Iterative optimization in quantum metrology and entanglement theory using semidefinite programming

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Outline

1. **Motivation**
   - What are entangled states useful for?
2. **Metrological gain and the optimal local Hamiltonian**
   - Metrological usefulness of a quantum state
   - Example for activation in small systems
   - Activation in the many-particle case
3. **Generalization of the ideas**
   - Calculational details for bipartite systems
   - Alternative method for finding the optimal Hamiltonian
   - Upper bounds instead of lower bounds
   - Wigner-Yanase skew information
   - The Computable Cross Norm-Realignment (CCNR) criterion

What are entangled states useful for?

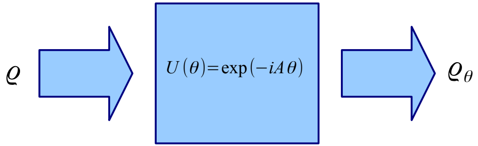
- Entanglement is needed for beating the shot-noise limit in quantum metrology.
- However, not all entangled states are more useful than separable states.
- Intriguing questions:
    - Can we decide which quantum state is more useful than separable states?
    - Can we activate the metrological usefulness of quantum states, if we use several copies?

Outline

The quantum Fisher information

- Cramér-Rao bound on the precision of parameter estimation

$$(\Delta\theta)^2 \geq \frac{1}{mF_Q[\varrho,A]},$$



where where m is the number of independent repetitions and $F_Q[\varrho, A]$ is the quantum Fisher information.

- The quantum Fisher information is

$$F_Q[\varrho, A] = 2\sum_{k,l} \frac{(\lambda_k - \lambda_l)^2}{\lambda_k + \lambda_l} |\langle k|A|l\rangle|^2,$$

where $\varrho = \sum_k \lambda_k |k\rangle\langle k|$.

The quantum Fisher information vs. entanglement

- Local Hamiltonians for linear interferometers

$$J_l = \sum_{n=1}^{N} j_l^{(n)}$$

for $l = x, y, z$.

- For separable states of N qubits

$$F_Q[\varrho, J_l] \le N, \qquad l = x, y, z.$$

L. Pezze, A. Smerzi, Phys. Rev. Lett. 102, 100401 (2009);
P. Hyllus, O. Gühne, A. Smerzi, Phys. Rev. A 82, 012337 (2010)

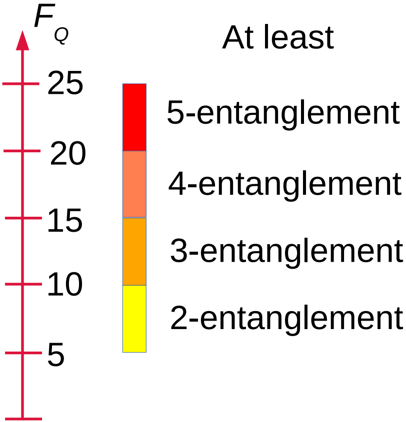
- For states with at most k-particle entanglement (tight bound if k is a divisor of N)

$$F_Q[\varrho, J_l] \le kN.$$

P. Hyllus *et al.*, Phys. Rev. A 85, 022321 (2012);
GT, Phys. Rev. A 85, 022322 (2012).

The quantum Fisher information vs. entanglement

5 spin-1/2 particles



(For simplicity, we used $F_Q[\varrho, J_l] \leq kN$, which is not tight.)

The quantum Fisher information vs. entanglement

- Let us consider the fraction of the QFI and the maximum QFI for separable states for N qubits

$$\frac{F_Q[\varrho, H]}{N}$$

- The maximum for separable states is N for any Hamiltonian of the form

$$H = \vec{c}^{(1)}\vec{\sigma}^{(1)} + \vec{c}^{(2)}\vec{\sigma}^{(2)} + \vec{c}^{(3)}\vec{\sigma}^{(3)} + ...,$$

  where $|\vec{c}^{(n)}| = 1$, and $\vec{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$.

