

Hysteresis-Based Modeling and Control of Piezoelectric Actuators for Precision Manipulation

Gerardo Flores, Ph.D.

Director, RAPTOR Lab

Associate Professor of Systems Engineering
Texas A&M International University

Winter School 2026

Roadmap

Part I — The Problem (10 min)

Why piezo actuators? The hysteresis barrier.
Measurement constraints.

Part II — Modeling Hysteresis (30 min)

Classical & Generalized Bouc–Wen. Dahl model.

Part III — Output Feedback Control (30 min)

Observers, robust control, separation principle.

Part IV — Advanced Topics (5 min)

Asymmetric hysteresis. Finite-time stabilization.
MPC.

Part V — Applications (5 min)

Distributed-parameter effects. Force control.

Part VI — Wrap-up (10 min)

Takeaways. Lessons. Open problems.

PART I

The Problem

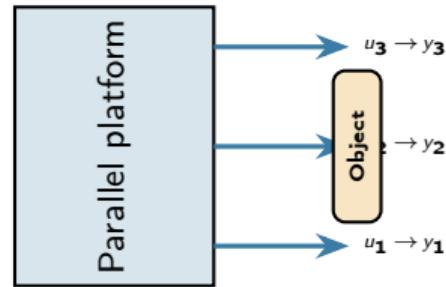
Why should I care?

~10 minutes

Precision Manipulation: Where Piezo Actuators Shine

Applications:

- Scanning probe microscopy & AFM
- Medical micro-robotics and micro-surgery
- Diesel injectors
- **Robotic hands** for manipulating fragile/deformable objects

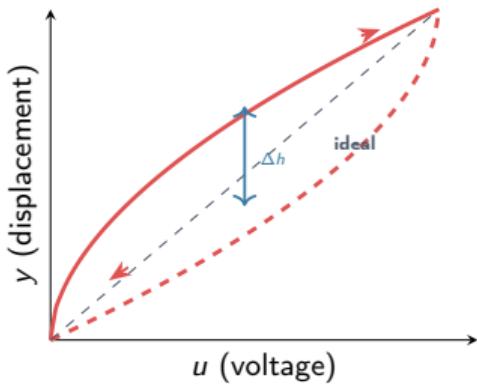


Why piezoelectric?

- Resolution down to **nanometers**
- Bandwidth: **hundreds of Hz to kHz**
- No mechanical play (solid-state)
- Direct electrical supply \Rightarrow easy integration

Piezoelectric robotic hand

Hysteresis: Not a Small Annoyance



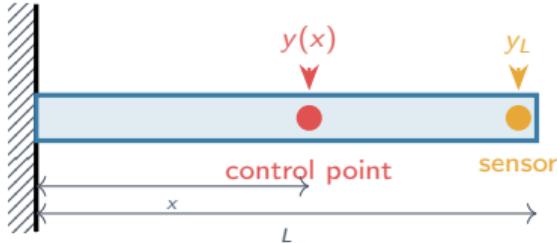
Impact on performance

- ① **Path-dependent output:** same voltage \Rightarrow different position depending on direction
- ② **Up to 60% amplitude error** in open loop
- ③ **Can destabilize feedback** if not modeled

Key point

Ignoring hysteresis is **not an option** for precision tasks.

Measurement Reality Check



Available:

- Position y at tip (sensor)
- Voltage u

Not available:

- Hysteresis state h
- Velocity \dot{y}
- Interaction force F
- Position at arbitrary point $y(x)$

Implication

We must **estimate what we cannot measure**. This motivates the entire **output-feedback framework**: nonlinear observers to reconstruct h , F , and hidden dynamics from y alone.

PART I — KEY MESSAGE

The difficulty is not control —
it's what you don't measure.

- ▶ Piezoelectric actuators offer unmatched precision, but hysteresis is the price
- ▶ Ignoring hysteresis: up to 60% error or instability
- ▶ Only position is measured — everything else must be estimated

PART II

Modeling Hysteresis

Without killing yourself

~30 minutes

How People Usually Deal with Hysteresis

Approach 1: Feedforward inversion

- Model hysteresis precisely
- Compute inverse $\mathcal{H}^{-1}[\cdot]$ and place in cascade
- ✗ No robustness vs. disturbances
- ✗ Model uncertainties accumulate

Approach 2: Treat as disturbance

- Use linear model + robust controller
- ✗ Fails when hysteresis is strong
- ✗ No formal stability guarantee

Approach 3: Direct model-based feedback

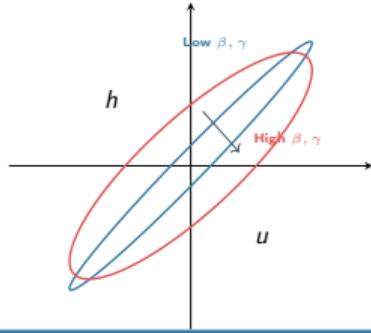
- ✓ Use explicit hysteresis model in control design
- ✓ Formal stability proofs
- ✓ Robust to disturbances
- This is our approach

Our philosophy

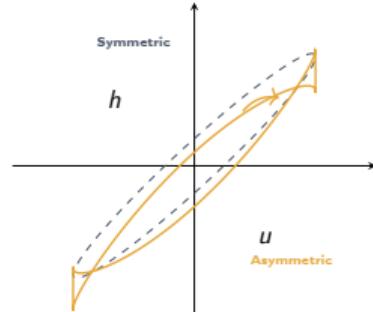
Model just enough hysteresis to make control possible.

