**Macroeconomics B**
Solution to problem set 5

1. (a) The consumer's maximization problem is

$$\max_{c_{1t},c_{2t+1}} \log c_{1t} + \frac{1}{1+\rho} \log c_{2t+1} \tag{65}$$

$$\text{s.t. } a_{1t} = y_{1t} - c_{1t} \tag{66}$$

$$a_{2t+1} = (1+\gamma)\, y_{1t} + (1+r)\, a_{1t} - c_{2t+1}, \tag{67}$$

$$\text{solvency } a_{2t+1} \geq 0. \tag{68}$$

where the solvency constraint requires end of life assets $a_{2t+1}$ to be non-negative. Given increasing marginal utility it is $a_{2t+1} = 0$ and

$$c_{2t+1} = (1+\gamma)\, y_{1t} + (1+r)\, a_{1t}. \tag{69}$$

Replacing for $c_{1t}$ and $c_{2t+1}$ in the objective function we can write the consumer problem as

$$\max_{a_{1t}} \log (y_{1t} - a_{1t}) + \frac{1}{1+\rho} \log \left[(1+\gamma)\, y_{1t} + (1+r)\, a_{1t}\right], \tag{70}$$

whose FOC is

$$\frac{1}{y_{1t} - a_{1t}} = \frac{1+r}{(1+\rho)\left[(1+\gamma)\, y_{1t} + (1+r)\, a_{1t}\right]}. \tag{71}$$

This can be rearranged as

$$(1+\rho)(1+\gamma)\, y_{1t} + (1+\rho)(1+r)\, a_{1t} = (1+r)(y_{1t} - a_{1t}). \tag{72}$$

Collecting terms in $a_{1t}$ we obtain

$$(2+\rho)(1+r)\, a_{1t} = (1+r)\left(1 - \frac{(1+\rho)(1+\gamma)}{1+r}\right) y_{1t} \tag{73}$$

or

$$s_{1t} = a_{1t} - 0 = \frac{1}{2+\rho}\left(1 - \frac{(1+\rho)(1+\gamma)}{1+r}\right) y_{1t}. \tag{74}$$

Individual saving is positive if

$$\frac{(1+\rho)(1+\gamma)}{1+r} < 1. \tag{75}$$

The term $(1+\rho)/(1+r)$ captures the consumption tilting motive. If such term is 1 but $\gamma > 0$ then saving is negative because of consumption smoothing (future income is higher than first period income). Hence, higher $\gamma$ reduces individual saving as it induces individuals to borrow more to bring the higher future income forward.

(b) Aggregate saving is the change in the aggregate stock of wealth

$$K_{t+1} - K_t = L_t s_{1t} - L_t s_{1t-1} = L_t \frac{1}{2+\rho} \left( 1 - \frac{(1+\rho)(1+\gamma)}{1+r} \right) (y_{1t} - y_{1t-1}) \tag{76}$$

or

$$K_{t+1} - K_t = L_t \frac{1}{2+\rho} \left( 1 - \frac{(1+\rho)(1+\gamma)}{1+r} \right) g y_{1t-1}. \tag{77}$$

A higher $\gamma$ depresses aggregate saving by reducing the individual marginal propensity to save out of their first period endowment. A higher $g$ increases aggregate saving by increasing the income and saving of the current generation relative to the previous one.

(c) Since higher growth means both that income increases more within lifetimes and across generations the net effect is ambiguous.

2. The only difference is that now net first period income is $W_t - \tau$ and second-period income equals the pension $\tau(1+n)$ rather than zero.

So the individual optimization problem is

$$\max_{c_{1t}, c_{2t+1}} \log c_{1t} + \frac{1}{1+\rho} \log c_{2t+1} \tag{78}$$

$$\text{s.t. } a_{1t} = W_t - \tau - c_{1t} \tag{79}$$

$$c_{2t+1} = \tau(1+n) + a_{1t}(1+r_{t+1}). \tag{80}$$

where we have already imposed solvency.

The associated intertemporal budget constraint is

$$c_{1t} + \frac{c_{2t+1}}{1+r_{t+1}} = W_t - \tau + \frac{\tau(1+n)}{1+r_{t+1}}. \tag{81}$$

Note that an individual lifetime resources are respectively higher/lower under a social security system if $n \gtrless r_{t+1}$. The Euler equation is

$$\frac{1}{c_{1t}} = \frac{1+r_{t+1}}{1+\rho} \frac{1}{c_{2t+1}}, \tag{82}$$

which implies

$$c_{2t+1} = \frac{1+r_{t+1}}{1+\rho} c_{1t}. \tag{83}$$

Replacing in the IBC and solving for $c_{1t}$ yields

$$c_{1t} = \frac{1+\rho}{2+\rho} \left[ W_t - \tau + \frac{\tau(1+n)}{1+r_{t+1}} \right]. \tag{84}$$

