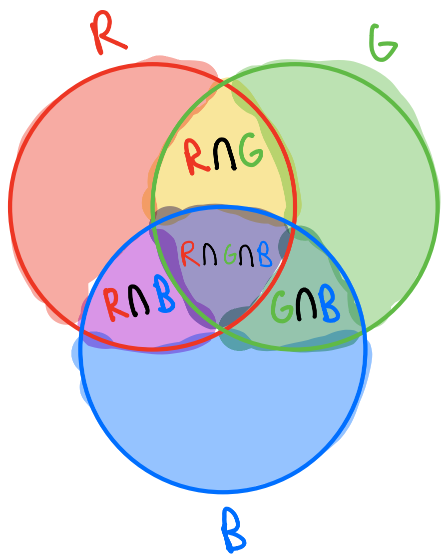
Counting Continued

Principle of Inclusion/Exclusion:



$$|R \cup G \cup B| =$$
$$|R| + |G| + |B|$$
$$- |R \cap G| - |R \cap B| - |G \cap B|$$
$$+ |R \cap G \cap B|$$

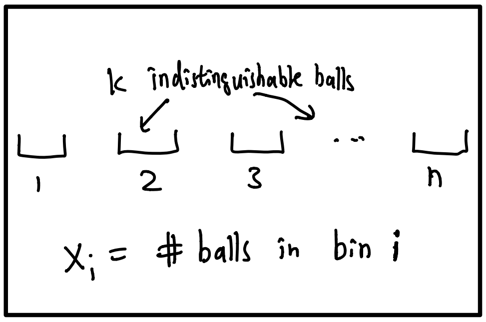
	$\lvert R\rvert$	$+\lvert G\rvert$	$+\lvert B\rvert$	$-\lvert R\cap G\rvert$	$-\lvert R\cap B\rvert$	$-\lvert G\cap B\rvert$	$+\lvert R\cap G\cap B\rvert$	# times counted $\lvert R\cup G\cup B\rvert$
	1							1
		1						1
			1					1
	1	1		−1				1
	1		1		−1			1
		1	1			−1		1
	1	1	1	−1	−1	−1	1	1

General Theorem:

$$|A_1 \cup A_2 \cup \cdots \cup A_n| = \sum_{k=1}^{n} (-1)^{k-1} \sum_{\substack{S \subseteq \{1,\dots,n\}; \\ |S|=k}} \left|\bigcap_{i\in S} A_i\right|$$

Counting Tips and Tricks:

- Exploit Symmetry
- Different ways to view same problem (combinatorial proofs)
- Pay attention to:
  - whether order matters
  - distinguishable vs. indistinguishable
  - w/ vs. w/o replacement
  - Stars and Bars: nonnegative vs. strictly positive



Nonnegative (vanilla case):

$$x_1 + x_2 + \dots + x_n = k \qquad x_i \geq 0$$

Strictly positive case, i.e. $x_i \geq 1$:

Let $y_i = x_i - 1 \longrightarrow y_i \geq 0$

$$y_1 + y_2 + \dots + y_n = k - n$$

"place a ball in every bin"

modified into a nonnegative case!!

2 The Count

(a) How many of the first 100 positive integers are divisible by 2, 3, or 5?

2: 50
3: +33
5: +20
6: −16
10: −10
15: −6
30: +3

$\boxed{74}$

(b) The Count is trying to choose his new 7-digit phone number. Since he is picky about his numbers, he wants it to have the property that the digits are non-increasing when read from left to right. For example, 9973220 is a valid phone number, but 9876545 is not. How many choices for a new phone number does he have?

$\{0,2,2,3,7,9,9\} \longleftrightarrow$ phone number

$$x_0 + x_1 + x_2 + \dots + x_9 = 7$$

$$\binom{7+10-1}{7} = \binom{16}{7}$$

(c) Now instead of non-increasing, they must be strictly decreasing. So 9983220 is no longer valid, while 9753210 is valid. How many choices for a new phone number does he nave now?

each digit picked at most once $\longrightarrow \binom{10}{7}$

3 CS70: The Musical

Edward, one of the previous head TA's, has been hard at work on his latest project, *CS70: The Musical*. It's now time for him to select a cast, crew, and directing team to help him make his dream a reality.

(a) First, Edward would like to select directors for his musical. He has received applications from $2n$ directors. Use this to provide a combinatorial argument that proves the following identity: $\binom{2n}{2} = 2\binom{n}{2} + n^2$

accept 2 out of 2n apps

n oldest ppl

n youngest ppl

$\binom{n}{2}$ only select old ppl

+

$\binom{n}{2}$ only select young ppl

+

n^2 select one from each group

(b) Edward would now like to select a crew out of n people, Use this to provide a combinatorial argument that proves the following identity: $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$ (this is called Pascal's Identity)

LHS: Select team of k out n ppl

RHS: n-1 ppl

1 person

Select 1. $\binom{n-1}{k-1}$

+

don't select 1. $\binom{n-1}{k}$

(c) There are n actors lined up outside of Edward's office, and they would like a role in the musical (including a lead role). However, he is unsure of how many individuals he would like to cast. Use this to provide a combinatorial argument that proves the following identity: $\sum_{k=1}^{n} k\binom{n}{k} = n2^{n-1}$

LHS

$k = 1$ to n

ppl in our show

$\binom{n}{k}$ → # ways to select k ppl out of n

$\times$

k → ~~Select~~ Select lead role

RHS

n → choose lead role

$\times\ 2^{n-1}$ → for each remaining person, accept or reject: binary decision (powerset)

(d) Generalizing the previous part, provide a combinatorial argument that proves the following identity: $\sum_{k=j}^{n} \binom{n}{k}\binom{k}{j} = 2^{n-j}\binom{n}{j}$.

LHS

$k = j$ to n

ppl in our show

$\binom{n}{k}$ → # ways to select k ppl out of n

$\times$

$\binom{k}{j}$ → choose $\underline{j}$ leads out of k ppl

RHS

$\binom{n}{j}$ → choose $\underline{j}$ lead roles

$\times\ 2^{n-j}$ → for each remaining person, accept or reject (binary decision)