_{j=0}^{\infty}$ is a basis of the Hilbert space, all the coeficients must be equal to 0, so

$$jc_j + d\sqrt{j+1}c_{j+1} = kc_j. \tag{89}$$

And at the end we get

$$c_{n+1} = \frac{(k-n)}{d\sqrt{n+1}}c_n. \tag{90}$$

Eq. 90 leads to a normalizable wave vector only if k is a non-negative integer. In this case, $c_l = 0$ for all $l > k$. Numerical evidence suggests that states $|d,k\rangle$ are close to saturating Eq. 1, but they are a bit worse than the states given by Eq. 72.

3.3.6 States minimizing $(\Delta N)^2$ for given $(\Delta a)^2$ and $\langle N\rangle$

Let us now look for states that minimize $(\Delta N)^2$ for given $(\Delta a)^2$ and $\langle N\rangle$. For that, we will follow an approach similar to the one presented in Ref. [31]. Since $(\Delta a)^2 = \langle N\rangle - \frac{1}{2}(\langle x\rangle^2 + \langle p\rangle^2)$, this task can be reformulated as looking for the states that minimize $(\Delta N)^2$ for given $\langle x\rangle = x_0$,

$\langle p \rangle = p_0$, and $\langle N \rangle = N_0$. Let us write the state as $|\Phi\rangle = \sum_n d_n |n\rangle$. Hence, we have to look for the minimum of the function

$$
\begin{aligned}
f(\{d_n\}, \lambda_N, \lambda_p, \lambda_x) = \langle N^2 \rangle - N_0^2 \\
+\lambda_x(\langle x \rangle - x_0) + \lambda_p(\langle p \rangle - p_0) + \lambda_N(\langle N \rangle - N_0),
\end{aligned}
\tag{91}
$$

where λ_k are Lagrange multipliers. Note that we do not include explicitly the $\langle \Phi | \Phi \rangle = 1$ condition in the function f. The minimum is given by one of the critical points for which all the partial derivatives are zero. Eq. 91 can be rewritten as

$$
f(\{d_n\}, \lambda_N, \lambda_p, \lambda_x) = \langle O(\lambda_N, \lambda_p, \lambda_x) \rangle_{|\Phi\rangle},
\tag{92}
$$

where O is the Hermitian operator

$$
O(\lambda_N, \lambda_p, \lambda_x) = (a^\dagger a)^2 + \lambda_N a^\dagger a + \Big(\frac{\lambda_x + i\lambda_p}{\sqrt{2}}\Big) a + \Big(\frac{\lambda_x - i\lambda_p}{\sqrt{2}}\Big) a^\dagger.
\tag{93}
$$

We have to look for $\{|\Phi\rangle^{(k)}, \lambda_N^{(k)}, \lambda_x^{(k)}, \lambda_p^{(k)}\}$ that minimize Eq. 92. Hence, the states $|\Phi\rangle^{(k)}$ must be the eigenstate of the operator (93) with the smallest eigenvalue (i.e., they have to be "ground states"). Note that the operator given in Eq. 93, appears as a system Hamiltonian in self-consistent calculations for the Bose-Hubbard model based on the Gutzwiller ansatz [32, 33].

4 Detection of entanglement

In this last section we will generalize Eq. 1 to a two-mode system, and we will derive an entanglement criterion based on it. First of all, we give some standard definitions (see [12]) for product, separable and entangled states:

In a quantum system composed of two subsystems, A and B, quantum states are described by density matrices in the tensor product space $\mathcal{H} = \mathcal{H_A} \otimes \mathcal{H_B}$. A quantum state ρ is said to be a "product state" if there exist states ρ^A for subsystem A and ρ^B for subsystem B such that

$$\rho = \rho^A \otimes \rho^B. \tag{94}$$

The state is called "separable" if there are convex weights p_i and product states ρ^A and ρ^B such that

$$\rho = \sum_i p_i \rho_i^A \otimes \rho_i^B \tag{95}$$

holds. Otherwise the state is called entangled.

