(ENKEMNA0302) Applied Linear Algebra

Dr. Gabor FACSKO, PhD
Senior Research Fellow
facskog@gamma.ttk.pte.hu

University of Pecs, Faculty of Sciences, Institute of Mathematics and Informatics, 7624 Pecs, Ifjusag utja 6.
Wigner Research Centre for Physics, Department of Space Physics and Space Technology, 1121 Budapest, Konkoly-Thege Miklos ut 29-33.
https://facsko.ttk.pte.hu

February 5, 2025

Course Objectives

- Terms and basic methods of Linear Algebra methodology
- Owning linear algebra knowledge to learn appropriate disciplines
- Using elementary methods of Linear Algebra to fix simple problems
- Approaching fixing problems with mathematical methods
- Seeking more knowledge in Mathematics
- Recognition independently when Linear Algebra could be used to solve problems

  ⋮

- I will follow the course thematics, however, I will also show you that it is useful
- You will write tests and take exams from the thematics of the course
- Let me know if you need something special for your studies

- All slides, thematics, video records, exercises, and tests will be uploaded to Teams
- Can you see the Teams group of the lecture? No Moodle this time

Requirements

- You will write tests based on the exercises of the practical courses. You can use everything during the test
- The minimum requirement is 41 % of both tests.
- Failed tests must be corrected
- You must take an ~~oral~~ written exam. You cannot use anything
- Grades: Insufficient/Fail (1): 0-40 %, Sufficient/Pass (2): 41-55 %, Average (3): 56-70 %, Good (4): 71-85 %, Excellent (5): 86-100 %.
- Mid-term test 1: March 13, mid-term test 2: May 8, retake tests: May 15, 2025.

Bibliography

Bernard Kolman and David Hill: Elementary Linear Algebra with Applications, 9th ed., Person, 2007

Philip N. Klein: Coding the Matrix: Linear Algebra through Applications to Computer Science, Newtonian Press 2013

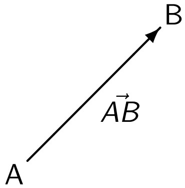
K. F. Riley, M. P. Hobson, S. J. Bence: Mathematical Methods for Physics and Engineering: A Comprehensive Guide, Cambridge University Press, 3rd. ed. (2006)

Scalars and vectors

- Scalars are quantities without directions. For example mass (m), speed ($|\mathbf{v}|$, *not* velocity), temperature (T), length, volume (V).
- Vectors are quantities with direction. For example: weight ($\mathbf{F}$), velocity ($\mathbf{v}$), position ($\mathbf{r}$), acceleration/deceleration ($\mathbf{a}$), rotation, rotational speed (ω).
- Two types of vectors:
  - Weight ($\mathbf{F}$), velocity ($\mathbf{v}$), position ($\mathbf{r}$), acceleration/deceleration ($\mathbf{a}$).
  - Rotation, rotational speed (ω).
- You can write vectors in three different ways: boldface ($\mathbf{v}$), underline ($\underline{\mathrm{v}}$), and arrow ($\vec{v}$)

Other definitions for vectors I

- Limited length line segments from Point A to Point B: $\vec{AB}$. The start point is Point A, and the end is Point B.



- Two vectors are equal if you can transform the first vector to the second vector using parallel shift/translation displacement.
- Or, if the lengths and directions of the vectors are the same.
- The vectors you can transform to each other form the group of additive vectors.

Other definitions for vectors II

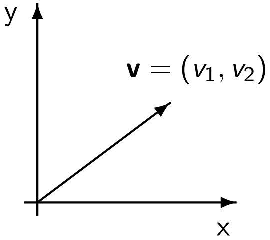
- Definition: Planar (V^2) or spatial vectors (V^3) are the group of those that you can transform to each other using translation (parallel displacement).

