Macroeconomics B

Solution to problem set 1

The recursive problem can be written in the following form

$$W(a_t, z_t) = \max_{\{c_t, a_{t+1}\}} u(c_t) + \beta E_t W(a_{t+1}, z_{t+1}) \tag{1}$$
$$\text{s.t. } a_{t+1} = (1+r)a_t + y_t - c_t,$$
$$a_t \text{ given}, \lim_{t\to\infty} \frac{a_t}{(1+r)^t} \geq 0,$$

where z_t is the appropriate state variable (to be determined) characterizing the evolution over time of the income process.

Replacing for c_t and maximizing with respect to a_{t+1} we obtain the FOC

$$u'(c_t) = \beta \frac{E_t \partial W(a_{t+1}, z_{t+1})}{\partial a_{t+1}}. \tag{2}$$

Shifting the envelope condition

$$\frac{\partial W(a_t, z_t)}{\partial a_t} = \beta(1+r)\frac{E_t \partial W(a_{t+1}, z_{t+1})}{\partial a_{t+1}} = (1+r)u'(c_t) \tag{3}$$

one period forward and using $\beta(1+r) = 1$ we obtain the Euler equation

$$u(c_t) = E_t u'(c_{t+1}). \tag{4}$$

Given that u' is linear we Euler equation can be rewritten as

$$c_t = E_t c_{t+1} \tag{5}$$

which we use in what follows.

1. Suppose labour income follows the stochastic process

$$y_t = \bar{y} + \varepsilon_t - \delta\varepsilon_{t-1}, \tag{6}$$

with ε_t white noise.

In choosing the state variables for the income process consider that y_t is a function of $(\varepsilon_t, \varepsilon_{t-1})$. Hence, $z_t = (\varepsilon_t, \varepsilon_{t-1})$ is a good candidate state variable for the income process (note that ε_t is sufficient to forecast future ε_{t+s}, $s > 0$, but then one would not fully know y_t without including ε_{t-1} separately).

Guess $c_t = \alpha_0 + \alpha_1 a_t + \alpha_2 \varepsilon_t + \alpha_3 \varepsilon_{t-1}$.

Replacing in the Euler equation and rearranging gives

$$a_{t+1} - a_t = \frac{\alpha_2 - \alpha_3}{\alpha_1}\varepsilon_t + \frac{\alpha_3}{\alpha_1}\varepsilon_{t-1}. \tag{7}$$

Replacing in the dynamic budget identity one obtains

$$a_{t+1} - a_t = ra_t + \bar{y} + \varepsilon_t - \delta\varepsilon_{t-1} - (\alpha_0 + \alpha_1 a_t + \alpha_2 \varepsilon_t + \alpha_3 \varepsilon_{t-1}). \tag{8}$$

Equating the RHS of the two equations yields

$$\frac{\alpha_2 - \alpha_3}{\alpha_1}\varepsilon_t + \frac{\alpha_3}{\alpha_1}\varepsilon_{t-1} = ra_t + \bar{y} + \varepsilon_t - \delta\varepsilon_{t-1} - (\alpha_0 + \alpha_1 a_t + \alpha_2\varepsilon_t + \alpha_3\varepsilon_{t-1}) \tag{9}$$

which requires $\alpha_0 = r,\ \alpha_1 = \bar{y},\ (\alpha_2 - \alpha_3) = r(1 - \alpha_2),\ \alpha_3 = -r(\alpha_3 + \delta)$ or

$$c_t = ra_t + \bar{y} + \frac{r}{1+r}\left[\left(1 - \frac{\delta}{1+r}\right)\varepsilon_t - \delta\varepsilon_{t-1}\right]. \tag{10}$$

It follows that

$$c_{t+1} - c_t = c_{t+1} - E_t c_{t+1} = \frac{r}{1+r}\left(1 - \frac{\delta}{1+r}\right)\varepsilon_{t+1} \tag{11}$$

and

$$s_t = \frac{1}{1+r}\left(\varepsilon_t - \delta\varepsilon_{t-1} + \frac{\delta r}{1+r}\varepsilon_t\right). \tag{12}$$

2. Suppose labour income follows the stochastic process

$$\Delta y_t = \lambda\Delta y_{t-1} + \varepsilon_t, \tag{13}$$

or, equivalently,

$$y_t = (1+\lambda)y_{t-1} - \lambda y_{t-2} + \varepsilon_t \tag{14}$$

with $0 \leq \lambda < 1$ and ε_t white noise.

To choose the state variables for the income process note that y_t is itself a second order Markov process. So the appropriate $z_t = \{y_t, y_{t-1}$. Therefore let's guess $c_t = \alpha_0 + \alpha_1 a_t + \alpha_2 y_t + \alpha_3 y_{t-1}$.

Replacing in the Euler equation

$$a_{t+1} - a_t = -\frac{\alpha_3 + \alpha_2\lambda}{\alpha_1}(y_t - y_{t-1}). \tag{15}$$

and in the dynamic budget identity

$$a_{t+1} - a_t = ra_t + y_t - (\alpha_0 + \alpha_1 a_t + \alpha_2 y_t + \alpha_3 y_{t-1}) \tag{16}$$

and equating we obtain

$$-\frac{\alpha_3 + \alpha_2\lambda}{\alpha_1}(y_t - y_{t-1}) = ra_t + y_t - (\alpha_0 + \alpha_1 a_t + \alpha_2 y_t + \alpha_3 y_{t-1}). \tag{17}$$

This is satisfied if $\alpha_0 = 0,\ \alpha_1 = r,\ -(\alpha_3 + \alpha_2\lambda) = r(1 - \alpha_2),\ (\alpha_3 + \alpha_2\lambda) = -r\alpha_3$ which implies

$$c_t = ra_t + \frac{1+r}{1+r-\lambda}y_t - \frac{\lambda}{1+r-\lambda}y_{t-1}. \tag{18}$$

It follows that

$$c_{t+1} - c_t = c_{t+1} - E_t c_{t+1} = \frac{1+r}{1+r-\lambda}(y_{t+1} - E_t y_{t+1}) = \frac{1+r}{1+r-\lambda}\varepsilon_{t+1}. \tag{19}$$

and

$$s_t = \left(1 - \frac{1+r}{1+r-\lambda}\right)y_t + \frac{\lambda}{1+r-\lambda}y_{t-1} = -\frac{\lambda}{1+r-\lambda}\Delta y_t. \tag{20}$$

If $\lambda = 0$ saving does not respond to income at all. This is not surprising given that in such case the income process is a random walk. Therefore saving cannot smooth shocks which are expected to be permanent. If $\lambda > 0$ saving is a decreasing function of income changes. The intuition is that the income *growth* process is persistent - it is an AR(1). So a positive income change today means a positive expected income growth tomorrow. Therefore saving responds negatively because income tomorrow is expected to increase relative to today.

