Cardinality

Two sets S and T have the same cardinality if we can find a bijection $f: S \to T$, OR if we can find 2 injections $f: S \to T$ and $g: T \to S$ (Cantor-Bernstein Thm)

Countability

$\left.\begin{array}{l}\mathbb{N} \\ \mathbb{Z} \\ \mathbb{Z}^+ \\ \mathbb{Q} \\ \mathbb{N}\times\mathbb{N}\end{array}\right\}$ Countable ↕ bijective with any subset of $\mathbb{N}$

$\left\{\begin{array}{l}\text{Finite} \\ \text{Countably Infinite} \\ \text{Uncountable}\end{array}\right.$

Countably Infinite, Uncountable } Infinite

Uncountable: $\mathbb{R}$, Cantor Set

Example: $\mathbb{Z}^+$: positive integers are countable

bijective $f: \mathbb{N} \to \mathbb{Z}^+$

$0 \to 1,\ 1 \to 2,\ 2 \to 3,\ \ldots$

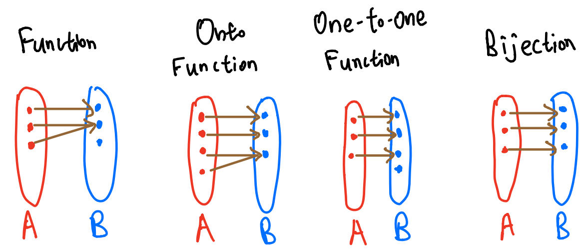
Fact: Subsets of countable sets always countable

Bijections

Mapping $f: A \to B$

onto: Every element in B has some $a \in A$ mapping to it

one-to-one: No two elements in A map to the same element in B



Cantor Diagonalization

$\mathbb{R}[0,1]$ is uncountable (so $\mathbb{R}$ is uncountable).

Assume $\mathbb{R}[0,1]$ was countable.

Then there is bijection $f: \mathbb{N} \to \mathbb{R}[0,1]$

$f(0) = 0.52149\ldots$ — $X \neq f(0)$ 1st digit doesn't match

$f(1) = 0.14162\ldots$ — $X \neq f(1)$ 2nd digit doesn't match

$f(2) = 0.94782\ldots$ — $X \neq f(2)$ 3rd digit doesn't match

$\vdots$

Diagonal digits $0.547\ldots$

$0\to1$, $1\to2$, $2\to3$, $3\to4$, $4\to5$, $5\to6$, $6\to7$, $7\to8$, $8\to1$, $9\to2$

Analyze $X = 0.658\ldots$

f doesn't map anything to X

$\Rightarrow$ contradiction (f not a bijection)

CS 70 Discrete Mathematics and Probability Theory
Spring 2021 **DIS 6B**

1 Hilbert's Hotel

You don't have any summer plans, so you decide to spend a few months working for a magical hotel with a countably infinite number of rooms. The rooms are numbered according to the natural numbers, and all the rooms are currently occupied. Assume that guests don't mind being moved from their current room to a new one, so long as they can get to the new room in a finite amount of time (i.e. guests can't be moved into a room infinitely far from their current one).

(a) A new guest arrives at the hotel. All the current rooms are full, but your manager has told you never to turn away a guest. How could you accommodate the new guest by shuffling other guests around? What if you instead had k guest arrive, for some fixed, positive $k \in \mathbb{Z}$?

$f(i) = i+k$

1 2 3 ... ...

$1 \dots k$ $k+1$

○ 2 3 4 ...

(b) Unfortunately, just after you've figured out how to accommodate your first $k+1$ guests, a countably infinite number of guests arrives in town on an infinitely long train. The guests on the train are sitting in seats numbered according to the natural numbers. How could you accommodate all the new guests?

