IV
Quantum Phase Estimation

To understand Quantum Phase Estimation (QPE), we begin by recalling the Quantum Fourier Transform (QFT), defined as:

$$|x\rangle \to \frac{1}{\sqrt{N}} \sum_{y=0}^{N-1} \omega_N^{-xy} |y\rangle \quad \text{with } \omega_N = e^{2\pi i/N}$$

Its inverse transformation can be written as:

$$\frac{1}{\sqrt{N}} \sum_{y} \omega_N^{xy} |y\rangle \to |x\rangle$$

QFT generalizes the idea of phase kickback. Unlike classical Boolean logic where operations may flip a sign (e.g., ± 1), QFT introduces more nuanced phase rotations—complex roots of unity:

$$e^{2\pi i k/N}$$

These values represent points on the unit circle in the complex plane. This enables fine-grained phase encoding, which is central to QPE.

If a unitary operator U has an eigenvector $|\psi\rangle$ with eigenvalue $e^{2\pi i\phi}$, then QPE enables the estimation of ϕ.

Although we cannot measure the phase directly, we can induce a phase kickback using controlled applications of powers of U and then apply QFT to extract ϕ into measurable qubits.

Suppose we can prepare an eigenstate $|\psi\rangle$ of a unitary U and apply controlled-U^{2^j} gates. If

$$U|\psi\rangle = e^{2\pi i\phi}|\psi\rangle \quad \Rightarrow \quad U^{2^j}|\psi\rangle = e^{2\pi i\cdot 2^j\phi}|\psi\rangle$$

then controlled unitaries act as:

$$CU(|0\rangle \otimes |\psi\rangle) = |0\rangle \otimes |\psi\rangle, \quad CU(|1\rangle \otimes |\psi\rangle) = |1\rangle \otimes U|\psi\rangle$$

Starting with the superposition state:

$$|\text{init}\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \otimes |\psi\rangle$$

After applying the controlled-U:

$$\frac{1}{\sqrt{2}}\left(|0\rangle \otimes |\psi\rangle + |1\rangle \otimes e^{2\pi i\phi}|\psi\rangle\right) = \frac{1}{\sqrt{2}}\left(|0\rangle + e^{2\pi i\phi}|1\rangle\right) \otimes |\psi\rangle$$

Extending to n control qubits, the combined state becomes:

$$\frac{1}{\sqrt{2^n}}\bigotimes_{j=0}^{n-1}\left(|0\rangle + e^{2\pi i\cdot 2^{n-1-j}\phi}|1\rangle\right) \tag{IV.1}$$

This encodes the binary representation of ϕ through relative phases of the qubit states.

Recall the binary expansion:

$$\phi = 0.\phi_1\phi_2\ldots\phi_n = \frac{\phi_1}{2} + \frac{\phi_2}{4} + \cdots + \frac{\phi_n}{2^n}$$

Each control qubit accumulates phase based on the significance of its bit: the highest qubit collects the most significant part of ϕ.

If ϕ has an exact n-bit binary representation, then applying the inverse QFT yields:

$$|\Psi\rangle = \frac{1}{\sqrt{2^n}} \sum_{x=0}^{2^n-1} e^{2\pi i \phi x} |x\rangle \quad \xrightarrow{\text{QFT}^{-1}} \quad |y\rangle \quad \text{where } y = 2^n \cdot \phi$$

Applying inverse QFT explicitly:

$$\text{QFT}^{-1} \left(\sum_{x=0}^{2^n-1} e^{2\pi i \phi x} |x\rangle \right) = \sum_{x=0}^{2^n-1} e^{2\pi i \phi x} \cdot \text{QFT}^{-1} |x\rangle$$

$$\text{QFT}^{-1} |x\rangle = \frac{1}{\sqrt{2^n}} \sum_{y=0}^{2^n-1} e^{-2\pi i x y / 2^n} |y\rangle$$

$$\Rightarrow \text{QFT}^{-1} |\Psi\rangle = \frac{1}{2^n} \sum_{y=0}^{2^n-1} \left(\sum_{x=0}^{2^n-1} e^{2\pi i x (\phi - y/2^n)} \right) |y\rangle$$

Thus, the amplitude for outcome y is:

$$A(y) = \frac{1}{2^n} \sum_{x=0}^{2^n-1} e^{2\pi i x (\phi - y/2^n)} \quad \Rightarrow \quad P(y) = |A(y)|^2$$

This means that $P(y)$ is maximized when $\frac{y}{2^n}$ is close to ϕ.

Define the phase error as:

$$\delta = \phi - \frac{y}{2^n} \quad \text{so that} \quad r = e^{2\pi i \delta}$$

We use the geometric series identity:

$$\sum_{x=0}^{2^n-1} r^x = \frac{1 - r^{2^n}}{1 - r} \quad \Rightarrow \quad P(y) = \left| \frac{1 - r^{2^n}}{2^n (1 - r)} \right|^2$$

When $\phi = \frac{y}{2^n}$ exactly, then $r = 1$ and $P(y) = 1$.
If not exact but close, we can show:

$$P(y) \geq \frac{4}{\pi^2} \approx 0.405$$

This guarantees a minimum of 40%.