Uncertainty relations with the variance and the quantum Fisher information

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Outline

1. **Motivation**
   - How can we improve uncertainty relations?
2. **Background**
   - Quantum Fisher information
   - Uncertainty relations
3. **Uncertainty relations with the variance and the QFI**
   - Uncertainty relations based on a convex roof of the bound
   - Uncertainty relations based on a concave roof of the bound
   - Simple observation to prove further relations
   - Further applications

How can we improve uncertainty relations?

- There are many approaches to improve uncertainty relations.
- We show a method that replaces the variance with the quantum Fisher information in some well known uncertainty relations.
- We use convex/concave roofs over the decompostions of the density matrix.

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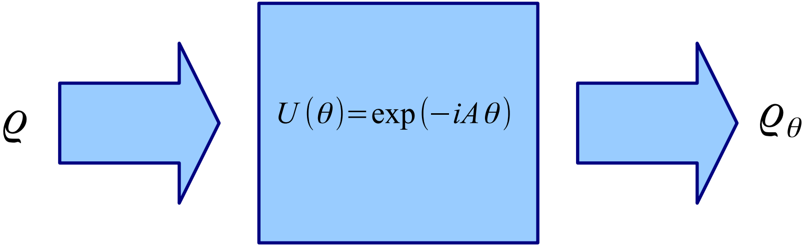
Quantum metrology

- Fundamental task in metrology



- We have to estimate θ in the dynamics

$$U = \exp(-iA\theta).$$

The quantum Fisher information

- Cramér-Rao bound on the precision of parameter estimation

$$(\Delta\theta)^2 \geq \frac{1}{mF_Q[\varrho, A]},$$

where $F_Q[\varrho, A]$ is the quantum Fisher information, and m is the number of independent repetitions.

- The quantum Fisher information is

$$F_Q[\varrho, A] = 2\sum_{k,l} \frac{(\lambda_k - \lambda_l)^2}{\lambda_k + \lambda_l} |\langle k|A|l\rangle|^2,$$

where $\varrho = \sum_k \lambda_k |k\rangle\langle k|$.

Most important characteristics used for estimation

The quantum Fisher information is the convex roof of the variance

$$F_Q[\varrho, A] = 4 \min_{\{p_k, |\psi_k\rangle\}} \sum_k p_k (\Delta A)^2{}_{\psi_k},$$

where

$$\varrho = \sum_k p_k |\psi_k\rangle\langle\psi_k|.$$

[GT, D. Petz, Phys. Rev. A 87, 032324 (2013); S. Yu, arXiv1302.5311 (2013);
GT, I. Apellaniz, J. Phys. A: Math. Theor. 47, 424006 (2014)]

Similar relation for the variance

The quantum Fisher information is the convex roof of the variance

$$(\Delta A)^2{}_{\varrho} = \max_{\{p_k, |\psi_k\rangle\}} \sum_k p_k (\Delta A)^2{}_{\psi_k},$$

where

$$\varrho = \sum_k p_k |\psi_k\rangle\langle\psi_k|.$$

[GT, D. Petz, Phys. Rev. A 87, 032324 (2013);
GT, I. Apellaniz, J. Phys. A: Math. Theor. 47, 424006 (2014)]

A single relation for the QFI and the variance

The previous statements can be concisely reformulated as follows. For any decomposition $\{p_k, |\psi_k\rangle\}$ of the density matrix ϱ we have

$$\frac{1}{4}F_Q[\varrho, A] \leq \sum_k p_k (\Delta A)^2{}_{\psi_k} \leq (\Delta A)^2{}_{\varrho},$$

where the upper and the lower bounds are both tight.

- Note that
$$F_Q[\varrho, A] \leq 4(\Delta A)^2{}_{\varrho},$$
where we have an equality for pure states.

- The QFI appears as a "pair" of variance.

