# (ENKEMNA0302) Applied Linear Algebra

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# Actualities

- I booked Room No. F07 in the Building F from 10:00 to 14:00 on May 15, 2025 for retake test.
- Are you interested in buildin a CubeSat?
- Recruit lecture will be held at UP Faculty of Engineering and Information Technology in Room B224 from 14:00 to 16:00 on Tuesday February 18, 2025.

# Requirements

- You will write tests based on the exercises of the practical courses. You can use everything during the test
- The minimum requirement is 41 % of both tests.
- Failed tests must be corrected
- You must take an ~~oral~~ written exam. You cannot use anything
- Grades: Insufficient/Fail (1): 0-40 %, Sufficient/Pass (2): 41-55 %, Average (3): 56-70 %, Good (4): 71-85 %, Excellent (5): 86-100 %.
- Mid-term test 1: March 13, mid-term test 2: May 8, retake tests: May 15, 2025.

**Bibliography**

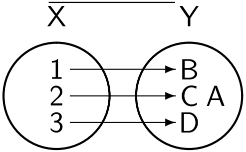
*Bernard Kolman and David Hill: Elementary Linear Algebra with Applications, 9th ed., Person, 2007*

*Philip N. Klein: Coding the Matrix: Linear Algebra through Applications to Computer Science, Newtonian Press 2013*

*K. F. Riley, M. P. Hobson, S. J. Bence: Mathematical Methods for Physics and Engineering: A Comprehensive Guide, Cambridge University Press, 3rd. ed. (2006)*

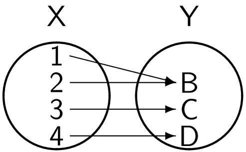
# Operators I

- Definition: The set is the sum of things. It is a fundamental term. You need a statement that collects the element. It means that you can decide whether an element is part of the set or not.
- Definition: The pair are sets consisting of two elements.
- Definition: Elements $e_1$ and $e_2$ consist of ordered pair if $\{e_1, \{e_2\}\}$. It sign is $(e_1, e_2)$.
- Definition: Relation is the set of ordered pairs.

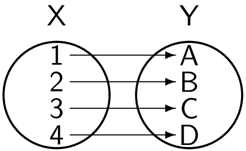



- Definition: The injection orders different elements (X) to different elements (Y).

# Operators II




- Definition: Surjections are those relations, that the values of the relation agree to the values of the set to order.




- Definition: Bijection is an injection and a surjection. All elements are related to all elements of the other set.

# Operators III

- Definition: The functions are such a set of ordered pairs in that one element shows up only once:

$$(\forall x)(\forall y_1)(\forall y_2)\left[(x, y_1) \in f \wedge (x, y_2) \in f \Rightarrow y_1 = y_2\right]$$

- Definition: $V$ are $U$ vectorspaces above $\mathbb{T}$ body. The $f : V \to U$ relation is linear if it is
  1. Additive, for all $v_1, v_2 \in V$ vectors $f(v_1 + v_2) = f(v_1) + f(v_2)$.
  2. Homogen, for all $v \in V$ vectors and $\lambda \in \mathbb{T}$ elements $f(\lambda v) = \lambda f(v)$.
- Definition: Operators are the linear vector-vector functions.
- Például:
  - Identical operator: $\mathbf{A} \cdot \mathbf{1} = \mathbf{A}$, for all $\mathbf{A}$ operators.
  - Null operator: $\mathbf{A} \cdot \mathbf{0} = \mathbf{0}$, for all $\mathbf{A}$ operators.

# Operators IV

- Mirror operators: $(\mathbf{A} \cdot \mathbf{M}) \cdot \mathbf{M} = \mathbf{A}$, for all $\mathbf{A}$ operators.
- Projection operator: $\mathbf{A} \cdot \mathbf{P} = \mathbf{P}$, for all $\mathbf{A}$ operator.
- Rotational operator: later.

- Operators could be multiplied on both sides.
- The representation of operators is the matrixes. See $\alpha_{ij} \in \mathbb{R}$ for all $i \in \{1, 2, \dots, m\}$ and $j \in \{1, 2, \dots, n\}$, where $m, n \in \mathbb{N}^+$. The

$$A = \begin{pmatrix} \alpha_{11} & \alpha_{12} & \cdots & \alpha_{1n} \\ \alpha_{21} & \alpha_{22} & \cdots & \alpha_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_{m1} & \alpha_{m2} & \cdots & \alpha_{mn} \end{pmatrix}$$

table is called $m \times n$ type matrix. The set of the $m \times n$ type matrixes is $M_{m \times n}$.

