Finite-Time Stabilization of the Generalized Bouc–Wen Model for Piezoelectric Systems

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***Abstract*—When designing controllers for piezoelectric systems with hysteresis, usually simplified models are used. This can lead to inaccuracies in the closed-loop system response. In this letter, we pose the problem of tracking stabilization of piezoelectric systems using the generalized Bouc-Wen model, a highly non-linear system rarely used to design controllers. Besides, we consider only partially knowledge of one hysteresis system parameter and external disturbances in all the system states. We propose an interconnected control composed of three parts for the solution: an observer, a virtual hysteresis control, and actuator control. It is demonstrated that the closed-loop system converges in finite time. Simulation experiments were carried out, demonstrating the effectiveness of our approach despite exogenous and unknown disturbances.**

***Index Terms*—Generalized Bouc-Wen model, hysteresis control, piezoelectric system, finite-time convergence.**

I. Introduction

WHEN it comes to high precision positioning applications, piezoelectric actuators are one of the widely employed. For example, they are used in scanning images based on scanning probe microscopy, diesel injectors, medical micro-robotics, or hard disk drivers [1], [2]. The reason for this recognition is the performances they can offer. Piezoelectric actuators could be used as their proper sensors thanks to a judicious combination of the direct and converse piezoelectric physical effects. This advantage is essential in applications where using external sensors for feedback control is difficult or impossible. In order to handle very fragile objects, we are studying robotic hands based on three piezoelectric "stack" actuators [3]. This letter aims to model and design a control law devoted to one of the actuators, as they have the same properties and are subject to the same environment, the manipulated object.

Despite the above features, piezoelectric actuators exhibit specific properties that could affect the final performances if not adequately accounted for. Among these harmful properties is the hysteresis nonlinearity found between the driving voltage and the output displacement. The hysteresis modifies the precision if not controlled. Moreover, instability could occur if a feedback controller against the hysteresis is poorly designed. Another less critical nonlinearity in piezoelectric actuators is the creep phenomenon, a slow drift of the output displacement when a constant input voltage is applied. The hysteresis and the creep have raised several studies in modeling and control. They can be categorized as feedforward and feedback control. Feedforward control of hysteresis and creep phenomena has the advantage of low cost and embarkability since no external sensor is used [4], [5]. Meanwhile, its principal handicap is the lack of robustness against model uncertainties and external disturbance. For feedback control, various techniques have been studied [6].

There are several models and approaches in the literature regarding hysteresis in piezoelectric actuators. They include the Prandtl-Ishlinskii approach, the Preisach approach, and the Bouc-Wen approach; the two formers are operator-based modeling while the latter is differential-based. The Bouc-Wen approach is fascinating because its models have specific structures allowing design feedback controllers to perform structural properties analysis such as stability [7]. The Bouc-Wen approach includes the classical Bouc-Wen (CBW) model, the generalized Bouc-Wen (GBW) model, and other extensions. The CBW is, however, limited to model symmetrical hysteresis only, while the GBW can be used to model hysteresis with asymmetry with more precision. In [3], we used the CBW to design controllers devoted to piezoelectric actuators for robotic hands, but the precision was limited due to the strong asymmetry of the hysteresis. Further, in [8], we used the GBW model for a model predictive control design. However, in both works, the performances in terms of stabilization time were not guaranteed. In this letter, we propose to design a controller for the generalized Bouc-Wen model of hysteresis to control the piezoelectric actuator with a finite-time stabilization feature. Additionally, to the hysteresis, the creep nonlinearity is accounted for in a disturbance part of the model.

The remainder of this letter is as follows. First, Section II presents the generalized Bouc-Wen model in the framework of the Hammerstein structure; besides, the problem statement is posed. The control design is divided into two parts: the first for the hysteresis and the second for the actuator system. This

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is given in Section III. Since the only available information is the input and output of the system, and in our approach, the hysteresis state is required for feedback, we present an observer in Section IV. A numerical simulation presented in the Supplementary Material verifies our control scheme. Finally, we present some concluding remarks in Section V.

