

# Quantum Wasserstein distance based on an optimization over separable states

## Quantum 7, 914 (2023)

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# Outline

## 1 Motivation

- Connecting Wasserstein distance to entanglement theory

## 2 Background

- Quantum Wasserstein distance
- Quantum Fisher information

## 3 Wasserstein distance and separable states

- Quantum Wasserstein distance based on an optimization over separable states
- Relation to entanglement conditions

# Motivation

- Many distance measures are maximal for orthogonal states.
- Recently, the Wasserstein distance appeared, which is different and this makes it very useful.
- For the quantum case, surprisingly, the self-distance can be nonzero.
- Can we connect these to entanglement theory and/or quantum metrology?

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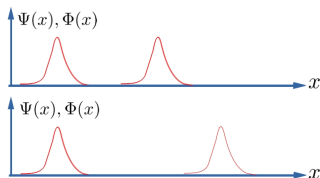
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## Wasserstein distance and separable states

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# An important property of the Wasserstein distance

- Many distance measures are maximal for orthogonal states, e.g., for the following state-pairs.



- In the second example, the two states are further apart from each other, based on common sense.
- Wasserstein distance can recognize this since it is related to the "cost of moving sand from a distribution to the other one."
- It can be used for machine learning.

# Quantum Wasserstein distance

- **Definition.**—The square of the distance between two quantum states described by the density matrices  $\varrho$  and  $\sigma$  is

$$D_{\text{DPT}}(\varrho, \sigma)^2 = \frac{1}{2} \min_{\varrho_{12}} \sum_{n=1}^N \text{Tr}[(H_n^T \otimes \mathbb{1} - \mathbb{1} \otimes H_n)^2 \varrho_{12}],$$

s. t.  $\varrho_{12} \in \mathcal{D},$   
 $\text{Tr}_2(\varrho_{12}) = \varrho^T,$   
 $\text{Tr}_1(\varrho_{12}) = \sigma,$

where  $\mathcal{D}$  is the set of density matrices, and  $H_n$  are Hermitian matrices.

- Note the relation to the representability problem.

G. De Palma and D. Trevisan,  
Quantum optimal transport with quantum channels,  
Ann. Henri Poincaré 22, 3199 (2021).

# Self-distance can be nonzero (unlike in the classical case)

- The self-distance of a state is

$$D_{\text{DPT}}(\varrho, \varrho)^2 = \sum_{n=1}^N I_{\varrho}(H_n),$$

where the Wigner-Yanase skew information is defined as

$$I_{\varrho}(H) = \text{Tr}(H^2 \varrho) - \text{Tr}(H \sqrt{\varrho} H \sqrt{\varrho}).$$

- This connects Wasserstein distance and quantum metrology.

G. De Palma and D. Trevisan,  
Quantum optimal transport with quantum channels,  
Ann. Henri Poincaré 22, 3199 (2021).



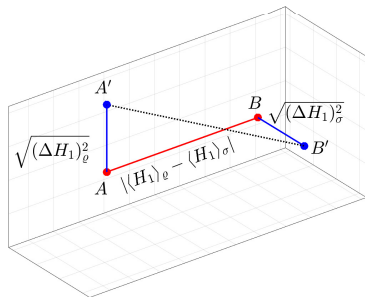
# How to compute the Wasserstein distance?

- The distance can be computed by semidefinite programming.  
→ This might be the reason that it has appeared recently.
- For a pure  $\varrho$  and a mixed  $\sigma$ , the distance is given as

$$\begin{aligned} D_{\text{DPT}}(\varrho, \sigma)^2 \\ = \frac{1}{2} \sum_{n=1}^N \left[ (\Delta H_n)_{\varrho}^2 + (\Delta H_n)_{\sigma}^2 + (\langle H_n \rangle_{\varrho} - \langle H_n \rangle_{\sigma})^2 \right], \end{aligned}$$

see the following figure, where  $(\Delta H_n)^2$  is the variance.

# How to compute the Wasserstein distance? II



- $N = 1$ , a single operator  $H_1$  is given.
- $\varrho$  is pure,  $\sigma$  is mixed.
- The quantum Wasserstein distance equals  $1/\sqrt{2}$  times the usual Euclidean distance between  $A'$  and  $B'$ .

GT and J. Pitrik,  
Quantum Wasserstein distance based on an optimization over separable states,  
Quantum 7, 914 (2023).

