Quantum states with a positive partial transpose are useful for metrology

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12 November 2019

Outline

1. **Motivation**
   - What are entangled states useful for?

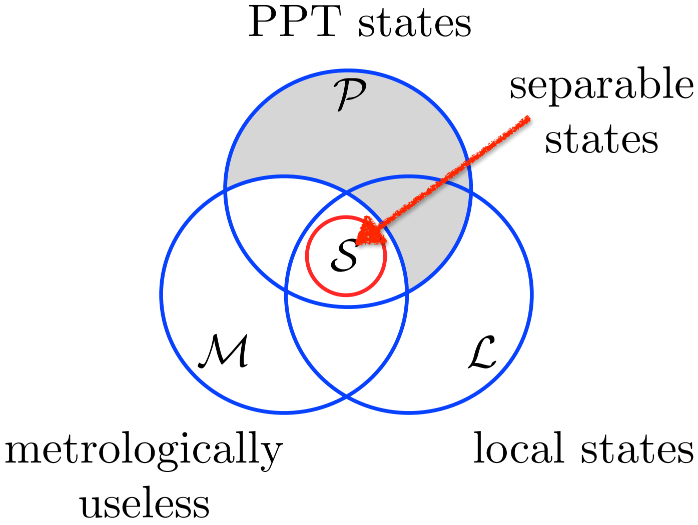
2. **Bacground**
   - Quantum Fisher information
   - Recent findings on the quantum Fisher information

3. **Maximizing the QFI for PPT states**
   - Results so far
   - Our results

What are entangled states useful for?

- Entangled states are useful, but not all of them are useful for some task.
- Entanglement is needed for beating the shot-noise limit in quantum metrology.
- Intriguing question: Are states with a positive partial transpose useful for metrology? Can they also beat the shot-noise limit?

What are entangled states useful for?



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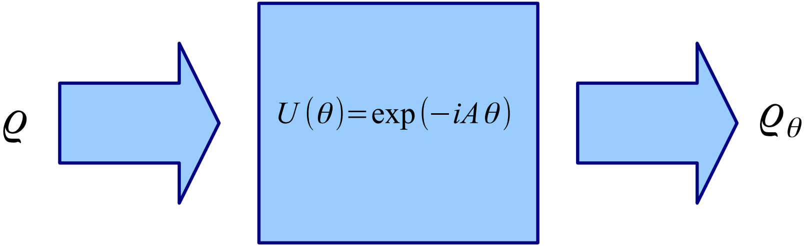
Quantum metrology

- Fundamental task in metrology



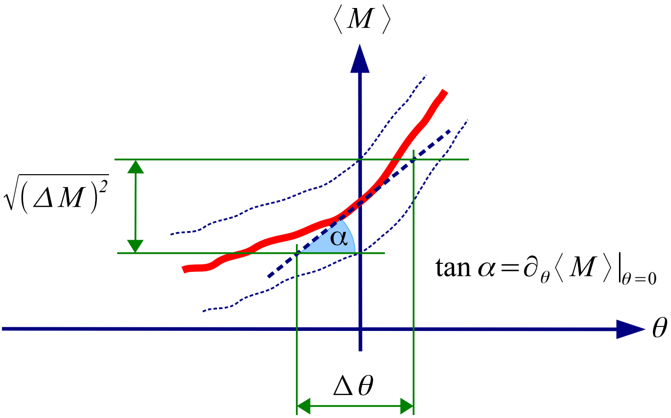
- We have to estimate θ in the dynamics

$$U = \exp(-iA\theta).$$

Precision of parameter estimation

- Measure an operator M to get the estimate θ. The precision is

$$(\Delta\theta)^2 = \frac{(\Delta M)^2}{|\partial_\theta \langle M \rangle|^2}.$$



The quantum Fisher information

- Cramér-Rao bound on the precision of parameter estimation

$$(\Delta\theta)^2 \geq \frac{1}{F_Q[\varrho, A]}, \qquad (\Delta\theta)^{-2} \leq F_Q[\varrho, A].$$

where $F_Q[\varrho, A]$ is the quantum Fisher information.