Macroeconomics B

Solution to problem set 2

1. Given $r = 0$ and $\beta = 1$, a consumer of age t solves

$$\max_{\{c_s, a_{s+1}\}} \sum_{s=t}^{60} u\left(c_s\right) \tag{21}$$

$$\text{s.t. } a_{s+1} = a_s + y_s - c_s, \tag{22}$$

$$a_t \text{ given, } c_s \geq 0, \tag{23}$$

$$\text{solvency constraint } a_{61} = 0. \tag{24}$$

An individual cannot leave a negative stock of wealth at the end of her 60th year and, under the assumption of increasing marginal utility, will not leave a positive stock of wealth either. The labour income process satisfies

$$y_t = \begin{cases} y_{t-1} + \varepsilon_t & \text{if } t \leq 40 \\ 0 & \text{if } t > 40. \end{cases} \tag{25}$$

(a) Replacing using the dynamic constraint (2) in the maximand (1) and maximizing with respect to a_{t+s+1} we obtain the Euler equation

$$u'\left(c_s\right) = E_s u'\left(c_{s+1}\right) \tag{26}$$

which, with quadratic utility, becomes

$$c_s = E_s c_{s+1}. \tag{27}$$

The intertemporal budget constraint can be written as

$$\sum_{s=t}^{60} c_s = a_t + \sum_{s=t}^{60} y_s + a_{61} \tag{28}$$

which using the solvency constraint $a_{61} = 0$ becomes

$$\sum_{s=t}^{60} c_s = a_t + \sum_{s=t}^{60} y_s. \tag{29}$$

The budget constraint holds at any moment in time with probability one, so it has to hold in expectation or

$$\sum_{s=t}^{60} E_t c_{t+s} = a_t + \sum_{s=t}^{60} E_t y_{t+s}. \tag{30}$$

Using the Euler equation (27) and the law of iterated expectations we have $E_t c_{t+s} = c_t$. Applying the law of iterated expectations to the income process we have

$$E_t y_s = \begin{cases} y_t & \text{if } s \leq 40 \\ 0 & \text{if } s > 40. \end{cases} \tag{31}$$

Hence, we can rewrite (30) as

$$\sum_{s=t}^{60} c_t = a_t + \sum_{s=t}^{40} y_t, \tag{32}$$

or

$$(60 - t + 1)\, c_t = a_t + y_t \sum_{s=t}^{40} 1 \tag{33}$$

The consumption function is

$$c_t = \begin{cases} \frac{a_t}{60-t+1} + \frac{40-t+1}{60-t+1} y_t & \text{if } t \leq 40 \\ \frac{a_t}{60-t+1} & \text{if } t > 40 \end{cases}. \tag{34}$$

Consumption spreads financial wealth and the present value of remaining labour income over the remaining lifetime $(60 - t + 1)$ in order to keep a flat expected consumption profile.

(b) Saving equals

$$s_t = r a_t + y_t - c_t = y_t - c_t, \tag{35}$$

given that $r = 0$. Hence, we have

$$s_t = \begin{cases} -\frac{a_t}{60-t+1} + \frac{20}{60-t+1} y_t & \text{if } t \leq 40 \\ -\frac{a_t}{60-t+1} & \text{if } t > 40 \end{cases} \tag{36}$$

Saving is a way to smooth consumption by carrying over purchasing power to the years in which income is expected to be zero. During retirement saving is negative as people run down their assets to keep consumption flat in the fact of zero labour income.

(c) Using the Euler equation (27) we can write

$$c_t - c_{t-1} = c_t - E_{t-1} c_t. \tag{37}$$

Replacing using the consumption function (34) we have

$$c_t - c_{t-1} = \begin{cases} \frac{40-t+1}{60-t+1} (y_t - y_{t-1}) & \text{if } t \leq 40 \\ 0 & \text{if } t > 40 \end{cases}, \tag{38}$$

or

$$c_t - c_{t-1} = \begin{cases} \frac{40-t+1}{60-t+1} \varepsilon_t & \text{if } t \leq 40 \\ 0 & \text{if } t > 40 \end{cases}. \tag{39}$$

So $k_t = \frac{40-t+1}{60-t+1}$, if $t \leq 40$ and zero otherwise. The innovation in permanent income in response to an innovation in labour income is a declining function of age t. In fact

$$\frac{dk_t}{dt} = \frac{-(60 - t + 1) + (40 - t + 1)}{(60 - t + 1)^2} = -\frac{20}{(60 - t + 1)^2} < 0, \text{ if } t \leq 40. \tag{40}$$

The intuition is that the lower the remaining working life the shorter the span of time for which a given innovation in labour income will persist, given that income drops to zero at retirement.

$$k_1 = 40/60 \sim 0.66 \tag{41}$$

$$k_{40} = 1/21 \sim 0.05. \tag{42}$$

(d) The average k_t is

$$\bar{k} = \frac{1}{60}\sum_{t=1}^{39}\frac{40-t+1}{60-t+1} = 0.295 \tag{43}$$

(e) If agents have infinite lifetimes and labour income follows a random walk it is

$$c_t - c_{t-1} = \frac{r}{1+r}\sum_{s=0}^{\infty}\frac{E_t y_{t+s} - E_{t-1}y_{t+s}}{(1+r)^s} = \varepsilon_t. \tag{44}$$

If labour income is a random walk, any income innovation is expected to be permanent. Consumption responds one-to-one to a permanent innovation in labour income. k is one in this case. With finite working lives, even a permanent income innovation is expected to end by the end of the working life hence it has a smaller impact on lifetime income.

Note that if the interest rate were positive k_t in part d. would be higher as agents could consume more of a given income innovation because they would get a positive (rather than a zero) interest rate on their saving. In fact, k_1 would be very close to 1. Yet, k_{39} would still be very close to 0.05 as the interest rate would be positive for just one year given that saving is negative for $t \geq 40$. Hence, with positive discounting the average response of consumption to a permanent innovation in labour income would be somewhere in between the infinite lifetime and the zero interest rate case. Anyway, with finite lifetimes *aggregate* consumption should respond less to an innovation in aggregate income than predicted by the permanent income theory with infinite lifetimes. One could work out exactly by how much less by using the more realistic income process used by Deaton.

