(ENKEMNA0302) Applied Linear Algebra

Dr. Gabor FACSKO, PhD
Senior Research Fellow
facskog@gamma.ttk.pte.hu

University of Pecs, Faculty of Sciences, Institute of Mathematics and Informatics, 7624 Pecs, Ifjusag utja 6.
Wigner Research Centre for Physics, Department of Space Physics and Space Technology, 1121 Budapest, Konkoly-Thege Miklos ut 29-33.
`https://facsko.ttk.pte.hu`

March 6, 2025

Diagonal matrixes I

- Diagonal matrixes: it is simple to do operations with them.
  Here $\mathbf{A} = diag\,(1,2,3)$ és $\mathbf{B} = diag\,(5,4,3)$. Then:

$$\mathbf{AB} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix} \begin{pmatrix} 5 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 3 \end{pmatrix} = \begin{pmatrix} 5 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & 9 \end{pmatrix},$$

$$\mathbf{A}^2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}^2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 9 \end{pmatrix}, \mathbf{A}^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{3} \end{pmatrix}$$

$$\mathbf{A}^k = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}^k = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2^k & 0 \\ 0 & 0 & 3^k \end{pmatrix}, \text{where } k \in \mathbb{Z}.$$

Diagonal matrixes II

Thesis: (Operations with diagonal matrixes) Here $\mathbf{A} = diag\,(a_1, a_2, \ldots, a_n)$, $\mathbf{B} = diag\,(b_1, b_2, \ldots, b_n)$ and $k \in \mathbb{Z}$. Then

1. $\mathbf{AB} = diag\,(a_1 b_1, a_2 b_2, \ldots, a_n b_n)$,
2. $\mathbf{A}^k = diag\,(a_1^k, a_2^k, \ldots, a_n^k)$, specialy
3. $\mathbf{A}^{-1} = diag\,(a_1^{-1}, a_2^{-1}, \ldots, a_n^{-1})$.

The (3) and if $k < 0$ the (2) operations can be done if $a_i \neq 0$, where $i = 1, 2, \ldots, n$.

Permutation Matrices I

- Permutation matrices and snakes are obtained by permuting the rows of diagonal matrices.
- Every permutation can be achieved by swapping element pairs. If we permute the rows of a matrix, we can do so by multiplying with elementary matrices that perform row swaps. The matrix obtained as the product of these elementary matrices can be derived from the identity matrix by executing the given row swaps.
- For example, performing the permutation $\{2, 4, 3, 1\}$ on the identity matrix $\mathbf{E}_n$ results in the following permutation matrix $\mathbf{P}$:

$$\mathbf{E}_n = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \overset{S_1 \leftrightarrow S_2}{\Longrightarrow} \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \overset{S_2 \leftrightarrow S_4}{\Longrightarrow} \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} = \mathbf{P}$$

Permutation Matrices II

- Multiplying any $4 \times m$ matrix from the left by $\mathbf{P}$ will rearrange its rows according to the given permutation. For example,

$$\mathbf{PA} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \\ a_{41} & a_{42} \end{pmatrix} = \begin{pmatrix} a_{21} & a_{22} \\ a_{41} & a_{42} \\ a_{31} & a_{32} \\ a_{11} & a_{12} \end{pmatrix}$$

- Defition: (Permutation matrix, snake) The matrixes created by permutation from the diagonal matrixes called snakes (or transversals). Snakes created from the identity matrix are called permutation matrixes.

Permutation Matrices III

- For example the following marixes are snakes, however, the last two matrixes are also permutation matrixes:

$$\begin{pmatrix} 0 & 5 & 0 \\ 0 & 0 & 9 \\ 3 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & \alpha & 0 & 0 \\ \gamma & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \beta \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

- The permutation matrixes are such a square a matrixes that has only 1 element and the other elemenst are 0.
- The snake is such a square matrix that has only one non-zero element in its each row and column.
- All snakes could be get from a diagonal matrix by switching columns.

