Macroeconomics B

Solution to problem set 6

1. Remember the generic Bellman equation for the value function of an agent in state ω receiving a flow of income $m(\omega)$

$$\rho V(\omega) = m(\omega) + \frac{E[V(\omega') - V(\omega)|\omega]}{dt} \tag{107}$$

or, with just a change in notation,

$$\rho V(\omega) = m(\omega) + \frac{E[dV(\omega)|\omega]}{dt}. \tag{108}$$

For an agent in state c (with a coconut tree) $m(\omega) = y$ and the Bellman equation is

$$\rho V(c) = y + \frac{E[dV(c)|c]}{dt}. \tag{109}$$

In example 2, having a coconut tree was a permanent state. Hence, $dV(c) = 0$ and equation (109) implies

$$\rho V(c) = y. \tag{110}$$

This is no longer true in the present question, since the agent may lose the coconut tree. The (instantaneous) expected capital gain associated with such a shock hitting the agent is

$$\frac{E[dV(c)|c]}{dt} = \frac{(1-\delta dt)V(c) + \delta dt V(n) - V(c)}{dt} = \delta(V(n) - V(c)). \tag{111}$$

The agent moves to state n (she has the asset) if she loses her tree.

Replacing in the Bellman equation (109) results in

$$\rho V(c) = y + \delta(V(n) - V(c)). \tag{112}$$

The Bellman equations for the other two states are unchanged. They are

$$\rho V(n) = z + s\left[V(o) - V(n)\right] \tag{113}$$

and

$$V(o) = \max_{i=\{1,0\}} iV(c) + (1-i)V(n) = \max\{V(c), V(n)\}. \tag{114}$$

We can distinguish two cases.

(a) $V(o) = V(c) \geq V(n)$, so that the agent always accepts to exchange her asset if given the opportunity to do so. Equation (113) then becomes

$$\rho V(n) = z + s\left[V(c) - V(n)\right] \tag{115}$$

This, together with (112), forms a system of two equations in the two unknowns $V(n), V(c)$. Subtracting one from the other and rearranging yields

$$V(c) - V(n) = \frac{y-z}{\rho+\delta+s} \tag{116}$$

and

$$\rho V(c) = y - \delta \frac{y-z}{\rho+\delta+s}, \tag{117}$$

$$\rho V(n) = z + s \frac{y-z}{\rho+\delta+s}, \tag{118}$$

This is indeed optimal if $V(c) \geq V(n)$ or, given equation (116), if $y \geq z$.

(b) $V(o) = V(n) > V(c)$, so that the agent never accepts to exchange her asset if given the opportunity to do so. Equation (118) then becomes

$$\rho V(n) = z + s[V(n) - V(n)] = z. \tag{119}$$

Replacing in (112) gives

$$\rho V(c) = y + \delta\left(\frac{z}{\rho} - V(c)\right) \tag{120}$$

or

$$V(c) = \frac{1}{\rho+\delta}\left[y + \frac{\delta z}{\rho}\right]. \tag{121}$$

The agent policy choice is indeed optimal if $V(n) > V(c)$ or if $z > y$.

The policy choice is unchanged relative to example 2. The value functions are different though. In particular, $V(c)$ is lower in case (a) and higher in case (b) relative to the same two cases in example 2, as the agent now loses the coconut tree with positive probability. This reduces her utility if $y > z$ and increases it if $z > y$.

2. Given that the two islands are identical apart from being in a boom or a recession, the minimum set of states has only two states $\{b, r\}$. Denote by $W(i)$ and $w(i)$ with $i = b, r$ respectively the value functions and wages for an agent in state i.

