(PTIA0301) Elementary Linear Algebra

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Announcements

- The group will write the second midterm test on December 6, 2024.
- A previous midterm test is available on Teams and Moodle. Topics:
  - ~~Operations and matrices~~
  - ~~Change of basis~~
  - ~~Solving matrix equations~~
  - Eigenvalue problem
  - ~~Linear mappings~~
- In practice, we are still covering matrix equations and solving systems of linear equations by matrix inversion.
- I will present the theory and solve examples in the next lectures as well.

Rappel - Linear Transformations I

- Definition: Let V_1 and V_2 be vector spaces. A function $\varphi : V_1 \to V_2$ is called a linear mapping if

$$\begin{aligned} \text{additive} : \varphi(\mathbf{a}+\mathbf{b}) &= \varphi(\mathbf{a}) + \varphi(\mathbf{b}) \\ \text{and homogeneous} : \varphi(\lambda\mathbf{a}) &= \lambda\varphi(\mathbf{a}), \end{aligned}$$

  where $\mathbf{a}, \mathbf{b} \in V_1$ and $\lambda \in \mathbb{R}$.
- Thesis: (Matrix Representation) A mapping $\varphi : \mathbb{R}^n \to \mathbb{R}^m$ is linear if and only if $\exists A \in \mathcal{M}_{m\times n}$ such that $\varphi(\mathbf{x}) = A \cdot \mathbf{x}$, where $\mathbf{x} \in \mathbb{R}^n$.
- Definition: Let V be a vector space. A mapping $\varphi : V \to V$ is called a linear transformation. The set of all linear transformations on V is denoted by $\mathcal{T}_V$.
- Definition: A mapping $f : V \to \mathbb{R}$ is called a linear form.

Rappel - Linear Transformations II

- Definition: A mapping $L : V \times V \to \mathbb{R}$ is called a bilinear form if it is linear in both arguments, that is,

$$\begin{aligned} L(\mathbf{x}+\mathbf{y},\mathbf{z}) &= L(\mathbf{x},\mathbf{z}) + L(\mathbf{y},\mathbf{z}) \\ L(\lambda\mathbf{x},\mathbf{y}) &= \lambda L(\mathbf{x},\mathbf{y}) \\ L(\mathbf{x},\mathbf{y}+\mathbf{z}) &= L(\mathbf{x},\mathbf{y}) + L(\mathbf{x},\mathbf{z}) \\ L(\mathbf{x},\lambda\mathbf{y}) &= \lambda L(\mathbf{x},\mathbf{y}), \end{aligned}$$

  where $\mathbf{x},\mathbf{y},\mathbf{z} \in V$ and $\lambda \in \mathbb{R}$.
- Thesis: A mapping $L : V \times V \to \mathbb{R}$ is a bilinear form if and only if (for a given basis) there exist unique coefficients $\alpha_{ik} \in \mathbb{R}$ such that $L(x,y) = \sum_{i=1}^{n} \sum_{k=1}^{n} \alpha_{ik} x_i y_k$.

Rappel - Linear Transformations III

- Consider the canonical basis of $\mathbb{R}^n$, denoted as $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$. In this case, $\alpha_{ik} = L(\mathbf{e}_i, \mathbf{e}_k)$. The matrix $A = (\alpha_{ik})_{n \times n}$ is called the matrix of the bilinear form L (with respect to the canonical basis).
- Definition: A bilinear form L is symmetric if $L(\mathbf{x}, \mathbf{y}) = L(\mathbf{y}, \mathbf{x})$ for all $\mathbf{x}, \mathbf{y} \in V$.
- Definition: Let L be a symmetric bilinear form on the vector space V. The function $Q(x) = L(x, x)$ is called a quadratic form.
- Definition: A quadratic form Q is positive definite if $\forall \mathbf{x} \neq \mathbf{0}$ we have $Q(\mathbf{x}) > 0$.
  Note: Q is positive semidefinite if $\forall \mathbf{x} \neq \mathbf{0}$ we have $Q(\mathbf{x}) \geq 0$ and $\exists \mathbf{y} \neq \mathbf{0}$ such that $Q(\mathbf{y}) = 0$. The concepts of negative definite and negative semidefinite are defined similarly.
- Definition: A symmetric bilinear form whose associated quadratic form is positive definite is called an inner product.
  Example: In the space $\mathbb{R}^3$, the scalar product is an inner product.

