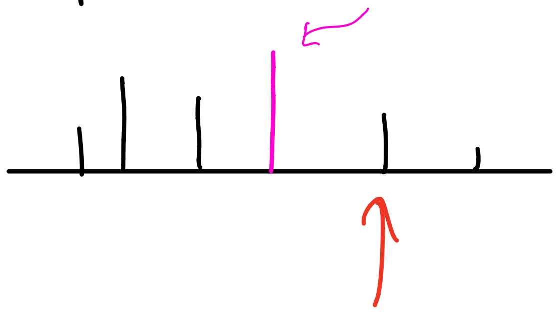
Continuous Probability

Discrete

PMF $P[X=i]$



$$P[X \le x] = \sum_{i \le x} P[X=i] = \sum_{i=-\infty}^{x} P(X=i)$$

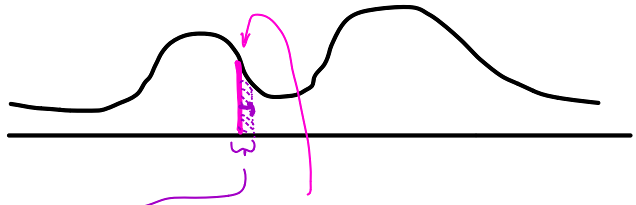
$$1 \ge P(X=a) \ge 0$$

$$\sum_a P(X=a) = 1$$

$$E[X] = \sum_a a \cdot P(X=a)$$

Continuous

p.d.f. $f_X(x)$



$$P[X=x]=0$$

$$P[x \le X \le x+dx] = \int_x^{x+dx} f(z)\,dz \approx f(x)\,dx$$

"probability per unit length"

c.d.f. $P[X \le x]$

$$= \int_{-\infty}^{x} f(z)\,dz$$

$$f(z) \ge 0$$

$$\int_{-\infty}^{\infty} f(z)\,dz = 1$$

$$E[X] = \int_{-\infty}^{\infty} z\, f(z)\,dz$$

1 Continuous Joint Densities

The joint probability density function of two random variables X and Y is given by $f(x,y) = Cxy$ for $0 \leq x \leq 1, 0 \leq y \leq 2$, and 0 otherwise (for a constant C).

(a) Find the constant C that ensures that $f(x,y)$ is indeed a probability density function.

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x,y)\,dy\,dx = 1$$

$$\int_0^1 \int_0^2 Cxy\,dy\,dx = 1$$

$$C = \frac{1}{\int_0^1 \int_0^2 xy\,dy} = \boxed{1}$$

(b) Find $f_X(x)$, the marginal distribution of X.

$$f_X(x) = \int_0^2 f(x,y)\,dy$$

$$= \int_0^2 xy\,dy = \boxed{2x}$$

(c) Find the conditional distribution of Y given $X = x$.

$$P(X=x \mid Y=y) = \frac{P(X=x \cap Y=y)}{P(Y=y)}$$

$$f(Y \mid X=x) = \frac{f(Y,X)}{f(X)} = \frac{xy}{2x}$$

$$= \boxed{y/2}$$

(d) Are X and Y independent?

Yes

$f(y) = \int_0^1 xy \, dx = \boxed{\frac{1}{2}y}$

$f(Y|X) = f(Y)$ ✓

or

$f(X|Y) = f(X)$

or

$f(X,Y) = f(X)f(Y)$

$P[A|B] = P[A]$

or

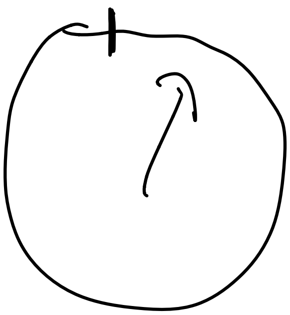
$P[B|A] = P[B]$

or

$P[A \cap B] = P[A]P[B]$

2 Uniform Distribution

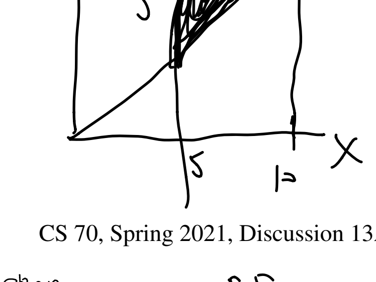
You have two fidget spinners, each having a circumference of 10. You mark one point on each spinner as a needle and place each of them at the center of a circle with values in the range $[0,10)$ marked on the circumference. If you spin both (independently) and let X be the position of the first spinner's mark and Y be the position of the second spinner's mark, what is the probability that $X \geq 5$, given that $Y \geq X$?



