Grothendieck's constant and local models for noisy entangled quantum states

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We relate the nonlocal properties of noisy entangled states to Grothendieck's constant, a mathematical constant appearing in Banach space theory. For two-qubit Werner states $\rho_p^W = p|\psi^-\rangle\langle\psi^-| + (1-p)\mathbb{1}/4$, we show that there is a local model for projective measurements if and only if $p \leqslant 1/K_G(3)$, where $K_G(3)$ is Grothendieck's constant of order 3. Known bounds on $K_G(3)$ prove the existence of this model at least for $p \lesssim 0.66$, quite close to the current region of Bell violation, $p \sim 0.71$. We generalize this result to arbitrary quantum states.

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I. INTRODUCTION

The impossibility of reproducing all correlations observed in composite quantum systems using models similar to that of Einstein, Podolsky, and Rosen (EPR) [1] was proven in 1964 by Bell. In his seminal work [2], Bell showed that all local models satisfy some conditions, the so-called Bell inequalities, but there are measurements on quantum states that violate a Bell inequality. Therefore, we say that quantum mechanics is nonlocal [3]. Experimental verification of Bell inequality violation closed the EPR debate, up to some technical loopholes [4].

From an operational point of view it is not difficult to define when a quantum state exhibits nonclassical correlations. Suppose that two parties Alice (A) and Bob (B) share a mixed quantum state ρ with support on $\mathcal{H}_A \otimes \mathcal{H}_B$, where $\mathcal{H}_A$ ($\mathcal{H}_B$) is the local Hilbert space of A's (B's) system. Then ρ contains quantum correlations when its preparation requires a nonlocal quantum resource. Conversely, a quantum state is classically correlated, or separable, when it can be prepared using only local quantum operations and classical communication (LOCC). From this definition, due to Werner [5], it follows that a quantum state ρ is separable if it can be expressed as a mixture of product states, $\rho = \sum_{i=1}^{N} p_i |\psi_A^i\rangle\langle\psi_A^i| \otimes |\psi_B^i\rangle\langle\psi_B^i|$. A state that cannot be written in this form has quantum correlations and is termed entangled. But the above definition, in spite of its clear physical meaning, is somewhat impractical. Tests to distinguish separable from entangled states are complicated [6], except when $d_A = 2$ and $d_B \leqslant 3$ [7], d_A and d_B denoting the dimensions of the local subsystems.

Violation of a Bell inequality by a quantum state is, in many situations, a witness of useful correlations [8]. In particular, Bell inequality violation is a witness of a quantum state's entanglement. Now, the question is, are all entangled states nonlocal? For the case of pure states, the answer is yes [9]: all entangled pure states violate the Clauser-Horne-Shimony-Holt (CHSH) inequality [10]. In 1989, Werner showed that the previous result cannot be generalized to mixed states [11]. He introduced what are now called Werner states, and gave a local hidden variables (LHV) model for measurement outcomes for some entangled states in this family [5]. Although the construction only worked for projective measurements, his result has since been extended to general measurements [12].

In spite of these partial results, it is in general extremely difficult to determine whether an entangled state has a local model or not [13], since (i) finding all Bell inequalities is a computationally hard problem [14,15] and (ii) the number of possible measurements is unbounded (see, however, [16] for recent progress). This question remains unanswered even in the simplest case of Werner states of two qubits. These are mixtures of the singlet $|\psi^-\rangle = (|01\rangle - |10\rangle)/\sqrt{2}$ with white noise of the form

$$\rho_p^W = p|\psi^-\rangle\langle\psi^-| + (1-p)\frac{\mathbb{1}}{4}. \tag{1}$$

It is known that Werner states are separable if and only if $p \leqslant 1/3$, admit a LHV model for all measurements for $p \leqslant 5/12$ [12], admit a LHV model for projective measurements for $p \leqslant 1/2$ [5], and violate the CHSH inequality for $p > 1/\sqrt{2}$ (see Fig. 1). However, the critical value of p, denoted p_c^W, at which two-qubit Werner states cease to be nonlocal under projective measurements is unknown. This question is particularly relevant from an experimental point of view, since p_c^W specifies the amount of noise the singlet tolerates before losing its nonlocal properties.