II. Modeling and Problem Statement

In this letter, we propose to use the *generalized Bouc-Wen model*, a high-nonlinear model that extends the hysteresis' classical Bouc-Wen model we employed in our previous works [3]. In contrast to the generalized Bouc-Wen model, the classical Bouc-Wen model is limited to symmetrical hysteresis. However, both models are static, meaning that the hysteresis behavior they model is static [9]. In this way, intending to account for the actuator dynamics, we employ a Hammerstein structure [3], where the static nonlinearity in Σ is the hysteresis, followed by a linear dynamics Ω [10]. Note that the Hammerstein structure has also been successfully used in various works for modeling nonlinearity and dynamics in piezoelectric actuators with, for instance, the classical Bouc-Wen or the Prandtl-Ishlinskii hysteresis models [3], [11]. Thus, for a SISO system of output, $y_h \in \mathbb{R}$ and input $u \in \mathbb{R}$, the generalized Bouc-Wen model with a Hammerstein structure susceptible to exogenous disturbances is proposed as follows,

$$\Sigma : \begin{cases} \dot{h} = \dot{u}(\alpha - |h|^m \Phi(u, \dot{u}, h)) + \delta_h(t) \\ \Phi(u, \dot{u}, h) = \beta_1 \mathrm{sgn}(\dot{u}h) + \beta_2 \mathrm{sgn}(\dot{u}u) \\ \qquad + \beta_3 \mathrm{sgn}(uh) + \beta_4 \mathrm{sgn}(\dot{u}) \\ \qquad + \beta_5 \mathrm{sgn}(h) + \beta_6 \mathrm{sgn}(u) \\ y_h = -h + d_p u \end{cases} \tag{1}$$

$$\Omega : \begin{cases} \dot{x} = Ax + B(\underbrace{d_p u - h}_{y_h}) + \delta_x(t) \\ y_x = Cx. \end{cases} \tag{2}$$

Subsystem Σ is the generalized Bouc-Wen model with internal hysteresis state $h \in \mathbb{R}$; parameters α, m and β_i for $i = 1, \dots, 6$ indicate the hysteresis width and asymmetry level [12]. The signum function $\mathrm{sgn}(\cdot)$ is defined as in [13, pp. 553]. The output equation y_h links the hysteresis dynamics with the actuator dynamics, as can be seen in (2). On the other hand, subsystem Ω models the linear actuator dynamics characterized by an internal state $x \in \mathbb{R}^n$ and described by the realization $(A, B, C, 0)$, where $A \in \mathbb{R}^{n\times n} < 0$, $B \in \mathbb{R}^{n\times 1}$ and $C \in \mathbb{R}^{1\times n}$; parameter d_p provides a general slope of the hysteresis; external disturbances (δ_h, δ_x) are considered acting on the system. Besides, for the piezoelectric actuator studied in this letter, it was shown in [12] that $m = 1$.

A. The Problem

By analyzing the system Σ, one can realize the difficulty of proposing Lyapunov-based controllers due to the nonlinearities in the hysteresis, the nature of the control input, and the unavailable state h. To cope with that, in our previous work [8], we suggested a model predictive control, a control technique based on a numerical minimization algorithm. In this letter, we pose the conditions to stabilize the system (Σ, Ω) and find a Lyapunov-based nonlinear controller capable of solving the tracking problem. With that aim, notice that,

$$|\Phi(u, \dot{u}, h)| \le \sum_{i=1}^{6} |\beta_i| \ \le\ q, \ \text{with} \ \ 0 < q \ll 1 \tag{3}$$

where for piezoelectric actuators we have the parameter $0 < q \ll 1$ [12], and it is clear that $\Phi(u, \dot{u}, h)$ is bounded for all the values of its arguments.

Assumption 1: The control input u and its first-time derivative are bounded. Moreover, there exists a controller u that makes the hysteresis bounded $\forall t \ge t_0$.

The voltage coming from a source or/and an amplifier is always saturated. Thus, the boundedness of u in Assumption 1 makes sense. For example, in the piezoelectric actuators, the voltage amplifier is limited to $\pm 200V$. Another physical motivation for boundedness of u is that piezoelectric actuators lose their piezoelectric property or even break down if they are given voltages above specific values; hence, one should limit their driving voltage. Finally, the boundedness of the first derivative of u comes from the fact that the voltage source and the amplifier have limited bandwidth or slew rate. Hence, the rate of u is also saturated by the technology.

Notice that one necessary condition, in order for the system (1) to be controllable, is that $|h|\Phi(u, \dot{u}, h) \neq \alpha$. This gives rise to the following assumption.

