# (PTIA0301) Elementary Linear Algebra

Dr. Gabor FACSKO, PhD
Senior Research Fellow
*facskog@gamma.ttk.pte.hu*

University of Pecs, Faculty of Sciences, Institute of Mathematics and Informatics, 7624 Pecs, Ifjusag utja 6.
Wigner Research Centre for Physics, Department of Space Physics and Space Technology, 1121 Budapest, Konkoly-Thege Miklos ut 29-33.
`https://facsko.ttk.pte.hu`

November 7, 2024

# Common Business

- I evaluated the tests of my Hungarian groups and solved the test examples again.
- Today we will practice.

# Transpose I

- Definition: The transpose of the $A = (\alpha_{ij})_{m \times n}$ matrix is the $A^T = (\alpha_{ji})_{m \times n}$. This means the change of the rows and columns. The transpose is the mirror of a square matrix.

$$A_{m\times n} = \begin{pmatrix} \alpha_{11} & \alpha_{12} & \cdots & \alpha_{1n} \\ \alpha_{21} & \alpha_{22} & \cdots & \alpha_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_{m1} & \alpha_{m2} & \cdots & \alpha_{mn} \end{pmatrix} \quad A^T_{n\times m} = \begin{pmatrix} \alpha_{11} & \alpha_{21} & \cdots & \alpha_{1m} \\ \alpha_{12} & \alpha_{22} & \cdots & \alpha_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_{1n} & \alpha_{2n} & \cdots & \alpha_{mn} \end{pmatrix}$$

- Examples for transpose.

# Matrix Operations I

- <u>Definition:</u> $A = (\alpha_{ij})_{m\times n}$ and $B = (\beta_{ij})_{m\times n}$ are two matrixes with same type, $\lambda \in \mathbb{R}$ a scalar. The sum of Matrixes $A$ and $B$ is Matrix $A + B = (\alpha_{ij} + \beta_{ij})_{m\times n}$, the $\lambda$ times Matrix $A$ is Matrix $\lambda A = (\lambda\alpha_{ij})_{m\times n}$.

$$A_{m\times n} = \begin{pmatrix} \alpha_{11} & \alpha_{12} & \cdots & \alpha_{1n} \\ \alpha_{21} & \alpha_{22} & \cdots & \alpha_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_{m1} & \alpha_{m2} & \cdots & \alpha_{mn} \end{pmatrix} B_{m\times n} = \begin{pmatrix} \beta_{11} & \beta_{12} & \cdots & \beta_{1n} \\ \beta_{21} & \beta_{22} & \cdots & \beta_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \beta_{m1} & \beta_{m2} & \cdots & \beta_{mn} \end{pmatrix}$$

$$A_{m\times n} + B_{m\times n} = \begin{pmatrix} \alpha_{11} + \beta_{11} & \alpha_{12} + \beta_{12} & \cdots & \alpha_{1n} + \beta_{1n} \\ \alpha_{21} + \beta_{21} & \alpha_{22} + \beta_{22} & \cdots & \alpha_{1n} + \beta_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_{m1} + \beta_{m1} & \alpha_{m2} + \beta_{m2} & \cdots & \alpha_{mn} + \beta_{mn} \end{pmatrix}$$

## Matrix Operations II

$$\lambda A_{m\times n} = \begin{pmatrix} \lambda\alpha_{11} & \lambda\alpha_{12} & \cdots & \lambda\alpha_{1n} \\ \lambda\alpha_{21} & \lambda\alpha_{22} & \cdots & \lambda\alpha_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \lambda\alpha_{m1} & \lambda\alpha_{m2} & \cdots & \lambda\alpha_{mn} \end{pmatrix}$$

The elements of the matrixes are added and multiplying by a scalar means to multiply all elements of the matrix by the scalar.

- Examples for matrix operations.

