Connectives:

"and" $\wedge$

"or" $\vee$

"not" $\neg$

"implies" $\Rightarrow$

Quantifiers:

"for all" $\forall$

"there exists" $\exists$

Implications: $P \Rightarrow Q$

"vacuously" true

P	Q	$P \Rightarrow Q$	$\neg P \vee Q$
F	F	T	T
F	T	T	T
T	F	F	F
T	T	T	T

Everyone in this room who goes to Stanford thinks Stanford > Berkeley

$\neg P \Rightarrow \neg Q$ (inverse)

$Q \Rightarrow P$ (converse)

$\neg Q \Rightarrow \neg P$ (contrapositive) $\equiv P \Rightarrow Q$

$P \Rightarrow Q \equiv \neg P \vee Q$

proof by truth table (notes)

De Morgan Laws

$$\neg (P \wedge Q) \equiv \neg P \vee \neg Q$$

$$\neg (P \vee Q) \equiv \neg P \wedge \neg Q$$

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1 Truth Tables

Determine whether the following equivalences hold, by writing out truth tables. Clearly state whether or not each pair is equivalent.

(a) $P \wedge (Q \vee P) \equiv P \wedge Q$ No

(b) $(P \vee Q) \wedge R \equiv (P \wedge R) \vee (Q \wedge R)$ Yes

(c) $(P \wedge Q) \vee R \equiv (P \vee R) \wedge (Q \vee R)$ Yes

P	Q	$Q \vee P$	$P \wedge (Q \vee P)$	$P \wedge Q$
F	F	F	F	F
F	T	T	F	F
T	F	T	T	F
T	T	T	T	T

2 Propositional Practice

Convert the following English sentences into propositional logic and the following propositions into English. State whether or not each statement is true with brief justification.

(a) There is a real number which is not rational.

(b) All integers are natural numbers or are negative, but not both.

(c) If a natural number is divisible by 6, it is divisible by 2 or it is divisible by 3.

(d) $(\forall x \in \mathbb{Z})\ (x \in \mathbb{Q})$

(e) $(\forall x \in \mathbb{Z})\ (((2 \mid x) \vee (3 \mid x)) \implies (6 \mid x))$

(f) $(\forall x \in \mathbb{N})\ ((x > 7) \implies ((\exists a, b \in \mathbb{N})\ (a + b = x)))$

"a divides b"

$a | b$ b is divisible by a

3 Converse and Contrapositive

$(\exists c \in \mathbb{Z})\ a \times c = b$

Consider the statement "if a natural number is divisible by 4, it is divisible by 2".

(a) Write the statement in propositional logic. Prove that it is true or give a counterexample.

(b) Write the inverse of the implication in English and in propositional logic. Prove that it is true or give a counterexample. (The inverse of an implication $P \implies Q$ is $\neg P \implies \neg Q$.)

(c) Write the converse of the implication in English and in propositional logic. Prove that it is true or give a counterexample.

(d) Write the contrapositive of the implication in English and in propositional logic. Prove that it is true or give a counterexample.

a) $(\forall x \in \mathbb{N})(4 | x \Rightarrow 2 | x)$ True $x = 4k = 2(2k)$ ∴ divisible by 2

False b) $(\forall x \in \mathbb{N})(4 \nmid x \Rightarrow 2 \nmid x)$ $x = 2$

False c) $(\forall x \in \mathbb{N})(2 | x \Rightarrow 4 | x)$ $4 \nmid 2$ but $2 | 2$

d) $(\forall x \in \mathbb{N})(2 \nmid x \Rightarrow 4 \nmid x)$ True

4 Equivalences with Quantifiers

Evaluate whether the expressions on the left and right sides are equivalent in each part, and briefly justify your answers.

(a)	$\forall x\ ((\exists y\ Q(x,y)) \Rightarrow P(x))$	$\forall x\ \exists y\ (Q(x,y) \Rightarrow P(x))$
(b)	$\neg\exists x\ \forall y\ (P(x,y) \Rightarrow \neg Q(x,y))$	$\forall x\ ((\exists y\ P(x,y)) \land (\exists y\ Q(x,y)))$
(c)	$\forall x\ \exists y\ (P(x) \Rightarrow Q(x,y))$	$\forall x\ (P(x) \Rightarrow (\exists y\ Q(x,y)))$

Not Eq
Not Eq
Equivalent

a) LHS: $\forall x\ (\neg(\exists y\ Q(x,y)) \lor P(x))$

$\equiv \forall x\ ((\forall y\ Q(x,y)) \lor P(x))$

RHS: $\forall x\ \exists y\ (\neg Q(x,y) \lor P(x))$

Not the same

b) LHS: $\forall x \exists y \overbrace{\neg [P(x,y) \Rightarrow \neg Q(x,y)]}^{\neg(\neg P(x,y) \lor \neg Q(x,y))}$

$\equiv \forall x \exists y \left(\underbrace{P(x,y) \land Q(x,y)}\right)$

RHS: $\forall x \left(\underline{\exists y\ P(x,y)}\right) \land \left(\underline{\exists y\ Q(x,y)}\right)$

not neccesarily same value of y

Not the same

$$\boxed{\neg(\forall x \; P(x))}$$

$$\equiv \neg(P(x_1) \wedge P(x_2) \wedge \ldots\ldots)$$

$$\equiv \neg P(x_1) \vee \neg P(x_2) \vee \ldots\ldots$$

$$\equiv \boxed{\exists x \, (\neg P(x))}$$

$$\boxed{\neg(\exists x \; P(x))}$$

$$\equiv \neg(P(x_1) \vee P(x_2) \vee \ldots\ldots)$$

$$\equiv \neg P(x_1) \wedge \neg P(x_2) \wedge \ldots\ldots$$

$$\equiv \boxed{\forall x \, \neg P(x)}$$