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Robertson-Schrödinger inequality

The Robertson-Schrödinger inequality is defined as

$$(\Delta A)^2{}_\varrho (\Delta B)^2{}_\varrho \geq \frac{1}{4}|L_\varrho|^2,$$

where the lower bound is given by

$$L_\varrho = \sqrt{|\langle\{A,B\}\rangle_\varrho - 2\langle A\rangle_\varrho\langle B\rangle_\varrho|^2 + |\langle C\rangle_\varrho|^2},$$

$\{A,B\} = AB + BA$ is the anticommutator, and we used the definition

$$C = i[A,B].$$

Important: L_ϱ is neither convex nor concave in ϱ.

Heisenberg uncertainty

The Heisenberg inequality is defined as

$$(\Delta A)^2{}_{\varrho}(\Delta B)^2{}_{\varrho} \geq \frac{1}{4}|\langle C\rangle_{\varrho}|^2,$$

where we used the definition

$$C = i[A, B].$$

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Robertson-Schrödinger inequality for ϱ_k

- Consider a decomposition to mixed states

$$\varrho = \sum_k p_k \varrho_k.$$

- For such a decomposition, for all ϱ_k the Robertson-Schrödinger inequality holds

$$(\Delta A)^2{}_{\varrho_k} (\Delta B)^2{}_{\varrho_k} \geq \frac{1}{4} |L_{\varrho_k}|^2.$$

- Let us consider the inequality

$$\left(\sum_k p_k a_k\right)\left(\sum_k p_k b_k\right) \geq \left(\sum_k p_k \sqrt{a_k b_k}\right)^2,$$

where $a_k, b_k \geq 0$.

Uncertainty with the variance and the QFI

- Hence, we arrive at
$$\left[\sum_k p_k (\Delta A)^2{}_{\varrho_k}\right]\left[\sum_k p_k (\Delta B)^2{}_{\varrho_k}\right] \geq \frac{1}{4}\left[\sum_k p_k L_{\varrho_k}\right]^2.$$
- We can choose the decomposition such that
$$\sum_k p_k (\Delta B)^2{}_{\varrho_k} = F_Q[\varrho, B]/4.$$
- Due to the concavity of the variance we also know that
$$\sum_k p_k (\Delta A)^2{}_{\varrho_k} \leq (\Delta A)^2.$$
- Hence, it follows that
$$(\Delta A)^2{}_{\varrho} F_Q[\varrho, B] \geq \left(\sum_k p_k L_{\varrho_k}\right)^2.$$
- In order to use the previous inequality, we need to know the decomposition $\{p_k, \varrho_k\}$ that minimizes $\sum_k p_k (\Delta B)^2{}_{\varrho_k}$.

Uncertainty with the variance and the QFI II

- Then, we can obtain the inequality

$$(\Delta A)^2{}_{\varrho} F_Q[\varrho, B] \geq \left(\min_{\{p_k, \varrho_k\}} \sum_k p_k L_{\varrho_k} \right)^2.$$

where there is a convex roof on the right-hand side.

- After simple algebra, we arrive at the improved Heisenberg-Robertson uncertainty

$$(\Delta A)^2{}_{\varrho} F_Q[\varrho, B] \geq |\langle C \rangle_{\varrho}|^2.$$

Uncertainty with the variance and the QFI IV

- The Heisenberg uncertainty

$$(\Delta A)^2{}_{\varrho}(\Delta B)^2{}_{\varrho} \geq \frac{1}{4}|\langle i[A,B]\rangle_{\varrho}|^2.$$

- The improved Heisenberg uncertinty

$$(\Delta A)^2{}_{\varrho}F_Q[\varrho, B] \geq |\langle i[A,B]\rangle_{\varrho}|^2.$$

- It has been derived originally with a different method in

F. Fröwis, R. Schmied, and N. Gisin, Phys. Rev. A 92, 012102 (2015).

Outline

Uncertainty relation based on a concave roof

- For any decomposition $\{p_k, \varrho_k\}$ we have

$$(\Delta A)^2(\Delta B)^2 \geq \frac{1}{4}\left(\sum_k p_k L_{\varrho_k}\right)^2,$$

where

$$L_\varrho = \sqrt{|\langle\{A, B\}\rangle_\varrho - 2\langle A\rangle_\varrho\langle B\rangle_\varrho|^2 + |\langle C\rangle_\varrho|^2}.$$

- We can even take a concave roof on the right-hand side

$$(\Delta A)^2{}_\varrho(\Delta B)^2{}_\varrho \geq \frac{1}{4}\left(\max_{\{p_k,\varrho_k\}} \sum_k p_k L_{\varrho_k}\right)^2.$$

- We prove that for qubits the above inequality is saturated for all states.