  P. Hyllus, O. Gühne, A. Smerzi, Phys. Rev. A 82, 012337 (2010)

Metrological usefulness

- Qudits are more complicated!
- Metrological gain for a given Hamiltonian

$$g_{\mathcal{H}}(\varrho) = \frac{\mathcal{F}_Q[\varrho, \mathcal{H}]}{\mathcal{F}_Q^{(\mathrm{sep})}(\mathcal{H})},$$

  where $\mathcal{F}_Q^{(\mathrm{sep})}(\mathcal{H})$ is the maximum of the QFI for separable states.
- Metrological gain

$$g(\varrho) = \max_{\mathrm{local}\mathcal{H}} g_{\mathcal{H}}(\varrho) = \max_{\mathrm{local}\mathcal{H}} \frac{\mathcal{F}_Q[\varrho, \mathcal{H}]}{\mathcal{F}_Q^{(\mathrm{sep})}(\mathcal{H})}.$$

  optimized over all local Hamiltonians

$$\mathcal{H} = H_1 + H_2 + ... + H_N.$$

- A state ϱ is entangled and metrologically useful if $g(\varrho) > 1$.
- The metrological gain is convex in the state.
  G. Toth, T. Vertesi, P. Horodecki, R. Horodecki, PRL 2020.

Metrologically useful *k*-entanglement

- If $g > k - 1$ then we have metrologically useful *k*-particle entanglement.

- We have *k*-particle entanglement, <u>and</u> the state is more useful than any states mentioned above.

  R. Trényi *et al.*, New J. Phys. 26, 023034 (2024).

The quantum metrological gain is an useful quantity!

- The metrological gain is connected to multipartite entanglement in a meaningful way.

- One could also calculate

$$\mathcal{F}_Q[\varrho, \mathcal{H}] - \mathcal{F}_Q^{(\mathrm{sep})}(\mathcal{H}),$$

  which could also be used for entanglement detection. However, the relation for multipartite entanglement is less direct.

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Method for maximizing g

- Metrological gain optimized over all local Hamiltonians

$$g(\varrho) = \max_{\mathrm{local}\,\mathcal{H}} \frac{\mathcal{F}_Q[\varrho, \mathcal{H}]}{\mathcal{F}_Q^{(\mathrm{sep})}(\mathcal{H})} \begin{array}{l} \leftarrow \text{metrological performance of } \varrho \\ \leftarrow \text{best metrological performance of} \\ \quad\ \text{separable states} \end{array}$$

- It is a fundamental quantity in metrology!
- Difficult to compute, since $\mathcal{H}$ is in both the numerator and the denominator!
- We reduce the problem to maximize $\mathcal{F}_Q$ over a set of local Hamiltonians.

Method for finding the optimal local Hamiltonian I

- Direct maximization of $\mathcal{F}_Q[\varrho,\mathcal{H}]$ over $\mathcal{H}$ is difficult: it is convex in $\mathcal{H}$.

- Let us consider the error propagation formula

$$(\Delta\theta)^2{}_M = \frac{(\Delta M)^2}{\langle i[M,\mathcal{H}]\rangle^2},$$

which provides a bound on the quantum Fisher information

$$\mathcal{F}_Q[\varrho,\mathcal{H}] \geq 1/(\Delta\theta)^2{}_M.$$

M. Hotta and M. Ozawa, Phys. Rev. A 2004; B. M. Escher, arXiv:1212.2533; K. Macieszczak, arXiv:1312.1356; F. Fröwis, R. Schmied, and N. Gisin, Phys. Rev. A 2015.

Method for finding the optimal Hamiltonian II

The maximum over local Hamiltonians can be obtained as

$$\max_{\textit{local } \mathcal{H}} \mathcal{F}_Q[\varrho,\mathcal{H}] = \max_{\textit{local } \mathcal{H}} \max_{M} \frac{\langle i[M,\mathcal{H}]\rangle_\varrho^2}{(\Delta M)^2}.$$

G. Toth, T. Vertesi, P. Horodecki, R. Horodecki, PRL 2020.

Iterative see saw methods (ISS)

- Iterative See Saw (ISS) has been used for optimizing over the state, rather than over $\mathcal{H}$:

  K. Macieszczak, arXiv:1312.1356; K. Macieszczak, M. Fraas, and R. Demkowicz-Dobrzański, New J. Phys. 16, 113002 (2014);
  GT, Vértesi, Phys. Rev. Lett. (2018): PPT states are metrologically useful.

