(PTIA0301) Elementary Linear Algebra

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Announcements

- The tests can be reviewed during consultin hours.
- The 3rd test will be held in December. Students can take them to pass or improve their grades.
- I should grade your tests during the first week.
- I am sorry that some of you had difficulties with vector operations and the triple product.
- The Sarrus rule applies to 2×2 and 3×3 matrices. It is faster than the expansion theorem. It is not applicable to 4×4 matrices.
- The Cramer rule makes it harder to make mistakes when solving systems of linear equations.

Linear Transformations I

- Definition: Let V_1 and V_2 be vector spaces. A function $\varphi : V_1 \to V_2$ is called a linear mapping if

$$\begin{aligned} \text{additive} : \varphi\left(\mathbf{a}+\mathbf{b}\right) &= \varphi\left(\mathbf{a}\right)+\varphi\left(\mathbf{b}\right) \\ \text{and homogeneous} : \varphi\left(\lambda\mathbf{a}\right) &= \lambda\varphi\left(\mathbf{a}\right), \end{aligned}$$

  where $\mathbf{a}, \mathbf{b} \in V_1$ and $\lambda \in \mathbb{R}$.
- Thesis: (Matrix Representation) A mapping $\varphi : \mathbb{R}^n \to \mathbb{R}^m$ is linear if and only if $\exists A \in \mathcal{M}_{m\times n}$ such that $\varphi\left(\mathbf{x}\right) = A \cdot \mathbf{x}$, where $\mathbf{x} \in \mathbb{R}^n$.
- Definition: Let V be a vector space. A mapping $\varphi : V \to V$ is called a linear transformation. The set of all linear transformations on V is denoted by $\mathcal{T}_V$.
- Definition: A mapping $f : V \to \mathbb{R}$ is called a linear form.

Linear Transformations II

- Definition: A mapping $L : V \times V \to \mathbb{R}$ is called a bilinear form if it is linear in both arguments, that is,

$$\begin{aligned} L(\mathbf{x}+\mathbf{y},\mathbf{z}) &= L(\mathbf{x},\mathbf{z}) + L(\mathbf{y},\mathbf{z}) \\ L(\lambda\mathbf{x},\mathbf{y}) &= \lambda L(\mathbf{x},\mathbf{y}) \\ L(\mathbf{x},\mathbf{y}+\mathbf{z}) &= L(\mathbf{x},\mathbf{y}) + L(\mathbf{x},\mathbf{z}) \\ L(\mathbf{x},\lambda\mathbf{y}) &= \lambda L(\mathbf{x},\mathbf{y}), \end{aligned}$$

  where $\mathbf{x},\mathbf{y},\mathbf{z} \in V$ and $\lambda \in \mathbb{R}$.
- Thesis: A mapping $L : V \times V \to \mathbb{R}$ is a bilinear form if and only if (for a given basis) there exist unique coefficients $\alpha_{ik} \in \mathbb{R}$ such that $L(x,y) = \sum_{i=1}^{n} \sum_{k=1}^{n} \alpha_{ik} x_i y_k$.

Linear Transformations III

- Consider the canonical basis of $\mathbb{R}^n$, denoted as $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$. In this case, $\alpha_{ik} = L(\mathbf{e}_i, \mathbf{e}_k)$. The matrix $A = (\alpha_{ik})_{n \times n}$ is called the matrix of the bilinear form L (with respect to the canonical basis).
- Definition: A bilinear form L is symmetric if $L(\mathbf{x}, \mathbf{y}) = L(\mathbf{y}, \mathbf{x})$ for all $\mathbf{x}, \mathbf{y} \in V$.
- Definition: Let L be a symmetric bilinear form on the vector space V. The function $Q(x) = L(x, x)$ is called a quadratic form.
- Definition: A quadratic form Q is positive definite if $\forall \mathbf{x} \neq \mathbf{0}$ we have $Q(\mathbf{x}) > 0$. Note: Q is positive semidefinite if $\forall \mathbf{x} \neq \mathbf{0}$ we have $Q(\mathbf{x}) \geq 0$ and $\exists \mathbf{y} \neq \mathbf{0}$ such that $Q(\mathbf{y}) = 0$. The concepts of negative definite and negative semidefinite are defined similarly.
- Definition: A symmetric bilinear form whose associated quadratic form is positive definite is called an inner product.
  Example: In the space $\mathbb{R}^3$, the scalar product is an inner product.

Linear Transformations IV

- A mapping $\varphi : \mathbb{R}^n \to \mathbb{R}^m$ is called an isomorphism if it is bijective.
- It can be shown that two vector spaces are isomorphic if and only if their dimensions are equal:
$$V_1 \cong V_2 \Leftrightarrow \dim V_1 = \dim V_2.$$

The End

Thank you for your attention!