Interesting quantum states (Lecture of the Quantum Information class of the Master in Quantum Science and Technology)

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Outline

Which quantum states are interesting?

- We have infinite possibilities to pick a quantum state in a multi-qubit system.

- We would like to find useful ones or states that have interesting symmetries.

Outline

Single particle states

- Pure states. The von Neumann entropy $S = 0$.
- Completely mixed state

$$\varrho_{\mathrm{cm}} = \frac{1}{d} \sum_{k=1}^{d} |k\rangle\langle k|.$$

The von Neumann entropy $S = \log d$, maximal.

Outline

1. **Interesting quantum states**
   - Motivation
   - A. Single particle states
   - B. Bipartite singlet state

Bipartite singlet state

- The two-qubit singlet state looks like

$$|\Psi_{\text{singlet}}\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle).$$

- We get the same form after any basis transformation (if we transform the bases of the two qubits in the same way).This can be seen as follows. Let us choose two vectors as

$$\begin{aligned} |v\rangle &= \alpha|0\rangle + \beta|1\rangle, \\ |v_\perp\rangle &= \beta^*|0\rangle - \alpha^*|1\rangle. \end{aligned}$$

  Clearly,

$$\langle v|v_\perp\rangle = 0,$$

  Then, simple algebra yields

$$\frac{1}{\sqrt{2}}(|v\rangle \otimes |v_\perp\rangle - |v_\perp\rangle \otimes |v\rangle) = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle).$$

  This is true for any $|v\rangle$ and $|v_\perp\rangle$.

Bipartite singlet state II

- Due to the independence from the choice of the local basis, it is invariant under a transformation of the type $U \otimes U$, apart from a global phase ϕ.

$$U \otimes U|\Psi_{\text{singlet}}\rangle = |\Psi_{\text{singlet}}\rangle \exp(-i\phi).$$

We can also say that

$$U \otimes U|\Psi_{\text{singlet}}\rangle\langle\Psi_{\text{singlet}}|(U \otimes U)^{\dagger} = |\Psi_{\text{singlet}}\rangle\langle\Psi_{\text{singlet}}|.$$

Hence,

$$U \otimes U|\Psi_{\text{singlet}}\rangle\langle\Psi_{\text{singlet}}| = |\Psi_{\text{singlet}}\rangle\langle\Psi_{\text{singlet}}|U \otimes U.$$

Thus, the density matrices of such states will commute with all $U \otimes U$:

$$[U \otimes U, |\Psi_{\text{singlet}}\rangle\langle\Psi_{\text{singlet}}|] = 0$$

for any U.

Bipartite singlet state III

- Let us consider some operators of the form

$$\sigma_{\vec{n}} = \sum_{l=x,y,z} n_l \sigma_l$$

  where $|\vec{n}| = 1$. For $\vec{n} = (1,0,0)$, $\sigma_{\vec{n}} = \sigma_x$. For $\vec{n} = (0,1,0)$, $\sigma_{\vec{n}} = \sigma_y$, and in general it is a generalization of the Pauli spin matrices to an arbitrary direction.

- Such operators all have eigenvalues ± 1. If you measure $\sigma_{\vec{n}}$ on party A and get a result, then if you also measure it on party B, you will get the opposite result. This is true for every $\sigma_{\vec{n}}$.

- This can be used in quantum communication to establish a bit sequence that is known only by Alice and Bob and by nobody else.

Bipartite singlet state IV

- For the singlet state

$$|\Psi_{\mathrm{singlet}}\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle).$$

  we have anticorrelations

$$\begin{aligned} \langle\sigma_x \otimes \sigma_x\rangle &= -1, \\ \langle\sigma_y \otimes \sigma_y\rangle &= -1, \\ \langle\sigma_z \otimes \sigma_z\rangle &= -1. \end{aligned}$$

- For the collective angular moment, we have

$$[\Delta(\sigma_x^{(1)} + \sigma_x^{(2))})]^2 = [\Delta(\sigma_y^{(1)} + \sigma_y^{(2))})]^2 = [\Delta(\sigma_z^{(1)} + \sigma_z^{(2))})]^2 = 0.$$

Bipartite singlet state V

- Why is it called a singlet? Remember the theory of angular momentum, triplet and singlet subspace.
- Alternatively, in quantum information, the maximally entangled state

$$|\Psi_{\mathrm{me}}\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$$

  can also be called singlet.
- A generalization for higher dimensions is the maximally entangled state

$$|\Psi_{\mathrm{me}}\rangle = \frac{1}{\sqrt{d}}\sum_{k=1}^{d}|kk\rangle.$$

- For the maximally entangled state, the reduced state is the completely mixed state

$$\mathrm{Tr}_A(|\Psi_{\mathrm{me}}\rangle\langle\Psi_{\mathrm{me}}|) = \mathrm{Tr}_B(|\Psi_{\mathrm{me}}\rangle\langle\Psi_{\mathrm{me}}|) = \frac{\mathbb{1}}{d}.$$

  Thus, if we have access only to one of the two subsystems, we know nothing.

Bipartite singlet state VI

- For the maximally entangled state

$$|\Psi_{\mathrm{me}}\rangle = \frac{1}{\sqrt{2}}\left(|00\rangle + |11\rangle\right)$$

we have

$$\begin{aligned} \langle \sigma_x \otimes \sigma_x \rangle &= +1, \\ \langle \sigma_y \otimes \sigma_y \rangle &= -1, \\ \langle \sigma_z \otimes \sigma_z \rangle &= +1. \end{aligned}$$

- For the collective angular moment, we have

$$[\Delta(\sigma_x^{(1)} - \sigma_x^{(2))})]^2 = [\Delta(\sigma_y^{(1)} + \sigma_y^{(2))})]^2 = [\Delta(\sigma_z^{(1)} - \sigma_z^{(2))})]^2 = 0.$$