Consumption when young increases if and only if the system increases the present value of lifetime income. The associated individual saving function is

$$s_{1t} = W_t - \tau - c_{1t} = \frac{1}{2+\rho} \left( W_t - \tau - \frac{\tau(1+n)(1+\rho)}{1+r_{t+1}} \right) \tag{85}$$

Let $A$ denote TFP. Given the Cobb-Douglas technology it is $W_t = A(1-\alpha)\tilde{k}_t^{\alpha}$, with $\tilde{k}_t = K_t/(AL_t)$, and $r_{t+1} = \alpha\tilde{k}_{t+1}^{\alpha-1} - \delta$. Hence individual saving is a function of the current stock of capital

$$s_{1t} = \frac{1}{2+\rho}\left(A(1-\alpha)\tilde{k}_t^{\alpha} - \tau - \frac{\tau(1+n)(1+\rho)}{1+\alpha\tilde{k}_{t+1}^{\alpha-1} - \delta}\right). \tag{86}$$

The stock of capital at $t+1$ equals the total saving of the young at time $t$ or

$$K_{t+1} = L_t\frac{1}{2+\rho}\left(A(1-\alpha)\tilde{k}_t^{\alpha} - \tau - \frac{\tau(1+n)(1+\rho)}{1+\alpha\tilde{k}_{t+1}^{\alpha-1} - \delta}\right) \tag{87}$$

or in per capita terms

$$\tilde{k}_{t+1} = \frac{1}{(2+\rho)(1+n)}\left(A(1-\alpha)\tilde{k}_t^{\alpha} - \tau - \frac{\tau(1+n)(1+\rho)}{1+\alpha\tilde{k}_{t+1}^{\alpha-1} - \delta}\right). \tag{88}$$

Rearranging yields,

$$\tilde{k}_{t+1} + \frac{\tau(1+n)(1+\rho)}{1+\alpha\tilde{k}_{t+1}^{\alpha-1} - \delta} = \frac{1}{(2+\rho)(1+n)}\left(A(1-\alpha)\tilde{k}_t^{\alpha} - \tau\right). \tag{89}$$

Equation (130) implies that, for any level of $\tilde{k}_t$, $k_{t+1}$ is lower and, the function has a lower slope, than if $\tau = 0$. A pay as you go social security system depresses capital formation. Intuition: private saving by the young generation falls and the government uses the contribution not to invest in physical capital but to pay the pensions of the currently old. Hence aggregate saving, and investment, fall at given $k_t$.

In steady state it has to be $\tilde{k}_{t+1} = \tilde{k}_t$. If $\tau > 0$ the solution to (108) is a lower steady state level of $\tilde{k}^*$ for the stable steady state equilibrium B (see Figure 1).

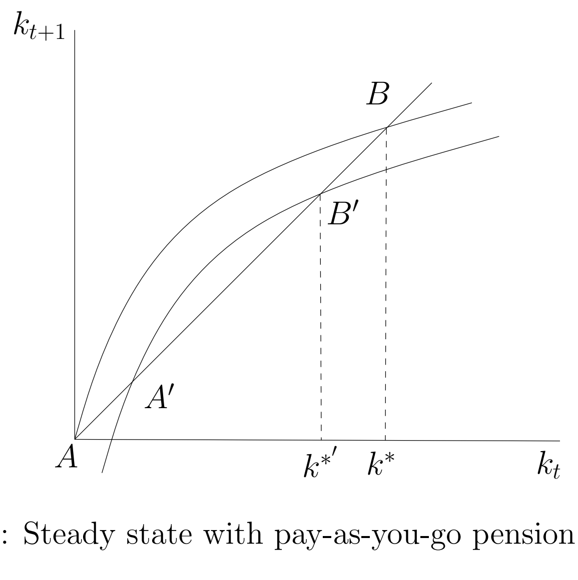


Figure 1: Steady state with pay-as-you-go pension system.

Finally, aggregate saving equals the change in the capital stock; i.e. $S_t = K_{t+1} - K_t = L_t s_{1t} - L_{t-1} s_{1t-1}$. In steady state, $s_{1t} = s_{1t-1} = s_1^*$ and we can write

$$S_t = L_{t-1}[(1+n)-1]s_1^* = L_{t-1}n\frac{1}{2+\rho}\left(A\,(1-\alpha)\,(\tilde{k}^*)^\alpha - \tau - \frac{\tau(1+n)(1+\rho)}{1+\alpha(\tilde{k}^*)^{\alpha-1}-\delta}\right)$$

Which is again lower than if $\tau = 0$.