



(PTIA0301) Elementary Linear Algebra

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Linear Transformations I

- ▶ Definition: Let V_1 and V_2 be vector spaces. A function $\varphi : V_1 \rightarrow V_2$ is called a linear mapping if

$$\begin{aligned} \text{additive : } \varphi(\mathbf{a} + \mathbf{b}) &= \varphi(\mathbf{a}) + \varphi(\mathbf{b}) \\ \text{and homogeneous : } \varphi(\lambda \mathbf{a}) &= \lambda \varphi(\mathbf{a}), \end{aligned}$$

where $\mathbf{a}, \mathbf{b} \in V_1$ and $\lambda \in \mathbb{R}$.

- ▶ Thesis: (Matrix Representation) A mapping $\varphi : \mathbb{R}^n \rightarrow \mathbb{R}^m$ is linear if and only if $\exists A \in \mathcal{M}_{m \times n}$ such that $\varphi(\mathbf{x}) = A \cdot \mathbf{x}$, where $\mathbf{x} \in \mathbb{R}^n$.
- ▶ Definition: Let V be a vector space. A mapping $\varphi : V \rightarrow V$ is called a linear transformation. The set of all linear transformations on V is denoted by \mathcal{T}_V .
- ▶ Definition: A mapping $f : V \rightarrow \mathbb{R}$ is called a linear form.

Linear Transformations II

- Definition: A mapping $L : V \times V \rightarrow \mathbb{R}$ is called a bilinear form if it is linear in both arguments, that is,

$$L(\mathbf{x} + \mathbf{y}, \mathbf{z}) = L(\mathbf{x}, \mathbf{z}) + L(\mathbf{y}, \mathbf{z})$$

$$L(\lambda \mathbf{x}, \mathbf{y}) = \lambda L(\mathbf{x}, \mathbf{y})$$

$$L(\mathbf{x}, \mathbf{y} + \mathbf{z}) = L(\mathbf{x}, \mathbf{y}) + L(\mathbf{x}, \mathbf{z})$$

$$L(\mathbf{x}, \lambda \mathbf{y}) = \lambda L(\mathbf{x}, \mathbf{y}),$$

where $\mathbf{x}, \mathbf{y}, \mathbf{z} \in V$ and $\lambda \in \mathbb{R}$.

- Thesis: A mapping $L : V \times V \rightarrow \mathbb{R}$ is a bilinear form if and only if (for a given basis) there exist unique coefficients $\alpha_{ik} \in \mathbb{R}$ such that
$$L(x, y) = \sum_{i=1}^n \sum_{k=1}^n \alpha_{ik} x_i y_k.$$

Linear Transformations III

- ▶ Consider the canonical basis of \mathbb{R}^n , denoted as $\{\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n\}$. In this case, $\alpha_{ik} = L(\mathbf{e}_i, \mathbf{e}_k)$. The matrix $A = (\alpha_{ik})_{n \times n}$ is called the matrix of the bilinear form L (with respect to the canonical basis).
- ▶ Definition: A bilinear form L is symmetric if $L(\mathbf{x}, \mathbf{y}) = L(\mathbf{y}, \mathbf{x})$ for all $\mathbf{x}, \mathbf{y} \in V$.
- ▶ Definition: Let L be a symmetric bilinear form on the vector space V . The function $Q(x) = L(x, x)$ is called a quadratic form.
- ▶ Definition: A quadratic form Q is positive definite if $\forall \mathbf{x} \neq \mathbf{0}$ we have $Q(\mathbf{x}) > 0$.
Note: Q is positive semidefinite if $\forall \mathbf{x} \neq \mathbf{0}$ we have $Q(\mathbf{x}) \geq 0$ and $\exists \mathbf{y} \neq \mathbf{0}$ such that $Q(\mathbf{y}) = 0$. The concepts of negative definite and negative semidefinite are defined similarly.
- ▶ Definition: A symmetric bilinear form whose associated quadratic form is positive definite is called an inner product.
Example: In the space \mathbb{R}^3 , the scalar product is an inner product.

Linear Transformations IV

- ▶ A mapping $\varphi : \mathbb{R}^n \rightarrow \mathbb{R}^m$ is called an isomorphism if it is bijective.
- ▶ It can be shown that two vector spaces are isomorphic if and only if their dimensions are equal:

$$V_1 \cong V_2 \Leftrightarrow \dim V_1 = \dim V_2.$$

The End

Thank you for your attention!