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(ONFIZ1-0401) Elementary Linear Algebra, Test 1

1. $\mathbf{a} = (-2, 0, -1)$, $\mathbf{b} = (0, -3, 1)$ és $\mathbf{c} = (2, 1, 1)$. Calculate the following expressions:

a.) $(\mathbf{a} - \mathbf{b})\,\mathbf{c} = \left[\begin{pmatrix}-2\\0\\-1\end{pmatrix} - \begin{pmatrix}0\\-3\\1\end{pmatrix}\right](2,1,1) = \begin{pmatrix}-2\\3\\-2\end{pmatrix}(2,1,1) = -2\cdot 2 + 3\cdot 1 - 2\cdot 1 = \underline{\underline{-3}}$

b.) $(\mathbf{b} + \mathbf{c}) \times \mathbf{a} = \left[\begin{pmatrix}0\\-3\\1\end{pmatrix} + \begin{pmatrix}2\\1\\1\end{pmatrix}\right] \times \begin{pmatrix}-2\\0\\-1\end{pmatrix} = \begin{pmatrix}2\\-2\\2\end{pmatrix} \times \begin{pmatrix}-2\\0\\-1\end{pmatrix} = \begin{pmatrix}-2\cdot(-1) - 2\cdot 0\\ 2\cdot(-2) - 2\cdot(-1)\\ 2\cdot 0 - (-2)(-2)\end{pmatrix} =$
$\begin{pmatrix}2\\-2\\-4\end{pmatrix} = \underline{\underline{2\begin{pmatrix}1\\-1\\-2\end{pmatrix}}}$

c.) $(\mathbf{a}, \mathbf{b}, \mathbf{c}) = (\mathbf{a} \times \mathbf{b})\,\mathbf{c} = \left[\begin{pmatrix}-2\\0\\-1\end{pmatrix} \times \begin{pmatrix}0\\-3\\1\end{pmatrix}\right] \cdot \begin{pmatrix}2\\1\\1\end{pmatrix} = \begin{pmatrix}0\cdot 1 - (-1)(-3)\\ -1\cdot 0 - (-2)\cdot 1\\ -2(-3) - 0\cdot 0\end{pmatrix} \cdot \begin{pmatrix}2\\1\\1\end{pmatrix} =$
$\begin{pmatrix}-3\\2\\6\end{pmatrix} \cdot \begin{pmatrix}2\\1\\1\end{pmatrix} = -3\cdot 2 + 2\cdot 1 + 6\cdot 1 = \underline{\underline{2}}$

d.) What is the angle of Vectors $\mathbf{a}$ and $\mathbf{b}$? $\cos\theta = \frac{\mathbf{ab}}{|\mathbf{a}||\mathbf{b}|} = \frac{-2\cdot 0 + 0\cdot(-3) + (-1)\cdot 1}{\sqrt{(-2)^2 + 0^2 + (-1)^2}\sqrt{0^2 + (-3)^2 + 1^2}} = \frac{-1}{\sqrt{5}\sqrt{10}} =$
$-\frac{1}{5\sqrt{2}} = \underline{\underline{-\frac{\sqrt{2}}{10}}}$

e.) Are Vectors $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ in the same plane?
We calculated previously the volume of the paralepipedon determined by **a**, **b**, **c** vectors, that is $(\mathbf{a} \times \mathbf{b})\,\mathbf{c}$ is not zero. Therefore, the three vectors are not in the same plane.

f.) Determine a perpendicular vector to Vector $\mathbf{b}$.
The **b** vector is in the palne of y and z axes. Therefore, all vectors paralel with x axis is perpendicular, for example the $(1, 0, 0)$ vector.

(8 point)

2. Calculate the determinant of the following matrixes:

$$\mathbf{A} = \begin{pmatrix}3 & -1\\-1 & 3\end{pmatrix} \quad \mathbf{B} = \begin{pmatrix}1 & -1 & -2\\-1 & 2 & 4\\-2 & 4 & 3\end{pmatrix} \quad \mathbf{C} = \begin{pmatrix}4 & 2 & -1 & 1\\2 & 1 & -1 & 2\\1 & 0 & 0 & 1\\-2 & -2 & 2 & 2\end{pmatrix}$$

$\det(\mathbf{A}) = \begin{vmatrix}3 & -1\\-1 & 3\end{vmatrix} = 3\cdot 3 - (-1)(-1) = 8$

$\det(\mathbf{B}) = \begin{vmatrix}1 & -1 & -2\\-1 & 2 & 4\\-2 & 4 & 3\end{vmatrix} = 1\cdot 2\cdot 3 + (-1)\cdot 4\cdot(-2) + (-2)(-1)4 - [(-2)\cdot 2\cdot(-2) + (-1)(-1)3 + 1\cdot 4\cdot 4] =$
$6 + 8 + 8 - (8 + 3 + 16) = 22 - 27 = \underline{\underline{-5}}$