Coordinates of vectors I

- The pairs/triplets of numbers ($\mathbb{R}^2$, or $\mathbb{R}^3$) can identify as vectors:

$$\mathbf{v} = (v_1, v_2) = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix},$$

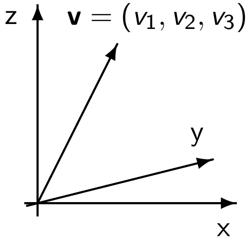
where $v_1 \in \mathbb{R}$, $v_2 \in \mathbb{R}$ are the components of the vector in 2D.



Coordinates of vectors II

$$\mathbf{v} = (v_1, v_2, v_3) = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix},$$

where $v_1 \in \mathbb{R}$, $v_2 \in \mathbb{R}$, $v_3 \in \mathbb{R}$ are the components of the vector in 3D.



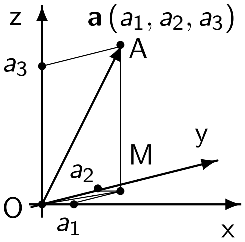
Length and equality of vectors I

- Definition: Two vectors are equal, and only equal if their origo centred representations are the same.
- In coordinates it means that $\mathbf{a}\,(a_1, a_2, a_3)$ and $\mathbf{b}\,(b_1, b_2, b_3)$ are equal, and only equal if $a_1 = b_1$, $a_2 = b_2$, and $a_3 = b_3$, where a_1, a_2, $a_3 \in \mathbb{R}$.

Length and equality of vectors II

- Thesis: The magnitude of the $\mathbf{a} = (a_1, a_2, a_3)$ vector is the following non-zero number:

$$|\mathbf{a}| = \|\mathbf{a}\| = \sqrt{a_1^2 + a_2^2 + a_3^2}.$$



Deduction: Points AOM form a right triangle, where the right angle is located at OMA$\angle$ angle. Appying the Pythagoras-thesis you got $|\mathbf{a}| = OM^2 + a_3^2$. Points O, $(a_1, 0, 0)$, $(0, a_2, 0)$ form a right triangle too, where the right angle is located at $[O, (a_1, 0, 0), (0, a_2, 0)]\angle$ angle according too the Pythagoras theorem: $OM^2 = a_1^2 + a_2^2$. After substituting the later equation into the first equation we got $|\mathbf{a}| = OM^2 = a_1^2 + a_2^2 + a_3^2$, therefore $\underline{\underline{|\mathbf{a}| = \sqrt{a_1^2 + a_2^2 + a_3^2}.}}$ *q. e. d.*

Length and equality of vectors III

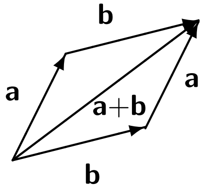
- The null vector has no length: $|\mathbf{0}\,(0,0,0)|=0$.

Multiply vector with a scalar, sum of vectors, and difference of vectors I

- <u>Definition:</u> Sum of vectors. If $\mathbf{a}\,(a_1, a_2, a_3)$ and $\mathbf{b}\,(b_1, b_2, b_3)$, then

$$\mathbf{a} + \mathbf{b} = (a_1 + b_1, a_2 + b_2, a_3 + b_3)\,,$$

where $a_1,\ a_2,\ a_3,,\ b_1,\ b_2,\ b_3 \in \mathbb{R}$.



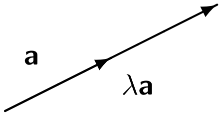
To add vectors $\mathbf{a}$ *and* $\mathbf{b}$ *you should transform vector* $\mathbf{b}$ *to the end point of the vector* $\mathbf{a}$*. Their sum* $(\mathbf{a} + \mathbf{b})$ *is the vector from the start point of* $\mathbf{a}$ *to the endpoint of the vector* $\mathbf{b}$*.*

Multiply vector with a scalar, sum of vectors, and difference of vectors II

- Definition: Multiplication of vectors with a scalar. If $\lambda \in \mathbb{R}$ and $\mathbf{a}\,(a_1, a_2, a_3)$, where $a_1,\ a_2,\ a_3 \in \mathbb{R}$, then

$$\lambda\mathbf{a} = (\lambda a_1, \lambda a_2, \lambda a_3)\,.$$

Mind if λ is 0, 1, -1, $<$1, or $>$1.



