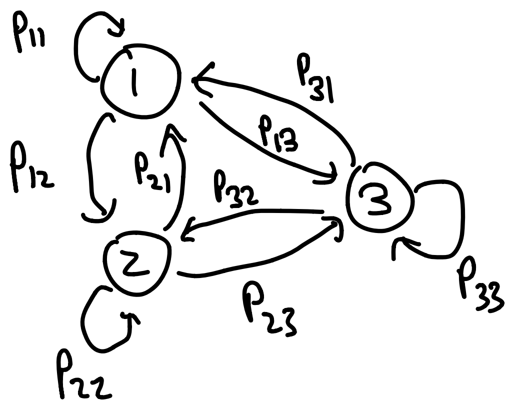
Markov Chains

Markov Property:

$$P[X_t=j \mid X_{t-1}, X_{t-2}, \dots, X_0] = P[X_t=j \mid X_{t-1}]$$

$$P[X_n=j \mid X_{n-1}=i] = P_{ij}$$



$$\begin{bmatrix} P_{11} & P_{12} & \cdots & P_{1n} \\ P_{21} & P_{22} & \cdots & P_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ P_{n1} & P_{n2} & \cdots & P_{nn} \end{bmatrix} = \boxed{P}$$

transition matrix

$$\pi_t = [\pi_t(1) \;\; \pi_t(2) \;\; \cdots \;\; \pi_t(n)] = [P(X_t=1) \;\; P(X_t=2) \cdots P(X_t=n)]$$

represents distribution at time t

$\pi_0 = [\pi_0(1) \;\; \pi_0(2) \;\; \cdots \;\; \pi_0(n)]$ is the initial distribution.

$$\pi_{t+1}(k) = P(X_{t+1}=k) = \sum_{j=1}^{n} P(X_t=j)\, P(X_{t+1}=k \mid X_t=j)$$

$$= \sum_{j=1}^{n} \pi_t(j)\, P_{jk}$$

$$\underbrace{[\pi_{t+1}(1) \;\; \pi_{t+1}(2) \;\; \cdots \;\; \pi_{t+1}(n)]} = \underbrace{[\pi_t(1) \;\; \pi_t(2) \;\; \cdots \;\; \pi_t(n)]} \underbrace{\begin{bmatrix} P_{11} & P_{12} & \cdots & P_{1n} \\ P_{21} & P_{22} & \cdots & P_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ P_{n1} & P_{n2} & \cdots & P_{nn} \end{bmatrix}}$$

$$\boxed{\pi_{t+1} = \pi_t \; P}$$

$$\pi_t = \pi_0 P^t$$

$$\overbrace{\qquad\qquad}^{\text{Linear System of Equations}}$$

$$\boxed{\text{Invariant/Stationary Dist } \pi \text{ satisfies } \pi = \pi P}$$

1 Markov Chain Basics

A Markov chain is a sequence of random variables X_n, $n = 0, 1, 2, \ldots$. Here is one interpretation of a Markov chain: X_n is the state of a particle at time n. At each time step, the particle can jump to another state. Formally, a Markov chain satisfies the Markov property:

$$\mathbb{P}(X_{n+1} = j \mid X_n = i, X_{n-1} = i_{n-1}, \ldots, X_0 = i_0) = \mathbb{P}(X_{n+1} = j \mid X_n = i), \tag{1}$$

for all n, and for all sequences of states $i_0, \ldots, i_{n-1}, i, j$. In other words, the Markov chain does not have any memory; the transition probability only depends on the current state, and not the history of states that have been visited in the past.

(a) In lecture, we learned that we can specify Markov chains by providing three ingredients: $\mathscr{X}$, P, and π_0. What do these represent, and what properties must they satisfy?

$\mathcal{X}$ = state space

$P \Rightarrow$ transition probabilities

$\pi_0 \Rightarrow$ vector representing initial distribution

(b) If we specify $\mathscr{X}$, P, and π_0, we are implicitly defining a sequence of random variables X_n, $n = 0, 1, 2, \ldots$, that satisfies (1). Explain why this is true.

RV's: $R = \begin{cases} - & \text{wp } p_1 \\ - & \text{wp } p_2 \\ \vdots & \end{cases}$

$X_0 = \begin{cases} 1 & \text{wp } \pi_0(1) \\ 2 & \text{wp } \pi_0(2) \\ \vdots & \\ n & \text{wp } \pi_0(n) \end{cases}$

$X_t = \begin{cases} 1 & \text{wp } \pi_t(1) \\ 2 & \text{wp } \pi_t(2) \\ \vdots & \\ n & \text{wp } \pi_t(n) \end{cases}$

(c) Calculate $\mathbb{P}(X_1 = j)$ in terms of π_0 and P. Then, express your answer in matrix notation. What is the formula for $\mathbb{P}(X_n = j)$ in matrix form?

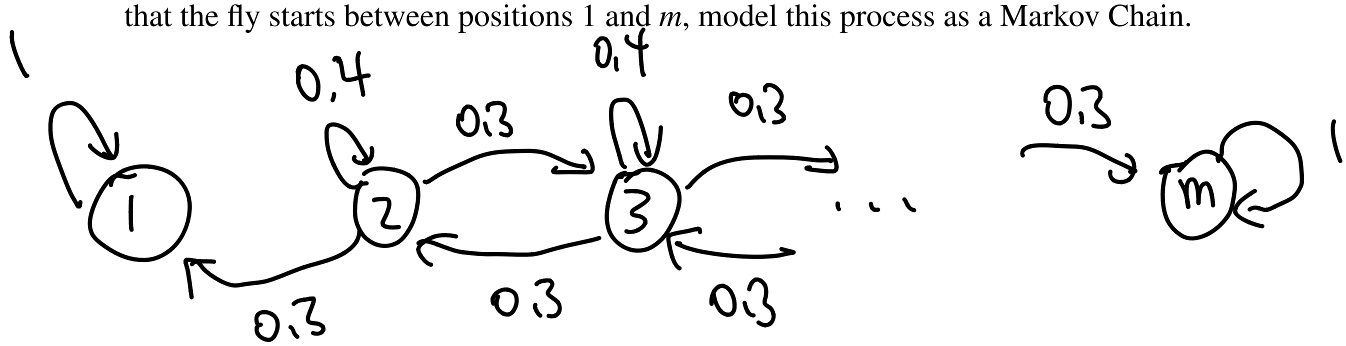
$$P(X_1=j) = \sum_{i \in \mathcal{X}} P(X_0=i)\, P(X_1=j \mid X_0=i)$$

