(ENKEMNA0302) Applied Linear Algebra

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Determinant of Square Matrixes I

- Leibnitz definition: If Matrix A is type $n \times n$, where $n > 1$ and $n \in \mathbb{N}$ (Square Matrix), then the determinant of Matrix A is the following number:

$$det\,(A) = \sum_{\{i_1, i_2, \ldots, i_n\} \in P_n} (-1)^{I(i_1, i_2, \ldots, i_n)} \alpha_{1i_1} \cdot \alpha_{2i_2} \cdot \cdots \cdot \alpha_{ni_n},$$

where the summary is for al the permutations of $1, 2, \ldots, n$ numbers, and $I\,(i_1, i_2, \ldots, i_n)$ means the number of inversions in the permutation $(i_1, i_2, \ldots, i_n)$:

$$\det\,(A)\,, \quad \begin{vmatrix} \alpha_{11} & \alpha_{12} & \cdots & \alpha_{1n} \\ \alpha_{21} & \alpha_{22} & \cdots & \alpha_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_{m1} & \alpha_{m2} & \cdots & \alpha_{mn} \end{vmatrix}, \quad |A|\,.$$

Determinant of Square Matrixes II

- The determinant is a number ordered to a square matrix. Its features are determined by the matrix. It is an extention of triple product to hiher dimensions. It is the pszeudoskalár component of multi-vectors that gives the magnitude and dirrection of elemental volumens.
- Axiomatical definition: $\mathbf{A} \in \mathbb{R}^{n \times n}$ is a square matrix and $\det : \mathbb{R}^{n \times n} \leftarrow \mathbb{R}$ function. The function $\det(\mathbf{A})$ is called the determinant of $\mathbf{A}^{n \times n}$ matrix if
  1. Homogeneous: $\det(\ldots \lambda_i \mathbf{a}_i \ldots) = \lambda_i \det(\ldots \mathbf{a}_i \ldots)$;
  2. Additive: $\det(\ldots \mathbf{a}_i + \mathbf{b}_i \ldots) = \det(\ldots \mathbf{a}_i \ldots) + \det(\ldots \mathbf{b}_i \ldots)$;
  3. Alternating: $\det(\ldots \mathbf{a}_i \ldots \mathbf{a}_j \ldots) = -\det(\ldots \mathbf{a}_j \ldots \mathbf{a}_i \ldots)$;
  4. The deteminant of the identity matrix is 1: $\det(\mathbf{E}_n) = 1$,

  where $\lambda_i \in \mathbb{R}$, and $\mathbf{a}_i, \mathbf{b}_i \in \mathbb{R}^n$ are the columb vectors of the $\mathbf{A}^{n \times n}$.
- This relation could be considered as a n-variables function above the columbs of the mátrix $\mathbb{R}^n \to \mathbb{R}$.

Determinant of Square Matrixes III

- These axioms determine this relation definitly. Another $\mathbb{R}^{n\times n} \to \mathbb{R}$ function with these four features are equivalent with det.
- Or, you can order unique value to any matrix with these rules.
- If $\mathbf{A} \in \mathbb{R}^{n\times n}$ then the determinant is n^{th} order determinant.
- The determinant is a functional. That is a relation that order scalars to functions.

Diagonal matrixes I

- Diagonal matrixes: it is simple to do operations with them.
  Here $\mathbf{A} = diag\,(1,2,3)$ és $\mathbf{B} = diag\,(5,4,3)$. Then:

$$\mathbf{AB} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix} \begin{pmatrix} 5 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 3 \end{pmatrix} = \begin{pmatrix} 5 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & 9 \end{pmatrix},$$

$$\mathbf{A}^2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}^2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 9 \end{pmatrix}, \mathbf{A}^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{3} \end{pmatrix}$$

$$\mathbf{A}^k = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}^k = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2^k & 0 \\ 0 & 0 & 3^k \end{pmatrix}, \text{where } k \in \mathbb{Z}.$$

Diagonal matrixes II

<u>Thesis:</u> (Operations with diagonal matrixes) Here $\mathbf{A} = diag\,(a_1, a_2, \ldots, a_n)$, $\mathbf{B} = diag\,(b_1, b_2, \ldots, b_n)$ and $k \in \mathbb{Z}$. Then

1. $\mathbf{AB} = diag\,(a_1 b_1, a_2 b_2, \ldots, a_n b_n)$,
2. $\mathbf{A}^k = diag\,\left(a_1^k, a_2^k, \ldots, a_n^k\right)$, specialy
3. $\mathbf{A}^{-1} = diag\,\left(a_1^{-1}, a_2^{-1}, \ldots, a_n^{-1}\right)$.

