Connectives:

"and" $\wedge$

"or" $\vee$

"not" $\neg$

"implies" $\Rightarrow$

Quantifiers:

"for all" $\forall$

"there exists" $\exists$

Implications:

$$P \Rightarrow Q \equiv \neg P \vee Q$$

P	Q	$P \Rightarrow Q$	$\neg P \vee Q$
F	F	T	T
F	T	T	T
T	F	F	F
T	T	T	T

$$P \Rightarrow Q \equiv \neg Q \Rightarrow \neg P$$ (contrapositive)

De Morgan Laws

$$\neg (P \wedge Q) \equiv \neg P \vee \neg Q$$

$$\neg (P \vee Q) \equiv \neg P \wedge \neg Q$$

CS 70 Discrete Mathematics and Probability Theory
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DIS 0A

$\text{True} \land (\underline{Q \lor \text{True}})$

1 Truth Tables

Determine whether the following equivalences hold, by writing out truth tables. Clearly state whether or not each pair is equivalent.

No (a) $P \land (Q \lor P) \equiv P \land Q$

Yes (b) $(P \lor Q) \land R \equiv (P \land R) \lor (Q \land R)$

Yes (c) $(P \land Q) \lor R \equiv (P \lor R) \land (Q \lor R)$

P	Q	$P \land (Q \lor P)$	$P \land Q$
F	F	F	F
F	T	F	F
T	F	T	F ✕
T	T	T	T

2 XOR

The truth table of XOR (denoted by $\oplus$) is as follows.

A	B	A $\oplus$ B	A ∨ B
F	F	F	F
F	T	T	T
T	F	T	T
T	T	F	T

1. Express XOR using only $(\land, \lor, \neg)$ and parentheses.

$(\neg A \land B) \lor (\neg B \land A)$

$(A \lor B) \land \neg(A \land B)$

2. Does $(A \oplus B)$ imply $(A \lor B)$? Explain briefly.

Whenever $(A \oplus B)$ is true, $(A \lor B)$ true $\boxed{\text{Yes}}$

3. Does $(A \lor B)$ imply $(A \oplus B)$? Explain briefly.

No $\quad (A \lor B) \not\Rightarrow (A \oplus B)$

$a|b$: "a divides b"
"b is divisible by a"

original implication: $P \Rightarrow Q$ contrapositive: $\neg Q \Rightarrow \neg P$

converse: $Q \Rightarrow P$ inverse: $\neg P \Rightarrow \neg Q$

3 Converse and Contrapositive

Consider the statement "if a natural number is divisible by 4, it is divisible by 2".

$(\forall x \in \mathbb{N})((4|x) \Rightarrow (2|x))$

T (a) Write the statement in propositional logic. Prove that it is true or give a counterexample.

F (b) Write the inverse of the implication in English and in propositional logic. Prove that it is true or give a counterexample. (The inverse of an implication $P \implies Q$ is $\neg P \implies \neg Q$.)

F (c) Write the converse of the implication in English and in propositional logic. Prove that it is true or give a counterexample.

T (d) Write the contrapositive of the implication in English and in propositional logic. Prove that it is true or give a counterexample.

4 Equivalences with Quantifiers

Evaluate whether the expressions on the left and right sides are equivalent in each part, and briefly justify your answers.

(a)	$\forall x\ \big((\exists y\ Q(x,y)) \Rightarrow P(x)\big)$	$\forall x\ \exists y\ \big(Q(x,y) \Rightarrow P(x)\big)$
(b)	$\neg\exists x\ \forall y\ \big(P(x,y) \Rightarrow \neg Q(x,y)\big)$	$\cancel{\forall x\ \big((\exists y\ P(x,y)) \wedge (\exists y\ Q(x,y))\big)}$
(c)	$\forall x\ \exists y\ \big(P(x) \Rightarrow Q(x,y)\big)$	$\forall x\ \big(P(x) \Rightarrow (\exists y\ Q(x,y))\big)$

Not equivalent

$$\neg \exists x \forall y (P(x,y) \Rightarrow \neg Q(x,y))$$

$$\equiv \neg \exists x \forall y (\neg P(x,y) \vee \neg Q(x,y))$$

$$\equiv \forall x \exists y (\neg(\neg P(x,y) \vee \neg Q(x,y)))$$

$$\equiv \forall x (\underbrace{\exists y (P(x,y) \wedge Q(x,y)}_{\text{Does NOT distribute!!}}))$$

$$\exists y\,[P(x,y) \land Q(x,y)]$$

vs

$$\exists y\,(P(x,y)) \land \exists y\,(Q(x,y)) \,;$$

$P(x,y) =$ "y is equal to $5x$"

$Q(x,y) =$ "y is equal to $10x$"