(ENKEMNA0302) Applied Linear Algebra

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Eigenvalue, Eigenvector, Eigenspace I

- Definition: Let V be a vector space over $\mathbb{R}$. Let $\varphi : V \to V$ be a linear mapping. If for a nonzero vector $\mathbf{a} \in V$ and a scalar $\lambda \in \mathbb{R}$, the equation $\varphi(\mathbf{a}) = \lambda\mathbf{a}$ holds, we say that $\mathbf{a}$ is an eigenvector of φ, and λ is the eigenvalue of φ corresponding to $\mathbf{a}$.
- Definition: Let $L_\lambda = \{\mathbf{a} \in V : \varphi(\mathbf{a}) = \lambda\mathbf{a}\}$ be the set of eigenvectors corresponding to λ, along with the zero vector. The set L_λ forms a subspace, and it is called the eigenspace corresponding to λ.
- Definition: (Determination of Eigenvalues) The characteristic polynomial of a matrix $A \in \mathcal{M}_{n\times n}$ is defined as the n^{th}-degree polynomial

$$f(x) = |A - xE_n| = \begin{vmatrix} a_{11} - x & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} - x & \dots & a_{2n} \\ \vdots & \vdots & \dots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} - x \end{vmatrix}.$$

Eigenvalue, Eigenvector, Eigenspace II

- Definition: Let φ be a linear transformation acting on $\mathbb{R}^n$, and let $A \in \mathcal{M}_{n\times n}$ be the matrix of φ with respect to the canonical basis. The characteristic polynomial of φ is defined as the characteristic polynomial of the matrix A.
- Definition: A number $\lambda \in \mathbb{R}$ is called a characteristic solution of the linear transformation φ if λ is a solution of the characteristic polynomial of φ.
- Thesis: A scalar λ is an eigenvalue of φ if and only if it is a characteristic solution of φ.
- Statement: (Subspaces of Eigenvectors). If $\mathbf{A}$ is a matrix and λ is an eigenvalue of $\mathbf{A}$, then the set of eigenvectors corresponding to λ, along with the zero vector, forms a subspace, which coincides with the null space of $\mathbf{A} - \lambda\mathbf{I}$.

  Proof: A nonzero vector $\mathbf{x}$ is an eigenvector corresponding to the eigenvalue λ if and only if it satisfies the equation $\mathbf{A}\mathbf{x} = \lambda\mathbf{x}$, that is, the equation $\mathbf{A}\mathbf{x} - \lambda\mathbf{x} = 0$, or equivalently, if it is a solution to the homogeneous linear equation $(\mathbf{A} - \lambda\mathbf{I})\,\mathbf{x} = 0$. This precisely means that $\mathbf{x}$ is an element of the null space of $\mathbf{A} - \lambda\mathbf{I}$.

Eigenvalue, Eigenvector, Eigenspace III

- Definition: (Eigenspace). The subspace formed by the eigenvectors of a square matrix $\mathbf{A}$ corresponding to an eigenvalue λ, along with the zero vector, is called the eigenspace corresponding to the eigenvalue λ.
- Statement: (Eigenvalues of Triangular Matrices). The eigenvalues of triangular matrices, and thus also of diagonal matrices, are equal to the elements of their main diagonal.

  Proof: If $\mathbf{A}$ is a triangular matrix, then $\mathbf{A} - \lambda\mathbf{I}$ is also a triangular matrix, and the determinant of a triangular matrix is the product of its diagonal elements. Therefore, the characteristic equation of the triangular matrix $\mathbf{A} = [a_{ij}]$ is

$$(a_{11} - \lambda)(a_{22} - \lambda)\ldots(a_{nn} - \lambda) = 0,$$

  whose roots are a_{ii} (for $i = 1, \ldots, n$). Thus, these are the eigenvalues of $\mathbf{A}$.

Eigenvalue, Eigenvector, Eigenspace IV

- Statement: (Determinant, Trace, and Eigenvalues). If the eigenvalues of an $n \times n$ matrix $\mathbf{A}$ are $\lambda_1, \ldots, \lambda_n$, then

$$\begin{aligned} \det(\mathbf{A}) &= \lambda_1 \lambda_2 \ldots \lambda_n \\ \mathit{trace}(\mathbf{A}) &= \lambda_1 + \lambda_2 + \cdots + \lambda_n \end{aligned}$$

  These values appear in the characteristic polynomial: the determinant corresponds to the constant term, while the trace is the coefficient of $(-\lambda)^{n-1}$.
  Proof: The factorized form of the characteristic polynomial is:

$$\det(\mathbf{A} - \lambda \mathbf{I}) = (\lambda_1 - \lambda)(\lambda_2 - \lambda) \ldots (\lambda_n - \lambda)$$

  Substituting $\lambda = 0$, we obtain

$$\det(\mathbf{A}) = \lambda_1 \lambda_2 \ldots \lambda_n.$$

  The proof of the statement regarding the trace is trivial.

