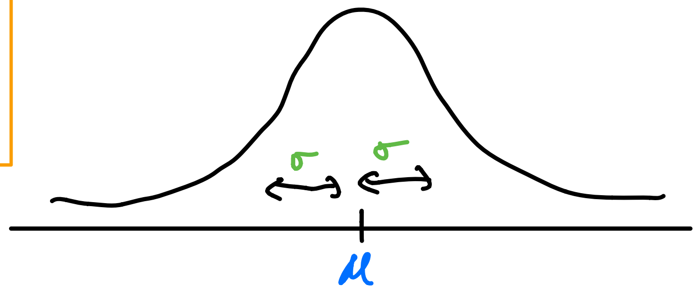
# Gaussians

$$X \sim \mathcal{N}(\mu, \sigma^2)$$

$$E[X] = \mu$$

$$Var(X) = \sigma^2$$

pdf $$\boxed{f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}}$$




# Transforms of Gaussians

indep. $$\begin{cases} X \sim \mathcal{N}(\mu_1, \sigma_1^2) \\ Y \sim \mathcal{N}(\mu_2, \sigma_2^2) \end{cases}$$

Normalizing: $$\frac{X-\mu_1}{\sigma_1} \sim \mathcal{N}(0,1)$$

$$\frac{Y-\mu_2}{\sigma_2} \sim \mathcal{N}(0,1)$$

$$\boxed{aX + bY + c \sim \mathcal{N}\left(a\mu_1 + b\mu_2 + c,\ a^2\sigma_1^2 + b^2\sigma_2^2\right)}$$

# Central Limit Theorem

iid $X_1, \dots, X_n$ s.t. $E[X_i] = \mu$, $Var(X_i) = \sigma^2$ } ANY DISTRIBUTION !!!

$$S = X_1 + \dots + X_n$$

$$E[S] = n\mu$$

$$Var[S] = n\sigma^2$$

$\downarrow$

Std. Dev of $S = \sqrt{n}\,\sigma$

$$\boxed{\frac{S - n\mu}{\sqrt{n}\,\sigma} \longrightarrow \mathcal{N}(0,1) \text{ as } n \to \infty}$$

Normalized Sum

CS 70 Discrete Mathematics and Probability Theory
Spring 2021 **Discussion 13B**

# 1 Interesting Gaussians

(a) If $X \sim N(0, \sigma_X^2)$ and $Y \sim N(0, \sigma_Y^2)$ are independent, then what is $\mathbb{E}\left[(X+Y)^k\right]$ for any *odd* $k \in \mathbb{N}$?

(b) Let $f_{\mu,\sigma}(x)$ be the density of a $N(\mu,\sigma^2)$ random variable, and let $X$ be distributed according to $\alpha f_{\mu_1,\sigma_1}(x)+(1-\alpha)f_{\mu_2,\sigma_2}(x)$ for some $\alpha \in [0,1]$. Please compute $\mathbb{E}[X]$ and $\mathrm{Var}[X]$. Is $X$ normally distributed?

## 2 Erasures, Bounds, and Probabilities

Alice is sending 1000 bits to Bob. The probability that a bit gets erased is $p$, and the erasure of each bit is independent of the others.

Alice is using a scheme that can tolerate up to one-fifth of the bits being erased. That is, as long as Bob receives at least 801 of the 1000 bits correctly, he can decode Alice's message.

In other words, Bob becomes unable to decode Alice's message only if 200 or more bits are erased. We call this a "communication breakdown", and we want the probability of a communication breakdown to be at most $10^{-6}$.

