Sets

Intersection: $A \cap B = \{x \mid (x \in A) \wedge (x \in B)\}$

"All elements in both A and B"

Union: $A \cup B = \{x \mid (x \in A) \vee (x \in B)\}$

"All elements in A or in B"

Complement: $A \setminus B = \{x \mid (x \in A) \wedge \neg(x \in B)\}$

"All elements in A and not in B"

Subset: $A \subseteq B$

("all elements in A are also in B")

Equality: $A = B$ if and only if

$A \subseteq B$ and $B \subseteq A$

Proofs:

Direct: Prove an implication $P \Rightarrow Q$ by assuming P true, then deriving Q is true.

Contraposition: Prove implication $P \Rightarrow Q$ by showing $\neg Q \Rightarrow \neg P$

Contradiction: Prove a statement S by assuming $\neg S$ and reaching a contradiction. Conclude S must be true (by Law of Excluded Middle).

Cases: Prove statement S by splitting into cases and showing S holds in each case.

Common Pitfalls to Watch Out For

- when proving a statement S, assuming S to be true from the beginning.
- Missing certain cases (e.g. divide by 0)
- Negative numbers in inequalities
- Getting too caught up in an approach!!!

CS 70 Discrete Mathematics and Probability Theory
Spring 2021 DIS 0B

1 Proof Practice

(a) Prove that $\forall n \in \mathbb{N}$, if n is odd, then n^2+1 is even. (Recall that n is odd if $n = 2k+1$ for some natural number k.)

$n = 2k+1 \quad k \in \mathbb{N}$

$n^2 = (2k+1)(2k+1) = 4k^2+4k+1$

$n^2+1 = 4k^2+4k+2 = 2(2k^2+2k+1)$

(b) Prove that $\forall x,y \in \mathbb{R}$, $\min(x,y) = (x+y-|x-y|)/2$. (Recall, that the definition of absolute value for a real number z, is

$$|z| = \begin{cases} z, & z \geq 0 \\ -z, & z < 0 \end{cases}$$

Case $x \geq y$ $\quad \min(x,y) = y$

$\min(x,y) = \frac{x+y-(x-y)}{2} = y$

Case $x < y$ $\quad \min(x,y) = x$

$\min(x,y) = \frac{x+y-(y-x)}{2} = x$

(c) Suppose $A \subseteq B$. Prove $\mathscr{P}(A) \subseteq \mathscr{P}(B)$. (Recall that $A' \in \mathscr{P}(A)$ if and only if $A' \subseteq A$.)

$A \subseteq B$

$(x \in A) \Rightarrow (x \in B)$

$A' \in \mathscr{P}(A)$

$A' \subseteq A$

$(x \in A') \Rightarrow (x \in A)$

$\Rightarrow (x \in B)$

$A' \subseteq B$

$A' \in \mathscr{P}(B)$

2 Preserving Set Operations

For a function f, define the image of a set X to be the set $f(X) = \{y \mid y = f(x) \text{ for some } x \in X\}$. Define the inverse image or preimage of a set Y to be the set $f^{-1}(Y) = \{x \mid f(x) \in Y\}$. Prove the following statements, in which A and B are sets. ~~By doing so, you will show that inverse images preserve set operations, but images typically do not.~~

Recall: For sets X and Y, $X = Y$ if and only if $X \subseteq Y$ and $Y \subseteq X$. To prove that $X \subseteq Y$, it is sufficient to show that $(\forall x)\ ((x \in X) \implies (x \in Y))$.

(a) $f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}(B)$.

1) $f^{-1}(A \cup B) \subseteq f^{-1}(A) \cup f^{-1}(B)$

$x \in f^{-1}(A \cup B)$

$f(x) \in A \cup B$

$(f(x) \in A) \vee (f(x) \in B)$

$(x \in f^{-1}(A)) \vee (x \in f^{-1}(B))$

$x \in f^{-1}(A) \cup f^{-1}(B)$

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2) $f^{-1}(A \cup B) \supseteq f^{-1}(A) \cup f^{-1}(B)$

$x \in f^{-1}(A) \cup f^{-1}(B)$

$(x \in f^{-1}(A)) \vee (x \in f^{-1}(B))$

$(f(x) \in A) \vee (f(x) \in B)$

$f(x) \in A \cup B$

$x \in f^{-1}(A \cup B)$

(b) $f(A \cup B) = f(A) \cup f(B)$.

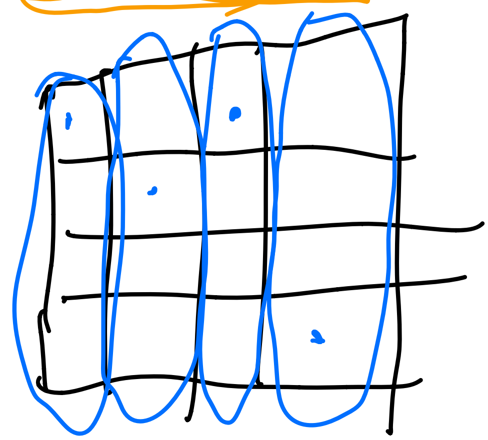
W- set of all ways

$(\forall w \in W)(\exists r \in w)$

3 Pebbles

Suppose you have a rectangular array of pebbles, where each pebble is either red or blue. Suppose that for every way of choosing one pebble from each column, there exists a red pebble among the chosen ones. Prove that there must exist an all-red column.

$P \Rightarrow Q$



Contraposition

$\neg Q \Rightarrow \neg P$

"If there is no all red column, there is some way of choosing 1 pebble from each column such that no red pebbles are chosen."