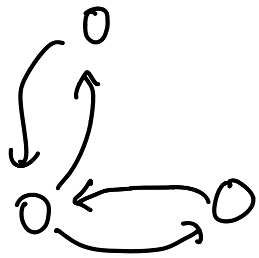
$$\boxed{\pi_{t+1} = \pi_t \; P}$$

$\pi_t = \pi_0 P^t$

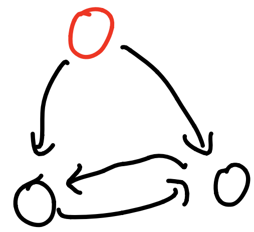
Markov Chains, cont.

Irreducibility: Any state can reach any other state.

("Reachability")



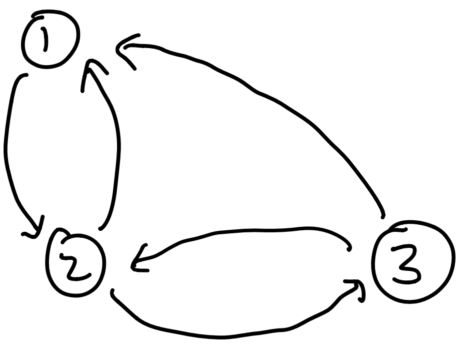
irreducible



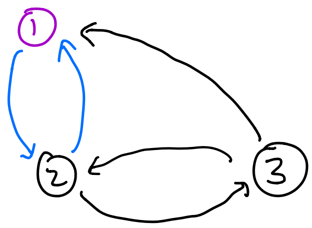
reducible

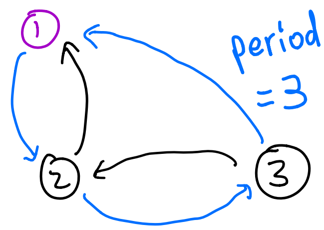
Periodicity: $d(i) = \gcd\{\text{all periods of state } i\}$

Ex.

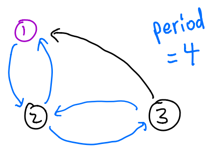


period $=2$





$d(1) = \gcd\{2, 3, \ldots\} = 1 \longrightarrow$ aperiodic



$\boxed{d(i) = 1 \iff \text{aperiodic}}$

> In an irreducible Markov chain, $d(i)$ is same for every state.

> Any irreducible MC with a self-loop is aperiodic.

Big Theorem for Markov Chains

1) Irreducibility guarrantees a unique invariant distribution π

$(\pi = \pi P)$

2) Irreducible + Aperiodic guarrantees convergence to π.

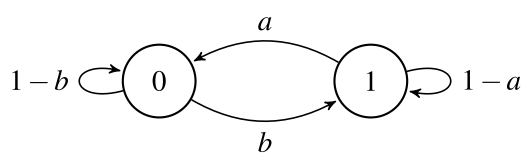
$$\lim_{t\to\infty} \pi_t = \lim_{t\to\infty} \pi_0 P^t = \pi$$

CS 70 Discrete Mathematics and Probability Theory
Spring 2021 **Discussion 12B**

1 Markov Chain Terminology

In this question, we will walk you through terms related to Markov chains.

1. (Irreducibility) A Markov chain is irreducible if, starting from any state i, the chain can transition to any other state j, possibly in multiple steps.
2. (Periodicity) $d(i) := \gcd\{n > 0 \mid P^n(i,i) = \mathbb{P}[X_n = i \mid X_0 = i] > 0\}$, $i \in \mathscr{X}$. If $d(i) = 1\ \forall i \in \mathscr{X}$, then the Markov chain is aperiodic; otherwise it is periodic.
3. (Matrix Representation) Define the transition probability matrix P by filling entry (i, j) with probability $P(i, j)$. → transition probabilities
4. (Invariance) A distribution π is invariant for the transition probability matrix P if it satisfies the following balance equations: $\boxed{\pi = \pi P}$