Physically, this definition discriminates between three scenarios. First, a product state is an uncorrelated state, where A and B own each a separate state. For non-product states there are two different kinds of correlation. Separable states are classically correlated. This means that for the production of a separable state only local operations and classical communication (LOCC) are necesary. A and B can, by classical communication, share a random number generator that produces the outcomes i with probabilities p_i. For each of the outcomes, they can agree to produce the state $\rho_i^A \otimes \rho_i^B$ locally. By this procedure they produce the state $\rho = \sum_i p_i \rho_i^A \otimes \rho_i^B$. This procedure is not specific for quantum theory, which justifies the notion of "classical" correlations. Otherwise, if a state is entangled, the correlations cannot originate from the classical procedure described above. In this sense entangled states are a typical feature of quantum mechanics.

4.1 Generalization to mixed product states in a two-mode system. Inequality for product states

In the following we will generalize Eq. 1 to a two-mode system, this is, we will derive an uncertainty relation similar to Eq. 1 for a system composed of two harmonic oscillators.

Starting from the inequality

$$\left[(\Delta N)^2 + \frac{1}{4}\right]\left[(\Delta a)^2 + \frac{1}{2}\right] \geq \frac{\langle N \rangle}{4} + \frac{1}{8},$$

wich we know that is true for one harmonic oscillator, we will look for a similar one for product states, that is, we are looking for a relation like

$$\left[(\Delta(N_1 + N_2))^2 + c_1\right]\left[(\Delta(a_1 - a_2))^2 + c_2\right] \geq c_3 \langle N \rangle + c_4, \tag{96}$$

and we will try to find the constants c_i. N_1, N_2, a_1 and a_2 are the number and destruction operators of subsystems 1 and 2. For doing this we will need to use the following property. If $\rho = \rho_1 \otimes \rho_2$, then

$$(\Delta(X_1 \pm X_2))^2_\rho = (\Delta X_1)^2_{\rho_1} + (\Delta X_2)^2_{\rho_2}, \tag{97}$$

where operator X_k acts in subsystem k. Rewriting the left-hand side of equation (97), we get

$$\begin{aligned}
&\Big[(\Delta N_1)^2_{\rho_1}+\frac{1}{4}-\frac{1}{4}\Big]\Big[(\Delta a_1)^2_{\rho_1}+\frac{1}{2}-\frac{1}{2}\Big]+\Big[(\Delta N_1)^2_{\rho_1}\Big]\Big[(\Delta a_2)^2_{\rho_2}+c_2\Big]\\
&+\Big[(\Delta N_2)^2_{\rho_2}+c_1\Big]\Big[(\Delta a_1)^2_{\rho_1}\Big]+\Big[(\Delta N_2)^2_{\rho_2}+c_1\Big]\Big[(\Delta a_2)^2_{\rho_2}+c_2\Big]\\
&\geq c_3\langle N_1\rangle_{\rho_1}+c_3\langle N_2\rangle_{\rho_2}+c_4. \qquad (98)
\end{aligned}$$

Now we will develop further the first term

$$\begin{aligned}
&\Big[(\Delta N_1)^2_{\rho_1}+\frac{1}{4}\Big]\Big[(\Delta a_1)^2_{\rho_1}+\frac{1}{2}\Big]-\frac{1}{2}\Big[(\Delta N_1)^2_{\rho_1}-\frac{1}{4}\Big]\\
&-\frac{1}{4}\Big[(\Delta a_1)^2_{\rho_1}+\frac{1}{2}\Big]+\frac{1}{8}+(\Delta N_1)^2_{\rho_1}\Big[(\Delta a_2)^2_{\rho_2}+c_2\Big]+(\Delta a_1)^2_{\rho_1}\Big[(\Delta N_2)^2_{\rho_2}+c_1\Big]\\
&+\Big[(\Delta N_2)^2_{\rho_2}+c_1\Big]\Big[(\Delta a_2)^2_{\rho_2}+c_2\Big]\geq c_3\langle N_1\rangle_{\rho_1}+c_3\langle N_2\rangle_{\rho_2}+c_4. \qquad (99)
\end{aligned}$$