Permutation Matrices IV

- You can get a snake from a diagonal matrix if you also permutate the columns.
- If $\mathbf{P}$ is a permutation matrix, then you can get $\mathbf{PA}$ from $\mathbf{A}$ by the same permutation of the rows that permutation leads from $\mathbf{E}$ to $\mathbf{P}$.
- <u>Thesis:</u> (Operations with Permutation Matrices). The product of any two permutation matrices of the same size, as well as any integer power of a permutation matrix, is also a permutation matrix. The inverse of a permutation matrix is equal to its transpose, i.e., if $\mathbf{P}$ is a permutation matrix, then $\mathbf{P}^{-1} = \mathbf{P}^{T}$.
  <u>Deduction:</u> Let $\mathbf{P}$ and $\mathbf{Q}$ be two permutation matrices. The row vectors of their product take the form $\mathbf{P}_{i*}\mathbf{Q}$, where $\mathbf{P}_{i*}$ corresponds to a standard unit vector, e.g., $\mathbf{P}_{i*} = \mathbf{e}_k$. In this case, only the element in the column that matches $\mathbf{e}_k$ is 1, and there is exactly one such column. Thus, in each row of the product matrix, there is exactly one entry equal to 1, while all others are 0.

Permutation Matrices V

A similar argument holds for the columns. The statement regarding multiplication implies the statement for positive integer powers. The case for negative integer exponents follows from considering the inverse.

Now, consider the product $\mathbf{P}\mathbf{P}^T$. The element $\left(\mathbf{P}\mathbf{P}^T\right)_{ii}$ is given by the dot product of the vector $\mathbf{P}_{i*}$ with $\left(\mathbf{P}^T\right)_{*i} = \mathbf{P}_{i*}$, which equals 1. Meanwhile,

$$\left(\mathbf{P}\mathbf{P}^T\right)_{ij} = (\mathbf{P})_{i*}\left(\mathbf{P}^T\right)_{*j} = (\mathbf{P})_{i*} \cdot (\mathbf{P})_{j*},$$

i.e., the (i, j)-th element of the product is the dot product of the i-th and j-th row vectors of $\mathbf{P}$, which is 0 because the 1s appear in different positions in different rows.

Permutation Matrices VI

- Example:

$$\mathbf{PP}^T = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Triangular Matrices

- Definition: (Triangular Matrix). A matrix in which all elements below the main diagonal are zero is called an upper triangular matrix, while a matrix in which all elements above the main diagonal are zero is called a lower triangular matrix. If all the elements on the main diagonal of a triangular matrix are 1, it is called a unit triangular matrix.
- Thesis: (Operations on Triangular Matrices). The sum, product, and inverse of an invertible upper triangular matrix are also upper triangular matrices. An analogous theorem holds for lower triangular matrices as well. A triangular matrix is invertible if and only if none of its diagonal elements are zero.
  Deduction: Trivial.

Symmetric and Skew-Symmetric Matrices I

- Definition: (Symmetric and Skew-Symmetric Matrices). A square matrix $\mathbf{A}$ is called symmetric if $\mathbf{A}^T = \mathbf{A}$, and it is called skew-symmetric if $\mathbf{A}^T = -\mathbf{A}$.
- Examples of symmetric and skew-symmetric matrices:

$$\mathbf{A} = \begin{pmatrix} 5 & 6 & 1 \\ 6 & 2 & 0 \\ 1 & 0 & 3 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 0 & 1 & -2 \\ -1 & 0 & 3 \\ 2 & -3 & 0 \end{pmatrix}, \mathbf{C} = \begin{pmatrix} 1 & 9 & 9 \\ -9 & 2 & 9 \\ -9 & -9 & 3 \end{pmatrix}$$

  $\mathbf{A}$ is symmetric, $\mathbf{B}$ is skew-symmetric, and $\mathbf{C}$ is neither.
- If $\mathbf{A}$ is skew-symmetric, then each element satisfies $a_{ij} = -a_{ji}$, meaning that for $i = j$, we have $a_{ii} = -a_{ii}$. This is only possible if $a_{ii} = 0$, meaning that the main diagonal of a skew-symmetric matrix consists entirely of zeros.

