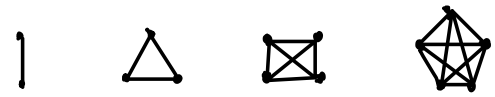
Graph Theory (continued)

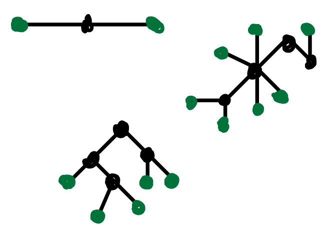
Complete Graphs: edges between <u>every</u> vertex pair.



K_2 K_3 K_4 K_5

$\Rightarrow \frac{n(n-1)}{2}$ edges in K_n

Trees: Four equivalent definitions:



1. Connected w/ no cycle
2. Connected w/ $|E| = |V| - 1$
3. connected, and removal of any edge disconnects G
4. No cycles, and adding any edge creates a cycle

Leaves: vertices with degree 1.

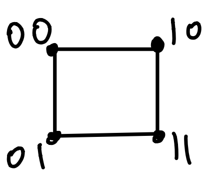
Hypercubes:

- nodes labeled by bitstrings of length n,
  (n is the <u>dimension</u> of the hypercube)
- edges exist between nodes that differ at exactly 1 position in bitstring

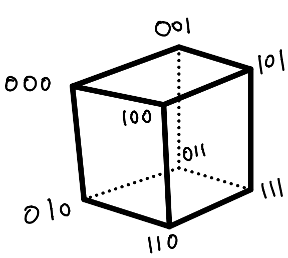
$\Rightarrow n \cdot 2^{n-1}$ edges

$n=1$

$n=2$



$n=3$



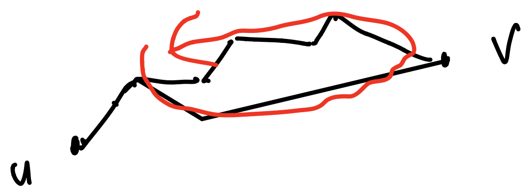
CS 70 Discrete Mathematics and Probability Theory
Spring 2021

DIS 2B

1 True or False

(a) Any pair of vertices in a tree are connected by exactly one path.

True



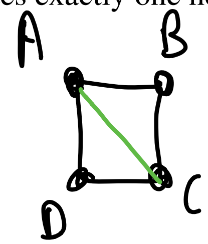
(b) A simple graph obtained by adding an edge between two vertices of a tree creates a cycle.

True

By def. of tree

(c) Adding an edge in a connected graph creates exactly one new cycle.

False



Inducting on graph components (vertices or edges)

- Rough Breakdown:

Base Case: Small (e.g. size 1 or 2) graph

Inductive Hypothesis: Assume works for all graphs up to size k

Inductive Step: Want to show for all graphs of size $k+1$

1) Consider any graph G of size $k+1$.

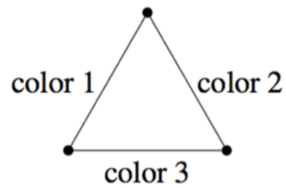
2) Remove a (vertex / edge) from G.

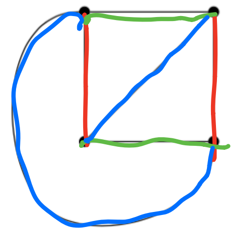
3) Remaining subgraph is size $k \Rightarrow$ Apply I.H.!!!

4) Add back removed component
   - show that adding back doesn't invalidate

3 Edge Colorings

An edge coloring of a graph is an assignment of colors to edges in a graph where any two edges incident to the same vertex have different colors. An example is shown on the left.



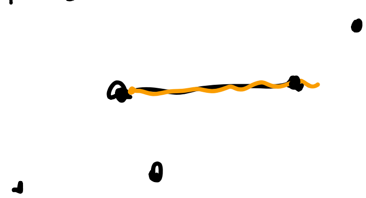


(a) Show that the 4 vertex complete graph above can be 3 edge colored. (Use the numbers $1,2,3$ for colors. A figure is shown on the right.)

(b) Prove that any graph with maximum degree $d \geq 1$ can be edge colored with $2d-1$ colors.

$P(n)$ = "any graph w/ n edges with max deg. d can be edge colored with 2d − 1 colors."

Base Case: $P(1)$

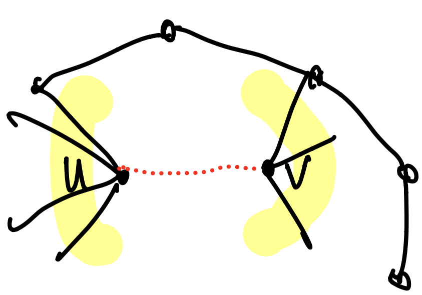


1

$2 \cdot 1 - 1 = 1$ ✓

IH: Assume $P(k)$

IS: Want to show $P(k+1)$



$G' = G - \{u, v\}$

G' satisfied by IH

G' can be colored w/ 2d−1 colors

v deg in G ≤ d

u deg in G ≤ d

v deg in $G' \leq d-1$ u deg in $G' \leq d-1$

v's edges use $d-1$ colors   u's edges use $d-1$ colors

$d-1+d-1 = 2d-2$ off-limits $\rightarrow$ adding back $\{u,v\}$ doesn't require new color

(c) Show that a tree can be edge colored with d colors where d is the maximum degree of any vertex.