# Matrix Operations III

- Definition: $A = (\alpha_{ij})_{m\times n}$ and $B = (\beta_{ij})_{n\times k}$ are two matrixes. The product of Matrixes $A$ and $B$ is Matrix $A \cdot B = (\gamma_{ij})_{m\times k}$, where
$$\gamma_{ij} = \sum_{l=1}^{n} \alpha_{il}\beta_{lj}.$$

  Or:
$$A_{m\times n} = \begin{pmatrix} \alpha_{11} & \alpha_{12} & \cdots & \alpha_{1n} \\ \alpha_{21} & \alpha_{22} & \cdots & \alpha_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_{i1} & \alpha_{i2} & \cdots & \alpha_{in} \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_{m1} & \alpha_{m2} & \cdots & \alpha_{mn} \end{pmatrix} \quad B_{n\times k} = \begin{pmatrix} \beta_{11} & \beta_{12} & \cdots & \beta_{1j} & \cdots & \beta_{1k} \\ \beta_{21} & \beta_{22} & \cdots & \beta_{2j} & \cdots & \beta_{2k} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ \beta_{n1} & \beta_{n2} & \cdots & \beta_{nj} & \cdots & \beta_{nk} \end{pmatrix}$$
$$A\cdot B_{m\times k} = \begin{pmatrix} \alpha_{11}\beta_{11} + \alpha_{12}\beta_{21} + \cdots + \alpha_{1n}\beta_{n1} & \alpha_{11}\beta_{12} + \alpha_{12}\beta_{22} + \cdots + \alpha_{1n}\beta_{n2} & \cdots & \alpha_{11}\beta_{1k} + \alpha_{12}\beta_{2k} + \cdots + \alpha_{1n}\beta_{nk} \\ \alpha_{21}\beta_{11} + \alpha_{22}\beta_{21} + \cdots + \alpha_{2n}\beta_{n1} & \alpha_{21}\beta_{12} + \alpha_{22}\beta_{22} + \cdots + \alpha_{2n}\beta_{n2} & \cdots & \alpha_{21}\beta_{1k} + \alpha_{22}\beta_{2k} + \cdots + \alpha_{2n}\beta_{nk} \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_{m1}\beta_{11} + \alpha_{m2}\beta_{21} + \cdots + \alpha_{mn}\beta_{n1} & \alpha_{m1}\beta_{12} + \alpha_{m2}\beta_{22} + \cdots + \alpha_{mn}\beta_{n2} & \cdots & \alpha_{m1}\beta_{1k} + \alpha_{m2}\beta_{2k} + \cdots + \alpha_{mn}\beta_{nk} \end{pmatrix}$$

- Examples for matrix multiplications.

# Matrix Operations IV

- For similar square matrixes the condition of matrix production is fulfilled and the product will be the same type. Therefore, there is an exponentiation of matrixes:

$$A^1 = A \quad \text{és} \quad A^m = AA^{m-1}$$

  where $(m \geq 2)$ és $A \in \mathcal{M}_{n \times n}$. Let us consider $A^0 = E_m$.
- Thesis: Equlations of matrix exponentation:

$$\begin{aligned} A^m A^k &= A^{m+k} \\ (A^m)^k &= A^{mk}, \end{aligned}$$

  ahol $m, k \in \mathbb{N}$.
  Deduction: It is trivial based on the definition of matrixproduct.
- Examples for matrix exponentation.

# Matrix Inversion I

- Definition: The $n^{th}$ order identity matrix is:

$$E_n = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}$$

- Thesis: For all $A \in \mathcal{M}_{n\times n}$: $A \cdot E_n = E_n \cdot A = A$, ormatrix $E_n$ is identity element of the $n \times n$ square matrixes for matrix production.
  Deduction: $A = (\alpha_{ij})_{n\times n}$ and $E_n = (\beta_{ij})_{n\times n}$ are two matrixes, where $\beta_{ij} = 1$, if $i = j$, otherwise it is zero. The product of Matrixes $A$ and $E_n$ is Matrix $A \cdot E_n = (\sum_{l=1}^{n} \alpha_{il}\beta_{lj})_{n\times n}$. It is Matrix $A = (\alpha_{ij})_{n\times n}$, because the definition of $b_{ij}$ erases all other elements than $\alpha_{ij}$.