2. With instantaneous utility function $u(c_t) = c_t^{1-\gamma}/(1-\gamma)$, the marginal utility is $u'(c_t) = c_t^{-\gamma}$. Hence the Euler equation is

$$c_t^{-\gamma} = \frac{1+r}{1+\rho}E_t\left(c_{t+1}^{-\gamma}\right), \tag{45}$$

which can be rearranged as

$$\frac{1+\rho}{1+r} = E_t\left(\frac{c_{t+1}^{-\gamma}}{c_t^{-\gamma}}\right). \tag{46}$$

The rest of the answer relies on the same kind of transformation used in the lecture to derive the consumption function. Since it is $x = e^{\log x}$, it is

$$E_t\left(\frac{c_{t+1}^{-\gamma}}{c_t^{-\gamma}}\right) = E_t\left(e^{\log\frac{c_{t+1}^{-\gamma}}{c_t^{-\gamma}}}\right) = E_t\left(e^{-\gamma(\log c_{t+1} - \log c_t)}\right) = E_t\left(e^{-\gamma\Delta\log c_{t+1}}\right), \tag{47}$$

where the second equality follows from the properties of the logarithm. Using equations (154) and (147) we can write

$$E_t\left(e^{-\gamma\Delta\log c_{t+1}}\right)=\frac{1+\rho}{1+r}. \tag{48}$$

By assumption $\Delta\log c_{t+1}$ has a normal distribution $N\left(\mu_t,\sigma_t\right)$ where $\mu_t=E_t\Delta\log c_{t+1}$. Since $-\gamma\Delta\log c_{t+1}$ is a constant scaling of a normal variable it has a normal distribution $N\left(-\gamma\mu_t,\left|\gamma\right|\sigma_t\right)$. From the result in handout 2, $E\left(e^x\right)=e^{\mu+\frac{\sigma^2}{2}}$ if x has a normal distribution $N\left(\mu,\sigma\right)$. Hence, (4) can be written as

$$e^{-\gamma\mu_t+\gamma^2\sigma_t^2/2}=\frac{1+\rho}{1+r}. \tag{49}$$

Taking logs of both sides we obtain

$$-\gamma\mu_t+\frac{\gamma^2\sigma_t^2}{2}=\log\left(1+\rho\right)-\log\left(1+r\right). \tag{50}$$

Remembering that μ_t is just a simbol for $E_t\Delta\log c_{t+1}$, we can write

$$-\gamma E_t\Delta\log c_{t+1}+\frac{\gamma^2\sigma_t^2}{2}=\log\left(1+\rho\right)-\log\left(1+r\right), \tag{51}$$

which can be rearranged as

$$E_t\Delta\log c_{t+1}=\frac{1}{\gamma}\left[\log\left(1+r\right)-\log\left(1+\rho\right)\right]+\frac{\gamma\sigma_t^2}{2}, \tag{52}$$

which tells you that even if $r=\rho$ the consumption profile has a positive slope as long as $\gamma,\sigma_t>0$. That is if income, hence consumption, is uncertain - $\sigma>0$ - and $\gamma>0$, the utility function displays prudence, there is a precautionary motive for saving.

Macroeconomics B
Solution to problem set 4

1. Denote by $u(C_i) = 1 - \frac{1}{C_i}$ the consumer's felicity function[1]. We use the general notation initial for ease of comparison with last week's lecture notes. We substitute the specific functional form later. The consumer problem is

$$\max_{C_1,C_2,a_2,e_2} u(C_1) + \frac{1}{1+\rho}\mathbb{E}_1 u(C_2) \tag{53}$$
$$\text{s.t. } Y = C_1 + e_2 P_1^o + a_2 P_1^f \tag{54}$$
$$C_2 = Y_2(1+e_2) + a_2. \tag{55}$$

2. One can replace for C_1 and C_2 using the two constraints (112)-(147) and maximize with respect to a_2 and e_2. The associated FOCs are

$$-u'(C_1)P_1^g + \frac{1}{1+\rho}\mathbb{E}_1 u'(C_2) = 0 \tag{56}$$
$$-u'(C_1)P_1^o + \frac{1}{1+\rho}\mathbb{E}_1[u'(C_2)Y_2] = 0. \tag{57}$$

Note that these are the same equations as last week once you note that the rate of return on the risk free asset is $1/P_1^f$ and the return on the risky one is (the random variable) Y_2/P_1^o.

3. Replacing for $u'(C_i) = \frac{1}{C_i^2}$ and rearranging we can write

$$P_1^f = \frac{1}{1+\rho}\mathbb{E}_1\left[\frac{C_1}{C_2}\right]^2 \tag{58}$$
$$P_1^o = \frac{1}{1+\rho}\mathbb{E}_1\left[\frac{C_1^2}{C_2}\right]. \tag{59}$$

Note that the last equation can be rewritten as

$$\frac{P_1^o}{C_1} = \frac{1}{1+\rho}\mathbb{E}_1\left[\frac{C_1}{C_2}\right]. \tag{60}$$

4. Imposing the equilibrium conditions in (117) and (118) yields

$$P_1^f = \frac{1}{1+\rho}\mathbb{E}_1\left[\frac{Y}{Y_2}\right]^2 = \frac{1}{1+\rho}\frac{1}{2}\left[\frac{(1-\sigma)^2}{(1+g)^2} + \frac{(1+\sigma)^2}{(1+g)^2}\right] = \frac{1}{1+\rho}\frac{1+\sigma^2}{(1+g)^2} \tag{61}$$
$$\frac{P_1^o}{Y} = \frac{1}{1+\rho}\mathbb{E}_1\left[\frac{Y}{Y_2}\right] = \frac{1}{1+\rho}\frac{1}{2}\left[\frac{1-\sigma}{1+g} + \frac{1+\sigma}{1+g}\right] = \frac{1}{1+\rho}\frac{1}{1+g}. \tag{62}$$

[1]Note that the utility function is CRRA with coefficient of relative risk aversion equal to 2.

Note that the risk free rate of return satisfies

$$1 + r = \frac{1}{P_1^f} = (1+\rho)\frac{(1+g)^2}{1+\sigma^2.} \tag{63}$$

It has to ensure that consumers optimally choose to consume their endowments in every period (neither save nor dissave). If $g = \sigma = 0$, endowments are flat across time and certain. It has to be $r = \rho$ for agents not to be willing to borrow or land. If $g > 0$ and $\sigma = 0$, it has to be $r > \rho$ in equilibrium, for consumption tilting to offset the desire to borrow against higher future income (consumption smoothing). If $g = 0$ and $\sigma > 0$ it is $r < \rho$ for consumption in equilibrium, for consumption tilting to offset the precautionary saving motive.

