Név: DR. FACSKÓ GÁBOR ISTVÁN Neptun kód: BFQQQY

(MATNA1902) Lineáris algebra 2. zárthelyi dolgozat

6. Adja meg meg az $\mathbf{a} = (1,0,0)$, a $\mathbf{b} = (0,1,0)$ és a $\mathbf{c} = (0,0,1)$ vektorokat az $(-2,0,1)$; $(0,1,-3)$; $(-1,2,2)$ bázisban. (10 pont)
7. Adja meg a következő pontokon átmenő sík egyenletét: $A\,(0,1,1)$, $B\,(1,0,1)$, $C\,(1,1,0)$. (10 pont)
8. Adja meg az alábbi mátrix sajátértékeit és a saját altereket, majd diagonizálja a mátrixot!

$$\mathbf{A} = \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}$$

(10 pont)

9. Az alábbi leképezések közül melyik lineáris? Adja meg a leképezés mátrixát is!

a.)

$$f\,(\mathbf{x}) = \begin{pmatrix} x_1 + 2x_2 \\ 2x_1 + x_2 \\ 3x_3 + 3_2 \\ x_3 \end{pmatrix} \; (\mathbf{x} \in \mathbb{R}^3)$$

b.)

$$g\,(\mathbf{x}) = \begin{pmatrix} 2x_1 \\ x_1 x_2 \\ 3x_2 \\ x_1 \end{pmatrix} \; (\mathbf{x} \in \mathbb{R}^2)$$

(10 pont)

10. Írja át az alábbi vektorokat ortogonális bázissá a Gram-Schmidt ortogonalizáció segítségével!

$$\begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} \begin{pmatrix} 3 \\ 0 \\ 0 \\ 1 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \\ 1 \\ 3 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

(10 pont)

A zárthelyi osztályzása: 0-20 pont: elégtelen (1), 21-27 pont: elégséges (2), 28-35 pont: közepes (3), 36-42 pont: jó (4) és 43-50 pont: jeles (5).

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Pécs, 2025. május 8.

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(6)

$$\lambda_1\begin{pmatrix}-2\\0\\1\end{pmatrix}+\lambda_2\begin{pmatrix}0\\1\\-3\end{pmatrix}+\lambda_3\begin{pmatrix}-1\\2\\2\end{pmatrix}=\begin{pmatrix}1\\0\\0\end{pmatrix}$$

$$\begin{cases}-2\lambda_1 + \quad -\lambda_3 = 1\\ \lambda_2 + 2\lambda_3 = 0 \rightarrow \boxed{\lambda_2 = -2\lambda_3}\\ \lambda_1 - 3\lambda_2 + 2\lambda_3 = 0\end{cases}$$

$$\begin{cases}-2\lambda_1 - \lambda_3 = 1\\ \lambda_1 + 6\lambda_3 + 2\lambda_3 = 0\end{cases}$$

$-\lambda_3 = +2\lambda_1 + 1$

$\lambda_1 + 8\lambda_3 = 0 \rightarrow \boxed{\lambda_1 = -8\lambda_3}$

$-\lambda_3 = -16\lambda_1 + 1$

$\cancel{\lambda_3 = -128\lambda_1}$

$-\lambda_3 = -16\lambda_3 + 1$

$\cancel{-15\lambda_3 = -1}$ $\lambda_3 = \frac{1}{15}$

$\cancel{\lambda_2 = -8\lambda_3}$

$\cancel{\lambda_1 = -8\lambda_3}$

e' $+\frac{16}{15} - \frac{1}{15} = +1$ ✓

$-\frac{2}{15} + \frac{2}{15} = 0$ ✓

$-\frac{8}{15} + \frac{6}{15} + \frac{2}{15} = 0$ ✓

$\boxed{\lambda_3 = \frac{1}{15}}$ $\lambda_2 = -\frac{2}{15}$ $\lambda_1 = -\frac{8}{15}$