$\det(\mathbf{C}) = \begin{vmatrix}4 & 2 & -1 & 1\\2 & 1 & -1 & 2\\1 & 0 & 0 & 1\\-2 & -2 & 2 & 2\end{vmatrix} = \begin{vmatrix}2 & -1 & 1\\1 & -1 & 2\\-2 & 2 & 2\end{vmatrix} - \begin{vmatrix}4 & 2 & -1\\2 & 1 & -1\\-2 & -2 & 2\end{vmatrix} =$
$\{2(-1)2 + (-1)2(-2) + 1\cdot 1\cdot 2 - [1(-1)(-2) + 1(-1)2 + 2\cdot 2\cdot 2]\} -$
$\{4\cdot 1\cdot 2 + 2(-1)(-2) + (-1)2(-2) - [(-1)1(-2) + 2\cdot 2\cdot 2 + 4(-1)(-2)]\} =$
$-4 + 4 + 2 - (2 - 2 + 8) - [8 + 4 + 4 - (2 + 8 + 8)] = -6 - (16 - 18) = \underline{\underline{-4}}$

(10 point)

3. Solve the following systems of linear equations:

a.)

$$
\begin{aligned}
x_1 - x_2 + 2x_3 &= 1 \\
2x_1 - x_2 + x_3 &= -2 \\
3x_1 + 5x_2 + 2x_3 &= 0
\end{aligned}
$$

(II)-2(I), (III)-3(I)

$$
\begin{aligned}
x_2 - 3x_3 &= -4 \\
8x_2 - 4x_3 &= -3
\end{aligned}
$$

(III)-8(II)

$$
20x_3 = 29
$$

`From the last equation:` $x_3 = \underline{\underline{\frac{29}{20}}}$. `From the first equation of the penultimate system of equations:` $x_2 = -4+3\frac{29}{20} = \underline{\underline{\frac{7}{20}}}$. `From the first equation of the original system of equations:` $x_1 = 1 + \frac{7}{20} - 2\frac{29}{20} = \underline{\underline{-\frac{31}{20}}}$.

b.)

$$
\begin{aligned}
-x_1 + 3x_2 + x_3 &= 1 \\
x_1 + 3x_2 + x_3 &= 0 \\
4x_1 + x_2 - 3x_3 &= 1
\end{aligned}
$$

(II)+(I), (III)+4(I)

$$
\begin{aligned}
6x_2 + 2x_3 &= 1 \\
13x_2 + x_3 &= 5
\end{aligned}
$$

13(I), 6(II)

$$
\begin{aligned}
78x_2 + 26x_3 &= 13 \\
78x_2 + 6x_3 &= 30
\end{aligned}
$$

(II)-(I)

$$
-20x_3 = 17
$$

`From the last equation:` $x_3 = \underline{\underline{-\frac{17}{20}}}$. `From the first equation of the second system of equations:` $x_2 = \frac{1}{6}\left[1 - 2 \cdot \left(-\frac{17}{20}\right)\right] = \underline{\underline{\frac{9}{20}}}$. `From the first equation of the original system of equations:` $x_1 = -3\frac{9}{20} + \frac{17}{20} = -\frac{10}{20} = \underline{\underline{-\frac{1}{2}}}$. (12 point)

4. Are independent linear Vectors $\mathbf{a} = (-1, 2, 1, 3)$, $\mathbf{b} = (0, 5, -2, 2)$, and $\mathbf{c} = (1, 1, 3, 1)$?

Vectors $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ are independent linearl if $\lambda_1\mathbf{a} + \lambda_2\mathbf{b} + \lambda_3\mathbf{c} = \mathbf{0}$ is satisfied then, and only then if $\lambda_1 = \lambda_2 = \lambda_3 = 0$. Therefore, we can multiply the vectors with these coefficients and write them as a system of linear equations:

$$\begin{aligned} -\lambda_1 + \lambda_3 &= 0 \\ 2\lambda_1 + 5\lambda_2 + \lambda_3 &= 0 \\ \lambda_1 - 2\lambda_2 + 3\lambda_3 &= 0 \\ 3\lambda_1 + 2\lambda_2 + \lambda_3 &= 0 \end{aligned}$$

From the first equation $\lambda_1 = \lambda_3$, that you can substitute to the other equations:

$$\begin{aligned} 3\lambda_1 + 5\lambda_2 &= 0 \\ 4\lambda_1 - 2\lambda_2 &= 0 \\ 4\lambda_1 + 2\lambda_2 &= 0 \end{aligned}$$