Hysteresis Models: Parameter Influence

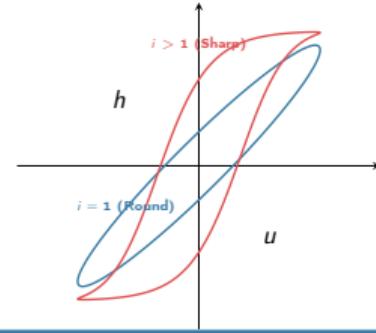
1. Classical Bouc–Wen



2. Generalized Bouc–Wen



3. Dahl Model



Parameters: α, β, γ

Controls amplitude and loop width (dissipation).
Always symmetric.

Parameters: \dots, ψ, δ_ν

Adds shape factors for asymmetry. Useful for specific piezo materials.

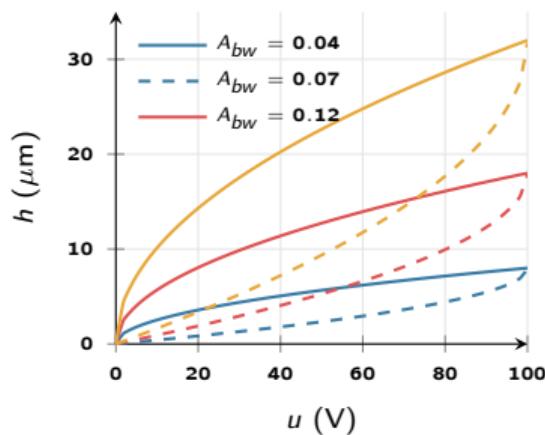
Parameters: σ, i

Shape factor i controls transition sharpness ("elasto-plastic" behavior).

Classical Bouc–Wen: Effect of Parameters

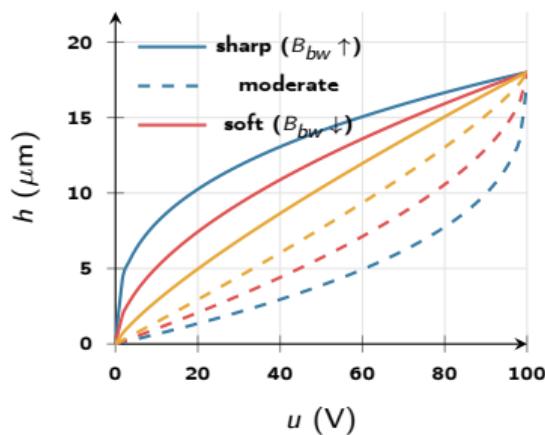
$$\dot{h} = A_{bw}\dot{u} - B_{bw}|\dot{u}|h - G_{bw}\dot{u}|h| \quad — \text{ Symmetric loops, shape controlled by } A_{bw}, B_{bw}, G_{bw}$$

Varying A_{bw} (amplitude)



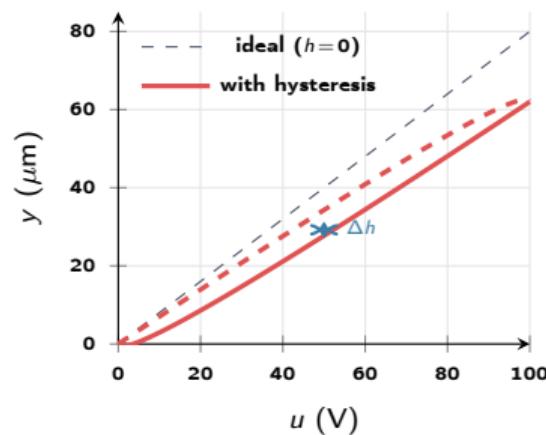
A_{bw} controls the **width** of the hysteresis loop (amplitude)

Varying B_{bw}, G_{bw} (shape)



B_{bw}, G_{bw} control the **shape**: sharp vs. gradual saturation

Full (u, y) map



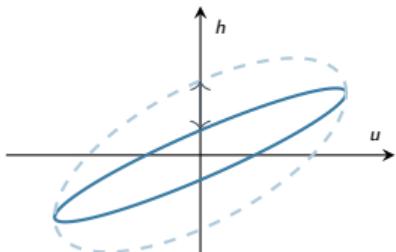
The output y deviates from the ideal line by the hysteresis h

Hysteresis Models: Visualizing the Parameters

1. Classical Bouc–Wen

$$\dot{h} = A_{bw}\dot{u} - B_{bw}|\dot{u}|h - G_{bw}\dot{u}|h|$$

Symmetric Variation

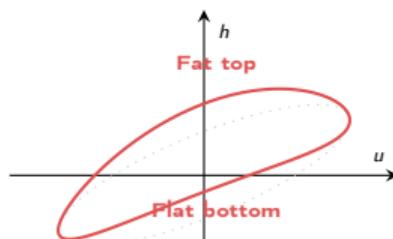


Parameters B_{bw}, G_{bw} control the width (dissipation). The loop remains perfectly symmetric.