5. The expected rate of return on the risk free asset satisfies

$$(1 + \mathbb{E}_1 r_1^e) = \mathbb{E}_1 \frac{Y_2}{P_1^o} = \frac{1}{2}\left[\frac{\frac{Y(1+g)}{(1-\sigma)}}{\frac{Y}{(1+\rho)(1+g)}} + \frac{\frac{Y(1+g)}{(1+\sigma)}}{\frac{Y}{(1+\rho)(1+g)}}\right] = \frac{(1+\rho)(1+g)^2}{(1-\sigma^2)} \tag{64}$$

and the risk-free premium (in ratio) is

$$\frac{1 + \mathbb{E}_1 r_1^e}{1+r} = \frac{1+\sigma^2}{1-\sigma^2}. \tag{65}$$

Macroeconomics B
Solution to problem set 4

1. (a) The consumer's maximization problem is

$$\max_{c_{1t},c_{2t+1}} \log c_{1t} + \frac{1}{1+\rho} \log c_{2t+1} \tag{66}$$

$$\text{s.t. } a_{1t} = y_{1t} - c_{1t} \tag{67}$$

$$a_{2t+1} = (1+\gamma)\, y_{1t} + (1+r)\, a_{1t} - c_{2t+1}, \tag{68}$$

$$\text{solvency } a_{2t+1} \geq 0. \tag{69}$$

where the solvency constraint requires end of life assets a_{2t+1} to be non-negative. Given increasing marginal utility it is $a_{2t+1} = 0$ and

$$c_{2t+1} = (1+\gamma)\, y_{1t} + (1+r)\, a_{1t}. \tag{70}$$

Replacing for c_{1t} and c_{2t+1} in the objective function we can write the consumer problem as

$$\max_{a_{1t}} \log (y_{1t} - a_{1t}) + \frac{1}{1+\rho} \log \left[(1+\gamma)\, y_{1t} + (1+r)\, a_{1t}\right], \tag{71}$$

whose FOC is

$$\frac{1}{y_{1t} - a_{1t}} = \frac{1+r}{(1+\rho)\left[(1+\gamma)\, y_{1t} + (1+r)\, a_{1t}\right]}. \tag{72}$$

This can be rearranged as

$$(1+\rho)(1+\gamma)\, y_{1t} + (1+\rho)(1+r)\, a_{1t} = (1+r)(y_{1t} - a_{1t}). \tag{73}$$

Collecting terms in a_{1t} we obtain

$$(2+\rho)(1+r)\, a_{1t} = (1+r)\left(1 - \frac{(1+\rho)(1+\gamma)}{1+r}\right) y_{1t} \tag{74}$$

or

$$s_{1t} = a_{1t} - 0 = \frac{1}{2+\rho}\left(1 - \frac{(1+\rho)(1+\gamma)}{1+r}\right) y_{1t}. \tag{75}$$

Individual saving is positive if

$$\frac{(1+\rho)(1+\gamma)}{1+r} < 1. \tag{76}$$

The term $(1+\rho)/(1+r)$ captures the consumption tilting motive. If such term is 1 but $\gamma > 0$ then saving is negative because of consumption smoothing (future income is higher than first period income). Hence, higher γ reduces individual saving as it induces individuals to borrow more to bring the higher future income forward.

(b) Aggregate saving is the change in the aggregate stock of wealth

$$K_{t+1} - K_t = L_t s_{1t} - L_t s_{1t-1} = L_t \frac{1}{2+\rho}\left(1 - \frac{(1+\rho)(1+\gamma)}{1+r}\right)(y_{1t} - y_{1t-1}) \tag{77}$$

or

$$K_{t+1} - K_t = L_t \frac{1}{2+\rho}\left(1 - \frac{(1+\rho)(1+\gamma)}{1+r}\right) g y_{1t-1}. \tag{78}$$

A higher γ depresses aggregate saving by reducing the individual marginal propensity to save out of their first period endowment. A higher g increases aggregate saving by increasing the income and saving of the current generation relative to the previous one.

(c) Since higher growth means both that income increases more within lifetimes and across generations the net effect is ambiguous.

2. The only difference is that now net first period income is $W_t - \tau$ and second-period income equals the pension $\tau(1+n)$ rather than zero.

So the individual optimization problem is

$$\max_{c_{1t}, c_{2t+1}} \log c_{1t} + \frac{1}{1+\rho} \log c_{2t+1} \tag{79}$$

$$\text{s.t. } a_{1t} = W_t - \tau - c_{1t} \tag{80}$$

$$c_{2t+1} = \tau(1+n) + a_{1t}(1 + r_{t+1}). \tag{81}$$

where we have already imposed solvency.

The associated intertemporal budget constraint is

$$c_{1t} + \frac{c_{2t+1}}{1+r_{t+1}} = W_t - \tau + \frac{\tau(1+n)}{1+r_{t+1}}. \tag{82}$$

Note that an individual lifetime resources are respectively higher/lower under a social security system if $n \gtrless r_{t+1}$. The Euler equation is

$$\frac{1}{c_{1t}} = \frac{1+r_{t+1}}{1+\rho}\frac{1}{c_{2t+1}}, \tag{83}$$

which implies

$$c_{2t+1} = \frac{1+r_{t+1}}{1+\rho} c_{1t}. \tag{84}$$

Replacing in the IBC and solving for c_{1t} yields

$$c_{1t} = \frac{1+\rho}{2+\rho}\left[W_t - \tau + \frac{\tau(1+n)}{1+r_{t+1}}\right]. \tag{85}$$

Consumption when young increases if and only if the system increases the present value of lifetime income. The associated individual saving function is

$$s_{1t} = W_t - \tau - c_{1t} = \frac{1}{2+\rho}\left(W_t - \tau - \frac{\tau(1+n)(1+\rho)}{1+r_{t+1}}\right) \tag{86}$$

Let A denote TFP. Given the Cobb-Douglas technology it is $W_t = A(1-\alpha)\tilde{k}_t^{\alpha}$, with $\tilde{k}_t = K_t/(AL_t)$, and $r_{t+1} = \alpha\tilde{k}_{t+1}^{\alpha-1} - \delta$. Hence individual saving is a function of the current stock of capital

$$s_{1t} = \frac{1}{2+\rho}\left(A(1-\alpha)\tilde{k}_t^{\alpha} - \tau - \frac{\tau(1+n)(1+\rho)}{1+\alpha\tilde{k}_{t+1}^{\alpha-1}-\delta}\right). \tag{87}$$

The stock of capital at $t+1$ equals the total saving of the young at time t or

$$K_{t+1} = L_t\frac{1}{2+\rho}\left(A(1-\alpha)\tilde{k}_t^{\alpha} - \tau - \frac{\tau(1+n)(1+\rho)}{1+\alpha\tilde{k}_{t+1}^{\alpha-1}-\delta}\right) \tag{88}$$

or in per capita terms

$$\tilde{k}_{t+1} = \frac{1}{(2+\rho)(1+n)}\left(A(1-\alpha)\tilde{k}_t^{\alpha} - \tau - \frac{\tau(1+n)(1+\rho)}{1+\alpha\tilde{k}_{t+1}^{\alpha-1}-\delta}\right). \tag{89}$$

Rearranging yields,

$$\tilde{k}_{t+1} + \frac{\tau(1+n)(1+\rho)}{1+\alpha\tilde{k}_{t+1}^{\alpha-1}-\delta} = \frac{1}{(2+\rho)(1+n)}\left(A(1-\alpha)\tilde{k}_t^{\alpha} - \tau\right). \tag{90}$$

Equation (131) implies that, for any level of $\tilde{k}_t$, k_{t+1} is lower and, the function has a lower slope, than if $\tau = 0$. A pay as you go social security system depresses capital formation. Intuition: private saving by the young generation falls and the government uses the contribution not to invest in physical capital but to pay the pensions of the currently old. Hence aggregate saving, and investment, fall at given k_t.

