(ENKEMNA0302) Applied Linear Algebra

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Operations with Block Matrices I

- Operations on extremely large matrices can be parallelized if the matrices are divided into blocks and the operations are performed on these smaller submatrices.
- When a matrix is divided into submatrices by horizontal and vertical lines, we say that this matrix is a block matrix composed of submatrices—also known as blocks.
- The rows and columns of a block matrix are called the block rows and block columns of the matrix.
- Block matrices are also referred to as hypermatrices, but this term is also used for multidimensional arrays. Therefore, we prefer to call them block matrices.

Operations with Block Matrices II

- The augmented matrix $[\mathbf{A}|\mathbf{b}]$ of a system of equations is a block matrix consisting of two blocks:

$$\left(\begin{array}{ccc|cc|c} 1 & 0 & 0 & 1 & 2 & 4 \\ 0 & 1 & 0 & 2 & 0 & 3 \\ 0 & 0 & 1 & 1 & 0 & 3 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

  This is the augmented matrix of a system of 5 equations with 5 unknowns. The first block column corresponds to the bound variables, the second to the free variables, and the third to the right-hand side of the system of equations. The second block row contains the zero rows.

Operations with Block Matrices III

- Statement: (Operations with Block Matrices). Multiplication of block matrices by a scalar and the addition of two block matrices partitioned in the same way can be performed block-wise, that is,

$$c\,[\mathbf{A}_{ij}] = [c\mathbf{A}_{ij}]\,, \quad [\mathbf{A}_{ij}] + [\mathbf{B}_{ij}] = [\mathbf{A}_{ij} + \mathbf{B}_{ij}]\,.$$

If $\mathbf{A} = [\mathbf{A}_{ik}]_{m\times t}$ and $\mathbf{B} = [\mathbf{B}_{kj}]_{t\times n}$ are two block matrices, and for every k, the number of columns of $\mathbf{A}_{ik}$ matches the number of rows of $\mathbf{B}_{kj}$, then the product $\mathbf{C} = \mathbf{AB}$ can be computed by applying the multiplication rule to blocks, that is, $\mathbf{C}$ is a block matrix where the block in the i-th block row and j-th block column is given by:

$$\mathbf{C}_{ij} = \sum_{k=1}^{t} \mathbf{A}_{ik}\mathbf{B}_{kj}\,.$$

Operations with Block Matrices IV

- Example of block matrix multiplication:

$$\left(\begin{array}{cc|c} 1 & 0 & 1 \\ \hline 2 & 1 & 1 \\ 0 & 3 & 1 \end{array}\right) \left(\begin{array}{c|c} 1 & 1 \\ 1 & 2 \\ \hline 0 & 1 \end{array}\right) = \left(\begin{array}{c|c} \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \end{pmatrix} \begin{pmatrix} 0 \end{pmatrix} & \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \end{pmatrix} + \begin{pmatrix} 1 \end{pmatrix} \begin{pmatrix} 1 \end{pmatrix} \\ \hline \begin{pmatrix} 2 & 1 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \end{pmatrix} \begin{pmatrix} 0 \end{pmatrix} & \begin{pmatrix} 2 & 1 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \end{pmatrix} \begin{pmatrix} 1 \end{pmatrix} \end{array}\right) =$$

$$\left(\begin{array}{c|c} \begin{pmatrix} 1 \end{pmatrix} & \begin{pmatrix} 2 \end{pmatrix} \\ \hline \begin{pmatrix} 3 \\ 3 \end{pmatrix} & \begin{pmatrix} 5 \\ 7 \end{pmatrix} \end{array}\right) = \left(\begin{array}{c|c} 1 & 2 \\ \hline 3 & 5 \\ 3 & 7 \end{array}\right).$$

Kronecker Product and the Vec Function I

- Certain block matrix operations cannot be derived from simple matrix operations.
- The *vec* function transforms an arbitrary matrix into a vector by stacking its column vectors on top of each other. If $\mathbf{A} = [\mathbf{a}_1 | \mathbf{a}_2 | \dots | \mathbf{a}_n]$, then

$$vec\,(\mathbf{A}) = \begin{pmatrix} \mathbf{a}_1 \\ \vdots \\ \mathbf{a}_n \end{pmatrix}$$

For example, if $\mathbf{A} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$, then $vec\,(\mathbf{A}) = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}$.

