(PTIA0301) Elementary Linear Algebra

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September 5, 2024

Course Objectives

- Terms and basic methods of Linear Algebra methodology
- Using elementary methods of Linear Algebra to fix simple problems
- Recognition when Linear Algebra could be used to solve problems

Requirements

- You will write tests based on the exercises of the practical courses.
- The minimum requirement is 40 % of both tests.
- I will offer a grade based on the test results for the colloquium if you get at least an Average (3) grade.
- If you do not like the offered grade, you can take a (written) exam.
- Grades: Insufficient/Fail (1): 0-40 %, Sufficient/Pass (2): 41-55 %, Average (3): 56-70 %, Good (4): 71-85 %, Excellent (5): 86-100 %.

Bibliography

Bernard Kolman and David Hill: Elementary Linear Algebra with Applications, 9th ed., Person, 2007

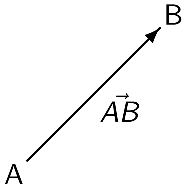
Henry Ricardo: A Modern Introduction to Linear Algebra, Taylor & Francis Group, LLC, 2010

Scalars and vectors

- Scalars are quantities without directions. For example mass (m), speed ($|\mathbf{v}|$, *not* velocity), temperature (T), length, volume (V).
- Vectors are quantities with direction. For example: weight ($\mathbf{F}$), velocity ($\mathbf{v}$), position ($\mathbf{r}$), acceleration/deceleration ($\mathbf{a}$), rotation, rotational speed (ω).
- Two types of vectors:
  - Weight ($\mathbf{F}$), velocity ($\mathbf{v}$), position ($\mathbf{r}$), acceleration/deceleration ($\mathbf{a}$).
  - Rotation, rotational speed (ω).
- You can write vectors in three different ways: boldface ($\mathbf{v}$), underline ($\underline{v}$), and arrow ($\vec{v}$)
- Weird association by **François Villon (1431– after 1463)**: *"Francis I am, which weighs me down, / born in Paris near Pontoise town, / and with a stretch of rope my pate / will learn for once my arse's weight."*

Other definitions for vectors I

- Limited length line segments from Point A to Point B: $\vec{AB}$. The start point is Point A, and the end is Point B.



- Two vectors are equal if you can transform the first vector to the second vector using parallel shift/translation displacement.
- Or, if the lengths and directions of the vectors are the same.
- The vectors you can transform to each other form the group of additive vectors.

Other definitions for vectors II

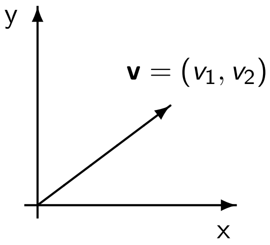
- Definition: Planar (V^2) or spatial vectors (V^3) are the group of those that you can transform to each other using translation (parallel displacement).

Coordinates of vectors I

- The pairs/triplets of numbers ($\mathbb{R}^2$, or $\mathbb{R}^3$) can identify as vectors:

$$\mathbf{v} = (v_1, v_2) = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix},$$

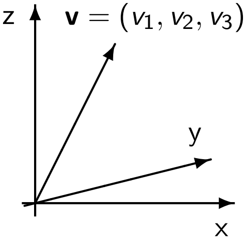
where $v_1 \in \mathbb{R}$, $v_2 \in \mathbb{R}$ are the components of the vector in 2D.



Coordinates of vectors II

$$\mathbf{v} = (v_1, v_2, v_3) = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix},$$

where $v_1 \in \mathbb{R}$, $v_2 \in \mathbb{R}$, $v_3 \in \mathbb{R}$ are the components of the vector in 3D.

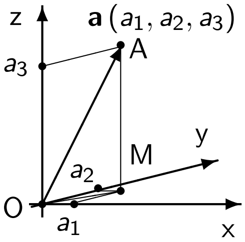


Length and equality of vectors I

- Definition: Two vectors are equal, and only equal if their origo centred representations are the same.
- In coordinates it means that $\mathbf{a}\,(a_1, a_2, a_3)$ and $\mathbf{b}\,(b_1, b_2, b_3)$ are equal, and only equal if $a_1 = b_1$, $a_2 = b_2$, and $a_3 = b_3$, where a_1, a_2, $a_3 \in \mathbb{R}$.