In steady state it has to be $\tilde{k}_{t+1} = \tilde{k}_t$. If $\tau > 0$ the solution to (109) is a lower steady state level of $\tilde{k}^*$ for the stable steady state equilibrium B (see Figure 1).

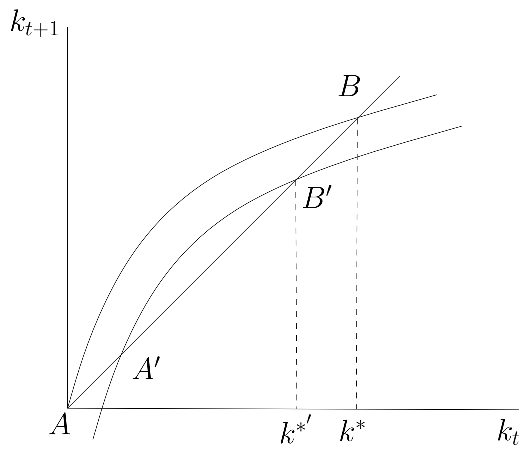


Figure 1: Steady state with pay-as-you-go pension system.

Finally, aggregate saving equals the change in the capital stock; i.e. $S_t = K_{t+1} - K_t = L_t s_{1t} - L_{t-1} s_{1t-1}$. In steady state, $s_{1t} = s_{1t-1} = s_1^*$ and we can write

$$S_t = L_{t-1}[(1+n) - 1]s_1^* = L_{t-1} n \frac{1}{2+\rho} \left(A\,(1-\alpha)\,(\tilde{k}^*)^{\alpha} - \tau - \frac{\tau(1+n)(1+\rho)}{1+\alpha(\tilde{k}^*)^{\alpha-1} - \delta} \right)$$

Which is again lower than if $\tau = 0$.

Macroeconomics B
Solution to problem set 5

1. The government tax on investment increases the cost of a unit of capital from 1 to $(1+\gamma)$. The profit tax reduce profits by a proportion τ. Hence, the firm maximizes

$$\max_{L_t, I_t, K_t} V_0 = \int_0^\infty (1-\tau)\left[A_t K_t^\alpha L_t^{1-\alpha} - w_t L_t - I_t(1+\gamma) - \frac{cI_t^2}{2}\right] e^{-rt} dt \quad (91)$$
$$\text{s.t. } K_0 \text{ given} \quad (92)$$
$$\dot{K}_t = I_t \quad (93)$$

The Lagrangean associated with the above problem is

$$\mathfrak{L}_0 = \int_0^\infty \left\{(1-\tau)\left[A_t K_t^\alpha L_t^{1-\alpha} - w_t L_t - I_t(1+\gamma) - \frac{cI_t^2}{2}\right] + q_t\left(I_t - \dot{K}_t\right)\right\} e^{-rt} dt. \quad (94)$$

q_t is the Lagrange multiplier associated with constraint (93) at each time t.

Integrating the term in $\dot{K}_t$ by parts[2] (94) can be written as

$$\mathfrak{L}_0 = \int_0^\infty \left\{(1-\tau)\left[A_t K_t^\alpha L_t^{1-\alpha} - w_t L_t - I_t(1+\gamma) - \frac{cI_t^2}{2}\right] + q_t I_t + \dot{q}_t K_t - r q_t K_t e^{-rt}\right\} dt \quad (95)$$
$$+ q_0 K_0 - \lim_{t\to\infty} q_t K_t e^{-rt}.$$

The necessary conditions for an optimum are

$$\frac{\partial \mathfrak{L}_0}{\partial L_t} = (1-\alpha) A_t K_t^\alpha L_t^{-\alpha} - w_t = 0 \quad (96)$$
$$\frac{\partial \mathfrak{L}_0}{\partial I_t} = -(1-\tau)\left[(1+\gamma) + cI_t\right] + q_t = 0 \quad (97)$$
$$\frac{\partial \mathfrak{L}_0}{\partial K_t} = (1-\tau)\alpha A_t K_t^{\alpha-1} L_t^{1-\alpha} + \dot{q}_t - r q_t = 0 \quad (98)$$
$$\lim_{t\to\infty} q_t K_t e^{-rt} = 0 \quad (99)$$

which together with equations (92) and (93) fully characterize the path for I_t and K_t.

Note that equation (99) implies[3]

$$\lim_{t\to\infty} q_t e^{-rt} = 0, \quad (100)$$

as $K_t > 0$ always on an optimal path.

[2] Integration by parts is discussed in the appendix to these lecture notes.

[3] On an optimal path K_t cannot converge to zero as $t \to \infty$. You can check that as $I_t \to 0$ (which is the case on a path approaching the steady state) the value of the firm V_0 becomes infinitely negative as K_t goes to zero.

We can rearrange equation (97) to write

$$I_t = c^{-1}\left(\frac{q_t}{1-\tau} - (1+\gamma)\right). \tag{101}$$

The firm invests if $q_t > (1-\tau)(1+\gamma)$. Compared to the case in which there are no taxes, for q_t given, the profit tax τ increases investment as the cost of capital is deductible from profits. On the other hand, the tax on investment γ reduces investment by increasing the cost of capital Replacing for I_t in (3) one obtains

$$\dot{K}_t = K_t c^{-1}\left(\frac{q_t}{1-\tau} - (1+\gamma)\right). \tag{102}$$

Assuming that the labour force is constant and equal to 1, equation (102) together with (98), (100) and the initial level of capital K_0 fully characterize the evolution of K_t and q_t on the optimal path. Equation (101) implies that given the current capital stock K_0, the *only* determinant of investment is q_t, the shadow value of capital.

q_t can be obtained by multiplying both sides of equation (98) by e^{-rt} and integrating forward. This gives

$$\int_0^\infty (\dot{q}_t - rq_t)\, e^{-rt}dt = \int_0^\infty \frac{dq_t e^{-rt}}{ds}dt = -\int_0^\infty \left((1-\tau)\,\alpha A_t K_t^{\alpha-1}\right) e^{-rt}dt \tag{103}$$

and using equation (100)

$$q_0 = \int_0^\infty \left((1-\tau)\,\alpha A_t K_t^{\alpha-1}\right) e^{-rt}dt. \tag{104}$$

The profit tax reduces the shadow value of capital as it reduces the net of tax revenue from the *marginal* unit of capital.