$P(n)$ = "any graph with n vertices w/ d max deg. can be colored using d colors."

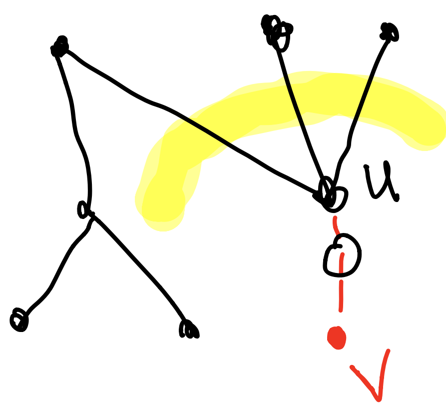
Base Case: $P(0)$ ✓

IH: Assume $P(k)$.

IS: Want to show $P(k+1)$

All trees have ≥ 2 leaves

$G' = G - v$



G' satisfies IH

G' can be d' colored where $d' \leq d$

$\Downarrow$

G' can be d colored

adding back v allows $\{u, v\}$ to be colored

u's deg in $G \leq d$

u's deg in $G - v \leq d-1$

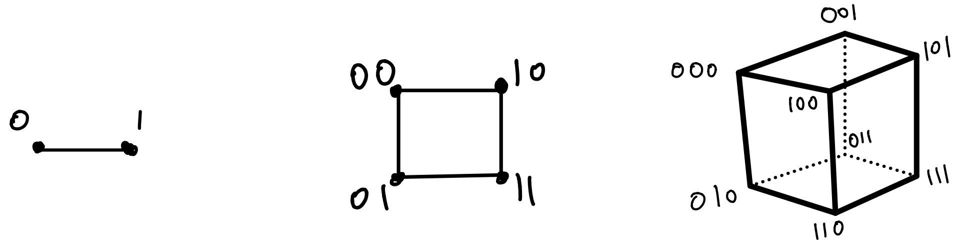
$\Downarrow$

$d-1$ colors

4 Hypercubes

The vertex set of the n-dimensional hypercube $G = (V, E)$ is given by $V = \{0,1\}^n$ (recall that $\{0,1\}^n$ denotes the set of all n-bit strings). There is an edge between two vertices x and y if and only if x and y differ in exactly one bit position. These problems will help you understand hypercubes.

(a) Draw 1-, 2-, and 3-dimensional hypercubes and label the vertices using the corresponding bit strings.

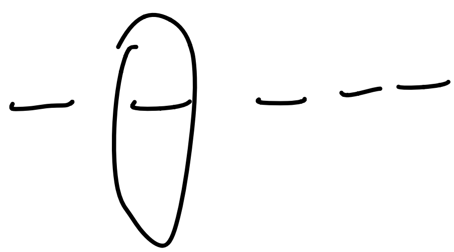


(b) Show that for any $n \geq 1$, the n-dimensional hypercube is bipartite.

$L = \{ v \mid v\text{'s label has an odd \# of 1's} \}$

$R = \{ v \mid v\text{'s label has an even \# of 1's} \}$

satisfies property of bipartiteness,



odd # of 1's

all edges

even # of 1's

2 Coloring Trees

Prove that all trees with at least 2 vertices are *bipartite*: the vertices can be partitioned into two groups so that every edge goes between the two groups.

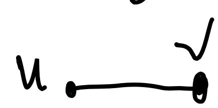
[*Hint:* Use induction on the number of vertices.]

$P(n)$ = "Any tree with n vertices, $n \geq 2$, is bipartite."

bipartite: L R $L \cup R = V$

No edge goes between any two vertices in the same set of (L or R).

Base Case: $P(2)$



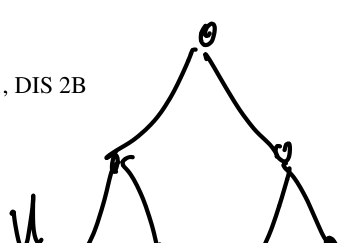
$L = \{u\}$, $R = \{v\}$ ✓

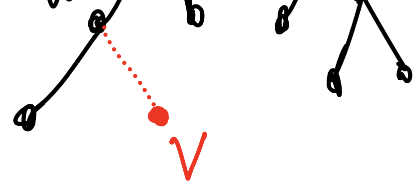
IH: Assume $P(k)$ for $k \geq 2$

IS: Want to show $P(k+1)$

$G' = G - V$

G' satisfies IH





$L = \{\ldots, u\}$

$R = \{\ldots\}$

$\downarrow$

Add back v

$L = \{\ldots, u\}$

$R = \{\ldots, v\}$