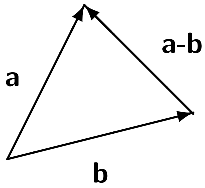
To multiply an $\mathbf{a}$ *vector by a* λ *number, you should draw a vector from the starting point of* $\mathbf{a}$ *vector in the direction of* $\mathbf{a}$ *that has* λ *length of the* $\mathbf{a}$ *vector.*

Multiply vector with a scalar, sum of vectors, and difference of vectors III

- Definition: Subtracktion of vectors. If $\mathbf{a}\,(a_1, a_2, a_3)$ and $\mathbf{b}\,(b_1, b_2, b_3)$, then

$$\mathbf{a} - \mathbf{b} = (a_1 - b_1, a_2 - b_2, a_3 - b_3)\,,$$

where $a_1,\ a_2,\ a_3,\ b_1,\ b_2,\ b_3 \in \mathbb{R}$.



To get the difference of vectors $\mathbf{a}$ *and* $\mathbf{b}$*, you should transfer to the vectors in a common start point. The difference of the vectors* $(\mathbf{a} - \mathbf{b})$ *starts from the end point of the vector* $\mathbf{b}$ *to the endpoint of vector* $\mathbf{a}$*.*

Multiply vector with a scalar, sum of vectors, and difference of vectors IV

- The sum of vectors is
  1. <u>Thesis:</u> Commutative $\mathbf{a} + \mathbf{b} = \mathbf{b} + \mathbf{a}$, where $\mathbf{a}, \mathbf{b} \in \mathbb{R}^3$.
     <u>Deduction:</u> $\mathbf{a} + \mathbf{b} = (a_1, a_2, a_3) + (b_1, b_2, b_3) = (a_1 + b_1, a_2 + b_2, a_3 + b_3) = (b_1 + a_1, b_2 + a_2, b_3 + a_3) = (b_1, b_2, b_3) + (a_1, a_2, a_3) = \underline{\underline{\mathbf{b} + \mathbf{a}}}_{\,q.e.d.}$
  2. <u>Thesis:</u> Associative $(\mathbf{a} + \mathbf{b}) + \mathbf{c} = \mathbf{a} + (\mathbf{b} + \mathbf{c})$, where $\mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathbb{R}^3$.
     <u>Deduction:</u> $(\mathbf{a} + \mathbf{b}) + \mathbf{c} = [(a_1, a_2, a_3) + (b_1, b_2, b_3)] + (c_1, c_2, c_3) =$
     $[(a_1 + b_1) + c_1, (a_2 + b_2) + c_2, (a_3 + b_3) + c_3] =$
     $[a_1 + (b_1 + c_1), a_2 + (b_2 + c_2), a_3 + (b_3 + c_3)] =$
     $(a_1, a_2, a_3) + [(b_1, b_2, b_3) + (c_1, c_2, c_3)] = \underline{\underline{\mathbf{a} + (\mathbf{b} + \mathbf{c})}}_{\,q.e.d.}$
  3. The null vector exists: $\exists \mathbf{0} \in \mathbb{R}^3$, where $\mathbf{a} + \mathbf{0} = \mathbf{a}$, and $\mathbf{a} \in \mathbb{R}^3$.
  4. All vectors have an inverse vector: $\forall \mathbf{a} \in \mathbb{R}^3\ \exists (-\mathbf{a}) \in \mathbb{R}^3$, where $\mathbf{a} + (-\mathbf{a}) = \mathbf{0}$.

