I
Quantum Computing Basics

This chapter introduces the minimum toolkit needed to read and write quantum circuits: bra–ket notation, tensor products, a small set of quantum gates, and two landmark constructions: Bell-state preparation and quantum teleportation.

1.1 Bra–Ket Notation and the Computational Basis

A single qubit is a unit vector in a two-dimensional complex vector space. The standard (computational) basis is written as

$$|0\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \qquad |1\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

In many physics contexts these are also associated with spin-down and spin-up (depending on convention). In this book, we will primarily interpret $|0\rangle$ and $|1\rangle$ as computational states.

A single-qubit pure state can be expressed as a linear combination of the computational basis states:

$$|\psi\rangle = \alpha |0\rangle + \beta |1\rangle, \qquad \alpha, \beta \in C, \qquad |\alpha|^2 + |\beta|^2 = 1.$$

A standard example is the balanced (equal-amplitude) superposition state:

$$|+\rangle = \frac{1}{\sqrt{2}}\big(|0\rangle + |1\rangle\big) = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}.$$

I.1.1 Multiple qubits and tensor products

For two qubits, the computational basis consists of four basis vectors,

$$|00\rangle = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad |01\rangle = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad |10\rangle = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \quad |11\rangle = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

These are tensor products of single-qubit basis states:

$$|00\rangle = |0\rangle \otimes |0\rangle, \quad |01\rangle = |0\rangle \otimes |1\rangle, \quad |10\rangle = |1\rangle \otimes |0\rangle, \quad |11\rangle = |1\rangle \otimes |1\rangle.$$

For example,

$$|0\rangle \otimes |1\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \otimes \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} = |01\rangle.$$

This notation means: the first qubit is in $|0\rangle$ and the second qubit is in $|1\rangle$.

1.1.2 Entanglement (a first preview)

Not every two-qubit state can be written as a simple tensor product of two single-qubit states. A canonical example is the Bell state

$$|\Phi^+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ 0 \\ \frac{1}{\sqrt{2}} \end{bmatrix}.$$

This is an entangled state: measurements of the two qubits are perfectly correlated.

1.2 Classical Logic Gates (Truth Tables)

Quantum gates generalize classical logic, but there is one key difference: classical logic gates are typically irreversible (information can be destroyed), while basic quantum gates must be reversible (unitary). We therefore begin with classical truth tables as intuition.

Buffer A buffer gate outputs the input unchanged:

A	Q
0	0
1	1

NOT A NOT gate flips the bit:

A	Q
0	1
1	0

AND

A	B	Q
0	0	0
0	1	0
1	0	0
1	1	1

OR

A	B	Q
0	0	0
0	1	1
1	0	1
1	1	1

NAND NAND is the negation of AND:

A	B	Q
0	0	1
0	1	1
1	0	1
1	1	0

I.3 Quantum Gates

Quantum gates are represented by unitary matrices. They act linearly on state vectors and preserve normalization.

Pauli-X (quantum NOT)

$$X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

It flips $|0\rangle \leftrightarrow |1\rangle$:

$$X|1\rangle = |0\rangle, \tag{I.1}$$

$$X|0\rangle = |1\rangle. \tag{I.2}$$

Pauli-Y

$$Y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}.$$

Pauli-Z (phase flip)

$$Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

This gate leaves $|0\rangle$ unchanged and flips the sign of $|1\rangle$.

Hadamard

$$H = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}.$$

It maps computational basis states into superpositions:

$$H|0\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) = |+\rangle, \tag{I.3}$$

$$H|1\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle) = |-\rangle. \tag{I.4}$$

Controlled-NOT (CNOT) CNOT acts on two qubits: a control and a target. In the basis $\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$,

$$\mathrm{CNOT} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

It flips the target iff the control is $|1\rangle$:

$$\mathrm{CNOT}|00\rangle = |00\rangle, \qquad \mathrm{CNOT}|01\rangle = |01\rangle, \tag{I.5}$$

$$\mathrm{CNOT}|10\rangle = |11\rangle, \qquad \mathrm{CNOT}|11\rangle = |10\rangle. \tag{I.6}$$

I.3.1 From Superposition to Entanglement (Hadamard + CNOT)

We now build our first maximally entangled two-qubit state. Initialize both qubits in $|00\rangle$. Apply a Hadamard on the first qubit:

$$|00\rangle \xrightarrow{H\otimes I} |+\rangle \otimes |0\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |10\rangle).$$

Then apply CNOT (control on the first qubit, target on the second):

$$\frac{1}{\sqrt{2}}(|00\rangle + |10\rangle) \xrightarrow{\text{CNOT}} \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) = |\Phi^+\rangle .$$

This state is entangled because the two qubits cannot be described independently: measuring one immediately determines the other.

Fig. I.1 shows the circuit. The code used to build the circuit and perform the measurements is available at `https://github.com/gfarhani/Quantum_Computing/tree/main/Chapter_1/Bell_State.ipynb`.

