(PTIA0301) Elementary Linear Algebra

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Common Business

- Next week we will write a test.

Cramer's rule I

- If the determinant of the coefficient matrix of the system of linear equations consists of n equations and has n unknowns is not zero $(det\,(A) \neq 0)$. The system of linear equations could be solved and its unique solution is:

$$x_k = \frac{\Delta_k}{|A|}, \left(k = 1, 2, \ldots, n \in \mathbb{N}^+\right)$$

where

$$\Delta_k = \begin{pmatrix} \alpha_{11} & \alpha_{12} & \cdots & \alpha_{1,k-1} & \beta_1 & \alpha_{1,k+11} & \cdots & \alpha_{1n} \\ \alpha_{21} & \alpha_{22} & \cdots & \alpha_{2,k-1} & \beta_2 & \alpha_{2,k+1} & \cdots & \alpha_{2n} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \\ \alpha_{m1} & \alpha_{m2} & \cdots & \alpha_{m,k-1} & \beta_n & \alpha_{m,k+1} & \cdots & \alpha_{mn} \end{pmatrix},$$

the free members are in the k^{th} column.

Cramer's rule II

- If $det\,(A) = 0$, however $\exists k \in \{1, 2, \ldots, n\}$, where $n \in \mathbb{N}$ and $\Delta_k \neq 0$, then the system of equations has no solution. However, if $det\,(A) = \Delta_k = 0\,(k = 1, 2, \ldots., n)$ 0 or infinity number of solutions might exist.

Cramer's rule III

- Solve the following system of linear equations using Cramèr' rule:

$$\begin{aligned} x_1 - 2x_2 - x_3 &= 6 \\ 2x_1 - 3x_2 + x_3 &= -1 \\ 3x_1 + x_2 + x_3 &= 5 \end{aligned}$$

$$\mathbf{A} = \begin{pmatrix} 1 & -2 & -3 \\ 2 & -3 & 1 \\ 3 & 1 & 1 \end{pmatrix} \quad \beta = \begin{pmatrix} 6 \\ -1 \\ 5 \end{pmatrix}$$

There is solution because $\det(\mathbf{A}) = -3 - 6 - 6 - 27 + 4 - 1 = -39 \neq 0$.

Cramer's rule IV

$$D_1 = \begin{vmatrix} 6 & -2 & -3 \\ -1 & -3 & 1 \\ 5 & 1 & 1 \end{vmatrix} = -18 - 10 + 3 - 45 - 2 - 6 = -78$$

Therefore,

$$x_1 = \frac{D_1}{\det(\mathbf{A})} = \frac{-78}{-39} = \underline{\underline{2}}.$$

Similarly,

$$D_2 = \begin{vmatrix} 1 & 6 & -3 \\ 2 & -1 & 1 \\ 3 & 5 & 1 \end{vmatrix} = -1 + 18 - 30 - 9 - 12 - 5 = -39$$

Cramer's rule V

Hence,

$$x_2 = \frac{D_2}{\det\left(\mathbf{A}\right)} = \frac{-39}{-39} = \underline{\underline{1}}.$$

$$D_3 = \begin{vmatrix} 1 & -2 & 6 \\ 2 & -3 & -1 \\ 3 & 1 & 5 \end{vmatrix} = -15 + 6 + 12 + 54 + 20 + 1 = 78$$

Therefore,

$$x_3 = \frac{D_3}{\det\left(\mathbf{A}\right)} = \frac{78}{-39} = \underline{\underline{-2}}.$$