And this expresion equals

$$\begin{aligned}
&\Big[(\Delta N_1)^2_{\rho_1}+\frac{1}{4}\Big]\Big[(\Delta a_1)^2_{\rho_1}+\frac{1}{2}\Big]-\frac{1}{2}(\Delta N_1)^2_{\rho_1}-\frac{1}{8}-\frac{1}{4}(\Delta a_1)^2_{\rho_1}-\frac{1}{8}+\frac{1}{8}\\
&+(\Delta N_1)^2_{\rho_1}\Big[(\Delta a_2)^2_{\rho_2}+c_2\Big]+(\Delta a_1)^2_{\rho_1}\Big[(\Delta N_2)^2_{\rho_2}+c_1\Big]\\
&+\Big[(\Delta N_2)^2_{\rho_2}+c_1\Big]\Big[(\Delta a_2)^2_{\rho_2}+c_2\Big]\geq c_3\langle N_1\rangle_{\rho_1}+c_3\langle N_2\rangle_{\rho_2}+c_4. \qquad (100)
\end{aligned}$$

We want to identify some terms in this equation, so we will write it in this way

$$\begin{aligned}
&\Big[(\Delta N_1)^2_{\rho_1}+\frac{1}{4}\Big]\Big[(\Delta a_1)^2_{\rho_1}+\frac{1}{2}\Big]-\frac{1}{2}(\Delta N_1)^2_{\rho_1}\\
&-\frac{1}{4}(\Delta a_1)^2_{\rho_1}-\frac{1}{8}+(\Delta N_1)^2_{\rho_1}(\Delta a_2)^2_{\rho_2}+c_2(\Delta N_1)^2_{\rho_1}+(\Delta a_1)^2_{\rho_1}(\Delta N_2)^2_{\rho_2}+c_1(\Delta a_1)^2_{\rho_1}\\
&+\Big[(\Delta N_2)^2_{\rho_2}+c_1\Big]\Big[(\Delta a_2)^2_{\rho_2}+c_2\Big]\geq c_3\langle N_1\rangle_{\rho_1}+c_3\langle N_2\rangle_{\rho_2}+c_4. \qquad (101)
\end{aligned}$$

Now we observe that if we set $c_1=\frac{1}{4}$ and $c_2=\frac{1}{2}$ the four terms with $(\Delta N_1)^2_{\rho_1}$ and $(\Delta a_1)^2_{\rho_1}$ cancell with each other and equation (101) becomes

$$\begin{aligned}
&\Big[(\Delta N_1)^2_{\rho_1}+\frac{1}{4}\Big]\Big[(\Delta a_1)^2_{\rho_1}+\frac{1}{2}\Big]+\Big[(\Delta N_2)^2_{\rho_2}+\frac{1}{4}\Big]\Big[(\Delta a_2)^2_{\rho_2}+\frac{1}{2}\Big]\\
&+(\Delta N_1)^2_{\rho_1}(\Delta a_2)^2_{\rho_2}+(\Delta N_2)^2_{\rho_2}(\Delta a_1)^2_{\rho_1}\geq c_3\langle N_1\rangle_{\rho_1}+c_3\langle N_2\rangle_{\rho_2}+c_4+\frac{1}{8}. \qquad (102)
\end{aligned}$$

But the term $(\Delta N_1)^2_{\rho_1}(\Delta a_2)^2_{\rho_2}+(\Delta N_2)^2_{\rho_2}(\Delta a_1)^2_{\rho_1}$ is always positive or 0

$$(\Delta N_1)^2_{\rho_1}(\Delta a_2)^2_{\rho_2}+(\Delta N_2)^2_{\rho_2}(\Delta a_1)^2_{\rho_1}\geq 0. \qquad (103)$$