$b = (0,1,0)$

$$\lambda_1 \cdot \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix} + \lambda_2 \cdot \begin{pmatrix} 0 \\ 1 \\ -3 \end{pmatrix} + \lambda_3 \begin{pmatrix} -1 \\ 2 \\ 2 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

$$-2\lambda_1 + \quad - \lambda_3 = 0 \rightarrow \boxed{\lambda_3 = -2\lambda_1 = \frac{2}{5}}$$

$$\lambda_2 + 2\lambda_3 = 1$$

$$\lambda_1 - 3 \cdot \lambda_2 + 2\lambda_3 = 0$$

$$\lambda_2 - 4\lambda_1 = 1 \rightarrow \lambda_2 = 1 + 4\lambda_1 = 1 - \frac{4}{5} =$$

$$\lambda_1 - 3\lambda_2 - 4\lambda_1 = 0 \qquad \boxed{\lambda_2 = \frac{1}{5}}$$

$$\lambda_1 - 3 - 12\lambda_1 - 4\lambda_1 = 0$$

$$-15\lambda_1 = 3 \rightarrow \boxed{\lambda_1 = -\frac{1}{5}}$$

P!
$$\frac{2}{5} - \frac{2}{5} = 0 \checkmark$$

$$\frac{1}{5} + \frac{4}{5} = 1 \checkmark$$

$$-\frac{1}{5} - \frac{3}{5} + \frac{4}{5} = 0 \checkmark$$

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6 folytatás

$$\lambda_1 \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix} + \lambda_2 \begin{pmatrix} 0 \\ 1 \\ -3 \end{pmatrix} + \lambda_3 \cdot \begin{pmatrix} -1 \\ 2 \\ 2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$-2\lambda_1 - \lambda_3 = 0 \rightarrow \boxed{\lambda_3 = -2\lambda_1 = \frac{2}{15}}$$

$$\lambda_2 + 2\lambda_3 = 0$$

$$\lambda_1 - 3\lambda_2 + 2\lambda_3 = 1$$

$$\lambda_2 - 4\lambda_1 = 0 \rightarrow \boxed{\lambda_2 = 4\lambda_1 = -\frac{4}{15}}$$

$$\lambda_1 - 3\lambda_2 - 4\lambda_1 = 1$$

$$-3\lambda_1 - 3\lambda_2 = 1$$

$$\cancel{-3\lambda_1} \quad -3\lambda_1 - 12\lambda_1 = 1$$

$$-15\lambda_1 = 1$$

$$\boxed{\lambda_1 = -\frac{1}{15}}$$

E:

$$\frac{2}{15} - \frac{2}{15} = 0 \checkmark$$

$$-\frac{4}{15} + \frac{4}{15} = 0 \checkmark$$

$$-\frac{1}{15} + \frac{12}{15} + \frac{4}{15} = 1 \checkmark$$

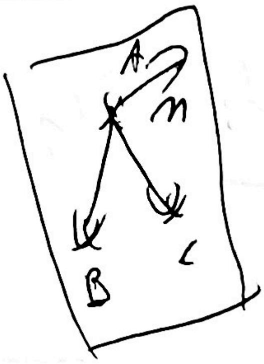
(7) $A(0,1,1)$
$B(1,0,1)$
$C(1,1,0)$

$S_{ABC} = ?$

$\vec{AB}(+1,-1,0)$

$\vec{AC}(1,0,-1)$

$$\vec{n} = \vec{AB} \times \vec{AC} = \begin{pmatrix}1\\-1\\0\end{pmatrix} \times \begin{pmatrix}1\\0\\-1\end{pmatrix} = \begin{pmatrix}(-1)(-1)-0\cdot0\\-1(-1)-0\cdot1\\1\cdot0-(-1)1\end{pmatrix} = \begin{pmatrix}\cancel{1}\\\cancel{+1}\\\cancel{1}\end{pmatrix} \begin{pmatrix}1\\1\\1\end{pmatrix}$$



