(ENKEMNA0302) Applied Linear Algebra

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Requirements

- You will write tests based on the exercises of the practical courses. You can use everything during the test
- The minimum requirement is 41 % of both tests.
- Failed tests must be corrected
- You must take an ~~oral~~ written exam. You cannot use anything
- Grades: Insufficient/Fail (1): 0-40 %, Sufficient/Pass (2): 41-55 %, Average (3): 56-70 %, Good (4): 71-85 %, Excellent (5): 86-100 %.
- Mid-term test 1: March 13, mid-term test 2: May 8, retake tests: May 15, 2025.

Bibliography

Bernard Kolman and David Hill: Elementary Linear Algebra with Applications, 9th ed., Person, 2007

Philip N. Klein: Coding the Matrix: Linear Algebra through Applications to Computer Science, Newtonian Press 2013

K. F. Riley, M. P. Hobson, S. J. Bence: Mathematical Methods for Physics and Engineering: A Comprehensive Guide, Cambridge University Press, 3rd. ed. (2006)

Determinant of Square Matrixes I

- Definition: Be $\alpha_{ij} \in \mathbb{R}$ for all $i \in \{1, 2, \ldots, m\}$ and $j \in \{1, 2, \ldots, n\}$, where $n, m \in \mathbb{N}$, $m > 1$ and $n > 1$. The Table

$$A = \begin{pmatrix} \alpha_{11} & \alpha_{12} & \cdots & \alpha_{1n} \\ \alpha_{21} & \alpha_{22} & \cdots & \alpha_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_{m1} & \alpha_{m2} & \cdots & \alpha_{mn} \end{pmatrix}$$

  is a $m \times n$ type matrix. The set of $m \times n$ type matrixes is $M_{m \times n}$.
- The diagonal of Matrix A is the set of $\{\alpha_{11}, \alpha_{22}, \ldots, \alpha_{nn}\}$.
- The indexes of α_{ij} element are the rowindex (i), and the columnindex (j).
- The Row i of the Matrix A is A_i, the Column j is A_j.

Determinant of Square Matrixes II

- Definition: If Matrix A is type $n \times n$, where $n > 1$ and $n \in \mathbb{N}$ (Square Matrix), then the determinant of Matrix A is the following number:

$$det\left(A\right) = \sum_{\{i_1,i_2,\ldots,i_n\}\in P_n} (-1)^{I(i_1,i_2,\ldots,i_n)}\,\alpha_{1i_1}\cdot\alpha_{2i_2}\cdot\cdots\cdot\alpha_{ni_n},$$

where the summary is for al the permutations of $1, 2, \ldots, n$ numbers, and $I\left(i_1, i_2, \ldots, i_n\right)$ means the number of inversions in the permutation $\left(i_1, i_2, \ldots, i_n\right)$:

$$\det\left(A\right), \quad \begin{vmatrix} \alpha_{11} & \alpha_{12} & \cdots & \alpha_{1n} \\ \alpha_{21} & \alpha_{22} & \cdots & \alpha_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_{m1} & \alpha_{m2} & \cdots & \alpha_{mn} \end{vmatrix}, \quad \left|A\right|.$$

Determinant of Square Matrixes III

- Calculation of a determinant using Laplace (or cofactor) expansion. Chess Board Rule:

$$\begin{vmatrix} + & - & + & \cdots \\ - & + & - & \cdots \\ + & - & + & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{vmatrix}$$

Go through the elements of the first row. You must create a subdeterminant by deleting the first row and the actual column. The original determinant is the sum of the multiplication of the elements of the first row and the new subdeterminant. The sign of the multiplications is taken from the Chess Board Rule.

$$\begin{vmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \end{vmatrix} = \alpha_{11} \cdot \begin{vmatrix} \alpha_{22} & \alpha_{23} \\ \alpha_{32} & \alpha_{33} \end{vmatrix} - \alpha_{12} \cdot \begin{vmatrix} \alpha_{21} & \alpha_{23} \\ \alpha_{31} & \alpha_{33} \end{vmatrix} + \alpha_{13} \cdot \begin{vmatrix} \alpha_{21} & \alpha_{22} \\ \alpha_{31} & \alpha_{32} \end{vmatrix} = =$$

$$\alpha_{11} \cdot \alpha_{22} \cdot \alpha_{33} - \alpha_{11} \cdot \alpha_{23} \cdot \alpha_{32} - \alpha_{12} \cdot \alpha_{21} \cdot \alpha_{33} + \alpha_{12} \cdot \alpha_{23} \cdot \alpha_{31} + \alpha_{13} \cdot \alpha_{21} \cdot \alpha_{32} - \alpha_{13} \cdot \alpha_{22} \cdot \alpha_{31}$$

Determinant of Square Matrixes IV

- Small matrixes could be calculated using the Sarrus rule. For 2×2 matrixes:

$$\begin{vmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{vmatrix} = \alpha_{11} \cdot \alpha_{22} - \alpha_{12} \cdot \alpha_{21}$$

  The elements of the main diagonal are multiplied, then the elements of the other diagonal are multiplied and subtracted from the previous multiplication.

  For 3×3 matrixes:

$$\begin{vmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \end{vmatrix} =$$

$$\alpha_{11} \cdot \alpha_{22} \cdot \alpha_{33} + \alpha_{12} \cdot \alpha_{23} \cdot \alpha_{31} + \alpha_{13} \cdot \alpha_{21} \cdot \alpha_{32} - \alpha_{13} \cdot \alpha_{22} \cdot \alpha_{31} - \alpha_{12} \cdot \alpha_{21} \cdot \alpha_{33} - \alpha_{23} \cdot \alpha_{32} \cdot \alpha_{11}$$

  These rules could be deducted using the Laplace expansion. (See the previous page for both cases.)