The (3) and if $k < 0$ the (2) operations can be done if $a_i \neq 0$, where $i = 1, 2, \ldots, n$.

Permutation Matrices I

- Permutation matrices and snakes are obtained by permuting the rows of diagonal matrices.
- Every permutation can be achieved by swapping element pairs. If we permute the rows of a matrix, we can do so by multiplying with elementary matrices that perform row swaps. The matrix obtained as the product of these elementary matrices can be derived from the identity matrix by executing the given row swaps.
- For example, performing the permutation $\{2,4,3,1\}$ on the identity matrix $\mathbf{E}_n$ results in the following permutation matrix $\mathbf{P}$:

$$\mathbf{E}_n = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \overset{S_1 \leftrightarrow S_2}{\Longrightarrow} \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \overset{S_2 \leftrightarrow S_4}{\Longrightarrow} \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} = \mathbf{P}$$

Permutation Matrices II

- Multiplying any $4 \times m$ matrix from the left by $\mathbf{P}$ will rearrange its rows according to the given permutation. For example,

$$\mathbf{PA} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \\ a_{41} & a_{42} \end{pmatrix} = \begin{pmatrix} a_{21} & a_{22} \\ a_{41} & a_{42} \\ a_{31} & a_{32} \\ a_{11} & a_{12} \end{pmatrix}$$

- Defition: (Permutation matrix, snake) The matrixes created by permutation from the diagonal matrixes called snakes (or transversals). Snakes created from the identity matrix are called permutation matrixes.

Permutation Matrices III

- For example the following marixes are snakes, however, the last two matrixes are also permutation matrixes:

$$\begin{pmatrix} 0 & 5 & 0 \\ 0 & 0 & 9 \\ 3 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & \alpha & 0 & 0 \\ \gamma & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \beta \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

- The permutation matrixes are such a square a matrixes that has only 1 element and the other elemenst are 0.
- The snake is such a square matrix that has only one non-zero element in its each row and column.
- All snakes could be get from a diagonal matrix by switching columns.

Permutation Matrices IV

- You can get a snake from a diagonal matrix if you also permutate the columns.
- If $\mathbf{P}$ is a permutation matrix, then you can get $\mathbf{PA}$ from $\mathbf{A}$ by the same permutation of the rows that permutation leads from $\mathbf{E}$ to $\mathbf{P}$.
- Thesis: (Operations with Permutation Matrices). The product of any two permutation matrices of the same size, as well as any integer power of a permutation matrix, is also a permutation matrix. The inverse of a permutation matrix is equal to its transpose, i.e., if $\mathbf{P}$ is a permutation matrix, then $\mathbf{P}^{-1} = \mathbf{P}^T$.
  Deduction: Let $\mathbf{P}$ and $\mathbf{Q}$ be two permutation matrices. The row vectors of their product take the form $\mathbf{P}_{i*}\mathbf{Q}$, where $\mathbf{P}_{i*}$ corresponds to a standard unit vector, e.g., $\mathbf{P}_{i*} = \mathbf{e}_k$. In this case, only the element in the column that matches $\mathbf{e}_k$ is 1, and there is exactly one such column. Thus, in each row of the product matrix, there is exactly one entry equal to 1, while all others are 0.

Permutation Matrices V

A similar argument holds for the columns. The statement regarding multiplication implies the statement for positive integer powers. The case for negative integer exponents follows from considering the inverse.

Now, consider the product $\mathbf{PP}^T$. The element $\left(\mathbf{PP}^T\right)_{ii}$ is given by the dot product of the vector $\mathbf{P}_{i*}$ with $\left(\mathbf{P}^T\right)_{*i} = \mathbf{P}_{i*}$, which equals 1. Meanwhile,

$$\left(\mathbf{PP}^T\right)_{ij} = (\mathbf{P})_{i*}\left(\mathbf{P}^T\right)_{*j} = (\mathbf{P})_{i*} \cdot (\mathbf{P})_{j*},$$

i.e., the (i, j)-th element of the product is the dot product of the i-th and j-th row vectors of $\mathbf{P}$, which is 0 because the 1s appear in different positions in different rows.

Permutation Matrices VI

- Example:

$$\mathbf{P}\mathbf{P}^T = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Triangular Matrices

- Definition: (Triangular Matrix). A matrix in which all elements below the main diagonal are zero is called an upper triangular matrix, while a matrix in which all elements above the main diagonal are zero is called a lower triangular matrix. If all the elements on the main diagonal of a triangular matrix are 1, it is called a unit triangular matrix.
- Thesis: (Operations on Triangular Matrices). The sum, product, and inverse of an invertible upper triangular matrix are also upper triangular matrices. An analogous theorem holds for lower triangular matrices as well. A triangular matrix is invertible if and only if none of its diagonal elements are zero.
  Deduction: Trivial.

