Uncertainty relations with the variance and the quantum Fisher information

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Outline

How can we improve uncertainty relations?

- There are many approaches to improve uncertainty relations.

- We show a method that replaces the variance with the quantum Fisher information in some well known uncertainty relations.

- We use convex/concave roofs over the decompostions of the density matrix.

Outline

Quantum metrology

- Fundamental task in metrology

$$\varrho \quad \longrightarrow \quad U(\theta) = \exp(-iA\theta) \quad \longrightarrow \quad \varrho_\theta$$
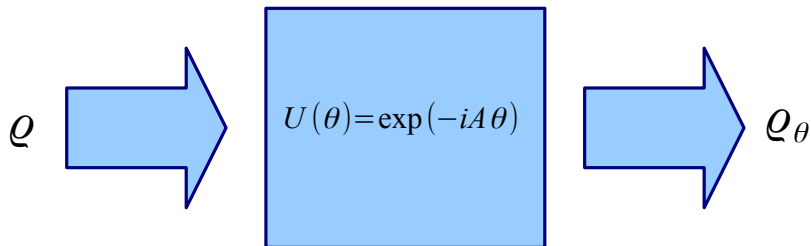
- We have to estimate θ in the dynamics

$$U = \exp(-iA\theta).$$

The quantum Fisher information

- Cramér-Rao bound on the precision of parameter estimation

$$(\Delta\theta)^2 \geq \frac{1}{mF_Q[\varrho, A]},$$

where $F_Q[\varrho, A]$ is the quantum Fisher information, and m is the number of independent repetitions.

- The quantum Fisher information is

$$F_Q[\varrho, A] = 2 \sum_{k,l} \frac{(\lambda_k - \lambda_l)^2}{\lambda_k + \lambda_l} |\langle k|A|l \rangle|^2,$$

where $\varrho = \sum_k \lambda_k |k\rangle\langle k|$.

Most important characteristics used for estimation

The quantum Fisher information is the convex roof of the variance

$$F_Q[\varrho, A] = 4 \min_{\{p_k, |\psi_k\rangle\}} \sum_k p_k (\Delta A)^2_{\psi_k},$$

where

$$\varrho = \sum_k p_k |\psi_k\rangle\langle\psi_k|.$$

[GT, D. Petz, Phys. Rev. A 87, 032324 (2013); S. Yu, arXiv1302.5311 (2013); GT, I. Apellaniz, J. Phys. A: Math. Theor. 47, 424006 (2014)]

- Extended convexity for non-unitary dynamics.
  [S. Alipour, A. T. Rezakhani, Phys. Rev. A 91, 042104 (2015).]

- Convex roof over purifications.
  [R. Demkowicz-Dobrzański, J. Kołodyński, M. Guţă, Nature Comm. 2012.]

Similar relation for the variance

The quantum Fisher information is the convex roof of the variance

$$(\Delta A)^2{}_\varrho = \max_{\{p_k, |\psi_k\rangle\}} \sum_k p_k (\Delta A)^2{}_{\psi_k},$$

where

$$\varrho = \sum_k p_k |\psi_k\rangle\langle\psi_k|.$$

[GT, D. Petz, Phys. Rev. A 87, 032324 (2013);
GT, I. Apellaniz, J. Phys. A: Math. Theor. 47, 424006 (2014)]

A single relation for the QFI and the variance

The previous statements can be concisely reformulated as follows. For any decomposition $\{p_k, |\psi_k\rangle\}$ of the density matrix ϱ we have

$$\frac{1}{4} F_Q[\varrho, A] \leq \sum_k p_k (\Delta A)^2{}_{\psi_k} \leq (\Delta A)^2{}_{\varrho},$$

where the upper and the lower bounds are both tight.

Outline

Robertson-Schrödinger inequality

The Robertson-Schrödinger inequality is defined as

$$(\Delta A)^2_{\varrho}(\Delta B)^2_{\varrho} \geq \tfrac{1}{4}|L_{\varrho}|^2,$$

where the lower bound is given by

$$L_{\varrho} = \sqrt{|\langle\{A, B\}\rangle_{\varrho} - 2\langle A\rangle_{\varrho}\langle B\rangle_{\varrho}|^2 + |\langle C\rangle_{\varrho}|^2},$$

$\{A, B\} = AB + BA$ is the anticommutator, and we used the definition

$$C = i[A, B].$$

Important: L_{ϱ} is neither convex nor concave in ϱ.

