Permutationally invariant quantum tomography

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Innsbruck, 30 May 2011

Outline

Why tomography is important?

- Many experiments aiming to create many-body entangled states.

- Quantum state tomography can be used to check how well the state has been prepared.

- However, the number of measurements scales exponentially with the number of qubits.

Outline

Physical systems

State-of-the-art in experiments

- 14 qubits with trapped cold ions
  T. Monz, P. Schindler, J.T. Barreiro, M. Chwalla, D. Nigg, W.A. Coish, M. Harlander, W. Haensel, M. Hennrich, R. Blatt, arxiv:1009.6126, 2010.

- 10 qubits with photons
  W.-B. Gao, C.-Y. Lu, X.-C. Yao, P. Xu, O. Gühne, A. Goebel, Y.-A. Chen, C.-Z. Peng, Z.-B. Chen, J.-W. Pan, Nature Physics, 6, 331 (2010).

Outline

Only local measurements are possible

Definition

A single local measurement setting is the basic unit of experimental effort.

A local setting means measuring operator $A^{(k)}$ at qubit k for all qubits.

| $A^{(1)}$ | $A^{(2)}$ | $A^{(3)}$ | ... | $A^{(N)}$ |

- All two-qubit, three-qubit correlations, etc. can be obtained.

$$\langle A^{(1)}A^{(2)}\rangle, \langle A^{(1)}A^{(3)}\rangle, \langle A^{(1)}A^{(2)}A^{(3)}\rangle...$$

Outline

Full quantum state tomography

- The density matrix can be reconstructed from 3^N measurement settings.

Example

For $N = 4$, the measurements are

1.	X	X	X	X
2.	X	X	X	Y
3.	X	X	X	Z
		...		
3^4.	Z	Z	Z	Z

- Note again that the number of measurements scales exponentially in N.

Outline

Experiments with ions and photons

- H. Haeffner, W. Haensel, C. F. Roos, J. Benhelm, D. Chek-al-kar, M. Chwalla, T. Koerber, U. D. Rapol, M. Riebe, P. O. Schmidt, C. Becher, O. Gühne, W. Dür, R. Blatt, Nature 438, 643-646 (2005).
- N. Kiesel, C. Schmid, G. Tóth, E. Solano, and H. Weinfurter, Phys. Rev. Lett. 98, 063604 (2007).

Outline

Approaches to solve the scalability problem

- If the state is expected to be of a certain form (MPS), we can measure the parameters of the ansatz.
  S.T. Flammia *et al.,* arxiv:1002.3839; M. Cramer, M.B. Plenio, arxiv:1002.3780; O. Landon-Cardinal *et al.,* arxiv:1002.4632.

- If the state is of low rank, we need fewer measurements.
  D. Gross et al., Phys. Rev. Lett. 105, 150401 (2010).

- We make tomography in a subspace of the density matrices (our approach).

Outline

Permutationally invariant part of the density matrix

Permutationally invariant part of the density matrix:

$$\varrho_{\mathrm{PI}} = \frac{1}{N!} \sum \Pi_k \varrho \Pi_{k,}^{\dagger}$$

where Π_k are all the permutations of the qubits.

- Related literature: Reconstructing ϱ_{PI} for spin systems.
  [G. M. D'Ariano *et al.*, J. Opt. B **5**, 77 (2003).]

- Photons in a single mode optical fiber are always in a permutationally invariant state. Small set of measurements are needed for their characterization (experiments).
  [R.B.A. Adamson *et al.*, Phys. Rev. Lett. **98**, 043601 (2007); R.B.A. Adamson *et al.*, Phys. Rev. A 2008; L. K. Shalm *et al.*, Nature **457**, 67 (2009).]