From the second equation $\lambda_2 = 2\lambda_1$. You can substitute this term to the rest of the equations:

$$\begin{aligned} 13\lambda_1 &= 0 \\ 8\lambda_1 &= 0 \end{aligned}$$

Both equation can be satisfied if $\lambda_1 = 0$. Hence, $\lambda_1 = \lambda_2 = \lambda_3 = 0$, therefore, the three vectors are independent linearly. (8 point)

5. Is subspace on $\mathbb{R}^3$ the $U = \{(x_1 + x_2, -x_1 - x_2, 4x_2) \,|x_1, x_2 \in \mathbb{R}\}$ set?
The U non-empty sub set of the vectorspace $\mathbb{R}^3$ is linear subspace, then and only then, if

$$\begin{aligned} \forall \mathbf{a}, \mathbf{b} \in U \qquad & \mathbf{a} + \mathbf{b} \in U \\ \forall \lambda \in \mathbb{R}, \forall \mathbf{a} \in U \qquad & \lambda \mathbf{a} \in U. \end{aligned}$$

If $\mathbf{a}, \mathbf{b} \in U$ then:

$$\begin{pmatrix} a_1 + a_2 \\ -a_1 - a_2 \\ 4a_2 \end{pmatrix} + \begin{pmatrix} b_1 + b_2 \\ -b_1 - b_2 \\ 4b_2 \end{pmatrix} = \begin{pmatrix} a_1 + a_2 + b_1 + b_2 \\ -a_1 - a_2 - b_1 - b_2 \\ 4a_2 + 4b_2 \end{pmatrix} = \begin{pmatrix} (a_1 + b_1) + (a_2 + b_2) \\ -(a_1 + b_1) - (a_2 + b_2) \\ 4(a_2 + b_2) \end{pmatrix} = \begin{pmatrix} c_1 + c_2 \\ -c_1 - c_2 \\ 4c_2 \end{pmatrix},$$

where $c_i = a_i + b_i$ and $i \in \{1, 2, 3\}$. Therefore, the $\mathbf{a} + \mathbf{b}$ vector constructed from the $\mathbf{a}, \mathbf{b} \in U$ vectors is also part of the U set.
If $\mathbf{a} \in L$ and $\lambda \in \mathbb{R}$. Then

$$\lambda \begin{pmatrix} a_1 + a_2 \\ -a_1 - a_2 \\ 4a_2 \end{pmatrix} = \begin{pmatrix} \lambda a_1 + \lambda a_2 \\ -\lambda a_1 - \lambda a_2 \\ 4\lambda 2 \end{pmatrix} = \begin{pmatrix} d_1 + d_2 \\ -d_1 - d_2 \\ 4d_2 \end{pmatrix}$$

where $d_i = \lambda a_i$ and $i \in \{1, 2, 3\}$. Therefore the $\lambda\mathbf{a}$ vector constituted from the $\mathbf{a} \in U$ vector and the $\lambda \in \mathbb{R}$ scalar is also part of the U set. It means that the U set is a linear subspace of the $\mathbb{R}^3$. (4 point)

6. Are independent linear Vectors $\mathbf{a} = (-1, 2, 1, 3)$, $\mathbf{b} = (0, 5, -2, 2)$, and $\mathbf{c} = (1, 1, 3, 1)$?

The task means that

$$\lambda_1 \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} + \lambda_2 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + \lambda_3 \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix},$$

or the $\mathbf{a}$ vector is constructed from the λ_1, λ_2, λ_3 times of the three vectors, respectively. szorosaként áll elö az. This is a linear syste of equations for the λ_1, λ_2, λ_3 triplet:

$$\begin{aligned} -\lambda_1 + \lambda_3 &= -1 \\ \lambda_1 + \lambda_2 + 2\lambda_3 &= 0 \\ \lambda_1 + \lambda_3 &= 0 \end{aligned}$$

From the third equations $\lambda_1 = -\lambda_3$. You can substitute this in the other equations:

$$\begin{aligned} -2\lambda_1 &= -1 \\ -\lambda_1 + \lambda_2 &= 0 \end{aligned}$$

From the first equation $\lambda_1 = \underline{\underline{\tfrac{1}{2}}}$, from the second equation $\lambda_2 = \lambda_1$. Hence, $\lambda_2 = \underline{\underline{\tfrac{1}{2}}}$ and $\lambda_3 = \underline{\underline{-\tfrac{1}{2}}}$. Therefore, $\mathbf{a} = \frac{1}{2} \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} + \frac{1}{2} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} - \frac{1}{2} \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$. (8 point)

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