MMSE aka conditional expectation $E[Y|X]$

Def: $E[Y|X=k] = \sum_a a\, P(Y=a|X=k) = \sum_a a \frac{P(Y=a \cap X=k)}{P(X=k)}$

Recall Expectation: "average value" of an RV

$E[Y] = \sum_a a\, P[Y=a]$

Conditional Expectation:

"average value" of an RV given the value of another

Key properties of conditional expectation:

linearity $E[aY + bZ|X] = a E[Y|X] + b E[Z|X]$

factoring $E[f(X) Y|X] = f(X) E[Y|X]$

Independence

For independent RVs X, Y $E[Y|X] = E[Y]$

Proof: $E[Y|X=k] = \sum_a a\, P(Y=a|X=k)$

$= \sum_a a\, P(Y=a)$ independence of X and Y

$= E[Y]$

Law of Total Expectation (aka Tower or Smoothing rule)

$$E[Y] = E\big[E[Y|X]\big]$$

1 Number Game

Sinho and Vrettos are playing a game where they each choose an integer uniformly at random from $[0,100]$, then whoever has the larger number wins (in the event of a tie, they replay). However, Vrettos doesn't like losing, so he's rigged his random number generator such that it instead picks randomly from the integers between Sinho's number and 100. Let S be Sinho's number and V be Vrettos' number.

(a) What is $\mathbb{E}[S]$?

$$E[S] = \sum_{i=0}^{100} i \cdot \frac{1}{101} = \frac{1}{101} \sum_{i=0}^{100} i = \frac{1}{\cancel{101}} \frac{100\cancel{(101)}}{2} = 50$$

$$S \sim \text{Uniform}\{0, 1, \ldots, 100\} \quad E[S] = \frac{0+100}{2} = 50$$

(b) What is $\mathbb{E}[V|S=s]$, where s is any constant such that $0 \le s \le 100$?

$$V|S \sim \text{Uniform}\{s, s+1, \ldots, 100\}$$

$$E[V|S] = \frac{S+100}{2} = 50 + \frac{S}{2}$$

(c) What is $\mathbb{E}[V]$?

$$E[V] = E\left[E[V|S]\right]$$

$$= E\left[50 + \frac{S}{2}\right]$$

$$= 50 + \frac{E[S]}{2} = \boxed{75}$$

2 Number of Ones

In this problem, we will revisit dice-rolling, except with conditional expectation.

$X = \#$ rolls

(a) If we roll a die until we see a 6, $\underbrace{\text{how many ones should we expect to see}}_{Y}$? Notice that this is the MMSE of the number of ones we see given that we've seen a 6.

Goal: $E[Y]$ $\qquad X \sim \text{Geometric}\left(\frac{1}{6}\right)$

$$E[X] = \frac{1}{1/6} = 6$$

$$E[Y \mid X] = (X-1) \cdot \frac{1}{5}$$

$R_1 \; R_2 \; \cdots \; R_{X-1} \quad R_X$

$\frac{1}{5} \qquad 6$

$$E[Y] = E[E[Y \mid X]]$$

$$= E\left[\frac{X-1}{5}\right] = \frac{E[X]-1}{5} = \boxed{1}$$

(b) If we roll a die until we see a number greater than 3, $\underbrace{\text{how many ones should we expect to see}}_{Y}$?

$X = \#$ rolls

Goal: $E[Y]$ $\qquad X \sim \text{Geometric}\left(\frac{1}{2}\right)$

$$E[X] = \frac{1}{1/2} = 2$$

$$E[Y \mid X] = (X-1) \cdot \frac{1}{3}$$

$R_1 \; R_2 \; \cdots \; R_{X-1} \quad R_X$

$\frac{1}{3} \qquad 4, 5, 6$

$$E[Y] = E[E[Y \mid X]]$$

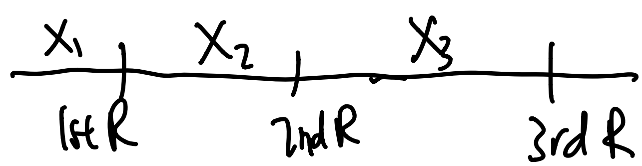
$$= E\left[\frac{X-1}{3}\right] = \frac{E[X]-1}{3} = \boxed{\frac{1}{3}}$$

3 Marbles in a Bag

We have r red marbles, b blue marbles, and g green marbles in the same bag. If we sample marbles with replacement until we get 3 red marbles (not necessarily consecutively), how many blue marbles should we expect to see? (*Hint*: It might be useful to use Law of Total Expectation, $E(Y) = E(E(Y|X))$)

$Y =$ # blue marbles we see

$X =$ # marbles drawn until 3rd red



$E[X] =$

$E[X_1] + E[X_2] + E[X_3]$

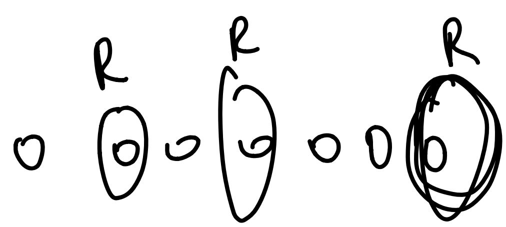
$= 3\left(1 + \frac{b+g}{r}\right)$

$X = X_1 + X_2 + X_3$

$X_1, X_2, X_3 \sim \text{Geometric}\left(\frac{r}{r+b+g}\right)$

$E[Y|X] =$

$(X-3) \cdot \frac{b}{g+b}$



$\frac{b}{g+b}$

$$E[Y] = E\left[E[Y|X]\right]$$

$$= E\left[(X-3)\frac{b}{g+b}\right] = \frac{b}{g+b}E[X] - \frac{3b}{g+b}$$

$$= \frac{b}{g+b}\left(3 + \frac{3(b+g)}{r}\right) - \frac{3b}{g+b}$$

$$= \cancel{\frac{3b}{g+b}} + \frac{3b}{r} - \cancel{\frac{3b}{g+b}} = \boxed{\frac{3b}{r}}$$