# Matrix Inversion II

- Definition: Square matrix $A \in \mathcal{M}_{n \times n}$ exists inverse if exist such a Matrix $B \in \mathcal{M}_{n \times n}$, that $AB = BA = E_n$. The inverse of Matrix $A$ is $A^{-1}$.
- Thesis: Matrix $A \in \mathcal{M}_{n \times n}$ exists inverse if only $\det(A) \neq 0$.
- Matrix $A \in \mathcal{M}_{n \times n}$ is regular if $\det(A) \neq 0$.
- Matrix $A \in \mathcal{M}_{n \times n}$ is singular if $\det(A) = 0$.

# Matrix Inversion III

- Inverse matrix calculation by elemental transformations:
  - Mulitplication of a row by a $\lambda \neq 0$ scalar.
  - Adding $\lambda$ times of a row to another row.
  - Changing of rows.

  If Matrix $A$ is a regular matrix, then the $(A|E_n)$ extended matrix could be transformed for $(E_n|B)$ form, where Matrix $B$ is the inverse of Matrix $A$.

  This transformation cannot be made for singular matrixes.

- Examples for matrix inversion by elemental transformations.

# Matrix Inversion IV

- Calculation of inverse matrix by subdeterminant.
  - You calculate the determinant of the matrix. The inverse exists if the determinant is not zero.
  - $A_{ij}$ is the subdeterminant for each element. The result must be transposed and divided by $\det(A)$ you get the inverse of Matrix $A$:

$$\left(A^{-1}\right)_{ij} = \frac{A_{ij}}{\det(A)}.$$

(The subdeterminant of Matrix $A$'s $\alpha_{ij}$ element is: $A_{ij} = (-1)^{i+j} D_{ij}$, where $D_{ij}$ is the determinant of the $(n-1) \times (n-1)$ matrix created by deleting the row and column of the element $\alpha_{ij}$.

- Examples for matrix inversion by subdeterminants.

# Matrix Inversion V

- Thesis: $A, B \in \mathcal{M}_{n\times n}$.
  1. If Matrixes $A$ and $B$ have invers, then $AB$ also has inverse and $(AB)^{-1} = B^{-1}A^{-1}$.
  2. $(AB)^{T} = B^{T}A^{T}$
  3. If $A$ has inverse, then $A^{T}$ also has inverse, and $\left(A^{T}\right)^{-1} = \left(A^{-1}\right)^{T}$.

- Examples for these statements.

# Matrix Rank I

- Definition: $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_s \in V$ are vectors. The rank of the $\{\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_s\}$ vector system is the dimension of the $\mathcal{L}(\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_s)$ subspace. Its sign is $\rho(\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_s)$.
- Thesis: The following transformation do not change the order of the $\{\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_s\}$ vector system:
  1. Multiplying a vectors by a $\lambda \neq 0$ scalar.
  2. Adding the vector multiplied by $\lambda$ to another vector.
  3. Eliminating a vector that is a linear combination of the remaining vectors.
  4. Changing the order of vectors.
- Definition: The rank of Matrix $A \in \mathcal{M}_{m \times n}$ is the rank of its row vector system.
- The rank of a matrix is equal to the common rank of the maximal ranked non-disappearing subdeterminants.

# Matrix Rank II

- The rank of a matrix is determined by transforming the matrix to trapezoid form by rank invariant transformations. You can change the columns. (A matrix has trapesoid shape if $\alpha_{ij} = 0$, $i > j$, and $\alpha_{ii} \neq 0$, where $(1 \leq i \leq min\{m, n\})$.) Rows and columns containing 0 could be deleted. The rank of the trapezoid matrix is the number of its rows.
- Examples of determination of the rank of a matrix.

# The End

Thank you for your attention!