Symmetric and Skew-Symmetric Matrices I

- Definition: (Symmetric and Skew-Symmetric Matrices). A square matrix $\mathbf{A}$ is called symmetric if $\mathbf{A}^T = \mathbf{A}$, and it is called skew-symmetric if $\mathbf{A}^T = -\mathbf{A}$.
- Examples of symmetric and skew-symmetric matrices:

$$\mathbf{A} = \begin{pmatrix} 5 & 6 & 1 \\ 6 & 2 & 0 \\ 1 & 0 & 3 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 0 & 1 & -2 \\ -1 & 0 & 3 \\ 2 & -3 & 0 \end{pmatrix}, \mathbf{C} = \begin{pmatrix} 1 & 9 & 9 \\ -9 & 2 & 9 \\ -9 & -9 & 3 \end{pmatrix}$$

  $\mathbf{A}$ is symmetric, $\mathbf{B}$ is skew-symmetric, and $\mathbf{C}$ is neither.
- If $\mathbf{A}$ is skew-symmetric, then each element satisfies $a_{ij} = -a_{ji}$, meaning that for $i = j$, we have $a_{ii} = -a_{ii}$. This is only possible if $a_{ii} = 0$, meaning that the main diagonal of a skew-symmetric matrix consists entirely of zeros.

Symmetric and Skew-Symmetric Matrices II

- Thesis: (Operations with (skew-)symmetric matrices). The sum, scalar multiple, and inverse of symmetric matrices are also symmetric. The sum, scalar multiple, and inverse of skew-symmetric matrices are also skew-symmetric.
  Deduction: Trivial.
- Thesis: (Decomposition into the sum of a symmetric and a skew-symmetric matrix). Every square matrix can be expressed as the sum of a symmetric and a skew-symmetric matrix. Specifically, for every square matrix $\mathbf{A}$:

$$\mathbf{A} = \frac{1}{2}\left(\mathbf{A} + \mathbf{A}^T\right) + \frac{1}{2}\left(\mathbf{A} - \mathbf{A}^T\right),$$

  where the first term in the sum is symmetric, and the second term is skew-symmetric.

Symmetric and Skew-Symmetric Matrices III

<u>Deduction:</u> Since a constant multiple of a symmetric matrix is also symmetric, it suffices to show that the matrix $\mathbf{A} + \mathbf{A}^T$ is symmetric:

$$\left(\mathbf{A} + \mathbf{A}^T\right)^T = \mathbf{A}^T + \left(\mathbf{A}^T\right)^T = \mathbf{A}^T + \mathbf{A} = \mathbf{A} + \mathbf{A}^T.$$

Similarly, the matrix $\mathbf{A} - \mathbf{A}^T$ is skew-symmetric:

$$\left(\mathbf{A} - \mathbf{A}^T\right)^T = \mathbf{A}^T - \left(\mathbf{A}^T\right)^T = \mathbf{A}^T - \mathbf{A} = -\left(\mathbf{A} - \mathbf{A}^T\right).$$

The sum of these two matrices gives $\mathbf{A}$:

$$\frac{1}{2}\left(\mathbf{A} + \mathbf{A}^T\right) + \frac{1}{2}\left(\mathbf{A} - \mathbf{A}^T\right) = \frac{1}{2}\mathbf{A} + \frac{1}{2}\mathbf{A}^T + \frac{1}{2}\mathbf{A} - \frac{1}{2}\mathbf{A}^T = \mathbf{A}.$$

Symmetric and Skew-Symmetric Matrices IV

- Thesis: ($\mathbf{A}^T\mathbf{A}$ and $\mathbf{A}\mathbf{A}^T$ are symmetric). The matrices $\mathbf{A}^T\mathbf{A}$ and $\mathbf{A}\mathbf{A}^T$ are symmetric for any matrix $\mathbf{A}$.
  Deduction: $\left(\mathbf{A}\mathbf{A}^T\right)^T = \left(\mathbf{A}^T\right)^T\mathbf{A}^T = \mathbf{A}\mathbf{A}^T$ and $\left(\mathbf{A}^T\mathbf{A}\right)^T = \mathbf{A}^T\left(\mathbf{A}^T\right)^T = \mathbf{A}^T\mathbf{A}$.

