(ENKEMNA0302) Applied Linear Algebra

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Orthonormal basis - orthogonal matrix I

- Orthogonal basis: the basis vectors are pairwise orthogonal, since in that case the scalar product of different basis vectors is 0. I
- Orthonormal basis: if the basis vectors are unit vectors.
- A set of pairwise orthogonal vectors is called an orthogonal basis (OB). An orthogonal system may contain zero vectors.
- A set of pairwise orthogonal unit vectors is called an orthonormal basis (ONB). Orthonormal bases do not contain zero vectors.
- From an orthogonal basis, an orthonormal basis can always be obtained by normalizing each basis vector (i.e., dividing each by its length).

Orthonormal basis - orthogonal matrix II

- Independence of orthogonal vectors: Wettl notes
- Best approximation in the case of an ONB: Wettl notes
- Definition: (Orthogonal and semi-orthogonal matrix). A real square matrix is called orthogonal if its column vectors or row vectors form an orthonormal system. If we do not require the matrix to be square, we speak of a semi-orthogonal matrix.
- Theorem: (Equivalent definitions of semi-orthogonal matrices) Let $m \geq n$ and $\mathbf{Q} \in \mathbb{R}^{m \times n}$. The following statements are equivalent:
  1. $\mathbf{Q}$ is semi-orthogonal,
  2. $\mathbf{Q}^T\mathbf{Q} = \mathbf{E}_n$.

  Similarly, for $m \leq n$, $\mathbf{Q}$ is semi-orthogonal if and only if $\mathbf{Q}\mathbf{Q}^T = \mathbf{E}_m$. Statement 2 expresses in algebraic terms that for $m \leq n$, $\mathbf{Q}$ is semi-orthogonal if and only if its transpose is its left inverse.

  Proof: Wettl notes.

Orthonormal basis - orthogonal matrix III

- Theorem: (Equivalent definitions of orthogonal matrices). Let $\mathbf{Q} \in \mathbb{R}^{n \times n}$. The following statements are equivalent:
  1. The column vectors of $\mathbf{Q}$ form an orthonormal system.
  2. $\mathbf{Q}^T\mathbf{Q} = \mathbf{E}_n$.
  3. $\mathbf{Q}^{-1} = \mathbf{Q}^T$.
  4. $\mathbf{Q}\mathbf{Q}^T = \mathbf{E}_n$.
  5. The row vectors of $\mathbf{Q}$ form an orthonormal system.

  Proof: Wettl notes.
- Theorem: (Matrix transformation associated with an orthogonal matrix) Let $\mathbf{Q} \in \mathbb{R}^{n \times n}$. The following statements are equivalent:
  1. $\mathbf{Q}$ is orthogonal.
  2. $|\mathbf{Qx}| = |\mathbf{x}|$ for all $\mathbf{x} \in \mathbb{R}^n$.
  3. $\mathbf{Qx} \cdot \mathbf{Qy} = \mathbf{x} \cdot \mathbf{y}$ for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$.

  Proof: Wettl notes.

Orthonormal basis - orthogonal matrix IV

- Theorem: (Properties of orthogonal matrices)
  1. If $\mathbf{Q}$ is a real orthogonal matrix, then $|\det(\mathbf{Q})| = 1$.
  2. The set of real $n \times n$ orthogonal matrices, denoted by $O(n)$, is closed under matrix multiplication and inversion.

  Proof: Wettl notes.
- Theorem: Every orthogonal matrix in $O(2)$ is either a rotation or a reflection across a line.
- The set of $n \times n$ real orthogonal matrices with determinant 1 is also closed under matrix multiplication and inversion. This set is denoted by $SO(n)$.

