

**Entanglement measures  
(How much is it entangled?)  
(Lecture of the Quantum Information class of  
the Master in Quantum Science and  
Technology)**

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## 1 Entanglement measures (How much is it entangled?)

- Motivation
- A. General quantum operation
- B. Local operations and classical communication (LOCC)
- C. Entanglement of formation
- D. Concurrence
- E. Entanglement of distillation
- F. Bound entanglement

# Entanglement measures

- After detecting entanglement, we have to ask how entangled the state is.
- It will turn out that entanglement is a resource.

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# General quantum operation

- The general quantum operation is defined as

$$\varrho' = \sum_k E_k \varrho E_k^\dagger$$

with

$$\sum_k E_k^\dagger E_k = 1.$$

- $E_k$  are Kraus operators.
- Generalized measurements, POVM (positive operator-valued measure).
- Special case: von Neumann measurements, when  $E_k$  are pairwise orthogonal projectors.
- Naimark's dilation theorem:  
general operation =  
von Neumann measurement on system+ancilla.

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# Local operations and classical communication (LOCC)

- LOCC are
  - local unitaries,
  - local von Neumann or POVM measurements,
  - local unitaries or measurements conditioned on measurement outcomes on the other party.
- Mathematical description of LOCC. Separable operations are a somewhat larger set, however, this set can easily be described.

$$\varrho' = \sum_k E_k^{(1)} \otimes E_k^{(2)} \varrho \left( E_k^{(1)} \otimes E_k^{(2)} \right)^\dagger$$

with

$$\sum_k \left( E_k^{(1)} \otimes E_k^{(2)} \right)^\dagger \left( E_k^{(1)} \otimes E_k^{(2)} \right) = 1.$$

# Local operations and classical communication (LOCC) II

- Stochastic Local Operations and Classical Communication (SLOCC):

$$|\Psi\rangle' \leftarrow E_k^{(1)} \otimes E_k^{(2)} |\Psi\rangle$$

It happens with some probability, not deterministic.

- LOCC cannot create entanglement. Separable states remain separable under LOCC.
- LOCC can create correlations.



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# Entropy of entanglement

- The von Neumann entropy is defined as

$$S(\varrho) = -\text{Tr}(\varrho \log_2 \varrho).$$

- It can be written with the eigenvalues of the density matrix as

$$S(\varrho) = -\sum_{k=1}^d \lambda_k \log_2 \lambda_k.$$

- For a pure state we have  $\lambda_k = \{1, 0, 0, \dots, 0\}$ , and thus it is zero.
- Its maximal is for the completely mixed state for which  $\lambda_k = \{\frac{1}{d}, \frac{1}{d}, \frac{1}{d}, \dots, \frac{1}{d}\}$ , and its value is  $\log_2 d$ .
- For a bipartite pure state, the **entropy of entanglement** is

$$E_E(|\Psi\rangle) = S(\text{Tr}_1(|\Psi\rangle\langle\Psi|)).$$

That is, it is the von Neumann entropy of the reduced state is an entanglement measure.

# Entropy of entanglement II

- Comments
  - It is one for two-qubit singlet states.
  - It is zero for product states.
  - It is invariant under  $U_1 \otimes U_2$ .

# Entanglement of formation

- For mixed states, the entanglement of formation is the convex roof of the von Neumann entropy of the reduced state.

$$E_F = \min_{|\Psi_k\rangle, p_k} \sum_k p_k E_E(|\Psi_k\rangle),$$

- The optimization is over all decompositions of the state of the type

$$\varrho = \sum_k p_k |\Psi_k\rangle\langle\Psi_k|.$$

- $E_F$  tells us, in the asymptotic limit, how many singlets we need to create the state.
- Is it easy to compute? No. For  $2 \times 2$  systems, there is an explicit formula with the concurrence. For larger systems, there is not a general method.

