LU decomposition

$\underline{A} = \underline{L} \cdot \underline{U}$ $\underline{L}$: lower $\triangle$

$\underline{U}$: upper $\triangle$

- $\nexists$ $\forall$ matrix
- $\neg$ unique
- $\exists \underline{A}^{-1}$ & $\underline{A} = \underline{L} \cdot \underline{U}$ exist $\Rightarrow$ $\underline{L}$ & $\underline{U}$ are unique

$\underline{U} = \underline{E}_3 \cdot \underline{E}_2 \cdot \underline{E}_1 \cdot \underline{A} \Rightarrow \underline{L} = \underline{E}_1^{-1} \cdot \underline{E}_2^{-1} \cdot \underline{E}_3^{-1}$

$$\begin{pmatrix} 4 & 8 & 4 & 8 \\ 2 & 6 & 4 & 4 \\ 1 & 3 & 2 & 4 \end{pmatrix} = \begin{pmatrix} 4 & 8 & 4 & 8 \\ 0 & 2 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 \\ \frac{1}{2} & 1 & 0 \\ \frac{1}{4} & \frac{1}{2} & 1 \end{pmatrix}$$

Example:

$$\underline{A} = \begin{pmatrix} 4 & 8 & 4 & 8 \\ 2 & 6 & 4 & 4 \\ 1 & 3 & 2 & 4 \end{pmatrix} \xrightarrow[(II) - \frac{1}{2}(I)]{} E_1 = \begin{pmatrix} 1 & 0 & 0 \\ -\frac{1}{2} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad \begin{pmatrix} 4 & 8 & 4 & 8 \\ 0 & 2 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix} = \underline{U}$$

$$\begin{pmatrix} 4 & 8 & 4 & 8 \\ 0 & 2 & 2 & 0 \\ 1 & 3 & 2 & 4 \end{pmatrix} \xrightarrow[(III) - \frac{1}{4}(I)]{} \underline{E_2} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -\frac{1}{4} & 0 & 1 \end{pmatrix} \qquad \begin{pmatrix} 1 & 0 & 0 \\ -\frac{1}{2} & 1 & 0 \\ -\frac{1}{4} & -\frac{1}{2} & 1 \end{pmatrix}^{-1} = \underline{L}$$

$$\begin{pmatrix} 4 & 8 & 4 & 8 \\ 0 & 2 & 2 & 0 \\ 0 & 1 & 1 & 2 \end{pmatrix} \xrightarrow[(III) - \frac{1}{2}(II)]{} \underline{E_3} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -\frac{1}{2} & 1 \end{pmatrix}$$

$$\underline{U} = \underbrace{\underline{E}_3 \cdot \underline{E}_2 \cdot \underline{E}_1}_{\underline{L}^{-1}} \underline{A} \quad / \; \underline{L} \cdot$$

$$\underline{A} = \underline{L} \cdot \underline{U} \quad ; \quad \underline{L} = \left(\underline{E}_3 \cdot \underline{E}_2 \cdot \underline{E}_1\right)^{-1} =$$

$$= \underline{E}_1^{-1} \cdot \underline{E}_2^{-1} \cdot \underline{E}_3^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ \frac{1}{2} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \frac{1}{4} & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & \frac{1}{2} & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ \frac{1}{2} & 1 & 0 \\ \frac{1}{4} & \frac{1}{2} & 1 \end{pmatrix}$$

$$\underline{B} = \begin{pmatrix} 4 & 8 & 8 \\ 2 & 6 & 4 \\ 1 & 3 & 4 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ \frac{1}{2} & 1 & 0 \\ \frac{1}{4} & \frac{1}{2} & 1 \end{pmatrix} \begin{pmatrix} 4 & 8 & 8 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

$$\begin{cases} 4x_1 + 8x_2 + 8x_3 = 8 \\ 2x_1 + 6x_2 + 4x_3 = 4 \\ x_1 + 3x_2 + 4x_3 = 4 \end{cases}$$

$$\begin{pmatrix} 1 & 0 & 0 \\ \frac{1}{2} & 1 & 0 \\ \frac{1}{4} & \frac{1}{2} & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} 8 \\ 4 \\ 4 \end{pmatrix}$$