Orthonormal basis - orthogonal matrix V

- Orthogonal transformations in 2D and 3D spaces. Orthogonal matrices can be described using rotations and reflections.
- Theorem: Every orthogonal matrix in $O(2)$ is either a rotation or a reflection across a line.
  Proof: Wettl notes.
- ~~Givens rotation, Householder reflection~~
- ~~Reflection of a vector into another~~
- Gram–Schmidt orthogonalization: studied last semester.
- QR decomposition: Just like LU decomposition compactly encodes the row operations needed to bring a matrix to triangular form, QR decomposition encodes the results of the orthogonalization process. This decomposition plays a crucial role in both the least squares method and in solving eigenvalue problems.

Orthonormal basis - orthogonal matrix VI

- Definition: (QR decomposition). Let $\mathbf{A}$ be a full column-rank real matrix. The decomposition $\mathbf{A} = \mathbf{QR}$ is called a QR decomposition (or reduced QR decomposition) if $\mathbf{Q}$ is a semi-orthogonal matrix of the same size as $\mathbf{A}$, and $\mathbf{R}$ is a square upper triangular matrix with positive elements on the diagonal.
- Theorem: (Existence and uniqueness of QR decomposition). Every real full column-rank matrix $\mathbf{A}$ has a QR decomposition, i.e., there exist a semi-orthogonal matrix $\mathbf{Q}$ and an upper triangular matrix $\mathbf{R}$ with positive diagonal elements such that $\mathbf{A} = \mathbf{QR}$. This decomposition is unique.
  Proof: Wettl notes.
- QR decomposition with primitive orthogonal transformations: Wettl notes.
- Optimal solution of a system of equations using QR decomposition: Wettl notes.
- Least squares using QR decomposition: Wettl notes.

Spaces over the Complex Field and Finite Fields I

- Complex vectors and spaces, scalar product of complex vectors.
- Definition: (Adjungate of a complex matrix). The adjungate (or Hermitian transpose) of a complex matrix $\mathbf{A}$ is the transpose of its elementwise conjugate. The adjungate of $\mathbf{A}$ is denoted by $\mathbf{A}^*$ or, after Hermite, by $\mathbf{A}^H$, thus $\mathbf{A}^H = \overline{\mathbf{A}}^T$.
- Definition: (Scalar product of complex vectors). For vectors $\mathbf{z} = (z_1, z_2, \ldots, z_n)$ and $\mathbf{w} = (w_1, w_2, \ldots, w_n)$ in $\mathbb{C}^n$, the scalar product is defined as

$$\mathbf{z} \cdot \mathbf{w} = \overline{z_1} w_1 + \overline{z_2} w_2 + \cdots + \overline{z_n} w_n$$

which is a complex scalar. Its matrix product form is $\mathbf{z} \cdot \mathbf{w} = \mathbf{z}^H \mathbf{w}$.

Spaces over the Complex Field and Finite Fields II

- Theorem: (Properties of the adjungate). Let $\mathbf{A}$ and $\mathbf{B}$ be complex matrices and let c be a complex number. Then:
  1. $\left(\mathbf{A}^H\right)^H = \mathbf{A}$,
  2. $(\mathbf{A}+\mathbf{B})^H = \mathbf{A}^H + \mathbf{B}^H$,
  3. $(c\mathbf{A})^H = \overline{c}\mathbf{A}^H$,
  4. $(\mathbf{AB})^H = \mathbf{B}^H\mathbf{A}^H$.
- The properties of the adjungate immediately imply:
- Theorem: (Properties of complex scalar product). Let $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{C}^n$ and let $c \in \mathbb{C}$. Then:
  1. $\mathbf{u} \cdot \mathbf{v} = \overline{\mathbf{v} \cdot \mathbf{u}}$,
  2. $\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w}$,
  3. $(c\mathbf{u}) \cdot \mathbf{v} = c\,(\mathbf{u} \cdot \mathbf{v})$ and $\mathbf{u} \cdot (c\mathbf{v}) = \overline{c}\,(\mathbf{u} \cdot \mathbf{v})$,
  4. $\mathbf{u} \cdot \mathbf{u} > 0$ if $\mathbf{u} \neq \mathbf{0}$, and $\mathbf{u} \cdot \mathbf{u} = \mathbf{0}$ if $\mathbf{u} = \mathbf{0}$.