(a) Use Chebyshev's inequality to upper bound $p$ such that the probability of a communications breakdown is at most $10^{-6}$.

i.i.d. $I_1, \ldots, I_{1000} \sim$ Bernoulli$(p)$

Var $= p(1-p)$

$D = I_1 + \cdots I_{1000}$

$E[D] = E[I_1] + \cdots + E[I_{1000}] = 1000p$

$\text{Var}(I_1 + \cdots + I_{1000}) = \text{Var}(I_1) + \cdots + \text{Var}(I_{1000}) = 1000p(1-p)$

$P(D \geq 200) \leq 10^{-6}$

$P(D - E[D] \geq 200 - 1000p) \leq 10^{-6}$

$\wedge\!|$

$\frac{\text{Var}(D)}{(200-1000p)^2} = \frac{1000p(1-p)}{(200-1000p)^2} \leq 10^{-6}$

$1000p(1-p) \leq 10^{-6}(200-1000p)^2$

$p(1-p) \leq 10^{-6}(1-5p)^2 \cdot 40$

$p \leq 3.998 \times 10^{-5}$

(b) As the CLT would suggest, approximate the fraction of erasures by a Gaussian random variable (with suitable mean and variance). Use this to find an approximate bound for $p$ such that the probability of a communications breakdown is at most $10^{-6}$.

You may use that $\Phi^{-1}(1-10^{-6}) \approx 4.753$.

$E[I_i] = p$

$Var = p(1-p)$

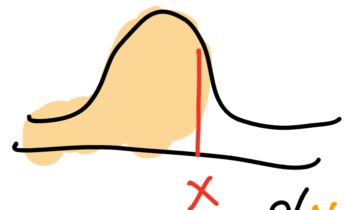
i.i.d. $I_1, \ldots, I_{1000} \sim$ Bernoulli$(p)$

$E[S] = E[I_1] + \ldots + E[I_{1000}] = 1000p$

$S = I_1 + \ldots I_{1000}$

$Var(S) = Var(I_1 + \ldots + I_{1000}) = 1000p(1-p)$

$$\frac{S - 1000p}{\sqrt{1000p(1-p)}} \approx N(0,1)$$




$P(X \leq x) = \Phi(x)$

$$P(S \geq 200) \leq 10^{-6}$$

$$P\left(\frac{S - 1000p}{\sqrt{1000p(1-p)}} \geq \frac{200 - 1000p}{\sqrt{1000p(1-p)}}\right) \leq 10^{-6}$$

$$1 - P\left(\frac{S - 1000p}{\sqrt{1000p(1-p)}} \leq \frac{200 - 1000p}{\sqrt{1000p(1-p)}}\right) \leq 10^{-6}$$

$$1 - \Phi\left(\frac{200 - 1000p}{\sqrt{1000p(1-p)}}\right) \leq 10^{-6}$$

$$\Phi\left(\frac{200 - 1000p}{\sqrt{1000p(1-p)}}\right) \geq 1 - 10^{-6}$$

$$\frac{200 - 1000p}{\sqrt{1000p(1-p)}} \geq 4.753$$

$$\boxed{p \leq 0.1468}$$

## 3 Suspicious Envelopes

There are two sealed envelopes. One containing $x$ dollars and the other one containing $2x$ dollars. You select one of the two envelopes at random (but, don't open it).

(a) According to the logic below, you should keep swapping the selected envelope with the other one indefinitely to improve your expected earning. Is something wrong in this logic?

**Logic:** Let $F$ and $S$ denote the the amount in the envelope you select at random and the other one, respectively.Then, $S = 2F$ or $\frac{F}{2}$, with equal probability $\frac{1}{2}$. Hence, $\mathbb{E}[S] = \frac{1}{2}\left(2F + \frac{F}{2}\right) = 1.25F$, and you are better off exchanging your selected (and still sealed) envelope with the the other sealed envelope. The same logic is applicable after the exchange, and swapping should continue ad infinitum.

(b) You are now allowed to pick one envelope and see how much cash is inside, and then based on this information, you can decide to switch envelopes or stick with the envelope you already have.

Can you come up with a strategy which will allow you to pick the envelope with more money, with probability strictly greater than $1/2$?