2. Generalized Bouc–Wen

$$\dot{h} = \alpha[\dot{u}(A_{bw} - (B_{bw} \operatorname{sgn}(\dot{u}) + G_{bw})|h|^n)]$$

Asymmetric Shape (α)

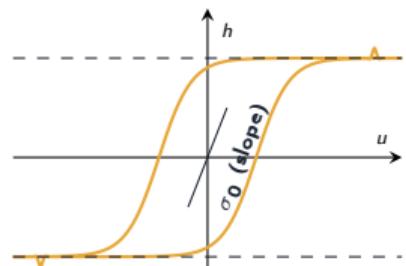


Parameter α breaks the symmetry. Note different curvatures for loading vs. unloading phases.

3. Dahl Model

$$\dot{h} = \sigma_0 \dot{u} (\mathbf{1} - \frac{h}{h_c} \operatorname{sgn}(\dot{u}))^i$$

Stiffness & Saturation



σ_0 sets the initial stiffness (slope). h_c sets the saturation force. Inherently asymmetric.

Run the Hysteresis Simulation Yourself

Open the interactive notebook:

github.com/gfloresc/winter-school-2026-piezo-control

Click: "Open in Google Colab/Jupyter Notebook/Binder"

No installation required

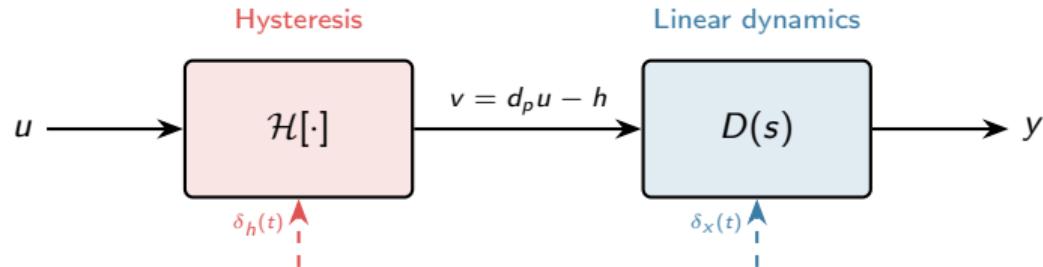
Runs in browser

Modify parameters live

Students can modify A_{bw} , B_{bw} , G_{bw} and visualize hysteresis in real time

The Hammerstein Structure

All our models share this architecture:



- u : driving voltage (control input)
- h : hysteresis state (**not measured**)
- $v = d_p u - h$: effective input to linear dynamics
- y : output displacement (**measured**)
- δ_h, δ_x : unknown disturbances / unmodeled terms

This structure is key: it separates the nonlinear hysteresis from the linear plant, enabling observer-based control.

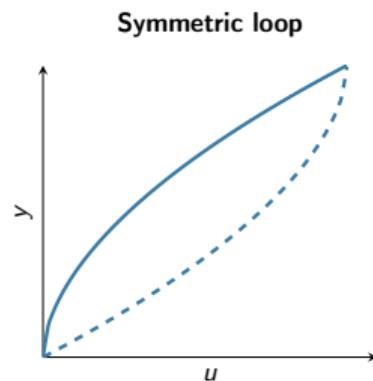
Classical Bouc–Wen (CBW) Model

The hysteresis dynamics:

$$\dot{h} = A_{bw} \dot{u} - B_{bw} |\dot{u}| h - G_{bw} \dot{u} |h| + \delta_h(t) \quad (1)$$

The full system in state-space:

$$\begin{cases} \dot{h} = A_{bw} \dot{u} - B_{bw} |\dot{u}| h - G_{bw} \dot{u} |h| + \delta_h \\ \dot{x} = Ax + B(d_p u - h) + \delta_x \\ y = Cx \end{cases} \quad (2)$$



Parameters:

- d_p : initial slope (piezo coefficient)
- A_{bw} : hysteresis amplitude
- B_{bw}, G_{bw} : shape parameters

Limitation

CBW models **symmetric** hysteresis only.
Real actuators are often asymmetric.

Generalized Bouc–Wen (GBW) Model

To capture **asymmetric** hysteresis, we extend the model:

$$\dot{h} = \alpha [\dot{u}(A_{bw} - (B_{bw} \operatorname{sgn}(\dot{u}) + G_{bw})|h|^n)] + \delta_h(t) \quad (3)$$

where α and n provide additional degrees of freedom for shaping the loop.

What changes from CBW:

- Additional shape parameter α controls asymmetry
- Exponent n controls sharpness of corners
- Nests the CBW as a special case ($\alpha = 1$, $n = 1$)

Used in:

- **Finite-time stabilization** [Flores & Rakotondrabe, L-CSS 2023a]
- **Model predictive control** [Flores, Aldana & Rakotondrabe, L-CSS 2022]

Tradeoff

More accurate \Rightarrow more complex control design.
Worth it when asymmetry is strong.

The Dahl Model — An Alternative for Asymmetry

The Dahl hysteresis model:

$$\dot{h} = \sigma_0 \dot{u} \left(1 - \frac{h}{h_c} \operatorname{sgn}(\dot{u}) \right)^i \quad (4)$$

where σ_0 is the initial stiffness, h_c is the saturation value, and i controls the curve shape.