Therefore we can put it out and the inequality will still hold. If we put it out we get

$$\Big[(\Delta N_1)^2_{\rho_1}+\frac{1}{4}\Big]\Big[(\Delta a_1)^2_{\rho_1}+\frac{1}{2}\Big]+\Big[(\Delta N_2)^2_{\rho_2}+\frac{1}{4}\Big]\Big[(\Delta a_2)^2_{\rho_2}+\frac{1}{2}\Big]\geq c_3\langle N_1\rangle_{\rho_1}+c_3\langle N_2\rangle_{\rho_2}+c_4+\frac{1}{8}. \qquad (104)$$

But we know that for each of the oscillators the following holds

$$\Big[(\Delta N_i)^2_{\rho_i} + \frac{1}{4}\Big]\Big[(\Delta a_i)^2_{\rho_i} + \frac{1}{2}\Big] \geq \frac{\langle N_i \rangle}{4} + \frac{1}{8}, i = 1, 2. \tag{105}$$

Now, if we set $c_3 = \frac{1}{4}$ and $c_4 = \frac{1}{8}$, inequality (104) is just the result of adding up (105) for each of the modes. So, we have choosen the following values for the c's

$$\begin{aligned} c_1 &= \frac{1}{4}, \\ c_2 &= \frac{1}{2}, \\ c_3 &= \frac{1}{4}, \\ c_4 &= \frac{1}{8}. \end{aligned} \tag{106}$$

And the inequality for the composed system reads

$$\Big[(\Delta(N_1 + N_2))^2_{\rho} + \frac{1}{4}\Big]\Big[(\Delta(a_1 - a_2))^2_{\rho} + \frac{1}{2}\Big] \geq \frac{\langle N \rangle_\rho}{4} + \frac{1}{8}, \tag{107}$$

$$(\rho = \rho_1 \otimes \rho_2).$$

4.2 Inequality for separable states

In the previous section we have shown that Eq. 107 is satisfied by product states. Now we will show that separable states also satisfy the same relation. A separable state is a state of the form

$$\rho = \sum_k p_k \rho_1^{(k)} \otimes \rho_2^{(k)}, \quad p_i \geq 0, \qquad \sum_k p_k = 1. \tag{108}$$

From now on we will write $\rho^{(k)} = \rho_1^{(k)} \otimes \rho_2^{(k)}$.

Now we will show that the previous inequality is also valid for separable states, that is, that for states ρ separables we have

$$\Big[(\Delta(N_1 + N_2))^2_{\rho} + \frac{1}{4}\Big]\Big[(\Delta(a_1 - a_2))^2_{\rho} + \frac{1}{2}\Big] \geq \frac{\langle N \rangle_\rho}{4} + \frac{1}{8}. \tag{109}$$

Making use of the following property

$$\begin{aligned} (\Delta X)^2_{\sum_k p_k \rho^{(k)}} &= \sum_k p_k \Big[(\Delta X)^2_{\rho^{(k)}} + (\langle X \rangle^2_{\rho^{(k)}} - \langle X \rangle^2_{\sum_k p_k \rho^{(k)}})\Big] \\ &= \sum_k p_k \Big[(\Delta X)^2_{\rho^{(k)}} + (\langle X \rangle_{\rho^{(k)}} - \langle X \rangle_\rho)^2\Big]. \end{aligned} \tag{110}$$

we come to

$$\begin{aligned} &\Big[(\Delta(N_1 + N_2))^2_{\rho} + \frac{1}{4}\Big]\Big[(\Delta(a_1 - a_2))^2_{\rho} + \frac{1}{2}\Big] \\ &= \Big[\sum_k p_k \Big[(\Delta(N_1 + N_2))^2_{\rho^{(k)}} + (\langle N_1 + N_2 \rangle_{\rho^{(k)}} - \langle N_1 + N_2 \rangle_\rho)^2\Big] + \frac{1}{4}\Big] \\ &\cdot \Big[\sum_k p_k \Big[(\Delta(a_1 - a_2))^2_{\rho^{(k)}} + (\langle a_1 - a_2 \rangle_{\rho^{(k)}} - \langle a_1 - a_2 \rangle_\rho)^2\Big] + \frac{1}{2}\Big]. \end{aligned} \tag{111}$$