Symmetric and Skew-Symmetric Matrices II

- Thesis: (Operations with (skew-)symmetric matrices). The sum, scalar multiple, and inverse of symmetric matrices are also symmetric. The sum, scalar multiple, and inverse of skew-symmetric matrices are also skew-symmetric.
- Thesis: (Decomposition into the sum of a symmetric and a skew-symmetric matrix). Every square matrix can be expressed as the sum of a symmetric and a skew-symmetric matrix. Specifically, for every square matrix $\mathbf{A}$:

$$\mathbf{A} = \frac{1}{2}\left(\mathbf{A} + \mathbf{A}^T\right) + \frac{1}{2}\left(\mathbf{A} - \mathbf{A}^T\right),$$

  where the first term in the sum is symmetric, and the second term is skew-symmetric.

Sherman-Morrison-Woodbury Theorem I

- Sherman–Morrison Formula: Suppose that the matrix $\mathbf{A} \in \mathbb{R}^{n\times n}$ is invertible, and $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$ are two vectors such that $1 + \mathbf{v}^T\mathbf{A}^{-1}\mathbf{u} \neq 0$. Then $\mathbf{A} + \mathbf{u}\mathbf{v}^T$ is invertible, and
$$\left(\mathbf{A} + \mathbf{u}\mathbf{v}^T\right)^{-1} = \mathbf{A}^{-1} - \frac{\mathbf{A}^{-1}\mathbf{u}\mathbf{v}^T\mathbf{A}^{-1}}{1 + \mathbf{v}^T\mathbf{A}^{-1}\mathbf{u}}.$$

- Thesis: (Sherman–Morrison–Woodbury Formula) The inverse of a rank-k correction of a matrix can be computed using the inverse of the original matrix with a rank-k correction:
$$(\mathbf{A} + \mathbf{UCV})^{-1} = \mathbf{A}^{-1} - \mathbf{A}^{-1}\mathbf{U}\left(\mathbf{C}^{-1} + \mathbf{V}\mathbf{A}^{-1}\mathbf{U}\right)^{-1}\mathbf{V}\mathbf{A}^{-1},$$
where $\mathbf{A}$, $\mathbf{U}$, $\mathbf{C}$, and $\mathbf{V}$ are matrices of dimensions $n \times n$, $n \times k$, $k \times k$, and $k \times n$, respectively.
Deduction: N/A

Sherman-Morrison-Woodbury Theorem II

- The above formula provides an efficient alternative for computing matrix inverses, i.e., solving linear systems. However, its numerical stability is not well understood.

Operations with Block Matrices I

- Operations on extremely large matrices can be parallelized if the matrices are divided into blocks and the operations are performed on these smaller submatrices.
- When a matrix is divided into submatrices by horizontal and vertical lines, we say that this matrix is a block matrix composed of submatrices—also known as blocks.
- The rows and columns of a block matrix are called the block rows and block columns of the matrix.
- Block matrices are also referred to as hypermatrices, but this term is also used for multidimensional arrays. Therefore, we prefer to call them block matrices.

Operations with Block Matrices II

- The augmented matrix $[\mathbf{A}|\mathbf{b}]$ of a system of equations is a block matrix consisting of two blocks:

$$\left(\begin{array}{ccc|cc|c} 1 & 0 & 0 & 1 & 2 & 4 \\ 0 & 1 & 0 & 2 & 0 & 3 \\ 0 & 0 & 1 & 1 & 0 & 3 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

  This is the augmented matrix of a system of 5 equations with 5 unknowns. The first block column corresponds to the bound variables, the second to the free variables, and the third to the right-hand side of the system of equations. The second block row contains the zero rows.