# Operators V

- The spur of the matrix is the set of $\{\alpha_{11}, \alpha_{22}, \ldots, \alpha_{nn}\}$.
- The first index of the elements $\alpha_{ij}$ is the rowindex $(i)$, the 2nd index is the column index $(j)$.
- The Row $i$ of the Matrix is $A_i$ , and the Column $j$of the matrix is $A_j$j.
- Determinant!!!

# Transpose I

- <u>Definition:</u> The transpose of the $A = (\alpha_{ij})_{m \times n}$ matrix is the $A^T = (\alpha_{ji})_{m \times n}$. This means the change of the rows and columns. The transpose is the mirror of a square matrix.

$$A_{m\times n} = \begin{pmatrix} \alpha_{11} & \alpha_{12} & \cdots & \alpha_{1n} \\ \alpha_{21} & \alpha_{22} & \cdots & \alpha_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_{m1} & \alpha_{m2} & \cdots & \alpha_{mn} \end{pmatrix} \quad A^T_{n\times m} = \begin{pmatrix} \alpha_{11} & \alpha_{21} & \cdots & \alpha_{1m} \\ \alpha_{12} & \alpha_{22} & \cdots & \alpha_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_{1n} & \alpha_{2n} & \cdots & \alpha_{mn} \end{pmatrix}$$

- Examples for transpose.

# Matrix Operations I

- <u>Definition:</u> $A = (\alpha_{ij})_{m\times n}$ and $B = (\beta_{ij})_{m\times n}$ are two matrixes with same type, $\lambda \in \mathbb{R}$ a scalar. The sum of Matrixes $A$ and $B$ is Matrix $A + B = (\alpha_{ij} + \beta_{ij})_{m\times n}$, the $\lambda$ times Matrix $A$ is Matrix $\lambda A = (\lambda\alpha_{ij})_{m\times n}$.

$$A_{m\times n} = \begin{pmatrix} \alpha_{11} & \alpha_{12} & \cdots & \alpha_{1n} \\ \alpha_{21} & \alpha_{22} & \cdots & \alpha_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_{m1} & \alpha_{m2} & \cdots & \alpha_{mn} \end{pmatrix} B_{m\times n} = \begin{pmatrix} \beta_{11} & \beta_{12} & \cdots & \beta_{1n} \\ \beta_{21} & \beta_{22} & \cdots & \beta_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \beta_{m1} & \beta_{m2} & \cdots & \beta_{mn} \end{pmatrix}$$

$$A_{m\times n} + B_{m\times n} = \begin{pmatrix} \alpha_{11} + \beta_{11} & \alpha_{12} + \beta_{12} & \cdots & \alpha_{1n} + \beta_{1n} \\ \alpha_{21} + \beta_{21} & \alpha_{22} + \beta_{22} & \cdots & \alpha_{1n} + \beta_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_{m1} + \beta_{m1} & \alpha_{m2} + \beta_{m2} & \cdots & \alpha_{mn} + \beta_{mn} \end{pmatrix}$$

# Matrix Operations II

$$\lambda A_{m\times n} = \begin{pmatrix} \lambda\alpha_{11} & \lambda\alpha_{12} & \cdots & \lambda\alpha_{1n} \\ \lambda\alpha_{21} & \lambda\alpha_{22} & \cdots & \lambda\alpha_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \lambda\alpha_{m1} & \lambda\alpha_{m2} & \cdots & \lambda\alpha_{mn} \end{pmatrix}$$

The elements of the matrixes are added and multiplying by a scalar means to multiply all elements of the matrix by the scalar.

- Examples for matrix operations.