Length and equality of vectors II

- Thesis: The magnitude of the $\mathbf{a} = (a_1, a_2, a_3)$ vector is the following non-zero number:
$$|\mathbf{a}| = \|\mathbf{a}\| = \sqrt{a_1^2 + a_2^2 + a_3^2}.$$



Deduction: Points AOM form a right triangle, where the right angle is located at OMA$\angle$ angle. Appying the Pythagoras-thesis you got $|\mathbf{a}| = OM^2 + a_3^2$. Points O, $(a_1, 0, 0)$, $(0, a_2, 0)$ form a right triangle too, where the right angle is located at $[O, (a_1, 0, 0), (0, a_2, 0)]\angle$ angle according too the Pythagoras theorem: $OM^2 = a_1^2 + a_2^2$. After substituting the later equation into the first equation we got $|\mathbf{a}| = OM^2 = a_1^2 + a_2^2 + a_3^2$, therefore $|\mathbf{a}| = \sqrt{a_1^2 + a_2^2 + a_3^2}$. *q. e. d.*

Length and equality of vectors III

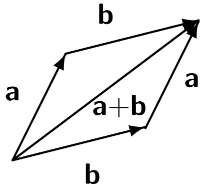
- The null vector has no length: $|\mathbf{0}\,(0,0,0)|=0$.

Multiply vector with a scalar, sum of vectors, and difference of vectors I

- Definition: Sum of vectors. If $\mathbf{a}\,(a_1, a_2, a_3)$ and $\mathbf{b}\,(b_1, b_2, b_3)$, then

$$\mathbf{a} + \mathbf{b} = (a_1 + b_1, a_2 + b_2, a_3 + b_3)\,,$$

where $a_1,\ a_2,\ a_3,,\ b_1,\ b_2,\ b_3 \in \mathbb{R}$.



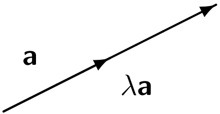
To add vectors $\mathbf{a}$ *and* $\mathbf{b}$ *you should transform vector* $\mathbf{b}$ *to the end point of the vector* $\mathbf{a}$*. Their sum* $(\mathbf{a} + \mathbf{b})$ *is the vector from the start point of* $\mathbf{a}$ *to the endpoint of the vector* $\mathbf{b}$*.*

Multiply vector with a scalar, sum of vectors, and difference of vectors II

- Definition: Multiplication of vectors with a scalar. If $\lambda \in \mathbb{R}$ and $\mathbf{a}\,(a_1, a_2, a_3)$, where a_1, a_2, $a_3 \in \mathbb{R}$, then

$$\lambda\mathbf{a} = (\lambda a_1, \lambda a_2, \lambda a_3)\,.$$

Mind if λ is 0, 1, -1, $<$1, or $>$1.



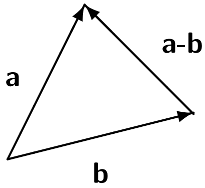
To multiply an **a** *vector by a* λ *number, you should draw a vector from the starting point of* **a** *vector in the direction of* **a** *that has* λ *length of the* **a** *vector.*

Multiply vector with a scalar, sum of vectors, and difference of vectors III

- Definition: Subtracktion of vectors. If $\mathbf{a}\,(a_1, a_2, a_3)$ and $\mathbf{b}\,(b_1, b_2, b_3)$, then

$$\mathbf{a} - \mathbf{b} = (a_1 - b_1, a_2 - b_2, a_3 - b_3)\,,$$

where $a_1,\ a_2,\ a_3,\ b_1,\ b_2,\ b_3 \in \mathbb{R}$.



To get the difference of vectors $\mathbf{a}$ *and* $\mathbf{b}$*, you should transfer to the vectors in a common start point. The difference of the vectors* $(\mathbf{a} - \mathbf{b})$ *starts from the end point of the vector* $\mathbf{b}$ *to the endpoint of vector* $\mathbf{a}$*.*