Operations with Block Matrices III

- Statement: (Operations with Block Matrices). Multiplication of block matrices by a scalar and the addition of two block matrices partitioned in the same way can be performed block-wise, that is,

$$c\,[\mathbf{A}_{ij}] = [c\mathbf{A}_{ij}]\,, \quad [\mathbf{A}_{ij}] + [\mathbf{B}_{ij}] = [\mathbf{A}_{ij} + \mathbf{B}_{ij}]\,.$$

If $\mathbf{A} = [\mathbf{A}_{ik}]_{m\times t}$ and $\mathbf{B} = [\mathbf{B}_{kj}]_{t\times n}$ are two block matrices, and for every k, the number of columns of $\mathbf{A}_{ik}$ matches the number of rows of $\mathbf{B}_{kj}$, then the product $\mathbf{C} = \mathbf{AB}$ can be computed by applying the multiplication rule to blocks, that is, $\mathbf{C}$ is a block matrix where the block in the i-th block row and j-th block column is given by:

$$\mathbf{C}_{ij} = \sum_{k=1}^{t} \mathbf{A}_{ik}\mathbf{B}_{kj}\,.$$

Operations with Block Matrices IV

- Example of block matrix multiplication:

$$\left(\begin{array}{cc|c} 1 & 0 & 1 \\ \hline 2 & 1 & 1 \\ 0 & 3 & 1 \end{array}\right) \left(\begin{array}{c|c} 1 & 1 \\ 1 & 2 \\ \hline 0 & 1 \end{array}\right) = \left(\begin{array}{c|c} \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \end{pmatrix} \begin{pmatrix} 0 \end{pmatrix} & \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \end{pmatrix} + \begin{pmatrix} 1 \end{pmatrix} \begin{pmatrix} 1 \end{pmatrix} \\ \hline \begin{pmatrix} 2 & 1 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \end{pmatrix} \begin{pmatrix} 0 \end{pmatrix} & \begin{pmatrix} 2 & 1 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \end{pmatrix} \begin{pmatrix} 1 \end{pmatrix} \end{array}\right) =$$

$$\left(\begin{array}{c|c} \begin{pmatrix} 1 \end{pmatrix} & \begin{pmatrix} 2 \end{pmatrix} \\ \hline \begin{pmatrix} 3 \\ 3 \end{pmatrix} & \begin{pmatrix} 5 \\ 7 \end{pmatrix} \end{array}\right) = \left(\begin{array}{c|c} 1 & 2 \\ \hline 3 & 5 \\ 3 & 7 \end{array}\right).$$

Kronecker Product and the Vec Function I

- Certain block matrix operations cannot be derived from simple matrix operations.
- The *vec* function transforms an arbitrary matrix into a vector by stacking its column vectors on top of each other. If $\mathbf{A} = [\mathbf{a}_1 | \mathbf{a}_2 | \dots | \mathbf{a}_n]$, then

$$vec\,(\mathbf{A}) = \begin{pmatrix} \mathbf{a}_1 \\ \vdots \\ \mathbf{a}_n \end{pmatrix}$$

For example, if $\mathbf{A} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$, then $vec\,(\mathbf{A}) = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}$.