- ISS-based method for an optimization over a general dynamics using its Choi- Jamiołkowski representation
  Y. L. Len, T. Gefen, A. Retzker, and J. Kołodyński, Quantum metrology with imperfect measurements, Nat. Commun. 13, 6971 (2022).

- An ISS-based method has also been used for optimizing over adaptive strategies, when coherently probing several independent quantum channels
  S. Kurdziałek, P. Dulian, J. Majsak, S. Chakraborty, and R. Demkowicz-Dobrzański, Quantum metrology using quantum combs and tensor network formalism, New J. Phys. 27, 013019 (2025).

Example: Maximally entangled state

- We consider the $d \times d$ maximally entangled state

$$|\Psi^{(\mathrm{me})}\rangle = \frac{1}{\sqrt{d}} \sum_{k=1}^{d} |k\rangle|k\rangle.$$

- The optimal Hamiltonian is

$$\mathcal{H}^{(\mathrm{me})} = D \otimes \mathbb{1} + \mathbb{1} \otimes D,$$

where

$$D = \mathrm{diag}(+1, -1, +1, -1, ...).$$

- We add white noise.

Numerical results

- The 3×3 isotropic state is useful if for the noise

$$p < \frac{25 - \sqrt{177}}{32} \approx 0.3655.$$

- Then, we have the following results for two copies.

	Analytic example	Numerics
Second copy	0.4164	0.4170

- In the case of two copies,
  the metrological usefulness has been activated in the spirit of P. Horodecki, M. Horodecki and R. Horodecki, PRL 1989!

G. Toth, T. Vertesi, P. Horodecki, R. Horodecki, PRL 2020.

Outline

Multicopy metrology without interaction

- M copies of a quantum state, all undergoing a dynamics governed by the Hamiltonian $\mathcal{H}$.

- For the quantum Fisher information we obtain

$$\mathcal{F}_Q[\varrho^{\otimes M}, \mathcal{H}^{\otimes M}] = M\mathcal{F}_Q[\varrho, \mathcal{H}],$$

  while the maximum for separable states also increases

$$\mathcal{F}_Q^{(\mathrm{sep})}(\mathcal{H}^{\otimes M}) = M\mathcal{F}_Q^{(\mathrm{sep})}(\mathcal{H}).$$

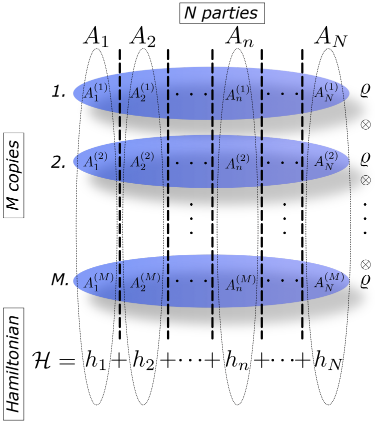
- The metrological gain does not change

$$g_{\mathcal{H}^{\otimes M}}(\varrho^{\otimes M}) = g_{\mathcal{H}}(\varrho).$$

- (Unbiased estimators.)

Multicopy metrology with interaction

- We need interaction between the copies.
- Weakly entangled states can reach maximal metrological usefulness in the many-copy case.



- Metrology with M copies of an N-partite quantum state ϱ.
- There is no interaction between particles corresponding to different parties.

R. Trényi *et al.*, New J. Phys. 26, 023034 (2024).

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Local Hamiltonian

- A local Hamiltonian is given as

$$\mathcal{H} = H_1 \otimes \mathbb{1}_2 + \mathbb{1}_1 \otimes H_2,$$

- For the quantum Fisher information

$$\mathcal{F}_Q[\varrho, c\mathcal{H}] = |c|^2 \mathcal{F}_Q[\varrho, \mathcal{H}]$$

  holds.
- Thus, we need to normalize, before we maximize $\mathcal{F}_Q$.
- We could normalize it with some norm of $\mathcal{H}$. We suggest to normalize it with

$$\mathcal{F}_Q^{(\mathrm{sep})}(\mathcal{H}) = \sum_{n=1,2} [\sigma_{\max}(H_n) - \sigma_{\min}(H_n)]^2,$$

  where $\sigma_{\max}(X)$ and $\sigma_{\min}(X)$ denote the maximal and minimal eigenvalues, respectively.

