Macroeconomics B
Solution to problem set 2

The recursive problem can be written in the following form

$$\begin{aligned} &W(a_t, z_t) = \max_{\{c_t, a_{t+1}\}} u(c_t) + \beta E_t W(a_{t+1}, z_{t+1}) \qquad (15)\\ &\text{s.t. } a_{t+1} = (1+r)\, a_t + y_t - c_t,\\ &a_t \text{ given, } \lim_{t\to\infty} \frac{a_t}{(1+r)^t} \geq 0, \end{aligned}$$

where z_t is the appropriate state variable (to be determined) characterizing the evolution over time of the income process.

Replacing for c_t and maximizing with respect to a_{t+1} we obtain the FOC

$$u'(c_t) = \beta \frac{E_t \partial W(a_{t+1}, z_{t+1})}{\partial a_{t+1}}. \qquad (16)$$

Shifting the envelope condition

$$\frac{\partial W(a_t, z_t)}{\partial a_t} = \beta(1+r) \frac{E_t \partial W(a_{t+1}, z_{t+1})}{\partial a_{t+1}} = (1+r)u'(c_t) \qquad (17)$$

one period forward and using $\beta(1+r) = 1$ we obtain the Euler equation

$$u(c_t) = E_t u'(c_{t+1}). \qquad (18)$$

Given that u' is linear we Euler equation can be rewritten as

$$c_t = E_t c_{t+1} \qquad (19)$$

which we use in what follows.

1. Suppose labour income follows the stochastic process

$$y_t = \bar{y} + \varepsilon_t - \delta \varepsilon_{t-1}, \qquad (20)$$

with ε_t white noise.

In choosing the state variables for the income process consider that y_t is a function of $(\varepsilon_t, \varepsilon_{t-1})$. Hence, $z_t = (y_t, \varepsilon_t)$ is a good candidate state variable for the income process.

Guess $c_t = \alpha_0 + \alpha_1 a_t + \alpha_2 y_t + \alpha_3 \varepsilon_t$.

Replacing in the Euler equation and rearranging gives

$$a_{t+1} - a_t = \frac{\alpha_2}{\alpha_1}(y_t - \bar{y}) + \frac{\delta\alpha_2 + \alpha_3}{\alpha_1}\varepsilon_t. \qquad (21)$$

Replacing in the dynamic budget identity one obtains

$$a_{t+1} - a_t = r a_t + y_t - (\alpha_0 + \alpha_1 a_t + \alpha_2 y_t + \alpha_3 \varepsilon_t). \qquad (22)$$

Equating the RHS of the two equations yields

$$\frac{\alpha_2}{\alpha_1}(y_t - \bar{y}) + \frac{\delta\alpha_2 + \alpha_3}{\alpha_1}\varepsilon_t = ra_t + y_t - (\alpha_0 + \alpha_1 a_t + \alpha_2 y_t + \alpha_3 \varepsilon_t) \tag{23}$$

which requires $\alpha_0 = \alpha_2/\alpha_1,\ \alpha_1 = r,\ \alpha_2/\alpha_1 = (1-\alpha_2),\ (\delta\alpha_2 + \alpha_3)/\alpha_1 = -\alpha_3$ or

$$c_t = ra_t + \frac{\bar{y}}{1+r} + \frac{r}{1+r}\left(y_t - \frac{\delta}{1+r}\varepsilon_t\right). \tag{24}$$

It follows that

$$c_{t+1} - c_t = c_{t+1} - E_t c_{t+1} = \frac{r}{1+r}\left(1 - \frac{\delta}{1+r}\right)\varepsilon_{t+1} \tag{25}$$

and

$$s_t = \frac{1}{1+r}\left(y_t - \bar{y} + \frac{\delta r}{1+r}\varepsilon_t\right). \tag{26}$$

2. Suppose labour income follows the stochastic process

$$\Delta y_t = \lambda \Delta y_{t-1} + \varepsilon_t, \tag{27}$$

or, equivalently,

$$y_t = (1+\lambda)y_{t-1} - \lambda y_{t-2} + \varepsilon_t \tag{28}$$

with $0 \le \lambda < 1$ and ε_t white noise.

To choose the state variables for the income process note that y_t is itself a second order Markov process. So the appropriate $z_t = \{y_t, y_{t-1}\}$. Therefore let's guess $c_t = \alpha_0 + \alpha_1 a_t + \alpha_2 y_t + \alpha_3 y_{t-1}$.

Replacing in the Euler equation

$$a_{t+1} - a_t = -\frac{\alpha_3 + \alpha_2\lambda}{\alpha_1}(y_t - y_{t-1}). \tag{29}$$

and in the dynamic budget identity

$$a_{t+1} - a_t = ra_t + y_t - (\alpha_0 + \alpha_1 a_t + \alpha_2 y_t + \alpha_3 y_{t-1}) \tag{30}$$

and equating we obtain

$$-\frac{\alpha_3 + \alpha_2\lambda}{\alpha_1}(y_t - y_{t-1}) = ra_t + y_t - (\alpha_0 + \alpha_1 a_t + \alpha_2 y_t + \alpha_3 y_{t-1}). \tag{31}$$

This is satisfied if $\alpha_0 = 0,\ \alpha_1 = r,\ -(\alpha_3 + \alpha_2\lambda) = r(1-\alpha_2),\ (\alpha_3 + \alpha_2\lambda) = -r\alpha_3$ which implies

$$c_t = ra_t + \frac{1+r}{1+r-\lambda}y_t - \frac{\lambda}{1+r-\lambda}y_{t-1}. \tag{32}$$

It follows that

$$c_{t+1} - c_t = c_{t+1} - E_t c_{t+1} = \frac{1+r}{1+r-\lambda}(y_{t+1} - E_t y_{t+1}) = \frac{1+r}{1+r-\lambda}\varepsilon_{t+1}. \tag{33}$$

and

$$s_t = \left(1 - \frac{1+r}{1+r-\lambda}\right)y_t + \frac{\lambda}{1+r-\lambda}y_{t-1} = -\frac{\lambda}{1+r-\lambda}\Delta y_t. \tag{34}$$

If $\lambda = 0$ saving does not respond to income at all. This is not surprising given that in such case the income process is a random walk. Therefore saving cannot smooth shocks which are expected to be permanent. If $\lambda > 0$ saving is a decreasing function of income changes. The intuition is that the income *growth* process is persistent - it is an AR(1). So a positive income change today means a positive expected income growth tomorrow. Therefore saving responds negatively because income tomorrow is expected to increase relative to today.