# Matrix Operations III

- Definition: $A = (\alpha_{ij})_{m\times n}$ and $B = (\beta_{ij})_{n\times k}$ are two matrixes. The product of Matrixes $A$ and $B$ is Matrix $A \cdot B = (\gamma_{ij})_{m\times k}$, where
$$\gamma_{ij} = \sum_{l=1}^{n} \alpha_{il}\beta_{lj}.$$
Or:
$$A_{m\times n} = \begin{pmatrix} \alpha_{11} & \alpha_{12} & \cdots & \alpha_{1n} \\ \alpha_{21} & \alpha_{22} & \cdots & \alpha_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_{i1} & \alpha_{i2} & \cdots & \alpha_{in} \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_{m1} & \alpha_{m2} & \cdots & \alpha_{mn} \end{pmatrix} \quad B_{n\times k} = \begin{pmatrix} \beta_{11} & \beta_{12} & \cdots & \beta_{1j} & \cdots & \beta_{1k} \\ \beta_{21} & \beta_{22} & \cdots & \beta_{2j} & \cdots & \beta_{2k} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ \beta_{n1} & \beta_{n2} & \cdots & \beta_{nj} & \cdots & \beta_{nk} \end{pmatrix}$$
$$A \cdot B_{m\times k} = \begin{pmatrix} \alpha_{11}\beta_{11} + \alpha_{12}\beta_{21} + \cdots + \alpha_{1n}\beta_{n1} & \alpha_{11}\beta_{12} + \alpha_{12}\beta_{22} + \cdots + \alpha_{1n}\beta_{n2} & \cdots & \alpha_{11}\beta_{1k} + \alpha_{12}\beta_{2k} + \cdots + \alpha_{1n}\beta_{nk} \\ \alpha_{21}\beta_{11} + \alpha_{22}\beta_{21} + \cdots + \alpha_{2n}\beta_{n1} & \alpha_{21}\beta_{12} + \alpha_{22}\beta_{22} + \cdots + \alpha_{2n}\beta_{n2} & \cdots & \alpha_{21}\beta_{1k} + \alpha_{22}\beta_{2k} + \cdots + \alpha_{2n}\beta_{nk} \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_{m1}\beta_{11} + \alpha_{m2}\beta_{21} + \cdots + \alpha_{mn}\beta_{n1} & \alpha_{m1}\beta_{12} + \alpha_{m2}\beta_{22} + \cdots + \alpha_{mn}\beta_{n2} & \cdots & \alpha_{m1}\beta_{1k} + \alpha_{m2}\beta_{2k} + \cdots + \alpha_{mn}\beta_{nk} \end{pmatrix}$$
- Examples for matrix multiplications.

# Matrix Operations IV

- For similar square matrixes the condition of matrix production is fulfilled and the product will be the same type. Therefore, there is an exponentiation of matrixes:

$$A^1 = A \quad \text{és} \quad A^m = AA^{m-1}$$

  where $(m \geq 2)$ és $A \in \mathcal{M}_{n\times n}$. Let us consider $A^0 = E_m$.
- Thesis: Equlations of matrix exponentation:

$$\begin{aligned} A^m A^k &= A^{m+k} \\ (A^m)^k &= A^{mk}, \end{aligned}$$

  ahol $m, k \in \mathbb{N}$.

  Deduction: It is trivial based on the definition of matrixproduct.
- Examples for matrix exponentation.

# Identity Matrix

- Definition: The $n^{th}$ order identity matrix is:

$$E_n = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}$$

- Thesis: For all $A \in \mathcal{M}_{n\times n}$: $A \cdot E_n = E_n \cdot A = A$, or matrix $E_n$ is identity element of the $n \times n$ square matrixes for matrix production.
  Deduction: $A = (\alpha_{ij})_{n\times n}$ and $E_n = (\beta_{ij})_{n\times n}$ are two matrixes, where $\beta_{ij} = 1$, if $i = j$, otherwise it is zero. The product of Matrixes $A$ and $E_n$ is Matrix $A \cdot E_n = (\sum_{l=1}^{n} \alpha_{il}\beta_{lj})_{n\times n}$. It is Matrix $A = (\alpha_{ij})_{n\times n}$, because the definition of $b_{ij}$ erases all other elements than $\alpha_{ij}$.

# Matrix Rank I

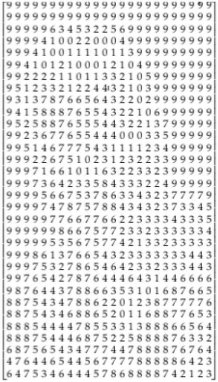
- Definition: $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_s \in V$ are vectors. The rank of the $\{\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_s\}$ vector system is the dimension of the $\mathcal{L}(\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_s)$ subspace. Its sign is $\rho(\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_s)$.
- Thesis: The following transformation do not change the order of the $\{\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_s\}$ vector system:
  1. Multiplying a vectors by a $\lambda \neq 0$ scalar.
  2. Adding the vector multiplied by $\lambda$ to another vector.
  3. Eliminating a vector that is a linear combination of the remaining vectors.
  4. Changing the order of vectors.
- Definition: The rank of Matrix $A \in \mathcal{M}_{m \times n}$ is the rank of its row vector system.
- The rank of a matrix is equal to the common rank of the maximal ranked non-disappearing subdeterminants.

