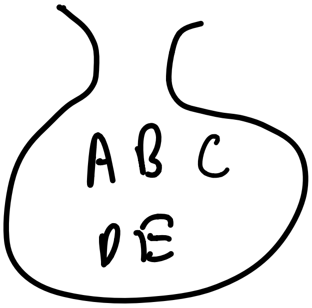
Bag of Letters



Want $k = 3$ letters from bag of $n = 5$ letters

$$\underbrace{\_\,x_1 \cdot \_\,x_2 \cdot \_\,x_3 \cdot \_\,x_4 \cdot \_\,x_5 \cdot \_\,x_6 \cdot \_\,x_7}_{\text{succession of choices} \Rightarrow \text{multiply}}$$

W/o replacement, order matters

$(A,B,C) \neq (B,C,A)$

succession of choices

$5 \cdot 4 \cdot 3 = \boxed{60}$

$$\boxed{\text{Permutation: } \frac{n!}{(n-k)!}}$$

$$\frac{5!}{(5-3)!} = \frac{5!}{2!} = \frac{5\cdot4\cdot3\cdot2\cdot1}{2\cdot1} = \boxed{60}$$

W/o replacement, order doesn't matter

$(A,B,C) = (B,C,A)$

$$\boxed{\text{Combination: } \binom{n}{k} = \frac{n!}{k!(n-k)!}}$$

$$\binom{5}{3} = \frac{5!}{3!(5-3)!} = \frac{5\cdot4\cdot3\cdot2\cdot1}{3\cdot2\cdot1\cdot 2\cdot1} = \boxed{10}$$

W/ replacement, order matters

(A,A,A) $\quad (B,A,A) \neq (A,B,A)$

succession of choices

$5 \cdot 5 \cdot 5 = \boxed{125}$

$$\underbrace{n\cdot n\cdot n\cdot n\cdot n \cdots}_{k \text{ times}} = \boxed{n^k}$$

$$5^3 = \boxed{125}$$

W/ replacement, order doesn't matter

$(B,A,A) = (A,B,A)$

$$\boxed{\text{Stars and Bars: } \binom{n+k-1}{k}}$$

$$x_A + x_B + x_C + x_D + x_E = 3$$

$$\binom{5+3-1}{3} = \binom{7}{3} = \boxed{35}$$

CS 70 Discrete Mathematics and Probability Theory
Spring 2021 **DIS 5B**

1 Clothing Argument

(a) There are four categories of clothings (shoes, trousers, shirts, hats) and we have ten distinct items in each category. How many distinct outfits are there if we wear one item of each category?

$10 \cdot 10 \cdot 10 \cdot 10 = 10^4 = 10000$

(b) How many outfits are there if we wanted to wear exactly two categories?

$\binom{4}{2} \cdot 10 \cdot 10 = 600$

(c) How many ways do we have of hanging four of our ten hats in a row on the wall? (Order matters.)

$10 \cdot 9 \cdot 8 \cdot 7 \qquad \frac{10!}{(10-4)!} = \frac{10!}{6!} = C$

(d) We can pack four hats for travels (order doesn't matter). How many different possibilities for packing four hats are there? Can you express this number in terms of your answer to part (c)?

$\binom{10}{4} = \frac{10!}{4!\,6!} = \boxed{\frac{C}{4!}}$ "overcount"

2 Strings

What is the number of strings you can construct given:

(a) n ones, and m zeroes?

$\frac{(n+m)!}{n!\,m!} = \binom{n+m}{n} = \binom{n+m}{m}$

(b) n_1 A's, n_2 B's and n_3 C's?

$\frac{(n_1+n_2+n_3)!}{n_1!\,n_2!\,n_3!}$

(c) $n_1, n_2, \ldots, n_k$ respectively of k different letters?

$$\frac{(n_1 + n_2 + \cdots + n_k)!}{n_1! \cdot n_2! \cdot n_3! \cdots n_k!}$$

3 Counting Game

RPG games are all about explore different mazes. Here is a weird maze: there are 2^n rooms, where each room is the vertex on a the n-dimensional hypercube, labeled by a n bit binary string.

For each room, there are n different doors, each door corresponding to an edge on the hypercube. If you are at room i, and choose door j, then you will go to room $i \oplus 2^j$ (flips the $j+1$-th bit in number i). $2^n - 1$ in binary: $\underbrace{11\ldots1}_{n}$

base 10	binary
63	00111111
3	00000011
19	00010011

(a) How many different shortest path are there from room 0 to room $2^n - 1$?

(b) How many different paths of $n+2$ steps are there to go from room 0 to room $2^n - 1$?

(c) If $n = 8$, how many different shortest pathes are there from room 0 to room 63 that pass through 3 and 19?

a) $0\cdots0 \to 1\cdots1$

$\boxed{n!}$

0: 0 0000000

3: 00000011

19: 00010011

63: 00111111

$2! \times 1! \times 3!$

c) $\boxed{12}$

b) pos: $n \cdots 2\ 1$

$0 \cdots 0\ 0$

$\downarrow$

$1 \cdots 1\ 1$

which k_i is 3 times?

n ways

$k_1\ k_2\ k_i \cdots k_n$

$1\ \ 1\ \ 3\ \ \ 1$

$$\frac{(n+2)!}{1! \cdot 1! \cdots 3! \cdots 1!} = \boxed{\frac{(n+2)!}{6}}$$

$$\boxed{\frac{n \cdot (n+2)!}{6}}$$