Local Hamiltonian II

The expression with the square root of the maximum for separable states

$$\sqrt{\mathcal{F}_Q^{(\mathrm{sep})}(\mathcal{H})}$$

is a seminorm for $\mathcal{H} \in \mathcal{L}$.

- $\sqrt{\mathcal{F}_Q^{(\mathrm{sep})}(\mathcal{H})} = 0$ if $\mathcal{H} = c\mathbb{1}$, where c is a real constant.
- $\sqrt{\mathcal{F}_Q^{(\mathrm{sep})}(\mathcal{H})} = 0$ does not imply that $\mathcal{H}$ is a zero matrix, thus it is not a norm, only a seminorm.

Method for finding the optimal Hamiltonian

- We define

$$g_{\mathcal{H}}(\varrho) = \frac{\mathcal{F}_Q[\varrho, \mathcal{H}]}{\mathcal{F}_Q^{(\mathrm{sep})}(\mathcal{H})}.$$

- We want to compute

$$g(\varrho) = \max_{\mathcal{H} \in \mathcal{L}} g_{\mathcal{H}}(\varrho).$$

- How can we avoid optimizing the numerator and the denominator?

Method for finding the optimal Hamiltonian

- Local Hamiltonians must fulfill

$$\sigma_{\min}(H_n) = -c_n, \quad \sigma_{\max}(H_n) = +c_n$$

for $n = 1, 2$. Then, for separable states we have

$$\mathcal{F}_Q^{(\mathrm{sep})}(\mathcal{H}) = 4(c_1^2 + c_2^2).$$

- Metrological gain for such local Hamiltonians

$$g_{c_1,c_2}(\varrho) = \max_{\mathcal{H} \in \mathcal{L}_{c_1,c_2}} \frac{\mathcal{F}_Q[\varrho, \mathcal{H}]}{4(c_1^2 + c_2^2)},$$

where we call $\mathcal{L}_{c_1,c_2}$ the set of local Hamiltonians satisfying the condition with c_1 and c_2.

- Finally, the maximal gain over all possible local Hamiltonians is given as

$$g(\varrho) = \max_{c_1,c_2} g_{c_1,c_2}(\varrho).$$

Method for finding the optimal Hamiltonian

- Change the constraints to inequalities

$$c_n \mathbb{1} \pm H_n \geq 0,$$

where $n = 1, 2$ and $c_n > 0$ is some constant.

- This way we make sure that

$$\sigma_{\min}(H_n) \geq -c_n, \quad \sigma_{\max}(H_n) \leq +c_n,$$

for $n = 1, 2$.
- We maximize a convex function over a convex set.
- the optimum is taken on the boundary of the set where the eigenvalues of H_n are $\pm c_n$. In this case,

$$H_n^2 = c_n^2 \mathbb{1}.$$

Method for finding the optimal Hamiltonian

- If two Hamiltonians, $\mathcal{H}'$ and $\mathcal{H}''$, are of the required form then their convex combination, i. e.,

$$\mathcal{H}_p = p\mathcal{H}' + (1-p)\mathcal{H}''$$

with $0 \leq p \leq 1$ is also of that form.

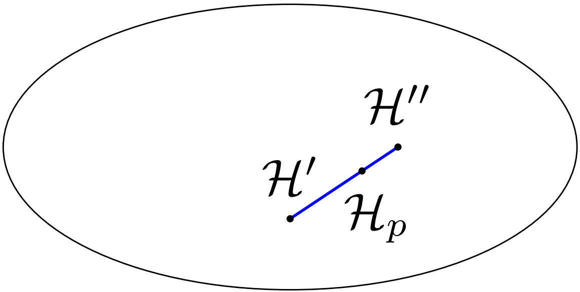

Figure: The convex set of local Hamiltonians. $\mathcal{H}_p$ is a "mixture" of $\mathcal{H}'$ and $\mathcal{H}''$.