Advantages:

- Naturally handles **non-symmetric** loops
- Bounded state: $|h| \leq h_c$
- Simpler structure \Rightarrow simpler observer design

When to use it:

- Strong asymmetry where CBW fails
- When boundedness simplifies stability analysis

Full system:

$$\begin{aligned} \dot{h} &= \sigma_0 \dot{u} \left(1 - \frac{h}{h_c} \operatorname{sgn}(\dot{u}) \right)^i \\ \ddot{y} &= -a_1 \dot{y} - a_2 y + b_0 (d_p u - h) \\ &\quad + \delta_y(t) \\ y_m &= y \end{aligned}$$

Used in: **ADRC position control** [Flores & Rakotondrabe, L-CSS 2023b]

Key result

Global asymptotic stability using an extended observer + active disturbance rejection.

Model Comparison: Which One to Use?

Property	CBW	GBW	Dahl
Symmetric hysteresis	✓	✓	✓
Asymmetric hysteresis	✗	✓	✓
Bounded state	requires proof	requires proof	built-in
Number of parameters	4	5–6	3
Control design complexity	moderate	high	moderate
Observer design	HGO, NNO	nonlinear	extended
Stability achieved	local exp.	finite-time	global asympt.

Part II — Meta-message

Model just enough hysteresis to make control possible.

The CBW is the workhorse; GBW when asymmetry matters; Dahl for simpler analysis.

~ Pause ~

Before we dive into control:

Which model would you choose and why?

PART III

Control with Almost No Measurements

The heart of the contribution

~30 minutes

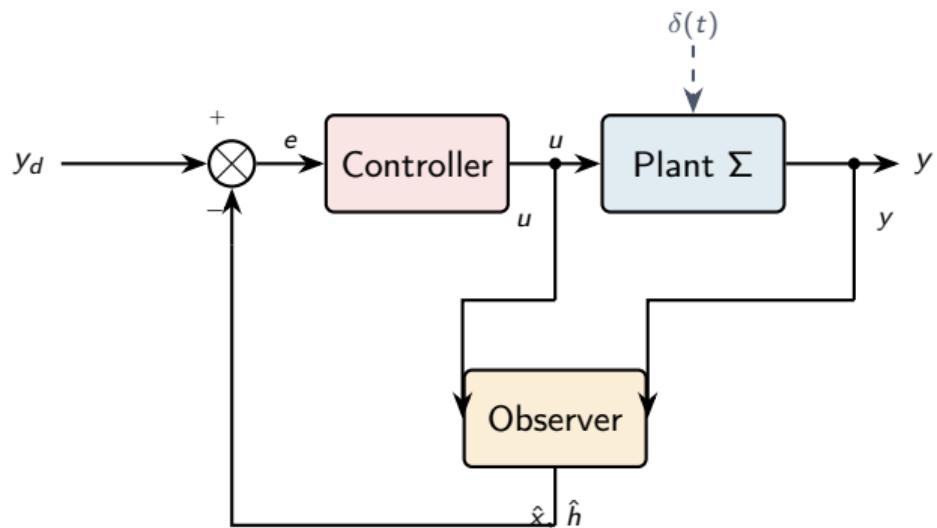
Output Feedback: The Core Idea

The situation:

- Only y (position) is measured
- h (hysteresis state) is hidden
- \dot{y} (velocity) is not available
- External forces F are unknown

Our solution:

- ① Design a **nonlinear observer** to estimate (\hat{x}, \hat{h}) from y
- ② Design a **control law** using the estimates (\hat{x}, \hat{h})
- ③ Prove stability of the **interconnected** system via separation principle



System Reformulation for Observer Design

Rewrite the system Σ in observer-canonical form. Setting $z = [z_1, z_2]^\top = [x, h]^\top$:

$$\boxed{\begin{aligned}\dot{z} &= \underbrace{\begin{pmatrix} 0 & -B \\ 0 & 0 \end{pmatrix}}_{\Phi} z + \underbrace{g(z, \dot{u})}_{\text{known nonlinearity}} + \underbrace{\delta(t)}_{\text{disturbance}} \\ y &= \bar{C} z = (1 \quad 0) z\end{aligned}} \tag{5}$$

$$\text{where } g(z, \dot{u}) = \begin{pmatrix} Az_1 + Bd_p u \\ f(h, \dot{u}) \end{pmatrix}, \quad \delta(t) = \begin{pmatrix} \delta_x \\ \delta_h \end{pmatrix}$$

Observability: Since $\text{rank } \mathcal{O} = \text{rank} \begin{pmatrix} \bar{C} \\ \bar{C}\Phi \end{pmatrix} = 2$ when $B \neq 0$, the system is **observable**.

This reformulation is the foundation for all observer designs that follow.

High-Gain Observer (HGO)

The observer:

$$\begin{aligned}\dot{\hat{z}}_1 &= A\hat{z}_1 + B(d_p u - \hat{z}_2) + \frac{k_1}{\varepsilon}(y - \hat{z}_1) \\ \dot{\hat{z}}_2 &= f(\hat{z}_2, \dot{u}) + \frac{k_2}{\varepsilon^2}(y - \hat{z}_1)\end{aligned}\tag{6}$$

where $\varepsilon > 0$ is small, and k_1, k_2 are chosen such that

$s^2 + k_1 s + k_2$ is Hurwitz.