Consider the value function of an agent in an island in recession. The agent can either stay and receive the wage $w(r)$ until the state changes or move to the other island which implies a one-off mobility cost c. The agent chooses optimally between the two options which implies

$$\rho W(r) = \max\{w(r) + \lambda[W(b) - W(r)], \rho(W(b) - c)\}. \tag{122}$$

The situation of an agent in an island in boom is symmetric

$$\rho W(b) = \max\{w(b) + \lambda[W(r) - W(b)], \rho(W(r) - c)\}. \tag{123}$$

As far as equation (122) is concerned there are two possibilities: either the first term or the second term in the max operator is larger.

(a) Consider first the latter case: $\rho W(r) = \rho(W(b) - c)$ or $W(r) = W(b) - c$. The optimal policy for an agent in state r is to move to the island in boom. $W(r) = W(b) - c$ implies $\rho W(b) > \rho(W(r) - c)$. The optimal policy for an agent in an island in boom is to remain in the island.

Therefore equation (122) becomes

$$\rho W(b) = w(b) + \lambda[W(r) - W(b)] \tag{124}$$

which implies

$$\rho W(b) = w(b) - \lambda c. \tag{125}$$

An agent in an island in boom receives the wage $w(b)$ while the boom lasts but switches to the other island and bears a mobility cost c every time a recession struck.

The policy in state r is optimal if $\rho W(r) = \rho(W(b) - c)$ or, from equation (122) if

$$w(r) + \lambda c \leq \rho(W(b) - c) = w(b) - (\lambda + \rho)c \tag{126}$$

or

$$w(b) - w(r) \geq (2\lambda + \rho)c; \tag{127}$$

if the wage in boom is large enough relative to the wage in recession to compensate workers for the mobility cost.

(b) Consider now the alternative case: $\rho W(r) > \rho(W(b) - c)$. The optimal policy for an agent in an island in recession is to stay. Equation (122) becomes

$$\rho W(r) = w(r) + \lambda[W(b) - W(r)]. \tag{128}$$

We need to distinguish two cases for equation (123).

i. $W(b) = W(r) - c$. The optimal policy for an agent in an island in boom is to relocate. This is symmetric to case (a) above. Agents always move to the island in recession.

Replacing in (128) yields

$$\rho W(r) = w(r) - \lambda c. \tag{129}$$

The policy is optimal in state b if

$$w(b) + \lambda c \leq w(r) - (2\lambda + \rho)c \tag{130}$$

or $w(r) - w(b) \geq (2\lambda + \rho)c$. If this condition is satisfies, the assumed policy in state r is also optimal. This is easily checked by noticing $W(b) = W(r) - c$ implies $W(r) > W(b) > W(b) - c$.

ii. Finally, $W(b) > W(r) - c$. The optimal policy for an agent on an island in a boom is not to relocate. Hence, agents never switch island.

The two Bellman equations are

$$\rho W(r) = w(r) + \lambda[W(b) - W(r)] \tag{131}$$

and

$$\rho W(b) = w(b) + \lambda[W(r) - W(b)]. \tag{132}$$

which imply

$$W(b) - W(r) = \frac{w(b) - w(r)}{r + 2\lambda}. \tag{133}$$

For the policies in each state to be optimal it has to be

$$c > W(b) - W(r) > -c \tag{134}$$

or

$$c > \frac{w(b) - w(r)}{r + 2\lambda} > -c. \tag{135}$$

which is the complement of the other two cases. The wage differential is not large enough to induce relocation from one island to the other.

Therefore, for employment to be positive in both islands, one of the following cases must apply.

- $$c \geq \frac{w(b) - w(r)}{r + 2\lambda} \geq 0. \tag{136}$$

  The wage is higher in boom. Workers never leave islands in boom. If they are in an island in recession: (1) they are indifferent between staying and moving if the wage differential exactly offsets the cost of relocating; (2) they do not relocate if the wage differential does not offset the cost of relocating.

- $$0 > \frac{w(b) - w(r)}{r + 2\lambda} \geq -c \tag{137}$$

  or

  $$c \geq \frac{w(r) - w(b)}{r + 2\lambda} \geq 0. \tag{138}$$

  Same as in the previous case but with the island in recession now paying a higher wage.