(PTIA0301) Elementary Linear Algebra

Dr. Gabor FACSKO, PhD
Senior Research Fellow
facskog@gamma.ttk.pte.hu

University of Pecs, Faculty of Sciences, Institute of Mathematics and Informatics, 7624 Pecs, Ifjusag utja 6.
Wigner Research Centre for Physics, Department of Space Physics and Space Technology, 1121 Budapest, Konkoly-Thege Miklos ut 29-33.
https://facsko.ttk.pte.hu

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Unitvector I

- Definition: The unit vectors are vectors with unit (1) length. The canonic basis of $\mathbb{R}^3$ is
$$\mathbf{i} = \mathbf{e}_1 = (1,0,0)\,, \mathbf{j} = \mathbf{e}_2 = (0,1,0)\,, \mathbf{k} = \mathbf{e}_3 = (0,0,1)\,.$$
- Thesis: For all $\mathbf{v}\,(v_1, v_2, v_3)$ 3D vector:
$$\mathbf{v} = v_1\mathbf{e}_1 + v_2\mathbf{e}_2 + v_3\mathbf{e}_3.$$
  Deduction:
  $\mathbf{v} = (v_1, v_2, v_3) = (v_1, 0, 0) + (0, v_2, 0) + (0, 0, v_3) = v_1\mathbf{e}_1 + v_2\mathbf{e}_2 + v_3\mathbf{e}_3.$
- If $\mathbf{v} \neq \mathbf{0}$, then $|\mathbf{v}| \neq 0$, therefore, it has a direction vector:
  Definition: The normal of the $|\mathbf{v}| \neq \mathbf{0}$ vector is $\frac{\mathbf{v}}{|\mathbf{v}|}$.

Unitvector II

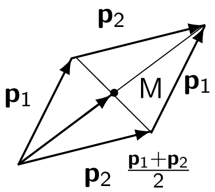
- The normal vector is a unit vector:

$$\left| \frac{\mathbf{v}}{|\mathbf{v}|} \right| = \frac{1}{|\mathbf{v}|} |\mathbf{v}| = 1.$$

Distance of points, equation of sphere I

- The M bisecting point of the segment between the $P_1\left(x_1, y_1, z_1\right)$ and the $P_2\left(x_2, y_2, z_2\right)$ points is

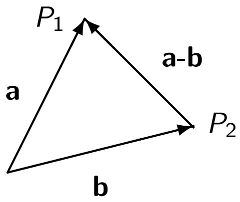
$$M\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}, \frac{z_1+z_2}{2}\right).$$



Contruct the vector addition by the Vectors $\mathbf{p_1}$, $\mathbf{p_2}$ *ending in Points* P_1, P_2, *respectively. The two* $\mathbf{p_1}$, *and the two* $\mathbf{p_2}$ *vectors are parallel, therefore, form a parallelogram. However, the diagonals of the parallelogram bisect each other into half, hence, Point M is at the half of the sum of the Vectors* $\mathbf{p_1}$ *and* $\mathbf{p_2}$. *Q. E. D.*

Distance of points, equation of sphere II

- The distance of the $P_1\,(x_1, y_1, z_1)$ and the $P_2\,(x_2, y_2, z_2)$ points is the length of the **a**, **b** vectors with ending point of P_1 and P_2: $|\mathbf{a} - \mathbf{b}|$.

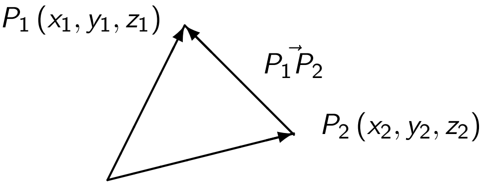


Distance of points, equation of sphere III

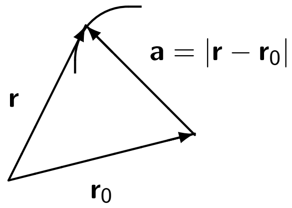
- Thesis: The distance of the $P_1\,(x_1, y_1, z_1)$ and the $P_2\,(x_2, y_2, z_2)$ points is

$$|P_1P_2| = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2}.$$

Statement: The distance of Point P_1 and P_2 is the difference of the vectors pointing into each point. The length of the difference vectors is the formula above.



Distance of points, equation of sphere IV



- <u>Thesis:</u> The equation of the sphere with a radius and (x_0, y_0, z_0) centre is

$$(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 = a^2.$$

<u>Deduction:</u> A sphere with radius a and $\mathbf{r}_0$ centre is the set of those points in 3D ($\mathbf{r}$), that are a distance from Point $\mathbf{r}_0$. It means that $|\mathbf{r} - \mathbf{r}_0| = a$. Therefore,
$\sqrt{(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2} = a$, or
$\underline{\underline{(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 = a^2}}_{\;q.\ e.\ d.}.$

Scalar multiplication of vectors I

- Definition. The scaler (or inner) multiplication of two vectors is

$$\mathbf{ab} = |\mathbf{a}|\,|\mathbf{b}|\cos\theta$$

where $\theta = (\mathbf{a},\mathbf{b})\angle$ és $\left(\mathbf{a},\mathbf{b} \in V^3\right)$.

