(PTIA0301) Elementary Linear Algebra

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Mátrixes I

- Definíció: The operators are linear vector-vector functions.
- Matrixes are the representations of operators. If $\alpha_{ij} \in \mathbb{R}$ for all $i \in \{1, 2, \ldots, m\}$ and $j \in \{1, 2, \ldots, n\}$, where $m, n \in \mathbb{N}^+$. Then the

$$A = \begin{pmatrix} \alpha_{11} & \alpha_{12} & \cdots & \alpha_{1n} \\ \alpha_{21} & \alpha_{22} & \cdots & \alpha_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_{m1} & \alpha_{m2} & \cdots & \alpha_{mn} \end{pmatrix}$$

  table is $m \times n$ type matrix. The set of $m \times n$ type matrixes is $M_{m \times n}$.
- The main diagonal of the matrix is the $\{\alpha_{11}, \alpha_{22}, \ldots, \alpha_{nn}\}$ set.
- The indexes of α_{ij} element are the first (i), the index of the row and the second (j), the index of the column.

Mátrixes II

- A_i is the i^{th} row of the matrix, A_j is the j^{th} column.

Matrix operations, inverse matrix, rank of matrix I

- Definition: The transponent of the $A = (\alpha_{ij})_{m \times n}$ matrix is the $A^T = (\alpha_{ji})_{m \times n}$. This means the change of the rows and columns. The transpose is the mirror of a square matrix.

$$A_{m \times n} = \begin{pmatrix} \alpha_{11} & \alpha_{12} & \cdots & \alpha_{1n} \\ \alpha_{21} & \alpha_{22} & \cdots & \alpha_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_{m1} & \alpha_{m2} & \cdots & \alpha_{mn} \end{pmatrix} \quad A^T_{n \times m} = \begin{pmatrix} \alpha_{11} & \alpha_{21} & \cdots & \alpha_{1m} \\ \alpha_{12} & \alpha_{22} & \cdots & \alpha_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_{1n} & \alpha_{2n} & \cdots & \alpha_{mn} \end{pmatrix}$$

Matrix operations, inverse matrix, rank of matrix II

- Definition: $A=(\alpha_{ij})_{m\times n}$ and $B=(\beta_{ij})_{m\times n}$ are two matrixes with same type, $\lambda\in\mathbb{R}$ a scalar. The sum of Matrixes A and B is Matrix $A+B=(\alpha_{ij}+\beta_{ij})_{m\times n}$, the λ times Matrix A is Matrix $\lambda A=(\lambda\alpha_{ij})_{m\times n}$.

$$A_{m\times n}=\begin{pmatrix}\alpha_{11}&\alpha_{12}&\cdots&\alpha_{1n}\\ \alpha_{21}&\alpha_{22}&\cdots&\alpha_{2n}\\ \vdots&\vdots&\ddots&\vdots\\ \alpha_{m1}&\alpha_{m2}&\cdots&\alpha_{mn}\end{pmatrix} B_{m\times n}=\begin{pmatrix}\beta_{11}&\beta_{12}&\cdots&\beta_{1n}\\ \beta_{21}&\beta_{22}&\cdots&\beta_{2n}\\ \vdots&\vdots&\ddots&\vdots\\ \beta_{m1}&\beta_{m2}&\cdots&\beta_{mn}\end{pmatrix}$$

$$A_{m\times n}+B_{m\times n}=\begin{pmatrix}\alpha_{11}+\beta_{11}&\alpha_{12}+\beta_{12}&\cdots&\alpha_{1n}+\beta_{1n}\\ \alpha_{21}+\beta_{21}&\alpha_{22}+\beta_{22}&\cdots&\alpha_{1n}+\beta_{2n}\\ \vdots&\vdots&\ddots&\vdots\\ \alpha_{m1}+\beta_{m1}&\alpha_{m2}+\beta_{m2}&\cdots&\alpha_{mn}+\beta_{mn}\end{pmatrix}$$

Matrix operations, inverse matrix, rank of matrix III

$$\lambda A_{m\times n} = \begin{pmatrix} \lambda\alpha_{11} & \lambda\alpha_{12} & \cdots & \lambda\alpha_{1n} \\ \lambda\alpha_{21} & \lambda\alpha_{22} & \cdots & \lambda\alpha_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \lambda\alpha_{m1} & \lambda\alpha_{m2} & \cdots & \lambda\alpha_{mn} \end{pmatrix}$$

The elements of the matrixes are added and multiplying by a scalar means to multiply all elements of the matrix by the scalar.