- The quantum Fisher information is

$$F_Q[\varrho, A] = 2 \sum_{k,l} \frac{(\lambda_k - \lambda_l)^2}{\lambda_k + \lambda_l} |\langle k|A|l\rangle|^2,$$

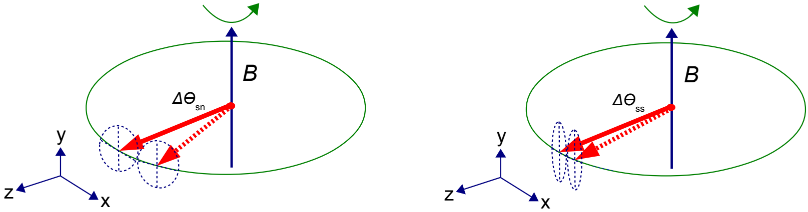
where $\varrho = \sum_k \lambda_k |k\rangle\langle k|$.

Special case $A = J_l$

- The operator A is defined as

$$A = J_l = \sum_{n=1}^{N} j_l^{(n)}, \quad l \in \{x, y, z\}.$$

- Magnetometry with a linear interferometer



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Properties of the Fisher information

Many bounds on the quantum Fisher information can be derived from these simple properties:

- For pure states, it equals four times the variance, $F[|\Psi\rangle\langle\Psi|, A] = 4(\Delta A)^2{}_{\Psi}$.
- For mixed states, it is convex.

The quantum Fisher information vs. entanglement

- For separable states

$$F_Q[\varrho, J_l] \le N, \qquad l = x, y, z.$$

[Pezze, Smerzi, Phys. Rev. Lett. 102, 100401 (2009); Hyllus, Gühne, Smerzi, Phys. Rev. A 82, 012337 (2010)]

- For states with at most k-particle entanglement (k is divisor of N)

$$F_Q[\varrho, J_l] \le kN.$$

[P. Hyllus *et al.*, Phys. Rev. A 85, 022321 (2012); GT, Phys. Rev. A 85, 022322 (2012)].

- Macroscopic superpositions (e.g, GHZ states, Dicke states)

$$F_Q[\varrho, J_l] \propto N^2,$$

[F. Fröwis, W. Dür, New J. Phys. 14 093039 (2012).]

Most important characteristics used for estimation

The quantum Fisher information is the convex roof of the variance

$$F_Q[\varrho, A] = 4 \min_{p_k, \Psi_k} \sum_k p_k (\Delta A)^2{}_k,$$

where

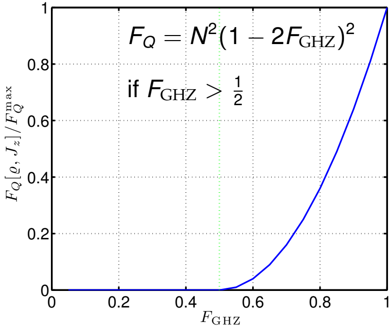
$$\varrho = \sum_k p_k |\Psi_k\rangle\langle\Psi_k|.$$

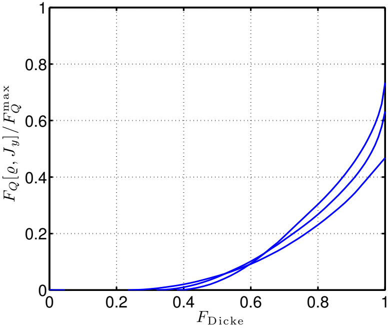
[GT, D. Petz, Phys. Rev. A 87, 032324 (2013); S. Yu, arXiv1302.5311 (2013); GT, I. Apellaniz, J. Phys. A: Math. Theor. 47, 424006 (2014)]

- Extended convexity for non-unitary dynamics.
  [S. Alipour, A. T. Rezakhani, Phys. Rev. A 91, 042104 (2015).]
- Convex roof over purifications.
  [R. Demkowicz-Dobrzański, J. Kołodyński, M. Guţă, Nature Comm. 2012.]

Witnessing the quantum Fisher information based on few measurements

- Let us bound the quantum Fisher information based on some measurements.





Quantum Fisher information vs. Fidelity with respect to (a) GHZ states and (b) Dicke states for $N = 4, 6, 12$.

[Apellaniz *et al.*, Phys. Rev. A 2017]

Continuouity of QFI and QFI for symmetric states

- Arbitrarily small entanglement can be used to get close to Heisenberg scaling.
- The difference between the QFI of two states can be bounded by the distance of the two states.
- Bound on the QFI with the geometric measure of entanglement.

[R. Augusiak, J. Kołodyński, A. Streltsov, M. N. Bera, A. Acín, M. Lewenstein, PRA 2016]

- Continuity in the non-unitary case:

[A. T. Rezakhani, S. Alipour, M. Hassani, PRA 2019]

- Random pure states of distinguishable particles typically do not lead to super-classical scaling of precision.
- Random states from the symmetric subspace typically achieve the optimal Heisenberg scaling.