Eigenvalue, Eigenvector, Eigenspace V

- Theorem: (Eigenspaces of 2×2 Symmetric Matrices). Let $\mathbf{A} \in \mathbb{R}^{2\times 2}$ be a symmetric matrix. Then:
  1. Every eigenvalue of $\mathbf{A}$ is real.
  2. $\mathbf{A}$ has two identical eigenvalues if and only if it is of the form $a\mathbf{I}$, in which case every vector in the plane is an eigenvector.
  3. If $\mathbf{A}$ has two distinct eigenvalues, then its eigenspaces are orthogonal to each other.

  Proof: The general form of a real symmetric 2×2 matrix is $\mathbf{A} = \begin{pmatrix} a & b \\ b & d \end{pmatrix}$, where $a, b, d \in \mathbb{R}$. Its characteristic equation is $\lambda^2 - (a+d)\lambda + (ad - b^2) = 0$. The discriminant of this equation is $D = (a+d)^2 - 4(ad - b^2) = (a-d)^2 + 4b^2 \geq 0$. Thus, the roots, which are the eigenvalues, are real. This proves (1). The two eigenvalues are equal if and only if $D = 0$, which occurs only when $a = d$ and $b = 0$, proving (2). The proof of (3) is trivial.

Eigenvalue, Eigenvector, Eigenspace VI

- (Determining All Eigenvalues and Eigenvectors of a Matrix) The eigenvalues and eigenvectors of a matrix can be determined in two steps:
  1. Solve the characteristic equation $\det(\mathbf{A} - \lambda\mathbf{I}) = 0$; its roots are the eigenvalues.
  2. For each eigenvalue λ, determine a basis for the null space of $\mathbf{A} - \lambda\mathbf{I}$. The nonzero vectors of this subspace are the eigenvectors corresponding to λ.
- Theorem: (Matrix Invertibility and the Eigenvalue 0). A matrix $\mathbf{A}$ is invertible if and only if 0 is not an eigenvalue.
  Proof: A matrix $\mathbf{A}$ is invertible if and only if $\det(\mathbf{A}) \neq 0$. This is equivalent to $\det(\mathbf{A} - 0\mathbf{I}) \neq 0$, which means that 0 is not an eigenvalue of $\mathbf{A}$.

Eigenvalue, Eigenvector, Eigenspace VII

- Theorem: (Eigenvalues of Special Matrices). Let $\mathbf{A}$ be an $n \times n$ real matrix. Then:

  1. If $\mathbf{A}$ is symmetric, all of its eigenvalues are real.
  2. If $\mathbf{A}$ is skew-symmetric, all of its eigenvalues are imaginary.

  Proof: (1) and (2) Let $(\lambda, \mathbf{x})$ be an eigenpair of $\mathbf{A}$. Multiplying both sides of the equation $\mathbf{A}\mathbf{x} = \lambda\mathbf{x}$ from the left by the adjoint (conjugate transpose) of $\mathbf{x}$ gives:

$$\mathbf{x}^H\mathbf{A}\mathbf{x} = \mathbf{x}^H\lambda\mathbf{x} = \lambda\,|\mathbf{x}|^2\,.$$

  Taking the adjoint (conjugate transpose) of both sides, and using the fact that since $\mathbf{A}$ is real, we have $\mathbf{A}^H = \mathbf{A}^T$:

$$\mathbf{x}^H\mathbf{A}^T\mathbf{x} = \overline{\lambda}\,|\mathbf{x}|^2\,.$$

Eigenvalue, Eigenvector, Eigenspace VIII

Let $\lambda = a + ib$. If $\mathbf{A}$ is symmetric, i.e., $\mathbf{A}^T = \mathbf{A}$, then $\lambda = \overline{\lambda}$, meaning $a + ib = a - ib$. Thus, the imaginary part of λ is 0, so λ is real. If $\mathbf{A}$ is skew-symmetric, i.e., $\mathbf{A}^T = -\mathbf{A}$, then $a + ib = -a + ib$, meaning the real part of λ is 0, so λ is purely imaginary.