$n_1(x-x_0)+n_2(y-y_0)+n_3(z-z_0)=0$

$S:$ ~~...~~ $\cancel{n\,x = n\,x_0}$ $\cancel{n_1x+n_2y+n_3z=}$

$\boxed{x+y+z=2}$

$\underline{n}\,\underline{x} = \underline{n}\,\underline{x_0}$

$(1,1,1)(x,y,z) = (1,1,1)(0,1,1)$

$\boxed{x+y+z=2}$

(9a) I $f(\underline{a}+\underline{b}) = \begin{pmatrix}\cancel{a_1+2a_2+a_1b_1+2b_2}\\\cancel{2a_1+a_2+2b_1+b_2}\\3a\end{pmatrix} = \begin{pmatrix}a_1+b_1+2(a_2+b_2)\\2(a_1+b_1)+(a_2+b_2)\\3(a_3+b_3)+3(a_2+b_2)\\a_3+b_3\end{pmatrix} =$

$$= \begin{pmatrix}(a_1+2a_2)+b_1+2b_2\\(2a_1+a_2)+(2b_1+b_2)\\(3a_3+3a_2)+(3b_3+3b_2)\\a_3+b_3\end{pmatrix} = \begin{pmatrix}a_1+2a_2\\2a_1+a_2\\3a_3+3a_2\\a_3\end{pmatrix} + \begin{pmatrix}b_1+2b_2\\2b_1+b_2\\3b_3+3b_2\\b_3\end{pmatrix} = f(\underline{a})+f(\underline{b})$$

(II) ~~$\lambda f(\underline{a})$~~
$f(\lambda\underline{a}) = \begin{pmatrix}\lambda a_1+2\lambda a_2\\2\lambda a_1+\lambda a_2\\3\lambda a_3+3\lambda a_2\\\ldots\end{pmatrix} = \lambda\cdot\begin{pmatrix}a_1+2a_2\\2a_1+a_2\\3a_3+3a_2\\a_3\end{pmatrix} = \lambda\cdot f(a)$ lin

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9a) folytatás

$$\begin{pmatrix} 1 & 2 & 0 \\ 1 & 1 & 0 \\ 0 & 3 & 3 \\ 0 & 0 & 1 \end{pmatrix}$$ a transzformációs mátrix

9b)

$$(a_1+b_1)(a_2+b_2) \to a_1 a_2 + a_1 b_2 + b_1 a_2 + b_1 b_2$$

$$g(\underline{a}+\underline{b}) = \begin{pmatrix} 2a_1 + 2b_1 \\ (a_1+b_1)(a_2+b_2) \\ 3a_2 + 3b_2 \\ a_1 + b_1 \end{pmatrix} = \begin{pmatrix} 2a_1 \\ a_1 a_2 \\ 3a_2 \\ a_1 \end{pmatrix} + \begin{pmatrix} 2b_1 \\ b_1 b_2 \\ 3b_2 \\ b_1 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ a_1 b_2 + b_1 a_2 \\ 0 \end{pmatrix}$$

$\neq$ lineáris tag

$\Downarrow$

$\neq$ lineáris transzformáció

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8)

$$\underline{A}\,\underline{x} = \lambda \underline{x}$$
$$(\underline{A} - \lambda \underline{E})\underline{x} = 0$$
$$\det(\underline{A} - \lambda \underline{E}) = 0$$
$$\begin{vmatrix} 3-\lambda & 1 \\ 1 & 3-\lambda \end{vmatrix} = 0$$
$$(3-\lambda)^2 - 1 = 0$$
$$\lambda^2 - 6\lambda + 8 = 0$$
$$(\lambda-2)(\lambda-4) = 0$$

$\lambda_1 = 2$

$$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$
$$x_1 = -x_2$$
$$\boxed{\alpha_{\lambda_1=2} = \left\{ \begin{pmatrix} t \\ -t \end{pmatrix} \middle| t \in \mathbb{R} \right\}} \to \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$\lambda_2 = 4$

$$\begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$
$$x_1 = x_2$$
$$\boxed{\alpha_{\lambda_2=4} = \left\{ \begin{pmatrix} t \\ t \end{pmatrix} \middle| t \in \mathbb{R} \right\}}$$
$$\downarrow$$
$$\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$C = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \to |C| = 2 \neq 0 \Rightarrow \exists\, C^{-1}$

$\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \xrightarrow[\text{al det}]{} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \xrightarrow[\text{adj}]{} \begin{pmatrix} 1 & +1 \\ -1 & 1 \end{pmatrix} \xrightarrow[\text{trasp}]{} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \to C^{-1} = \frac{1}{2}\begin{pmatrix} 1 & - \\ 1 & 1 \end{pmatrix}$