Now we use another useful relation, which is a particular case of the Cauchy-Schwarz inequallity

$$\Big(\sum_k p_k A_k\Big)\Big(\sum_l p_l B_l\Big) \geq \Big(\sum_k p_k \sqrt{A_k B_k}\Big)^2, \tag{112}$$

and we can write

$$\begin{aligned}
&\Big[\sum_k p_k \Big[(\Delta(N_1+N_2))^2_{\rho^{(k)}} + (\langle N_1+N_2\rangle_{\rho^{(k)}} - \langle N_1+N_2\rangle_\rho)^2\Big] + \frac{1}{4}\Big] \\
&\cdot\Big[\sum_k p_k \Big[(\Delta(a_1-a_2))^2_{\rho^{(k)}} + (\langle a_1-a_2\rangle_{\rho^{(k)}} - \langle a_1-a_2\rangle_\rho)^2\Big] + \frac{1}{2}\Big] \\
&\geq \Big(\sum_k p_k \sqrt{\Big[(\Delta N_1+N_2)^2_{\rho^{(k)}} + \frac{1}{4} + (\langle N_1+N_2\rangle_{\rho^{(k)}} - \langle N_1+N_2\rangle_\rho)^2\Big]} \\
&\cdot\sqrt{\Big[(\Delta a_1-a_2)^2_{\rho^{(k)}} + \frac{1}{2} + (\langle a_1-a_2\rangle_{\rho^{(k)}} - \langle a_1-a_2\rangle_\rho)^2\Big]}\Big)^2 \\
&\geq \Big(\sum_k p_k \sqrt{\Big[(\Delta N_1+N_2)^2_{\rho^{(k)}} + \frac{1}{4}\Big]\Big[(\Delta a_1-a_2)^2_{\rho^{(k)}} + \frac{1}{2}\Big] + \frac{1}{2}\Big(\langle N_1+N_2\rangle_{\rho^{(k)}} - \langle N_1+N_2\rangle_\rho\Big)^2}\Big)^2.
\end{aligned}$$

In the last step all the terms that we didn't write are greater than or equal to 0. But now we know that for each state of the form $\rho^{(k)} = \rho_1^{(k)} \otimes \rho_2^{(k)}$ the following holds:

$$\Big[(\Delta N_1+N_2)^2_{\rho^{(k)}} + \frac{1}{4}\Big]\Big[(\Delta a_1-a_2)^2_{\rho^{(k)}} + \frac{1}{2}\Big] \geq \frac{N_k}{4} + \frac{1}{8}, \tag{113}$$

where $N_k = \langle N_1+N_2\rangle_{\rho^{(k)}}$. Taking this into account, we can write

$$\begin{aligned}
&\Big(\sum_k p_k \sqrt{\Big[(\Delta N_1+N_2)^2_{\rho^{(k)}} + \frac{1}{4}\Big]\Big[(\Delta a_1-a_2)^2_{\rho^{(k)}} + \frac{1}{2}\Big] + \frac{1}{2}\Big(\langle N_1+N_2\rangle_{\rho^{(k)}} - \langle N_1+N_2\rangle_\rho\Big)^2}\Big)^2 \\
&\geq \Big(\sum_k p_k \sqrt{\Big[\frac{N_k}{4} + \frac{1}{8}\Big] + \frac{1}{2}(\langle N_1+N_2\rangle_{\rho^{(k)}} - \langle N_1+N_2\rangle_\rho)^2}\Big)^2.
\end{aligned} \tag{114}$$