Kronecker Product and the Vec Function II

- Let $\mathbf{A}$ be an $m \times n$ matrix and $\mathbf{B}$ a $p \times q$ matrix. Their Kronecker product (also called the tensor product) is the $mp \times nq$ matrix denoted by $A \otimes B$, which has the block matrix form:

$$\mathbf{A} \otimes \mathbf{B} = \begin{pmatrix} a_{11}\mathbf{B} & a_{12}\mathbf{B} & \dots & a_{1n}\mathbf{B} \\ a_{21}\mathbf{B} & a_{22}\mathbf{B} & \dots & a_{2n}\mathbf{B} \\ \vdots & \vdots & \dots & \vdots \\ a_{m1}\mathbf{B} & a_{m2}\mathbf{B} & \dots & a_{mn}\mathbf{B} \end{pmatrix}.$$

- For example,

$$\begin{pmatrix} -1 & 2 \\ 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} 0 & 1 & 2 \\ 3 & 3 & 3 \end{pmatrix} = \begin{pmatrix} 0 & -1 & -2 & 0 & 2 & 4 \\ -3 & -3 & -3 & 6 & 6 & 6 \\ 0 & 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 3 & 3 & 3 \end{pmatrix}.$$

Kronecker Product and the Vec Function III

- Theorem: (Properties of the Kronecker Product). Given the matrices $\mathbf{A}_{m\times n}$, $\mathbf{B}_{m\times n}$, $\mathbf{C}_{p\times q}$, and $\mathbf{D}_{r\times s}$, we have:
  1. $(\mathbf{A}+\mathbf{B})\otimes\mathbf{C}=\mathbf{A}\otimes\mathbf{C}+\mathbf{B}\otimes\mathbf{C}$, $\mathbf{C}\otimes(\mathbf{A}+\mathbf{B})=\mathbf{C}\otimes\mathbf{A}+\mathbf{C}\otimes\mathbf{B}$,
  2. $(\mathbf{A}\otimes\mathbf{C})\otimes\mathbf{D}=\mathbf{A}\otimes(\mathbf{C}\otimes\mathbf{D})$,
  3. $(\mathbf{A}\otimes\mathbf{C})^T=\mathbf{C}^T\otimes\mathbf{A}^T$.

  Proof: By definition, e.g., for (2):

$$(\mathbf{AXB})_{*j}=\mathbf{AXB}_{*j}=\sum_{i=1}^{n}(b_{ij}\mathbf{A})\,\mathbf{X}_{*i}=(b_{1j}\mathbf{A}|\ldots|b_{nj}\mathbf{A})\,vec\,(\mathbf{X})=\left(\mathbf{B}^T\otimes\mathbf{A}\right)_{*j}vec\,(\mathbf{X})\,.$$

Hypermatrixes I

- Certain data can be well organized in arrays of dimension higher than 2.
- <u>Definition:</u> (Hypermatrix). Let $n_1, n_2, \ldots, n_d \in \mathbb{N}^+$ and let S be an arbitrary set (e.g., $S = \mathbb{R}, \mathbb{Q}, \mathbb{N}, \mathbb{Z} \ldots$). A dth-order (or d-dimensional) $n_1 \times n_2 \times \cdots \times n_d$-type hypermatrix is a mapping of the form

$$\mathbf{A} : \{1, \ldots, n_1\} \times \{1, \ldots, n_2\} \times \cdots \times \{1, \ldots, n_d\} \to S$$

The element $\mathbf{A}(i_1, i_2, \ldots, i_d)$ is denoted by $a_{i_1 i_2 \ldots i_d}$, which is an element of a d-dimensional table, and similarly to matrices, we can write:

$$\mathbf{A} = (a_{i_1 i_2 \ldots i_d})_{i_1, i_2, \ldots, i_d}^{n_1, n_2, \ldots, n_d} = 1, \text{or simply } \mathbf{A} = (a_{i_1 i_2 \ldots i_d}).$$

If $n_1 = n_2 = \cdots = n_d = n$, then we refer to a hyper-cube matrix.