[M. Oszmaniec, R. Augusiak, C. Gogolin, J. Kołodyński, A. Acín, M. Lewenstein, PRX 2016]

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Results so far concerning metrologically useful PPT states

- Bound entangled states with PPT and some non-PPT partitions.
- Violates an entanglement criterion with three QFI terms.

  [P. Hyllus, W. Laskowski, R. Krischek, C. Schwemmer, W. Wieczorek, H. Weinfurter, L. Pezze, and A. Smerzi, PRA 85, 022321 (2012).]

- Non-unlockable bound entangled states with PPT and some non-PPT partitions.
- Violates the criterion with a single QFI term, better than shot-noise limit.

  [Ł. Czekaj, A. Przysiężna, M. Horodecki, P. Horodecki, Phys. Rev. A 92, 062303 (2015).]

on nonlocality [43]) to answer would be, Is there any family of quantum states that allows for a general Local Hidden Variables (LHV) model but can be used to obtain sub-shot-noise (i.e., better than classical) quantum metrology? This question is related to another question (especially in the context of both general requirements in quantum metrology [26] and recent results on nonlocality [43]) regarding whether there is any chance for sub-shot-noise metrology for states obeying the PPT condition with respect to *any* cut. While the present result

Outline

Our results

We look for bipartite PPT entangled states and multipartite states that are PPT with respect to all partitions.

Maximizing the QFI for PPT states: brute force

- Maximize the QFI for PPT states. Remember

$$F_Q[\varrho, A] = 2\sum_{k,l} \frac{(\lambda_k - \lambda_l)^2}{\lambda_k + \lambda_l} |\langle k|A|l\rangle|^2,$$

  where $\varrho = \sum_k \lambda_k |k\rangle\langle k|$.

- Difficult to maximize a convex function over a convex set. The maximum is taken on the boundary of the set.

- Not guaranteed to find the global maximum.

- Note: Finding the *minimum* is possible!

Maximizing the QFI for PPT state: our method

- We mentioned that the QFI gives a bound on the precision of the parameter estimation

$$F_Q[\varrho, A] \geq \frac{1}{(\Delta\theta)^2} = \frac{|\partial_\theta \langle M \rangle|^2}{(\Delta M)^2} = \frac{\langle i[M, A] \rangle^2}{(\Delta M)^2} \quad \text{(dynamics is } U = e^{-iA\theta}\text{)}.$$

- The bound is sharp

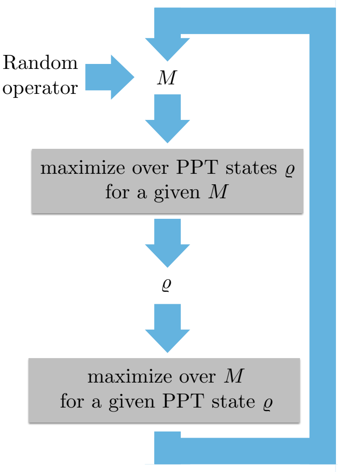
$$F_Q[\varrho, A] = \max_M \frac{\langle i[M, A] \rangle_\varrho^2}{(\Delta M)^2}.$$

[M. G. Paris, Int. J. Quantum Inform. 2009. Used, e.g., in F. Fröwis, R. Schmied, and N. Gisin, 2015; I. Appelaniz *et al.*, NJP 2015.]

The maximum for PPT states can be obtained as

$$\max_{\varrho \text{ is PPT}} F_Q[\varrho, A] = \max_{\varrho \text{ is PPT}} \max_M \frac{\langle i[M, A] \rangle_\varrho^2}{(\Delta M)^2}.$$

Sew-saw algorithm for maximizing the precision



Similar iterative approach was used for maximzing over ϱ for noisy states: [Macieszczak, arXiv:1312.1356v1;

Macieszczak, Fraas, Demkowicz- Dobrzanski, NJP 2014.]

Maximize over PPT states for a given M

Best precision for PPT states for a given operator M can be obtained by a semidefinite program.

Proof.—Let us define first

$$\begin{aligned} f_M(X,Y) = \min_{\varrho} \quad & \mathrm{Tr}(M^2\varrho), \\ \text{s.t.} \quad & \varrho \geq 0, \varrho^{\mathrm{T}k} \geq 0 \text{ for all } k, \mathrm{Tr}(\varrho) = 1, \\ & \langle i[M,A]\rangle = X \text{ and } \langle M\rangle = Y. \end{aligned}$$

The best precsion for a given M and for PPT states is

$$(\Delta\theta)^2 = \min_{X,Y} \frac{f_M(X,Y) - Y^2}{X^2}.$$

The state giving the best precision is $\varrho_{\mathrm{PPTopt}}$.

