Pendulum Cart System

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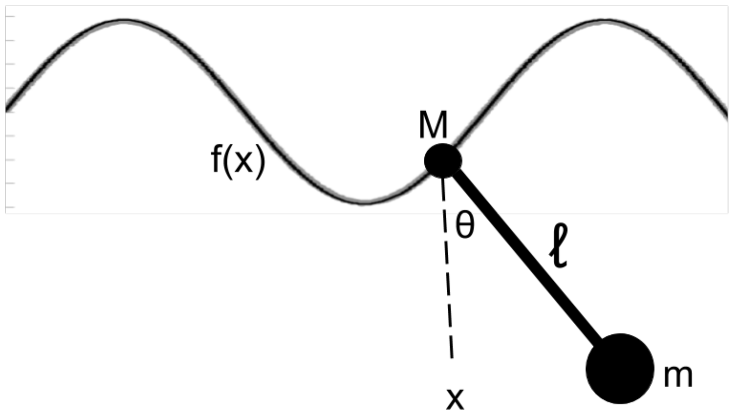
October 27, 2019

1 Introduction

Goal The pendulum cart system is intended to demonstrate the motion of a system derived using Lagrangian mechanics. This system was chosen as it meets all of the following criteria:

- The system should have the potential to be chaotic: A tiny change in the initial conditions should result in a much larger and relatively unpredictable change as the system evolves.
- It should not be possible to easily derive the equations of motion using Newtonian mechanics.
- It should be possible to easily create a visual representation of the system over time.
- The system should theoretically conserve energy, allowing for a way to check the accuracy of the simulation.

Description of system Let f(x) be an arbitrary continuous, twice differentiable function on a 2 dimensional plane. Along f(x) there is a "cart" or a point of mass M which is allowed to travel freely along the path f(x). Attached to the "cart" is another point of mass m, which is attached to the original mass by a weightless rod of length ℓ. The secondary mass and rod are allowed to rotate freely around the first mass, essentially creating a pendulum. The two variable coordinates in this system are the x position of the first mass (x) and the angle between the two masses measured from the -y axis (θ).



2 Finding the Lagrangian

2.1 Definitions of Points

$$M_x = x$$
$$M_y = f(x)$$
$$m_x = x + \ell sin(\theta)$$
$$m_y = f(x) - \ell \cos(\theta)$$

2.2 Derivatives of Points

$$\dot{M}_x = \dot{x}$$
$$\dot{M}_y = f'(x)\dot{x}$$
$$\dot{m_x} = \dot{x} + \ell \cos(\theta)\dot{\theta}$$
$$\dot{m_y} = f'(x)\dot{x} + \ell sin(\theta)\dot{\theta}$$

2.3 Kinetic Energy of System

$$T = \frac{M}{2}(\dot{x}^2 + (f'(x)\dot{x})^2) + \frac{m}{2}((\dot{x} + \ell\cos(\theta)\dot{\theta})^2 + (f'(x)\dot{x} + \ell sin(\theta)\dot{\theta})^2)$$

$$T = \frac{M}{2}(\dot{x}^2 + f'(x)^2\dot{x}^2) + \frac{m}{2}(\dot{x}^2 + 2\dot{x}\ell\cos(\theta)\dot{\theta} + \ell^2 cos^2(\theta)\dot{\theta}^2 + f'(x)^2\dot{x}^2 + 2f'(x)\dot{x}\ell sin(\theta)\dot{\theta} + \ell^2 sin^2(\theta)\dot{\theta}^2)$$

$$T = \frac{M}{2}\dot{x}^2(1 + f'(x)^2) + \frac{m}{2}(\dot{x}^2 + 2\dot{x}\ell\cos(\theta)\dot{\theta} + f'(x)^2\dot{x}^2 + 2f'(x)\dot{x}\ell sin(\theta)\dot{\theta} + \ell^2\dot{\theta}^2)$$

$$T = \frac{M}{2}\dot{x}^2(1 + f'(x)^2) + \frac{m}{2}\dot{x}^2[1 + f'(x)^2] + \frac{m}{2}(2\dot{x}\dot{\theta}\ell[\cos(\theta) + f'(x)sin(\theta)] + \ell^2\dot{\theta}^2)$$

$$T = \frac{M+m}{2}\dot{x}^2(1 + f'(x)^2) + m\dot{x}\dot{\theta}\ell(\cos(\theta) + f'(x)sin(\theta)) + \frac{m}{2}\ell^2\dot{\theta}^2$$