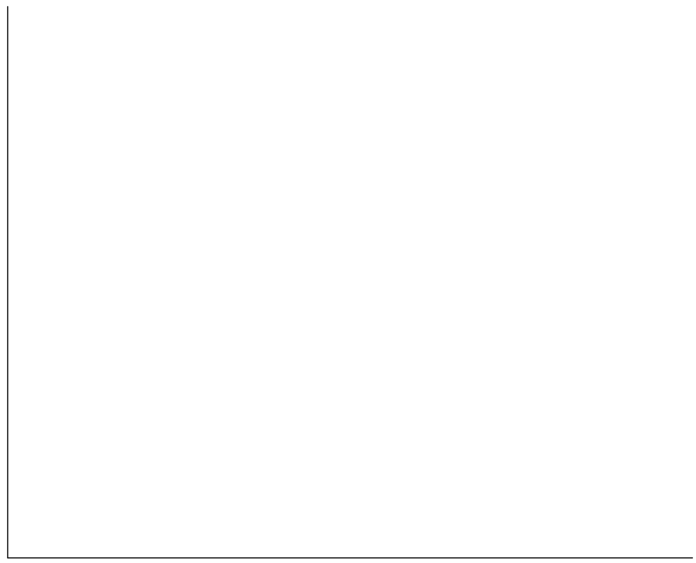
The steady state loci are

$$\dot{K} = 0 \qquad q_t = (1-\tau)(1+\gamma) \tag{105}$$

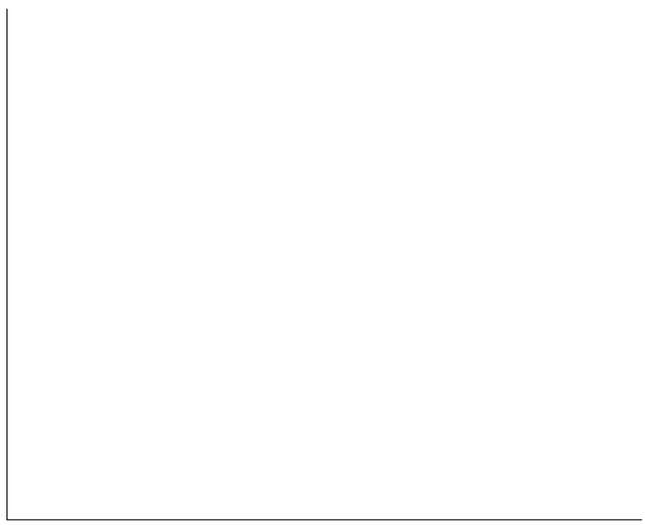
$$\dot{q} = 0 \qquad q_t = \frac{(1-\tau)\,\alpha A_t K_t^{\alpha-1}}{r}. \tag{106}$$

Eliminating q_t from these last two equations one can solve for the steady state capital stock which satisfies $r(1+\gamma) = \alpha A_t K_t^{\alpha-1}$. The steady state capital stock is unaffected from the tax on profit τ but is reduced by the tax on investment γ.

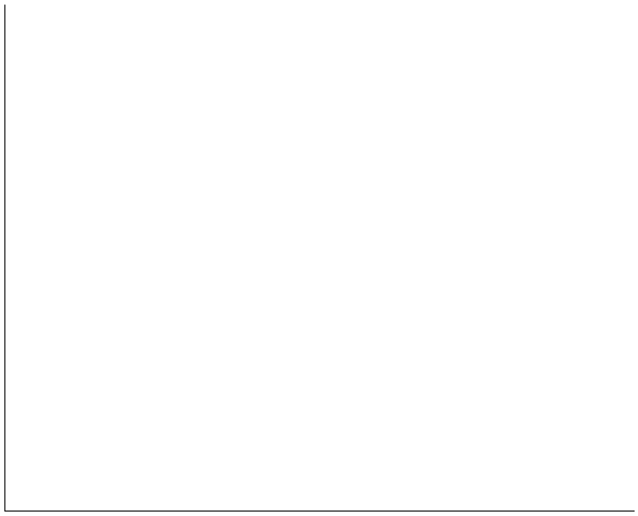
(a) The capital stock falls discretely, this raises the marginal product of capital and q_t and firms invest at a decreasing rate until the steady state is reached again.



(b) Both the $\dot{q} = 0$ and $\dot{K} = 0$ loci shift down by the same amount. The shadow price of capital falls but firms have a higher incentive to invest as the cost of capital is now deductible. If the firm is originally in steady state nothing happens.

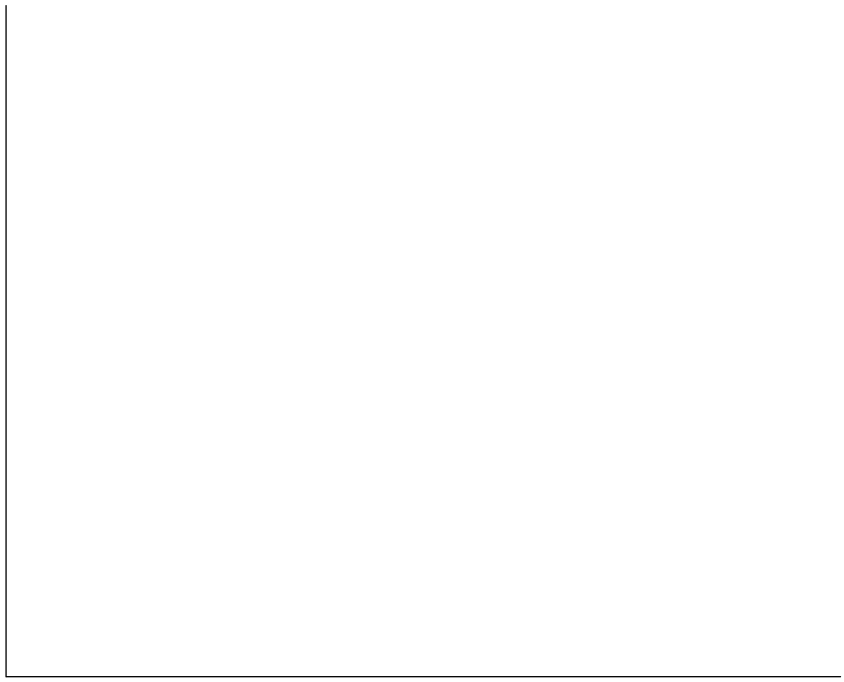


(c) The $\dot{K} = 0$ locus shifts up. The firm invests less at given q_t as the cost of capital is higher. The direct return from selling a unit of installed capital is $(1 + \gamma)$. Since the firm uninstalls capital on the optimal path and the marginal product of capital is decreasing q_t increases on the optimal path.