Maximize over M for a given PPT state

For a state ϱ, the best precision is obtained with the operator given by the symmetric logarithmic derivative

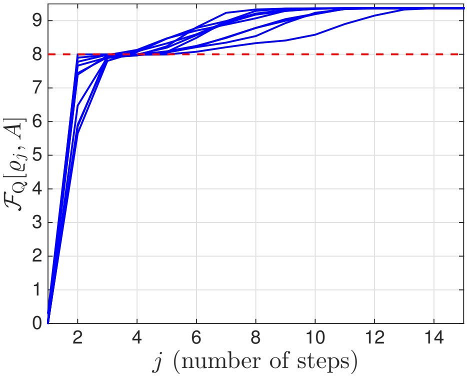
$$M = 2i \sum_{k,l} \frac{\lambda_k - \lambda_l}{\lambda_k + \lambda_l} |k\rangle\langle l| \langle k|A|l\rangle,$$

where $\varrho = \sum_k \lambda_k |k\rangle\langle k|$.

Convergence of the method

The precision cannot get worse with the iteration!

Convergence of the method II



Generation of the 4×4 bound entangled state.

(blue) 10 attempts. After 15 steps, the algorithm converged.

(red) Maximal quantum Fisher information for separable states.

Robustness of the states

$$\varrho(p) = (1-p)\varrho + p\varrho_{\rm noise}$$

- Robustness of entanglement: the maximal p for which $\varrho(p)$ is entangled for any separable $\varrho_{\rm noise}$.
  [Vidal and Tarrach, PRA 59, 141 (1999).]

- Robustness of metrological usefulness: the maximal p for which $\varrho(p)$ outperforms separable state for any separable $\varrho_{\rm noise}$.

Robustness of the states II

System	A	$\mathcal{F}_{\mathrm{Q}}[\varrho, A]$	$\mathcal{F}_{\mathrm{Q}}^{(\mathrm{sep})}$	$p_{\text{white noise}}$
four qubits	J_z	4.0088	4	0.0011
three qubits	$j_z^{(1)} + j_z^{(2)}$	2.0021	2	0.0005
2×4 (three qubits, only $1:23$ is PPT)	$j_z^{(1)} + j_z^{(2)}$	2.0033	2	0.0008

Multiqubit states

Robustness of the states III

d	$\mathcal{F}_Q[\varrho, A]$	$p_{\text{white noise}}$	$p_{\text{noise}}^{\text{LB}}$
3	8.0085	0.0006	0.0003
4	9.3726	0.0817	0.0382
5	9.3764	0.0960	0.0361
6	10.1436	0.1236	0.0560
7	10.1455	0.1377	0.0086
8	10.6667	0.1504	0.0670
9	10.6675	0.1631	0.0367
10	11.0557	0.1695	0.0747
11	11.0563	0.1807	0.0065
12	11.3616	0.1840	0.0808

- $d \times d$ systems.
- Maximum of the quantum Fisher information for separable states is 8.
- The operator A is not the usual J_z.

Robustness of the states IV: 4×4 bound entangled PPT state

Let us define the following six states
$|\Psi_1\rangle = (|0,1\rangle + |2,3\rangle)/\sqrt{2}$, $|\Psi_2\rangle = (|1,0\rangle + |3,2\rangle)/\sqrt{2}$,
$|\Psi_3\rangle = (|1,1\rangle + |2,2\rangle)/\sqrt{2}$, $|\Psi_4\rangle = (|0,0\rangle - |3,3\rangle)/\sqrt{2}$,
$|\Psi_5\rangle = (1/2)(|0,3\rangle + |1,2\rangle) + |2,1\rangle/\sqrt{2}$,
$|\Psi_6\rangle = (1/2)(-|0,3\rangle + |1,2\rangle) + |3,0\rangle/\sqrt{2}$.

Our state is a mixture

$$\varrho_{4\times 4} = p\sum_{n=1}^{4} |\Psi_n\rangle\langle\Psi_n| + q\sum_{n=5}^{6} |\Psi_n\rangle\langle\Psi_n|,$$

where $q = (\sqrt{2}-1)/2$ and $p = (1-2q)/4$. We consider the operator

$$A = H \otimes \mathbb{1} + \mathbb{1} \otimes H,$$

where $H = \mathrm{diag}(1, 1, -1, -1)$.