Kronecker Product and the Vec Function II

- Let $\mathbf{A}$ be an $m \times n$ matrix and $\mathbf{B}$ a $p \times q$ matrix. Their Kronecker product (also called the tensor product) is the $mp \times nq$ matrix denoted by $A \otimes B$, which has the block matrix form:

$$\mathbf{A} \otimes \mathbf{B} = \begin{pmatrix} a_{11}\mathbf{B} & a_{12}\mathbf{B} & \dots & a_{1n}\mathbf{B} \\ a_{21}\mathbf{B} & a_{22}\mathbf{B} & \dots & a_{2n}\mathbf{B} \\ \vdots & \vdots & \dots & \vdots \\ a_{m1}\mathbf{B} & a_{m2}\mathbf{B} & \dots & a_{mn}\mathbf{B} \end{pmatrix}.$$

- For example,

$$\begin{pmatrix} -1 & 2 \\ 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} 0 & 1 & 2 \\ 3 & 3 & 3 \end{pmatrix} = \begin{pmatrix} 0 & -1 & -2 & 0 & 2 & 4 \\ -3 & -3 & -3 & 6 & 6 & 6 \\ 0 & 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 3 & 3 & 3 \end{pmatrix}.$$

Kronecker Product and the Vec Function III

- Theorem: (Properties of the Kronecker Product). Given the matrices $\mathbf{A}_{m\times n}$, $\mathbf{B}_{m\times n}$, $\mathbf{C}_{p\times q}$, and $\mathbf{D}_{r\times s}$, we have:
  1. $(\mathbf{A}+\mathbf{B})\otimes\mathbf{C}=\mathbf{A}\otimes\mathbf{C}+\mathbf{B}\otimes\mathbf{C}$, $\mathbf{C}\otimes(\mathbf{A}+\mathbf{B})=\mathbf{C}\otimes\mathbf{A}+\mathbf{C}\otimes\mathbf{B}$,
  2. $(\mathbf{A}\otimes\mathbf{C})\otimes\mathbf{D}=\mathbf{A}\otimes(\mathbf{C}\otimes\mathbf{D})$,
  3. $(\mathbf{A}\otimes\mathbf{C})^T=\mathbf{C}^T\otimes\mathbf{A}^T$.

  Proof: By definition, e.g., for (2):

$$(\mathbf{AXB})_{*j}=\mathbf{AXB}_{*j}=\sum_{i=1}^{n}(b_{ij}\mathbf{A})\,\mathbf{X}_{*i}=(b_{1j}\mathbf{A}|\ldots|b_{nj}\mathbf{A})\,vec\,(\mathbf{X})=\left(\mathbf{B}^T\otimes\mathbf{A}\right)_{*j}vec\,(\mathbf{X})\,.$$

Hypermatrixes I

- Certain data can be well organized in arrays of dimension higher than 2.
- Definition: (Hypermatrix). Let $n_1, n_2, \ldots, n_d \in \mathbb{N}^+$ and let S be an arbitrary set (e.g., $S = \mathbb{R}, \mathbb{Q}, \mathbb{N}, \mathbb{Z} \ldots$). A dth-order (or d-dimensional) $n_1 \times n_2 \times \cdots \times n_d$-type hypermatrix is a mapping of the form

$$\mathbf{A} : \{1, \ldots, n_1\} \times \{1, \ldots, n_2\} \times \cdots \times \{1, \ldots, n_d\} \to S$$

The element $\mathbf{A}(i_1, i_2, \ldots, i_d)$ is denoted by $a_{i_1 i_2 \ldots i_d}$, which is an element of a d-dimensional table, and similarly to matrices, we can write:

$$\mathbf{A} = (a_{i_1 i_2 \ldots i_d})_{i_1, i_2, \ldots, i_d}^{n_1, n_2, \ldots, n_d} = 1, \text{or simply } \mathbf{A} = (a_{i_1 i_2 \ldots i_d})\,.$$

If $n_1 = n_2 = \cdots = n_d = n$, then we refer to a hyper-cube matrix.