Matrix Inversion I

- Definition: The n^{th} order identity matrix is:

$$E_n = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}$$

- Thesis: For all $A \in \mathcal{M}_{n \times n}$: $A \cdot E_n = E_n \cdot A = A$, ormatrix E_n is identity element of the $n \times n$ square matrixes for matrix production.
  Deduction: $A = (\alpha_{ij})_{n \times n}$ and $E_n = (\beta_{ij})_{n \times n}$ are two matrixes, where $\beta_{ij} = 1$, if $i = j$, otherwise it is zero. The product of Matrixes A and E_n is Matrix $A \cdot E_n = (\sum_{l=1}^{n} \alpha_{il}\beta_{lj})_{n \times n}$. It is Matrix $A = (\alpha_{ij})_{n \times n}$, because the definition of b_{ij} erases all other elements than α_{ij}.

Matrix Inversion II

- Definition: Square matrix $A \in \mathcal{M}_{n \times n}$ exists inverse if exist such a Matrix $B \in \mathcal{M}_{n \times n}$, that $AB = BA = E_n$. The inverse of Matrix A is A^{-1}.
- Thesis: Matrix $A \in \mathcal{M}_{n \times n}$ exists inverse if only $\det(A) \neq 0$.
- Matrix $A \in \mathcal{M}_{n \times n}$ is regular if $\det(A) \neq 0$.
- Matrix $A \in \mathcal{M}_{n \times n}$ is singular if $\det(A) = 0$.

Matrix Inversion III

- Inverse matrix calculation by elemental transformations:
  - Mulitplication of a row by a $\lambda \neq 0$ scalar.
  - Adding λ times of a row to another row.
  - Changing of rows.

  If Matrix A is a regular matrix, then the $(A|E_n)$ extended matrix could be transformed for $(E_n|B)$ form, where Matrix B is the inverse of Matrix A.

  This transformation cannot be made for singular matrixes.

- Examples for matrix inversion by elemental transformations.

Matrix Inversion IV

- Calculation of inverse matrix by subdeterminant.
  - You calculate the determinant of the matrix. The inverse exists if the determinant is not zero.
  - A_{ij} is the subdeterminant for each element. The result must be transposed and divided by $\det(A)$ you get the inverse of Matrix A:

$$\left(A^{-1}\right)_{ij} = \frac{A_{ij}}{\det(A)}.$$

(The subdeterminant of Matrix A's α_{ij} element is: $A_{ij} = (-1)^{i+j} D_{ij}$, where D_{ij} is the determinant of the $(n-1) \times (n-1)$ matrix created by deleting the row and column of the element α_{ij}.

- Examples for matrix inversion by subdeterminants.

Matrix Inversion V

- Thesis: $A, B \in \mathcal{M}_{n \times n}$.
  1. If Matrixes A and B have invers, then AB also has inverse and $(AB)^{-1} = B^{-1}A^{-1}$.
  2. $(AB)^T = B^T A^T$
  3. If A has inverse, then A^T also has inverse, and $\left(A^T\right)^{-1} = \left(A^{-1}\right)^T$.

- Examples for these statements.

Sherman-Morrison-Woodbury Theorem I

- Sherman–Morrison Formula: Suppose that the matrix $\mathbf{A} \in \mathbb{R}^{n\times n}$ is invertible, and $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$ are two vectors such that $1 + \mathbf{v}^T\mathbf{A}^{-1}\mathbf{u} \neq 0$. Then $\mathbf{A} + \mathbf{u}\mathbf{v}^T$ is invertible, and
$$\left(\mathbf{A} + \mathbf{u}\mathbf{v}^T\right)^{-1} = \mathbf{A}^{-1} - \frac{\mathbf{A}^{-1}\mathbf{u}\mathbf{v}^T\mathbf{A}^{-1}}{1 + \mathbf{v}^T\mathbf{A}^{-1}\mathbf{u}}.$$
- Thesis: (Sherman–Morrison–Woodbury Formula) The inverse of a rank-k correction of a matrix can be computed using the inverse of the original matrix with a rank-k correction:
$$(\mathbf{A} + \mathbf{UCV})^{-1} = \mathbf{A}^{-1} - \mathbf{A}^{-1}\mathbf{U}\left(\mathbf{C}^{-1} + \mathbf{V}\mathbf{A}^{-1}\mathbf{U}\right)^{-1}\mathbf{V}\mathbf{A}^{-1},$$
where $\mathbf{A}$, $\mathbf{U}$, $\mathbf{C}$, and $\mathbf{V}$ are matrices of dimensions $n \times n$, $n \times k$, $k \times k$, and $k \times n$, respectively.
Deduction: N/A

Sherman-Morrison-Woodbury Theorem II

- The above formula provides an efficient alternative for computing matrix inverses, i.e., solving linear systems. However, its numerical stability is not well understood.

The End

Thank you for your attention!