Indicators

For event E occurring with probability p,

I is an indicator of E if $I = \begin{cases} 1 & \text{when } E \text{ occurs} \\ 0 & \text{o/w} \end{cases}$

Notice: I's distribution is $I = \begin{cases} 1 & \text{w.p. } p \\ 0 & \text{o/w} \end{cases}$

$\Uparrow$

$I \sim \text{Bernoulli}(p)$

Important: For any Bernoulli / Indicator RV $X \sim \text{Bernoulli}(p)$

$$\boxed{E[X]} = \sum_k k P(X=k) = 1 \cdot p + 0 \cdot (1-p) = \boxed{p}$$

Linearity of Expectation

For **ANY** (not neccesarily independent!) Random Variables $X_1, X_2, \dots, X_n$,

$$E[X_1 + X_2 + \dots + X_n] = E[X_1] + E[X_2] + \dots + E[X_n]$$

An example:

Suppose we have i.i.d. $X_1, X_2, \dots, X_n \sim \text{Bernoulli}(p)$

$$Y = (X_1 + X_2 + \dots + X_n) \sim \text{Bin}(n, p)$$

What's $E[Y]$?

Method 1 (Definition of Exp.)

$$E[Y] = \sum_k k P(Y=k) = \sum_{k=0}^{n} k \binom{n}{k} p^k (1-p)^{n-k}$$

$$\begin{aligned}
\mu &= \sum_{k=0}^{n} k \binom{n}{k} p^k (1-p)^{n-k} \\
&= np \sum_{k=0}^{n} k \frac{(n-1)!}{(n-k)!k!} p^{k-1} (1-p)^{(n-1)-(k-1)} \\
&= np \sum_{k=1}^{n} \frac{(n-1)!}{((n-1)-(k-1))!(k-1)!} p^{k-1} (1-p)^{(n-1)-(k-1)} \\
&= np \sum_{k=1}^{n} \binom{n-1}{k-1} p^{k-1} (1-p)^{(n-1)-(k-1)} \\
&= np \sum_{\ell=0}^{n-1} \binom{n-1}{\ell} p^{\ell} (1-p)^{(n-1)-\ell} \\
&= np \sum_{\ell=0}^{m} \binom{m}{\ell} p^{\ell} (1-p)^{m-\ell} \\
&= np(p + (1-p))^m \\
&= np
\end{aligned}$$

Method 2 (Linearity of Exp.) ☆

$$E[Y] = E\left[\sum_{i=1}^{n} X_i\right] = \sum_{i=1}^{n} E[X_i] = \sum_{i=1}^{n} p = \boxed{np}$$

CS 70 Discrete Mathematics and Probability Theory
Spring 2021 **DIS 10A**

1 Linearity

Solve each of the following problems using linearity of expectation. Explain your methods clearly.

(a) In an arcade, you play game A 10 times and game B 20 times. Each time you play game A, you win with probability 1/3 (independently of the other times), and if you win you get 3 tickets (redeemable for prizes), and if you lose you get 0 tickets. Game B is similar, but you win with probability 1/5, and if you win you get 4 tickets. What is the expected total number of tickets you receive?

$$\underbrace{I_1, \dots, I_{10}}_{\text{10 times we play A}}, \underbrace{I_{11}, \dots, I_{30}}_{\text{20 times we play B}}$$

$$T = 3(I_1 + \dots + I_{10}) + 4(I_{11} + \dots + I_{30})$$

$$E[T] = 3(E[I_1] + \dots + E[I_{10}]) + 4(E[I_{11}] + \dots + E[I_{30}])$$

$$= 3(10 \cdot 1/3) + 4(20 \cdot \tfrac{1}{5}) = \boxed{26}$$

(b) A monkey types at a 26-letter keyboard with one key corresponding to each of the lower-case English letters. Each keystroke is chosen independently and uniformly at random from the 26 possibilities. If the monkey types 1 million letters, what is the expected number of times the sequence "book" appears?

b o o k

1 2 …. 999997 1000000

I_1 I_2 I_3

$$E[I_i] = p = \frac{1}{26^4}$$

$$T = I_1 + I_2 + \dots + I_{999997}$$

$$E[T] = E[I_1] + E[I_2] + \dots + E[I_{999997}]$$

$$= \boxed{999997 \cdot \frac{1}{26^4}}$$

2 Ball in Bins

You are throwing k balls into n bins. Let X_i be the number of balls thrown into bin i.