In this paper, we exploit the connection between correlation Bell inequalities and Grothendieck's constant [17], first

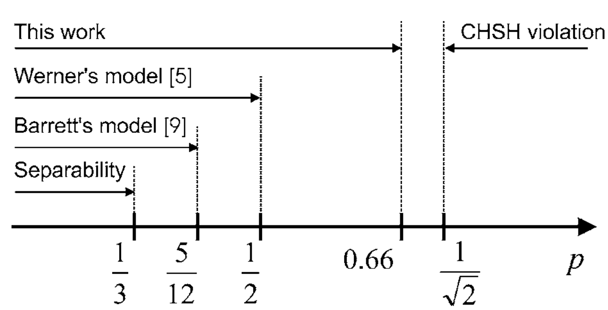

FIG. 1. Nonlocal properties of two-qubit Werner states ρ_p^W. Werner's local model works up to $p = 1/2$, while the CHSH inequality is violated when $p > 2^{-1/2} \sim 0.71$. Here, we prove the existence of a local model for projective measurements when $p \lesssim 0.66$.

noticed by Tsirelson [18], to prove the existence of a local model for several noisy entangled states. We first demonstrate that p_c^W is related to a generalization of this constant, namely, $p_c^W = 1/K_G(3)$, where $K_G(3)$ is Grothendieck's constant of order 3 [19]. The exact value of $K_G(3)$ is unknown, but known bounds establish that $0.6595 \leq p_c^W \leq 1/\sqrt{2}$. Thus, we close more than three-quarters of the gap between Werner's result and the known region of Bell inequality violation (see Fig. 1). Next, we show that if Alice (or Bob) is restricted to make measurements in a plane of the Poincaré sphere, then there is an explicit LHV model for all $p \leq 1/K_G(2) = 1/\sqrt{2}$. This improves on the bound of Larsson, who constructed a LHV model for planar measurements for $p \leq 2/\pi$ [20]. Thus, in the case of planar projective measurements, violation of the CHSH inequality completely characterizes the nonlocality of two-qubit Werner states.

In the case of *traceless two-outcome* observables, we can extend our results to mixtures of an arbitrary state ρ on $\mathbb{C}^d \otimes \mathbb{C}^d$ with the identity, of the form [21]

$$\rho_p = p\rho + (1-p)\frac{\mathbb{1}}{d^2}. \tag{2}$$

Denote by $p_c(\rho)$ the maximum value of p for which there exists a LHV model for the joint correlation of traceless two-outcome observables on ρ_p, and define

$$p_c^d = \min_{\rho} p_c(\rho), \quad p_c = \lim_{d\to\infty} p_c^d. \tag{3}$$

Then $p_c = 1/K_G$ where K_G is Grothendieck's constant. Again, the exact value of K_G is unknown, but known bounds imply $0.5611 \leq p_c \leq 0.5963$.

Finally, we discuss the opposite question of finding Bell inequalities better than the CHSH inequality at detecting the nonlocality of ρ_p^W, or, more generally, of Bell diagonal states [22]. In particular, we show that none of the I_{nn22} Bell inequalities introduced in Ref. [23] is better than the CHSH inequality for these states.

Before proving our results, we require some notation. We write a two-outcome measurement by Alice (Bob) as $\{A^+, A^-\}$ ($\{B^+, B^-\}$), where the projectors $A^\pm$ correspond to measurement outcomes ± 1. We define the *observable* corresponding to Alice's (Bob's) measurement as $A = A^+ - A^-$ ($B = B^+ - B^-$). An observable A is *traceless* if $\mathrm{tr}A = 0$, or equivalently $\mathrm{tr}A^- = \mathrm{tr}A^+$. The *joint correlation* of Alice and Bob's measurement results, denoted α and β, respectively, is