Assumption 2: The inequality $\alpha > |h|\Phi(u, \dot{u}, h)$ holds $\forall (u, \dot{u}, h)$ for piezoelectric actuators [12].

Before presenting our first main result, we consider the following assumptions.

Assumption 3: The external disturbances are bounded as $|\delta_h(t)| \le c_1$, $|\delta_x(t)| \le c_2$, with $c_1, c_2 \in \mathbb{R}_{>0}$.

Problem 1: Let us consider the system (Σ, Ω) with error definition,

$$e_h = h - h_d, \ \ e_x = x - x_d, \tag{4}$$

where x_d is the actuator desired trajectory, and h_d is the virtual control input for the system Ω (also, it is considered the desired hysteresis). The problem is achieving the error state (e_h, e_x) to zero as quickly as possible, despite the knowledge of parameters β_i in the subsystem Σ. In this way, we only consider the knowledge of the hysteresis system's parameters α and d_p.

Remark 1: Since there is no sensor to measure the hysteresis state, the idea is to design an observer with the available state information (x in Ω). Once we get an estimate of h, denoted by $\hat{h}$, it is possible to use such a signal for feedback [14]. That is one of the keys to our control design.

Remark 2: Notice that the connection between system Σ and Ω is given by dynamic relations depending on h, in particular, the hysteresis output, $y_h = -h + d_p u$. Therefore, we take advantage of it and take h as a virtual control for subsystem Ω. Once we designed such virtual control, now defined as h_d, it is used as the reference for system Σ.

III. Control Design

A. Hysteresis Stabilization

The control strategy involves designing a control law for $\dot{u}$, and then we integrate such an algorithm to find the real control u. We begin by computing the error dynamics of the first error in (4) as follows,

$$\dot{e}_h = \dot{u}(\alpha - |e_h + h_d|\Phi(u, \dot{u}, e_h + h_d)) + \delta_h(t) - \dot{h}_d. \quad (5)$$

The first main result of this letter is presented in the following proposition.

Proposition 1: Let us take into account Assumptions 1-3 and consider the hysteresis error dynamics (5) with the control law:

$$\dot{u} = -\frac{k_h}{\alpha - q|e_h + h_d|}\mathrm{sgn}(e_h) + \frac{\dot{h}_d}{\alpha - q|e_h + h_d|} \quad (6)$$

where $k_h \in \mathbb{R}_{>0}$. Then, the equilibrium point $e_h = 0$ of the closed-loop system is global finite-time stable with convergence time given by $t \leq \frac{\sqrt{2}}{k_h}V^{1/2}(e_h(0))$, where V is a Lyapunov function given by $V = \frac{1}{2}e_h^2$. [1]

Proof: Notice that in the control (6) we do not use the knowledge of $\Phi(u, \dot{u}, h)$ in (1), just an upper approximate bounded q given in (3). Thus, the hysteresis error dynamics (5) together with control (6) is given by,

$$\dot{e}_h = \underbrace{\left(\frac{\alpha - |e_h + h_d|\Phi(u, \dot{u}, e_h + h_d)}{\alpha - |e_h + h_d|q}\right)}_{f(t)}(-k_h\mathrm{sgn}(e_h) + \dot{h}_d) + \delta_h(t) - \dot{h}_d. \quad (7)$$

Notice that the term above, $f(t)$, includes the assumed unknown terms β_i in the numerator, and is considered as a partially known function of time. Note that $f(t) = 1$ as long as $e + h_d = 0$. The case $\Phi(u, \dot{u}, e_h + h_d) = q$ is omitted since q in general is an upper bounded of $\Phi(u, \dot{u}, e_h + h_d)$ and not necessarily its maximum possible value. Besides, thanks to (3)

$$1 \leq f(t) \quad (8)$$

always holds. Thus, to this point, we can conclude that $f(t)$ is always positive and varies with time. Therefore, we rewrite the closed-loop system (7) as a time-varying system:

$$\dot{e}_h = -k_h f(t)\mathrm{sgn}(e_h) + f(t)\dot{h}_d + \delta_h(t) - \dot{h}_d. \quad (9)$$