So, if we write $N = \langle N_1+N_2\rangle_\rho$, we have

$$\Big[(\Delta N_1+N_2)^2_\rho + \frac{1}{4}\Big]\Big[(\Delta a_1-a_2)^2_\rho + \frac{1}{2}\Big] \geq \Big(\sum_k p_k \sqrt{\Big[\frac{N_k}{4} + \frac{1}{8}\Big] + \frac{1}{2}(N_k - N)^2}\Big)^2. \tag{115}$$

Now we will show that the right-hand side of inequality (115) gets its minnimum when $N_k = N \ \forall k$, so that we will have shown that the smallest value that $[(\Delta N_1+N_2)^2_\rho+\frac{1}{4}][(\Delta a_1-a_2)^2_\rho+\frac{1}{2}]$ can take is reached when all the N_k's are equal to each other. To show this we have to minimize the right-hand side constrained to the condition $\sum_m N_m = N$, and show that the minimmum is reached when $N_k = N$. Instead of minimizing the right-hand side we will minimize its (positive) square root, because this way the algebra will be easier and the minimmum of both functions will be reached in the same place, because $\sqrt{x}$ is an increasing function of x. For doing this we will use Lagrange multipliers. We define the auxiliary function:

$$g(N_k,\lambda) = \Big[\sum_k p_k \sqrt{\frac{N_k}{4} + \frac{1}{8} + \frac{1}{2}(N_k - N)^2}\Big] - \lambda\Big(\sum_k p_k N_k - N\Big), \tag{116}$$

where λ is a Lagrange multiplier. Now we have to perform the derivatives with respect to all the N_m's and with respect to λ, equal them to 0 and solve the resulting system of equations.

$$\frac{\partial g}{\partial N_m} = 0 \quad \forall m,$$
$$\frac{\partial g}{\partial \lambda} = 0.$$

The equations that we get are the following

$$\frac{\partial g}{\partial \lambda} = 0 \Rightarrow \sum_j p_j N_j = N,$$
$$\frac{\partial g}{\partial N_m} = 0 \Rightarrow \lambda = \frac{\frac{1}{4} + N_m - N}{2\sqrt{\frac{N_m}{4} + \frac{1}{8} + \frac{1}{2}(N_m - N)^2}} \quad \forall m. \tag{117}$$

If λ defined this way was one-to-one as a function of N_m, then Eq. 117 would imply $N_m = N_n, m \neq n$. Let's show that this is the case: A sufficient condition for λ to be one-to-one is that $\frac{\partial \lambda}{\partial N_m} \neq 0$ in the region where the N_m's are defined ($N_m \geq 0$). If we perform the derivative we obtain

$$\frac{\partial \lambda}{\partial N_m} = \frac{2\sqrt{\frac{N_m}{4} + \frac{1}{8} + \frac{1}{2}(N_m - N)^2} - (\frac{1}{4} + N_m - N)\frac{\frac{1}{4} + N_m - N}{\sqrt{\frac{N_m}{4} + \frac{1}{8} + \frac{1}{2}(N_m - N)^2}}}{4\left[\frac{N_m}{4} + \frac{1}{8} + \frac{1}{2}(N_m - N)^2\right]}. \tag{118}$$

From Eq. 118 we get

$$\frac{\partial \lambda}{\partial N_m} = 0 \Rightarrow 2\sqrt{\frac{N_m}{4} + \frac{1}{8} + \frac{1}{2}(N_m - N)^2} = (\frac{1}{4} + N_m - N)\frac{\frac{1}{4} + N_m - N}{\sqrt{\frac{N_m}{4} + \frac{1}{8} + \frac{1}{2}(N_m - N)^2}}. \tag{119}$$

