(PTIA0301) Elementary Linear Algebra

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Vector Space I

- <u>Definition:</u> The set $V \neq \emptyset$ a vector space above $\mathbb{R}$ set, if there isan "$+$" operation with the following features:

$$
\begin{aligned}
\mathbf{a} + \mathbf{b} &= \mathbf{b} + \mathbf{a}, \text{where} \left(\mathbf{a}, \mathbf{b} \in V\right) \\
\left(\mathbf{a} + \mathbf{b}\right) + \mathbf{c} &= \mathbf{a} + \left(\mathbf{b} + \mathbf{c}\right), \text{where} \left(\mathbf{a}, \mathbf{b}, \mathbf{c} \in V\right) \\
\exists \mathbf{0} \in V \ \text{ that } \ \mathbf{a} + \mathbf{0} &= \mathbf{a} \ \ \forall \mathbf{a} \in V \\
\forall \mathbf{a} \in V \exists \left(-\mathbf{a}\right) \in V : \mathbf{a} + \left(-\mathbf{a}\right) &= \mathbf{0},
\end{aligned}
$$

Vector Space II

furthermore, for all $\lambda \in \mathbb{R}$ and for all $\mathbf{a} \in V$ the term $\lambda\mathbf{a} \in V$ exists, and the following features are fullfilled:

$$\begin{aligned}
\lambda\left(\mu\mathbf{a}\right) &= \left(\lambda\mu\right)\mathbf{a}, \text{where}\left(\mathbf{a} \in V, \lambda, \mu \in \mathbb{R}\right) \\
\lambda\left(\mathbf{a}+\mathbf{b}\right) &= \lambda\mathbf{a}+\lambda\mathbf{b}, \text{where}\left(\mathbf{a}, \mathbf{b} \in V, \lambda \in \mathbb{R}\right) \\
\left(\lambda+\mu\right)\mathbf{a} &= \lambda\mathbf{a}+\mu\mathbf{a}, \text{where}\left(\mathbf{a} \in V \text{ és}\lambda, \mu \in \mathbb{R}\right) \\
\forall\mathbf{a} \in V\text{--re}1 \cdot \mathbf{a} &= \mathbf{a}.
\end{aligned}$$

These features are true for the V^2 and V^3 set, however the $\mathbb{R}^n = \mathbb{R} \times \mathbb{R} \times \cdots \times \mathbb{R}$ seat are vector space too. Furtheremore, the set of the maximum n^{th} ordered polynomes $\left(R_n\left[x\right]\right)$ are vector spaces too.

Vector Space III

- Definition: The V vector space's non empty subset L is a linear subspace if L also is a vector space with the operations in V.
- Thesis: The V vector space's L non-empty set is a linear subspace if and only if the following conditions are fulfilled:

$$\begin{aligned} \forall \mathbf{a}, \mathbf{b} \in L \quad &\text{esetén} \quad \mathbf{a} + \mathbf{b} \in L \\ \forall \lambda \in \mathbb{R}, \forall \mathbf{a} \in L \quad &\text{esetén} \quad \lambda \mathbf{a} \in L. \end{aligned}$$

- Definition: $H \neq \emptyset$ is subset of the V vector space. The subspace generated by H is the minimal size of such a subset of V that contains H. (Or it is a subset of all subspace containing H.) Sign: $\mathcal{L}(H)$.
- Alway exists a minimal subspace containing H that is the intersection of all subspaces containing H.

Vector Space IV

- Definition: Set H is a generator system of V vector space if $\mathcal{L}(H) = V$.
- Definition: V vector space is generated finitely if it has a generator system containing a finite number of elements.
- Comment: The $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ vector system is a generator system of a finitely generated V vector system if all elements of the V set could be written as the linear combination of $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ vectors.
- Definition: The linear independent genarator system of the V vector space is a basis of the V vector space.
- Tétel: In a finitely generated vector space all bases have the same cardinality.
- Definition: The common element number of the basis of a vector space is called dimension. Sign: $dim(V)$.

Vector Space V

- Definition: $\{\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_n\}$ is a basis in V and $\mathbf{a} \in V$. Those $\lambda_1, \lambda_2, \ldots, \lambda_n$ numbers for which $\mathbf{a} = \lambda_1 \mathbf{b}_1 + \lambda_2 \mathbf{b}_2 + \cdots + \lambda_n \mathbf{b}_n$ we call coordinates the $\mathbf{a}$ vector for the $\{\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_n\}$ basis.
- Definition: If $\mathbf{v} \in V^2 \setminus \{\mathbf{0}\}$, or $\mathbf{v} \in V^3 \setminus \{\mathbf{0}\}$. Then the $l = \{\alpha \mathbf{v} : \alpha \in \mathbb{R}\}$ set is a line (crossing the origo).
- Definition: If $\mathbf{u}, \mathbf{v} \in V^3 \setminus \{\mathbf{0}\}$ és $\nexists \lambda \in \mathbb{R}$, so $\mathbf{u} = \lambda \mathbf{v}$. Then the $L = \{\alpha \mathbf{u} + \beta \mathbf{v} : \alpha, \beta \in \mathbb{R}\}$ set a plane (crossing the origo).
- The set $L = \{\alpha_1 \mathbf{u}_1 + \alpha_2 \mathbf{u}_2 + \cdots + \alpha_m \mathbf{u}_m : \alpha_1, \ldots, \alpha_m \in \mathbb{R}\}$ is just the linear subspace generated by $\{\mathbf{u}_1, \ldots, \mathbf{u}_m\}$. If $\mathbf{u}_1, \ldots, \mathbf{u}_m$ are linear independent, then the set L is am mD subspace- Then the set $K = \{\alpha_1 \mathbf{u}_1 + \alpha_2 \mathbf{u}_2 + \cdots + \alpha_m \mathbf{u}_m + \mathbf{v} : \alpha_1, \ldots, \alpha_m, \in \mathbb{R}\}$ is an mD affin subspace. All affine subspace can be constructed in the $K = L + \mathbf{v}$ form, where L is a linear subspace and $\mathbf{v} \in V$.

Vector Space VI

- The equation of a plane which has the $\mathbf{n}\left(n_1, n_2, n_3\right)$ normal vector and crossed the point $P\left(p_1, p_2, p_3\right)$ is $n_1x + n_2y + n_3z = n_1p_1 + n_2p_2 + n_3p_3$.
- <u>Definition:</u> The (Minkowski-)sum of set is $A + B = \{a + b : a \in A, b \in B\}$.
- <u>Állítás:</u> The sum and intersection of subspaces are $L_1 + L_2$ and $L_1 \bigcap L_2$ is altér.
- <u>Definition:</u> The $L_1 + L_2$ is a direct sum if $L_1 \bigcap L_2 = \{\mathbf{0}\}$.

The End

Thank you for your attention!