Rappel - Linear Transformations IV

- A mapping $\varphi : \mathbb{R}^n \to \mathbb{R}^m$ is called an isomorphism if it is bijective.
- It can be shown that two vector spaces are isomorphic if and only if their dimensions are equal:

$$V_1 \cong V_2 \Leftrightarrow \dim V_1 = \dim V_2.$$

Gram-Schmidt Orthogonalization I

- Definition: A vector space E is called Euclidean if it is equipped with an inner product $\cdot$. (In this case, a norm naturally exists: $\|\mathbf{x}\| = \mathbf{x} \cdot \mathbf{x}$.)
- Definition: A system of vectors is called orthogonal if the vectors are pairwise perpendicular, i.e., $\mathbf{v}_i \cdot \mathbf{v}_j = 0$, where $(i \neq j)$.
- Definition: A system of vectors is called orthonormal if it consists of pairwise perpendicular unit vectors, i.e., $\mathbf{v}_i \cdot \mathbf{v}_j = 0$, where $(i \neq j)$, and $\|\mathbf{v}_i\| = 1$ for $(i = 1, 2, \ldots, n)$.
- Thesis: Let $\{\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_n\}$ be a basis of the Euclidean space E. Then, up to multiplication by ± 1, there uniquely exists an orthonormal basis $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$ in E such that

$$\mathcal{L}\left(\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_k\right) = \mathcal{L}\left(\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_k\right),$$

  where $k = 1, 2, \ldots, n$.

Gram-Schmidt Orthogonalization II

- Orthogonalization procedure:
  1. Set $\mathbf{e}_1^{'} = \mathbf{b}_1$ and $\mathbf{e}_1 = \frac{\mathbf{e}_1^{'}}{\|\mathbf{e}_1^{'}\|}$.
  2. Compute the vectors $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_k$.
  3. Finally,

$$\mathbf{e}_{k+1}^{'} = \mathbf{b}_{k+1} - (\mathbf{b}_{k+1} \cdot \mathbf{e}_1)\,\mathbf{e}_1 - (\mathbf{b}_{k+1} \cdot \mathbf{e}_2)\,\mathbf{e}_2 - \cdots - (\mathbf{b}_{k+1} \cdot \mathbf{e}_k)\,\mathbf{e}_k,$$

and

$$\mathbf{e}_{k+1} = \frac{\mathbf{e}_{k+1}^{'}}{\|\mathbf{e}_{k+1}^{'}\|}.$$

Eigenvalue, Eigenvector I

- Definition: Let V be a vector space over $\mathbb{R}$. Let $\varphi : V \to V$ be a linear mapping. If for a nonzero vector $\mathbf{a} \in V$ and a scalar $\lambda \in \mathbb{R}$, the equation $\varphi(\mathbf{a}) = \lambda\mathbf{a}$ holds, we say that $\mathbf{a}$ is an eigenvector of φ, and λ is the eigenvalue of φ corresponding to $\mathbf{a}$.
- Definition: Let $L_\lambda = \{\mathbf{a} \in V : \varphi(\mathbf{a}) = \lambda\mathbf{a}\}$ be the set of eigenvectors corresponding to λ, along with the zero vector. The set L_λ forms a subspace, and it is called the eigenspace corresponding to λ.
- Definition: (Determination of Eigenvalues) The characteristic polynomial of a matrix $A \in \mathcal{M}_{n\times n}$ is defined as the n^{th}-degree polynomial

$$f(x) = |A - xE_n| = \begin{vmatrix} a_{11} - x & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} - x & \dots & a_{2n} \\ \vdots & \vdots & \dots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} - x \end{vmatrix}.$$

Eigenvalue, Eigenvector II

- Definition: Let φ be a linear transformation acting on $\mathbb{R}^n$, and let $A \in \mathcal{M}_{n \times n}$ be the matrix of φ with respect to the canonical basis. The characteristic polynomial of φ is defined as the characteristic polynomial of the matrix A.
- Definition: A number $\lambda \in \mathbb{R}$ is called a characteristic solution of the linear transformation φ if λ is a solution of the characteristic polynomial of φ.
- Thesis: A scalar λ is an eigenvalue of φ if and only if it is a characteristic solution of φ.

The End

Thank you for your attention!