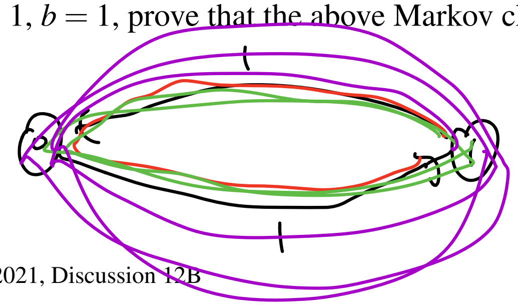


(a) For what values of a and b is the above Markov chain irreducible? Reducible?

$a = 0$ or $b = 0$: reducible

otherwise : irreducible

(b) For $a = 1$, $b = 1$, prove that the above Markov chain is periodic.



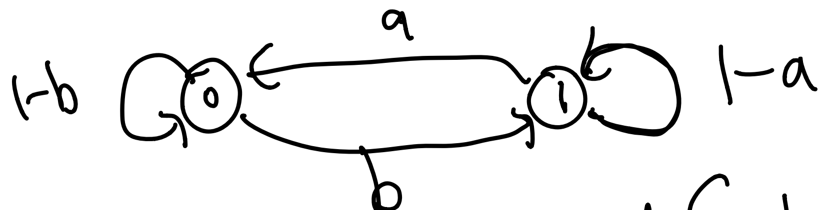
irreducuble

$d(0) = d(1)$

$= \gcd\{\text{periods of state } 1\}$

$= \gcd\{2, 4, 6, \dots\} = 2$

(c) For $0 < a < 1$, $0 < b < 1$, prove that the above Markov chain is aperiodic.



$d(1) = gcd\{1, \ldots,\} = 1$

(d) Construct a transition probability matrix using the above Markov chain.

(e) Write down the balance equations for this Markov chain and solve them. Assume that the Markov chain is irreducible.

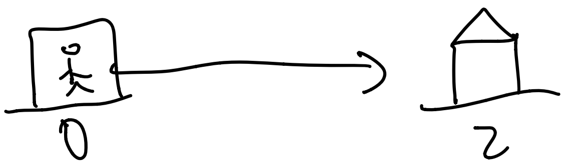
2 Allen's Umbrella Setup

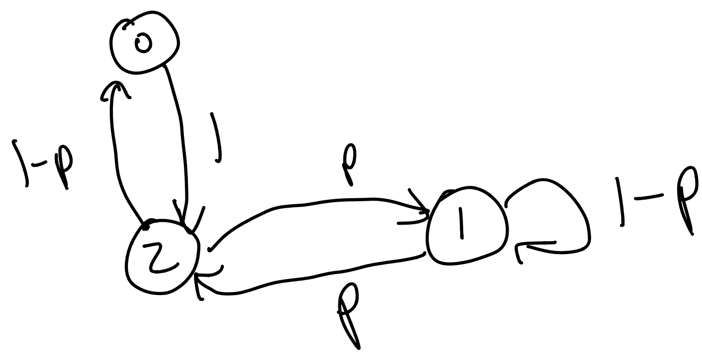
Every morning, Allen walks from his home to Soda, and every evening, Allen walks from Soda to his home. Suppose that Allen has two umbrellas in his possession, but he sometimes leaves his umbrellas behind. Specifically, before leaving from his home or Soda, he checks the weather. If it is raining outside, he will bring his umbrella (that is, if there is an umbrella where he currently is). If it is not raining outside, he will forget to bring his umbrella. Assume that the probability of rain is p.

