Sets

Intersection: $A \cap B = \{x \mid (x \in A) \wedge (x \in B)\}$

"All elements in both A and B"

Union: $A \cup B = \{x \mid (x \in A) \vee (x \in B)\}$

"All elements in A or in B"

Complement: $A \setminus B = \{x \mid (x \in A) \wedge \neg(x \in B)\}$

"All elements in A and not in B"

Subset: $A \subseteq B$

("all elements in A are also in B")

Equality: $A = B$ if and only if

$A \subseteq B$ and $B \subseteq A$

Proofs:

Direct — Prove an implication $P \Rightarrow Q$ by assuming P true, then deriving Q is true.

Contraposition — Prove implication $P \Rightarrow Q$ by showing $\neg Q \Rightarrow \neg P$

Contradiction — Prove a statement S by assuming $\neg S$ and reaching a contradiction. Conclude S must be true (by Law of Excluded Middle).

Cases — Prove statement S by splitting into cases and showing S holds in each case.

Common Pitfalls to Watch Out For

- when proving a statement S, assuming S to be true from the beginning.
- Missing certain cases (e.g. divide by 0)
- Negative numbers in inequalities
- Getting too caught up in an approach!!!

CS 70 Spring 2022 — Discrete Mathematics and Probability Theory — Koushik Sen and Satish Rao — DIS 0B

1 Contraposition

Prove the statement "if $a+b<c+d$, then $a<c$ or $b<d$".

$P \Rightarrow Q$

$\neg Q \Rightarrow \neg P$

$(a \geq c) \wedge (b \geq d) \Rightarrow a+b \geq c+d$

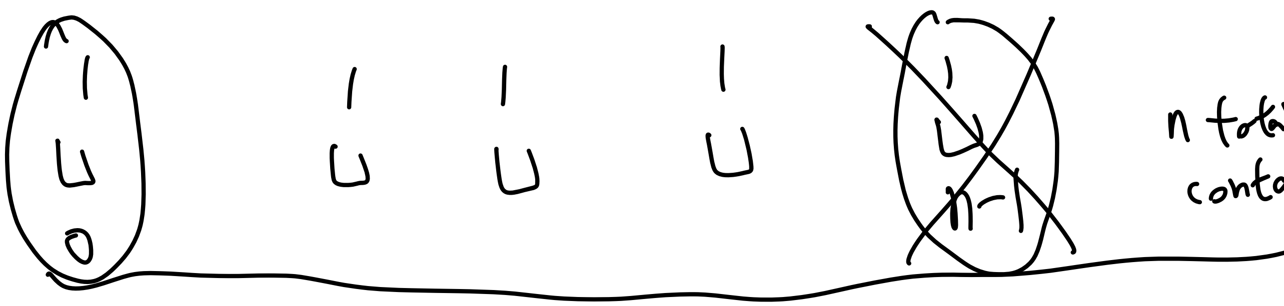
$$\begin{array}{r} a \geq c \\ +\; b \geq d \\ \hline a+b \geq c+d \end{array}$$

Contrapositive true

→ original statement is true.

2 Numbers of Friends

Prove that if there are $n \geq 2$ people at a party, then at least 2 of them have the same number of friends at the party. Assume that friendships are always reciprocated: that is, if Alice is friends with Bob, then Bob is also friends with Alice.
(Hint: The Pigeonhole Principle states that if n items are placed in m containers, where $n > m$, at least one container must contain more than one item. You may use this without proof.)



n total containers

Assume that $\neg S$

all people have diff. # of friends

→ contradiction w/ the reciprocity property of friendship

$\neg P \equiv \neg(\forall w \, \exists p \; R)$

Contraposition

$\neg Q \implies \neg P$

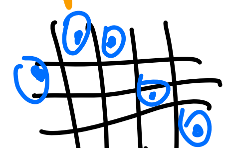
3 Pebbles

$\equiv \exists w \, \forall p \; \neg R$

Suppose you have a rectangular array of pebbles, where each pebble is either red or blue. Suppose that for every way of choosing one pebble from each column, there exists a red pebble among the chosen ones. Prove that there must exist an all-red column. Q

If there is no all-red column,

$\neg P$ then there is a way to choose a pebble from each column such that all pebbles chosen are blue.



4 Preserving Set Operations

For a function f, define the image of a set X to be the set $f(X) = \{y \mid y = f(x) \text{ for some } x \in X\}$. Define the inverse image or preimage of a set Y to be the set $f^{-1}(Y) = \{x \mid f(x) \in Y\}$. Prove the following statements, in which A and B are sets.

Recall: For sets X and Y, $X = Y$ if and only if $X \subseteq Y$ and $Y \subseteq X$. To prove that $X \subseteq Y$, it is sufficient to show that $(\forall x)\ ((x \in X) \implies (x \in Y))$.

(a) $f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}(B)$.

(b) $f(A \cup B) = f(A) \cup f(B)$.

a) $f^{-1}(A \cup B) \subseteq f^{-1}(A) \cup f^{-1}(B)$ ✓

Consider $x \in f^{-1}(A \cup B)$

$f(x) \in A \cup B$ by definition of preimage

$(f(x) \in A) \vee (f(x) \in B)$ def of Union

$(x \in f^{-1}(A)) \vee (x \in f^{-1}(B))$ def. preimage

$x \in f^{-1}(A) \cup f^{-1}(B)$ def. Union

$f^{-1}(A) \cup f^{-1}(B) \subseteq f^{-1}(A \cup B)$ ✓

Consider $x \in f^{-1}(A) \cup f^{-1}(B)$

$$\left(x \in f^{-1}(A)\right) \vee \left(x \in f^{-1}(B)\right)$$ def. of Union

$$\left(f(x) \in A\right) \vee \left(f(x) \in B\right)$$ def. of preimage

$$f(x) \in A \cup B$$ def. of Union

$$x \in f^{-1}(A \cup B)$$ def. of preimage

$$\Rightarrow \quad f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}(B)$$

$\square$

$$f(X) = \{ y \mid \underline{y = f(x) \text{ for some } x \in X} \}$$

$$\underline{\exists x \in X \quad y = f(x)}$$

"set of all elements y where
y = f(x) for some x in set X"

$$f^{-1}(Y) = \{ x \mid f(x) \in Y \}$$

"set of all elements x where
f(x) is in the set Y."