Heisenberg uncertainty

The Heisenberg inequality is defined as

$$(\Delta A)^2{}_\varrho (\Delta B)^2{}_\varrho \geq \tfrac{1}{4} |\langle C \rangle_\varrho|^2,$$

where we used the definition

$$C = i[A, B].$$

The two inequalities together

We have two inequalities

$$(\Delta A)^2{}_\varrho (\Delta B)^2{}_\varrho \geq \tfrac{1}{4}|L_\varrho|^2 \geq \tfrac{1}{4}|\langle C\rangle_\varrho|^2.$$

The Heisenberg uncertainty can be saturated only if

$$|L_\varrho| = |\langle C\rangle_\varrho|.$$

Outline

New inequality from decompositions

- Consider a decomposition to mixed states

$$\varrho = \sum_k p_k \varrho_k.$$

- For such a decomposition, for all ϱ_k the Robertson-Schrödinger inequality holds

$$(\Delta A)^2_{\varrho_k} (\Delta B)^2_{\varrho_k} \geq \tfrac{1}{4} |L_{\varrho_k}|^2.$$

- Let us consider the inequality

$$\left(\sum_k p_k a_k \right) \left(\sum_k p_k b_k \right) \geq \left(\sum_k p_k \sqrt{a_k b_k} \right)^2,$$

where $a_k, b_k \geq 0$.

Uncertainty with the variance and the QFI

- Hence, we arrive at

$$\left[\sum_k p_k (\Delta A)^2{}_{\varrho_k}\right]\left[\sum_k p_k (\Delta B)^2{}_{\varrho_k}\right] \geq \tfrac{1}{4}\left[\sum_k p_k L_{\varrho_k}\right]^2.$$

- We can choose the decomposition such that

$$\sum_k p_k (\Delta B)^2{}_{\varrho_k} = F_Q[\varrho, B]/4.$$

- Due to the concavity of the variance we also know that

$$\sum_k p_k (\Delta A)^2{}_{\varrho_k} \leq (\Delta A)^2.$$

- Hence, it follows that

$$(\Delta A)^2{}_{\varrho} F_Q[\varrho, B] \geq \left(\sum_k p_k L_{\varrho_k}\right)^2.$$

- In order to use the previous inequality, we need to know the decomposition $\{p_k, \varrho_k\}$ that minimizes $\sum_k p_k (\Delta B)^2{}_{\varrho_k}$.

Uncertainty with the variance and the QFI II

- We can have a inequality where we do not need to know that decomposition

$$(\Delta A)^2{}_\varrho F_Q[\varrho, B] \geq \left(\min_{\{p_k, \varrho_k\}} \sum_k p_k L_{\varrho_k} \right)^2.$$

- On the right-hand side, the bound is defined based on a convex roof.

- It can be shown that we can move to pure state decompositions.

- We know that

$$L_{\psi_k} \geq |\langle C \rangle_{\psi_k}|$$

holds.

Uncertainty with the variance and the QFI II

- Then, we can obtain the inequality

$$(\Delta A)^2{}_\varrho F_Q[\varrho, B] \geq \left(\min_{\{p_k, |\psi_k\rangle\}} \sum_k p_k |\langle C \rangle_{\psi_k}| \right)^2,$$

- Using

$$\sum_k p_k |\langle C \rangle_{\psi_k}| \geq \left| \sum_k p_k \langle C \rangle_{\psi_k} \right| \equiv |\langle C \rangle_\varrho|,$$

we arrive at the improved Heisenberg-Robertson uncertainty

$$(\Delta A)^2{}_\varrho F_Q[\varrho, B] \geq |\langle C \rangle_\varrho|^2.$$

Uncertainty with the variance and the QFI III

- The Heisenberg uncertainty

$$(\Delta A)^2_{\varrho}(\Delta B)^2_{\varrho} \geq \tfrac{1}{4}|\langle i[A, B]\rangle_{\varrho}|^2.$$

- The improved Heisenberg uncertinty

$$(\Delta A)^2_{\varrho}F_Q[\varrho, B] \geq |\langle i[A, B]\rangle_{\varrho}|^2.$$

- It has been derived originally with a different method in
  F. Fröwis, R. Schmied, and N. Gisin, Phys. Rev. A 92, 012102 (2015).