# Matrix Rank II

- The rank of a matrix is determined by transforming the matrix to trapezoid form by rank invariant transformations. You can change the columns. (A matrix has trapesoid shape if $\alpha_{ij} = 0$, $i > j$, and $\alpha_{ii} \neq 0$, where $(1 \leq i \leq min\{m, n\})$.) Rows and columns containing 0 could be deleted. The rank of the trapezoid matrix is the number of its rows.

- Examples of determination of the rank of a matrix.

# Image processing I



- Matrix $\mathbf{A}_{m\times n}$ is a representation of an $m \times n$ greyscale image.
- All matrix elements gives color from the $\{0, 1, \ldots, k\}$ range, where 0 is black, $k-1$ is white and $k$ means transparency.
- The $\mathbf{B}_{m\times n}$ background is transparent.
- Let us construct the $\mathbf{A} \odot \mathbf{B} = [a_{ij} \odot b_{ij}]$ operations that copies the 2nd image to the background of the first image.
- Formula: $[a_{ij} \odot b_{ij}] = \begin{cases} b_{ij}, & \text{if } a_{ij} = k. \\ a_{ij}, & \text{otherwise.} \end{cases}$
- Let us use the $x \mapsto \lfloor x \rfloor$ function, that means to rounding to the lower integer value.
- If $a \in [0, k]$, then $0 \le a/k \le 1$, therefore $\lfloor a/k \rfloor$ is 0, or 1.

# Image processing II

$$
\begin{bmatrix}
9&9&9&9&9&9&9&9&9&9&9&9&9&9&9&9&9&9&9&9&9&9&9&9\\
9&9&9&9&9&9&9&9&9&9&9&9&9&9&9&9&9&9&9&9&9&9&9&9\\
9&9&9&9&9&6&3&4&5&3&2&2&5&6&9&9&9&9&9&9&9&9&9&9\\
9&9&9&9&4&1&0&0&2&2&0&0&0&4&9&9&9&9&9&9&9&9&9&9\\
9&9&9&4&1&0&0&1&1&1&1&0&1&1&3&9&9&9&9&9&9&9&9&9\\
9&9&4&1&0&1&2&1&0&0&0&1&2&1&0&4&9&9&9&9&9&9&9&9\\
9&9&2&2&2&2&1&1&0&1&1&3&3&2&1&0&5&9&9&9&9&9&9&9\\
9&5&1&2&3&3&2&1&2&2&4&4&3&2&1&0&3&9&9&9&9&9&9&9\\
9&3&1&3&7&8&7&6&6&5&6&4&3&2&2&0&2&9&9&9&9&9&9&9\\
9&4&1&5&8&8&8&7&6&5&5&4&3&2&2&1&0&6&9&9&9&9&9&9\\
9&5&2&5&8&8&7&6&5&5&5&4&4&3&2&2&1&3&7&9&9&9&9&9\\
9&9&2&3&6&7&7&6&5&5&4&4&4&0&0&0&3&3&5&9&9&9&9&9\\
9&9&5&1&4&6&7&7&7&5&4&3&1&1&1&1&2&3&4&9&9&9&9&9\\
9&9&9&2&2&6&7&5&1&0&2&3&1&2&3&2&2&3&3&9&9&9&9&9\\
9&9&9&7&1&6&6&1&0&1&1&6&3&2&2&3&3&2&3&9&9&9&9&9\\
9&9&9&7&3&6&4&2&3&3&5&8&4&3&3&3&2&2&4&9&9&9&9&9\\
9&9&9&9&5&6&6&7&5&3&7&8&6&3&3&4&3&2&3&7&7&7&7&9\\
9&9&9&9&7&4&7&8&7&5&7&8&8&4&3&4&3&2&3&7&3&3&4&5\\
9&9&9&9&9&7&7&6&6&7&7&6&6&2&2&3&3&3&3&4&3&3&3&5\\
9&9&9&9&9&9&8&6&6&7&5&7&7&2&3&3&2&3&3&3&3&3&3&4\\
9&9&9&9&9&5&3&5&6&7&5&7&7&4&2&1&3&3&2&3&3&3&3&3\\
9&9&9&8&6&1&3&7&6&6&5&4&3&2&3&3&3&3&3&3&3&4&4&3\\
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9&8&7&6&4&4&3&7&8&8&6&6&3&5&3&1&0&1&6&8&7&6&6&5\\
8&8&7&5&4&3&4&7&8&8&6&2&2&0&1&2&3&8&7&7&7&7&7&6\\
8&8&7&5&4&3&4&6&8&8&6&5&2&0&1&1&6&8&8&7&7&6&5&3\\
8&8&8&5&4&4&4&4&7&8&5&5&3&3&1&3&8&8&8&6&6&5&6&4\\
8&8&8&7&5&4&4&4&6&8&7&5&2&2&5&8&8&8&8&7&6&3&3&2\\
6&8&7&5&6&5&4&3&4&7&7&7&4&4&7&8&8&8&8&7&6&7&6&4\\
4&7&6&4&4&6&5&4&4&5&6&7&7&7&7&8&8&8&8&8&6&4&2&3\\
6&4&7&5&3&4&6&4&4&4&5&7&8&6&8&8&8&8&7&4&2&1&2&3
\end{bmatrix}
$$

