Entanglement theory (entangled/not entangled) (Lecture of the Quantum Information class of the Master in Quantum Science and Technology)

Géza Tóth

Theoretical Physics, University of the Basque Country (UPV/EHU), Bilbao, Spain
Donostia International Physics Center (DIPC), San Sebastián, Spain
IKERBASQUE, Basque Foundation for Science, Bilbao, Spain
Wigner Research Centre for Physics, Budapest, Hungary

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Outline

1. **Entanglement theory (entangled/not entangled)**
   - Motivation
   - A. Bipartite case
     - Pure states
     - Mixed states

Entanglement detection

- We would like to distinguish entangled states from separable states.

- The problem is very difficult, there are no general methods.

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1. **Entanglement theory (entangled/not entangled)**
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Separability and entanglement of pure states

- if the pure state is a product state then it is separable. If it is not a product state then it is entangled.

- If the reduced state
$$\varrho_1 = \mathrm{Tr}_2(|\Psi\rangle\langle\Psi|)$$
is pure then the state is a product state, otherwise it is entangled. In other words, if
$$\mathrm{Tr}\{[\mathrm{Tr}_2(|\Psi\rangle\langle\Psi|)]^2\} = 1$$
then the state is a product state.

Separability and entanglement for pure states I

- A quantum state is called separable if it can be written as a convex sum of product states as

$$\varrho = \sum_k p_k \varrho_k^{(1)} \otimes \varrho_k^{(2)},$$

where p_k form a probability distribution ($p_k > 0, \sum_k p_k = 1$), and $\varrho_k^{(n)}$ are single-qudit density matrices. A state that is not separable is called entangled.

- R. F. Werner, 1989:

with the density matrix $W = \sum_{r=1}^{n} p_r W_r^1 \otimes W_r^2$, i.e., W is a convex combination of product states. Expectation

Separability and entanglement of mixed states II

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Quantum states with Einstein-Podolsky-Rosen correlations admitting a hidden-variable model

Reinhard F. Werner*

Dublin Institute for Advanced Studies, 10 Burlington Road, Dublin 4, Ireland

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A state of a composite quantum system is called classically correlated if it can be approximated by convex combinations of product states, and Einstein-Podolsky-Rosen correlated otherwise. Any classically correlated state can be modeled by a hidden-variable theory and hence satisfies all generalized Bell's inequalities. It is shown by an explicit example that the converse of this statement is false.

I. INTRODUCTION

Consider a composite quantum system described in a Hilbert space $\mathcal{H}=\mathcal{H}^1\otimes\mathcal{H}^2$. An uncorrelated state of this system is given by a density matrix W [i.e., an operator $W\in\mathcal{B}(\mathcal{H})$ with $W\geq 0$ and tr $W=1$] in $\mathcal{H}$ of the form $W=W^1\otimes W^2$ for two density matrices $W^i\in\mathcal{B}(\mathcal{H}_i)$. This is equivalent to saying that the expectation value $\mathrm{tr}(WA_1\otimes A_2)$ for the joint measurement of observables $A^i\in\mathcal{B}(\mathcal{H}^i)$ $(i=1,2)$ on the respective subsystems always factorizes, i.e.,

$$\begin{aligned}\mathrm{tr}(WA^1\otimes A^2)&=\mathrm{tr}(W\cdot A^1\otimes\mathbb{1})\mathrm{tr}(W\cdot\mathbb{1}\otimes A^2)\\&=\mathrm{tr}(W^1A^1)\mathrm{tr}(W^2A^2)\ .\end{aligned}$$

Such uncorrelated states can be prepared very easily by using two preparing devices for systems 1 and 2, which

it can be approximated (e.g., in trace norm) by density matrices of the form (1). States that are not classically correlated have been called *EPR correlated*[1] to emphasize the crucial role of such states in the Einstein-Podolsky-Rosen paradox, and for the violations of Bell's inequalities (see below). EPR correlation and classical correlation are defined as a property of the density matrix W. Since there are usually very different ways of preparing the same state W, classical correlation does not mean that the state has actually been prepared in the manner described, but only that its statistical properties can be reproduced by a classical mechanism.

The terminology "classically correlated" is further justified by the observation that in classical probability theory all states have this property. States in probability theory are given by probability measures, and the state of a composite system is given by a probability measure on a

Separability and entanglement of pure states III

Comments:

- For pure states it is easy to decide whether a state is separable of not. For mixed states, it is very hard.
- Hand waving meaning of the definition above: with probability p_k a machine produced the product state $\varrho_k^{(1)} \otimes \varrho_k^{(2)}$.
- The two parties (i.e., 1 and 2) can be far from each other (i.e., on the Moon and on Earth).
- No real quantum dynamics is needed between the two parties to create the separable state.
- Local Operation and Classical Communiation (LOCC) cannot create an entangled state from a separable one.

Separability and entanglement of pure states IV

Comments (continued)

- Let us see the following two maximally entangled states

$$|\Phi^{\pm}\rangle = \frac{1}{\sqrt{2}}(|00\rangle \pm |11\rangle).$$

  An equal mixture of these states is

$$\frac{1}{2}\left(|00\rangle\langle 00| + |11\rangle\langle 11|\right),$$

  which is separable.

- Thus, if we mix two entangled states, we might end up with a separable state.

Separability and entanglement of pure states V

Comments (continued)

- Separable states can be correlated. For example, the state

$$\frac{1}{2}\left(|00\rangle\langle 00| + |11\rangle\langle 11|\right)$$

  has nonzero correlations, however, it is separable.

- This can be seen noting that

$$\langle \sigma_z \otimes \sigma_z \rangle = +1.$$

  We can also say that

$$\langle \sigma_z \otimes \sigma_z \rangle - \langle \sigma_z \otimes \mathbb{1} \rangle \langle \mathbb{1} \otimes \sigma_z \rangle = +1.$$