$$\langle \alpha\beta \rangle = \mathrm{tr}(A \otimes B\rho). \tag{4}$$

Alice's *local marginal* is specified by $\langle \alpha \rangle = \mathrm{tr}(A \otimes \mathbb{1}\rho)$, and similarly for Bob. Together, $\langle \alpha\beta \rangle$, $\langle \alpha \rangle$, and $\langle \beta \rangle$ define the full probability distribution for two-outcome measurements on ρ. A LHV model for the full probability distribution is one that gives the same values $\langle \alpha\beta \rangle$, $\langle \alpha \rangle$, and $\langle \beta \rangle$ as quantum theory. A LHV model for the joint correlation is one that gives the same joint correlation $\langle \alpha\beta \rangle$, but not necessarily the correct marginals. In the qubit case, the projective measurements applied by the parties are specified by the direction of their Stern-Gerlach apparatuses, given by normalized three-dimensional real vectors $\vec{a}$ and $\vec{b}$: $A = \vec{a} \cdot \vec{\sigma}$ and $B = \vec{b} \cdot \vec{\sigma}$.

II. WERNER STATES

Let us first consider the case of Werner states (1). For projective measurements on ρ_p^W, LHV simulation of the joint correlation is sufficient to reproduce the full probability distribution. This follows from the lemma given below.

Lemma 1. Suppose that there is a LHV model L that gives joint correlation $\langle \alpha\beta \rangle_L$. Then there is a LHV model L' with the same joint correlation and uniform marginals: $\langle \alpha\beta \rangle_{L'} = \langle \alpha\beta \rangle_L$, $\langle \alpha \rangle_{L'} = \langle \beta \rangle_{L'} = 0$.

Proof. Let α and β be the outputs generated by the LHV L (dependent on the hidden variables and measurement choices). Define a new LHV L' by augmenting the hidden variables of L with an additional random bit $c \in \{-1, 1\}$. In L', Alice outputs $c\alpha$ and Bob $c\beta$. ■

Therefore, the analysis of the nonlocal properties of Werner states under projective measurements can be restricted to Bell inequalities involving only the joint correlation. Actually, this holds for any Bell diagonal state, under projective measurements, since $\mathrm{tr}_A\rho = \mathrm{tr}_B\rho = \mathbb{1}/2$ for all these states, so all projective measurements give uniform marginals. In the Bell scenarios we consider, Alice and Bob each choose from m observables, specified by $\{A_1, \ldots, A_m\}$ and $\{B_1, \ldots, B_m\}$. We can write a generic correlation Bell inequality as

$$\left| \sum_{i,j=1}^{m} M_{ij} \langle \alpha_i \beta_j \rangle \right| \leq 1, \tag{5}$$

where $M = (M_{ij})$ is an $m \times m$ matrix of real coefficients defining the Bell inequality. The matrix M is normalized such that the local bound is achieved by a deterministic local model, i.e.,

$$\max_{a_i = \pm 1, b_j = \pm 1} \left| \sum_{i,j=1}^{m} M_{ij} a_i b_j \right| = 1. \tag{6}$$

For the singlet state, $\langle \alpha_i \beta_j \rangle_{\psi^-} = -\vec{a}_i \cdot \vec{b}_j$. We obtain the maximum ratio of Bell inequality violation for the singlet state, denoted Q, by maximizing over normalized Bell inequalities, and taking the limit as the number of settings goes to infinity:

$$Q = \lim_{m\to\infty} \sup_{M_{ij}} \max_{\vec{a}_i, \vec{b}_j} \left| \sum_{i,j=1}^{m} M_{ij} \vec{a}_i \cdot \vec{b}_j \right|. \tag{7}$$

Since all joint correlations vanish for the maximally mixed state, it follows that the critical point at which two-qubit Werner states do not violate any Bell inequality is $p_c^W = 1/Q$.

