(ENKEMNA0302) Applied Linear Algebra

Dr. Gabor FACSKO, PhD
Senior Research Fellow
facskog@gamma.ttk.pte.hu

University of Pecs, Faculty of Sciences, Institute of Mathematics and Informatics, 7624 Pecs, Ifjusag utja 6.
Wigner Research Centre for Physics, Department of Space Physics and Space Technology, 1121 Budapest, Konkoly-Thege Miklos ut 29-33.
`https://facsko.ttk.pte.hu`

April 2, 2025

Eigenvalues, Eigenvectors, Diagonalization of Linear Transformations I

- Eigenvalues and eigenvectors of linear transformations.
  1. Mirroring of plane vectors across a line (or reflection of points across a line passing through the origin).
  2. Orthogonal projection of plane vectors onto a line (or orthogonal projection of points onto a line passing through the origin).
  3. Rotation of space vectors around a line by an angle different from an integer multiple of 180^o.
  4. Orthogonal projection of space vectors onto a plane.
  5. Mirroring of space vectors across a plane.

  All these transformations are linear.

Eigenvalues, Eigenvectors, Diagonalization of Linear Transformations II

1. In mirroring across a line, only vectors parallel and perpendicular to the line are mapped to scalar multiples of themselves: parallel vectors remain unchanged, while perpendicular ones are mapped to their negations. Thus, the eigenspace corresponding to eigenvalue 1 consists of the vectors parallel to the line, while the eigenspace corresponding to eigenvalue -1 consists of the perpendicular vectors. The corresponding statement for points follows from their position vectors.
2. In orthogonal projection onto a line, vectors parallel to the line remain unchanged, while perpendicular vectors are mapped to the zero vector. Thus, the eigenspace for eigenvalue 1 consists of the vectors parallel to the line, while the eigenspace for eigenvalue 0 consists of the perpendicular vectors.
3. Rotation of space vectors around an axis leaves the vectors parallel to the axis unchanged. If the rotation angle is not an integer multiple of 180^o, no other vectors are mapped to scalar multiples of themselves. Thus, the only eigenvalue is 1, with an eigenspace consisting of the vectors parallel to the rotation axis.

Eigenvalues, Eigenvectors, Diagonalization of Linear Transformations III

4. Orthogonal projection of space vectors onto a plane leaves all vectors in the plane unchanged while mapping perpendicular vectors to the zero vector. Thus, the two eigenvalues are 1 and 0, with the eigenspaces consisting of the plane's vectors and the perpendicular vectors, respectively.
5. The two eigenvalues are 1 and -1, with the eigenspaces consisting of the plane's vectors and the perpendicular vectors, respectively.

A linear transformation can have different matrix representations depending on the chosen basis, but its eigenvalues remain the same because a vector's image depends only on the transformation itself, not on the chosen basis.

Eigenvalues, Eigenvectors, Diagonalization of Linear Transformations IV

- Diagonalization of a linear transformation. Diagonalizing the transformations above.
  1. Let **a** be a direction vector of the reflection line, and let **b** be a nonzero vector perpendicular to it. Then, the reflection transformation **T** satisfies $\mathbf{Ta} = \mathbf{a}$ and $\mathbf{Tb} = -\mathbf{b}$. In the basis $\{\mathbf{a}, \mathbf{b}\}$, the matrix of **T** is:
  $$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$
  2. Let **a** be a direction vector of the projection line, and let **b** be a nonzero vector perpendicular to it. Then, the projection transformation **P** satisfies $\mathbf{Pa} = \mathbf{a}$ and $\mathbf{Pb} = \mathbf{0}$. In the basis $\{\mathbf{a}, \mathbf{b}\}$, the matrix of **P** is:
  $$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

Eigenvalues, Eigenvectors, Diagonalization of Linear Transformations V

3. This transformation does not have a real diagonal matrix because it has only a single real eigenvector space, which is one-dimensional: the subspace spanned by the axis direction vector. The plane perpendicular to the rotation axis is not an eigenspace but is an invariant subspace, allowing a "nearly diagonal" form. If $\mathbf{a}$ is a direction vector of the rotation axis and $\{\mathbf{b}, \mathbf{c}\}$ is an orthonormal basis of the perpendicular plane, where $\mathbf{b}$ is rotated by 90^o to $\mathbf{c}$, then in the basis $\{\mathbf{a}, \mathbf{b}, \mathbf{c}\}$, the matrix of the rotation transformation $\mathbf{F}$ is:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & -\sin\alpha \\ 0 & \sin\alpha & \cos\alpha \end{pmatrix},$$

since $\mathbf{Fa} = \mathbf{a}$, $\mathbf{Fb} = \cos\alpha\mathbf{b} + \sin\alpha\mathbf{c}$, and $\mathbf{Fc} = -\sin\alpha\mathbf{b} + \cos\alpha\mathbf{c}$.

