(PTIA0301) Elementary Linear Algebra

Dr. Gabor FACSKO, PhD
Senior Research Fellow
facskog@gamma.ttk.pte.hu

University of Pecs, Faculty of Sciences, Institute of Mathematics and Informatics, 7624 Pecs, Ifjusag utja 6.
Wigner Research Centre for Physics, Department of Space Physics and Space Technology, 1121 Budapest, Konkoly-Thege Miklos ut 29-33.
https://facsko.ttk.pte.hu

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Common Business

- Sorry, I have flu and do not want to spread it. Therefore, I give this lecture online.
- On September 27, 2024, during the Researcher's Night I visited the UP Faculty of Engineering and Information Technology.
  - You can take courses there and learn new topics. The Faculty has its beach, by the way.
  - I would like to establish a space weather forecast centre at UP Faculty of Science with the support of the European Space Agency (ESA)
  - I also want to build a student picosatellite (PecsSat, a cube with 10 cm edges) and later a bit larger

These topics will prove many BSc/MSc/PhD and so-called Scientific Student theses (TDK, `https://www.ttk.pte.hu/hallgatok/tdk/`)

Gauss Elimination Method for Calculating Determinants I

- Definition: Matrix $A = (\alpha_{ij})_{n \times n}$ is upper triangulat matrix if $\alpha_{ij} = 0$ for all $i > j$. (The elements under the main diagonal are zero.)
- Thesis: The determinant of an upper triangular matrix is the multiplication of the elements of the main diagonal of the matrix.
  Deduction: For matrixes 2×2 and 3×3 are trivial based on the Sarrus rules. Using the Laplace expansion, all multiplications will be zero except the main diagonal.

Gauss Elimination Method for Calculating Determinants II

- The purpose of the Gaussian elimination is to convert the matrix to an upper triangular matrix, whose determinants are the same as the determinant of the original matrix.
  1. If it is necessary, set $\alpha_{11} \neq 0$ by changing rows. (The sign for the determinant will be changed if you change rows.)
  2. Adding the first row multiplied by a suitable constant you get $\alpha_{21}, \alpha_{32}, \ldots, \alpha_{n1} = 0$.
  3. If it is necessary set $\alpha_{22} \neq 0$ by changing rows.
  4. Adding the 2nd times constant to the rows $3, 4, \ldots, n$ you get $\alpha_{32}, \alpha_{42}, \ldots, \alpha_{n2} = 0$.

  Proceed until all elements are zero under the main diagonal.

Gauss Elimination Method for Calculating Determinants III

- Look at this determinant and calculate its value using Laplace Expansion...:

$$\begin{vmatrix} 1 & 3 & 2 & -1 \\ -2 & -3 & 0 & 2 \\ -1 & 3 & 6 & 1 \\ -1 & 2 & 0 & 0 \end{vmatrix} = +1 \times \begin{vmatrix} 3 & 2 & -1 \\ -3 & 0 & 2 \\ 3 & 6 & 1 \end{vmatrix} + 2 \times \begin{vmatrix} 1 & 2 & -1 \\ -2 & 0 & 2 \\ -1 & 6 & 1 \end{vmatrix} =$$

$$+1 \times (0 + 12 + 18 - 0 + 6 - 36) + 2 \times (0 - 4 + 12 - 0 + 4 - 12) = 0 + 2 \times 0 = 0$$

Gauss Elimination Method for Calculating Determinants IV

- ... and Gauss elimination:

$$\begin{vmatrix} 1 & 3 & 2 & -1 \\ -2 & -3 & 0 & 2 \\ -1 & 3 & 6 & 1 \\ -1 & 2 & 0 & 0 \end{vmatrix} \to \begin{vmatrix} 1 & 3 & 2 & -1 \\ 0 & 3 & 4 & 0 \\ 0 & 6 & 8 & 0 \\ 0 & 5 & 2 & -1 \end{vmatrix} \to \begin{vmatrix} 1 & 3 & 2 & -1 \\ 0 & 3 & 4 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 2 - \frac{5}{3} \times 4 & -1 - \frac{5}{3} \times 0 \end{vmatrix} \to$$

$$\begin{vmatrix} 1 & 3 & 2 & -1 \\ 0 & 3 & 4 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{8}{3} & -1 \end{vmatrix} = - \begin{vmatrix} 1 & 3 & 2 & -1 \\ 0 & 3 & 4 & 0 \\ 0 & 0 & -\frac{8}{3} & -1 \\ 0 & 0 & 0 & 0 \end{vmatrix} = 0$$

