Macroeconomics B
Solution to problem set 3

1. Given $r = 0$ and $\beta = 1$, a consumer of age t solves

$$\max_{\{c_s, a_{s+1}\}} \sum_{s=t}^{60} u(c_s) \tag{35}$$

$$\text{s.t. } a_{s+1} = a_s + y_s - c_s, \tag{36}$$

$$a_t \text{ given, } c_s \geq 0, \tag{37}$$

$$\text{solvency constraint } a_{61} = 0. \tag{38}$$

An individual cannot leave a negative stock of wealth at the end of her 60th year and, under the assumption of increasing marginal utility, will not leave a positive stock of wealth either. The labour income process satisfies

$$y_t = \begin{cases} y_{t-1} + \varepsilon_t & \text{if } t \leq 40 \\ 0 & \text{if } t > 40. \end{cases} \tag{39}$$

(a) Replacing using the dynamic constraint (2) in the maximand (1) and maximizing with respect to a_{t+s+1} we obtain the Euler equation

$$u'(c_s) = E_s u'(c_{s+1}) \tag{40}$$

which, with quadratic utility, becomes

$$c_s = E_s c_{s+1}. \tag{41}$$

The intertemporal budget constraint can be written as

$$\sum_{s=t}^{60} c_s = a_t + \sum_{s=t}^{60} y_s + a_{61} \tag{42}$$

which using the solvency constraint $a_{61} = 0$ becomes

$$\sum_{s=t}^{60} c_s = a_t + \sum_{s=t}^{60} y_s. \tag{43}$$

The budget constraint holds at any moment in time with probability one, so it has to hold in expectation or

$$\sum_{s=t}^{60} E_t c_{t+s} = a_t + \sum_{s=t}^{60} E_t y_{t+s}. \tag{44}$$

Using the Euler equation (41) and the law of iterated expectations we have $E_t c_{t+s} = c_t$. Applying the law of iterated expectations to the income process we have

$$E_t y_s = \begin{cases} y_t & \text{if } s \leq 40 \\ 0 & \text{if } s > 40. \end{cases} \tag{45}$$

Hence, we can rewrite (44) as

$$\sum_{s=t}^{60} c_t = a_t + \sum_{s=t}^{40} y_t, \tag{46}$$

or

$$(60 - t + 1)\, c_t = a_t + y_t \sum_{s=t}^{40} 1 \tag{47}$$

The consumption function is

$$c_t = \begin{cases} \frac{a_t}{60-t+1} + \frac{40-t+1}{60-t+1} y_t & \text{if } t \leq 40 \\ \frac{a_t}{60-t+1} & \text{if } t > 40 \end{cases} . \tag{48}$$

Consumption spreads financial wealth and the present value of remaining labour income over the remaining lifetime $(60 - t + 1)$ in order to keep a flat expected consumption profile.

(b) Saving equals

$$s_t = r a_t + y_t - c_t = y_t - c_t, \tag{49}$$

given that $r = 0$. Hence, we have

$$s_t = \begin{cases} -\frac{a_t}{60-t+1} + \frac{20}{60-t+1} y_t & \text{if } t \leq 40 \\ -\frac{a_t}{60-t+1} & \text{if } t > 40 \end{cases} \tag{50}$$

Saving is a way to smooth consumption by carrying over purchasing power to the years in which income is expected to be zero. During retirement saving is negative as people run down their assets to keep consumption flat in the fact of zero labour income.

(c) Using the Euler equation (41) we can write

$$c_t - c_{t-1} = c_t - E_{t-1} c_t. \tag{51}$$

Replacing using the consumption function (48) we have

$$c_t - c_{t-1} = \begin{cases} \frac{40-t+1}{60-t+1} (y_t - y_{t-1}) & \text{if } t \leq 40 \\ 0 & \text{if } t > 40 \end{cases} , \tag{52}$$

or

$$c_t - c_{t-1} = \begin{cases} \frac{40-t+1}{60-t+1} \varepsilon_t & \text{if } t \leq 40 \\ 0 & \text{if } t > 40 \end{cases} . \tag{53}$$

So $k_t = \frac{40-t+1}{60-t+1}$, if $t \leq 40$ and zero otherwise. The innovation in permanent income in response to an innovation in labour income is a declining function of age t. In fact

$$\frac{dk_t}{dt} = \frac{-(60 - t + 1) + (40 - t + 1)}{(60 - t + 1)^2} = -\frac{20}{(60 - t + 1)^2} < 0, \text{ if } t \leq 40. \tag{54}$$

The intuition is that the lower the remaining working life the shorter the span of time for which a given innovation in labour income will persist, given that income drops to zero at retirement.