Multiply vector with a scalar, sum of vectors, and difference of vectors V

- The vector multiplication with scalar is
  - Thesis: Multiplication of vectors by a scalar is associative, $\lambda\left(\mu\mathbf{a}\right)=\left(\lambda\mu\right)\mathbf{a}$, where $\mathbf{a}\in\mathbb{R}^3,\lambda,\mu\in\mathbb{R}$.
    Deduction: $\lambda\left(\mu\mathbf{a}\right)=\lambda\left[\mu\left(a_1,a_2,a_3\right)\right]=\lambda\left(\mu a_1,\mu a_2,\mu a_3\right)=\left(\lambda\mu a_1,\lambda\mu a_2,\lambda\mu a_3\right)=\left(\lambda\mu\right)\left(a_1,a_2,a_3\right)=\underline{\underline{\left(\lambda\mu\right)\mathbf{a}}}_{\;q.e.d.}$
  - Thesis: Addition of vectors is distributive for the multiplication by a scalar, $\lambda\left(\mathbf{a}+\mathbf{b}\right)=\lambda\mathbf{a}+\lambda\mathbf{b}$, where$\mathbf{a},\mathbf{b}\in\mathbb{R}^3,\lambda\in\mathbb{R}$.
    Deduction: $\lambda\left(\mathbf{a}+\mathbf{b}\right)=\lambda\left[\left(a_1,a_2,a_3\right)+\left(b_1,b_2,b_3\right)\right]=\lambda\left(a_1+b_1,a_2+b_2,a_3+b_3\right)=\left[\lambda\left(a_1+b_1\right),\lambda\left(a_2+b_2\right),\lambda\left(a_3+b_3\right)\right]=\left(\lambda a_1+\lambda b_1,\lambda a_2+\lambda b_2,\lambda a_3+\lambda b_3\right)=\left(\lambda a_1,\lambda a_2,\lambda a_3\right)+\left(\lambda b_1,\lambda b_2,\lambda b_3\right)=\lambda\left(a_1,a_2,a_3\right)+\lambda\left(b_1,b_2,b_3\right)=\underline{\underline{\lambda\mathbf{a}+\lambda\mathbf{b}}}_{\;q.\;e.\;d}$

Multiply vector with a scalar, sum of vectors, and difference of vectors VI

- Thesis: The addition of scalars is distributive for multiplication by a vector, $(\lambda + \mu)\,\mathbf{a} = \lambda\mathbf{a} + \mu\mathbf{a}$, where $\mathbf{a} \in \mathbb{R}^3, \lambda, \mu \in \mathbb{R}$.
  Deduction: $(\lambda + \mu)\,\mathbf{a} = (\lambda + \mu)\,(a_1, a_2, a_3) = [(\lambda + \mu)\,a_1, (\lambda + \mu)\,a_2, (\lambda + \mu)\,a_3] = (\lambda a_1 + \mu a_1, \lambda a_2 + \mu a_2, \lambda a_3 + \mu a_3) = (\lambda a_1, \lambda a_2, \lambda a_3) + (\mu a_1, \mu a_2, \mu a_3) = \lambda\,(a_1, a_2, a_3) + \mu\,(a_1, a_2, a_3) = \underline{\underline{\lambda\mathbf{a} + \mu\mathbf{a}}}$ *q. e. d*
- $\forall \in \mathbf{a} \cdot 1 = \mathbf{a}$, where $\mathbf{a} \in \mathbb{R}^3$.