Multiply vector with a scalar, sum of vectors, and difference of vectors IV

- The sum of vectors is
  1. Thesis: Commutative $\mathbf{a}+\mathbf{b}=\mathbf{b}+\mathbf{a}$, where $\mathbf{a},\mathbf{b}\in\mathbb{R}^3$.
     Deduction: $\mathbf{a}+\mathbf{b}=(a_1,a_2,a_3)+(b_1,b_2,b_3)=(a_1+b_1,a_2+b_2,a_3+b_3)=(b_1+a_1,b_2+a_2,b_3+a_3)=(b_1,b_2,b_3)+(a_1,a_2,a_3)=\underline{\underline{\mathbf{b}+\mathbf{a}}}$ $_{q.e.d.}$
  2. Thesis: Associative $(\mathbf{a}+\mathbf{b})+\mathbf{c}=\mathbf{a}+(\mathbf{b}+\mathbf{c})$, where $\mathbf{a},\mathbf{b},\mathbf{c}\in\mathbb{R}^3$.
     Deduction: $(\mathbf{a}+\mathbf{b})+\mathbf{c}=[(a_1,a_2,a_3)+(b_1,b_2,b_3)]+(c_1,c_2,c_3)=$
     $[(a_1+b_1)+c_1,(a_2+b_2)+c_2,(a_3+b_3)+c_3]=$
     $[a_1+(b_1+c_1),a_2+(b_2+c_2),a_3+(b_3+c_3)]=$
     $(a_1,a_2,a_3)+[(b_1,b_2,b_3)+(c_1,c_2,c_3)]=\underline{\underline{\mathbf{a}+(\mathbf{b}+\mathbf{c})}}$ $_{q.e.d.}$
  3. The null vector exists: $\exists\mathbf{0}\in\mathbb{R}^3$, where $\mathbf{a}+\mathbf{0}=\mathbf{a}$, and $\mathbf{a}\in\mathbb{R}^3$.
  4. All vectors have an inverse vector: $\forall\mathbf{a}\in\mathbb{R}^3\ \exists(-\mathbf{a})\in\mathbb{R}^3$, where $\mathbf{a}+(-\mathbf{a})=\mathbf{0}$.

Multiply vector with a scalar, sum of vectors, and difference of vectors V

- The vector multiplication with scalar is
  - Thesis: Multiplication of vectors by a scalar is associative, $\lambda(\mu\mathbf{a}) = (\lambda\mu)\mathbf{a}$, where $\mathbf{a} \in \mathbb{R}^3, \lambda, \mu \in \mathbb{R}$.
    Deduction: $\lambda(\mu\mathbf{a}) = \lambda[\mu(a_1, a_2, a_3)] = \lambda(\mu a_1, \mu a_2, \mu a_3) = (\lambda\mu a_1, \lambda\mu a_2, \lambda\mu a_3) = (\lambda\mu)(a_1, a_2, a_3) = \underline{\underline{(\lambda\mu)\mathbf{a}}}$ $_{q.e.d.}$
  - Thesis: Addition of vectors is distributive for the multiplication by a scalar, $\lambda(\mathbf{a}+\mathbf{b}) = \lambda\mathbf{a} + \lambda\mathbf{b}$, where$\mathbf{a}, \mathbf{b} \in \mathbb{R}^3, \lambda \in \mathbb{R}$.
    Deduction: $\lambda(\mathbf{a}+\mathbf{b}) = \lambda[(a_1, a_2, a_3)+(b_1, b_2, b_3)] = \lambda(a_1+b_1, a_2+b_2, a_3+b_3) = [\lambda(a_1+b_1), \lambda(a_2+b_2), \lambda(a_3+b_3)] = (\lambda a_1 + \lambda b_1, \lambda a_2 + \lambda b_2, \lambda a_3 + \lambda b_3) = (\lambda a_1, \lambda a_2, \lambda a_3) + (\lambda b_1, \lambda b_2, \lambda b_3) = \lambda(a_1, a_2, a_3) + \lambda(b_1, b_2, b_3) = \underline{\underline{\lambda\mathbf{a} + \lambda\mathbf{b}}}$ $_{q.\ e.\ d}$

Multiply vector with a scalar, sum of vectors, and difference of vectors VI

- Thesis: The addition of scalars is distributive for multiplication by a vector, $(\lambda + \mu)\,\mathbf{a} = \lambda\mathbf{a} + \mu\mathbf{a}$, where $\mathbf{a} \in \mathbb{R}^3$, $\lambda, \mu \in \mathbb{R}$.
  Deduction: $(\lambda + \mu)\,\mathbf{a} = (\lambda + \mu)\,(a_1, a_2, a_3) = [(\lambda + \mu)\,a_1, (\lambda + \mu)\,a_2, (\lambda + \mu)\,a_3] = (\lambda a_1 + \mu a_1, \lambda a_2 + \mu a_2, \lambda a_3 + \mu a_3) = (\lambda a_1, \lambda a_2, \lambda a_3) + (\mu a_1, \mu a_2, \mu a_3) = \lambda\,(a_1, a_2, a_3) + \mu\,(a_1, a_2, a_3) = \underline{\underline{\lambda\mathbf{a} + \mu\mathbf{a}}}$ $_{q.\ e.\ d}$
- $\forall \in \mathbf{a} \cdot 1 = \mathbf{a}$, where $\mathbf{a} \in \mathbb{R}^3$.

The End

Thank you for your attention!