As first noticed by Tsirelson, the previous formulation of the Bell inequality problem is closely related to the definition of Grothendieck's inequality and Grothendieck's constant, K_G (see [18] for details). Grothendieck's inequality first arose in Banach space theory, particularly in the theory of p-summing operators [24]. We shall need a refinement of his constant, which can be defined as follows [17]:

Definition 1. For any integer $n \geq 2$, Grothendieck's constant of order n, denoted $K_G(n)$, is the smallest number with the following property: Let M be any $m \times m$ matrix for which

$$\left| \sum_{i,j=1}^{m} M_{ij} a_i b_j \right| \leq 1, \tag{8}$$

for all real numbers $a_1, \dots, a_m, b_1, \dots, b_m \in [-1, +1]$. Then

$$\left| \sum_{i,j=1}^{m} M_{ij} \vec{a}_i \cdot \vec{b}_j \right| \leq K_G(n) \tag{9}$$

for all unit vectors $\vec{a}_1, \dots, \vec{a}_m, \vec{b}_1, \dots, \vec{b}_m$ in $\mathbb{R}^n$.

Definition 2. Grothendieck's constant is defined as

$$K_G = \lim_{n\to\infty} K_G(n). \tag{10}$$

The best bounds currently known for K_G are $1.6770 \leq K_G \leq \pi/[2\ln(1+\sqrt{2})] = 1.7822$ [25]. The lower bound is due to Reeds and, independently, Davies [26], while the upper bound is due to Krivine [19].

It follows immediately from the first definition that the maximal Bell violation for the singlet state (7) is $K_G(3)$. We have therefore proved the following theorem.

Theorem 1. There is a LHV model for projective measurements on the Werner state ρ_p^W if and only if $p \leq p_c^W = 1/K_G(3)$.

It is known that $\sqrt{2} \leq K_G(3) \leq 1.5163$. The lower bound follows from the CHSH inequality; the upper bound is again due to Krivine [19]. He shows that $K_G(3) \leq \pi/(2c_3)$ where c_3 is the unique solution of

$$\frac{\sqrt{c_3}}{2}\int_0^{c_3} t^{-3/2}\sin t \, dt = 1 \tag{11}$$

in the interval $[0, \pi/2]$. Numerically we find that $c_3 \approx 1.0360$. This implies $K_G(3) \leq 1.5163$ and $p_c^W \geq 0.6595$. Furthermore, it turns out that an explicit LHV model emerges from Krivine's upper bound on $K_G(3)$, and the details are presented in [27].

Another result follows from Krivine's work.

Theorem 2. If Alice's projective measurements are restricted to a plane in the Poincaré sphere, then there is a LHV model for ρ_p^W if and only if $p \leq 1/\sqrt{2}$.

Proof. In this case, the vectors $\vec{a}_i$ in Eq. (7) are two dimensional. Since the quantum correlation depends only on the projection of $\vec{b}_j$ onto $\vec{a}_i$, we can assume that the vectors $\vec{b}_j$ lie in the same plane. It follows that $p_c^W = 1/K_G(2)$ for planar measurements, and Krivine has shown that $K_G(2)$ is equal to $\sqrt{2}$ [19]. ■

Again Krivine's proof can be adapted to give an explicit LHV model for planar measurements, valid for $p \leq 1/\sqrt{2}$ [27].

III. GENERALIZATION TO HIGHER DIMENSION

It is possible to extend these results to general states of the form (2), if we restrict our analysis to correlation Bell inequalities of traceless two-outcome observables. Admittedly, this analysis is far from sufficient. Indeed it does not allow us to determine whether the full probability distribution admits a LHV model even in the case of two-outcome measurements, since the most general Bell inequalities have terms that depend on marginal probabilities [23]. Mindful of this caveat, we now prove the existence of LHV models for the joint correlation of the states (2). To make the connection with Grothendieck's constant, we start with a representation of quantum correlations as dot products, first noted by Tsirelson [18]. It is sufficient to restrict attention to the case of pure states, since we can obtain a LHV for a mixed state ρ by decomposing it into a convex sum of pure states, and taking a convex combination of the LHV's for those pure states.

Lemma 2. Suppose Alice and Bob measure observables A and B on a pure quantum state $|\psi\rangle \in \mathbb{C}^d \otimes \mathbb{C}^d$. Then we can associate a real unit vector $\vec{a} \in \mathbb{R}^{2d^2}$ with A (independent of B), and a real unit vector $\vec{b} \in \mathbb{R}^{2d^2}$ with B (independent of A) such that $\langle \alpha\beta\rangle_\psi = \vec{a}\cdot\vec{b}$. Moreover, if $|\psi\rangle$ is maximally entangled, then we can assume the vectors $\vec{a}$ and $\vec{b}$ lie in $\mathbb{R}^{d^2-1}$.