To go further in the proof, we desire to know the upper bound of $f(t)$. For that, let consider the following inequalities that hold for $f(t)$ in (7):

$$\alpha - q|\max(e_h + h_d)| \leq |\alpha - q|e_h + h_d|| \leq \alpha, \quad (10)$$

also notice that,

$$|\alpha - \Phi(\cdot)|e_h + h_d|| \leq \alpha + q|\max(e_h + h_d)| \quad (11)$$

holds. By combining inequalities (8), (10) and (12), it is clear that the following inequalities hold,

$$1 \leq f(t) \leq \rho, \ \text{ for } \rho = \frac{\alpha + q|\max(e_h + h_d)|}{\alpha - q|\max(e_h + h_d)|}. \quad (12)$$

[1] Notice that this control uses h for feedback according to (4). Since h is not available, one needs to get an estimate of it given by $\hat{h}$, which is provided in Proposition 3.

Let us continue with the stability analysis. For that, observe that $-f(t)$ is Hurwitz for all $t \geq 0$. According to [15] Theorem 1, to prove the stability of the closed-loop time-varying system (9), one must construct a Lyapunov function proposed as in the conditions of the Proposition 1, whose time-derivative along solutions of the closed loop system (9) is given as,

$$\dot{V} = e_h\big(-k_h f(t)\mathrm{sgn}(e_h) + f(t)\dot{h}_d + \delta_h(t) - \dot{h}_d\big). \quad (13)$$

At this point we consider Assumption 3 and the fact that we know the parameter $\gamma = \max\{\dot{h}_d\}$ since one chooses the desired signal (the virtual control explained in the following subsection). Then, (13) is simplified as,

$$\begin{aligned}\dot{V} &\leq -k_h f(t)|e_h| + f(t)|\dot{h}_d||e_h| + c_1|e_h| + \dot{h}_d||e_h| \\ &\leq -\underbrace{(k_h f(t) - \gamma f(t) - c_1 - \gamma)}_{\kappa(t)}|e_h| \end{aligned} \quad (14)$$

which is negative definite as long as the inequality holds:

$$k_h > \gamma\left(1 + \frac{c_1/\gamma + 1}{\min\{f(t)\}}\right) = 2\gamma + c_1, \quad (15)$$

where we have used (8) and the fact that we must find the greater positive gain k_h such that it fits with all the possible values of $f(t)$.

To prove the global finite-time stabilization, consider the last inequality in (14) and notice from the proposed Lyapunov function $V = \frac{1}{2}e_h^2$ that $V^{1/2} = \frac{1}{\sqrt{2}}|e_h|$, and then $\dot{V}_h \leq -\sqrt{2}\kappa(t)V^{1/2}$. Furthermore, since $\kappa(t)$ varies with time due to the presence of $f(t)$, one can use the inequality (15) and get, $\dot{V}_h \leq -\sqrt{2}k_d V^{1/2}$. The last differential inequity can be solved by separation of variables and get $V^{1/2}(e_h(t)) - V^{1/2}(e_h(0)) \leq -\frac{k_h}{\sqrt{2}}t$, then, since $V^{1/2}(e_h(t)) \geq 0$ and $V^{1/2}(e_h(t)) \leq V^{1/2}(e_h(0))$, the term $V^{1/2}(e_h(t)) = 0$ up to t as given in Proposition 1, and the proof is complete. ■

B. Control for the Actuator Dynamics

In this subsection, we design the control algorithm to stabilize, in finite-time, the equilibrium $e_x = x - x_d = 0$ corresponding to the error dynamics of the linear disturbed actuator Ω. We consider that h acts as a virtual controller for such a system for that goal. Therefore, we name h_d to that virtual controller in the following. We begin by considering the error dynamics of e_x defined in (4) and subsystem Ω as,

$$\dot{e}_x = A(e_x + x_d) + Bd_p u - Bh_d + \delta_x(t) - \dot{x}_d. \quad (16)$$

In this case, the user defines the desired signal x_d, and its first-time derivative $\dot{x}_d$ and corresponds to the desired actuator dynamics. Before presenting our main result, We employ a class of linear saturation functions defined next.

Definition 1 [16], [17]: Let consider $L, M \in \mathbb{R}_{>0}$ with $L \leq M$. A linear saturation function for (L, M) is given by $\sigma : \mathbb{R} \to \mathbb{R}$ being continuous and non-decreasing, satisfying:

1) $s\sigma(s) > 0$ for all $s \neq 0$;
2) $\sigma(s) = s$ when $|s| \leq L$; and
3) $|\sigma(s)| \leq M$ for all $s \in \mathbb{R}$.

