(ENKEMNA0302) Applied Linear Algebra

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LU Decomposition - cont'd I

- (Matrix inversion using LU decomposition) Invert the following matrix using its LU decomposition:

$$\begin{pmatrix} 4 & 8 & 8 \\ 2 & 6 & 4 \\ 1 & 3 & 4 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ \frac{1}{2} & 1 & 0 \\ \frac{1}{4} & \frac{1}{2} & 1 \end{pmatrix} \begin{pmatrix} 4 & 8 & 8 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

LU Decomposition - cont'd II

Using the LU decomposition of the matrix $\mathbf{B}$, we first solve the matrix equation $\mathbf{LY} = \mathbf{E}$:

$$\begin{pmatrix} 1 & 0 & 0 \\ \frac{1}{2} & 1 & 0 \\ \frac{1}{4} & \frac{1}{2} & 1 \end{pmatrix} \begin{pmatrix} y_{11} & y_{12} & y_{13} \\ y_{21} & y_{22} & y_{23} \\ y_{31} & y_{32} & y_{33} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Multiplying by the first row of $\mathbf{L}$, we get: $(y_{11}, y_{12}, y_{13}) = (1, 0, 0)$. Multiplying by the second row, we have $\frac{1}{2}(y_{11}, y_{12}, y_{13}) + (y_{21}, y_{22}, y_{23}) = (0, 1, 0)$. After substitution, we get $(y_{21}, y_{22}, y_{23}) = \left(-\frac{1}{2}, 1, 0\right)$. Finally, multiplying by the third row gives: $\frac{1}{4}(y_{11}, y_{12}, y_{13}) + \frac{1}{2}(y_{21}, y_{22}, y_{23}) + (y_{31}, y_{32}, y_{33})$, which, after substitution, gives the third row of $\mathbf{Y}$, so $(y_{31}, y_{32}, y_{33}) = \left(0, -\frac{1}{2}, 1\right)$.

LU Decomposition - cont'd III

Next, in the same way, by simple substitution, we can solve $\mathbf{UX} = \mathbf{Y}$, that is, the matrix equation

$$\begin{pmatrix} 4 & 8 & 8 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ -\frac{1}{2} & 1 & 0 \\ 0 & -\frac{1}{2} & 1 \end{pmatrix}$$

is also solvable, with the solution

$$\mathbf{X} = \begin{pmatrix} \frac{3}{4} & -\frac{1}{2} & -1 \\ -\frac{1}{4} & \frac{1}{2} & 0 \\ 0 & -\frac{1}{4} & \frac{1}{2} \end{pmatrix}.$$

LU Decomposition - cont'd IV

- The computational cost of LU decomposition is the same as that of Gaussian elimination, i.e., for an n-order matrix, the computational complexity is approximately $2n^3/3$.
- Advantages of LU decomposition:
  1. Since the LU decomposition of the coefficient matrix of a system of equations does not require the right-hand side of the system, it can be used in cases where the right-hand side is not yet known or where there are multiple different right-hand sides to work with.
  2. Knowing the LU decomposition makes it possible to perform matrix-related calculations more quickly than otherwise, such as finding the matrix inverse or, as we will learn later, determining the determinant.
  3. As mentioned earlier, LU decomposition is very memory-efficient, and there are specific classes of matrices (e.g., band matrices or sparse matrices) for which there exist faster algorithms for LU decomposition than the standard elimination method.

LU Decomposition - cont'd V

4. Computer algebra systems operate in such a way that if a matrix needs to undergo a computation that can be solved by LU decomposition (or the PLU decomposition to be discussed in the next section), they will use it. This means that if later another computation needs to be done with the same matrix, knowing this decomposition can significantly speed up the process.

The End

Thank you for your attention!