# Recent efforts to prove that it is a distance

- For any  $\varrho, \tau$ , and  $\sigma$  the modified triangle inequality holds

$$D_{\text{DPT}}(\varrho, \sigma) \leq D_{\text{DPT}}(\varrho, \tau) + D_{\text{DPT}}(\tau, \tau) + D_{\text{DPT}}(\tau, \sigma).$$

- Conjecture: a modified version of the quantum optimal transport defined by

$$d(\varrho, \omega) := \sqrt{D_{\text{DPT}}^2(\varrho, \omega) - [D_{\text{DPT}}^2(\varrho, \varrho) + D_{\text{DPT}}^2(\omega, \omega)]/2}.$$

is a metric.

G. De Palma and D. Trevisan, *Ann. Henri Poincaré* 22, 3199 (2021).

- Triangle inequality for quantum Wasserstein divergences

$$d(\tau, \varrho) + d(\varrho, \omega) \leq d(\tau, \omega)$$

holds for any mixed  $\tau, \omega$ , any pure  $\varrho$  and any quadratic cost + strong numerical evidence for general states.

G. Bunth, J. Pitrik, T. Titkos, and D. Virosztek, *arxiv:2402.13150*.

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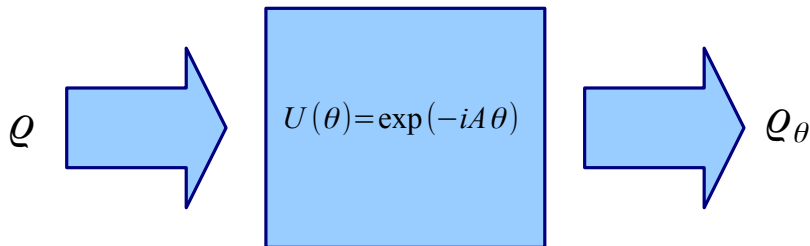
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# Quantum metrology

- Fundamental task in metrology



- We have to estimate  $\theta$  in the dynamics

$$U = \exp(-iA\theta).$$

# The quantum Fisher information

- Cramér-Rao bound on the precision of parameter estimation

$$(\Delta\theta)^2 \geq \frac{1}{mF_Q[\varrho, A]},$$

where  $F_Q[\varrho, A]$  is the **quantum Fisher information**, and  $m$  is the number of independent repetitions.

- The quantum Fisher information is

$$F_Q[\varrho, A] = 2 \sum_{k,l} \frac{(\lambda_k - \lambda_l)^2}{\lambda_k + \lambda_l} |\langle k|A|l \rangle|^2,$$

where  $\varrho = \sum_k \lambda_k |k\rangle\langle k|$ .

# Formula based on convex roofs

The quantum Fisher information is the convex roof of the variance times four

$$F_Q[\varrho, A] = 4 \min_{\{p_k, |\psi_k\rangle\}} \sum_k p_k (\Delta A)_{\psi_k}^2,$$

where

$$\varrho = \sum_k p_k |\psi_k\rangle\langle\psi_k|.$$

GT, D. Petz, Phys. Rev. A 87, 032324 (2013); S. Yu, arXiv1302.5311 (2013);

GT, I. Apellaniz, J. Phys. A: Math. Theor. 47, 424006 (2014).

# Formula based on an optimization in the two-copy space

- Two-copy formulation for the variance

$$(\Delta H)^2_\Psi = \text{Tr}(\Omega|\Psi\rangle\langle\Psi| \otimes |\Psi\rangle\langle\Psi|),$$

where we define the operator

$$\Omega = H^2 \otimes \mathbb{1} - H \otimes H.$$

We can reformulate the convex roof as

$$\begin{aligned} F_Q[\varrho, H] = \min_{\varrho_{12}} \quad & 4\text{Tr}(\Omega\varrho_{12}), \\ \text{s. t.} \quad & \varrho_{12} \in \mathcal{S}', \\ & \text{Tr}_2(\varrho_{12}) = \varrho. \end{aligned}$$

Here  $\mathcal{S}'$  is the set of symmetric separable states.

GT, T. Moroder, and O. Gühne,  
Evaluating convex roof entanglement measures,  
Phys. Rev. Lett. 114, 160501 (2015).