Unitvector I

- Definition: The unit vectors are vectors with unit (1) length. The canonic basis of $\mathbb{R}^3$ is
$$\mathbf{i} = \mathbf{e}_1 = (1,0,0)\,, \mathbf{j} = \mathbf{e}_2 = (0,1,0)\,, \mathbf{k} = \mathbf{e}_3 = (0,0,1)\,.$$
- Thesis: For all $\mathbf{v}\,(v_1, v_2, v_3)$ 3D vector:
$$\mathbf{v} = v_1\mathbf{e}_1 + v_2\mathbf{e}_2 + v_3\mathbf{e}_3.$$
  Deduction:
  $\mathbf{v} = (v_1, v_2, v_3) = (v_1, 0, 0) + (0, v_2, 0) + (0, 0, v_3) = v_1\mathbf{e}_1 + v_2\mathbf{e}_2 + v_3\mathbf{e}_3.$
- If $\mathbf{v} \neq \mathbf{0}$, then $|\mathbf{v}| \neq 0$, therefore, it has a direction vector:
  Definition: The normal of the $|\mathbf{v}| \neq \mathbf{0}$ vector is $\frac{\mathbf{v}}{|\mathbf{v}|}$.

Unitvector II

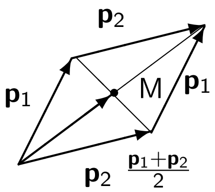
- The normal vector is a unit vector:

$$\left|\frac{\mathbf{v}}{|\mathbf{v}|}\right| = \frac{1}{|\mathbf{v}|}\,|\mathbf{v}| = 1.$$

Distance of points, equation of sphere I

- The M bisecting point of the segment between the $P_1\left(x_1, y_1, z_1\right)$ and the $P_2\left(x_2, y_2, z_2\right)$ points is

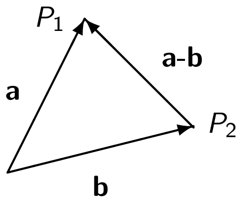
$$M\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}, \frac{z_1+z_2}{2}\right).$$



Contruct the vector addition by the Vectors $\mathbf{p_1}$, $\mathbf{p_2}$ *ending in Points* P_1, P_2, *respectively. The two* $\mathbf{p_1}$, *and the two* $\mathbf{p_2}$ *vectors are parallel, therefore, form a parallelogram. However, the diagonals of the parallelogram bisect each other into half, hence, Point* M *is at the half of the sum of the Vectors* $\mathbf{p_1}$ *and* $\mathbf{p_2}$. *Q. E. D.*

Distance of points, equation of sphere II

- The distance of the $P_1(x_1, y_1, z_1)$ and the $P_2(x_2, y_2, z_2)$ points is the length of the **a**, **b** vectors with ending point of P_1 and P_2: $|\mathbf{a} - \mathbf{b}|$.

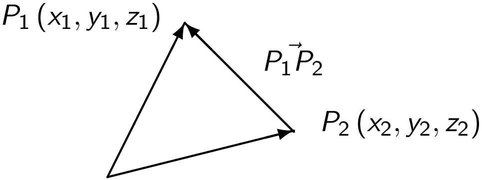


Distance of points, equation of sphere III

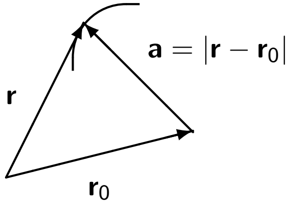
- Thesis: The distance of the $P_1(x_1, y_1, z_1)$ and the $P_2(x_2, y_2, z_2)$ points is

$$|P_1P_2| = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2}.$$

Statement: The distance of Point P_1 and P_2 is the difference of the vectors pointing into each point. The length of the difference vectors is the formula above.



Distance of points, equation of sphere IV



- Thesis: The equation of the sphere with a radius and (x_0, y_0, z_0) centre is

$$(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 = a^2.$$

Deduction: A sphere with radius a and $\mathbf{r}_0$ centre is the set of those points in 3D ($\mathbf{r}$), that are a distance from Point $\mathbf{r}_0$. It means that $|\mathbf{r} - \mathbf{r}_0| = a$. Therefore, $\sqrt{(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2} = a$, or $\underline{\underline{(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 = a^2}}_{\;q.\ e.\ d.}$.