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# Entanglement of formation

- For two qubits,  $E_F$  can be calculated explicitly (Wootters, 1997).
- The **concurrence** is defined as

$$C(\varrho) = \max(0, \lambda_1 - \lambda_2 - \lambda_3 - \lambda_4),$$

where  $\lambda_k$ 's are, in a decreasing order, the eigenvalues of

$$R = \sqrt{\sqrt{\varrho} \tilde{\varrho} \sqrt{\varrho}},$$

and

$$\tilde{\varrho} = (\sigma_y \otimes \sigma_y) \varrho^* (\sigma_y \otimes \sigma_y).$$

We also need that

$$\epsilon(c) = H_2\left(\frac{1 + \sqrt{1 - c^2}}{2}\right).$$

Here

$$H_2 = -x \log_2 x - (1 - x) \log_2 (1 - x).$$

- Then,  $E_F$  can be obtained as

$$E_F(\varrho) = \epsilon(C(\varrho)).$$

# Entanglement of formation II

- Special case: for pure states

$$C(|\Psi\rangle) = |\langle\Psi|\tilde{\Psi}\rangle| = 2|a_{11}a_{22} - a_{12}a_{21}|,$$

where

$$|\Psi\rangle = \begin{pmatrix} a_{11} \\ a_{12} \\ a_{21} \\ a_{22} \end{pmatrix}.$$

Scott Hill and William K. Wootters, Entanglement of a Pair of Quantum Bits, Phys. Rev. Lett. 78, 5022 (1997).

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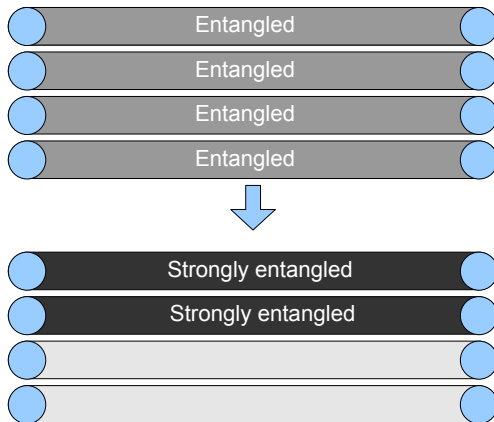
# Entanglement of distillation

- $E_D$  tells us, how many singlets we can obtain from the state with LOCC. In general,

$$E_F \geq E_D.$$

- Note that local operation and classical communication means that we have several copies and we can act on the copies locally.

# Entanglement of distillation II



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# Bound entanglement

- There are states that need entangled particles to be created, but singlets cannot be distilled from them.
- All PPT entangled states are like that. (That is, all entangled states that are not detected by the Peres-Horodecki criterion.)

# Bound entanglement II

- Let us again remember the flip operator

$$F|k\rangle|l\rangle = |l\rangle|k\rangle$$

It has eigenvalues  $\pm 1$ .

- The maximally entangled state

$$|\Psi_{\text{me}}\rangle = \frac{1}{\sqrt{d}} \sum_{k=1}^d |k\rangle|k\rangle.$$

# Bound entanglement III

- We can show that

$$|\Psi_{\text{me}}\rangle\langle\Psi_{\text{me}}| = \frac{1}{d} \sum_{k,l} |k\rangle\langle l| \otimes |k\rangle\langle l|,$$

$$|\Psi_{\text{me}}\rangle\langle\Psi_{\text{me}}|^{T_1} = \frac{1}{d} \sum_{k,l} |k\rangle\langle l| \otimes |l\rangle\langle k| \equiv \frac{F}{d}.$$

- Now we show that PPT states have a small overlap with the maximally entangled state. For PPT states, the fidelity with respect to the maximally entangled state is

$$\text{Tr}(|\Psi_{\text{me}}\rangle\langle\Psi_{\text{me}}|\varrho) = \text{Tr}(|\Psi_{\text{me}}\rangle\langle\Psi_{\text{me}}|^{T_1}\varrho^{T_1}) = \frac{1}{d}\text{Tr}(F\varrho^{T_1}) \leq \frac{1}{d},$$

since  $\varrho^{T_1} \geq 0$  and  $F$  has  $\pm 1$  eigenvalues.

## Bound entanglement IV

- Thus, PPT states have a small fidelity with respect to the maximally entangled state. Even LOCC operations cannot increase this.
- A simple product state can reach  $1/d$

$$\text{Tr}(|\Psi_{\text{me}}\rangle\langle\Psi_{\text{me}}||11\rangle\langle 11|) = \frac{1}{d}.$$