Conditions for saturation

- Conditions for saturating the relation with the simple bound

$$(\Delta A)^2_{\varrho} F_Q[\varrho, B] \geq \left(\min_{\{p_k, \varrho_k\}} \sum_k p_k L_{\varrho_k} \right)^2 \geq |\langle C \rangle_{\varrho}|^2.$$

- We have to have equality on the right-hand side.

- The optimal decomposotion can be made with pure components Ψ_k. Then, for all k, l we must have

$$\tfrac{1}{2} \langle \{A, B\} \rangle_{\psi_k} - \langle A \rangle_{\psi_k} \langle B \rangle_{\psi_k} = 0,$$

$$\begin{aligned}
(\Delta A)^2_{\psi_k} &= (\Delta A)^2_{\psi_l}, \\
(\Delta B)^2_{\psi_k} &= (\Delta B)^2_{\psi_l}, \\
|\langle C \rangle_{\psi_k}| &= |\langle C \rangle_{\varrho}|,
\end{aligned}$$

etc.

Outline

Cramér-Rao bound

- Error propagation formula

$$(\Delta\theta)^2_A = \frac{(\Delta A)^2}{|\partial_\theta \langle A \rangle|^2} = \frac{(\Delta A)^2}{|\langle C \rangle|^2}.$$

- Then the precision of the estimation is bounded as

$$(\Delta\theta)^2 \geq \frac{1}{m} \min_A (\Delta\theta)^2{}_A,$$

where m is the number of independent repetitions.

- Cramér-Rao bound based on a convex roof

$$(\Delta\theta)^2 \geq \frac{1}{m} \times \frac{1}{\underbrace{4 \min_{\{p_k, |\psi_k\rangle\}} \left[\sum_k p_k (\Delta B)^2{}_{\psi_k} \right]}_{F_Q[\varrho, B]}}.$$

Outline

Uncertainty relation based on a concave roof

- For any decomposition $\{p_k, \varrho_k\}$ we have

$$(\Delta A)^2 (\Delta B)^2 \geq \frac{1}{4} \left(\sum_k p_k L_{\varrho_k} \right)^2,$$

where

$$L_\varrho = \sqrt{|\langle \{A, B\} \rangle_\varrho - 2 \langle A \rangle_\varrho \langle B \rangle_\varrho|^2 + |\langle C \rangle_\varrho|^2}.$$

- We can even take a concave roof on the right-hand side

$$(\Delta A)^2{}_\varrho (\Delta B)^2{}_\varrho \geq \frac{1}{4} \left(\max_{\{p_k, \varrho_k\}} \sum_k p_k L_{\varrho_k} \right)^2.$$

- We prove that for qubits the above inequality is saturated for all states.

Any decomposition leads to a valid bound

- A simple inequality that is valid

$$(\Delta A)^2{}_\varrho (\Delta B)^2{}_\varrho \geq \frac{1}{4} \left(\sum_k \lambda_k L_{|k\rangle} \right)^2,$$

  if we have an eigendecompostion

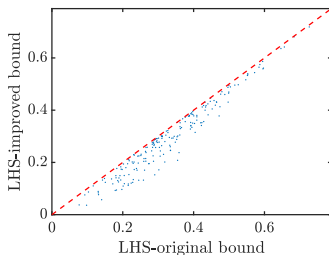$$\varrho = \sum_k \lambda_k |k\rangle \langle k|.$$

- We can even look for concave roof numerically.
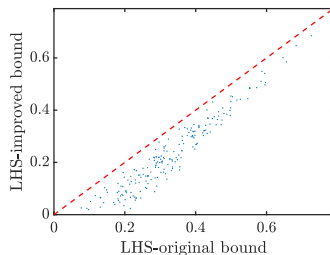
Numerical example

- For $d = 3$

$$(\Delta J_x)^2{}_\varrho (\Delta J_y)^2{}_\varrho \geq \frac{1}{4} \left(\max_{\{p_k, \varrho_k\}} \sum_k p_k L_{\varrho_k} \right)^2 .$$

- Eigenvalues J_x and J_y are $-1, 0, +1$.