Scalar multiplication of vectors I

- Definition: The scalar (or inner) multiplication of two vectors is

$$\mathbf{ab} = |\mathbf{a}|\,|\mathbf{b}|\cos\theta$$

  where $\theta = (\mathbf{a}, \mathbf{b})\angle$ és $\left(\mathbf{a}, \mathbf{b} \in V^3\right)$.
- Note that $\mathbf{aa} = |\mathbf{a}|^2$.
- The features of scalar multiplication
  1. Thesis: The scalar multiplication of vector is commutative: $\mathbf{ab} = \mathbf{ba}$, where $\left(\mathbf{a}, \mathbf{b} \in V^3\right)$.
     Deduction: If $\theta = (\mathbf{a}, \mathbf{b})\angle$, then $\mathbf{ab} = |\mathbf{a}| \cdot |\mathbf{b}|\cos\theta = |\mathbf{b}| \cdot |\mathbf{a}|\cos\theta = \underline{\underline{\mathbf{ba}}}$ *q. e. d.*
  2. Thesis: The scalar multiplication of vectors is distributive: $(\mathbf{a} + \mathbf{b})\,\mathbf{c} = \mathbf{ac} + \mathbf{bc}$, where $\left(\mathbf{a}, \mathbf{b}, \mathbf{c} \in V^3\right)$.
     Deduction: If $\theta = (\mathbf{a}, \mathbf{b})\angle$, then
     $(\mathbf{a} + \mathbf{b})\,\mathbf{c} = |\mathbf{a} + \mathbf{b}|\,|\mathbf{c}|\cos\theta = |\mathbf{a}|\,|\mathbf{c}|\cos\theta + |\mathbf{b}|\,|\mathbf{c}|\cos\theta = \underline{\underline{\mathbf{ac} + \mathbf{bc}}}$ *q. e. d.*

Scalar multiplication of vectors II

3. Thesis: The scalar multiplication of vectors is homogenous, $(\lambda \mathbf{a})\,\mathbf{b} = \lambda\,(\mathbf{ab})$, where $\lambda \in \mathbb{R}$ and $\left(\mathbf{a}, \mathbf{b} \in V^3\right)$.
   Deduction: If $\theta = (\mathbf{a}, \mathbf{b})\,\angle$, then
   $(\lambda \mathbf{a})\,\mathbf{b} = |\lambda \mathbf{a}|\,|\mathbf{b}|\cos\theta = \lambda\,|\mathbf{a}|\,|\mathbf{b}|\cos\theta = \underline{\underline{\lambda\,(\mathbf{ab})}}$ *q. e. d.*
4. The scalar multiplication of vectors is positive definit, $\mathbf{aa} \geq 0$, where $\left(\mathbf{a} \in V^3\right)$ and $\mathbf{aa} = 0 \Leftrightarrow \mathbf{a} = \mathbf{0}$.

You can deduct the statements above using the following thesis.

Scalar multiplication of vectors III

- Thesis: The scalar multiplication of $\mathbf{a} = (a_1, a_2, a_3)$ and $\mathbf{b} = (b_1, b_2, b_3)$ vectors is

$$\mathbf{ab} = a_1 b_1 + a_2 b_2 + a_3 b_3.$$

Deduction: Based on the definition of scalar multiplication, furthermore $\cos 90^\circ = 0$, and $\cos 0^\circ = 1$ you can see that

$$\mathbf{e}_i \mathbf{e}_j = \begin{cases} 1, & \text{ha } i = j. \\ 0, & \text{ha } i \neq j. \end{cases}$$