Uncertainty relations with two variances and the QFI

- Let us start from the relation for pure states

$$(\Delta J_x)^2 + (\Delta J_y)^2 + (\Delta J_z)^2 \geq j,$$

where J_l are the spin components fulfilling

$$J_x^2 + J_y^2 + J_z^2 = j(j+1)\mathbb{1}.$$

- Based on similar ideas we arriving at

$$(\Delta J_x)^2 + (\Delta J_y)^2 + \tfrac{1}{4} F_Q[\varrho, J_z] \geq j.$$

- See parallel publication in

S.-H. Chiew and M. Gessner, Phys. Rev. Research 4, 013076 (2022).

Outline

Simple observation to prove further relations

- Let us consider a relation

$$\underbrace{(\Delta A)^2{}_\varrho}_{\text{variance}} \geq \underbrace{g(\varrho)}_{\text{convex in } \varrho},$$

  which is true for pure states.

- If $g(\varrho)$ is convex in density matrices, then

$$\frac{1}{4}F_Q[\varrho, A] \geq g(\varrho)$$

  holds for mixed states.

- *Proof.* $\frac{1}{4}F_Q[\varrho, A]$ is given as a convex roof of the variance.

- It is the largest convex function that equals $(\Delta A)^2{}_\varrho$ for all pure states. ∎

Extreme spin squeezing

- For a particle with spin-j

$$\underbrace{(\Delta J_x)^2}_{\text{variance}} \geq \underbrace{jF_j(\langle J_z\rangle/j)}_{\text{convex in } \varrho}$$

holds, where $F_j(X)$ is a convex function defined as

$$F_j(X) = \min_{\varrho:\langle J_z\rangle = Xj} \frac{(\Delta J_x)^2}{j}.$$

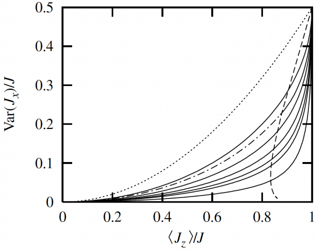

FIG. 1. Maximal squeezing for different values of J. The curves starting at the origin represent the minimum obtainable variance as a function of the mean spin. Starting from above, the curves represent $J = 1/2$, 1, 3/2, 2, 3, 4, 5, and 10. The dotted curve for $J = 1/2$ is the limit identified in Ref. [11]. The solid

Application for extreme spin squeezing

- The metrological usefulness of a state is bounded with the spin-length as
$$F_Q[\varrho, J_x] \geq 4jF_j(\langle J_z \rangle / j).$$

- *Proof.* For the components of the angular momentum
$$(\Delta J_x)^2 \geq jF_j(\langle J_z \rangle / j)$$
holds. Then, it follows
$$\frac{1}{4}F_Q[\varrho, J_x] \geq jF_j(\langle J_z \rangle / j).$$
∎

Can also be seen based on I. Apellaniz, M. Kleinmann, O. Gühne, and G. Tóth, Phys. Rev. A 95, 032330 (2017).

Outline

Further applications

- Metrological usefulness based on the violation of entanglement conditions.
- Cramér-Rao bound as a convex roof.
- Conditions on the saturation of the improved Heisenberg uncertainty.

Summary

- We showed how to derive new uncertainty relations with the variance and the quantum Fisher information based on simple convexity arguments.

See:

Géza Tóth and Florian Fröwis,

Uncertainty relations with the variance and the quantum Fisher information based on convex decompositions of density matrices,

Phys. Rev. Research 4, 013075 (2022).

THANK YOU FOR YOUR ATTENTION!

QUANTERA
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