- Definition: $A = (\alpha_{ij})_{m\times n}$ and $B = (\beta_{ij})_{n\times k}$ are two matrixes. The product of Matrixes A and B is Matrix $A \cdot B = (\gamma_{ij})_{m\times k}$, where

$$\gamma_{ij} = \sum_{l=1}^{n} \alpha_{il}\beta_{lj}.$$

Matrix operations, inverse matrix, rank of matrix IV

Or:

$$A_{m\times n} = \begin{pmatrix} \alpha_{11} & \alpha_{12} & \cdots & \alpha_{1n} \\ \alpha_{21} & \alpha_{22} & \cdots & \alpha_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_{i1} & \alpha_{i2} & \cdots & \alpha_{in} \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_{m1} & \alpha_{m2} & \cdots & \alpha_{mn} \end{pmatrix} \quad B_{n\times k} = \begin{pmatrix} \beta_{11} & \beta_{12} & \cdots & \beta_{1j} & \cdots & \beta_{1k} \\ \beta_{21} & \beta_{22} & \cdots & \beta_{2j} & \cdots & \beta_{2k} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ \beta_{n1} & \beta_{n2} & \cdots & \beta_{nj} & \cdots & \beta_{nk} \end{pmatrix}$$

$$A{\cdot}B_{m\times k} = \begin{pmatrix} \alpha_{11}\beta_{11}+\alpha_{12}\beta_{21}+\cdots+\alpha_{1n}\beta_{n1} & \alpha_{11}\beta_{12}+\alpha_{12}\beta_{22}+\cdots+\alpha_{1n}\beta_{n2} & \cdots & \alpha_{11}\beta_{1k}+\alpha_{12}\beta_{2k}+\cdots+\alpha_{1n}\beta_{nk} \\ \alpha_{21}\beta_{11}+\alpha_{22}\beta_{21}+\cdots+\alpha_{2n}\beta_{n1} & \alpha_{21}\beta_{12}+\alpha_{22}\beta_{22}+\cdots+\alpha_{2n}\beta_{n2} & \cdots & \alpha_{21}\beta_{1k}+\alpha_{22}\beta_{2k}+\cdots+\alpha_{2n}\beta_{nk} \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_{m1}\beta_{11}+\alpha_{m2}\beta_{21}+\cdots+\alpha_{mn}\beta_{n1} & \alpha_{m1}\beta_{12}+\alpha_{m2}\beta_{22}+\cdots+\alpha_{mn}\beta_{n2} & \cdots & \alpha_{m1}\beta_{1k}+\alpha_{m2}\beta_{2k}+\cdots+\alpha_{mn}\beta_{nk} \end{pmatrix}$$

Matrix operations, inverse matrix, rank of matrix V

- Definition: The n^{th} order identity matrix is:

$$E_n = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}$$

- Thesis: For all $A \in \mathcal{M}_{n\times n}$: $A \cdot E_n = E_n \cdot A = A$, ormatrix E_n is identity element of the $n \times n$ square matrixes for matrix production.
  Deduction: $A = (\alpha_{ij})_{n\times n}$ and $E_n = (\beta_{ij})_{n\times n}$ are two matrixes, where $\beta_{ij} = 1$, if $i = j$, otherwise it is zero. The product of Matrixes A and E_n is Matrix $A \cdot E_n = \left(\sum_{l=1}^{n} \alpha_{il}\beta_{lj}\right)_{n\times n}$. It is Matrix $A = (\alpha_{ij})_{n\times n}$, because the definition of b_{ij} erases all other elements than α_{ij}.