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# Quantum Wasserstein distance based on an optimization over separable states

- **Definition**—We can also define

$$D_{\text{sep}}(\varrho, \sigma)^2 = \frac{1}{2} \min_{\varrho_{12}} \sum_{n=1}^N \text{Tr}[(H_n^T \otimes \mathbb{1} - \mathbb{1} \otimes H_n)^2 \varrho_{12}],$$

s. t.  $\varrho_{12} \in \mathcal{S},$   
 $\text{Tr}_2(\varrho_{12}) = \varrho^T,$   
 $\text{Tr}_1(\varrho_{12}) = \sigma,$

where  $\mathcal{S}$  is the set of separable states.

GT and J. Pitrik,

Quantum Wasserstein distance based on an optimization over separable states,  
Quantum 7, 914 (2023).

# Quantum Wasserstein distance based on an optimization over separable states II

- For two-qubits, **it is computable numerically** with semidefinite programming.
- For systems of larger dimensions, one can obtain a **very good lower bound** based on an optimization over states with a positive partial transpose (PPT).
- Even better lower bounds can be obtained.

P. Horodecki, Phys. Lett. A 232, 333 (1997);

A. Peres, Phys. Rev. Lett. 77, 1413 (1996);

A. C. Doherty, P. A. Parrilo, and F. M. Spedalieri, Phys. Rev. A 69, 022308 (2004).

# Self-distance

- The self-distance for  $N = 1$  is

$$D_{\text{sep}}(\varrho, \varrho)^2 = \frac{1}{4} F_Q[\varrho, H_1].$$

- Note that

$$I_{\varrho}(A) \leq \frac{1}{4} F_Q[\varrho, A] \leq (\Delta A)_{\varrho}^2.$$

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# Entanglement of $\varrho_{12}$

- In general,

$$D_{\text{sep}}(\varrho, \sigma) \geq D_{\text{DPT}}(\varrho, \sigma).$$

- If the relation

$$D_{\text{sep}}(\varrho, \sigma) > D_{\text{DPT}}(\varrho, \sigma)$$

holds, then all the optimal  $\varrho_{12}$  couplings for  $D_{\text{DPT}}(\varrho, \sigma)$  are entangled.

- Thus, an entangled  $\varrho_{12}$  can be cheaper than a separable one.

# Comparison of the two types of Wasserstein distance

- Let us consider the distance between two single-qubit mixed states

$$\varrho = \frac{1}{2}|1\rangle\langle 1|_x + \frac{1}{2} \cdot \frac{\mathbb{1}}{2},$$

and

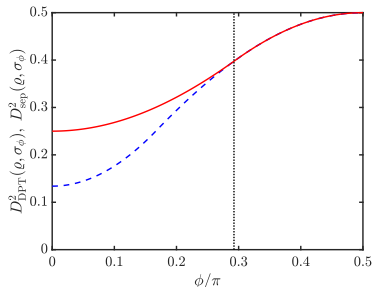
$$\sigma_\phi = e^{-i\frac{\sigma_y}{2}\phi} \varrho e^{+i\frac{\sigma_y}{2}\phi},$$

and

$$H_1 = \sigma_z.$$

$$F_Q[\varrho, \sigma_z]/4 = 0.25 \rightarrow$$

$$I_\varrho(\sigma_z) \approx 0.13 \rightarrow$$



# Bounds on the distance

- **Entanglement condition:** Let us choose a set of  $H_n$  such that

$$\frac{1}{2} \sum_n \langle (H_n^T \otimes \mathbb{1} - \mathbb{1} \otimes H_n)^2 \rangle \geq \text{const.}$$

holds for separable states.

- E. g.,  $\{H_n\} = \{j_x, j_y, j_z\}$  and "const." =  $j$ .

- If the inequality

$$D_{\text{DPT}}(\varrho, \sigma) < \text{const.}$$

holds, then all optimal  $\varrho_{12}$  states for  $D_{\text{DPT}}(\varrho, \sigma)$  are entangled.

- Then, we will have a **minimal distance**

$$D_{\text{sep}}(\varrho, \sigma) \geq \text{const.}$$



# Summary

- For the quantum Wasserstein distance, we restrict the optimization to separable states.
- Then, the self-distance equals the quantum Fisher information over four.
- We found a fundamental connection from quantum optimal transport to quantum entanglement theory and quantum metrology.

G. Tóth and J. Pitrik, [Quantum 7, 914 \(2023\)](#).

THANK YOU FOR YOUR ATTENTION!

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