Random Variables

Sample Space Ω: set of all possible outcomes $\{w_1, w_2, \dots, w_n\}$

Ω:

$w_1 = TTT$	$w_5 = HTT$
$w_2 = TTH$	$w_6 = HTH$
$w_3 = THT$	$w_7 = HHT$
$w_4 = THH$	$w_8 = HHH$

$P[TTT] = \frac{1}{8}$

$\vdots$

$P[HHH] = \frac{1}{8}$

Probability Space

Scenario:

3 fair Coin Flips

outcomes $\rightarrow$ real numbers

Random Variable $X: \Omega \rightarrow \mathbb{R}$

Expectation: "Average value" of X

$$E[X] = \sum_a a\, P[X=a]$$

"$X=a$" is an event

example:

$$X: \begin{cases} w_1 \to 3 \\ w_2 \to 2 \\ w_3 \to 2 \\ w_4 \to 1 \\ w_5 \to 2 \\ w_6 \to 1 \\ w_7 \to 1 \\ w_8 \to 0 \end{cases}$$

("number of tails")

$$E[X] = 0 \cdot P[X=0] + 1 \cdot P[X=1] + 2 \cdot P[X=2] + 3 \cdot P[X=3]$$
$$= 0 \cdot P(w_8) + 1\,(P(w_4) + P(w_6) + P(w_7))$$
$$+ 2 \cdot (P(w_2) + P(w_3) + P(w_5)) + 3 \cdot P(w_1)$$
$$= 0 + \frac{3}{8} + \frac{3}{4} + \frac{3}{8} = \frac{3}{2} = 1.5$$

Notable RVs:

Bernoulli (p) — represents single coin flip of prob. p.

$$\boxed{\begin{aligned} P[X=1] &= p \\ P[X=0] &= 1-p \end{aligned}}$$

Bin (n,p) — represents n coin flips of prob. p.

$$\underbrace{T \quad H \quad T \quad H \quad H}_{n}$$

$$\boxed{P[X=i] = \binom{n}{i} p^i (1-p)^{n-i}}$$

$i \in \{0, 1, \dots, n\}$

1 Head Count

Consider a coin with $\mathbb{P}(\text{Heads}) = 2/5$. Suppose you flip the coin 20 times, and define X to be the number of heads.

(a) Name the distribution of X and what its parameters are.

$$X \sim \text{Bin}\left(20, \frac{2}{5}\right)$$

(b) What is $\mathbb{P}(X = 7)$?

$$P[X=7] = \binom{20}{7}\left(\frac{2}{5}\right)^7 \left(\frac{3}{5}\right)^{13}$$

(c) What is $\mathbb{P}(X \geq 1)$? Hint: You should be able to do this without a summation.

$$1 - P[X=0] = 1 - \binom{20}{0}\left(\frac{2}{5}\right)^0\left(\frac{3}{5}\right)^{20}$$

(d) What is $\mathbb{P}(12 \leq X \leq 14)$?

$$P[X=12] + P(X=13) + P[X=14)$$

$$\sum_{i=12}^{14} \binom{20}{i}\left(\frac{2}{5}\right)^i\left(\frac{3}{5}\right)^{20-i}$$

2 Family Planning

Mr. and Mrs. Brown decide to continue having children until they either have their first girl or until they have three children. Assume that each child is equally likely to be a boy or a girl, independent of all other children, and that there are no multiple births. Let G denote the numbers of girls that the Browns have. Let C be the total number of children they have.

(a) Determine the sample space, along with the probability of each sample point.

w_i	$P(w_i)$	C	G
g	1/2	1	1
bg	1/4	2	1
bbg	1/8	3	1
bbb	1/8	3	0

Ω

(b) Compute the joint distribution of G and C. Fill in the table below.

	$C=1$	$C=2$	$C=3$
$G=0$	0	0	1/8
$G=1$	1/2	1/4	1/8

$(C=1)\cap(G=0)=\emptyset$

(c) Use the joint distribution to compute the marginal distributions of G and C and confirm that the values are as you'd expect. Fill in the tables below.

$\mathbb{P}(G=0)$	1/8
$\mathbb{P}(G=1)$	7/8

$\mathbb{P}(C=1)$	$\mathbb{P}(C=2)$	$\mathbb{P}(C=3)$
1/2	1/4	1/4

(d) Are G and C independent?

$P(C=1 \cap G=0)=0$

$P(C=1)\,P(G=0)=\frac{1}{8}\cdot\frac{1}{2}\neq 0$

NOT INDEP.

(e) What is the expected number of girls the Browns will have? What is the expected number of children that the Browns will have?

$E[G]=\frac{1}{2}\cdot 1+\frac{1}{4}\cdot 1+\frac{1}{8}\cdot 1+\frac{1}{8}\cdot 0$

$=\boxed{\frac{7}{8}}$

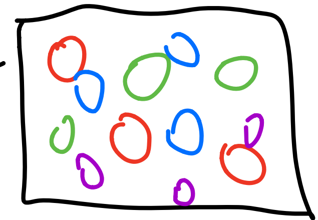
$E[C]=\frac{1}{2}\cdot 1+\frac{1}{4}\cdot 2+\frac{1}{8}\cdot 3+\frac{1}{8}\cdot 3=\boxed{\frac{7}{4}}$

3 How Many Queens?

You shuffle a standard 52-card deck, before drawing the first three cards from the top of the pile. Let X denote the number of queens you draw.

(a) What is $\mathbb{P}(X=0)$, $\mathbb{P}(X=1)$, $\mathbb{P}(X=2)$ and $\mathbb{P}(X=3)$?

Ω



$$P(X=0) = \frac{\binom{4}{0}\binom{48}{3}}{\binom{52}{3}}$$

$$P(X=1) = \frac{\binom{4}{1}\binom{48}{2}}{\binom{52}{3}}$$

$$P(X=2) = \frac{\binom{4}{2}\binom{48}{1}}{\binom{52}{3}}$$

$$P(X=3) = \frac{\binom{4}{3}\binom{48}{0}}{\binom{52}{3}}$$

(b) What do your answers you computed in part a add up to?

1

(c) Compute $\mathbb{E}(X)$ from the definition of expectation.

$$0 \cdot P(X=0) + 1 \cdot P(X=1) + 2 \cdot P(X=2) + 3 \cdot P(X=3)$$

(d) Let X_i be an indicator random variable that equals 1 if the ith card a is queen and 0 otherwise. Are the X_i indicators independent?

NOT INDEP.

Check: $P(X_1 \cap X_2) \overset{?}{\neq} P(X_1)\,P(X_2)$

$$\frac{1}{13} \cdot \frac{3}{51} = \frac{1}{221} \qquad \frac{1}{13} \cdot \frac{1}{13} = \frac{1}{169}$$