(ENKEMNA0302) Applied Linear Algebra

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Schedule I

- **Classes: April 16-17, 30, and May 7, 2025.**
- Szinguláris érték, szinguláris vektor, SVD, PCA.
- ~~Mátrixok összehasonlítása, pozitív mátrixok, nemnegatív mátrixok, irreducibilis mátrixok, SMRC, NMF.~~
- Reakcióegyenletek sztöchiometrikus rendezése.
- ~~Lineáris programozási feladatok mátrixaritmetikai megoldhatósága. (MLF?)~~
- ~~Power of matrices. Applications: linear recursions, power of incidence matrixes.~~
- ~~Gram-Schmidt ortogonalization.~~ Fourier-series.
- ~~Further applications. (MLF?)~~

Discrete Fourier Transformation I

- The complex form of the Fourier series

$$\sum_{n=-\infty}^{\infty} c_n e^{nit}$$

and its partial sums

$$\sum_{n=0}^{N-1} c_n e^{nit} = c_0 + c_1 e^{it} + c_2 e^{2it} + \cdots + c_{N-1} e^{(N-1)it}$$

play an important role in the description of periodic functions and functions defined on a bounded domain. The above sum is called a discrete Fourier sum.

Discrete Fourier Transformation II

- Statement: (Substitution values of the Fourier sum). The mapping that assigns to the coefficients of the Fourier sum the substitution values of the Fourier sum at the points $0, \frac{2\pi}{N}, \frac{4\pi}{N}, \ldots, \frac{2(N-1)\pi}{N}$ dividing the interval $[0, 2\pi\ ]$ into N equal parts is linear, and its matrix is $\left[e^{\frac{2\pi i}{N}mn}\right]$, where $(0 \leq m, n < N)$.

  Proof: Wettl notes.

Discrete Fourier Transformation III

- Theorem: (Properties of Fourier matrices). Let N be a positive integer, $\epsilon = e^{2\pi i/N}$, $\omega = \overline{\epsilon} = e^{-2\pi i/N}$. The matrices $\mathbf{\Phi}_{N,\epsilon}$ and $\mathbf{\Phi}_{N,\omega}$ are Fourier matrices with the following properties:
  1. Any Fourier matrix has its k-th and $N-k$-th rows as complex conjugates of each other. For even N, the $N/2$-th row vector is $(1,-1,1,-1,\dots)$.
  2. The two Fourier matrices are conjugates of each other and also adjoint of each other, i.e., $\mathbf{\Phi}_{N,\omega} = \mathbf{\Phi}_{N,\epsilon}^H$ and $\mathbf{\Phi}_{N,\epsilon} = \overline{\mathbf{\Phi}}_{N,\omega} = \mathbf{\Phi}_{N,\omega}^H$.
  3. $\mathbf{\Phi}_{N,\epsilon}\mathbf{\Phi}_{N,\omega} = N\mathbf{E}_N$, thus both $\mathbf{\Phi}_{N,\epsilon}$ and $\mathbf{\Phi}_{N,\omega}$ are invertible:

$$\mathbf{\Phi}_{N,\epsilon}^{-1} = \frac{1}{N}\mathbf{\Phi}_{N,\omega}, \quad \mathbf{\Phi}_{N,\omega}^{-1} = \frac{1}{N}\mathbf{\Phi}_{N,\epsilon},$$

  Moreover, $\frac{1}{\sqrt{N}}\mathbf{\Phi}_{N,\epsilon}$ and $\frac{1}{\sqrt{N}}\mathbf{\Phi}_{N,\omega}$ are unitary matrixes.

  Proof: Wettl notes.