Proof. Let $|a\rangle = A \otimes \mathbb{1}_B |\psi\rangle$ and $|b\rangle = \mathbb{1}_A \otimes B|\psi\rangle$. Then $\langle\alpha\beta\rangle = \langle a|b\rangle$, $\langle a|a\rangle = \langle b|b\rangle = 1$. Denote the components of $|a\rangle$ as a_i where $i = 1, 2, \dots, d^2$, and similarly for $|b\rangle$. We now define a $2d^2$-dimensional real vector $\vec{a} = (\mathrm{Re}\, a_1, \mathrm{Im}\, a_1, \mathrm{Re}\, a_2, \mathrm{Im}\, a_2, \dots, \mathrm{Re}\, a_{d^2}, \mathrm{Im}\, a_{d^2})$, and similarly $\vec{b} = (\mathrm{Re}\, b_1, \mathrm{Im}\, b_1, \mathrm{Re}\, b_2, \mathrm{Im}\, b_2, \dots, \mathrm{Re}\, b_{d^2}, \mathrm{Im}\, b_{d^2})$. Then $\vec{a}\cdot\vec{a} = \vec{b}\cdot\vec{b} = 1$ and $\langle\alpha\beta\rangle = \vec{a}\cdot\vec{b}$ (because $\langle a|b\rangle$ is real).

If $|\psi\rangle$ is maximally entangled, we can assume $|\psi\rangle = |\psi^+\rangle = (1/\sqrt{d})\Sigma_{i=1}^d |ii\rangle$. We calculate $\langle\alpha\beta\rangle_{\psi^+} = \mathrm{tr}_A(AB^t)/d$ where B^t is the transpose of B. Introduce a (d^2-1)-dimensional basis g_i for traceless operators on $\mathcal{H}_A$, normalized such that $\mathrm{tr}(g_i g_j) = d\delta_{ij}$. Let $A = \Sigma_i a_i g_i$, $B^t = \Sigma_i b_i g_i$, which define the vectors $\vec{a}$ and $\vec{b}$. Squaring these definitions and taking the trace gives $\Sigma_i a_i^2 = \Sigma_i b_i^2 = 1$. Finally, $\mathrm{tr}(AB^t) = d\Sigma_i a_i b_j$, which implies that $\langle\alpha\beta\rangle = \Sigma_i a_i b_i = \vec{a}\cdot\vec{b}$.

The converse of Lemma 2 is also true: all dot products of normalized vectors $\vec{a}, \vec{b} \in \mathbb{R}^n$ are realized as observables on $|\psi^+\rangle$, where $n = 2\lfloor \log_2 d\rfloor + 1$ and $\lfloor x\rfloor$ denotes the largest integer less than or equal to x. This result was derived by Tsirelson in Ref. [18]. For the sake of completeness, we state it here without proof (see [18] for the details).

Theorem 3. Let $\{\hat{a}_i\}_{i=1}^m$ and $\{\hat{b}_j\}_{j=1}^m$ be sets of unit vectors in $\mathbb{R}^n$. Let $d = 2^{\lfloor n/2\rfloor}$ and $|\psi^+\rangle$ be a maximally entangled state on $\mathbb{C}^d \otimes \mathbb{C}^d$. Then there are observables $A_1 \dots, A_m$ and $B_1 \dots, B_m$ on $\mathbb{C}^d$ such that

$$\langle\alpha_i\rangle = \langle\psi^+|A_i \otimes \mathbb{1}|\psi^+\rangle = 0, \tag{12}$$

$$\langle\beta_j\rangle = \langle\psi^+|\mathbb{1} \otimes B_j|\psi^+\rangle = 0, \tag{13}$$

$$\langle\alpha_i\beta_j\rangle = \langle\psi^+|A_i \otimes B_j|\psi^+\rangle = \hat{a}_i \cdot \hat{b}_j, \tag{14}$$

for all $1 \leq i, j \leq m$.