Key assumption on disturbances: The disturbance $\delta(t) = [\delta_x, \delta_h]^\top$ satisfies $\|\delta\| \leq \varrho \|\tilde{z}\|$, $\varrho \in \mathbb{R}^+$ i.e., the perturbation is bounded by the observation error itself.

Result: Under this assumption, the observation error $\tilde{z} = \hat{z} - z$ is **locally exponentially stable**.

Intuition:

- Small $\varepsilon \Rightarrow$ fast convergence
- But too small \Rightarrow noise amplification
- Practical tradeoff: choose ε based on noise level

Physical meaning

$\hat{z}_1 \rightarrow x$: estimated displacement state

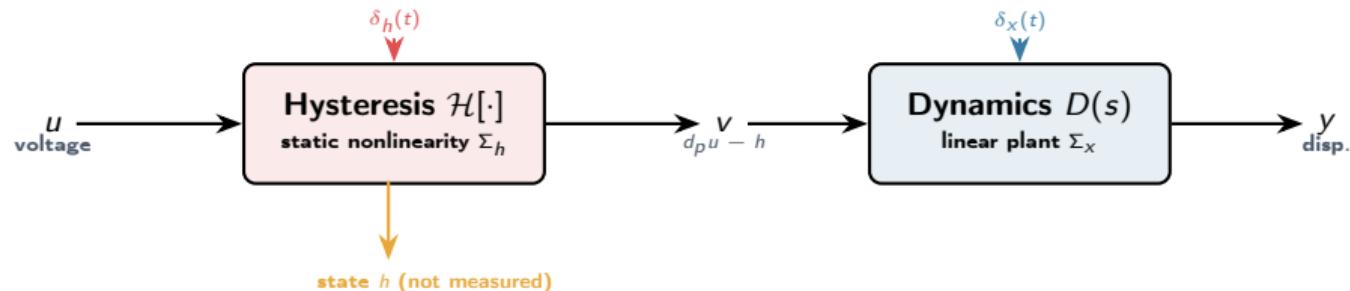
$\hat{z}_2 \rightarrow h$: **estimated hysteresis** — the key unmeasured variable

Why this assumption matters

It means the disturbance vanishes as the observer converges — this is what enables exponential stability rather than just bounded error.

The Hammerstein Structure: How We Model the Actuator

A piezoelectric actuator has two physical parts: a **transduction stage** (electrical → mechanical, contains hysteresis) followed by the **mechanical structure** (linear dynamics). This cascade is a **Hammerstein structure**.



In state-space form:

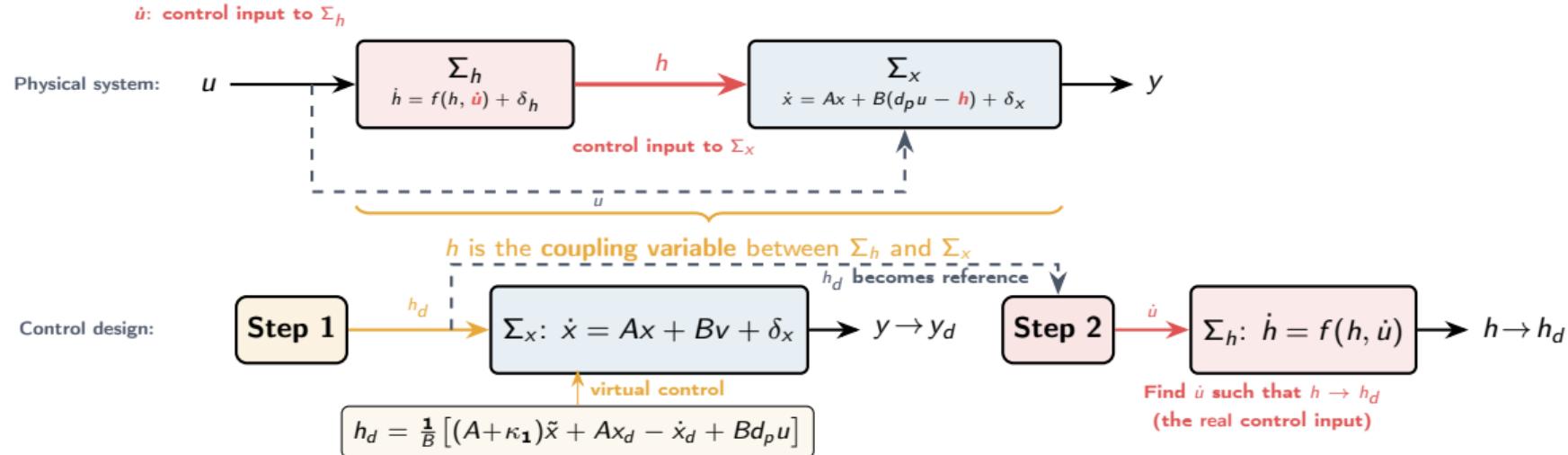
$$\Sigma : \begin{cases} \dot{h} = f(h, \dot{u}) + \delta_h(t) & \Sigma_h \\ \dot{x} = Ax + B \underbrace{(d_p u - h)}_v + \delta_x(t) & \Sigma_x \\ y = Cx & \text{output} \end{cases}$$

Why this structure matters:

- **Separates** the nonlinear hysteresis from the linear plant — each analyzed independently
- $v = d_p u - h$ is **not accessible** — we only measure y
- Enables: observer design and *virtual control*
 - ★ CBW, GBW, Dahl **all fit** this structure — only $f(h, \dot{u})$ changes

The Virtual Control Idea: Why h Becomes a Design Variable

Recall: the hysteresis output h is the *only link* between Σ_h and Σ_x in the Hammerstein structure.



