

'The set of finite length strings is countable'

CS 70  
Spring 2021

Discrete Mathematics and Probability Theory

DIS 7A

## 1 Countability and the Halting Problem

Prove the Halting Problem using the set of all programs and inputs.

- a) What is a reasonable representation for a computer program? Using this definition, show that the set of all programs are countable. (Hint: Python Code)

Any program is uniquely represented  
by a finite length string  
↓  
countable

- b) We consider only finite-length inputs. Show that the set of all inputs are countable.

↓  
finite-length strings  
↓  
countable

- c) Assume that you have a program that tells you whether or not a given program halts on a specific input. Since the set of all programs and the set of all inputs are countable, we can enumerate them and construct the following table.

	$x_1$	$x_2$	$x_3$	$x_4$	...
$p_1$	H	L	H	L	...
$p_2$	L	L	L	H	...
$p_3$	H	L	H	L	...
$p_4$	L	H	L	L	...
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

An  $H$  (resp.  $L$ ) in the  $i$ th row and  $j$ th column means that program  $p_i$  halts (resp. loops) on input  $x_j$ . Now write a program that is not within the set of programs in the table above.

$P(x_i)$

If  $p_i$  halts on  $x_i$ ,  
loop forever

If  $p_i$  doesn't halt on  $x_i$ ,  
halt

return

d) Find a contradiction in part a and part c to show that the halting problem can't be solved.

Contradiction  $\rightarrow$  halting problem  
cannot be solved.

## 2 Fixed Points

Consider the problem of determining if a function  $F$  has any fixed points. That is, given a function  $F$  that takes inputs from some (possibly infinite) set  $\mathcal{X}$ , we want to know if there is any input  $x \in \mathcal{X}$  such that  $F(x)$  outputs  $x$ . Prove that this problem is undecidable.

## 3 Hello World!

Determine the computability of the following tasks. If it's not computable, write a reduction or self-reference proof. If it is, write the program.

- (a) You want to determine whether a program  $P$  on input  $x$  prints "Hello World!". Is there a computer program that can perform this task? Justify your answer.

- (b) You want to determine whether a program  $P$  prints "Hello World!" before running the  $k$ th line in the program. Is there a computer program that can perform this task? Justify your answer.
- (c) You want to determine whether a program  $P$  prints "Hello World!" in the first  $k$  steps of its execution. Is there a computer program that can perform this task? Justify your answer.