(PTIA0301) Elementary Linear Algebra

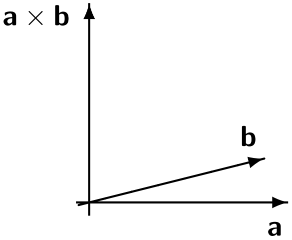
Dr. Gabor FACSKO, PhD
Senior Research Fellow
facskog@gamma.ttk.pte.hu

University of Pecs, Faculty of Sciences, Institute of Mathematics and Informatics, 7624 Pecs, Ifjusag utja 6.
Wigner Research Centre for Physics, Department of Space Physics and Space Technology, 1121 Budapest, Konkoly-Thege Miklos ut 29-33.
https://facsko.ttk.pte.hu

September 19, 2024

Vector Product I

- Definition: The system consists of from $\{\mathbf{a}, \mathbf{b}, \mathbf{c}\}$ non-zero vectors is a right-handed system if from the endpoint of $\mathbf{c}$, $\mathbf{a}$ could be rotated to the direction of the $\mathbf{b}$ by less than 180^o angle in anti-clockward direction.



- Definition: The vectorial product of non-zero Vectors $\mathbf{a}$ and $\mathbf{b}$ is that $\mathbf{a} \times \mathbf{b}$ vector, which length is $|\mathbf{a} \times \mathbf{b}| = |\mathbf{a}|\,|\mathbf{b}| \sin\theta$, where $\theta = (\mathbf{a}, \mathbf{b})\angle$. The Vector $\mathbf{a} \times \mathbf{b}$ is perpendicular to Vectors $\mathbf{a}$ and $\mathbf{b}$, furthermore $\{\mathbf{a}, \mathbf{b}, \mathbf{a} \times \mathbf{b}\}$ consist of a right-handed system. Finally, $\mathbf{0} \times \mathbf{a} = \mathbf{0}$, where $\left(\mathbf{a} \in V^3\right)$.

Vector Product II

- Features of the vector product
  1. Thesis: The vectorial product is anticommutative, $\mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a}$, where $\left(\mathbf{a}, \mathbf{b} \in V^3\right)$.
     Deduction: It is trivial based on the definition of the right-handed system.
  2. Thesis: The vectorial product is homogen, $(\lambda \mathbf{a}) \times \mathbf{b} = \lambda (\mathbf{a} \times \mathbf{b})$, where $\mathbf{a}, \mathbf{b} \in V^3$ and $\lambda \in \mathbb{R}$.
     Deduction: $|(\lambda \mathbf{a}) \times \mathbf{b}| = |\lambda \mathbf{a}| \, |\mathbf{b}| \sin\theta = \lambda \, |\mathbf{a} \times \mathbf{b}|$, ahol $\theta = (\lambda \mathbf{a}, \mathbf{b}) \angle$.
     The direction of vectors agrees because Vector $\mathbf{a}$ is parallel to Vector $\lambda \mathbf{a}$.
  3. Thesis: The vectorial product is dissociative, $(\mathbf{a} + \mathbf{b}) \times \mathbf{c} = \mathbf{a} \times \mathbf{c} + \mathbf{b} \times \mathbf{c}$, where $\left(\mathbf{a}, \mathbf{b}, \mathbf{c} \in V^3\right)$.
     Deduction: Later, based on components.
- Definition: Non-zero Vectors $\mathbf{a}$ and $\mathbf{b}$ are parallel if $\exists \lambda \in \mathbb{R}$, and $\mathbf{a} = \lambda \mathbf{b}$. Its sign is $\mathbf{a} \parallel \mathbf{b}$.

Vector Product III

- All vector multiplied itself is zero-vecdtor, $\mathbf{a} \times \mathbf{a} = \mathbf{0}$ $\forall \mathbf{a} \in V^3$-re. esetén.
- Furthermore $\mathbf{a} \times \mathbf{b} = \mathbf{0} \Leftrightarrow \mathbf{a} \parallel \mathbf{b}$, or at least one of Vectors $\mathbf{a}, \mathbf{b}$ is a null-vector.
- It is easy to prove that

$$\begin{aligned} \mathbf{e}_1 \times \mathbf{e}_2 &= \mathbf{e}_3 \\ \mathbf{e}_2 \times \mathbf{e}_3 &= \mathbf{e}_1 \\ \mathbf{e}_3 \times \mathbf{e}_1 &= \mathbf{e}_2. \end{aligned}$$

- The vectorial product with components is $\mathbf{a} \times \mathbf{b} = (a_2 b_3 - a_3 b_2)\,\mathbf{e}_1 + (a_3 b_1 - a_1 b_3)\mathbf{e}_2 + (a_1 b_2 - a_2 b_1)\,\mathbf{e}_3$.
- $|\mathbf{a} \times \mathbf{b}|$ is equla to the area of the paralelogram determined by $\mathbf{a}$ and $\mathbf{b}$, because $|\mathbf{a}|$ is the basis of the paralelogram and $|\mathbf{b}|\,|\sin\theta|$ is its height, where $\theta = (\mathbf{a}, \mathbf{b})\angle$.