  Proof: See Wettl notes.

Spaces over the Complex Field and Finite Fields III

- Distinguished subspaces of complex matrices: see Wettl notes.
- Self-adjoint matrices: Just as the adjoint is the extension of the transpose (considering complex scalar product), the self-adjungate matrix is the extension of the symmetric matrix. A symmetric matrix is equal to its own transpose, while a self-adjungate matrix is equal to its own adjungate.
- A complex matrix $\mathbf{A}$ is self-adjungate if $\mathbf{A}^H = \mathbf{A}$.
- Self-adjungate matrices are also called Hermitian matrices.
- The diagonal of a self-adjungate matrix contains only real numbers, as only real numbers are equal to their own conjugate.
- Every real symmetric matrix is self-adjungate, since real numbers are equal to their own conjugate. Since non-real complex numbers are not equal to their conjugates, a complex symmetric matrix is self-adjungate if and only if all its elements are real.

Spaces over the Complex Field and Finite Fields IV

- Distance and orthogonal projection in complex spaces: The scalar product allows the definition of distance and orthogonality between complex vectors.
- The length or absolute value of a complex vector $\mathbf{u} \in \mathbb{C}^n$ is $|\mathbf{u}| = \sqrt{\mathbf{u} \cdot \mathbf{u}}$; the distance between two vectors equals the length of their difference: for vectors $\mathbf{u}, \mathbf{v} \in \mathbb{C}^n$, $d(\mathbf{u}, \mathbf{v}) = |\mathbf{u} - \mathbf{v}|$. Two vectors are considered orthogonal if their scalar product is 0.
- Theorem: (Cauchy–Bunyakovsky–Schwarz inequality). For all $\mathbf{u}, \mathbf{v} \in \mathbb{C}^n$:

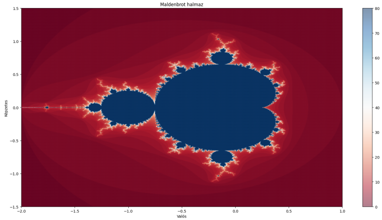
$$|\mathbf{u} \cdot \mathbf{v}| \leq |\mathbf{u}| \, |\mathbf{v}| \, .$$

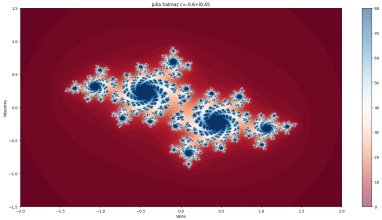
  Equality holds if and only if $\mathbf{u}$ and $\mathbf{v}$ are linearly dependent, i.e., one is a scalar multiple of the other.

Spaces over the Complex Field and Finite Fields V

- The complex analogue of orthogonal matrices is the unitary matrix.
- Definition: (Unitary matrix). A square complex matrix $\mathbf{U}$ is unitary if $\mathbf{U}^H\mathbf{U} = \mathbf{E}$.
- Just as in the case of orthogonal matrices, one can prove that a matrix $\mathbf{U} \in \mathbb{C}^{n\times n}$ is unitary if any of the following equivalent conditions hold:
  1. $\mathbf{U}\mathbf{U}^H = \mathbf{E}$,
  2. $\mathbf{U}^{-1} = \mathbf{U}^H$,
  3. The column vectors of $\mathbf{U}$ form an orthonormal basis with respect to the complex scalar product,
  4. The row vectors of $\mathbf{U}$ form an orthonormal basis with respect to the complex scalar product,
  5. $|\mathbf{U}\mathbf{x}| = |\mathbf{x}|$ for every vector $\mathbf{x} \in \mathbb{C}^n$,
  6. $\mathbf{U}\mathbf{x} \cdot \mathbf{U}\mathbf{y} = \mathbf{x} \cdot \mathbf{y}$.

Complex numbers are ~~cool~~ beautiful





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Thank you for your attention!