Induction: Powerful tool when working in natural numbers $\mathbb{N}$.

Principle: To prove $(\forall n \in \mathbb{N})\ P(n)$

1. Base Case   Show $P(0)$ holds true.

   Smallest case

2. Induction Hypothesis

   For some $k \geq 0$, assume $P(k)$ holds true.

3. Induction Step

   Using the Induction Hypothesis $P(k)$, prove that $P(k+1)$ is true.

Shows $P(k) \Rightarrow P(k+1)$

How does this show $P(n)$ true for all natural n?

$P(0)$ is true by Base Case

$P(0) \Rightarrow P(1)$ by IH/IS, so $P(1)$ is true

$P(1) \Rightarrow P(2)$ by IH/IS, so $P(2)$ is true

$P(2) \Rightarrow P(3)$ by IH/IS, so $P(3)$ is true

$\vdots$

$P(k) \Rightarrow P(k+1)$ by IH/IS, so $P(k+1)$ is true

$\vdots$

and so on! (falling chain of dominoes)

1 Fibonacci for Home

$F_1 = 1$
$F_2 = 1$
$F_3 = 1 + 1 = 2$
$F_4 = 2 + 1 = 3$

Recall, the Fibonacci numbers, defined recursively as

$F_1 = 1$, $F_2 = 1$, and $F_n = F_{n-2} + F_{n-1}$.

Prove that every third Fibonacci number is even. For example, $F_3 = 2$ is even and $F_6 = 8$ is even.

$P(i)$ = "the $(3i)$th Fib. number is even."

Base Case: $P(1)$ is true

Induction Hypothesis:

Assume $P(i)$ holds true for some $i \geq 0$

"The $(3i)$th Fib. number is even"

$F_{3i} = 2x$

Inductive Step:

Want to show: $P(i+1)$ holds.

"the $(3(i+1))$th fib number is even"

$$F_{3(i+1)} = F_{3i+3} = F_{3i+2} + F_{3i+1}$$
$$= F_{3i+1} + F_{3i} + F_{3i+1}$$
$$= 2F_{3i+1} + F_{3i}$$
$$= 2F_{3i+1} + 2x = 2(F_{3i+1} + x)$$

✓

2 Make It Stronger

Let $x \geq 1$ be a real number. Use induction to prove that for all positive integers n, all of the entries in the matrix

$$\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}^n$$

are $\leq xn$. (Hint 1: Find a way to strengthen the inductive hypothesis! Hint 2: Try writing out the first few powers.)

$$\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}^1 = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}^2 = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2x \\ 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}^3 = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}^2\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 3x \\ 0 & 1 \end{pmatrix}$$

<u>Base Case</u>: $P(1)$ is true

<u>I.H.</u>: Assume the <u>strengthened</u> hypothesis:

$$\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}^n = \begin{pmatrix} 1 & nx \\ 0 & 1 \end{pmatrix}$$

<u>IS</u>:
$$\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}^{n+1} = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}^n \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & nx \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & (n+1)x \\ 0 & 1 \end{pmatrix}$$

A Satisfying Insight about Strong Induction

Suppose we want to show $(\forall n \in \mathbb{N})\, P(n)$ via induction.

Define a new proposition:

$P'(k) =$ "$P(0), P(1), \ldots, P(k)$ are all true."

Weak induction on $P'(k)$ is equivalent to Strong induction on $P(k)$!!!

Furthermore, proving $(\forall n \in \mathbb{N})\, P'(n)$ also proves $(\forall n \in \mathbb{N})\, P(n)$!!!

3 Binary Numbers

via ~~Strong induction~~

Prove that every positive integer n ~~can be written in binary~~. In other words, prove that we can write

$$n = c_k \cdot 2^k + c_{k-1} \cdot 2^{k-1} + \cdots + c_1 \cdot 2^1 + c_0 \cdot 2^0,$$

where $k \in \mathbb{N}$ and $c_k \in \{0, 1\}$.

Base Case: $P(1)$

$1 = 1 \cdot 2^0$ ✓

I.H.: Assume for every number $1 \leq i \leq k$,

$P(i) =$ "i can be written in the above form"

holds true.

I.S.: $P(k+1)$: $k+1$ can be written in the above form?

k+1 is odd: k is even

For k, $c_0 = 0$

Take the representation of k,
and change $c_0 = 1$ ✓

k+1 is even:

$P\left(\frac{k+1}{2}\right)$: $\frac{k+1}{2}$ can be represented
in the above form.

$$\frac{k+1}{2} = c_j \cdot 2^j + c_{j-1} \cdot 2^{j-1} + \dots + c_0 2^0$$

$$k+1 = c_j \cdot 2^{j+1} + c_{j-1} 2^j + \dots + c_0 \cdot 2^1 + 0 \cdot 2^0$$

✓