$$k_1 = 40/60 \sim 0.66 \tag{55}$$

$$k_{40} = 1/21 \sim 0.05. \tag{56}$$

(d) The average k_t is

$$\bar{k} = \frac{1}{60}\sum_{t=1}^{39}\frac{40-t+1}{60-t+1} = 0.295 \tag{57}$$

(e) If agents have infinite lifetimes and labour income follows a random walk it is

$$c_t - c_{t-1} = \frac{r}{1+r}\sum_{s=0}^{\infty}\frac{E_t y_{t+s} - E_{t-1}y_{t+s}}{(1+r)^s} = \varepsilon_t. \tag{58}$$

If labour income is a random walk, any income innovation is expected to be permanent. Consumption responds one-to-one to a permanent innovation in labour income. k is one in this case. With finite working lives, even a permanent income innovation is expected to end by the end of the working life hence it has a smaller impact on lifetime income.

Note that if the interest rate were positive k_t in part d. would be higher as agents could consume more of a given income innovation because they would get a positive (rather than a zero) interest rate on their saving. In fact, k_1 would be very close to 1. Yet, k_{39} would still be very close to 0.05 as the interest rate would be positive for just one year given that saving is negative for $t \geq 40$. Hence, with positive discounting the average response of consumption to a permanent innovation in labour income would be somewhere in between the infinite lifetime and the zero interest rate case. Anyway, with finite lifetimes *aggregate* consumption should respond less to an innovation in aggregate income than predicted by the permanent income theory with infinite lifetimes. One could work out exactly by how much less by using the more realistic income process used by Deaton.

2. With instantaneous utility function $u(c_t) = c_t^{1-\gamma}/(1-\gamma)$, the marginal utility is $u'(c_t) = c_t^{-\gamma}$. Hence the Euler equation is

$$c_t^{-\gamma} = \frac{1+r}{1+\rho}E_t\left(c_{t+1}^{-\gamma}\right), \tag{59}$$

which can be rearranged as

$$\frac{1+\rho}{1+r} = E_t\left(\frac{c_{t+1}^{-\gamma}}{c_t^{-\gamma}}\right). \tag{60}$$

The rest of the answer relies on the same kind of transformation used in the lecture to derive the consumption function. Since it is $x = e^{\log x}$, it is

$$E_t \left(\frac{c_{t+1}^{-\gamma}}{c_t^{-\gamma}} \right) = E_t \left(e^{\log \frac{c_{t+1}^{-\gamma}}{c_t^{-\gamma}}} \right) = E_t \left(e^{-\gamma(\log c_{t+1} - \log c_t)} \right) = E_t \left(e^{-\gamma \Delta \log c_{t+1}} \right), \tag{61}$$

where the second equality follows from the properties of the logarithm. Using equations (177) and (170) we can write

$$E_t \left(e^{-\gamma \Delta \log c_{t+1}} \right) = \frac{1+\rho}{1+r}. \tag{62}$$

By assumption $\Delta \log c_{t+1}$ has a normal distribution $N(\mu_t, \sigma_t)$ where $\mu_t = E_t \Delta \log c_{t+1}$. Since $-\gamma \Delta \log c_{t+1}$ is a constant scaling of a normal variable it has a normal distribution $N(-\gamma \mu_t, |\gamma| \sigma_t)$. From the result in handout 2, $E(e^x) = e^{\mu + \frac{\sigma^2}{2}}$ if x has a normal distribution $N(\mu, \sigma)$. Hence, (4) can be written as

$$e^{-\gamma \mu_t + \gamma^2 \sigma_t^2 / 2} = \frac{1+\rho}{1+r}. \tag{63}$$

Taking logs of both sides we obtain

$$-\gamma \mu_t + \frac{\gamma^2 \sigma_t^2}{2} = \log(1+\rho) - \log(1+r). \tag{64}$$

Remembering that μ_t is just a simbol for $E_t \Delta \log c_{t+1}$, we can write

$$-\gamma E_t \Delta \log c_{t+1} + \frac{\gamma^2 \sigma_t^2}{2} = \log(1+\rho) - \log(1+r), \tag{65}$$

which can be rearranged as

$$E_t \Delta \log c_{t+1} = \frac{1}{\gamma} \left[\log(1+r) - \log(1+\rho) \right] + \frac{\gamma \sigma_t^2}{2}, \tag{66}$$

which tells you that even if $r = \rho$ the consumption profile has a positive slope as long as $\gamma, \sigma_t > 0$. That is if income, hence consumption, is uncertain - $\sigma > 0$ - and $\gamma > 0$, the utility function displays prudence, there is a precautionary motive for saving.