2.4 Potential Energy of System

$$V = Mgf(x) + mg(f(x) - \ell\cos(\theta))$$

$$V = Mgf(x) + mgf(x) - mg\ell\cos(\theta)$$

2.5 Lagrangian

$$\mathscr{L} = T - V$$

$$\mathscr{L} = \frac{M+m}{2}\dot{x}^2(1 + f'(x)^2) + m\dot{x}\dot{\theta}\ell(\cos(\theta) + f'(x)sin(\theta)) + \frac{m}{2}\ell^2\dot{\theta}^2 - Mgf(x) - mgf(x) + mg\ell\cos(\theta)$$

3 Solving the Euler-Lagrange Equation

3.1 Solving Equation for x

3.1.1 Solving for $\frac{\partial\mathscr{L}}{\partial\dot{x}}$

$$\frac{\partial\mathscr{L}}{\partial\dot{x}} = (M+m)\dot{x}(1 + f'(x)^2) + m\ell(\dot{\theta}\cos(\theta) + \dot{\theta}f'(x)\sin(\theta))$$

$$\frac{\partial\mathscr{L}}{\partial\dot{x}} = (M+m)\dot{x} + (M+m)\dot{x}f'(x)^2 + m\ell\dot{\theta}\cos\theta + m\ell\dot{\theta}f'(x)\sin\theta$$

3.1.2 Solving for $\frac{d}{dt}\frac{\partial\mathscr{L}}{\partial\dot{x}}$

$$\begin{aligned}\frac{d}{dt}\frac{\partial\mathscr{L}}{\partial\dot{x}} = (M+m)\ddot{x} + [(M+m)\ddot{x}f'(x)^2 + 2(M+m)\dot{x}f'(x)f''(x)\dot{x}] + [ml\ddot{\theta}\cos\theta - m\ell\dot{\theta}^2\sin\theta] \\ +[m\ell\ddot{\theta}f'(x)\sin\theta + m\ell\dot{\theta}f''(x)\dot{x}\sin\theta + m\ell\dot{\theta}f'(x)\cos\theta\dot{\theta}]\end{aligned}$$

3.1.3 Solving for $\frac{\partial\mathscr{L}}{\partial x}$

$$\frac{\partial\mathscr{L}}{\partial x} = (M+m)\dot{x}^2f'(x)f''(x) + m\dot{x}\ell\dot{\theta}\sin\theta f''(x) - Mgf'(x) - mgf'(x)$$

3.1.4 Final equation

$$
\begin{aligned}
&(M+m)\ddot{x}+(M+m)\ddot{x}f'(x)^2+2(M+m)\dot{x}^2f'(x)f''(x)+m\ell\ddot{\theta}\cos\theta-m\ell\dot{\theta}^2\sin\theta \\
&+m\ell\ddot{\theta}f'(x)\sin\theta+m\ell\dot{\theta}^2f'(x)\cos\theta=(M+m)\dot{x}^2f'(x)f''(x)-Mgf'(x)-mgf'(x)
\end{aligned}
$$

3.2 Solving for θ

3.2.1 Solving for $\frac{\partial\mathscr{L}}{\partial\dot{\theta}}$

$$\frac{\partial\mathscr{L}}{\partial\dot{\theta}}=m\ell(\dot{x}\cos(\theta)+\dot{x}f'(x)\sin(\theta))+m\ell^2\dot{\theta}$$

3.2.2 Solving for $\frac{d}{dt}\frac{\partial\mathscr{L}}{\partial\dot{x}}$

$$\frac{d}{dt}\frac{\partial\mathscr{L}}{\partial\dot{\theta}}=[m\ell\ddot{x}\cos\theta-m\ell\dot{x}\sin\theta\dot{\theta}]+[m\ell\ddot{x}f'(x)\sin\theta+m\ell\dot{x}^2f''(x)\sin\theta+m\ell\dot{x}f'(x)\cos\theta\dot{\theta}]+m\ell^2\ddot{\theta}$$

3.2.3 Solving for $\frac{\partial\mathscr{L}}{\partial\theta}$

$$\frac{\partial\mathscr{L}}{\partial\theta}=-m\dot{x}\ell\dot{\theta}\sin\theta+m\dot{x}\ell\dot{\theta}f'(x)\cos\theta-mg\ell\sin\theta$$