Therefore: $\mathbf{ab} = (\mathbf{a}_1 \mathbf{e}_1 + a_2 \mathbf{e}_2 + a_3 \mathbf{e}_3)(b_1 \mathbf{e}_1 + b_2 \mathbf{e}_2 + b_3 \mathbf{e}_3) =$
$a_1 b_1 \mathbf{e}_1 \mathbf{e}_1 + a_1 b_2 \mathbf{e}_1 \mathbf{e}_2 + a_1 b_3 \mathbf{e}_1 \mathbf{e}_3 + a_2 b_1 \mathbf{e}_2 \mathbf{e}_1 + a_2 b_2 \mathbf{e}_2 \mathbf{e}_2 + a_2 b_3 \mathbf{e}_2 \mathbf{e}_3 + a_3 b_1 \mathbf{e}_3 \mathbf{e}_1 +$
$a_3 b_2 \mathbf{e}_3 \mathbf{e}_2 + a_3 b_3 \mathbf{e}_3 \mathbf{e}_3 = \underline{\underline{a_1 b_1 + a_2 b_2 + a_3 b_3}}\ _{q.e.d.}$

Scalar multiplication of vectors IV

- Thesis: The angle of two non-zero vectors ($\mathbf{a} = (a_1, a_2, a_3)$ and $\mathbf{b} = (b_1, b_2, b_3)$) is

$$\cos\theta = \frac{\mathbf{ab}}{|\mathbf{a}| \cdot |\mathbf{b}|}$$

- Definition: The $\mathbf{a}$ and $\mathbf{b}$ vectors are orthogonal (perpendicular) if $\mathbf{ab} = 0$.
- Definition: The perpendicular projection ($proj_{\mathbf{b}}\mathbf{a}$) of $\mathbf{a}$ vector to $\mathbf{b}$ vector is that $\mathbf{b}$ directed vector that ends in the point that is determined by a perpendicular line to $\mathbf{b}$ vector.
- Thesis: If $\left(\mathbf{a}, \mathbf{b} \in V^3\right)$, then

$$proj_{\mathbf{b}}\mathbf{a} = \frac{\mathbf{ab}}{|\mathbf{b}|^2}\mathbf{b}.$$

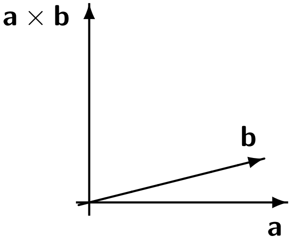
Scalar multiplication of vectors V

- If **b** unit vector has unit length, then the formula is simple:

$$proj_{\mathbf{b}}\mathbf{a} = (\mathbf{ab})\,\mathbf{b}.$$

Vector Product I

- Definition: The system consists of from $\{\mathbf{a}, \mathbf{b}, \mathbf{c}\}$ non-zero vectors is a right-handed system if from the endpoint of $\mathbf{c}$, $\mathbf{a}$ could be rotated to the direction of the $\mathbf{b}$ by less than 180^o angle in anti-clockward direction.



- Definition: The vectorial product of non-zero Vectors $\mathbf{a}$ and $\mathbf{b}$ is that $\mathbf{a} \times \mathbf{b}$ vector, which length is $|\mathbf{a} \times \mathbf{b}| = |\mathbf{a}|\,|\mathbf{b}| \sin\theta$, where $\theta = (\mathbf{a}, \mathbf{b})\angle$. The Vector $\mathbf{a} \times \mathbf{b}$ is perpendicular to Vectors $\mathbf{a}$ and $\mathbf{b}$, furthermore $\{\mathbf{a}, \mathbf{b}, \mathbf{a} \times \mathbf{b}\}$ consist of a right-handed system. Finally, $\mathbf{0} \times \mathbf{a} = \mathbf{0}$, where $\left(\mathbf{a} \in V^3\right)$.