Eigenvalues, Eigenvectors, Diagonalization of Linear Transformations VI

4. The plane onto which we project corresponds to eigenvalue 1. Choosing a basis $\{\mathbf{a}, \mathbf{b}\}$ in the plane and letting $\mathbf{c}$ be a nonzero vector perpendicular to it, we get $\mathbf{Ta} = \mathbf{a}$, $\mathbf{Tb} = \mathbf{b}$, and $\mathbf{Tc} = \mathbf{0}$, so the matrix of $\mathbf{T}$ is:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

5. The plane across which we reflect corresponds to eigenvalue 1. Choosing a basis $\{\mathbf{a}, \mathbf{b}\}$ in the plane and letting $\mathbf{c}$ be a nonzero vector perpendicular to it, we get $\mathbf{Ta} = \mathbf{a}$, $\mathbf{Tb} = \mathbf{b}$, and $\mathbf{Tc} = -\mathbf{c}$, so the matrix of $\mathbf{T}$ is:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

LU Decomposition I

- (LU Decomposition) Suppose that a matrix $\mathbf{A}$ can be transformed into an upper triangular matrix $\mathbf{U}$ using only row operations where a multiple of one row is added to a row below it. Each such elementary row operation corresponds to an elementary matrix that is lower triangular. Thus, there exist elementary lower triangular matrices $\mathbf{E}_1, \ldots, \mathbf{E}_k$ such that

$$\mathbf{E}_k \ldots \mathbf{E}_1 \mathbf{A} = \mathbf{U}.$$

From this, we obtain

$$\mathbf{A} = (\mathbf{E}_k \ldots \mathbf{E}_1)^{-1} \mathbf{U},$$

where $(\mathbf{E}_k \ldots \mathbf{E}_1)^{-1}$ is the inverse of a product of lower triangular matrices, which is also a lower triangular matrix. Moreover, in each of these matrices, including their product and its inverse, the diagonal consists entirely of ones.

LU Decomposition II

- Definition: (LU Decomposition). We say that the factorization of an $m \times n$ matrix $\mathbf{A}$ into the form $\mathbf{A} = \mathbf{LU}$ is an LU decomposition (LU factorization or LU decomposition) if $\mathbf{L}$ is a unit lower triangular matrix (i.e., ones on the diagonal and zeros above it), and $\mathbf{U}$ is an upper triangular matrix.
- Not every matrix has an LU decomposition.
- The LU decomposition is not unique.
- However, it can be shown that if $\mathbf{A}$ is invertible and has an LU decomposition, then it is unique.

LU Decomposition III

- Example of computing an LU decomposition: Using elementary row operations, transform the matrices

$$\mathbf{A} = \begin{pmatrix} 4 & 8 & 4 & 8 \\ 2 & 6 & 4 & 4 \\ 1 & 3 & 2 & 4 \end{pmatrix} \text{ and } \mathbf{B} = \begin{pmatrix} 4 & 8 & 8 \\ 2 & 6 & 4 \\ 1 & 3 & 4 \end{pmatrix}$$

  to upper triangular form, then use these steps to express an LU decomposition for each matrix.
  Perform Gaussian elimination step by step, writing out the elementary transformation matrices.
  $\vdots$

LU Decomposition IV

Thus, $\mathbf{E}_3\mathbf{E}_2\mathbf{E}_1\mathbf{A} = \mathbf{U}$, and multiplying by the inverse matrix $(\mathbf{E}_3\mathbf{E}_2\mathbf{E}_1)^{-1} = \mathbf{E}_1^{-1}\mathbf{E}_2^{-1}\mathbf{E}_3^{-1}$ gives $\mathbf{A} = \left(\mathbf{E}_1^{-1}\mathbf{E}_2^{-1}\mathbf{E}_3^{-1}\right)\mathbf{U}$. We obtain the lower triangular matrix $\mathbf{L} = \mathbf{E}_1^{-1}\mathbf{E}_2^{-1}\mathbf{E}_3^{-1}$. Invert the triangular matrices and multiply them together!