Vector Space I

- Definition: The set $V \neq \emptyset$ a vector space above $\mathbb{R}$ set, if there isan "+" operation with the following features:

$$\begin{aligned}
\mathbf{a}+\mathbf{b} &= \mathbf{b}+\mathbf{a}, \text{where} \left(\mathbf{a},\mathbf{b} \in V\right) \\
(\mathbf{a}+\mathbf{b})+\mathbf{c} &= \mathbf{a}+(\mathbf{b}+\mathbf{c})\,, \text{where} \left(\mathbf{a},\mathbf{b},\mathbf{c} \in V\right) \\
\exists \mathbf{0} \in V \ \ \text{that} \ \ \mathbf{a}+\mathbf{0} &= \mathbf{a} \ \ \forall \mathbf{a} \in V \\
\forall \mathbf{a} \in V \exists \left(-\mathbf{a}\right) \in V : \mathbf{a}+\left(-\mathbf{a}\right) &= \mathbf{0},
\end{aligned}$$

Vector Space II

furthermore, for all $\lambda \in \mathbb{R}$ and for all $\mathbf{a} \in V$ the term $\lambda \mathbf{a} \in V$ exists, and the following features are fullfilled:

$$\begin{aligned}
\lambda(\mu \mathbf{a}) &= (\lambda\mu)\,\mathbf{a}, \text{where}\,(\mathbf{a} \in V, \lambda, \mu \in \mathbb{R}) \\
\lambda(\mathbf{a}+\mathbf{b}) &= \lambda\mathbf{a}+\lambda\mathbf{b}, \text{where}\,(\mathbf{a},\mathbf{b} \in V, \lambda \in \mathbb{R}) \\
(\lambda+\mu)\,\mathbf{a} &= \lambda\mathbf{a}+\mu\mathbf{a}, \text{where}\,(\mathbf{a} \in V\ \text{és}\lambda, \mu \in \mathbb{R}) \\
\forall \mathbf{a} \in V\text{−re}1 \cdot \mathbf{a} &= \mathbf{a}.
\end{aligned}$$

These features are true for the V^2 and V^3 set, however the $\mathbb{R}^n = \mathbb{R} \times \mathbb{R} \times \cdots \times \mathbb{R}$ seat are vector space too. Furtheremore, the set of the maximum n^{th} ordered polynomes $(R_n[x])$ are vector spaces too.

Vector Space III

- Question: What is the Cartesian product? What is a polynomial?
- Definition: Cartesian product of sets A and B is such a set, that consists of all ordered pairs, in which the first member is an element from the set A and the second member is an element from the set B. Or

$$A \times B = \{(a, b) \mid a \in A \wedge b \in B\}.$$

- Example:

  $\{$John, Ann$\}$ $\times$ $\{$Smith, Turner, Carpenter$\}$ $=$ $\{$John Smith, Ann Smith, John Turner, Ann Turner, Johan Carpenter, Ann Carpenter$\}$.
- You can generalize this Cartesian product for $3, 4, \ldots, n \in \mathbb{N}^+$ members, where$n \geq 2$.

Vector Space IV

- Definition: Polinome is a sum where you can find the multiplication of numbers and variables over non-negative power. Example:

$$\begin{aligned} p(x, y, z, u) &= 5x^4y^6 - 3xz^3 + 11y^{15}u^7 \\ q(x) &= 2x^2 + 6x + 9 \\ r(x, y) &= x^3 + 3x^2y + 3x^2y + y^3 \end{aligned}$$

Or:

$$P(x) = \sum_{i=0}^{n} a_i x^i = a_0 + a_1 x + a_2 x^2 + \cdots + a_{n-1} x^{n-1} + a_n x^n, n \in \mathbb{N}^+.$$

Here x is the polinome's variable and n is its order. There are polynomes with multiple variables. Their order the maximal number that we get summarizes the order of the variable. All powers must be non-zero numbers.

Vector Space V

- Definition: The V vector space's non empty subset L is a linear subspace if L also is a vector space with the operations in V.
- Thesis: The V vector space's L non-empty set is a linear subspace if and only if the following conditions are fulfilled:

$$\begin{aligned} \forall \mathbf{a}, \mathbf{b} \in L \quad &\text{esetén} \quad \mathbf{a} + \mathbf{b} \in L \\ \forall \lambda \in \mathbb{R}, \forall \mathbf{a} \in L \quad &\text{esetén} \quad \lambda \mathbf{a} \in L. \end{aligned}$$

- Definition: $H \neq \emptyset$ is subset of the V vector space. The subspace generated by H is the minimal size of such a subset of V that contains H. (Or it is a subset of all subspace containing H.) Sign: $\mathcal{L}(H)$.
- Alway exists a minimal subspace containing H that is the intersection of all subspaces containing H.