The key insight

Treat h as if you could choose it (virtual control h_d) to stabilize Σ_x . Then design the actual u to steer $h \rightarrow h_d$ through Σ_h .

Used across all papers

CBW: h_d stabilizes $\tilde{x} \rightarrow 0$, \dot{u} drives $h \rightarrow h_d$
GBW+FTS: h_d with saturation σ_1
Force: h_d via barrier-Lyapunov

Robust Nonlinear Control Law

Step 1 — Virtual hysteresis control. Define the desired hysteresis signal:

$$h_d = d_p u - \frac{1}{B} \left[\dot{y}_d - (A + \lambda_1) \hat{z}_1 + \lambda_1 y_d \right] \quad (7)$$

where $\lambda_1 > 0$ is a design parameter.

Step 2 — Actual control input. From the hysteresis model, compute u to drive $\hat{h} \rightarrow h_d$:

$$u(t) = \int_0^t \frac{1}{A_{bw} - (B_{bw} \operatorname{sgn}(\dot{u}) + G_{bw}) |\hat{h}|} [\dot{h}_d + \lambda_2 (\hat{h} - h_d)] d\tau \quad (8)$$

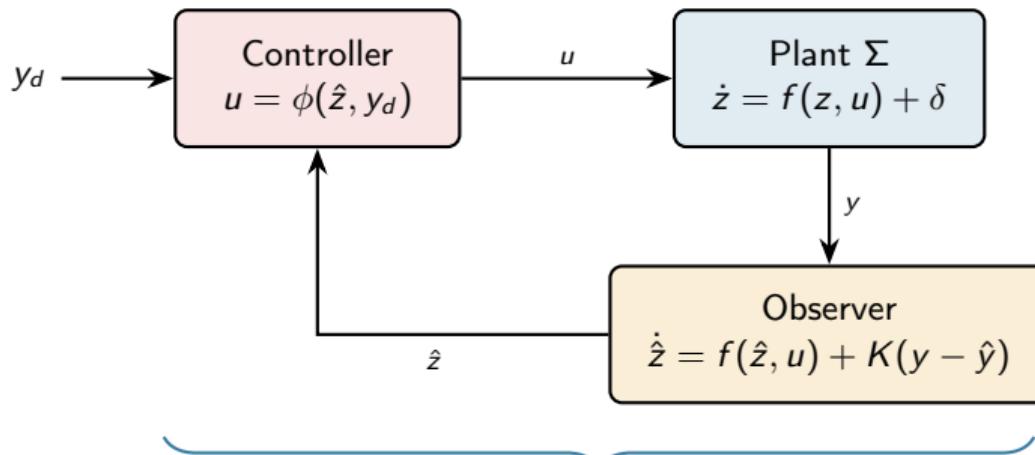
Stability result

Under Assumptions 1–2, the closed-loop system with HGO is **locally exponentially stable**.

Key feature

Only y is used for feedback. The control is an explicit function of Bouc–Wen parameters \Rightarrow **model-based robustness**.

The Separation Principle



- Observer error $\tilde{z} \rightarrow 0$ at rate $\mathcal{O}(1/\varepsilon)$
- Tracking error $e = y - y_d \rightarrow 0$ once \tilde{z} is small
- The interconnection preserves stability properties

Results: What Do We Get?

Position tracking:

- Hysteresis **completely removed** in closed loop
- The (y_d, y) map becomes a straight line
- Tested with sinusoidal and multi-frequency references

Disturbance rejection:

- Robust up to **37%** of output range
- External forces treated as bounded disturbances
- Multiple parameter sets tested

Stability guarantees:

Paper	Guarantee
CBW + HGO	local exp. stable
GBW + FTS	finite-time conv.
GBW + MPC	tracking + robust
Dahl + ADRC	global asympt.
CBW + BLF	asympt. stable

Key achievement

Formal stability from **position-only** measurements.

Execute the Control System

Run the complete control implementation:

github.com/gfloresc/winter-school-2026-piezo-control

Click: "Open in Google Colab/Jupyter Notebook/Binder"

Load control model

Execute simulation

Visualize results

Students can run PID vs Robust control and observe tracking performance in real time

PART III — KEY MESSAGE

A good model + a good observer =
control with almost no sensors.

The observer reconstructs the invisible hysteresis state.

The controller cancels it.

The separation principle glues them together with formal guarantees.