iid $X, Y \sim \text{Uniform}[0,10]$

$P[X \geq 5 \mid Y \geq X]$

$$= \frac{P[X \geq 5 \cap Y \geq X]}{P[Y \geq X]} = \frac{\int_5^{10}\int_x^{10} \frac{1}{100}\, dy\, dx}{1/2}$$



$$= \frac{\int_5^{10}\int_5^{y} \frac{1}{100}\, dx\, dy}{1/2}$$

$$= \frac{1/8}{1/2} = \boxed{1/4}$$

$\int_5^{10}\int_x^{10} \frac{1}{100}\, dy\, dx = \frac{5 \cdot 5}{2} \cdot \frac{1}{100} = \frac{1}{8}$

3 Exponential Practice

(a) Let $X_1, X_2 \sim \text{Exponential}(\lambda)$ be independent, $\lambda > 0$. Calculate the density of $Y := X_1 + X_2$. [*Hint*: One way to approach this problem would be to compute the CDF of Y and then differentiate the CDF.]

$$f_Y(y) = \frac{d}{dy} F(y) \qquad f(x_1, x_2) = f(x_1) f(x_2)$$

$$F(y) = P(Y \le y) = P(X_1 \le y,\ X_2 \le y - X_1)$$

$$= \int_0^y \int_0^{y-x_1} \lambda e^{-\lambda x_1} \lambda e^{-\lambda x_2} \, dx_2 \, dx_1$$

$$= \lambda^2 \int_0^y e^{-\lambda x_1} \left(\frac{1}{\lambda} - \frac{1}{\lambda} e^{-\lambda(y-x_1)} \right) dx_1$$

$$= \lambda \int_0^y \left(e^{-\lambda x_1} - e^{-\lambda y} \right) dx_1$$

$$= \lambda \left[-\frac{1}{\lambda} e^{-\lambda x_1} - x_1 e^{-\lambda y} \Big|_{x_1=0}^{y} \right]$$

$$-\frac{1}{\lambda} e^{-\lambda x_2} \Big|_0^{y-x_1}$$

$$-\frac{1}{\lambda} e^{-\lambda(y-x_1)} + \frac{1}{\lambda}$$

$$= -e^{-\lambda y} - \lambda y e^{-\lambda y} + 1$$

$$f_Y(y) = \frac{d}{dy} F(y) = \lambda e^{-\lambda y} - \left(\lambda e^{-\lambda y} - \lambda^2 y e^{-\lambda y} \right) = \boxed{\lambda^2 y e^{-\lambda y}}$$

(b) Let $t > 0$. What is the density of X_1, conditioned on $X_1 + X_2 = t$? [*Hint*: Once again, it may be helpful to consider the CDF $\mathbb{P}(X_1 \leq x \mid X_1 + X_2 = t)$. To tackle the conditioning part, try conditioning instead on the event $\{X_1 + X_2 \in [t, t+\varepsilon]\}$, where $\varepsilon > 0$ is small.]

$$F(X_1 \mid X_1 + X_2 = t) = P(X_1 \leq x \mid X_1 + X_2 = t)$$

$$= \frac{P(X_1 \leq x,\ X_1 + X_2 = t)}{P(X_1 + X_2 = t)}$$

$$= \frac{P(X_1 \leq x,\ X_1 + X_2 \in [t, t+\varepsilon])}{P(X_1 + X_2 \in [t, t+\varepsilon])}$$

$$= \frac{\int_0^x \int_{t-x_1}^{t-x_1+\varepsilon} f(x_1, x_2)\, dx_2\, dx_1}{f(X_1 + X_2)\,\varepsilon}$$

$$= \frac{\int_0^x \int_{t-x_1}^{t-x_1+\varepsilon} \lambda e^{-\lambda x_1} \lambda e^{-\lambda x_2}\, dx_2\, dx_1}{\lambda^2 y e^{-\lambda y}\, \varepsilon}$$

(see sols for algebra)

$$= \frac{x}{t} \longrightarrow f_{X_1}(x \mid X_1 + X_2 = t) = \boxed{\frac{1}{t}}$$

$$f(x_1 \mid x_1 + x_2 = t)$$

$$= \frac{f_{x_1}(x_1)\, f_{x_2}(t - x_1)}{f_Y(x_1 + x_2)}$$

$$= \frac{\lambda e^{-\lambda x_1} \; \lambda e^{-\lambda(t - x_1)}}{\lambda^2 t e^{-\lambda t}}$$

$$= \boxed{\frac{1}{t}}$$