$e:\ C \cdot C^{-1} = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} = \underline{\underline{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}}}$ ✓

$C^{-1} \cdot A \cdot C = \frac{1}{2}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix} C = \frac{1}{2}\begin{pmatrix} 2 & -2 \\ 4 & 4 \end{pmatrix} \cdot \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} =$

$= \frac{1}{2}\begin{pmatrix} 4 & 0 \\ 0 & 8 \end{pmatrix} = \underline{\underline{\begin{pmatrix} 2 & 0 \\ 0 & 4 \end{pmatrix}}}$

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(10) $\underline{e_1'} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} \quad \boxed{e_1 = \frac{e_1'}{|e_1'|} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}}$ ✓

$e_2' = b_2 - (b_2 \cdot e_1)\, e_1 = \begin{pmatrix} 3 \\ 0 \\ 0 \\ 1 \end{pmatrix} - \left[\begin{pmatrix} 3 \\ 0 \\ 0 \\ 1 \end{pmatrix} \cdot \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}\right] \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} =$

$= \begin{pmatrix} 3 \\ 0 \\ 0 \\ 1 \end{pmatrix} - \frac{1}{2} \cdot 4 \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 0 \\ 0 \\ 1 \end{pmatrix} - \begin{pmatrix} 2 \\ 0 \\ 0 \\ 2 \end{pmatrix} = \underline{\begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix}}$

$\boxed{e_2 = \frac{e_2'}{|e_2'|} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix}}$ ✓

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(10) folytatás

$$e_3' = b_3 - (b_3 \cdot e_1) e_1 - (b_3 \cdot e_2) e_2 = \begin{pmatrix} 2 \\ 1 \\ 1 \\ 3 \end{pmatrix} - \left[\begin{pmatrix} 2 \\ 1 \\ 1 \\ 3 \end{pmatrix} \cdot \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right] \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

$$- \left[\begin{pmatrix} 2 \\ 1 \\ 1 \\ 3 \end{pmatrix} \cdot \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix} \right] \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 1 \\ 3 \end{pmatrix} - \frac{1}{2} \cdot 5 \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} - \frac{1}{2} \cdot (-1) \begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix} =$$

$$= \begin{pmatrix} 2 \\ 1 \\ 1 \\ 3 \end{pmatrix} - \begin{pmatrix} \frac{5}{2} \\ 0 \\ 0 \\ \frac{5}{2} \end{pmatrix} + \begin{pmatrix} \frac{1}{2} \\ 0 \\ 0 \\ -\frac{1}{2} \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}$$

$$\boxed{e_3 = \frac{e_3'}{|e_3'|} = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}} \checkmark$$

$$e_4' = b_4 - (b_4 \cdot e_1) \cdot e_1 - (b_4 \cdot e_2) \cdot e_2 - (b_4 \cdot e_3) \cdot e_3 =$$

$$= \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} - \left[\begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} \cdot \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right] \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} - \left[\begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} \cdot \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix} \right] \frac{1}{\sqrt{2}} \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix} - \frac{1}{\sqrt{2}} \left[\begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} \cdot \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix} \right] \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} - \frac{1}{2} \cdot 1 \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} - \frac{1}{2} \cdot 1 \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix} - \frac{1}{2} \cdot 1 \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix} =$$

$$= \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} - \begin{pmatrix} \frac{1}{2} \\ 0 \\ 0 \\ \frac{1}{2} \end{pmatrix} - \begin{pmatrix} \frac{1}{2} \\ 0 \\ 0 \\ -\frac{1}{2} \end{pmatrix} - \begin{pmatrix} 0 \\ 1/2 \\ 1/2 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ \frac{1}{2} \\ -1/2 \\ 0 \end{pmatrix}$$

$$\boxed{e_4 = \frac{e_4'}{|e_4'|}} = \cancel{\begin{pmatrix} 0 \\ 0 \\ -1 \\ 0 \end{pmatrix}} \qquad \frac{\cancel{\frac{1}{2}}(0, 1, -1, 0)}{\cancel{\frac{1}{2}}\sqrt{2}} = \boxed{\frac{\sqrt{2}}{2} \begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \end{pmatrix}} \checkmark$$