Negativity

Apart from making calculations for PPT bound entangled states, we can also make calculations for states with given minimal eigenvalues of the partial transpose, or for a given negativity.

[G. Vidal and R. F. Werner, PRA 65, 032314 (2002).]

Entanglement

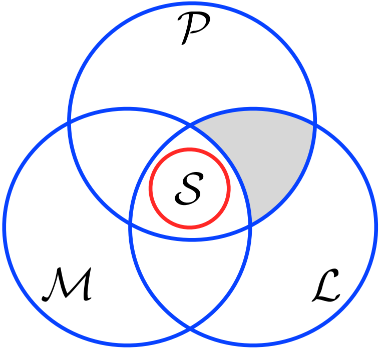
Bipartite state	Entanglement
3×3	0.0003
4×4	0.0147
5×5	0.0239
6×6	0.0359
7×7	0.0785
UPB 3×3	0.0652
Breuer 4×4	0.1150

Convex roof of the linear entanglement entropy. The entanglement is also shown for the 3×3 state based on unextendible product bases (UPB) and for the Breuer state with a parameter $\lambda = 1/6$.

[G. Tóth, T. Moroder, and O. Gühne, PRL 114, 160501 (2015).]

Metrologically useful quantum states with LHV models (PPT)

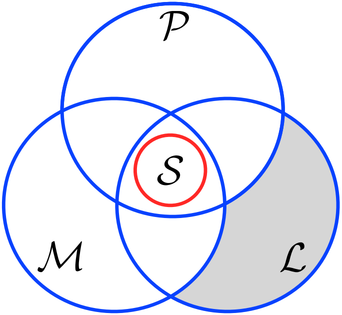
Consider the 2×4 state listed before. Possible to construct numerically a LHV model for the state.



[F. Hirsch, M. T. Quintino, T. Vértesi, M. F. Pusey, and N. Brunner, PRL 2016;
D. Cavalcanti, L. Guerini, R. Rabelo, and P. Skrzypczyk, PRL 2016.]

Metrologically useful quantum states with LHV models (non-PPT)

- Two-qubit Werner state $p|\Psi^-\rangle\langle\Psi^-| + (1-p)\mathbb{1}/4$, with $|\Psi^-\rangle = (|01\rangle - |10\rangle)/\sqrt{2}$.
- Better for metrology than separable states ($\mathcal{F}_{\mathrm{Q}} > 2$) for $p > 0.3596$.
- They violate a Bell inequality for $p > 0.3171$.



[F. Hirsch, M. T. Quintino, T. Vértesi, M. Navascués, N. Brunner, Quantum 2017; A. Acín, N. Gisin, B. Toner, PRA 2006.]

Cluster states

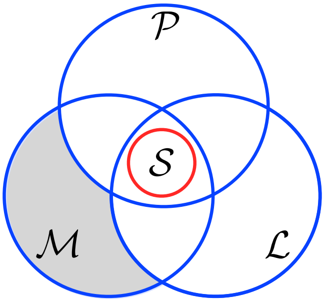
Cluster states: resource in measurement-based quantum computing

[R. Raussendorf and H. J. Briegel, PRL 2001.]

- Fully entangled pure states.
- Violate a Bell inequality

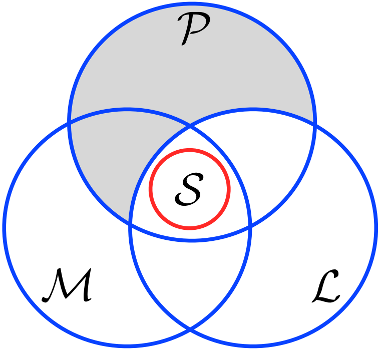
  [V. Scarani, A. Acín, E. Schenck, M. Aspelmeyer, PRA 2005; O. Gühne, GT, P. Hyllus, H. J. Briegel, PRL 2005; GT, O. Gühne, and H. J. Briegel, PRA 2006.]
- Metrologically not useful

  [P. Hyllus, O. Gühne, and A. Smerzi, PRA 2010.]



Non-local PPT states

Counterexample for the Peres conjecture



[T. Vértesi and N. Brunner, Nature Communications 2015.]

Summary

- We presented quantum states with a positive partial transpose with respect to all bipartitions that are useful for metrology.

See:

Géza Tóth and Tamás Vértesi,

Quantum states with a positive partial transpose are useful for metrology,

Phys. Rev. Lett. 120, 020506 (2018).

THANK YOU FOR YOUR ATTENTION!

European Research Council

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