Permutationally invariant part of the density matrix II

Examples for permutationally invariant quantum states:

- States of the symmetric subspace, like

$$(|00\rangle + |11\rangle)/\sqrt{2}.$$

- States of the anti-symmetric subspace, like

$$(|01\rangle - |10\rangle)/\sqrt{2}.$$

- Mixture of such states.
- White noise

$$\frac{1}{2^N}(|0\rangle\langle0| + |1\rangle\langle1|)^{\otimes N}.$$

- Symmetric Dicke states mixed with white noise.
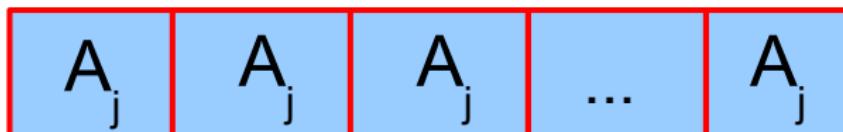
Main results

Features of our method:

1. Is for spatially separated qubits.

2. Needs the minimal number of measurement settings.

3. Uses the measurements that lead to the smallest uncertainty possible of the elements of ϱ_{PI}.

4. Gives an uncertainty for the recovered expectation values and density matrix elements.

5. If ϱ_{PI} is entangled, so is ϱ. Can be used for entanglement detection!

Measurements

- We measure the same observable A_j on all qubits. (Necessary for optimality.)

A_j	A_j	A_j	...	A_j

- Each qubit observable is defined by the measurement directions $\vec{a}_j$ using $A_j = a_{j,x}X + a_{j,y}Y + a_{j,z}Z$.

Number of measurement settings:

$$\mathcal{D}_N = \binom{N+2}{N} = \frac{1}{2}(N^2 + 3N + 2).$$

What do we get from the measurements?

Which quantities do we need:

We obtain the expectation values for

$$\langle (A_j^{\otimes(N-n)} \otimes \mathbb{1}^{\otimes n})_{\mathrm{PI}} \rangle$$

for $j = 1, 2, .., \mathcal{D}_N$ and $n = 0, 1, ..., N$.

For example, for $N = 3$ we have

$$\langle A_j \otimes \mathbb{1} \otimes \mathbb{1} + \mathbb{1} \otimes A_j \otimes \mathbb{1} + \mathbb{1} \otimes \mathbb{1} \otimes A_j \rangle,$$

$$\langle A_j \otimes A_j \otimes \mathbb{1} + \mathbb{1} \otimes A_j \otimes A_j + A_j \otimes \mathbb{1} \otimes A_j \rangle,$$

$$\langle A_j \otimes A_j \otimes A_j \rangle.$$

How do we obtain the Bloch vector elements?

A Bloch vector element can be obtained as

$$\underbrace{\langle (X^{\otimes k} \otimes Y^{\otimes l} \otimes Z^{\otimes m} \otimes \mathbb{1}^{\otimes n})_{\mathrm{PI}} \rangle}_{\text{Bloch vector elements}} = \sum_{j=1}^{\mathcal{D}_N} \underbrace{c_j^{(k,l,m)}}_{\text{coefficients}} \times \underbrace{\langle (A_j^{\otimes (N-n)} \otimes \mathbb{1}^{\otimes n})_{\mathrm{PI}} \rangle}_{\text{Measured data}}.$$

- Coefficients are not unique if $n > 0$.

Uncertainties

The uncertainty of the reconstructed Bloch vector element is

$$\mathcal{E}^2[(X^{\otimes k} \otimes Y^{\otimes l} \otimes Z^{\otimes m} \otimes \mathbb{1}^{\otimes n})_{\mathrm{PI}}] = \sum_{j=1}^{\mathcal{D}_N} |c_j^{(k,l,m)}|^2 \mathcal{E}^2[(A_j^{\otimes(N-n)} \otimes \mathbb{1}^{\otimes n})_{\mathrm{PI}}].$$

- For a fixed set of A_j, we have a formula to find the $c_j^{(k,l,m)}$'s giving the minimal uncertainty.