$\to y_1 = 8$

$\frac{1}{2} y_1 + y_2 = 4 \to y_2 = 0$

$\frac{1}{4} y_1 + \frac{1}{2} y_2 + y_3 = 4 \to y_3 = 2$

$$\begin{pmatrix} 4 & 8 & 8 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 8 \\ 0 \\ 2 \end{pmatrix}$$

$2x_3 = 2 \to \underline{x_3 = 1}$

$2x_2 = 0 \to \underline{x_2 = 0}$

$4x_1 + 8x_2 + 8x_3 = 8$

$4x_1 + 0 + 8 = 8$

$\underline{x_1 = 0}$

$$\underline{x} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 8 \\ 0 \\ 2 \end{pmatrix}$$

$x_3 = 1$

$\to x_2 = 0$

$= 8$

$= 8$

$$\underline{A}\,\underline{x} = \underline{b}$$

$$\underline{L} \cdot \underline{U}\,\underline{x} = \underline{b} \quad \to \quad \boxed{\underline{L}\,\underline{y} = \underline{b}}$$

$$\underline{U}\,\underline{x} = \underline{y}$$

$$\underline{B} = \begin{pmatrix} 4 & 8 & 8 \\ 2 & 6 & 4 \\ 1 & 3 & 4 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ \frac{1}{2} & 1 & 0 \\ \frac{1}{4} & \frac{1}{2} & 1 \end{pmatrix} \cdot \begin{pmatrix} 4 & 8 & 8 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

$$y_{11} = 1$$

$$\frac{1}{2} y_{12} + x_{22} = 1$$

$$\frac{1}{4} x_{13} + \frac{1}{2} x_{23} + x_{33} = 1$$

$$\begin{pmatrix} 1 & 0 & 0 \\ \frac{1}{2} & 1 & 0 \\ \frac{1}{4} & \frac{1}{2} & 1 \end{pmatrix} \begin{pmatrix} y_{11} & y_{12} & y_{13} \\ y_{21} & x_{22} & x_{23} \\ y_{31} & x_{32} & x_{33} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} y_{11} & y_{12} & y_{13} \\ \frac{1}{2} y_{11} + y_{21} & \frac{1}{2} x_{12} + x_{22} & \frac{1}{2} y_{13} + x_{23} \\ \frac{1}{4} y_{11} + \frac{1}{2} y_{21} + x_{31} & \frac{1}{4} x_{12} + \frac{1}{2} y_{22} + y_{33} & \frac{1}{4} y_{13} + \frac{1}{2} y_{23} + y_{33} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$x_{33} = 1$

$$\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$\underline{A} \cdot \overset{= \underline{A}^{-1}}{\underline{X}} = \underline{E}$$

$$\underline{L} \cdot \underline{U} \cdot \underline{X} = \underline{E} \quad \Leftrightarrow \quad \begin{array}{l} \underline{L}\,\underline{Y} = \underline{E} \\ \underline{U}\,\underline{X} = \underline{Y} \end{array}$$

$$\underline{L} = \begin{pmatrix} 1 & 0 & 0 \\ -\frac{1}{3} & 1 & 0 \\ 0 & -\frac{1}{2} & 1 \end{pmatrix}$$

$$\underline{U}\,\underline{X} = \underline{Y}$$

$$\underline{Y} = \begin{pmatrix} \frac{3}{4} & -\frac{1}{2} & -1 \\ -\frac{1}{4} & \frac{1}{2} & 0 \\ 0 & \frac{1}{4} & \frac{1}{2} \end{pmatrix}$$