Outline

See saw methods in science

- General idea of the see-saw iteration used in physics and engineering
$$\max_{\vec{v}} \vec{v}^T R \vec{v} = \max_{\vec{v},\vec{w}} \vec{v}^T R \vec{w}.$$
R is a positive semidefinite symmetric matrix with real values.

- Let us see some further applications in quantum metrology and entanglement theory.

Alternative method for finding the optimal Hamiltonian

Maximize the QFI using the quantum Fisher matrix elements as

$$\max_{\mathcal{H}\in\mathcal{L}_{c_1,c_2}} \mathcal{F}_Q[\varrho,\mathcal{H}] = \max_{\mathcal{H},\mathcal{K}\in\mathcal{L}_{c_1,c_2}} \mathcal{F}_Q[\varrho,\mathcal{H},\mathcal{K}].$$

Here, we define

$$\mathcal{F}_Q[\varrho,\mathcal{H},\mathcal{K}] = \sum_{kl} 2\frac{(\lambda_k-\lambda_l)^2}{\lambda_k+\lambda_l}\left[\mathcal{H}_{kl}^{\mathrm{r}}\mathcal{K}_{kl}^{\mathrm{r}} + \mathcal{H}_{kl}^{\mathrm{i}}\mathcal{K}_{kl}^{\mathrm{i}}\right].$$

Á. Lukács, R. Trényi, T. Vértesi, GT, Quantum Sci. Technol. 11, 015042 (2026)

Method for finding the optimal Hamiltonian

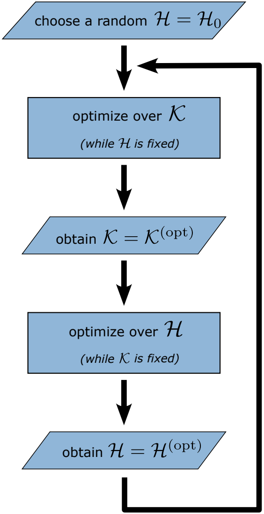

Figure: We can optimize the quantum Fisher information over local Hamiltonians for a fixed probe state with an iterative see-saw (ISS) method.

Outline

Upper bound with SDP

- The maximization
$$\max_{\vec{v}} \vec{v}^T R \vec{v}$$
is replaced by the Shor relaxation maximizing
$$\mathrm{Tr}(RX),$$
where the Hermitian X is constrained as
$$X \leq \vec{v}\vec{v}^T,$$
and it does not have to be rank-1.
- We can also add further and further conditions on X that lead to lower and lower values for the maximum.

Upper bound with SDP

- Method of moments. The variables correspond to the products of the original variables.
- In this case, the conditions are

$$H_n^2 = c_n^2 \mathbb{1},$$

for $n = 1, 2$. All eigenvalues of H_n are $\pm c_n$.

- In quadratic programming, we can also use the constraint

$$H_1^2 + H_2^2 \leq \mathbb{1}.$$

Á. Lukács *et al.*, Quantum Sci. Technol. (2026)

Outline

Wigner-Yanase skew information

- The Wigner-Yanase skew information is defined as

$$I_\varrho(\mathcal{H}) = \mathrm{Tr}(\mathcal{H}^2\varrho) - \mathrm{Tr}(\mathcal{H}\sqrt{\varrho}\mathcal{H}\sqrt{\varrho}).$$

- We maiximize over $\mathcal{H}$. It is clear how to maximize it with a see-saw method:

$$\max_{\mathcal{H},\mathcal{K}\in\mathcal{L}_{c_1,c_2}} I_\varrho(\mathcal{H}) = \mathrm{Tr}\left[\frac{1}{2}(\mathcal{H}\mathcal{K}+\mathcal{K}\mathcal{H})\varrho\right] - \mathrm{Tr}(\mathcal{H}\sqrt{\varrho}\mathcal{K}\sqrt{\varrho}).$$

- Here, instead of $\mathcal{H}\mathcal{K}$ we use $\frac{1}{2}(\mathcal{H}\mathcal{K}+\mathcal{K}\mathcal{H})$, since the latter is Hermitian.