Hypermatrixes II

- The set of all $n_1 \times n_2 \times \cdots \times n_d$-type hypermatrices formed from the elements of S is denoted by $S^{n_1 \times n_2 \times \ldots n_d}$.
- Second-order hypermatrices coincide with matrices.
- The elements of third-order hypermatrices can be described by slicing them according to the third index. Each slice is a matrix, and these matrices are written next to each other separated by vertical lines. For example, the general form of a $4 \times 2 \times 3$-type hypermatrix is:

$$\left(\begin{array}{cc|cc|cc} a_{111} & a_{121} & a_{112} & a_{122} & a_{113} & a_{123} \\ a_{211} & a_{221} & a_{212} & a_{222} & a_{213} & a_{223} \\ a_{311} & a_{321} & a_{312} & a_{322} & a_{313} & a_{323} \\ a_{411} & a_{421} & a_{412} & a_{422} & a_{413} & a_{423} \end{array}\right)$$

Hypermatrixes III

- Addition of two hypermatrices of the same type and scalar multiplication of a hypermatrix follow element-wise operations similar to matrices:

$$\left(a_{i_1 i_2 \ldots i_d}\right) + \left(b_{i_1 i_2 \ldots i_d}\right) = \left(a_{i_1 i_2 \ldots i_d} + b_{i_1 i_2 \ldots i_d}\right),$$
$$c\left(a_{i_1 i_2 \ldots i_d}\right) = \left(ca_{i_1 i_2 \ldots i_d}\right).$$

- Definition: (Transpose of a hypermatrix). Let π be a permutation of the set $\{1, 2, \ldots, d\}$. The π-transpose of the dth-order hypermatrix $\mathbf{A} = \left(a_{i_1 i_2 \ldots i_d}\right) \in S^{n_1 \times n_2 \times \ldots n_d}$ is defined as:

$$\mathbf{A}^{\pi} = \left(a_{i_{\pi(1)} i_{\pi(2)} \ldots i_{\pi(d)}}\right) \in S^{n_{\pi(1)} \times n_{\pi(2)} \times \ldots n_{\pi(d)}}$$

  A hyper-cube matrix $\mathbf{A} \in S^{n \times n \times \cdots \times n}$ is symmetric if for all permutations π, we have $\mathbf{A}^{\pi} = \mathbf{A}$, and it is skew-symmetric if $\mathbf{A}^{\pi} = sgn(\pi)\,\mathbf{A}$, where $sgn(\pi) = -1$ for odd permutations and 1 for even permutations.

Hypermatrixes IV

- Accordingly, the general form of $2 \times 2 \times 2$ hypermatrices and symmetric hypermatrices is: $\left(\begin{array}{cc|cc} a_{111} & a_{121} & a_{112} & a_{122} \\ a_{211} & a_{221} & a_{212} & a_{222} \end{array}\right)$, $\left(\begin{array}{cc|cc} a & b & b & c \\ b & c & c & d \end{array}\right)$

  The general form of $3 \times 3 \times 3$ hypermatrices, symmetric, and skew-symmetric hypermatrices is:

$$\left(\begin{array}{ccc|ccc|ccc} a_{111} & a_{121} & a_{131} & a_{112} & a_{122} & a_{132} & a_{113} & a_{123} & a_{133} \\ a_{211} & a_{221} & a_{231} & a_{212} & a_{222} & a_{232} & a_{213} & a_{223} & a_{233} \\ a_{311} & a_{321} & a_{331} & a_{312} & a_{322} & a_{332} & a_{313} & a_{323} & a_{333} \end{array}\right),$$

$$\left(\begin{array}{ccc|ccc|ccc} a & b & c & b & d & e & c & e & f \\ b & d & e & d & g & h & e & h & i \\ c & e & f & e & h & i & f & i & j \end{array}\right), \left(\begin{array}{ccc|ccc|ccc} 0 & 0 & 0 & 0 & 0 & -a & 0 & a & 0 \\ 0 & 0 & a & 0 & 0 & 0 & -a & 0 & 0 \\ 0 & -a & 0 & a & 0 & 0 & 0 & 0 & 0 \end{array}\right),$$

  where $a, b, c, d, e, f, g, h, i, j \in S$ are not necessarily distinct elements.

The End

Thank you for your attention!