Vector Product II

- Features of the vector product
  1. Thesis: The vectorial product is anticommutative, $\mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a}$, where $\left(\mathbf{a}, \mathbf{b} \in V^3\right)$.
     Deduction: It is trivial based on the definition of the right-handed system.
  2. Thesis: The vectorial product is homogen, $(\lambda \mathbf{a}) \times \mathbf{b} = \lambda (\mathbf{a} \times \mathbf{b})$, where $\mathbf{a}, \mathbf{b} \in V^3$ and $\lambda \in \mathbb{R}$.
     Deduction: $|(\lambda \mathbf{a}) \times \mathbf{b}| = |\lambda \mathbf{a}| \, |\mathbf{b}| \sin\theta = \lambda \, |\mathbf{a} \times \mathbf{b}|$, ahol $\theta = (\lambda \mathbf{a}, \mathbf{b}) \angle$.
     The direction of vectors agrees because Vector $\mathbf{a}$ is parallel to Vector $\lambda \mathbf{a}$.
  3. Thesis: The vectorial product is dissociative, $(\mathbf{a} + \mathbf{b}) \times \mathbf{c} = \mathbf{a} \times \mathbf{c} + \mathbf{b} \times \mathbf{c}$, where $\left(\mathbf{a}, \mathbf{b}, \mathbf{c} \in V^3\right)$.
     Deduction: Later, based on components.
- Definition: Non-zero Vectors $\mathbf{a}$ and $\mathbf{b}$ are parallel if $\exists \lambda \in \mathbb{R}$, and $\mathbf{a} = \lambda \mathbf{b}$. Its sign is $\mathbf{a} \parallel \mathbf{b}$.

Vector Product III

- All vector multiplied itself is zero-vecdtor, $\mathbf{a} \times \mathbf{a} = \mathbf{0}\ \forall \mathbf{a} \in V^3$-re. esetén.
- Furthermore $\mathbf{a} \times \mathbf{b} = \mathbf{0} \Leftrightarrow \mathbf{a} \parallel \mathbf{b}$, or at least one of Vectors $\mathbf{a}, \mathbf{b}$ is a null-vector.
- It is easy to prove that

$$\begin{aligned} \mathbf{e}_1 \times \mathbf{e}_2 &= \mathbf{e}_3 \\ \mathbf{e}_2 \times \mathbf{e}_3 &= \mathbf{e}_1 \\ \mathbf{e}_3 \times \mathbf{e}_1 &= \mathbf{e}_2. \end{aligned}$$

- The vectorial product with components is $\mathbf{a} \times \mathbf{b} = (a_2 b_3 - a_3 b_2)\,\mathbf{e}_1 + (a_3 b_1 - a_1 b_3)\mathbf{e}_2 + (a_1 b_2 - a_2 b_1)\,\mathbf{e}_3$.
- $|\mathbf{a} \times \mathbf{b}|$ is equla to the area of the paralelogram determined by $\mathbf{a}$ and $\mathbf{b}$, because $|\mathbf{a}|$ is the basis of the paralelogram and $|\mathbf{b}|\,|\sin\theta|$ is its height, where $\theta = (\mathbf{a}, \mathbf{b})\,\angle$.

Triple product

- Definition: The triple product of Vectors $\mathbf{a}, \mathbf{b}, \mathbf{c} \in V^3$ is

$$(\mathbf{a}, \mathbf{b}, \mathbf{c}) = (\mathbf{a} \times \mathbf{b})\, \mathbf{c}.$$

- If $\mathbf{a}, \mathbf{b}, \mathbf{c}$ constist of a right-handed system, then $(\mathbf{a}, \mathbf{b}, \mathbf{c})$ is the volume of the Parallelepiped of $\mathbf{a}, \mathbf{b}, \mathbf{c}$ vectors. Otherwise, you got the volume -1 times.
- It is easy to prove that

$$(\mathbf{a}, \mathbf{b}, \mathbf{c}) = (\mathbf{b}, \mathbf{c}, \mathbf{a}) = (\mathbf{c}, \mathbf{a}, \mathbf{b}) = -(\mathbf{a}, \mathbf{c}, \mathbf{b}) = -(\mathbf{c}, \mathbf{b}, \mathbf{a}) = -(\mathbf{b}, \mathbf{a}, \mathbf{c}).$$

The End

Thank you for your attention!