



# (PTIA0301) Elementary Linear Algebra

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# Common Business

- ▶ I will evaluate and grade the tests during the holiday.
- ▶ I will upload the solution of the homework exercises during the holiday.
- ▶ Neptun, Moodle, and Teams page starts on <https://www.pte.hu!!!>
- ▶ Use Neptun code (my Neptun code is BQQQFY) and password (crack my BitWarden for password).
- ▶ Your PTE e-mail addresses are: *bqqqfy@tr.pte.hu*, and *facsko.gabor@pte.hu*.
- ▶ You can use Outlook, MS Office (O365), and Matlab as a citizen of UP.

# Transpose I

- Definition: The transpose of the  $A = (\alpha_{ij})_{m \times n}$  matrix is the  $A^T = (\alpha_{ji})_{n \times m}$ . This means the change of the rows and columns. The transpose is the mirror of a square matrix.

$$A_{m \times n} = \begin{pmatrix} \alpha_{11} & \alpha_{12} & \cdots & \alpha_{1n} \\ \alpha_{21} & \alpha_{22} & \cdots & \alpha_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_{m1} & \alpha_{m2} & \cdots & \alpha_{mn} \end{pmatrix} \quad A_{n \times m}^T = \begin{pmatrix} \alpha_{11} & \alpha_{21} & \cdots & \alpha_{1m} \\ \alpha_{12} & \alpha_{22} & \cdots & \alpha_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_{1n} & \alpha_{2n} & \cdots & \alpha_{nm} \end{pmatrix}$$

# Matrix Operations I

- Definition:  $A = (\alpha_{ij})_{m \times n}$  and  $B = (\beta_{ij})_{m \times n}$  are two matrixes with same type,  $\lambda \in \mathbb{R}$  a scalar. The sum of Matrixes  $A$  and  $B$  is Matrix  $A + B = (\alpha_{ij} + \beta_{ij})_{m \times n}$ , the  $\lambda$  times Matrix  $A$  is Matrix  $\lambda A = (\lambda \alpha_{ij})_{m \times n}$ .

$$A_{m \times n} = \begin{pmatrix} \alpha_{11} & \alpha_{12} & \cdots & \alpha_{1n} \\ \alpha_{21} & \alpha_{22} & \cdots & \alpha_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_{m1} & \alpha_{m2} & \cdots & \alpha_{mn} \end{pmatrix} \quad B_{m \times n} = \begin{pmatrix} \beta_{11} & \beta_{12} & \cdots & \beta_{1n} \\ \beta_{21} & \beta_{22} & \cdots & \beta_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \beta_{m1} & \beta_{m2} & \cdots & \beta_{mn} \end{pmatrix}$$

$$A_{m \times n} + B_{m \times n} = \begin{pmatrix} \alpha_{11} + \beta_{11} & \alpha_{12} + \beta_{12} & \cdots & \alpha_{1n} + \beta_{1n} \\ \alpha_{21} + \beta_{21} & \alpha_{22} + \beta_{22} & \cdots & \alpha_{2n} + \beta_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_{m1} + \beta_{m1} & \alpha_{m2} + \beta_{m2} & \cdots & \alpha_{mn} + \beta_{mn} \end{pmatrix}$$

## Matrix Operations II

$$\lambda A_{m \times n} = \begin{pmatrix} \lambda\alpha_{11} & \lambda\alpha_{12} & \cdots & \lambda\alpha_{1n} \\ \lambda\alpha_{21} & \lambda\alpha_{22} & \cdots & \lambda\alpha_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \lambda\alpha_{m1} & \lambda\alpha_{m2} & \cdots & \lambda\alpha_{mn} \end{pmatrix}$$

The elements of the matrixes are added and multiplying by a scalar means to multiply all elements of the matrix by the scalar.

## Matrix Operations III

- Definition:  $A = (\alpha_{ij})_{m \times n}$  and  $B = (\beta_{ij})_{n \times k}$  are two matrixes. The product of Matrixes  $A$  and  $B$  is Matrix  $A \cdot B = (\gamma_{ij})_{m \times k}$ , where

$$\gamma_{ij} = \sum_{l=1}^n \alpha_{il} \beta_{lj}.$$

Or:

$$A_{m \times n} = \begin{pmatrix} \alpha_{11} & \alpha_{12} & \cdots & \alpha_{1n} \\ \alpha_{21} & \alpha_{22} & \cdots & \alpha_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_{i1} & \alpha_{i2} & \cdots & \alpha_{in} \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_{m1} & \alpha_{m2} & \cdots & \alpha_{mn} \end{pmatrix} \quad B_{n \times k} = \begin{pmatrix} \beta_{11} & \beta_{12} & \cdots & \beta_{1j} & \cdots & \beta_{1k} \\ \beta_{21} & \beta_{22} & \cdots & \beta_{2j} & \cdots & \beta_{2k} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ \beta_{n1} & \beta_{n2} & \cdots & \beta_{nj} & \cdots & \beta_{nk} \end{pmatrix}$$

$$A \cdot B_{m \times k} = \begin{pmatrix} \alpha_{11}\beta_{11} + \alpha_{12}\beta_{21} + \cdots + \alpha_{1n}\beta_{n1} & \alpha_{11}\beta_{12} + \alpha_{12}\beta_{22} + \cdots + \alpha_{1n}\beta_{n2} & \cdots & \alpha_{11}\beta_{1k} + \alpha_{12}\beta_{2k} + \cdots + \alpha_{1n}\beta_{nk} \\ \alpha_{21}\beta_{11} + \alpha_{22}\beta_{21} + \cdots + \alpha_{2n}\beta_{n1} & \alpha_{21}\beta_{12} + \alpha_{22}\beta_{22} + \cdots + \alpha_{2n}\beta_{n2} & \cdots & \alpha_{21}\beta_{1k} + \alpha_{22}\beta_{2k} + \cdots + \alpha_{2n}\beta_{nk} \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_{m1}\beta_{11} + \alpha_{m2}\beta_{21} + \cdots + \alpha_{mn}\beta_{n1} & \alpha_{m1}\beta_{12} + \alpha_{m2}\beta_{22} + \cdots + \alpha_{mn}\beta_{n2} & \cdots & \alpha_{m1}\beta_{1k} + \alpha_{m2}\beta_{2k} + \cdots + \alpha_{mn}\beta_{nk} \end{pmatrix}$$