Optimization for A_j

- We have to find $\mathcal{D}_N$ measurement directions $\vec{a}_j$ on the Bloch sphere minimizing the variance

$$(\mathcal{E}_{\text{total}})^2 = \sum_{k+l+m+n=N} \mathcal{E}^2 \left[(X^{\otimes k} \otimes Y^{\otimes l} \otimes Z^{\otimes m} \otimes \mathbb{1}^{\otimes n})_{\text{PI}} \right] \times \left(\frac{N!}{k!l!m!n!} \right).$$

Summary of algorithm

Obtaining the "total uncertainty" for given measurements

$\left.\begin{array}{ll} \varrho_0, & \text{the state we expect} \\ A_j, & \text{what we measure} \end{array}\right\} \Rightarrow \text{BOX \#1} \Rightarrow (\mathcal{E}_{\text{total}})^2$

Evaluating the experimental results

$\left.\begin{array}{c} \text{measurement results} \\ A_j \end{array}\right\} \Rightarrow \text{BOX \#2} \Rightarrow \left\{\begin{array}{c} \text{Bloch vector elements} \\ \text{variances} \end{array}\right.$

How much is the information loss?

Estimation of the fidelity $F(\varrho, \varrho_{\mathrm{PI}})$:

$$F(\varrho, \varrho_{\mathrm{PI}}) \geq \langle P_{\mathrm{s}} \rangle_{\varrho}^2 \equiv \langle P_{\mathrm{s}} \rangle_{\varrho_{\mathrm{PI}}}^2,$$

where P_{s} is the projector to the *N*-qubit symmetric subspace.

- $F(\varrho, \varrho_{\mathrm{PI}})$ can be estimated only from ϱ_{PI}!

- Proof: using the theory of angular momentum for qubits.

- Similar formalism appear concerning handling multi-copy qubit states:

  [J. I. Cirac, A. K. Ekert, C. Macchiavello, Optimal purification of single qubits PRL 1999.]

  [E. Bagan *et al.*, PRA 2006;
  G. Sentís, E. Bagan, J. Calsamiglia, R. Muñoz-Tapia, Multi-copy programmable discrimination of general qubit states, PRA 2010.]

Outline

Simple example: XY PI tomography

- Let us assume that we want to know only the expectation values of operators of the form

$$\langle A(\phi)^{\otimes N} \rangle$$

where

$$A(\phi) = \cos(\phi)\sigma_x + \sin(\phi)\sigma_y.$$

- The space of such operators has dimension $N + 1$. We have to choose $\{\phi_j\}_{j=1}^{N+1}$ angles for the $\{A_j\}_{j=1}^{N+1}$ operators we have to measure.

Simple example: XY PI tomography II

- Let us assume that we measure

$$\langle A_j^{\otimes N} \rangle$$

for $j = 1, 2, ..., N + 1$.

- Reconstructed values and uncertainties
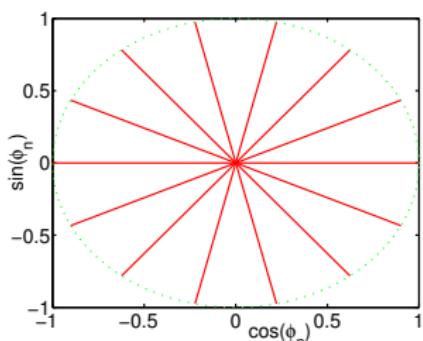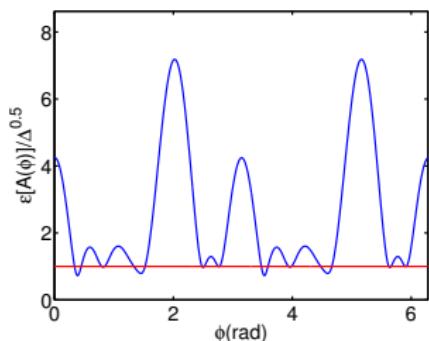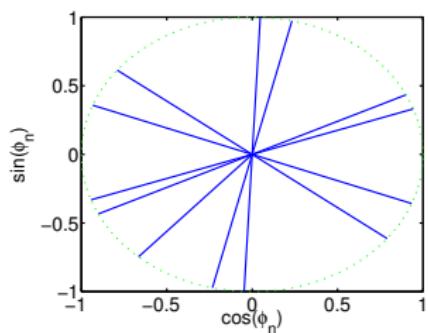
$$\underbrace{\langle A(\phi)^{\otimes N} \rangle}_{\text{Reconstructed}} = \sum_{j=1}^{N+1} \underbrace{c_j^{(\phi)}}_{\text{coefficients}} \times \underbrace{\langle A_j^{\otimes N} \rangle}_{\text{Measured data}}.$$

$$\mathcal{E}^2[A(\phi)] = \sum_{j=1}^{N+1} |c_j^{(\phi)}|^2 \mathcal{E}^2(A_j^{\otimes N}).$$

- Let us assume that all of these measurements have a variance Δ^2.

