Discrete Math Fundamentals

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CS 70

§1 Sets

Sets are collections of elements. A brief list of notable attributes/definitions:

- Equality - Equal sets contain the same elements; order/repeats don't matter.
- Cardinality - The size (number of elements) of a set. $\emptyset$ has cardinality 0.
- Subsets - $A \subseteq B$ if and only if every member of A is in B.
  - If it also happens that $A \neq B$, then A is a *proper subset* of B, denoted $A \subset B$.
- Intersection - $A \cap B$ is the set of all members common between A and B.
- Union - $A \cup B$ is the set of all members that are in at least one of A or B.
- Relative Complement - $B \setminus A$ is the set of all elements in B but not in A.
- Significant Sets
  - $\mathbb{N}$, the Naturals - includes 0, i.e. $\{0, 1, 2, \dots\}$.
  - $\mathbb{Z}$, the Integers.
  - $\mathbb{Q}$, the Rationals - $\{\frac{a}{b} \mid (a, b \in \mathbb{Z}) \wedge (b \neq 0)\}$.
  - $\mathbb{R}$, the Reals.
  - $\mathbb{C}$, the Complex Numbers.
- Cartesian Product - $A \times B = \{(a, b) \mid (a \in A) \wedge (b \in B)\}$.
  - In words: the Cartesian product of two sets A and B is the set of all pairs (a, b) where a is in set A and b is in set B.
- Power Set - $\wp(A)$ is the set of all subsets of A.

> **Remark 1.1.** If A has size $|A|$, then $|\wp(A)| = 2^{|A|}$, since each of A's elements *could or could not* (2 possibilities) be in a subset of A.

§2 Propositional Logic

Propositions are statements with a truth value. Some important things to remember:

- Connectives: "and" ($\wedge$), "or" ($\vee$), "not" ($\neg$), "implies" ($\implies$)
- Quantifiers: "for all" ($\forall$), "there exists" ($\exists$)

- $P \implies Q \equiv \neg P \vee Q$ (verifiable with truth tables)
- $P \implies Q \equiv \neg Q \implies \neg P$ (contrapositive)
- $\forall x(\exists y P(x,y)) \not\equiv \exists y(\forall x P(x,y))$
  - Different quantifiers cannot be switched/interchanged!

Example 2.1

Consider $P(x,y)$ as the statement $y > x$. The left side becomes "For every x, there exists a y greater than x" (true statement). The right side is "There is a number y greater than every x" (false statement).

- DeMorgan's Laws
  1. $\neg(P \wedge Q) \equiv \neg P \vee \neg Q$
  2. $\neg(P \vee Q) \equiv \neg P \wedge \neg Q$
  3. $\neg(\forall x P(x)) \equiv \exists x \neg P(x)$
  4. $\neg(\exists x P(x)) \equiv \forall x \neg P(x)$

§3 Proofs

Proofs are important because they assure that an implication or statement is true. The main types of proofs are:

- Direct - Prove an implication $P \implies Q$ by assuming P is true and through a series of implications, deriving that Q is true.
- Contraposition - Prove an implication $P \implies Q$ by proving its contrapositive, $\neg Q \implies \neg P$ (which is logically equivalent), is true.
- Contradiction - Prove a statement P by first assuming $\neg P$, then reaching a contradiction, and thus concluding P must be true by the law of the excluded middle.
- Cases - Prove a statement P by splitting P into cases and proving that P holds true in each case.

Developing proof-writing ability takes lots of practice! Common mistakes include:

- When attempting to prove a claim P, assuming P is true from the start.
- Missing cases, such as division by 0.
- Negative numbers with inequalities (don't forget that multiplying by a negative flips the inequality direction).

§4 Induction

Induction is another proof technique, similar in essence to recursion. It can be a powerful tool when working with *natural numbers*.

Proposition 4.1 (Principle of Induction)

To prove the statement $P(n)$ holds for all natural number values of n:

1. Base Case - Show that $P(0)$ holds.

2. Inductive Hypothesis - For some $k \geq 0$, assume $P(k)$ holds.

3. Inductive Step - Using the Inductive Hypothesis $P(k)$, prove that $P(k+1)$ is true, i.e. $P(k) \implies P(k+1)$.

Remark 4.2. This will effectively prove $P(n)$ holds for all naturals n.

- $P(0)$ is true (by Base Case).
- $P(0) \implies P(1)$ is true (by IS), so $P(1)$ is true;
- $P(1) \implies P(2)$ is true (by IS), so $P(2)$ is true;

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- $P(k) \implies P(k+1)$ is true (by IS), so $P(k+1)$ is true;

  and so on. The statements are shown to be true like a falling chain of dominoes, giving us that $P(n)$ is true for all naturals $n \in \{0, 1, 2, \ldots\}$.

Strong induction is when we modify the induction hypothesis to assume for *all* $i \leq k$, $P(i)$ holds true (i.e., $P(0), P(1), \ldots, P(k)$ are all true). In certain proofs this may help us prove $P(k+1)$ in the inductive step more easily.

Remark 4.3. It can be easy to get confused by the seeming distinction between strong and weak induction. However, they are in essence the exact same thing:

- Say we wanted to show some proposition $P(n)$ to be true for all natural n via induction.
  - Let us define a new proposition $P'(n) =$ "$P(0), P(1), \ldots, P(n)$ are all true."
  - Strong induction on $P(n)$—*which is equivalent to weak induction on* $P'(n)$—then proves $P'(n)$ is true for all naturals (thereby proving $P(n)$ is true for all naturals).
- Another way to see this relationship is by extending the domino analogy:
  - With strong induction we are using "all dominoes up to the kth domino have fallen" to show "the $(k+1)$th domino will fall."
  - With weak induction we are only using "the kth domino falling" to show that "the $(k+1)$th domino will fall."
    - ∗ However, the fact that "the kth domino has fallen" means that all dominoes before it have already fallen anyways!