$$= \sum_{i \in \mathcal{X}} \pi_0(i)\, P_{ij}$$

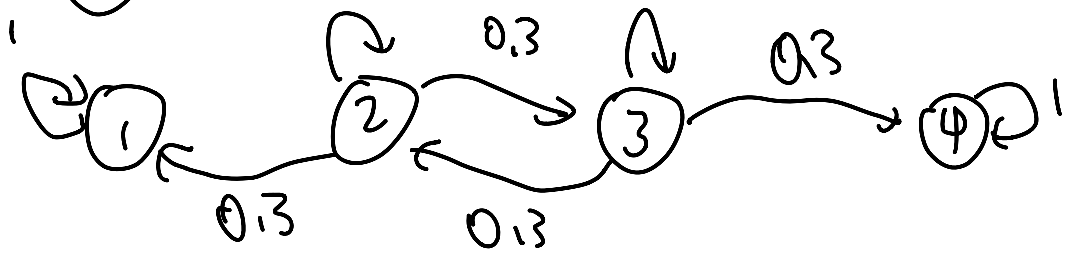
$$\pi_n = \pi_0 P^n$$

2 Can it be a Markov Chain?

(a) A fly flies in a straight line in unit-length increments. Each second it moves to the left with probability 0.3, right with probability 0.3, and stays put with probability 0.4. There are two spiders at positions 1 and m and if the fly lands in either of those positions it is captured. Given that the fly starts between positions 1 and m, model this process as a Markov Chain.



(b) Take the same scenario as in the previous part with $m = 4$. Let $Y_n = 0$ if at time n the fly is in position 1 or 2 and let $Y_n = 1$ if at time n the fly is in position 3 or 4. Is the process Y_n a Markov chain? No



$$P[X_t = j \mid X_{t-1}, X_{t-2}, \ldots, X_0] = P[X_t = j \mid X_{t-1}]$$

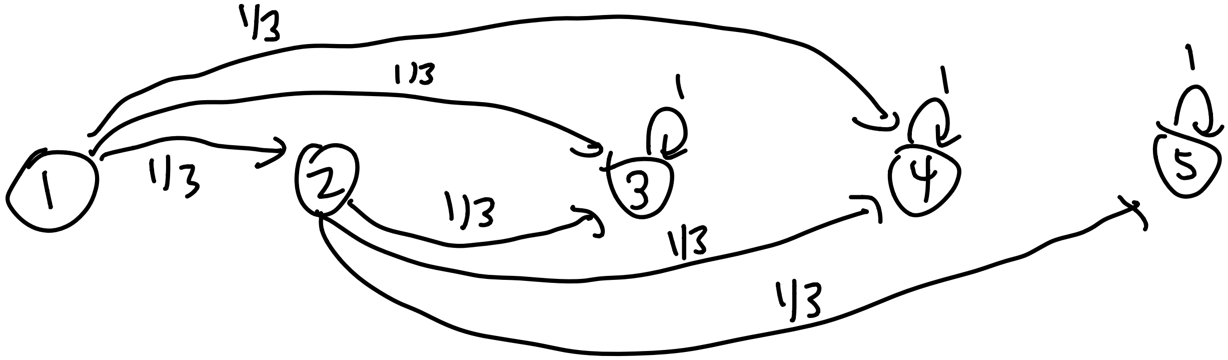
$$P[Y_2 = 0 \mid Y_1 = 1, Y_0 = 0]$$

$$\neq P[Y_2 = 0 \mid Y_1 = 1, Y_0 = 1]$$

Markov property is not satisfied.

3 Skipping Stones

We consider a simple Markov chain model for skipping stones on a river, but with a twist: instead of trying to make the stone travel as far as possible, you want the stone to hit a target. Let the set of states be $\mathscr{X} = \{1,2,3,4,5\}$. State 3 represents the target, while states 4 and 5 indicate that you have overshot your target. Assume that from states 1 and 2, the stone is equally likely to skip forward one, two, or three steps forward. If the stone starts from state 1, compute the probability of reaching our target before overshooting, i.e. the probability of $\{3\}$ before $\{4,5\}$.



$\alpha(i) = P(\text{visit 3 before } 4,5 \mid X_t = i)$

$\alpha(1) = \frac{1}{3}\cdot\alpha(2) + \frac{1}{3}\cdot\alpha(3) + \frac{1}{3}\alpha(4)$

$\alpha(2) = \frac{1}{3}\alpha(3) + \frac{1}{3}\alpha(4) + \frac{1}{3}\alpha(5)$

$\alpha(3) = 1$

$\alpha(4) = 0$

$\alpha(5) = 0$

$\alpha(2) = \frac{1}{3}$

$\alpha(1) = \frac{1}{3}\cdot\frac{1}{3} + \frac{1}{3}\cdot 1$

$= \boxed{\frac{4}{9}}$