Matrix operations, inverse matrix, rank of matrix VI

- Definition: Square matrix $A \in \mathcal{M}_{n\times n}$ exists inverse if exist such a Matrix $B \in \mathcal{M}_{n\times n}$, that $AB = BA = E_n$. The inverse of Matrix A is A^{-1}.
- Thesis: Matrix $A \in \mathcal{M}_{n\times n}$ exists inverse if only $\det(A) \neq 0$.
- Matrix $A \in \mathcal{M}_{n\times n}$ is regular if $\det(A) \neq 0$.
- Matrix $A \in \mathcal{M}_{n\times n}$ is singular if $\det(A) = 0$.
- Inverse matrix calculation by elemental transformations:
  - Mulitplication of a row by a $\lambda \neq 0$ scalar.
  - Adding λ times of a row to another row.
  - Changing of rows.

  If Matrix A is a regular matrix, then the $(A|E_n)$ extended matrix could be transformed for $(E_n|B)$ form, where Matrix B is the inverse of Matrix A.

  This transformation cannot be made for singular matrixes.

Matrix operations, inverse matrix, rank of matrix VII

- Calculation of inverse matrix by subdeterminant.
  - You calculate the determinant of the matrix. The inverse exists if the determinant is not zero.
  - A_{ij} is the subdeterminant for each element. The result must be transposed and divided by $\det(A)$ you get the inverse of Matrix A:

$$\left(A^{-1}\right)_{ij} = \frac{A_{ij}}{\det(A)}.$$

(The subdeterminant of Matrix A's α_{ij} element is: $A_{ij} = (-1)^{i+j} D_{ij}$, where D_{ij} is the determinant of the $(n-1) \times (n-1)$ matrix created by deleting the row and column of the element α_{ij}.

Matrix operations, inverse matrix, rank of matrix VIII

- Thesis: $A, B \in \mathcal{M}_{n\times n}$.
  1. If Matrixes A and B have invers, then AB also has inverse and $(AB)^{-1} = B^{-1}A^{-1}$.
  2. $(AB)^T = B^T A^T$
  3. If A has inverse, then A^T also has inverse, and $\left(A^T\right)^{-1} = \left(A^{-1}\right)^T$.
- For similar square matrixes the condition of matrix production is fulfilled and the product will be the same type. Therefore, there is an exponentation of matrixes:

$$A^1 = A \quad \text{és} \quad A^m = AA^{m-1}$$

where $(m \geq 2)$ és $A \in \mathcal{M}_{n\times n}$. Let us consider $A^0 = E_m$.

Matrix operations, inverse matrix, rank of matrix IX

- Thesis: Equlations of matrix exponentation:

$$\begin{aligned} A^m A^k &= A^{m+k} \\ (A^m)^k &= A^{mk}, \end{aligned}$$

  ahol $m, k \in \mathbb{N}$.
  Deduction: It is trivial based on the definition of matrixproduct.
- Definition: $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_s \in V$ are vectors. The rank of the $\{\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_s\}$ vector system is the dimension of the $\mathcal{L}(\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_s)$ subspace. Its sign is $\rho(\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_s)$.

Matrix operations, inverse matrix, rank of matrix X

- Thesis: The following transformation do not change the order of the $\{\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_s\}$ vector system:
  1. Multiplying a vectors by a $\lambda \neq 0$ scalar.
  2. Adding the vector multiplied by λ to another vector.
  3. Eliminating a vector that is a linear combination of the remaining vectors.
  4. Changing the order of vectors.
- Definition: The rank of Matrix $A \in \mathcal{M}_{m \times n}$ is the rank of its vector system.
- The rank of a matrix is determined by transforming the matrix to trapezoid form by rank invariant transformations. You can change the columns. (A matrix has trapesoid shape if $\alpha_{ij} = 0$, $i > j$, and $\alpha_{ii} \neq 0$, where $(1 \leq i \leq min\{m, n\})$.) Rows and columns containing 0 could be deleted. The rank of the trapezoid matrix is the number of its rows.
- The rank of a matrix is equal to the common rank of the maximal ranked non-disappearing subdeterminants.

The End

Thank you for your attention!