Using the eigendecomposition   Numerical search

Outline

Uncertainty relations with a variance and the QFI

- Similar ideas work even for a sum of two variances. For example, for a continuous variable system

$$(\Delta x)^2 + (\Delta p)^2 \geq 1$$

holds, where x and p are the position and momentum operators.

- Hence, for any decompositions of the density matrix it follows that

$$\sum_k p_k (\Delta x)^2{}_{\psi_k} + \sum_k p_k (\Delta p)^2{}_{\psi_k} \geq 1.$$

- For p we choose the decomposition that leads to the minimal value for the average variance, i.e., the QFI over four.

- Then, since $\sum_k p_k (\Delta x)^2{}_{\psi_k} \leq (\Delta x)^2$ holds, it follows that

$$(\Delta x)^2 + \tfrac{1}{4} F_Q[\varrho, p] \geq 1.$$

Uncertainty relations with two variances and the QFI

- Let us start from the relation for pure states

$$(\Delta J_x)^2 + (\Delta J_y)^2 + (\Delta J_z)^2 \geq j,$$

where J_l are the spin components fulfilling

$$J_x^2 + J_y^2 + J_z^2 = j(j+1)\mathbb{1}.$$

- Based on similar ideas we arriving at

$$(\Delta J_x)^2 + (\Delta J_y)^2 + \tfrac{1}{4}F_Q[\varrho, J_z] \geq j.$$

- See parallel publication in

  S.-H. Chiew and M. Gessner, Phys. Rev. Research 4, 013076 (2022).

Uncertainty relations with two variances and the QFI II

- For a spin-j particle, the following inequality bounds from below the metrological usefulness of the state

$$F_Q[\varrho, J_z] \geq 4j - 4(\Delta J_x)^2 - 4(\Delta J_y)^2 =: B_{FQ}.$$

- For instance, this can be used in BEC of two-state atoms to bound $F_Q[\varrho, J_z]$ with the variances.

Outline

Simple observation to prove further relations

- Let us consider a relation

$$\underbrace{(\Delta A)^2_{\varrho}}_{\text{variance}} \geq \underbrace{g(\varrho)}_{\text{convex in } \varrho} \ ,$$

  which is true for pure states.

- If $g(\varrho)$ is convex in density matrices, then

$$\frac{1}{4} F_Q[\varrho, A] \geq g(\varrho)$$

  holds for mixed states.

- *Proof.* $\frac{1}{4} F_Q[\varrho, A]$ is given as a convex roof of the variance.

- It is the largest convex function that equals $(\Delta A)^2_{\varrho}$ for all pure states. ∎

Repeating the proof for the two variances and the QFI

- We rewrite the relation with three variances as

$$\underbrace{(\Delta J_x)^2}_{\text{variance}} \geq \underbrace{j - (\Delta J_y)^2 - (\Delta J_z)^2}_{\text{convex in } \varrho},$$

- The right-hand side is convex in ϱ and the left-hand side is a variance.

- Hence,

$$\frac{1}{4} F_Q[\varrho, J_z] \geq j - (\Delta J_x)^2 - (\Delta J_y)^2.$$

Extreme spin squeezing

- For a particle with spin-j

$$\underbrace{(\Delta J_x)^2}_{\text{variance}} \geq \underbrace{jF_j(\langle J_z\rangle/j)}_{\text{convex in } \varrho}$$

holds, where $F_j(X)$ is a convex function defined as

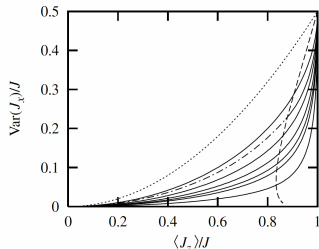$$F_j(X) = \min_{\varrho:\langle J_z\rangle=Xj} \frac{(\Delta J_x)^2}{j}.$$

FIG. 1. Maximal squeezing for different values of J. The curves starting at the origin represent the minimum obtainable variance as a function of the mean spin. Starting from above, the curves represent $J = 1/2, 1, 3/2, 2, 3, 4, 5,$ and 10. The dotted curve for $J = 1/2$ is the limit identified in Ref. [11]. The solid

Application for extreme spin squeezing

- The metrological usefulness of a state is bounded with the spin-length as

$$F_Q[\varrho, J_x] \geq 4j F_j(\langle J_z \rangle / j).$$

- *Proof.* For the components of the angular momentum

$$(\Delta J_x)^2 \geq j F_j(\langle J_z \rangle / j)$$

holds. Then, it follows

$$\frac{1}{4} F_Q[\varrho, J_x] \geq j F_j(\langle J_z \rangle / j).$$

■

Application for extreme spin squeezing II

- It is clear that if $\langle J_z \rangle > 0$ then $F_j(\langle J_z \rangle / j) > 0$. Hence, it follows that $F_Q[\varrho, J_x] > 0$.