3.2.4 Final equation

$$
\begin{aligned}
&m\ell\ddot{x}\cos\theta-m\ell\dot{x}\sin\theta\dot{\theta}+m\ell\ddot{x}f'(x)\sin\theta+m\ell\dot{x}^2f''(x)\sin\theta+m\ell\dot{x}f'(x)\cos\theta\dot{\theta}+m\ell^2\ddot{\theta} \\
&\qquad=-m\dot{x}\ell\dot{\theta}\sin\theta+m\dot{x}\ell\dot{\theta}f'(x)\cos\theta-mg\ell\sin\theta
\end{aligned}
$$

$$m\ell\ddot{x}\cos\theta+m\ell\ddot{x}f'(x)\sin\theta+m\ell\dot{x}^2f''(x)\sin\theta+m\ell^2\ddot{\theta}=-mg\ell\sin\theta$$

4 Finding the Equations of Motion

The equations of motion can be derived from the two equations from the previous sections:

$$m\ell\ddot{x}\cos\theta+m\ell\ddot{x}f'(x)\sin\theta+m\ell\dot{x}^2f''(x)\sin\theta+m\ell^2\ddot{\theta}=-mg\ell\sin\theta$$

and

$$
\begin{aligned}
&(M+m)\ddot{x}+(M+m)\ddot{x}f'(x)^2+2(M+m)\dot{x}^2f'(x)f''(x)+m\ell\ddot{\theta}\cos\theta-m\ell\dot{\theta}^2\sin\theta \\
&+m\ell\ddot{\theta}f'(x)\sin\theta+m\ell\dot{\theta}^2f'(x)\cos\theta=(M+m)\dot{x}^2f'(x)f''(x)-Mgf'(x)-mgf'(x)
\end{aligned}
$$

4.1 Isolating $\ddot{x}$ and $\ddot{\theta}$

$$\ddot{x} = [-2(M+m)\dot{x}^2 f'(x)f''(x) - m\ell\ddot{\theta}\cos\theta + m\ell\dot{\theta}^2\sin\theta - m\ell\ddot{\theta}f'(x)\sin\theta - m\ell\dot{\theta}^2 f'(x)\cos\theta$$
$$+(M+m)\dot{x}^2 f'(x)f''(x) - Mgf'(x) - mgf'(x)]/(M+m)(1+f'(x)^2)$$

$$\ddot{\theta} = \frac{-g\sin\theta - \ddot{x}\cos\theta - \ddot{x}f'(x)\sin\theta - \dot{x}^2 f''(x)\sin\theta}{\ell}$$

4.2 Substitute $\ddot{x}$ and $\ddot{\theta}$ Into Each Other and Solve

$$\ddot{x} = [-2(M+m)\dot{x}^2 f'(x)f''(x) + mg\sin\theta\cos\theta + m\dot{x}^2 f''(x)\sin\theta\cos\theta + m\ell\dot{\theta}^2\sin\theta + mgf'(x)\sin^2\theta$$
$$+m\dot{x}^2 f'(x)f''(x)\sin^2\theta - m\ell\dot{\theta}^2 f'(x)\cos\theta + (M+m)\dot{x}^2 f'(x)f''(x) - Mgf'(x) - mgf'(x)]$$
$$/[(M+m)(1+f'(x)^2) - m\cos^2\theta - mf'(x)\sin\theta\cos\theta - mf'(x)\cos\theta\sin\theta - mf'(x)^2\sin^2\theta]$$

$$\ddot{\theta} = [-g\sin\theta - \frac{\cos\theta + f'(x)\sin\theta}{(M+m)(1+f'(x)^2)}[-2(M+m)\dot{x}^2 f'(x)f''(x) + m\ell\dot{\theta}^2\sin\theta$$
$$-m\ell\dot{\theta}^2 f'(x)\cos\theta + (M+m)\dot{x}^2 f'(x)f''(x) - mgf'(x) - Mgf'(x)] - \dot{x}^2 f''(x)\sin\theta]$$
$$/[\ell - \frac{m\ell}{(M+m)(1+f'(x)^2)}(\cos^2\theta + f'(x)\sin\theta\cos\theta + f'(x)\sin\theta\cos\theta + f'(x)^2\sin^2\theta)]$$