Le Verrier-Souriau algorithm I

The Algorithm [edit]

The objective is to calculate the coefficients c_k of the characteristic polynomial of the $n \times n$ matrix A,

$$p_A(\lambda) \equiv \det(\lambda I_n - A) = \sum_{k=0}^{n} c_k \lambda^k ,$$

where, evidently, $c_n = 1$ and $c_0 = (-1)^n \det A$.

The coefficients c_{n-i} are determined by induction on i, using an auxiliary sequence of matrices

$$\begin{aligned} M_0 &\equiv 0 & c_n &= 1 & (k &= 0) \\ M_k &\equiv AM_{k-1} + c_{n-k+1} I & c_{n-k} &= -\frac{1}{k}\mathrm{tr}(AM_k) & k &= 1, \ldots, n\,. \end{aligned}$$

Thus,

$$M_1 = I\,, \quad c_{n-1} = -\mathrm{tr}A = -c_n \mathrm{tr}A;$$
$$M_2 = A - I\mathrm{tr}A, \quad c_{n-2} = -\frac{1}{2}\Big(\mathrm{tr}A^2 - (\mathrm{tr}A)^2\Big) = -\frac{1}{2}(c_n \mathrm{tr}A^2 + c_{n-1}\mathrm{tr}A);$$
$$M_3 = A^2 - A\mathrm{tr}A - \frac{1}{2}\Big(\mathrm{tr}A^2 - (\mathrm{tr}A)^2\Big) I,$$
$$c_{n-3} = -\tfrac{1}{6}\Big((\mathrm{tr}\,A)^3 - 3\,\mathrm{tr}(A^2)(\mathrm{tr}\,A) + 2\,\mathrm{tr}(A^3)\Big) = -\frac{1}{3}(c_n \mathrm{tr}A^3 + c_{n-1}\mathrm{tr}A^2 + c_{n-2}\mathrm{tr}A);$$

etc.,[9][10] ...;

$$M_m = \sum_{k=1}^{m} c_{n-m+k} A^{k-1}\,,$$

$$c_{n-m} = -\frac{1}{m}(c_n \mathrm{tr}A^m + c_{n-1}\mathrm{tr}A^{m-1} + \ldots + c_{n-m+1}\mathrm{tr}A) = -\frac{1}{m}\sum_{k=1}^{m} c_{n-m+k}\mathrm{tr}A^k\,; \ldots$$

Observe $A^{-1} = -M_n/c_0 = (-1)^{n-1} M_n/\det A$ terminates the recursion at λ. This could be used to obtain the inverse or the determinant of A.

Derivation [edit]

The proof relies on the modes of the adjugate matrix, $B_k \equiv M_{n-k}$, the auxiliary matrices encountered. This matrix is defined by

$$(\lambda I - A)B = I\, p_A(\lambda)$$

and is thus proportional to the resolvent

$$B = (\lambda I - A)^{-1} I\, p_A(\lambda)\,.$$

It is evidently a matrix polynomial in λ of degree $n-1$. Thus,

$$B \equiv \sum_{k=0}^{n-1} \lambda^k B_k = \sum_{k=0}^{n} \lambda^k M_{n-k},$$

where one may define the harmless $M_0 \equiv 0$.

Inserting the explicit polynomial forms into the defining equation for the adjugate, above,

$$\sum_{k=0}^{n} \lambda^{k+1} M_{n-k} - \lambda^k (AM_{n-k} + c_k I) = 0\,.$$

Now, at the highest order, the first term vanishes by $M_0=0$; whereas at the bottom order (constant in λ, from the defining equation of the adjugate, above),

$$M_n A = B_0 A = c_0\,,$$

so that shifting the dummy indices of the first term yields

$$\sum_{k=1}^{n} \lambda^k \Big(M_{1+n-k} - AM_{n-k} + c_k I\Big) = 0\,,$$

which thus dictates the recursion

$$\therefore \qquad M_m = AM_{m-1} + c_{n-m+1} I\,,$$

for $m=1,\ldots,n$. Note that ascending index amounts to descending in powers of λ, but the polynomial coefficients c are yet to be determined in terms of the Ms and A.