- Note that

$$\mathcal{F}_Q[\varrho,\mathcal{H}] \geq 4I_\varrho(\mathcal{H}).$$

Á. Lukács *et al.*, Quantum Sci. Technol. (2026)

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The Computable Cross Norm-Realignment (CCNR) criterion

- The trace norm is defined as

$$||X||_{\mathrm{tr}} = \mathrm{Tr}\left(\sqrt{XX^\dagger}\right).$$

- Dual relation

$$||X||_{\mathrm{tr}} = \max_{Y \in \mathbb{R}^{m\times m}: YY^T \leq \mathbb{1}} \mathrm{Tr}(X^T Y).$$

- The maximization of $||X||_{\mathrm{tr}}$ over a set S can then be written as

$$\max_{X \in S} ||X||_{\mathrm{tr}} = \max_{X \in S} \max_{Y \in \mathbb{R}^{m\times m}: YY^T \leq \mathbb{1}} \mathrm{Tr}(X^T Y),$$

  which can be calculated by a see-saw.
- We have to maximize $\mathrm{Tr}(X^T Y)$ alternatingly by X and Y.

The Computable Cross Norm-Realignment (CCNR) criterion

The CCNR criterion says that for every bipartite separable state ϱ we have

$$||R(\varrho)||_{\mathrm{tr}} \leq 1,$$

where $R(\varrho)$ is the realigned matrix obtained by a certain permutation of the elements of ϱ.

- If the inequality is violated then the state is entangled.
- The criterion can detect PPT bound entangled states not detected by the Peres-Horodecki crietrion.
- Clearly, the larger the left-hand side, the larger the violation.

CCNR criterion

Dimension $d_1 \times d_2$	Maximum of $\|\|R(\varrho)\|\|_{\mathrm{tr}}$
2×2	1
2×4	1
3×3	1.1891
3×4	1.2239
4×4	1.5
5×5	1.5
6×6	1.5881

Table: The largest values for $||R(\varrho)||_{\mathrm{tr}}$ for $d \times d$ PPT states for various d. The results were obtained by numerical maximization.

Á. Lukács *et al.*, Quantum Sci. Technol. (2026)

CCNR criterion

We present the 4×4 bound entangled state for which the violation of the CCNR criterion is maximal. The state is

$$\varrho_{\rm CCNR} = \sum_{i=1}^{4} p_i |\Psi_i\rangle\langle\Psi_i|_{AB} \otimes \varrho_{A'B'}^{(i)},$$

where the probabilities are

$$p_1 = p_2 = p_3 = 1/6, \quad p_4 = 1/2,$$

where the four Bell states are defined as

$$|\Phi^{\pm}\rangle = \frac{1}{\sqrt{2}}(|00\rangle \pm |11\rangle), \quad |\Psi^{\pm}\rangle = \frac{1}{\sqrt{2}}(|01\rangle \pm |10\rangle),$$

and the components for A' and B' are given as

$$\begin{aligned}
\varrho_{A'B'}^{(1)} &= |\Psi^+\rangle\langle\Psi^+|, \quad \varrho_{A'B'}^{(2)} = |\Psi^-\rangle\langle\Psi^-|, \\
\varrho_{A'B'}^{(3)} &= |\Phi^+\rangle\langle\Phi^+|, \quad \varrho_{A'B'}^{(4)} = \left(\mathbb{1} - |\Phi^-\rangle\langle\Phi^-|\right)/3.
\end{aligned}$$

For the state we have

$$||R(\varrho_{\rm CCNR})||_{\rm tr} = 1.5.$$

Summary

- We discussed methods for finding the optimal Hamiltonian for a quantum state using see-saw iterations.
- We discussed some other problems where similar see-saws could be used.

Á. Lukács, R. Trényi, T. Vértesi, and G. Tóth, Iterative optimization in quantum metrology and entanglement theory using semidefinite programming, Quantum Sci. Technol. 11, 015042 (2026)

R. Trényi, Á. Lukács, P. Horodecki, R. Horodecki, T. Vértesi, and G. Tóth, Activation of metrologically useful genuine multipartite entanglement, New J. Phys. 26, 023034 (2024).

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