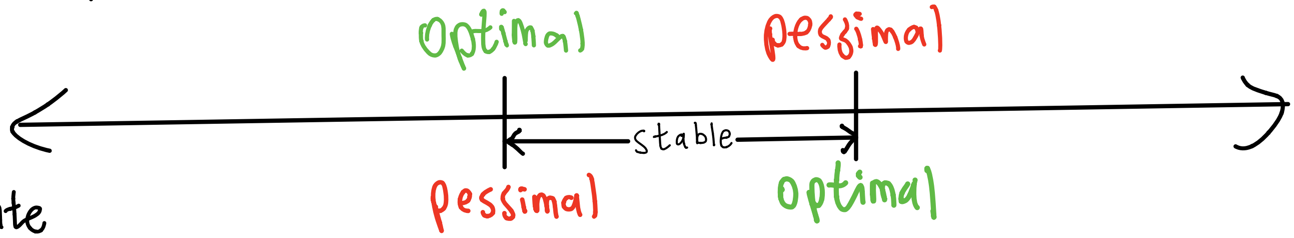
Optimality & Pessimality

Job

$J_x : C_1 > \cdots > C_i > \cdots > C_j > \cdots > C_n$



Candidate

$C_y : J_n < \cdots < J_j < \cdots < J_i < \cdots < J_1$

1 Optimal Candidates

In the notes, we proved that the propose-and-reject algorithm always outputs the job-optimal pairing. However, we never explicitly showed why it is guaranteed that putting every job with its optimal candidate results in a pairing at all. Prove by contradiction that no two jobs can have the same optimal candidate. (Note: your proof should not rely on the fact that the propose-and-reject algorithm outputs the job-optimal pairing.)

Assume for sake of contradiction:

J's optimal is C

J*'s optimal is C

Case 1: Preference of C: $J > J^*$

(J^*, C) (J, C')

J: $C > C'$

C, J are rogue

$\longrightarrow$ contradiction

Case 2: Preference of C: $J^* > J$

(J, C) (J^*, C')

J^*: $C > C'$

C, J^* is rogue

→ contradiction

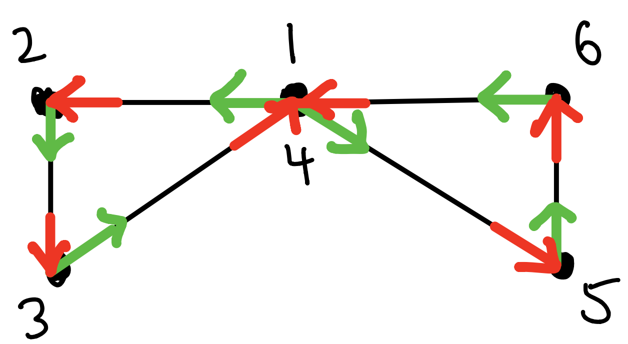
Graph Theory

- Vertices (Nodes), V
- Edges, E
- degree of a vertex = # of adjacent edges

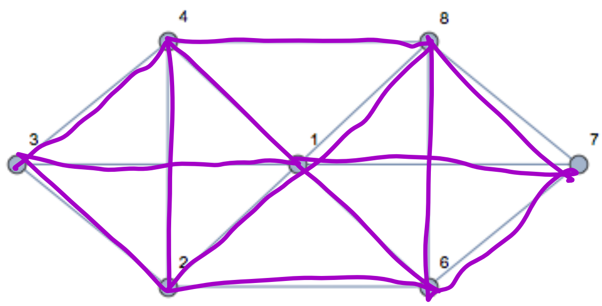
		Start = End
No Repeats	(Simple) path	Cycle
Repeated vertex or Edge	Walk	Tour

Eulerian Tour: Tour visiting each edge exactly once

- exists if and only if G is connected and all vertices have even degree



2 Eulerian Tour and Eulerian Walk



(a) Is there an Eulerian tour in the graph above? If no, give justification. If yes, provide an example.

No

(b) Is there an Eulerian walk in the graph above? An Eulerian walk is a walk that uses each edge exactly once. If no, give justification. If yes, provide an example.