Discrete Fourier Transformation IV

- $[\mathbf{\Phi}_{N,\epsilon}]_{kn} = \epsilon^{kn}$ and $[\mathbf{\Phi}_{N,\omega}]_{kn} = \omega^{kn}$, where $(0 \leq k,\ n < N)$. These are the Fourier matrices and are complex conjugates of each other. Written explicitly:

$$\mathbf{\Phi}_{N,\epsilon} = \begin{pmatrix} 1 & 1 & \dots & 1 \\ 1 & \epsilon & \dots & \epsilon^{N-1} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & \epsilon^{N-1} & \dots & \epsilon^{(N-1)^2} \end{pmatrix}$$

$$\mathbf{\Phi}_{N,\omega} = \begin{pmatrix} 1 & 1 & \dots & 1 \\ 1 & \omega & \dots & \omega^{N-1} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & \omega^{N-1} & \dots & \omega^{(N-1)^2} \end{pmatrix}$$

Discrete Fourier Transformation V

- (Discrete Fourier Transform) The discrete Fourier transform is a linear mapping $\mathbb{C}^N \to \mathbb{C}^N$ that maps the vector of sampled values of a complex function to the vector of coefficients of its trigonometric components.
- Earlier we expressed the values of the function at given points using the coefficients of a Fourier sum. Let's reverse it! Assume we know the values of a function f at N distinct points, and we are given N linearly independent functions. We are looking for the coefficients of the linear combination of those functions which agree with f at the given points. We start from the function

$$f(t) = \frac{1}{N} \sum_{n=0}^{N-1} c_n e^{nit}$$

with sample points dividing the interval $[0, 2\pi]$ into N equal parts:

Discrete Fourier Transformation VI

$2k\pi/N$ for $k = 0, 1, \ldots, N-1$. The inverse mapping $(c_0, c_1, \ldots, c_{N-1}) \mapsto (y_0, y_1, \ldots, y_{N-1})$ is what we will call the discrete Fourier transform. Its matrix is $\mathbf{\Phi}_{N,\omega}$, which we will denote by $\mathbf{F}_N$. In this interpretation, the function f is not needed at all – we are simply associating one N-tuple of numbers with another!

- Definition: (Discrete Fourier Transform (DFT)). The mapping $F_N : \mathbb{C}^N \to \mathbb{C}^N : \mathbf{x} \mapsto \mathbf{X} = \mathbf{F}_N\mathbf{x}$ is called the discrete Fourier transform.
- The discrete Fourier transform is thus the matrix transformation associated with $\mathbf{F}_N = \mathbf{\Phi}_{N,\omega}$.

Discrete Fourier Transformation VII

- Expanding the transformation coordinate-wise:

$$X_k = \sum_{n=0}^{N-1} x_n e^{\frac{-2\pi i}{N} kn} = \sum_{n=0}^{N-1} x_n \omega^{kn}, \ \left(\omega = e^{-\frac{2\pi i}{N}}\right).$$

- The F_N transformation in matrix multiplication form:

$$F_N : \begin{pmatrix} x_0 \\ x_1 \\ \vdots \\ x_{N-1} \end{pmatrix} \mapsto \begin{pmatrix} X_0 \\ X_1 \\ \vdots \\ X_{N-1} \end{pmatrix} = \begin{pmatrix} 1 & 1 & \dots & 1 \\ 1 & \omega & \dots & \omega^{N-1} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & \omega^{N-1} & \dots & \omega^{(N-1)^2} \end{pmatrix} \begin{pmatrix} x_0 \\ x_1 \\ \vdots \\ x_{N-1} \end{pmatrix}.$$

Discrete Fourier Transformation VIII

- From now on, we denote the dimension of the input vector by a capital N, and the image vector is the capitalized version of the input, i.e., $\mathbf{x}$ maps to $\mathbf{X}$, $\mathbf{y}$ to $\mathbf{Y}$, etc., with coordinates indexed from 0 to $N-1$.
- Some matrix values:

$$\mathbf{F}_1 = [()\,, \mathbf{F}_2 = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \mathbf{F}_4 = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -i & -1 & i \\ 1 & -1 & 1 & -1 \\ 1 & i & -1 & -i \end{pmatrix},$$