Triple product

- Definition: The triple product of Vectors $\mathbf{a}, \mathbf{b}, \mathbf{c} \in V^3$ is

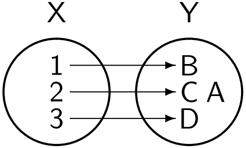
$$(\mathbf{a}, \mathbf{b}, \mathbf{c}) = (\mathbf{a} \times \mathbf{b})\, \mathbf{c}.$$

- If $\mathbf{a}, \mathbf{b}, \mathbf{c}$ constist of a right-handed system, then $(\mathbf{a}, \mathbf{b}, \mathbf{c})$ is the volume of the Parallelepiped of $\mathbf{a}, \mathbf{b}, \mathbf{c}$ vectors. Otherwise, you got the -1 times of the volume.
- It is easy to prove that

$$(\mathbf{a}, \mathbf{b}, \mathbf{c}) = (\mathbf{b}, \mathbf{c}, \mathbf{a}) = (\mathbf{c}, \mathbf{a}, \mathbf{b}) = -(\mathbf{a}, \mathbf{c}, \mathbf{b}) = -(\mathbf{c}, \mathbf{b}, \mathbf{a}) = -(\mathbf{b}, \mathbf{a}, \mathbf{c})\,.$$

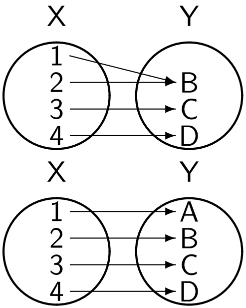
Operators I

- Definition: The set is the sum of things. It is a fundamental term. You need a statement that collects the element. It means that you can decide whether an element is part of the set or not.
- Definition: The pair are sets consisting of two elements.
- Definition: Elements e_1 and e_2 consist of ordered pair if $\{e_1, \{e_2\}\}$. It sign is (e_1, e_2).
- Definition: Relation is the set of ordered pairs.



- Definition: The injection orders different elements (X) to different elements (Y).

Operators II



- Definition: Surjections are those relations, that the values of the relation agree to the values of the set to order.
- Definition: Bijection is an injection and a surjection. All elements are related to all elements of the other set.

Operators III

- Definition: The functions are such a set of ordered pairs in that one element shows up only once:

$$(\forall x)\,(\forall y_1)\,(\forall y_2)\,[(x, y_1) \in f \wedge (x, y_2) \in f \Rightarrow y_1 = y_2]$$

- Definition: V are U vectorspaces above $\mathbb{T}$ body. The $f : V \to U$ relation is linear if it is
  1. Additive, for all $v_1, v_2 \in V$ vectors $f\,(v_1 + v_2) = f\,(v_1) + f\,(v_2)$.
  2. Homogen, for all $v \in V$ vectors and $\lambda \in \mathbb{T}$ elements $f\,(\lambda v) = \lambda f\,(v)$.
- Definition: Operators are the linear vector-vector functions.
- Például:
  - Identical operator: $\mathbf{A} \cdot \mathbf{1} = \mathbf{A}$, for all $\mathbf{A}$ operators.
  - Null operator: $\mathbf{A} \cdot \mathbf{0} = \mathbf{0}$, for all $\mathbf{A}$ operators.

Operators IV

- Mirror operators: $(\mathbf{A} \cdot \mathbf{T}) \cdot \mathbf{T} = \mathbf{A}$, for all $\mathbf{A}$ operators.
- Projection operator: $\mathbf{A} \cdot \mathbf{P} = \mathbf{P}$, for all $\mathbf{A}$ operator.
- Rotational operator: later.

- Operators could be multiplied on both sides.
- The representation of operators is the matrixes. See $\alpha_{ij} \in \mathbb{R}$ for all $i \in \{1, 2, \ldots, m\}$ and $j \in \{1, 2, \ldots, n\}$, where $m, n \in \mathbb{N}^+$. The

$$A = \begin{pmatrix} \alpha_{11} & \alpha_{12} & \cdots & \alpha_{1n} \\ \alpha_{21} & \alpha_{22} & \cdots & \alpha_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_{m1} & \alpha_{m2} & \cdots & \alpha_{mn} \end{pmatrix}$$

table is called $m \times n$ type matrix. The set of the $m \times n$ type matrixes is $M_{m \times n}$.

Operators V

- The spur of the matrix is the set of $\{\alpha_{11}, \alpha_{22}, \ldots, \alpha_{nn}\}$.
- The first index of the elements α_{ij} is the rowindex (i), the 2nd index is the column index (j).
- The Row i of the Matrix is A_i , and the Column $j$of the matrix is A_jj.
- Determinant!!!

The End

Thank you for your attention!