Note that in our case, the stipulation that the observables be traceless ensures that their outcomes are random on the maximally mixed state. The next theorem follows from Lemma 2 and Theorem 3.

Theorem 4. Let ρ be a state on $\mathbb{C}^d \otimes \mathbb{C}^d$ and define ρ_p and p_c^d as in Eqs. (2) and (3). Then

$$\frac{1}{K_G(2d^2)} \leqslant p_c^d \leqslant \frac{1}{K_G(2\lfloor \log_2 d \rfloor + 1)}. \tag{15}$$

In other words, there is always a LHV model for the joint correlation of traceless two-outcome observables on ρ_p for $p \leqslant 1/K_G(2d^2)$ and there is a state (in fact, the maximally entangled state on $\lfloor \log_2 d \rfloor$ qubits) such that the joint correlation is nonlocal for $p > 1/K_G(2\lfloor \log_2 d \rfloor + 1)$.

Corollary 1. The threshold noise for the joint correlation of two-outcome traceless observables is $p_c = 1/K_G$.

This follows from the previous theorem, taking the limit $d \rightarrow \infty$. The known bounds imply $0.5611 \leqslant p_c \leqslant 0.5963$. Compare this to p_s, the threshold noise at which the state ρ_p is guaranteed separable: while p_s decreases with dimension at least as $1/(1+d)$ [28], p_c approaches a constant. In the case of two-qubit systems, we can be more specific, because projective measurements are traceless and have two outcomes.

Corollary 2. Suppose ρ is an arbitrary state on $\mathbb{C}^2 \otimes \mathbb{C}^2$. Then there is a LHV model for the joint correlation on $\rho_p = p\rho + (1-p)\mathbb{1}/4$ for $p \leqslant 1/K_G(8)$. In particular, $K_G(8) \leqslant 1.6641$ [19,27], which implies there is a LHV model for $p \leqslant 0.6009$.

For maximally entangled states, marginals of traceless observables are uniform, so Lemmas 1 and 2 imply the following.

Theorem 5. Let $\rho_p = p|\psi^+\rangle\langle\psi^+| + (1-p)\mathbb{1}/d^2$ where $|\psi^+\rangle$ is a maximally entangled state in $\mathbb{C}^d \otimes \mathbb{C}^d$. Then there is a LHV for the full probability distribution arising from traceless observables for $p \leqslant 1/K_G(d^2-1)$.

IV. BELL INEQUALITIES FOR WERNER STATES

Just as upper bounds on $K_G(n)$ yield LHV models, lower bounds yield Bell inequalities. The case of Werner states appears of particular interest: at present, there is no Bell inequality better than the CHSH inequality at detecting the nonlocality of ρ_p^W [29]. This and other approaches to construct new Bell inequalities will be presented in [27]. Unfortunately, none of these inequalities could be proven to be better than the CHSH inequality. It is remarkable how difficult it is to enlarge this region of Bell violation or, equivalently, to show that $K_G(3) > K_G(2) = \sqrt{2}$. Actually, in the case of random marginal probabilities, as for Bell diagonal states under projective measurements, no improvement over the CHSH inequality can be obtained using $3 \times n$ measurements [30].

A similar result can also be proven for the whole family of the so-called I_{nn22} [23] Bell inequalities. These are specified by a matrix of zeros and ±1 as follows:

$$I_{nn22} = \left(\begin{array}{c|cccccc} & -1 & 0 & \cdots & \cdots & \cdots & 0 \\ \hline -(n-1) & 1 & \cdots & \cdots & \cdots & \cdots & 1 \\ -(n-2) & 1 & \cdots & \cdots & \cdots & 1 & -1 \\ -(n-3) & 1 & \cdots & \cdots & 1 & -1 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ -1 & 1 & 1 & -1 & 0 & \cdots & 0 \\ 0 & 1 & -1 & 0 & \cdots & \cdots & 0 \end{array} \right) \tag{16}$$

All the coefficients in the first column (row) refer to Alice's (Bob's) marginal probabilities, while the rest of the terms are for joint probabilities. Only one of the two possible outcomes, say +1, appears in the inequality and its local bound is always zero. For example, when $n=2$, and denoting $p(a_i,b_j) = p(a_i = +1, b_j = +1)$, I_{2222} reads

$$p(a_1,b_1) + p(a_1,b_2) + p(a_2,b_1) - p(a_2,b_2) - p(a_1 = +1) - p(b_1 = +1) \leqslant 0, \tag{17}$$

which is equivalent to the CHSH inequality.