$f(i) = 2i-1$ or $f(i)=2i$

$f(j) = 2j$ $f(j)=2j-1$

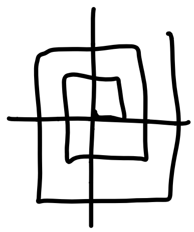
(c) Thanks to a (literally) endless stream of positive TripAdvisor reviews, word of the infinite hotel gets around quickly. Soon enough you find out that a countably infinite number of trains have arrived in town. Each is of infinite length, and carries a countably infinite number of passengers. How would you accommodate all the new passengers?

$\mathbb{N} \times \mathbb{N} \to \mathbb{N}$

$\mathbb{Z} \times \mathbb{Z}$

$(x, y) \to k \to 2k$

$f(i) \to 2i-1$



2 Countability Practice

(a) Do $(0,1)$ and $\mathbb{R}_+ = (0,\infty)$ have the same cardinality? If so, either give an explicit bijection (and prove that it is a bijection) or provide an injection from $(0,1)$ to $(0,\infty)$ and an injection from $(0,\infty)$ to $(0,1)$ (so that by Cantor-Bernstein theorem the two sets will have the same cardinality). If not, then prove that they have different cardinalities.

$(0,1) \to [0,\infty)$ $\quad f(x) = x$

$(0,\infty) \to (0,1)$ $\quad f(x) = \frac{1}{x+1}$

Cantor Bernstein

$\downarrow$

Same Cardinality

$f(x) = f(y) \Rightarrow x = y$

(b) Is the set of strings over the English alphabet countable? (Note that the strings may be arbitrarily long, but each string has finite length. Also the strings need not be real English words.) If so, then provide a method for enumerating the strings. If not, then use a diagonalization argument to show that the set is uncountable.

Yes

26 letters

length		Numbering
1	26	1 to 26
2	26^2	27 to $(27+26^2)$
3	26^3	⋮
⋮		

$f: \text{Strings} \to \mathbb{N}$

(c) Consider the previous part, except now the strings are drawn from a countably infinite alphabet $\mathscr{A}$. Does your answer from before change? Make sure to justify your answer.

Countable

letters used	length
first 1	1, 2, ...
first 2	1, 2, ...
first 3	1, 2, ...

$\mathcal{A} = \{a_1, a_2, \dots\}$

$\mathbb{N} \times \mathbb{N} \to \mathbb{N}$

3 Counting Functions

Are the following sets countable or uncountable? Prove your claims.

(a) The set of all functions f from $\mathbb{N}$ to $\mathbb{N}$ such that f is non-decreasing. That is, $f(x) \le f(y)$ whenever $x \le y$.

Uncountable

	0	1	2	...
f_0	$f_0(0)$	$f_0(1)$	$f_0(2)$	...
f_1	$f_1(0)$	$f_1(1)$	$f_1(2)$	...
f_2	$f_2(0)$	$f_2(1)$	$f_2(2)$	...
$\vdots$	$\vdots$			

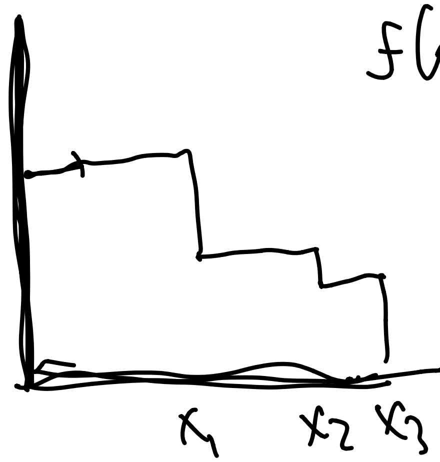
$g(0) > f_0(0)$

$g(1) > \max(f_1(1), g(0))$

$g(2) > \max(f_2(2), g(1))$

(b) The set of all functions f from $\mathbb{N}$ to $\mathbb{N}$ such that f is non-increasing. That is, $f(x) \ge f(y)$ whenever $x \le y$.

Countable



$f(0) = y$

↑ Finite

Finite # places where f decreases

$\mathbb{N}^y \to$ countable

$\underbrace{\mathbb{N} \times \mathbb{N}}_{\text{"enumeratable"}} \times \mathbb{N} \times \cdots \times \mathbb{N}$