Vector Space VI

- Definition: Set H is a generator system of V vector space if $\mathcal{L}(H) = V$.
- Definition: V vector space is generated finitely if it has a generator system containing a finite number of elements.
- Comment: The $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ vector system is a generator system of a finitely generated V vector system if all elements of the V set could be written as the linear combination of $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ vectors.
- Definition: The linear independent genarator system of the V vector space is a basis of the V vector space.
- Tétel: In a finitely generated vector space all bases have the same cardinality.
- Definition: The common element number of the basis of a vector space is called dimension. Sign: $dim(V)$.

Vector Space VII

- <u>Definition:</u> $\{\mathbf{b}_1, \mathbf{b}_2, \dots, \mathbf{b}_n\}$ is a basis in V and $\mathbf{a} \in V$. Those $\lambda_1, \lambda_2, \dots, \lambda_n$ numbers for which $\mathbf{a} = \lambda_1 \mathbf{b}_1 + \lambda_2 \mathbf{b}_2 + \dots + \lambda_n \mathbf{b}_n$ we call coordinates the $\mathbf{a}$ vector for the $\{\mathbf{b}_1, \mathbf{b}_2, \dots, \mathbf{b}_n\}$ basis.
- <u>Definition:</u> If $\mathbf{v} \in V^2 \setminus \{\mathbf{0}\}$, or $\mathbf{v} \in V^3 \setminus \{\mathbf{0}\}$. Then the $l = \{\alpha \mathbf{v} : \alpha \in \mathbb{R}\}$ set is a line (crossing the origo).
- <u>Definition:</u> If $\mathbf{u}, \mathbf{v} \in V^3 \setminus \{\mathbf{0}\}$ és $\nexists \lambda \in \mathbb{R}$, so $\mathbf{u} = \lambda \mathbf{v}$. Then the $L = \{\alpha \mathbf{u} + \beta \mathbf{v} : \alpha, \beta \in \mathbb{R}\}$ set a plane (crossing the origo).
- The set $L = \{\alpha_1 \mathbf{u}_1 + \alpha_2 \mathbf{u}_2 + \dots + \alpha_m \mathbf{u}_m : \alpha_1, \dots, \alpha_m \in \mathbb{R}\}$ is just the linear subspace generated by $\{\mathbf{u}_1, \dots, \mathbf{u}_m\}$. If $\mathbf{u}_1, \dots, \mathbf{u}_m$ are linear independent, then the set L is am mD subspace- Then the set $K = \{\alpha_1 \mathbf{u}_1 + \alpha_2 \mathbf{u}_2 + \dots + \alpha_m \mathbf{u}_m + \mathbf{v} : \alpha_1, \dots, \alpha_m, \in \mathbb{R}\}$ is an mD affin subspace. All affine subspace can be constructed in the $K = L + \mathbf{v}$ form, where L is a linear subspace and $\mathbf{v} \in V$.

Vector Space VIII

- The equation of a plane which has the $\mathbf{n}\left(n_1, n_2, n_3\right)$ normal vector and crossed the point $P\left(p_1, p_2, p_3\right)$ is $n_1 x + n_2 y + n_3 z = n_1 p_1 + n_2 p_2 + n_3 p_3$.
- Definition: The (Minkowski-)sum of set is $A + B = \{a + b : a \in A, b \in B\}$.
- Állítás: The sum and intersection of subspaces are $L_1 + L_2$ and $L_1 \bigcap L_2$ is altér.
- Definition: The $L_1 + L_2$ is a direct sum if $L_1 \bigcap L_2 = \{\mathbf{0}\}$.

The End

Thank you for your attention!