- Numerical example for $N = 6$.

Random directions ϕ_j Uncertainty of $A(\phi)^{\otimes N}$ Uniform directions
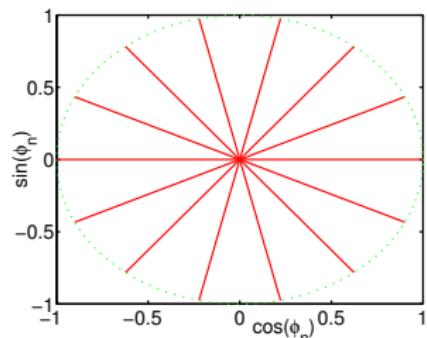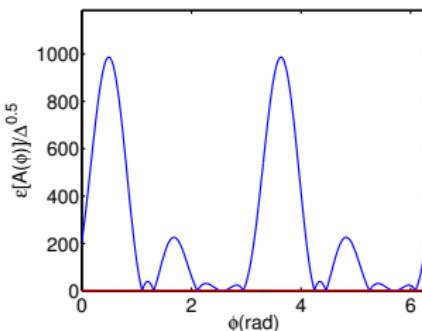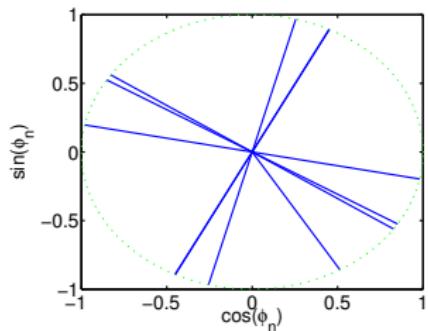
- Numerical example for $N = 6$. This random choice is even worse ...

Random directions ϕ_j Uncertainty of $A(\phi)^{\otimes N}$ Uniform directions

Outline

4-qubit Dicke state, optimized settings (exp.)

(a) Sym. correlation

PI tomogrpahy with optimized settings vs. full tomogrpahy

(b)

a_z

a_x

a_y
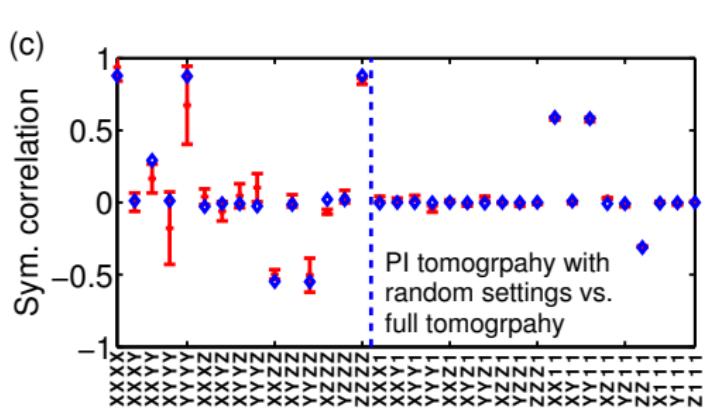
The measured correlations
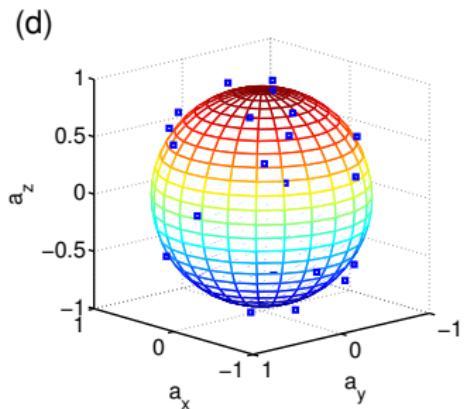
$\vec{a_j}$ measurement directions

Random settings (exp.)