PART IV

When Things Get Serious

Beyond the basic framework

~5 minutes

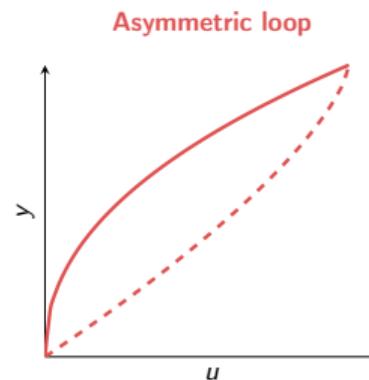
Asymmetric Hysteresis: Why Classical Models Fail

The problem:

- Real PZT actuators produce **asymmetric** loops
- CBW assumes symmetry \Rightarrow model mismatch
- Controller designed for CBW may **lose stability** on real hardware

Solutions explored:

- ① **GBW model** — additional parameters capture asymmetry at the cost of complexity
- ② **Dahl model** — intrinsically handles asymmetry with fewer parameters



The ascending and descending branches have different curvatures — CBW cannot capture this.

Finite-Time Stabilization

Why finite-time?

- Asymptotic: $e(t) \rightarrow 0$ as $t \rightarrow \infty$ (never quite reaches zero)
- **Finite-time:** $\exists T^* < \infty$ such that $e(t) = 0$ for all $t \geq T^*$
- Critical for pick-and-place, precision assembly

Our approach [Flores & Rakotondrabe, L-CSS 2023a]:

- ① GBW model for asymmetric hysteresis
- ② Three-part interconnected control:
 - Observer for (\hat{x}, \hat{h})
 - Virtual hysteresis control h_d
 - Actuator control u with finite-time terms

Result:

Theorem

Under bounded disturbances $\delta(t)$ and partial knowledge of one hysteresis parameter, the closed-loop tracking error converges to zero in **finite time** T^* .

Practical implication:

Exact zero error in finite time \Rightarrow well-defined task completion guarantees.

Model Predictive Control with Hysteresis

Idea: Use GBW model as the **prediction model** inside an MPC framework.

[Flores, Aldana & Rakotondrabe, L-CSS 2022]:

Architecture:

- ① Nonlinear observer estimates (\hat{x}, \hat{h})
- ② NMPC solves at each step:

$$\min_{\{u_k\}} \sum_{k=0}^{N-1} \|y_{k+1} - y_{d,k+1}\|^2 + R\|u_k\|^2$$

subject to GBW dynamics

- ③ Hysteresis-aware prediction \Rightarrow **hysteresis removed in closed loop**

Key results:

- Hysteresis **completely eliminated**
- Tested with $\pm 20\%$ parameter variation
- Robust tracking maintained

Why it works

Once you have a good model and a good observer, MPC becomes **viable**. The model does the heavy lifting.

PART IV — KEY MESSAGE

Modeling enables advanced control —
not the other way around.

PART V

Real Manipulation Scenarios

Where applied people reconnect

~5 minutes

Distributed-Parameter Effects

The problem:

The sensor measures at $x = L$, but the control objective may be at $x \neq L$.

[Trejo, Flores & Rakotondrabe, IFAC 2023]:

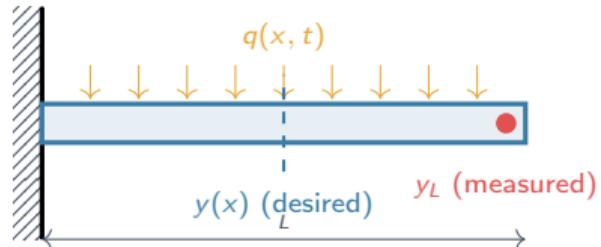
Extend CBW with Euler–Bernoulli beam theory:

$$EI \frac{\partial^4 y}{\partial x^4} + \rho A_s \frac{\partial^2 y}{\partial t^2} = q(x, t) \quad (9)$$

The hysteresis enters through the distributed load $q(x, t)$ from the piezoelectric effect.

Observers compared:

- High-Gain Observer (HGO)
- Neural Network Observer (NNO)
- NNO showed **better performance**



Challenge

The sensor is **not** at the control point. The distributed-parameter model bridges this gap.

Force Control with Deformable Objects

Scenario: Robotic hand manipulates a deformable object. Must control **force**, not just position.

[Flores & Rakotondrabe, L-CSS 2023c]:

Combined model:

- Actuator: CBW hysteresis + linear dynamics
- Object: nonlinear deformation model $\psi[y]$
- Interaction force: $F = \psi[y]$

Control approach:

- ① Output feedback (only F measured)
- ② **Barrier-Lyapunov function** \Rightarrow force error stays within prescribed bounds
- ③ **Saturation-based control** \Rightarrow bounded input

Result:

Theorem

The force-tracking error remains inside a **predefined set** for all $t \geq 0$, and the equilibrium is asymptotically stable.

Significance:

- Aggressive disturbance rejection
- Bounded control \Rightarrow implementable
- Safe manipulation of fragile objects

Object Characterization via Piezoelectric Manipulation

[Flores & Rakotondrabe, Nonlinear Dynamics 2025]:

Idea: Use the controlled actuator to **identify** the object's mechanical properties.

Approach:

- ① Control position precisely (CBW + output feedback)
- ② Estimate interaction force F using observers **without** force sensor
- ③ Extract force-deformation curve F vs. y
- ④ Identify object model parameters

Key innovation:

No object model needed *a priori*. The actuator becomes a **sensing tool** for characterizing unknown objects.