This equality holds only when $N = -\frac{3}{8}$. But N must be a positive number, so, in the interval we are interested in ($N \geq 0$), we will always have $\partial\lambda/\partial N_m \neq 0$. So we have shown that λ is a one-to-one function of N_m, and therefore the minimmum of $\left(\sum_k p_k \sqrt{\left[\frac{N_k}{4} + \frac{1}{8}\right] + \frac{1}{2}(N_k - N)^2}\right)^2$ is reached when $N_m = N, \quad \forall m$. If $N_m = N$, and taking into account that $\sum_k p_k = 1$, we have

$$\left(\sum_k p_k \sqrt{\frac{N_k}{4} + \frac{1}{8} + \frac{1}{2}(N_k - N)^2}\right)^2 = \frac{N}{4} + \frac{1}{8} = \frac{\sum_k p_k N_k}{4} + \frac{1}{8}$$
$$= \frac{\sum_k p_k \langle N_1 + N_2 \rangle_{\rho^{(k)}}}{4} + \frac{1}{8} = \frac{\langle N_1 + N_2 \rangle_\rho}{4} + \frac{1}{8}. \tag{120}$$

Therefore, we have shown that

$$\left[(\Delta(N_1 + N_2))^2_\rho + \frac{1}{4}\right]\left[(\Delta(a_1 - a_2))^2_\rho + \frac{1}{2}\right] \geq \frac{\langle N_1 + N_2 \rangle_\rho}{4} + \frac{1}{8} \tag{121}$$

with

$$\rho = \sum_k p_k \rho_1^{(k)} \otimes \rho_2^{(k)}. \tag{122}$$

That is, the inequality holds for any non-entangled state.

5 Discussion

Let us discuss the use of quantum states that minimize $(\Delta N)^2$ for given $\langle N\rangle$ and $(\Delta a)^2$. They present a trade-off between two requirements: the smallest possible variance of a randomly chosen quadrature component and the smallest possible particle number variance. In a sense, they are similar to states minimizing $(\Delta N)^2$ for given $(\Delta\phi)^2$. The latter present a trade-off between smallest possible variance for phase measurements and particle number measurements.

Clearly, the right hand side of Eq. 1 is not a constant, but it is a quantity that can be controlled well in many systems. Moreover, note that the measurement of $(\Delta a)^2$ does not require measuring variances of x and p if we use $(\Delta a)^2 = \langle N\rangle - |\langle a\rangle|^2 = \langle N\rangle - \frac{1}{2}(\langle x\rangle^2 + \langle p\rangle^2)$.

A single trapped ion seems to be a good candidate for testing our inequalities and realizing quantum states that saturate it [34, 35, 36]. For this system, x and p are the physical position and momentum coordinates, and N determines the energy of the ion.

The uncertainty relation Eq. 1 can also be experimentally verified in a single mode electromagnetic field. The two orthogonal quadrature components can be measured for example with homodyne detection [37]. It does not influence the result which two orthogonal components we choose to measure.

Bose-Einstein condensates of alkali atoms seem to be also a possible candidate for experiments [38, 35]. It is usual to talk about number squeezing in multi-well Bose-Einstein condensates in the sense that increasing the barrier hight between the wells decreases the number fluctuation within the wells [39]. Here, one has to note that for cold atoms the particle number is conserved. Because of that, for a single bosonic mode of cold atoms, superpositions of states with different particle numbers are not allowed. For this reason, for pure states $(\Delta N)^2 = 0$. It is possible to mix states with different particle numbers making $(\Delta N)^2 > 0$. However, even for such states $\langle a\rangle = 0$ and $(\Delta a)^2 = \langle N\rangle$, which makes our inequalities trivial in such systems.

While we cannot create particles in a single mode, we can move particles from one mode to another one. Thus, it is instructive to consider two-mode systems of cold atoms. The two modes can be realized with atoms in a double well or with a single Bose-Einstein condensate of two state atoms. Let us denote the annihilation operators of the two modes by a_1 and a_2, respectively. The corresponding particle numbers are N_1 and N_2. If $\langle N_1\rangle \ll \langle N_2\rangle$ and $(\Delta N_2)^2 \ll \langle N_2\rangle^2$ then with the substitution

$$\begin{aligned} a &\rightarrow \frac{a_1 a_2^\dagger}{\sqrt{\langle N_2\rangle}}, \\ N &\rightarrow N_1, \end{aligned} \tag{123}$$

the uncertainty relations (1) and (64) can be tested.