(a) What is $\mathbb{E}[X_i]$?

$I_1, I_2, \ldots, I_K$ for each ball

$X_i = I_1 + \cdots + I_K$

$E[X_i] = E[I_1] + \cdots + E[I_K] = \boxed{K \cdot \frac{1}{n}}$

(b) What is the expected number of empty bins?

$I_1, I_2, \ldots, I_n$ for each of the n bins

$B = I_1 + I_2 + \cdots + I_n$

$E[B] = E[I_1] + \cdots + E[I_n] = \boxed{n\left(1 - \frac{1}{n}\right)^K}$

(c) Define a collision to occur when two balls land in the same bin (if there are n balls in a bin, count that as $n-1$ collisions). What is the expected number of collisions?

n $\max(0, n-1) \rightarrow K - M$

$M = n - B$

$\rightarrow K - n + B$

$K - n + E[B] = \boxed{K - n + n\left(1 - \frac{1}{n}\right)^K}$

3 Swaps and Cycles

We'll say that a permutation $\pi = (\pi(1), \ldots, \pi(n))$ contains a *swap* if there exist $i, j \in \{1, \ldots, n\}$ so that $\pi(i) = j$ and $\pi(j) = i$.

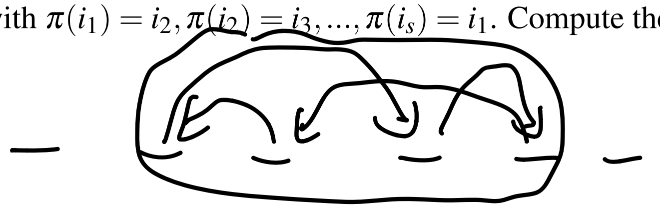
(a) What is the expected number of swaps in a random permutation?

$$I_{i,j} \Rightarrow \frac{(n-2)!}{n!}$$

$$X = \sum_{i,j \in \{1,\dots,n\}} I_{i,j}$$

$$E[X] = \sum_{i,j \in \{1,\dots,n\}} E[I_{i,j}] = \sum_{i,j \in \{1,\dots,n\}} \frac{(n-2)!}{n!} = \binom{n}{2}\frac{\cancel{(n-2)!}}{\cancel{n!}} = \boxed{\frac{1}{2}}$$

(b) In the same spirit as above, we'll say that π contains a *s-cycle* if there exist $i_1,\dots,i_s \in \{1,\dots,n\}$ with $\pi(i_1) = i_2, \pi(i_2) = i_3, \dots, \pi(i_s) = i_1$. Compute the expectation of the number of s-cycles.



$$I_{\{i_1,i_2,\dots,i_s\}} \Rightarrow \frac{(n-s)!}{n!} \qquad \binom{n}{s}\cdot\frac{s!}{s}$$

$$X = \sum_{\{i_1,i_2,\dots,i_s\}} I_{\{i_1,i_2,\dots,i_s\}}$$

$$E(X) = \sum_{\{i_1,i_2,\dots,i_s\}} E[I_{\{i_1,i_2,\dots,i_s\}}] = \sum_{\{i_1,i_2,\dots,i_s\}} \frac{(n-s)!}{n!}$$

$$= \binom{n}{s}\cdot\frac{s!}{s}\cdot\frac{(n-s)!}{n!} = \frac{\cancel{n!}}{\cancel{s!}\cancel{(n-s)!}}\cdot\frac{\cancel{s!}}{s}\cdot\frac{\cancel{(n-s)!}}{\cancel{n!}} = \boxed{\frac{1}{s}}$$