Determinant of Square Matrixes V

- <u>Thesis:</u> $\mathbf{a} \times \mathbf{b} = (a_2 b_3 - a_3 b_2) \cdot \mathbf{e}_1 + (a_3 b_1 - a_1 b_3) \cdot \mathbf{e}_2 + (a_1 b_2 - a_2 b_1) \cdot \mathbf{e}_3 =$
$\begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} \cdot \mathbf{e}_1 + \begin{vmatrix} a_3 & a_1 \\ b_3 & b_1 \end{vmatrix} \cdot \mathbf{e}_2 + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \cdot \mathbf{e}_3 = \begin{vmatrix} e_1 & e_2 & e_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$
- <u>Thesis:</u> The Triple Product could be written in the determinant form:

$$(\mathbf{a}, \mathbf{b}, \mathbf{c}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

- The determinant above gives the volume for the Paralepipedon generated by **a**, **b**, **c** vectors.

Determinant of Square Matrixes VI

- Some elementary features of determinants:
  - If you change the order of two rows (or columns), then the determinant of the new matrix will be the opposite of the determinant of the original matrix: $det\,(A) = -det\,(A)$.
  - If two rows (or columns) of a matrix are the same, its determinant is zero.
  - If a row (or column) of Matrix A is multiplied by α number, then the determinant of the new matrix will be $\alpha \cdot det\,(A)$.
  - If Matrixes A, B, C differ only in Row i (or Column i) as follows: $\mathbf{c}_i = \mathbf{a}_i + \mathbf{b}_i$. Then $det\,(C) = det\,(A) + det\,(B)$.
  - If you add a row (or column) of Matrix A its other row (or column) multiplied by a constant number, the determinant of the new matrix will be the same of the determinant of the original matrix.

These features will be useful for calculating the determinants.

Determinant of Square Matrixes VII

- Thesis: (Laplace (or cofactor) expansion) The D_{ij} the determinant of the $(n-1) \times (n-1)$-es matrix creating by deleting the row and column contains α_{ij}. This determinant is the subdeterminant of the A matrix bound to the α_{ij} element. The subdeterminant bonded to the α_{ij} element if A matrix is:

$$A_{ij} = (-1)^{i+j} D_{ij}.$$

Thesis: (Laplace expansion by row i): For all $i \in \{1, 2, \dot{;} n\}$

$$det(A) = \sum_{j=1}^{n} \alpha_{ij} A_{ij},$$

where $n \in \mathbb{N}$ and $n > 1$.

Determinant of Square Matrixes VIII

Thesis: (Laplace expansion by column j): For all $j \in \{1, 2, ;n\}$

$$det(A) = \sum_{i=1}^{n} \alpha_{ij} A_{ij},$$

where $n \in \mathbb{N}$ and $n > 1$.

Gaussian elimination

- Definition: Matrix $A = (\alpha_{ij})_{n \times n}$ is upper triangulat matrix if the elements under the main diagonal are zero.
- Thesis: The determinant of an upper triangular matrix is the multiplication of the elements of the main diagonal of the matrix.
- The purpose of the Gaussian elimination is to convert the matrix to an upper triangular matrix, whose determinants are the same as the determinant of the original matrix.
  1. If it is necessary, set $\alpha_{11} \neq 0$ by changing rows. (The sign for the determinant will be changed if you change rows.)
  2. Adding the first row multiplied by a suitable constant you get $\alpha_{21}, \alpha_{32}, \ldots, \alpha_{n1} = 0$.
  3. If it is necessary set $\alpha_{22} \neq 0$ by changing rows.
  4. Adding the 2nd times constant to the rows $3, 4, \ldots, n$ you get $\alpha_{32}, \alpha_{42}, \ldots, \alpha_{n2} = 0$.

  Proceed until all elements are zero under the main diagonal.

Special marixes I

- Diagonal matrixes: it is simple to do operations with them.
  Here $\mathbf{A} = diag\,(1,2,3)$ és $\mathbf{B} = diag\,(5,4,3)$. Then:

$$\mathbf{AB} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix} \begin{pmatrix} 5 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 3 \end{pmatrix} = \begin{pmatrix} 5 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & 9 \end{pmatrix},$$

$$\mathbf{A}^2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}^2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 9 \end{pmatrix}, \mathbf{A}^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{3} \end{pmatrix}$$

$$\mathbf{A}^k = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}^k = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2^k & 0 \\ 0 & 0 & 3^k \end{pmatrix}, \text{where } k \in \mathbb{Z}.$$

Special marixes II

<u>Thesis:</u> (Operations with diagonal matrixes) Here $\mathbf{A} = diag\,(a_1, a_2, \dots, a_n)$, $\mathbf{B} = diag\,(b_1, b_2, \dots, b_n)$ and $k \in \mathbb{Z}$. Then

1. $\mathbf{AB} = diag\,(a_1 b_1, a_2 b_2, \dots, a_n b_n)$,
2. $\mathbf{A}^k = diag\,\left(a_1^k, a_2^k, \dots, a_n^k\right)$, specialy
3. $\mathbf{A}^{-1} = diag\,\left(a_1^{-1}, a_2^{-1}, \dots, a_n^{-1}\right)$.

The (3) and if $k < 0$ the (2) operations can be done if $a_i \neq 0$, where $i = 1, 2, \dots, n$.

- Comming soon...
  - Permutation matrixes and snakes
  - Triangular matrixes
  - Symmetric and skew-symmetric matrixes

The End

Thank you for your attention!