It is also possible to find relations that do not require such approximations. For the two-mode system, inequalities similar to Eq. 64 and Eq. 1 can be found using the Schwinger representations of the angular momentum operators

$$J_l = \frac{1}{2}\begin{pmatrix} a_1^\dagger \\ a_2^\dagger \end{pmatrix}^T \sigma_l \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}, \tag{124}$$

for $l = x, y, z$ where σ_l are the Pauli spin matrices. Let us define an operator that is an analogue of a in the two mode-system as

$$\widetilde{a} = J_x - iJ_y \equiv a_1 a_2^\dagger. \tag{125}$$

With this definition, we have

$$\begin{aligned} (|\Delta\widetilde{a}|^2) &= \frac{1}{2}[(\Delta\widetilde{a})^2 + (\Delta\widetilde{a}^\dagger)^2] = (\Delta J_x)^2 + (\Delta J_y)^2, \\ |\langle\widetilde{a}\rangle|^2 &= \langle J_x\rangle^2 + \langle J_y\rangle^2. \end{aligned} \tag{126}$$

Using Eq. (126) and the Heisenberg uncertainty relation $(\Delta J_k)^2(\Delta J_l)^2 \geq \frac{1}{4}|\langle J_m\rangle|^2$, we obtain the analogue of Eq. 64

$$(\Delta N_1)^2(|\Delta\widetilde{a}|^2) \geq \frac{1}{4}|\langle\widetilde{a}\rangle|^2. \tag{127}$$

Adding $\frac{1}{4}(|\Delta\widetilde{a}|^2)$ to both sides of Eq. (127) and using

$$J_x^2 + J_y^2 = \frac{1}{2}(N_1 + 1)(N_2 + 1) - \frac{1}{2}, \tag{128}$$

we obtain an analogue of Eq. 1

$$\left[(\Delta N_1)^2 + \frac{1}{4}\right](|\Delta\widetilde{a}|^2) \geq \frac{1}{8}\left\langle (N_1 + 1)(N_2 + 1)\right\rangle - \frac{1}{8}. \tag{129}$$

As mentioned above, number squeezing with Bose-Einstein condensates in a double-well can occur if the barrier between the wells increases [39]. Eq. 1 bounds the number variance of a well in such systems [40].

Finally, in statistical physics of bosonic systems, $\langle\Psi(x)\rangle$, i.e., the expectation value of the field operator, plays the role of the order parameter. In this context, our findings present a quantitative relationship between the variance of the field operator and the variance of the particle density $\Psi(x)^\dagger\Psi(x)$.

6 Summary

In this work, we have studied the possibility of using the annihilation operator as an alternative to the phase operator, in the context of a quantum one-dimensional harmonic oscilator. The use of a phase operator is difficult because, due to the Pauli theorem, it is not possible to construct a Hermitian phase operator, so all the approaches proposed by several authors must take certain compromises. We have briefly revisited some of those approaches. To avoid defining a non-well behavied phase operator we propose to use the annihilation operator, and we construct an uncertainty relation for N and a, which plays the role of a relation for N and ϕ. We argue that states close to saturate this unequallity are similar (or close) to Minimmum Uncertainty States presented by J.A. Vaccaro and D.T. Pegg in [9]. Several families of states close to saturate it are presented, they are plotted in the $(\Delta a)^2 - (\Delta N)^2$ plane and they are shown to be a trade-off between coherent states and number states, that is, you can travel continuously from a coherent state to a Fock state by changing the parameter of the family. Then this inequality is extended to a two mode-system inequality and it is used to detect quantum entanglement. Finally several quantum optical systems are proposed in which our inequality could be tested. In future, it would be interesting to improve the bounds in the uncertainty relations presented in this work.

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Article

Number-operator-annihilation-operator uncertainty as an alternative for the number-phase uncertainty relation

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