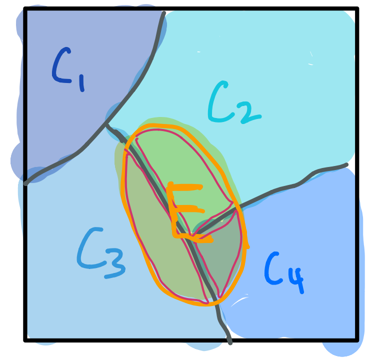
Discrete Probability (cont.)



Total Prob. Law

$$\boxed{P[E] = \sum P[\underbrace{E \cap C_i}_{\text{"pieces"}}] = \sum P[E \mid C_i]\, P[C_i]}$$

Bayes:

$$\boxed{P[A \mid B] = \frac{P[B \mid A]\, P[A]}{P[B]}}$$

Independence:

Events A and B are independent iff:

$$P[A \cap B] = P[A]\,P[B] \underset{\text{definitions}}{\overset{\text{equivalent}}{\Longleftrightarrow}} \begin{cases} P[A \mid B] = P[A] \\ P[B \mid A] = P[B] \end{cases}$$

Mutual Independence $\Rightarrow$ Pairwise Independence

$$P[A_1 \cap A_2 \cap \cdots \cap A_n] = P[A_1]\,P[A_2] \cdots P[A_n]$$

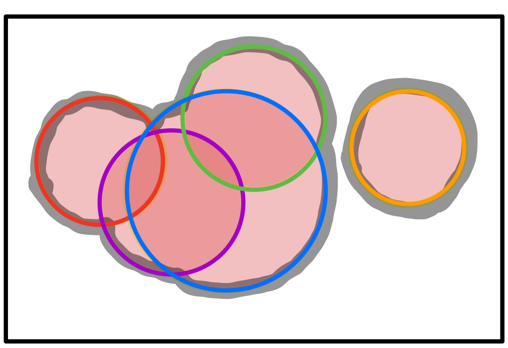
$$\left.\begin{aligned} &P[A_i \cap A_j] \\ &= P[A_i]\,P[A_j] \end{aligned}\right\} \text{for all pairs } i, j$$

Mutual Independence $\not\Leftarrow$ Pairwise Independence

Union Bound

$$\boxed{P\left[\bigcup_{i=1}^{n} A_i\right] \le \sum_{i=1}^{n} P[A_i]}$$

$$\boxed{P[A_1 \cup A_2 \cup \cdots \cup A_n] \le P[A_1] + P[A_2] + \cdots + P[A_n]}$$



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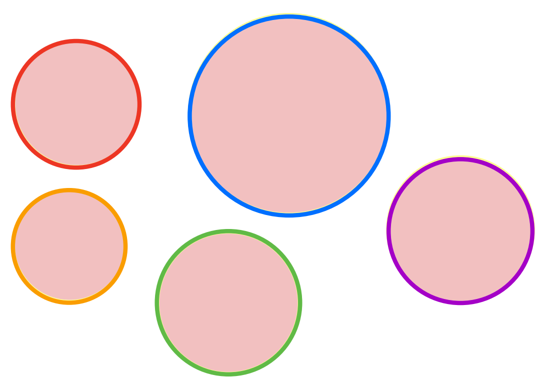
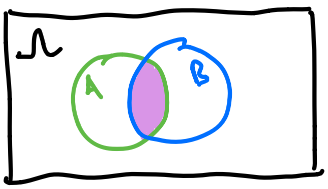


CS 70 Spring 2021 — Discrete Mathematics and Probability Theory — DIS 9A

1 Probability Potpourri

Prove a brief justification for each part.

(a) For two events A and B in any probability space, show that $\mathbb{P}(A \setminus B) \geq \mathbb{P}(A) - \mathbb{P}(B)$.



$$\begin{aligned} & P[A] = \left(P[A \setminus B] + \cancel{P[A \cap B]}\right) \\ - \; & P[B] = \left(P[B \setminus A] + \cancel{P[A \cap B]}\right) \\ \hline & P[A] - P[B] = P[A \setminus B] - P[B \setminus A] \\ & \leq P[A \setminus B] \end{aligned}$$

(b) Suppose $\mathbb{P}(D \mid C) = \mathbb{P}(D \mid \overline{C})$, where $\overline{C}$ is the complement of C. Prove that D is independent of C.