This can be easiest achieved through the following auxiliary equation (Hou, 1998),

$$\lambda \frac{\partial p_A(\lambda)}{\partial \lambda} - np = \mathrm{tr}\, AB\,.$$

This is but the trace of the defining equation for B by dint of Jacobi's formula

Le Verrier-Souriau algorithm II

This is but the trace of the defining equation for B by dint of Jacobi's formula,

$$\frac{\partial p_A(\lambda)}{\partial \lambda} = p_A(\lambda) \sum_{m=0}^{\infty} \lambda^{-(m+1)} \operatorname{tr} A^m = p_A(\lambda) \ \operatorname{tr} \frac{I}{\lambda I - A} \equiv \operatorname{tr} B \,.$$

Inserting the polynomial mode forms in this auxiliary equation yields

$$\sum_{k=1}^{n} \lambda^k \left(k c_k - n c_k - \operatorname{tr} A M_{n-k} \right) = 0 \,,$$

so that

$$\sum_{m=1}^{n-1} \lambda^{n-m} \left(m c_{n-m} + \operatorname{tr} A M_m \right) = 0 \,,$$

and finally

$$\therefore \qquad c_{n-m} = -\frac{1}{m} \operatorname{tr} A M_m \,.$$

This completes the recursion of the previous section, unfolding in descending powers of λ.

Further note in the algorithm that, more directly,

$$M_m = A M_{m-1} - \frac{1}{m-1} (\operatorname{tr} A M_{m-1}) I \,,$$

and, in comportance with the Cayley–Hamilton theorem,

$$\operatorname{adj}(A) = (-1)^{n-1} M_n = (-1)^{n-1} (A^{n-1} + c_{n-1} A^{n-2} + \ldots + c_2 A + c_1 I) = (-1)^{n-1} \sum_{k=1}^{n} c_k A^{k-1} \,.$$

The final solution might be more conveniently expressed in terms of complete exponential Bell polynomials as

$$c_{n-k} = \frac{(-1)^{n-k}}{k!} \mathcal{B}_k \Big(\operatorname{tr} A, -1! \ \operatorname{tr} A^2, 2! \ \operatorname{tr} A^3, \ldots, (-1)^{k-1} (k-1)! \ \operatorname{tr} A^k \Big) .$$

ample [edit]

$$A = \begin{bmatrix} 3 & 1 & 5 \\ 3 & 3 & 1 \\ 4 & 6 & 4 \end{bmatrix}$$

$$M_0 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad c_3 = 1$$

$$M_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad A\,M_1 = \begin{bmatrix} 3 & 1 & 5 \\ 3 & 3 & 1 \\ 4 & 6 & 4 \end{bmatrix} \qquad c_2 = -\frac{1}{1} 10 = -10$$

$$M_2 = \begin{bmatrix} -7 & 1 & 5 \\ 3 & -7 & 1 \\ 4 & 6 & -6 \end{bmatrix} \qquad A\,M_2 = \begin{bmatrix} 2 & 26 & -14 \\ -8 & -12 & 12 \\ 6 & -14 & 2 \end{bmatrix} \qquad c_1 = -\frac{1}{2}(-8) = 4$$

$$M_3 = \begin{bmatrix} 6 & 26 & -14 \\ -8 & -8 & 12 \\ 6 & -14 & 6 \end{bmatrix} \qquad A\,M_3 = \begin{bmatrix} 40 & 0 & 0 \\ 0 & 40 & 0 \\ 0 & 0 & 40 \end{bmatrix} \qquad c_0 = -\frac{1}{3} 120 = -40$$

Furthermore, $M_4 = A\,M_3 + c_0\,I = 0$, which confirms the above calculations.

The characteristic polynomial of matrix A is thus $p_A(\lambda) = \lambda^3 - 10\lambda^2 + 4\lambda - 40$; the determinant of A is $\det(A) = (-1)^3 c_0 = 40$; the trace is 10=−$c_2$; and the inverse of A is

$$A^{-1} = -\frac{1}{c_0} M_3 = \frac{1}{40} \begin{bmatrix} 6 & 26 & -14 \\ -8 & -8 & 12 \\ 6 & -14 & 6 \end{bmatrix} = \begin{bmatrix} 0.15 & 0.65 & -0.35 \\ -0.20 & -0.20 & 0.30 \\ 0.15 & -0.35 & 0.15 \end{bmatrix}.$$

An equivalent but distinct expression [edit]

A compact determinant of an $m \times m$-matrix solution for the above Jacobi's formula may alternatively determine the coefficients c,[11][12]

$$c_{n-m} = \frac{(-1)^m}{m!} \begin{vmatrix} \operatorname{tr} A & m-1 & 0 & \cdots & 0 \\ \operatorname{tr} A^2 & \operatorname{tr} A & m-2 & \cdots & 0 \\ \vdots & \vdots & & & \vdots \\ \operatorname{tr} A^{m-1} & \operatorname{tr} A^{m-2} & \cdots & \cdots & 1 \\ \operatorname{tr} A^m & \operatorname{tr} A^{m-1} & \cdots & \cdots & \operatorname{tr} A \end{vmatrix} .$$

The End

Thank you for your attention!