Induction: Powerful tool when working in natural numbers $\mathbb{N}$.

Principle: To prove $(\forall n \in \mathbb{N})\, P(n)$

1. **Base Case** Show $P(0)$ holds true.

Shows $P(k) \Rightarrow P(k+1)$

2. **Induction Hypothesis**

For some $k \geq 0$, assume $P(k)$ holds true.

3. **Induction Step**

Using the Induction Hypothesis $P(k)$, prove that $P(k+1)$ is true.

How does this show $P(n)$ true for all natural n?

$P(0)$ is true by Base Case

$P(0) \Rightarrow P(1)$ by IH/IS, so $P(1)$ is true

$P(1) \Rightarrow P(2)$ by IH/IS, so $P(2)$ is true

$P(2) \Rightarrow P(3)$ by IH/IS, So $P(3)$ is true

$\vdots$

$P(k) \Rightarrow P(k+1)$ by IH/IS, So $P(k+1)$ is true

$\vdots$

and so on! (falling chain of dominoes)

A Satisfying Insight about Strong Induction

Suppose we want to show $(\forall n \in \mathbb{N})\, P(n)$ via induction.

Define a new proposition:

$P'(k) =$ " $P(0), P(1), \ldots, P(k)$ are all true."

$$P'(k) = P(0) \wedge P(1) \wedge \cdots \wedge P(k)$$

Weak induction on $P'(k)$ is equivalent to

Strong induction on $P(k)$

Furthermore, proving $(\forall n \in \mathbb{N})\, P'(n)$ also proves

$$(\forall n \in \mathbb{N})\, P(n)$$

1 Natural Induction on Inequality

Prove that if $n \in \mathbb{N}$ and $x > 0$, then $(1+x)^n \geq 1+nx$.

BC: $n=0$: $(1+x)^0 \geq 1+0x$

$1 \geq 1$ ✓

IH: Assume for $n=k$,

$(1+x)^k \geq 1+kx$

IS: Want to show: $(1+x)^{k+1} \geq 1+(k+1)x$.

$(1+x)^{k+1} = (1+x)^k(1+x) \geq (1+kx)(1+x)$

$= 1+kx^2+(k+1)x$

$\geq 1+(k+1)x$

2 Make It Stronger

Suppose that the sequence $a_1, a_2, \ldots$ is defined by $a_1 = 1$ and $a_{n+1} = 3a_n^2$ for $n \geq 1$. We want to prove that

$$a_n \leq 3^{(2^n)}$$

for every positive integer n.

(a) Suppose that we want to prove this statement using induction. Can we let our inductive hypothesis be simply $a_n \leq 3^{(2^n)}$? Attempt an induction proof with this hypothesis to show why this does not work.

IS: Want to show $a_{n+1} \leq 3^{(2^{n+1})}$

$a_{n+1} = 3a_n^2 \leq 3 \times \left(3^{(2^n)}\right)^2 = 3^{2^{n+1}+1}$

(b) Try to instead prove the statement $a_n \leq 3^{(2^n-1)}$ using induction.

IS: Want to show $a_{n+1} \leq 3^{(2^{n+1}-1)}$

$a_{n+1} = 3a_n^2 \leq 3 \times \left(3^{(2^n-1)}\right)^2 = 3^{(2^{n+1}-2+1)}$

$= 3^{(2^{n+1}-1)}$

(c) Why does the hypothesis in part (b) imply the overall claim?

$a_n \leq 3^{2^n-1} = \frac{1}{3} \times 3^{2^n}$

$\leq 3^{2^n}$

3 Binary Numbers

Prove that every positive integer n can be written in binary. In other words, prove that we can write

$$n = c_k \cdot 2^k + c_{k-1} \cdot 2^{k-1} + \cdots + c_1 \cdot 2^1 + c_0 \cdot 2^0,$$

where $k \in \mathbb{N}$ and $c_i \in \{0,1\}$ for all $i \leq k$.

BC: $1 = 1 \times 2^0$

IH: For $1 \leq n \leq m$, n is representable in binary

IS: Want to show $n = m+1$ is representable in binary

$m+1 = c_k \cdot 2^k + \dots + c_1 \cdot 2^1 + c_0 2^0$

If $m+1$ is odd, m is even

$m = c_k \cdot 2^k + \dots + c_1 \cdot 2^1 + 0 \cdot 2^0$

$m+1 = c_k \cdot 2^k + \dots + c_1 2^1 + 1 \cdot 2^0$

If $m+1$ is even, $\frac{m+1}{2}$ is binary-representable

$\frac{m+1}{2} = c_k \cdot 2^k + \dots + c_0 \cdot 2^0$

$m+1 = c_k \cdot 2^{k+1} + c_{k-1} 2^k + \dots + c_0 \cdot 2^1 + 0 \cdot 2^0$

4 Fibonacci for Home

Recall, the Fibonacci numbers, defined recursively as

$F_1 = 1$, $F_2 = 1$, and $F_n = F_{n-2} + F_{n-1}$.

Prove that every third Fibonacci number is even. For example, $F_3 = 2$ is even and $F_6 = 8$ is even.

BC: $F_3 = 2$, 2 even

IH: Assume for $n = k$, F_{3k} is even

IS: Want to show for $n = k+1$, F_{3k+3} is even

$F_{3k+3} = F_{3k+2} + F_{3k+1} = (F_{3k+1} + F_{3k}) + F_{3k+1}$

$= \underbrace{F_{3k}}_{\text{even by IH}} + \underbrace{2F_{3k+1}}_{\text{even multiple of 2}}$