Validation:

- Nonlinear viscoelastic object
- Gaussian noise added to all channels
- Force estimation tracks ground truth
- Experimental validation included

Implication

The hysteresis framework enables not just control, but also **perception** — estimating what the actuator touches.

PART VI

Wrap-up & Big Picture

~10 minutes

Key Takeaways

1. Hysteresis must be modeled

Ignoring it \Rightarrow 60% error or instability.
CBW/GBW/Dahl provide control-friendly representations.

2. Output feedback is viable

Nonlinear observers reconstruct the hidden hysteresis state from position measurements alone.

3. Robustness is achievable

Formal stability guarantees (exponential, asymptotic, finite-time) with few sensors.

4. The framework extends

Force control, object characterization, distributed parameters — same core ideas.

Unified message: Hysteresis-aware modeling \rightarrow observer \rightarrow control \rightarrow formal guarantees

Publications Map

Paper	Model	Control	Stability
L-CSS 2022a	CBW	HGO + nonlinear	local exp.
L-CSS 2022b	GBW	observer + NMPC	tracking
L-CSS 2023a	GBW	FTS (3-part)	finite-time
L-CSS 2023b	Dahl	ext. obs. + ADRC	global asympt.
L-CSS 2023c	CBW + object	BLF + saturation	asymptotic
IFAC 2023	CBW (distributed)	HGO vs NNO	separation
Nonlinear Dyn. 2025	CBW + force est.	obs. + identification	robust

Lessons for Young Researchers

Don't overmodel

A model is a **tool**, not a goal. Model just enough to enable the control you need. CBW covers 80% of cases.

Control \neq tuning gains

If your “control design” is tuning PID gains until it works, you have no guarantees. A model-based approach gives you **provable** robustness.

Don't ignore physics

The Hammerstein structure, boundedness of h , the separation principle — these come from **understanding the physics**, not from optimization.

Write the proof first

If you can prove stability on paper, the simulation will confirm it. If you start with simulation, you may never prove anything.

Open Problems & Future Directions

Learning + Physics:

- Can neural networks learn the *residual* after Bouc–Wen?
- Physics-informed neural networks for hysteresis
- Online adaptation of model parameters

Soft Robots:

- Piezo-driven soft actuators with distributed hysteresis
- Infinite-dimensional control challenges
- Model reduction strategies

Hybrid Data-Driven Models:

- Combine Bouc–Wen structure with data-driven corrections
- Guarantees + adaptability
- Transfer learning across actuator families

Multi-Actuator Coordination:

- Three-finger robotic hand: coupled hysteresis
- Cooperative manipulation with deformable objects
- Decentralized vs. centralized control

Thank You

Gerardo Flores, Ph.D.

RAPTOR Lab — Texas A&M International University

gerardo.flores@tamiu.edu

All papers available on IEEE Xplore and Springer

Slides prepared for Winter School 2026

Backup Slides

Backup: CBW Parameter Sets Used in Simulations

Parameter	Set 1	Set 2	Set 3	Units
d_p	0.16	0.16	0.16	$\mu\text{m}/\text{V}$
A_{bw}	0.07	0.09	0.05	$\mu\text{m}/\text{V}$
B_{bw}	0.02	0.03	0.015	μm^{-1}
G_{bw}	0.01	0.015	0.008	μm^{-1}
A	-86.28	-86.28	-86.28	s^{-1}
B	7.95×10^7	7.95×10^7	7.95×10^7	s^{-2}

Multiple parameter sets demonstrate **robustness** of the control framework to model uncertainties.

Backup: Extended Observer for Dahl Model

Extended state:

$$\bar{x} = \begin{pmatrix} y \\ \dot{y} \\ \Delta(t) \end{pmatrix}, \quad \Delta(t) = -a_1\dot{y} - a_2y + b_0(d_p u - h) + \delta_y$$

Observer:

$$\dot{\hat{x}} = \bar{A}\hat{x} + \bar{B}u + L(\varepsilon)(y - \hat{y}) \quad (10)$$

where $L(\varepsilon) = \begin{pmatrix} k_1/\varepsilon \\ k_2/\varepsilon^2 \\ k_3/\varepsilon^3 \end{pmatrix}$ and the ADRC law:

$$u = \frac{1}{b_0 d_p} \left[\ddot{y}_d + \lambda_1(\dot{y}_d - \hat{y}) + \lambda_0(y_d - y) - \hat{\Delta} \right] \quad (11)$$

Result: Global asymptotic stability of tracking error.

Backup: Barrier-Lyapunov Function for Force Control

To constrain the force error $e_F = F - F_d$ within $|e_F| < k_b$:

BLF candidate:

$$V = \frac{1}{2} \ln \frac{k_b^2}{k_b^2 - e_F^2} \quad (12)$$

Properties:

- $V \rightarrow \infty$ as $|e_F| \rightarrow k_b \Rightarrow$ error **never** leaves the set $(-k_b, k_b)$
- Combined with saturation-based control \Rightarrow **bounded input**
- Asymptotic stability proven via Lyapunov analysis

This combines constrained control (BLF) with bounded actuation (saturation), a powerful combination for safe manipulation.