(c) The measured correlations

(d) $\vec{a}_j$ measurement directions

Density matrices (exp.)
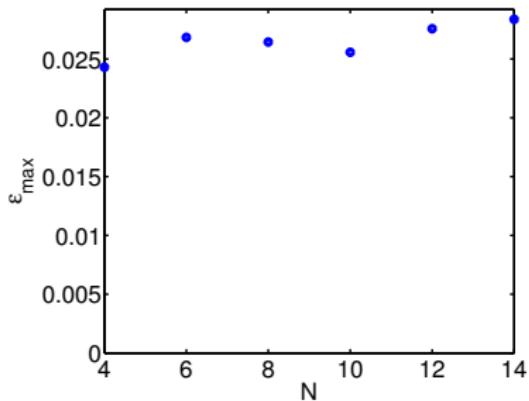
PI tomography for larger systems

- We determined the optimal A_j for p.i. tomography for $N = 4, 6, ..., 14$. The maximal squared uncertainty of the Bloch vector elements is

$$\epsilon_{\max}^2 = \max_{k,l,m,n} \mathcal{E}^2[(X^{\otimes k} \otimes Y^{\otimes l} \otimes Z^{\otimes m} \otimes \mathbb{1}^{\otimes n})_{\text{PI}}]$$

(Total count is the same as in the experiment: 2050.)
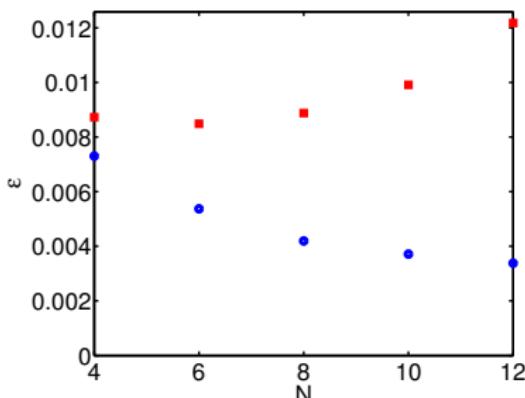
Expectation values directly from measured data

- Operator expectation values can be recovered directly from the measurement data as

$$\langle Op \rangle = \sum_{j=1}^{\mathcal{D}_N} \sum_{n=1}^{N} c_{j,n}^{Op} \langle (A_j^{\otimes(N-n)} \otimes \mathbb{1}^{\otimes n})_{\mathrm{PI}} \rangle,$$

  where the $c_{j,n}^{Op}$ are constants.

- $Op = |D_N^{(N/2)}\rangle\langle D_N^{(N/2)}|$, blue: $\varrho_0 \propto \mathbb{1}$, red: upper bound for any ϱ_0.

Comparison with other methods for efficient tomography

- If a state is detected as entangled, it is surely entangled. No assumption is used concerning the form of the quantum state.

- Expectation values of all permutationally invariant operators are the same for ϱ and ϱ_{PI}.

- Typically, it can be used in experiments in which permutationally invariant states are created.

Comparison with other methods for efficient tomography II

- Individual addressing of the qubits is not needed.

- That is, it is enough ti measure the collective operator

$$\sum_{n=1}^{N} A_j^{(n)}.$$

- This has a clear advantage for many physical systems.

Participants in the project

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Summary

- We discussed permutationally invariant tomography for large multi-qubits systems.

- It paves the way for quantum experiments with more than $6 - 8$ qubits.

See:
G. Tóth, W. Wieczorek, D. Gross, R. Krischek, C. Schwemmer, and H. Weinfurter, Permutationally invariant quantum tomography, Phys. Rev. Lett. 105, 250403 (2010).

THANK YOU FOR YOUR ATTENTION!

How many settings we need?

- Expectation values of $(X^{\otimes k} \otimes Y^{\otimes l} \otimes Z^{\otimes m} \otimes \mathbb{1}^{\otimes n})_{\mathrm{PI}}$ are needed.

- For a given n, the dimension of this subspace is $\mathcal{D}_{(N-n)}$ (simple counting).

- Operators with different n are orthogonal to each other.

- Every measurement setting gives a single real degree of freedom for each subspace

- Hence the number of settings cannot be smaller than the largest dimension, which is $\mathcal{D}_N$.