Yes

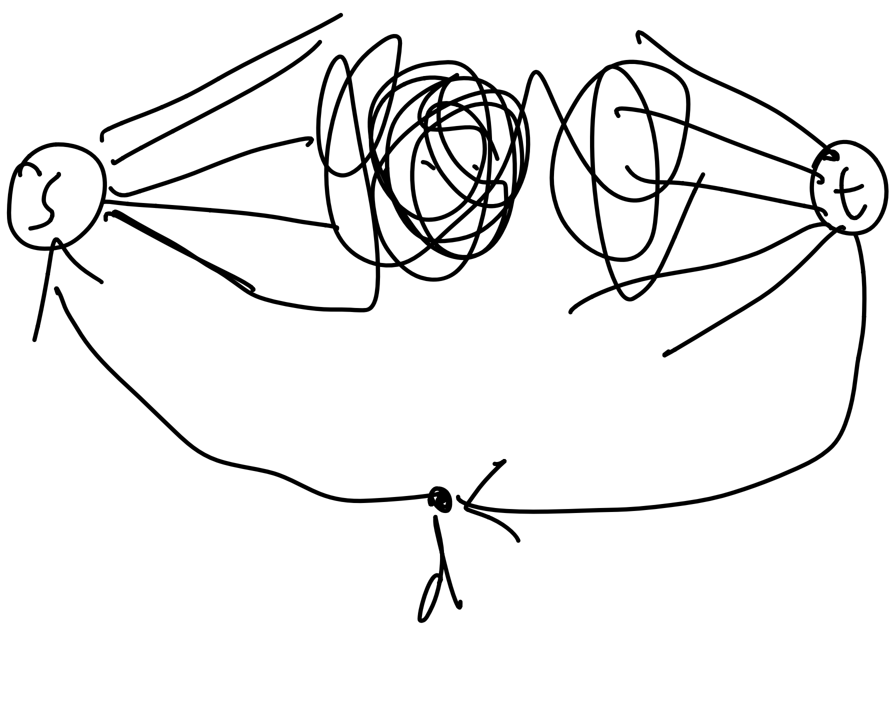
(c) What is the condition that there is an Eulerian walk in an undirected graph? Briefly justify your answer.

Exists if up to 2 vertices have odd degree, and connected.

If 0 vertices odd degree: ✓

If 1 vertex odd degree: impossible

If 2 vertices odd degree:



Graph Theory Proofs

☆ Induction ☆

- induct on vertices or edges
- Rough Breakdown:

<u>Base Case</u>: Small (e.g. size 1 or 2) graph

<u>Inductive Hypothesis</u>: Assume works for <u>all</u> graphs up to size k

<u>Inductive Step</u>: Want to show for <u>all</u> graphs of size k+1

Consider <u>any</u> graph G of size k+1.

Remove a (vertex / edge) from G.

Remaining subgraph is size k $\Rightarrow$ Apply I.H.!!!

Add back removed component

- show that adding back doesn't invalidate

3 Not everything is normal: Odd-Degree Vertices

Claim: Let $G = (V, E)$ be an undirected graph. The number of vertices of G that have odd degree is even.

Prove the claim above using:

(i) Direct proof (e.g., counting the number of edges in G). *Hint: in lecture, we proved that* $\sum_{v \in V} \deg v = 2|E|$.

$$\sum_{v \in V} \deg v = 2|E|$$

$$\sum_{v \in V_{odd}} \deg v + \sum_{v \in V_{even}} \deg v = 2|E|$$

$$\underbrace{\sum_{v \in V_{odd}} \underbrace{\deg v}_{\text{odd}}}_{\text{even}} = \underbrace{2|E|}_{\text{even}} - \underbrace{\sum_{v \in V_{even}} \deg v}_{\text{even}}$$

must have even # of odd degree vertices

(ii) Induction on $m = |E|$ (number of edges)

Base case:

I.H.:

Assume statement holds for all graphs up to m edges.

I.S.:

W.T.S. true for all graphs with m+1 edges.

Consider any graph G with m+1 edges.

$G - \{u, v\} \Rightarrow G'$ (m edges)

$\downarrow$

G' has an even number of odd degree vertices. IH

u even, v even in G':

$\rightarrow$ u odd, v odd in G $\Rightarrow$ $+2$ odd degree vertices

u even, v odd in G':

$\rightarrow$ u odd, v even in G $\Rightarrow$ $+0$ odd degree vertices

v even, u odd in G':

$\rightarrow$ v odd, u even in G $\Rightarrow$ $+0$ odd degree vertices

u odd, v odd in G':

$\rightarrow$ u even, v even in G $\Rightarrow$ -2 odd degree vertices

(iii) Induction on $n = |V|$ (number of vertices)