## Matrix Operations IV

- ▶ For similar square matrixes the condition of matrix production is fulfilled and the product will be the same type. Therefore, there is an exponentiation of matrixes:

$$A^1 = A \quad \text{és} \quad A^m = AA^{m-1}$$

where  $(m \geq 2)$  és  $A \in \mathcal{M}_{n \times n}$ . Let us consider  $A^0 = E_m$ .

- ▶ Thesis: Equations of matrix exponentation:

$$\begin{aligned} A^m A^k &= A^{m+k} \\ (A^m)^k &= A^{mk}, \end{aligned}$$

where  $m, k \in \mathbb{N}$ .

# Matrix Inversion I

- ▶ Definition: The  $n^{th}$  order identity matrix is:

$$E_n = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}$$

- ▶ Thesis: For all  $A \in \mathcal{M}_{n \times n}$ :  $A \cdot E_n = E_n \cdot A = A$ , or matrix  $E_n$  is identity element of the  $n \times n$  square matrixes for matrix production.



## Matrix Inversion II

- ▶ Definition: Square matrix  $A \in \mathcal{M}_{n \times n}$  exists inverse if exist such a Matrix  $B \in \mathcal{M}_{n \times n}$ , that  $AB = BA = E_n$ . The inverse of Matrix  $A$  is  $A^{-1}$ .
- ▶ Thesis: Matrix  $A \in \mathcal{M}_{n \times n}$  exists inverse if only  $\det(A) \neq 0$ .
- ▶ Matrix  $A \in \mathcal{M}_{n \times n}$  is regular if  $\det(A) \neq 0$ .
- ▶ Matrix  $A \in \mathcal{M}_{n \times n}$  is singular if  $\det(A) = 0$ .

## Matrix Inversion III

► Inverse matrix calculation by elemental transformations:

- Multiplication of a row by a  $\lambda \neq 0$  scalar.
- Adding  $\lambda$  times of a row to another row.
- Changing of rows.

If Matrix  $A$  is a regular matrix, then the  $(A|E_n)$  extended matrix could be transformed for  $(E_n|B)$  form, where Matrix  $B$  is the inverse of Matrix  $A$ .

This transformation cannot be made for singular matrixes.

## Matrix Inversion IV

- ▶ Calculation of inverse matrix by subdeterminant.
  - ▶ You calculate the determinant of the matrix. The inverse exists if the determinant is not zero.
  - ▶  $A_{ij}$  is the subdeterminant for each element. The result must be transposed and divided by  $\det(A)$  you get the inverse of Matrix  $A$ :

$$(A^{-1})_{ij} = \frac{A_{ij}}{\det(A)}.$$

(The subdeterminant of Matrix  $A$ 's  $\alpha_{ij}$  element is:  $A_{ij} = (-1)^{i+j} D_{ij}$ , where  $D_{ij}$  is the determinant of the  $(n-1) \times (n-1)$  matrix created by deleting the row and column of the element  $\alpha_{ij}$ ).

# Matrix Inversion V

► Thesis:  $A, B \in \mathcal{M}_{n \times n}$ .

1. If Matrixes  $A$  and  $B$  have invers, then  $AB$  also has inverse and  $(AB)^{-1} = B^{-1}A^{-1}$ .
2.  $(AB)^T = B^T A^T$
3. If  $A$  has inverse, then  $A^T$  also has inverse, and  $(A^T)^{-1} = (A^{-1})^T$ .

# Matrix Rank I

- ▶ Definition:  $\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_s \in V$  are vectors. The rank of the  $\{\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_s\}$  vector system is the dimension of the  $\mathcal{L}(\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_s)$  subspace. Its sign is  $\rho(\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_s)$ .
- ▶ Thesis: The following transformation do not change the order of the  $\{\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_s\}$  vector system:
  1. Multiplying a vectors by a  $\lambda \neq 0$  scalar.
  2. Adding the vector multiplied by  $\lambda$  to another vector.
  3. Eliminating a vector that is a linear combination of the remaining vectors.
  4. Changing the order of vectors.
- ▶ Definition: The rank of Matrix  $A \in \mathcal{M}_{m \times n}$  is the rank of its row vector system.
- ▶ The rank of a matrix is equal to the common rank of the maximal ranked non-disappearing subdeterminants.

## Matrix Rank II

- ▶ The rank of a matrix is determined by transforming the matrix to trapezoid form by rank invariant transformations. You can change the columns. (A matrix has trapezoid shape if  $\alpha_{ij} = 0$ ,  $i > j$ , and  $\alpha_{ii} \neq 0$ , where  $(1 \leq i \leq \min\{m, n\})$ .) Rows and columns containing 0 could be deleted. The rank of the trapezoid matrix is the number of its rows.

# The End

Thank you for your attention!