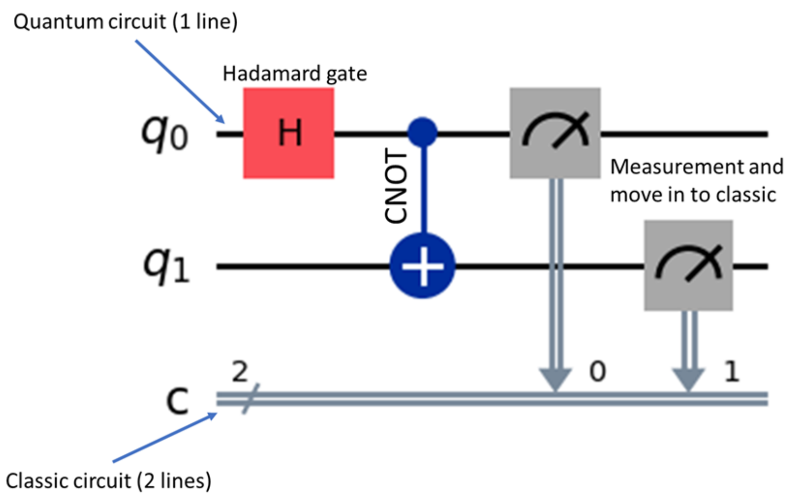

Figura I.1: A quantum circuit that prepares a Bell state $|\Phi^+\rangle = (|00\rangle + |11\rangle)/\sqrt{2}$.

I.4 Teleportation

Quantum teleportation transfers an unknown quantum state using (i) one shared entangled pair and (ii) two classical bits. Importantly, it does not copy the state: this is consistent with the no-cloning theorem.

We use three qubits. Alice holds q_1 and q_2, Bob holds q_3. Alice wants Bob to end up with the state

$$|\psi\rangle = \alpha |0\rangle + \beta |1\rangle ,$$

initially stored in q_1.

I.4.1 Measurement outcomes and Bob's correction

Alice performs a Bell-basis measurement on her two qubits, producing two classical bits $(m_1, m_2) \in \{00, 01, 10, 11\}$. Bob then applies a simple correction on q_3:

(m_1, m_2)	Bob applies on q_3
00	I
01	X
10	Z
11	ZX

The goal is that after Bob's correction, the state of q_3 is exactly $|\psi\rangle$.

I.4.2 Compute the joint state step by step

Initially, q_2 and q_3 are prepared in $|00\rangle$, so the full state is

$$|\Psi_0\rangle = |\psi\rangle_1 |0\rangle_2 |0\rangle_3 .$$

Alice and Bob first create a Bell pair between q_2 and q_3:

$$|\beta_{00}\rangle_{23} = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle).$$

The combined three-qubit state becomes

$$\begin{aligned} |\Psi_1\rangle &= |\psi\rangle_1 \otimes |\beta_{00}\rangle_{23} \\ &= (\alpha\,|0\rangle + \beta\,|1\rangle) \otimes \frac{1}{\sqrt{2}}\,(|00\rangle + |11\rangle) \\ &= \frac{1}{\sqrt{2}}\Big(\alpha\,|000\rangle + \alpha\,|011\rangle + \beta\,|100\rangle + \beta\,|111\rangle\Big), \end{aligned} \tag{I.7}$$

where the ordering is $|q_1 q_2 q_3\rangle$.

Next Alice applies CNOT with control q_1 and target q_2:

$$\begin{aligned} |\Psi_2\rangle &= \mathrm{CNOT}_{1\to 2}\,|\Psi_1\rangle \\ &= \frac{1}{\sqrt{2}}\Big(\alpha\,|000\rangle + \alpha\,|011\rangle + \beta\,|110\rangle + \beta\,|101\rangle\Big). \end{aligned} \tag{I.8}$$

Then Alice applies a Hadamard gate to q_1:

$$|\Psi_3\rangle = H_1\,|\Psi_2\rangle\,. \tag{I.9}$$

Expanding $H\,|0\rangle = (|0\rangle + |1\rangle)/\sqrt{2}$ and $H\,|1\rangle = (|0\rangle - |1\rangle)/\sqrt{2}$ yields

$$\begin{aligned} |\Psi_3\rangle = \frac{1}{2}\Big[\, &|00\rangle\,\big(\alpha\,|0\rangle + \beta\,|1\rangle\,\big) + |01\rangle\,\big(\alpha\,|1\rangle + \beta\,|0\rangle\,\big) \\ &+ |10\rangle\,\big(\alpha\,|0\rangle - \beta\,|1\rangle\,\big) + |11\rangle\,\big(\alpha\,|1\rangle - \beta\,|0\rangle\,\big)\Big] \end{aligned} \tag{I.10}$$

Equivalently, we can write this in terms of Pauli corrections applied to Bob's qubit:

$$|\Psi_3\rangle = \frac{1}{2}\left[|00\rangle\;(I\,|\psi\rangle) + |01\rangle\;(X\,|\psi\rangle) + |10\rangle\;(Z\,|\psi\rangle) + |11\rangle\;(XZ\,|\psi\rangle)\right]. \tag{I.11}$$

Therefore, once Alice measures (m_1, m_2), Bob knows exactly which Pauli operator to apply so that q_3 becomes $|\psi\rangle$.

Fig. I.2 shows the teleportation circuit, and code to reproduce it can be found in `https://github.com/gfarhani/Quantum_Computing/tree/main/Chapter_1/Teleportation.ipynb`.

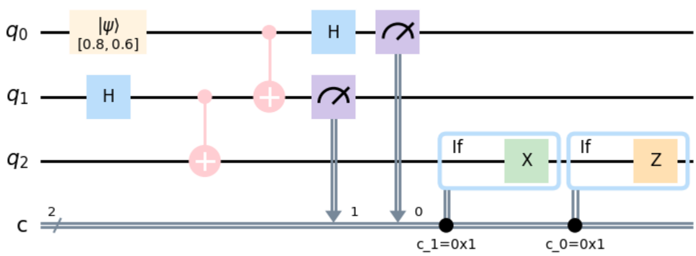

Figura I.2: Quantum teleportation circuit. Alice measures her two qubits and sends two classical bits to Bob, who applies the appropriate correction.