2. This can be seen as a reduction in γ in question 1. This shifts the $\dot{K} = 0$ locus down at time T. At time T the firm needs to be on the saddle path converging to the new steady state. So, it jumps now onto the unique path that leads to the saddle path at time T.

   (a) The jump is from point a to B which would reach C on the new saddle path at time T. The firm disinvests before time T as it is optimal to sell capital today and reinstall at lower cost from T onwards. From T onwards the capital stock would be increasing.



(b) At time T there is a second unexpected event. The $\dot{K} = 0$ locus shifts back up to its original position and the firm jumps from C on the new saddle path to D on the old saddle path.

Macroeconomics B
Solution to problem set 6

1. Remember the generic Bellman equation for the value function of an agent in state ω receiving a flow of income $m(\omega)$

$$\rho V(\omega) = m(\omega) + \frac{E[V(\omega') - V(\omega)|\omega]}{dt} \tag{107}$$

or, with just a change in notation,

$$\rho V(\omega) = m(\omega) + \frac{E[dV(\omega)|\omega]}{dt}. \tag{108}$$

For an agent in state c (with a coconut tree) $m(\omega) = y$ and the Bellman equation is

$$\rho V(c) = y + \frac{E[dV(c)|c]}{dt}. \tag{109}$$

In example 2, having a coconut tree was a permanent state. Hence, $dV(c) = 0$ and equation (109) implies

$$\rho V(c) = y. \tag{110}$$

This is no longer true in the present question, since the agent may lose the coconut tree. The (instantaneous) expected capital gain associated with such a shock hitting the agent is

$$\frac{E[dV(c)|c]}{dt} = \frac{(1-\delta dt)V(c) + \delta dt V(n) - V(c)}{dt} = \delta(V(n) - V(c)). \tag{111}$$

The agent moves to state n (she has the asset) if she loses her tree.

Replacing in the Bellman equation (109) results in

$$\rho V(c) = y + \delta(V(n) - V(c)). \tag{112}$$

The Bellman equations for the other two states are unchanged. They are

$$\rho V(n) = z + s\left[V(o) - V(n)\right] \tag{113}$$

and

$$V(o) = \max_{i=\{1,0\}} iV(c) + (1-i)V(n) = \max\left\{V(c), V(n)\right\}. \tag{114}$$

We can distinguish two cases.

(a) $V(o) = V(c) \geq V(n)$, so that the agent always accepts to exchange her asset if given the opportunity to do so. Equation (113) then becomes

$$\rho V(n) = z + s\left[V(c) - V(n)\right] \tag{115}$$

This, together with (112), forms a system of two equations in the two unknowns $V(n), V(c)$. Subtracting one from the other and rearranging yields

$$V(c) - V(n) = \frac{y - z}{\rho + \delta + s} \tag{116}$$

and

$$\rho V(c) = y - \delta \frac{y - z}{\rho + \delta + s}, \tag{117}$$

$$\rho V(n) = z + s \frac{y - z}{\rho + \delta + s}, \tag{118}$$

This is indeed optimal if $V(c) \geq V(n)$ or, given equation (116), if $y \geq z$.

(b) $V(o) = V(n) > V(c)$, so that the agent never accepts to exchange her asset if given the opportunity to do so. Equation (118) then becomes

$$\rho V(n) = z + s[V(n) - V(n)] = z. \tag{119}$$

Replacing in (112) gives

$$\rho V(c) = y + \delta \left(\frac{z}{\rho} - V(c) \right) \tag{120}$$

or

$$V(c) = \frac{1}{\rho + \delta} \left[y + \frac{\delta z}{\rho} \right]. \tag{121}$$

The agent policy choice is indeed optimal if $V(n) > V(c)$ or if $z > y$.

The policy choice is unchanged relative to example 2. The value functions are different though. In particular, $V(c)$ is lower in case (a) and higher in case (b) relative to the same two cases in example 2, as the agent now loses the coconut tree with positive probability. This reduces her utility if $y > z$ and increases it if $z > y$.

2. Given that the two islands are identical apart from being in a boom or a recession, the minimum set of states has only two states $\{b, r\}$. Denote by $W(i)$ and $w(i)$ with $i = b, r$ respectively the value functions and wages for an agent in state i.

   Consider the value function of an agent in an island in recession. The agent can either stay and receive the wage $w(r)$ until the state changes or move to the other island which implies a one-off mobility cost c. The agent chooses optimally between the two options which implies

$$\rho W(r) = \max\{w(r) + \lambda[W(b) - W(r)], \rho(W(b) - c)\}. \tag{122}$$

   The situation of an agent in an island in boom is symmetric

$$\rho W(b) = \max\{w(b) + \lambda[W(r) - W(b)], \rho(W(r) - c)\}. \tag{123}$$

   As far as equation (122) is concerned there are two possibilities: either the first term or the second term in the max operator is larger.

(a) Consider first the latter case: $\rho W(r) = \rho(W(b) - c)$ or $W(r) = W(b) - c$. The optimal policy for an agent in state r is to move to the island in boom. $W(r) = W(b) - c$ implies $\rho W(b) > \rho(W(r) - c)$. The optimal policy for an agent in an island in boom is to remain in the island.

Therefore equation (122) becomes

$$\rho W(b) = w(b) + \lambda[W(r) - W(b)] \tag{124}$$

which implies

$$\rho W(b) = w(b) - \lambda c. \tag{125}$$

An agent in an island in boom receives the wage $w(b)$ while the boom lasts but switches to the other island and bears a mobility cost c every time a recession struck.

The policy in state r is optimal if $\rho W(r) = \rho(W(b) - c)$ or, from equation (122) if

$$w(r) + \lambda c \leq \rho(W(b) - c) = w(b) - (\lambda + \rho)c \tag{126}$$

or

$$w(b) - w(r) \geq (2\lambda + \rho)c; \tag{127}$$

if the wage in boom is large enough relative to the wage in recession to compensate workers for the mobility cost.