Linear Combination, System of Linear Equations I

- Definition: $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n \in V^3$ are vectors and $\lambda_1, \lambda_2, \ldots, \lambda_n \in \mathbb{R}$ are scalars. The linear combination of vectors $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n$ with coefficients $\lambda_1, \lambda_2, \ldots, \lambda_n$ are:

$$\lambda_1 \mathbf{a}_1 + \lambda_2 \mathbf{a}_2 + \cdots + \lambda_n \mathbf{a}_n.$$

- Definition: The linear combinations of equations are their sum multiplied by a real coefficient.
- Definition: $\alpha_{ij} \in \mathbb{R}$ and $\beta_i \in \mathbb{R}$, where $(1 \leq i \leq m, 1 \leq j \leq n)$ és $m, n \in \mathbb{N}^+$. The following system of equations is a system of linear equations:

$$\left.\begin{array}{l} \alpha_{11}x_1 + \alpha_{12}x_2 + \cdots + \alpha_{1n}x_n = \beta_1 \\ \alpha_{21}x_1 + \alpha_{22}x_2 + \cdots + \alpha_{2n}x_n = \beta_2 \\ \vdots \\ \alpha_{m1}x_1 + \alpha_{m2}x_2 + \cdots + \alpha_{mn}x_n = \beta_m \end{array}\right\}$$

Linear Combination, System of Linear Equations II

- Definition: The coefficient matrix of a system of linear equations is the following:

$$\begin{pmatrix} \alpha_{11} & \alpha_{12} & \cdots & \alpha_{1n} \\ \alpha_{21} & \alpha_{22} & \cdots & \alpha_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_{m1} & \alpha_{m2} & \cdots & \alpha_{mn} \end{pmatrix}$$

Gauss Elimination Method for Solving System of Linear Equations

- Definition: Two systems of linear equations are equivalent if their sets of solutions are the same.
- Thesis: The following transformations of a system of linear equations result in an equivalent system of linear equations:
  1. Multiplying an equation by $\lambda \neq 0$, where $\lambda \in \mathbb{R}$.
  2. Adding the λ times of an equation to another equation, where $\lambda \in \mathbb{R}$.
  3. Erasing an equation that is the linear combination of the remaining equations.
  4. Changing of the order of equations.
  5. Changing of the order of x_is with their coefficients (λ_i).

  Solving the system of linear equations using Gauss-elimination means that you transform the equation for trapezoid form ($\alpha_{ij} = 0$ for all $i > j$).

Cramer's rule I

- If the determinant of the coefficient matrix of the system of linear equations consists of n equations and has n unknowns is not zero $(det\,(A) \neq 0)$, then the system of linear equations could be solved and its only and unique solution is:

$$x_k = \frac{\Delta_k}{|A|}, \left(k = 1, 2, \ldots, n \in \mathbb{N}^+\right)$$

where

$$\Delta_k = \begin{pmatrix} \alpha_{11} & \alpha_{12} & \cdots & \beta_1 & \ldots & \alpha_{1n} & \\ \alpha_{21} & \alpha_{22} & \cdots & \beta_2 & \ldots & \alpha_{2n} & \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \alpha_{m1} & \alpha_{m2} & \cdots & \beta_n & \ldots & \alpha_{mn} & \end{pmatrix},$$

the free members are in the k^{th} column.

Cramer's rule II

- If $det\,(A) = 0$, however $\exists k \in \{1, 2, \ldots, n\}$, where $n \in \mathbb{N}$ and $\Delta_k \neq 0$, then the system of equations has no solution. However, if $det\,(A) = \Delta_k = 0\,(k = 1, 2, \ldots., n)$ 0 or infinity number of solutions might exist.

Linear Independence

- The Vectors $\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_n \in V^3$, where $(\lambda_i \in \mathbb{R}, i \in \{1, 2, \dots, n\}, n \in \mathbb{N}^+)$, and

$$\lambda_1 \mathbf{a}_1 + \lambda_2 \mathbf{a}_2 + \cdots + \lambda_n \mathbf{a}_n = \mathbf{0},$$

  are independent lineary if the equation could be satisfied if $\lambda_1 = \lambda_2 = \cdots = \lambda_n = 0$. Otherwise, if there are non-zero $\lambda_1, \lambda_2, \dots, \lambda_n$, that fullfil $\lambda_1 \mathbf{a}_1 + \lambda_2 \mathbf{a}_2 + \cdots + \lambda_n \mathbf{a}_n = \mathbf{0}$, then we say that the Vectors $\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_n$ are dependent lineary. In this latter case, one of the vectors can be written as the linear combination of the other vectors.

The End

Thank you for your attention!