$$\mathbf{L} = \begin{pmatrix} 1 & 0 & 0 \\ \frac{1}{2} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \frac{1}{4} & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & \frac{1}{2} & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ \frac{1}{2} & 1 & 0 \\ \frac{1}{4} & \frac{1}{2} & 1 \end{pmatrix}.$$

The product of the elementary matrices can be obtained by copying the elements below the main diagonal. $\mathbf{L}$ is a lower triangular matrix. Thus, the LU decomposition of $\mathbf{A}$ is:

$$\begin{pmatrix} 4 & 8 & 4 & 8 \\ 2 & 6 & 4 & 4 \\ 1 & 3 & 2 & 4 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ \frac{1}{2} & 1 & 0 \\ \frac{1}{4} & \frac{1}{2} & 1 \end{pmatrix} \begin{pmatrix} 4 & 8 & 4 & 8 \\ 0 & 2 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}.$$

LU Decomposition V

Since no operations were performed between the columns during the transformation of $\mathbf{A}$, and the matrix $\mathbf{B}$ is obtained from $\mathbf{A}$ by removing the third column, the LU decomposition of $\mathbf{B}$ directly follows from the previous decomposition:

$$\begin{pmatrix} 4 & 8 & 8 \\ 2 & 6 & 4 \\ 1 & 3 & 4 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ \frac{1}{2} & 1 & 0 \\ \frac{1}{4} & \frac{1}{2} & 1 \end{pmatrix} \begin{pmatrix} 4 & 8 & 8 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}.$$

- Algorithm for computing an LU decomposition: Wettl notes
- Existence and uniqueness of LU decomposition: Wettl notes

LU Decomposition VI

- Solving systems of equations using LU decomposition: If we know the LU decomposition of a matrix $\mathbf{A}$, then solving the system $\mathbf{Ax} = \mathbf{b}$ is straightforward. The solution involves solving the systems $\mathbf{Ly} = \mathbf{b}$ and $\mathbf{Ux} = \mathbf{y}$. If $\mathbf{x}$ is the solution to $\mathbf{Ax} = \mathbf{b}$, then $\mathbf{LUx} = \mathbf{b}$, and setting $\mathbf{y} = \mathbf{Ux}$ gives $\mathbf{Ly} = \mathbf{b}$. Conversely, if $\mathbf{y}$ solves $\mathbf{Ly} = \mathbf{b}$ and $\mathbf{x}$ solves $\mathbf{Ux} = \mathbf{y}$, then substituting $\mathbf{y}$ yields $\mathbf{Ax} = \mathbf{b}$.

$$\mathbf{Ax} = \mathbf{b} \text{ is solvable} \Leftrightarrow \mathbf{Ly} = \mathbf{b},\ \ \mathbf{Ux} = \mathbf{y} \text{ is solvable.}$$

  From the form of $\mathbf{L}$ and $\mathbf{U}$, it follows that the systems of equations $\mathbf{Ly} = \mathbf{b}$ and $\mathbf{Ux} = \mathbf{y}$ can be solved by simple back-substitution.

LU Decomposition VII

- (Solving the system of equations using LU decomposition) Solve the following system of equations!

$$\begin{aligned} 4x_1 + 8x_2 + 8x_3 &= 8 \quad (1) \\ 2x_1 + 6x_2 + 4x_3 &= 4 \quad (2) \\ x_1 + 3x_2 + 4x_3 &= 4 \quad (3) \end{aligned}$$

Since we know the LU decomposition of the coefficient matrix, we use it to first solve the system $\mathbf{Ly} = \mathbf{b}$:

$$\begin{pmatrix} 1 & 0 & 0 \\ \frac{1}{2} & 1 & 0 \\ \frac{1}{4} & \frac{1}{2} & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} 8 \\ 4 \\ 4 \end{pmatrix}.$$

LU Decomposition VIII

From this, we get $y_1 = 8$, and substituting this into the second equation gives $y_2 = 0$, then substituting these values into the third equation gives $y_3 = 2$. Next, we solve the system $\mathbf{Ux} = \mathbf{y}$, which has the form

$$\begin{pmatrix} 4 & 8 & 8 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 8 \\ 0 \\ 2 \end{pmatrix}.$$

By simple back-substitution, we get $x_3 = 1$, $x_2 = 0$, and $x_1 = 1$. The solution is $\mathbf{x} = (0, 0, 1)$.

The End

Thank you for your attention!