Discrete Fourier Transformation IX

$$\mathbf{F}_8 = \begin{pmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & \frac{1-i}{\sqrt{2}} & -i & \frac{-1-i}{\sqrt{2}} & -1 & \frac{-1+i}{\sqrt{2}} & i & \frac{1+i}{\sqrt{2}} \\
1 & -i & -1 & i & 1 & -i & -1 & i \\
1 & \frac{-1-i}{\sqrt{2}} & i & \frac{1-i}{\sqrt{2}} & -1 & \frac{1+i}{\sqrt{2}} & -i & -1 \\
1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\
1 & \frac{-1+i}{\sqrt{2}} & -i & \frac{1+i}{\sqrt{2}} & -1 & \frac{1-i}{\sqrt{2}} & i & \frac{-1-i}{\sqrt{2}} \\
1 & i & -1 & i & 1 & i & -1 & i \\
1 & \frac{1+i}{\sqrt{2}} & i & \frac{-1+i}{\sqrt{2}} & -1 & \frac{-1-i}{\sqrt{2}} & -i & \frac{1-i}{\sqrt{2}}
\end{pmatrix}$$

Discrete Fourier Transformation X

- Theorem: (Properties of the DFT). Consider the discrete Fourier transformation F_N, and let the image of the vector $\mathbf{x} = (x_0, x_1, \ldots, x_{N-1})$ be $\mathbf{X} = (X_0, X_1, \ldots, X_{N-1})$. Then:
  1. The image of a constant vector is an impulse vector (whose coordinates are all 0 except the zeroth one), and vice versa, specifically:
  $$F_N(c, c, \ldots, c) = (Nc, 0, \ldots, 0), \; F_N(c, 0, \ldots, 0) = (c, c, \ldots, c).$$
  where $c \in \mathbb{C}$ is an arbitrary constant.
  2. If $\mathbf{x}$ is a real vector, then $X_{N-k} = \overline{X_k}$.
  3. The transformation F_N is invertible, and its inverse (IDFT) can be written in several forms:
  $$\mathbf{x} = F_N^{-1}\mathbf{X} = \frac{1}{N}\mathbf{\Phi}_{N,\epsilon}\mathbf{X}, \; x_k = \frac{1}{N}\sum_{n=0}^{N-1} X_n \epsilon^{kn} = \frac{1}{N}\sum_{n=0}^{N-1} X_n e^{\frac{2\pi i}{N}kn}.$$

Proof: Wettl notes.

Discrete Fourier Transformation XI

- (Computing the DFT) Determine the discrete Fourier transform of the vector $\mathbf{x} = (1, i, i, 2)$!
  N=4, so:

$$\mathbf{X} = F_4\mathbf{x} = \mathbf{F}_4\mathbf{x} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -i & -1 & i \\ 1 & -1 & 1 & -1 \\ 1 & i & -1 & -i \end{pmatrix} \begin{pmatrix} 1 \\ i \\ i \\ 2 \end{pmatrix} = \begin{pmatrix} 3+2i \\ 2+i \\ -1 \\ -3i \end{pmatrix}$$

- (Filtering periodic components) In technical applications, it is common for a signal described by a periodic function to be affected by high-frequency noise, which we want to "filter out" afterward. This can easily be done with a DFT-IDFT pair.

Discrete Fourier Transformation XII

- The general model of filtering consists of three steps:
  1. DFT
  2. "filtering"
  3. IDFT

  The "filtering" is a transformation mapping the vector $\mathbf{X}$ to $\hat{\mathbf{X}}$.
- Example: Filtering high-frequency components: see Wettl notes.
- (Fast Fourier Transform) To compute the discrete Fourier transform, that is, the matrix multiplication with the n-th order Fourier matrix, requires n^2 multiplications. Any algorithm that performs this transformation in $O(n \log n)$, i.e., in a number of steps proportional to $n \log n$, is called a fast Fourier transform.

Discrete Fourier Transformation XIII

- Theorem: (Fast Fourier Transform). There exists an algorithm that computes the discrete Fourier transform of an N-dimensional vector using at most $O\left(N \log_2 N\right)$ arithmetic operations.
  Proof: Wettl notes.
- Convolution of vectors:

$$(f * g)(n) = \sum_{k \in D} f(k)\, g(n-k) \, .$$

- Convolution of vectors appears in many contexts: from polynomial multiplication to transformations where a coordinate must be replaced by a fixed linear combination of its neighbors. It can be efficiently computed using the fast Fourier transform, since convolution becomes multiplication in the Fourier domain.

The End

Thank you for your attention!