- Note that $\mathbf{aa} = |\mathbf{a}|^2$.
- The features of scalar multiplication
    1. Thesis: The scalar multiplication of vector is commutative: $\mathbf{ab} = \mathbf{ba}$, where $\left(\mathbf{a},\mathbf{b} \in V^3\right)$.
       Deduction: If $\theta = (\mathbf{a},\mathbf{b})\angle$, then $\mathbf{ab} = |\mathbf{a}|\cdot|\mathbf{b}|\cos\theta = |\mathbf{b}|\cdot|\mathbf{a}|\cos\theta = \underline{\underline{\mathbf{ba}}}$ *q. e. d.*
    2. Thesis: The scalar multiplication of vectors is distributive: $(\mathbf{a}+\mathbf{b})\,\mathbf{c} = \mathbf{ac}+\mathbf{bc}$, where $\left(\mathbf{a},\mathbf{b},\mathbf{c} \in V^3\right)$.
       Deduction: If $\theta = (\mathbf{a},\mathbf{b})\angle$, then
       $(\mathbf{a}+\mathbf{b})\,\mathbf{c} = |\mathbf{a}+\mathbf{b}|\,|\mathbf{c}|\cos\theta = |\mathbf{a}|\,|\mathbf{c}|\cos\theta + |\mathbf{b}|\,|\mathbf{c}|\cos\theta = \underline{\underline{\mathbf{ac}+\mathbf{bc}}}$ *q. e. d.*

Scalar multiplication of vectors II

3. <u>Thesis:</u> The scalar multiplication of vectors is homogenous, $(\lambda \mathbf{a})\,\mathbf{b} = \lambda\,(\mathbf{ab})$, where $\lambda \in \mathbb{R}$ and $\left(\mathbf{a}, \mathbf{b} \in V^3\right)$.
   <u>Deduction:</u> If $\theta = (\mathbf{a}, \mathbf{b})\,\angle$, then
   $(\lambda \mathbf{a})\,\mathbf{b} = |\lambda \mathbf{a}|\,|\mathbf{b}|\cos\theta = \lambda\,|\mathbf{a}|\,|\mathbf{b}|\cos\theta = \underline{\underline{\lambda\,(\mathbf{ab})}}$ *q. e. d.*
4. The scalar multiplication of vectors is positive definit, $\mathbf{aa} \geq 0$, where $\left(\mathbf{a} \in V^3\right)$ and $\mathbf{aa} = 0 \Leftrightarrow \mathbf{a} = \mathbf{0}$.

You can deduct the statements above using the following thesis.

Scalar multiplication of vectors III

- Thesis: The scalar multiplication of $\mathbf{a} = (a_1, a_2, a_3)$ and $\mathbf{b} = (b_1, b_2, b_3)$ vectors is

$$\mathbf{ab} = a_1 b_1 + a_2 b_2 + a_3 b_3.$$

Deduction: Based on the definition of scalar multiplication, furthermore $\cos 90^\circ = 0$, and $\cos 0^\circ = 1$ you can see that

$$\mathbf{e}_i \mathbf{e}_j = \begin{cases} 1, & \text{ha } i = j. \\ 0, & \text{ha } i \neq j. \end{cases}$$

Therefore: $\mathbf{ab} = (\mathbf{a}_1 \mathbf{e}_1 + a_2 \mathbf{e}_2 + a_3 \mathbf{e}_3)(b_1 \mathbf{e}_1 + b_2 \mathbf{e}_2 + b_3 \mathbf{e}_3) =$
$a_1 b_1 \mathbf{e}_1 \mathbf{e}_1 + a_1 b_2 \mathbf{e}_1 \mathbf{e}_2 + a_1 b_3 \mathbf{e}_1 \mathbf{e}_3 + a_2 b_1 \mathbf{e}_2 \mathbf{e}_1 + a_2 b_2 \mathbf{e}_2 \mathbf{e}_2 + a_2 b_3 \mathbf{e}_2 \mathbf{e}_3 + a_3 b_1 \mathbf{e}_3 \mathbf{e}_1 +$
$a_3 b_2 \mathbf{e}_3 \mathbf{e}_2 + a_3 b_3 \mathbf{e}_3 \mathbf{e}_3 = \underline{\underline{a_1 b_1 + a_2 b_2 + a_3 b_3}}$ *q.e.d.*

Scalar multiplication of vectors IV

- Thesis: The angle of two non-zero vectors ($\mathbf{a} = (a_1, a_2, a_3)$ and $\mathbf{b} = (b_1, b_2, b_3)$) is

$$\cos\theta = \frac{\mathbf{ab}}{|\mathbf{a}| \cdot |\mathbf{b}|}$$

- Definition: The $\mathbf{a}$ and $\mathbf{b}$ vectors are orthogonal (perpendicular) if $\mathbf{ab} = 0$.
- Definition: The perpendicular projection ($proj_{\mathbf{b}}\mathbf{a}$) of $\mathbf{a}$ vector to $\mathbf{b}$ vector is that $\mathbf{b}$ directed vector that ends in the point that is determined by a perpendicular line to $\mathbf{b}$ vector.
- Thesis: If $\left(\mathbf{a}, \mathbf{b} \in V^3\right)$, then

$$proj_{\mathbf{b}}\mathbf{a} = \frac{\mathbf{ab}}{|\mathbf{b}|^2}\mathbf{b}.$$

Scalar multiplication of vectors V

- If $\mathbf{b}$ unit vector has unit length, then the formula is simple:

$$proj_{\mathbf{b}}\mathbf{a} = (\mathbf{ab})\,\mathbf{b}.$$

The End

Thank you for your attention!