Hypermatrixes II

- The set of all $n_1 \times n_2 \times \cdots \times n_d$-type hypermatrices formed from the elements of S is denoted by $S^{n_1 \times n_2 \times \ldots n_d}$.
- Second-order hypermatrices coincide with matrices.
- The elements of third-order hypermatrices can be described by slicing them according to the third index. Each slice is a matrix, and these matrices are written next to each other separated by vertical lines. For example, the general form of a $4 \times 2 \times 3$-type hypermatrix is:

$$\left(\begin{array}{cc|cc|cc} a_{111} & a_{121} & a_{112} & a_{122} & a_{113} & a_{123} \\ a_{211} & a_{221} & a_{212} & a_{222} & a_{213} & a_{223} \\ a_{311} & a_{321} & a_{312} & a_{322} & a_{313} & a_{323} \\ a_{411} & a_{421} & a_{412} & a_{422} & a_{413} & a_{423} \end{array}\right)$$

Hypermatrixes III

- Addition of two hypermatrices of the same type and scalar multiplication of a hypermatrix follow element-wise operations similar to matrices:

$$\left(a_{i_1 i_2 \dots i_d}\right) + \left(b_{i_1 i_2 \dots i_d}\right) = \left(a_{i_1 i_2 \dots i_d} + b_{i_1 i_2 \dots i_d}\right),$$
$$c\left(a_{i_1 i_2 \dots i_d}\right) = \left(c a_{i_1 i_2 \dots i_d}\right).$$

- Definition: (Transpose of a hypermatrix). Let π be a permutation of the set $\{1, 2, \dots, d\}$. The π-transpose of the dth-order hypermatrix $\mathbf{A} = \left(a_{i_1 i_2 \dots i_d}\right) \in S^{n_1 \times n_2 \times \dots n_d}$ is defined as:

$$\mathbf{A}^{\pi} = \left(a_{i_{\pi(1)} i_{\pi(2)} \dots i_{\pi(d)}}\right) \in S^{n_{\pi(1)} \times n_{\pi(2)} \times \dots n_{\pi(d)}}$$

A hyper-cube matrix $\mathbf{A} \in S^{n \times n \times \dots \times n}$ is symmetric if for all permutations π, we have $\mathbf{A}^{\pi} = \mathbf{A}$, and it is skew-symmetric if $\mathbf{A}^{\pi} = sgn(\pi)\mathbf{A}$, where $sgn(\pi) = -1$ for odd permutations and 1 for even permutations.

Hypermatrixes IV

- Accordingly, the general form of $2 \times 2 \times 2$ hypermatrices and symmetric hypermatrices is: $\left(\begin{array}{cc|cc} a_{111} & a_{121} & a_{112} & a_{122} \\ a_{211} & a_{221} & a_{212} & a_{222} \end{array}\right)$, $\left(\begin{array}{cc|cc} a & b & b & c \\ b & c & c & d \end{array}\right)$

  The general form of $3 \times 3 \times 3$ hypermatrices, symmetric, and skew-symmetric hypermatrices is:

$$\left(\begin{array}{ccc|ccc|ccc} a_{111} & a_{121} & a_{131} & a_{112} & a_{122} & a_{132} & a_{113} & a_{123} & a_{133} \\ a_{211} & a_{221} & a_{231} & a_{212} & a_{222} & a_{232} & a_{213} & a_{223} & a_{233} \\ a_{311} & a_{321} & a_{331} & a_{312} & a_{322} & a_{332} & a_{313} & a_{323} & a_{333} \end{array}\right),$$

$$\left(\begin{array}{ccc|ccc|ccc} a & b & c & b & d & e & c & e & f \\ b & d & e & d & g & h & e & h & i \\ c & e & f & e & h & i & f & i & j \end{array}\right), \left(\begin{array}{ccc|ccc|ccc} 0 & 0 & 0 & 0 & 0 & -a & 0 & a & 0 \\ 0 & 0 & a & 0 & 0 & 0 & -a & 0 & 0 \\ 0 & -a & 0 & a & 0 & 0 & 0 & 0 & 0 \end{array}\right),$$

  where $a, b, c, d, e, f, g, h, i, j \in S$ are not necessarily distinct elements.

The End

Thank you for your attention!