(b) Consider now the alternative case: $\rho W(r) > \rho(W(b) - c)$. The optimal policy for an agent in an island in recession is to stay. Equation (122) becomes

$$\rho W(r) = w(r) + \lambda[W(b) - W(r)]. \tag{128}$$

We need to distinguish two cases for equation (123).

i. $W(b) = W(r) - c$. The optimal policy for an agent in an island in boom is to relocate. This is symmetric to case (a) above. Agents always move to the island in recession.

Replacing in (128) yields

$$\rho W(r) = w(r) - \lambda c. \tag{129}$$

The policy is optimal in state b if

$$w(b) + \lambda c \leq w(r) - (2\lambda + \rho)c \tag{130}$$

or $w(r) - w(b) \geq (2\lambda + \rho)c$. If this condition is satisfies, the assumed policy in state r is also optimal. This is easily checked by noticing $W(b) = W(r) - c$ implies $W(r) > W(b) > W(b) - c$.

ii. Finally, $W(b) > W(r) - c$. The optimal policy for an agent on an island in a boom is not to relocate. Hence, agents never switch island.

The two Bellman equations are

$$\rho W(r) = w(r) + \lambda[W(b) - W(r)] \tag{131}$$

and

$$\rho W(b) = w(b) + \lambda[W(r) - W(b)]. \tag{132}$$

which imply

$$W(b) - W(r) = \frac{w(b) - w(r)}{r + 2\lambda}. \tag{133}$$

For the policies in each state to be optimal it has to be

$$c > W(b) - W(r) > -c \tag{134}$$

or

$$c > \frac{w(b) - w(r)}{r + 2\lambda} > -c. \tag{135}$$

which is the complement of the other two cases. The wage differential is not large enough to induce relocation from one island to the other.

Therefore, for employment to be positive in both islands, one of the following cases must apply.

- $$c \geq \frac{w(b) - w(r)}{r + 2\lambda} \geq 0. \tag{136}$$

  The wage is higher in boom. Workers never leave islands in boom. If they are in an island in recession: (1) they are indifferent between staying and moving if the wage differential exactly offsets the cost of relocating; (2) they do not relocate if the wage differential does not offset the cost of relocating.

- $$0 > \frac{w(b) - w(r)}{r + 2\lambda} \geq -c \tag{137}$$

  or

  $$c \geq \frac{w(r) - w(b)}{r + 2\lambda} \geq 0. \tag{138}$$

  Same as in the previous case but with the island in recession now paying a higher wage.

Macroeconomics B
Solution to problem set 7

Let us introduce both unemployment benefit financing and proportional wage subsidies in the standard model. Since we are interested only in the steady state equilibrium we drop time indices. Define market tightness $\theta = v/u$, where v is the stock of unfilled vacancies and u the number of unemployed workers and the unemployment rate. The vacancy filling rate is

$$q(\theta) = \frac{m(u, v)}{v} \tag{139}$$

and the job finding rate

$$p(\theta) = \theta q(\theta) = \frac{m(u, v)}{u}. \tag{140}$$

The Beveridge curve - the steady state flow equilibrium condition - is

$$\lambda(1-u) = \theta q(\theta) u. \tag{141}$$

This fully determines the unemployment rate for given θ. So, we are looking for a reduced form equation for θ (the JC curve). The Bellman equations for the permanent income from unemployment U and employment E are respectively

$$rU = z + b + \theta q(\theta)(E - U) \tag{142}$$

and

$$rE = w + \lambda(U - E), \tag{143}$$

where w is the wage of an employed worker and r is the discount rate. The asset value of a vacant V and a filled job J are respectively

$$rV = -pc + q(\theta)(J - V) \tag{144}$$

and

$$rJ = p - w(1-t) + \lambda(V - \tau - J). \tag{145}$$

Free entry by firms requires $V = 0$. Equation (144) then implies

$$J = J - V = \frac{pc}{q(\theta)}. \tag{146}$$

This allows to write equation (145) as

$$(r+\lambda)\frac{pc}{q(\theta)} = p - w(1-t) - \lambda\tau, \tag{147}$$

which gives market tightness as a function of the (yet unknown) equilibrium wage w.

It remains to determine the equilibrium wage. The equilibrium wage is given by the Nash bargaining solution to

$$\max_w (J - V)^{1-\beta}(E - U)^{\beta}, \tag{148}$$

where both V and U are taken as given by a firm-worker pair. The first order condition for such a problem is

$$-(1-\beta)(J-V)^{-\beta}(1-t)(E-U)^{\beta}+\beta(E-U)^{\beta-1}(J-V)^{1-\beta}=0, \tag{149}$$

as it is $\partial J/\partial w=-(1-t)$ and $\partial E/\partial w=1$. This can be rearranged as

$$\frac{E-U}{\beta}=\frac{(J-V)}{(1-t)(1-\beta)}, \tag{150}$$

Rearranging and equating to the value of $(E-U)$ obtained from subtracting (142) from (143) one obtains

$$E-U=\frac{\beta}{(1-\beta)(1-t)}\frac{pc}{q(\theta)}=\frac{w-z-b}{r+\lambda+\theta q(\theta)}, \tag{151}$$

which implies

$$w=z+b+\frac{\beta(r+\lambda+\theta q(\theta))}{(1-t)(1-\beta)}\frac{pc}{q(\theta)} \tag{152}$$

A wage subsidy increases the wage at constant θ. Replacing for the wage and J in (145) using (152), (146) and $V=0$, we obtain the reduced form job creation condition (JC curve)

$$(r+\lambda)\frac{pc}{q(\theta)}=p-(z+b)(1-t)-\frac{\beta}{(1-\beta)}(r+\lambda+\theta q(\theta))\frac{pc}{q(\theta)}-\lambda\tau, \tag{153}$$

which can be rearranged as

$$0=(1-\beta)(p-(z+b)(1-t)-\lambda\tau)-(r+\lambda+\beta\theta q(\theta))\frac{pc}{q(\theta)} \tag{154}$$

which fully determines θ. This together with the flow equilibrium condition (141) fully pins down the equilibrium unemployment rate and number of vacancies.

1. So (experience-rated) unemployment benefits reduces job creation through two channels: an increase in the return to being unemployed (the term associated with b) and a fall in the expected return from a vacancy stemming from the tax on firing τ.

2. A *proportional* wage subsidy both results in higher wages (equation 152) and higher job creation (equation (154)). The intuition is that, as long as $z+b>0$, the firm and the worker share the subsidy that the government provides, as the utility from leisure is *not* subsidized. If $z+b=0$ instead the subsidy fully accrues to the worker as the wage and the rent are proportional to each other as can be seen from equation (152). Interpreting t as a tax on wages $(t<0)$, financing unemployment benefits through a proportional tax on wages depresses job creation despite the fact that workers do reduce their wage demands. Job creation is unaffected only if $z=0$.