- Thus, if the z-component of the angular momentum has a non-zero expectation value then the state can be used for metrology with the Hamiltonian J_x.

Outline

CV systems

- Bosonic mode, canonical x and p operators. For coherent states, $|\alpha\rangle$

$$(\Delta x)^2 = (\Delta p)^2 = \frac{1}{2}$$

holds. For mixtures of coherent states

$$\varrho_{\mathrm{mc}} = \sum_k p_k |\alpha_k\rangle\langle\alpha_k|$$

due to the concavity of the variance and the convexity of the QFI

$$(\Delta x)^2, (\Delta p)^2 \geq \frac{1}{2}, \quad F_Q[x, \varrho], F_Q[p, \varrho] \leq 2.$$

CV systems II

- For a mixture of products of coherent states $\alpha_k^{(l)}$ of the form

$$\varrho_{\text{sepc}} = \sum_k p_k |\alpha_k^{(1)}\rangle\langle\alpha_k^{(1)}| \otimes |\alpha_k^{(2)}\rangle\langle\alpha_k^{(2)}|$$

  for the the collective quantites

$$[\Delta(x_1 \pm x_2)]^2 \geq 1; \quad [\Delta(p_1 \pm p_2)]^2 \geq 1.$$

  Moreover,

$$F_Q[\varrho, p_1 \pm p_2] \leq 4; \quad F_Q[\varrho, x_1 \pm x_2] \leq 4.$$

  For such states the multi-variable Glauber-Sudarshan P function is non-negative.

CV systems III

- Consider entanglement detection in two-mode systems with uncertainty relations.

- A well-known entanglement criterion is

$$[\Delta(x_1 + x_2)]^2 + [\Delta(p_1 - p_2)]^2 \geq 2.$$

If a quantum state violates it, then it is entangled.

L.-M. Duan, G. Giedke, J. I. Cirac, and P. Zoller, Phys. Rev. Lett. 84, 2722 (2000);
R. Simon, Phys. Rev. Lett. 84, 2726 (2000).

CV systems IV

- For a two-mode state, the following uncertainty relation holds

$$[\Delta(x_1 + x_2)]^2 + [\Delta(p_1 - p_2)]^2 \geq \\ 4/F_Q[\varrho, p_1 + p_2] + 4/F_Q[\varrho, x_1 - x_2].$$

As a consequence, states violating the entanglement condition are metrologically more useful than states that are the mixtures of products of coherent states.

- *Proof.* We start from the relations

$$[\Delta(x_1 + x_2)]^2 F_Q[\varrho, p_1 + p_2] \geq 4,$$
$$[\Delta(p_1 - p_2)]^2 F_Q[\varrho, x_1 - x_2] \geq 4.$$

- Then, in both inequalities we divide by the term containing the QFI. Finally, we sum the two resulting inequalities.

CV systems V

- The violation of the entanglement criterion given implies the violation of one of the inequalities for the QFI.

- Thus, violation of the uncertainty relation-based entanglement condition also means that the state has larger metrological usefulness than mixtures of products of coherent states. ∎

- We did not prove that violating the entanglement condition leads to larger metrological usefulness than that of separable states in general.

- Even for pure product states $F_Q[\varrho, x_1 \pm x_2]$ or $F_Q[\varrho, p_1 \pm p_2]$ can be arbitrarily large for two bosonic modes.

Summary

- We showed how to derive new uncertainty relations with the variance and the quantum Fisher information based on simple convexity arguments.

See:

Géza Tóth and Florian Fröwis,

Uncertainty relations with the variance and the quantum Fisher information based on convex decompositions of density matrices,

Phys. Rev. Research 4, 013075 (2022).

THANK YOU FOR YOUR ATTENTION!

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