$X_t = 0$
$X_{t+1} = 2$

\# of umbrellas at Allen's current location (at each timestep)

Soda (a) Model this as a Markov chain. What is $\mathscr{X}$? Write down the transition matrix.





$$\begin{array}{c|ccc} & 0 & 1 & 2 \\ \hline 0 \\ 1 \\ 2 \end{array} \quad \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1-p & p \\ 1-p & p & 0 \end{bmatrix}$$

(b) What is the transition matrix after 2 trips? n trips? Determine if the distribution of X_n converges to the invariant distribution, and compute the invariant distribution. Determine the long-term fraction of time that Allen will walk through rain with no umbrella.

$\boxed{\text{Yes}}$

$$\pi_2 = \pi_0 \cdot \boxed{P^2}$$

$$\pi_n = \pi_0 \boxed{P^n}$$

(Connected graph)

$\Downarrow$

irreducible + aperiodic (b/c State 1 has a self-loop)

$$\pi = \pi P$$

$$[\pi(0) \quad \pi(1) \quad \pi(2)] = [\pi(0) \quad \pi(1) \quad \pi(2)] \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1-p & p \\ 1-p & p & 0 \end{bmatrix}$$

$$\pi(0) + \pi(1) + \pi(2) = 1$$

$$[\pi(0) \quad \pi(1) \quad \pi(2)] = \boxed{\left[\frac{1-p}{3-p} \quad \frac{1}{3-p} \quad \frac{1}{3-p}\right]}$$

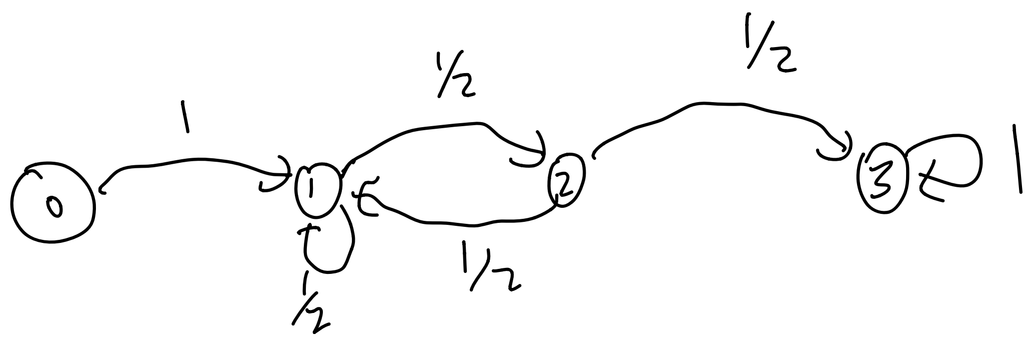
Fraction of time: $\boxed{\frac{1-p}{3-p} \cdot p}$

3 Consecutive Flips

Suppose you are flipping a fair coin (one Head and one Tail) until you get the same side 3 times (Heads, Heads, Heads) or (Tails, Tails, Tails) in a row.

(a) Construct an Markov chain that describes the situation with a start state and end state.

$X = \{\#$ in a row up to cur$\}$



(b) Given that you have flipped a (Tails, Heads) so far, what is the expected number of flips to see the same side three times? $\boxed{6}$

$\beta(i)$ = expected # steps to state 3 from state i

$$\beta(0) = 1 + 1 \cdot \beta(1)$$

$$TH \longrightarrow \beta(1) = 1 + \tfrac{1}{2}\beta(1) + \tfrac{1}{2} \cdot \beta(2)$$

$$\beta(2) = 1 + \tfrac{1}{2}\beta(1) + \tfrac{1}{2}\beta(3)$$

$$\beta(3) = 0$$

$$\beta(2) = 1 + \tfrac{1}{2}\beta(1)$$

$$\beta(1) = 1 + \tfrac{1}{2}\beta(1) + \tfrac{1}{2} + \tfrac{1}{4}\beta(1)$$

$\boxed{\beta(0) = 7}$ $\boxed{\beta(1) = 6}$ $\boxed{\beta(2) = 4}$

(c) What is the expected number of flips to see the same side three times, beginning at the start state?

$$\boxed{\beta(0) = 7}$$