$P[D] = P[D \mid C] \leftarrow$ Want to show

$$\begin{aligned} P[D] &= P[D \mid C]\,P[C] + P[D \mid \bar{C}]\,P[\bar{C}] \\ &= P[D \mid C]\,P[C] + P[D \mid C]\,P[\bar{C}] \\ &= P[D \mid C]\,\underbrace{\left(P[C] + P[\bar{C}]\right)}_{1} \\ &= P[D \mid C] \end{aligned}$$

(c) If A and B are disjoint, does that imply they're independent?

$$P[A] \overset{?}{=} \underbrace{P[A|B]}_{0}$$

Not independent when $P[A] \neq 0$
or $P[B] \neq 0$

2 Aces

Consider a standard 52-card deck of cards:

(a) Find the probability of getting an ace or a red card, when drawing a single card.

A B

$$P[A \cup B] = \frac{|A \cup B|}{|\Omega|} = \frac{4+26-2}{52} = \frac{7}{13}$$

(b) Find the probability of getting an ace or a spade, but not both, when drawing a single card.

$\underbrace{\text{ace or a spade}}_{A}$, but $\underbrace{\text{not}}_{B}$

$$\frac{15}{52}$$

$A \setminus B$

$$(4+13-1)-1 = 15$$

(c) Find the probability of getting the ace of diamonds when drawing a 5 card hand.

$$1 - \frac{51}{52} \cdot \frac{\cancel{50}}{\cancel{51}} \cdot \frac{\cancel{49}}{\cancel{50}} \cdot \frac{\cancel{48}}{\cancel{49}} \cdot \frac{47}{\cancel{48}} = \frac{5}{52}$$

$$\frac{\binom{51}{4}}{\binom{52}{5}} = \frac{5}{52}$$

(d) Find the probability of getting exactly 2 aces when drawing a 5 card hand.

$$\frac{\binom{4}{2}\binom{48}{3}}{\binom{52}{5}}$$

(e) Find the probability of getting at least 1 ace when drawing a 5 card hand.

$$1 - \frac{\binom{48}{5}}{\binom{52}{5}}$$

≥ 1 ace | 0 aces

(f) Find the probability of getting at least 1 ace or at least 1 heart when drawing a 5 card hand.

$$1 - \frac{\binom{36}{5}}{\binom{52}{5}}$$

$13+4-1$

$= 16$ cards that are aces or hearts

3 Balls and Bins

Throw n balls into n labeled bins one at a time.

(a) What is the probability that the first bin is empty?

$$\left(1-\frac{1}{n}\right)^n$$

(b) What is the probability that the first k bins are empty?

$$\left(1-\frac{k}{n}\right)^n$$

(c) Let A be the event that at least k bins are empty. Notice that there are $m = \binom{n}{k}$ sets of k bins out of the total n bins. If we assune A_i is the event that the i^{th} set of k bins is empty. Then we can write A as the union of A_i's.

$$A = \bigcup_{i=1}^{m} A_i.$$

Write the union bound for the probability A.

$$P[A] = P\left[\bigcup_{i=1}^{m} A_i\right] \leq \sum_{i=1}^{m} P[A_i]$$

$$P[A_1 \cup A_2 \cup \cdots \cup A_m] \qquad P[A_1] + P[A_2] + \cdots + P[A_m]$$

(d) Use the union bound to give an upper bound on the probability A from part (c).

$$P[A] \leq \sum_{i=1}^{m} P[A_i]$$

$$= \sum_{i=1}^{m} \left(1-\frac{k}{n}\right)^n = m\left(1-\frac{k}{n}\right)^n$$

$$= \binom{n}{k}\left(1-\frac{k}{n}\right)^n$$

(e) What is the probability that the second bin is empty given that the first one is empty?

A B

$$P[A|B] = \frac{P[A \cap B]}{P[B]} = \frac{\left(1-\frac{2}{n}\right)^n}{\left(1-\frac{1}{n}\right)^n} = \left(\frac{n-2}{n-1}\right)^n$$

(f) Are the events that "the first bin is empty" and "the first two bins are empty" independent? No

A B

$$P[A|B] = 1 \neq P[A] = \left(1-\frac{1}{n}\right)^n$$

(g) Are the events that "the first bin is empty" and "the second bin is empty" independent? No

A B

$$P[A \cap B] = \left(1-\frac{2}{n}\right)^n$$

$$P[A] \cdot P[B] = \left(1-\frac{1}{n}\right)^n \left(1-\frac{1}{n}\right)^n$$

$$= \left(1-\frac{1}{n}\right)^{2n}$$

not equivalent