- Or $\lfloor a/k \rfloor$ is 1, if only $a = k$, or the pixel is transparent, otherwise it is 0.
- $1 - \lfloor a/k \rfloor$ is 0, if only $a = k$, or otherwise it is 1.
- Therefore, $a \odot b = \left\lfloor \frac{a}{k} \right\rfloor b + \left(1 - \left\lfloor \frac{a}{k} \right\rfloor\right) a$ is the desired operation.
- (Top) $32 \times 24$ matrix. (Bottom left) Original image. (Bottom middle) Background. (Bottom right) The result of the operation.

# Moment of inertia

- Mass times square of distance from the axis.
- For a rigid object of $N$ point masses $m_k$, the moment of inertia tensor is given by

$$\mathbf{I} = \begin{pmatrix} I_{11} & I_{12} & I_{13} \\ I_{21} & I_{22} & I_{23} \\ I_{31} & I_{32} & I_{33} \end{pmatrix},$$

where

$$I_{ij} = \sum_{k=1}^{N} m_k \left( \|\mathbf{r}_k\|^2 \delta_{ij} - x_i^{(k)} x_j^{(k)} \right)$$

and $i, j \in \{1, 2, 3\}$, $\mathbf{r}_k = \left( x_1^{(k)}, x_2^{(k)}, x_3^{(k)} \right)$ is the vector to the point mass $m_k$ from the point about which the tenzor is calculated, and $\delta_{ij}$ is the Kronecker delta.

# Transformation matrixes I

- Rotational matrix in 2D:

$$\begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}.$$

- Rotational matrixes in 3D around z, x, y axes:

$$\begin{pmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 \\ 0 & \cos\alpha & -\sin\alpha \\ 0 & \sin\alpha & \cos\alpha \end{pmatrix} \begin{pmatrix} \cos\alpha & 0 & \sin\alpha \\ 0 & 1 & 0 \\ -\sin\alpha & 0 & \cos\alpha \end{pmatrix}.$$

- Mirror of the vectors of the plan for the $\alpha/2$ angular line:

$$\begin{pmatrix} \cos\alpha & \sin\alpha \\ \sin\alpha & -\cos\alpha \end{pmatrix}.$$

# Transformation matrixes II

- Mirror of the vectors of the 3D space for the **n** normal vector planes:

$$\mathbf{M} = \mathbf{I} - 2\mathbf{n} \otimes \mathbf{n}^T.$$

- Perpendicular projection to a line with **b** direction vector:

$$\mathbf{P} = \frac{1}{\mathbf{b}\mathbf{b}^T}\mathbf{b} \otimes \mathbf{b}^T.$$

- Perpendicular projection to the plan with **n** normal vector:

$$\mathbf{P} = \mathbf{I} - \mathbf{n} \otimes \mathbf{n}^T.$$

# Transformation matrixes III

- Shift by $(a, b)$ vector in 2D:
$$\begin{pmatrix} 1 & 0 & a \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix}.$$
- Shift by $(a, b, c)$ vector in 3D:
$$\begin{pmatrix} 1 & 0 & 0 & a \\ 0 & 1 & 0 & b \\ 0 & 0 & 1 & c \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

# Comming soon. . .

- Diagonal matrixes
- Permutation matrixes and snakes
- Triangular matrixes
- Symmetric and skew-symmetric matrixes

# The End

Thank you for your attention!