Theorem 6. Consider the set of I_{nn22} Bell inequalities, for n two-outcome settings. Then, if a Bell diagonal state violates any of these inequalities with projective measurements, it also violates the CHSH inequality.

Proof. Our proof takes advantage of the fact that all marginal probabilities for projective measurements on Bell diagonal states are fully random. Thus, when dealing with these states, one can put all the terms in the first row and column of (16) equal to 1/2. In order to avoid confusion, we denote by I_n' the I_{nn22} inequalities where the local terms have been replaced by 1/2.

We start our proof with the simplest nontrivial case I_{3322}. For Bell diagonal states, it can be written as

$$I_3' = \frac{1}{2}(I_2'(1213) + I_2'(1223) + I_2'(1312) + I_2'(2312)) \leqslant 0, \tag{18}$$

where the arguments of $I_2'(ijkl)$ are the measurements that appear in the I_2' inequality, i and j for Alice, and k and l for Bob. From this identity we have that the violation of I_3' implies that at least one of the I_2' inequalities is violated too. This procedure can be generalized for all n: the idea is to express I_n' in terms of I_2' inequalities using the joint probability terms with a negative sign in (16). For example, when $n=4$ one has

$$I_4' = \frac{1}{3}\Big(I_2'(1214) + I_2'(1224) + I_2'(1234) + I_2'(1313) + I_2'(1323) + I_2'(2313) + I_2'(2323) + I_2'(1412) + I_2'(2412) + I_2'(3412) + p(a_3,b_3) - \frac{1}{2} \Big) \leqslant 0. \tag{19}$$

Note that since all local probabilities are equal to 1/2, $p(a_3,b_3) - 1/2$ is never positive. Thus, whenever $I_4' > 0$, at least one of the I_2' inequalities appearing in (19) is violated. For arbitrary n, I_n' can always be written as

$$I'_n = \frac{1}{n-1}\left[\sum_{i=1}^{s_1(n)} I'_2 + \sum_{i=1}^{s_2(n)}\left(p(a,b) - \frac{1}{2}\right)\right] \leqslant 0, \qquad (20)$$

i.e., the sum of $s_1(n)$ I'_2 inequalities and $s_2(n)$ negative terms $p(a_i,b_j)-1/2$, up to an $n-1$ factor. Some patient calculation shows that $s_1(n)=n(n^2-1)/6$ and $s_2(n)=(n-1)(n-2)(n-3)/6$. Thus, if a Bell diagonal state violates I_{nn22}, it also violates a CHSH inequality. Consequently, none of these inequalities enlarge the known region of Bell violation for Werner states.

After seeing these results, one would be tempted to conjecture that the CHSH violation provides a necessary and sufficient condition for detecting the nonlocality of Bell diagonal states, and in particular of Werner states. This result, however, would imply that $K_G(3)=K_G(2)=\sqrt{2}$, which seems unlikely. Actually, one can find in [25] an explicit construction with 20 settings showing that $K_G(5)\geqslant 10/7>\sqrt{2}$. More recently, one of us has shown that $K_G(4)>\sqrt{2}$ as well [27].

V. CONCLUSIONS

In this work, we have exploited the connection between Bell correlation inequalities and Grothendieck's constants to prove the existence of LHV models for several noisy entangled states. In the case of Werner states, one can demonstrate the existence of a local model for projective measurements up to $p\sim 0.66$, close to the known region of Bell violation. Although we only proved here the existence of the LHV models, the correspondence between noise thresholds and Grothendieck's constants can also be exploited to construct the explicit models. Indeed, these can be extracted from (the proofs of) Krivine's upper bounds on $K_G(n)$. The details are presented in Ref. [27].

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