I
Quantum Computing Basics

Bra–Ket Notation

At fist we introduce some notations for quantum computing, this starts with notation for spin up and spin down here:

$$|1\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad \text{(spin up)}, \qquad |0\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \text{(spin down)}.$$

Each of these represents one qubit. Using $|1\rangle$ and $|0\rangle$ we can define single qubit superposition as:

$$\begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \frac{1}{\sqrt{2}} \big(|0\rangle + |1\rangle \big).$$

Based on the same logic we can define higher dimensions for example, the computational basis for two qubits is:

$$|00\rangle = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad |01\rangle = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad |10\rangle = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \quad |11\rangle = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

which is equivalent to:

$$|00\rangle = |0\rangle \otimes |0\rangle\,, \quad |01\rangle = |0\rangle \otimes |1\rangle\,, \quad |10\rangle = |1\rangle \otimes |0\rangle\,, \quad |11\rangle = |1\rangle \otimes |1\rangle\,.$$

The above notion is a tensor product notion. One example of tensor product is:

$$|0\rangle \otimes |1\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \otimes \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} = |01\rangle\,.$$

This means we have two qubits: the first is in $|0\rangle$ and the second is in $|1\rangle$.

Similarly, we can have entanglement:

$$\begin{bmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ 0 \\ \frac{1}{\sqrt{2}} \end{bmatrix} = \frac{1}{\sqrt{2}}\big(|00\rangle + |11\rangle\big), \qquad \text{(superposition + entanglement).}$$

Classical Logic Gates (Truth Tables)

To better understand the Quantum gates, we start from the classical logic gates. Hence, we introduce the classical logic gates:

Buffer Buffer gate passes the same bit as is (this is a classical bit so the out put (Q) can be either 0 or 1. Depending on the input (A) for the Buffer gate, we get the same bit with no change:

A	Q
0	0
1	1

NOT NOT gate changes flips the 0 to 1, and 1 to 0:

A	Q
0	1
1	0

AND Gate AND is based on two inputs and one classical bit for output:

A	B	Q
0	0	0
0	1	0
1	0	0
1	1	1

OR Gate OR is also based on two inputs and one output:

A	B	Q
0	0	0
0	1	1
1	0	1
1	1	1

NAND Gate NAND is the flipped output of gate AND:

A	B	Q
0	0	1
0	1	1
1	0	1
1	1	0

Quantum Gates

Now, we focus our attention into the quantum gates,

Pauli-X (NOT): The Pauli matrix: $X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, in practice performs the same logical flip as the classical NOT gate.

$$X|1\rangle = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} = |0\rangle \,, \tag{I.1}$$

$$X|0\rangle = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} = |1\rangle \,. \tag{I.2}$$

Pauli-Y

$$Y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}. \tag{I.3}$$

Pauli-Z

$$Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}. \tag{I.4}$$

Hadamard Hadamard is one the most important gates in quantum computing:

$$H = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}. \tag{I.5}$$

Hadamard makes superposition! For example, when it is applied into $|0\rangle$, the output becomes a superposition of $|0\rangle$ and $|1\rangle$:

$$H|0\rangle = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \frac{1}{\sqrt{2}} \left(|0\rangle + |1\rangle\right), \tag{I.6}$$

$$H|1\rangle = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \frac{1}{\sqrt{2}} \left(|0\rangle - |1\rangle\right). \tag{I.7}$$

Conditional NOT (CNOT) Operates on two qubits, its matrix form is:

$$\mathrm{CNOT} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}. \tag{I.8}$$

Here is the operation of CNOT on $|01\rangle$:

$$\mathrm{CNOT}\,|01\rangle = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} = |01\rangle\,. \tag{I.9}$$

Hence we have:

$$\mathrm{CNOT}\,|00\rangle = |00\rangle\,, \qquad \mathrm{CNOT}\,|01\rangle = |01\rangle\,, \tag{I.10}$$
$$\mathrm{CNOT}\,|10\rangle = |11\rangle\,, \qquad \mathrm{CNOT}\,|11\rangle = |10\rangle\,. \tag{I.11}$$

With out going through the computation also we can use the control logic which is, if the control (first bit) is 0, the target (second bit) remains unchanged, and if the control is 1, then the target will flip.

From Superposition to Entanglement (Hadamard + CNOT)

To demonstrate demonstrate how entanglement arises, we will build our first two-qubit entangled state: a Bell state. We begin by introducing a few basic circuit conventions. In Fig. I.1, the horizontal wires represent qubit lines in the circuit, and we initialize both qubits in the state $|0\rangle$.

Starting from the two-qubit state $|00\rangle$, we apply a Hadamard gate to the first qubit q_1. This creates the superposition state $|+\rangle$:

$$H\,|0\rangle = |+\rangle = \frac{1}{\sqrt{2}}\big(|0\rangle + |1\rangle\big). \tag{I.12}$$

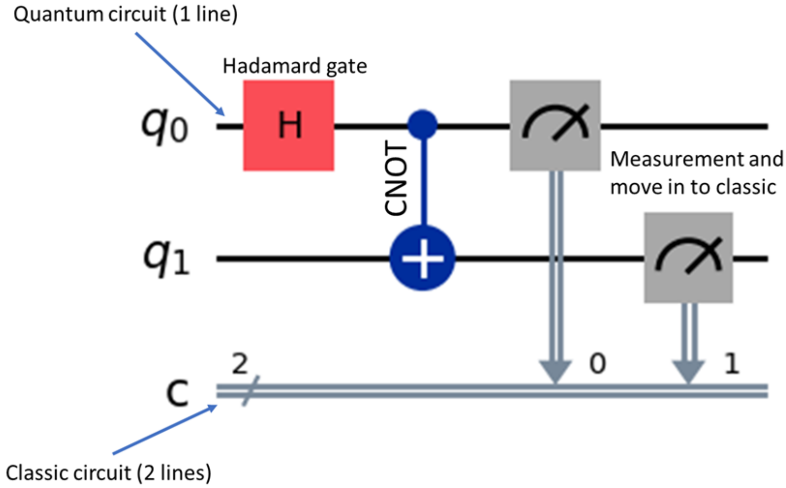

Figura I.1: Schematics of a quantum circuit that prepares a Bell state.

Including the second qubit (still in $|0\rangle$), the joint state becomes

$$\begin{aligned} |+\rangle \otimes |0\rangle &= \frac{1}{\sqrt{2}}\big(|0\rangle \otimes |0\rangle + |1\rangle \otimes |0\rangle\big) \\ &= \frac{1}{\sqrt{2}}\big(|00\rangle + |10\rangle\big). \end{aligned} \tag{I.13}$$

Next, we apply a CNOT gate with the first qubit as the control and the second as the target:

$$\mathrm{CNOT}\left[\frac{1}{\sqrt{2}}\big(|00\rangle + |10\rangle\big)\right] = \frac{1}{\sqrt{2}}\big(|00\rangle + |11\rangle\big). \tag{I.14}$$

Because CNOT flips the target qubit only when the control qubit is $|1\rangle$, the two basis states become correlated. The resulting state

$$\frac{1}{\sqrt{2}}\big(|00\rangle + |11\rangle\big)$$

is a maximally entangled Bell state: $|00\rangle$ and $|11\rangle$ occur with equal probability upon measurement. Once a measurement is performed (the measurement symbol is indicated in Fig. I.1), the quantum state collapses to one of these classical outcomes.

The code used to build the circuit and perform the measurements is available at `https://github.com/gfarhani/Quantum_Computing/tree/main/Chapter_1/Bell_State.ipynb`.