Now, we are ready to present our second main result.

Proposition 2: Let consider Assumption 3 and a linear saturation function $\{\sigma_1\}$ with corresponding positive constants $\{(L_1, M_1)\}$ where $L_1 \le M_1$. Then, the bounded virtual control,

$$h_d = \frac{1}{B}\big(\sigma_1(k_{x_1}e_x) + k_{x_2}\mathrm{sgn}e_x + A(e_x + x_d) + Bd_pu - \dot{x}_d\big) \tag{17}$$

with $M_1 > k_{x_2} > c_2$, results in global finite-time stability for system (16) with convergence time

$$t \le \frac{1}{k_{x_1}}\log\left(\frac{\sqrt{2}k_{x_1}}{(k_{x_2} - c_2)}W^{1/2}(e_x(0)) + 1\right). \tag{18}$$

Proof: Let $W = \frac{1}{2}e_x^2$ a candidate Lyapunov function whose time-derivative along the trajectories of (16) together with control (17) is computed as follows,

$$\begin{aligned}\dot{W} &= e_x\big(A(e_x + x_d) + Bd_pu - \big[\sigma_1(k_{x_1}e_x) + k_{x_2}\mathrm{sgn}e_x \\ &\quad + A(e_x + x_d) + Bd_pu - \dot{x}_d\big] + \delta_x(t) - \dot{x}_d\big) \\ &\le -e_x\sigma_1(k_{x_1}e_x) - k_{x_2}|e_x| + c_2|e_x|,\end{aligned} \tag{19}$$

where we have claimed Assumption 3. Let consider the evolution of the state e_x when $e_x \notin Q$ where, $Q = \{e_x : |e_x| \le L_1\}$. In that case, by claiming point 1) of Definition 1 applied to σ_1, it is clear that $\dot{W} < 0$ for all $e_x \notin Q$. And thus, $e_x \in Q$ in finite time and remains in Q hereafter. Then, after e_x has entered in Q, it follows that

$$\begin{aligned}\dot{W} &\le -k_{x_1}|e_x|^2 - \big(k_{x_2} - c_2\big)|e_x| \\ &\le -2k_{x_1}W - \sqrt{2}\big(k_{x_2} - c_2\big)W^{1/2}.\end{aligned} \tag{20}$$

To find the stabilization time, we solve (20) for t by integration first using partial fractions as follows,

$$\begin{aligned}&\frac{1}{\sqrt{2}(x_{x_2} - c_2)}\int_{W(e(0))}^{W(e(t))}\frac{dW}{W^{1/2}} \\ &-\frac{2k_{x_1}}{\sqrt{2}(k_{x_2} - c_2)}\int_{W(e(0))}^{W(e(t))}\frac{dW}{2k_{x_1}W^{1/2} + \sqrt{2}(k_{x_2} - c_2)} \\ &\le -\int_0^t dt.\end{aligned} \tag{21}$$

Then, we need to find t such that $W^{1/2}(e(t)) = 0$, by using the fact that the Lyapunov function fulfill the inequality $W^{1/2}(e(t)) \le W^{1/2}(e(0))$. Then, after simplifying the above expression, the solution is given as in the Proposition 2. ■

Remark 3: The saturation function $\sigma_1(k_{x_1}e_x)$, which is linear in the region $|e_x| \le L_1$, helps to deal with perturbations that are very far from the origin and weaker near the origin. To cope with the latter, we implement the term $k_{x2}\mathrm{sgn}e_x$, which can support a strong perturbation near the origin.

IV. Observer

Recall that the hysteresis state h is not available. However, our control (6) requires it, or at least an estimate of it. The above motivates us to design the observer presented below. For that, we consider the following assumption.

Assumption 4: The external disturbances are bounded as follows, $|\delta_x(t)| \le \varpi|e_1|$, $|\delta_h(t)| \le \varrho|e_2|$, where $\varrho, \varpi \in \mathbb{R}_{>0}$, and (e_1, e_2) are the observer errors defined as in (25).

Now, we are ready to present our observer design.

Proposition 3: Let us consider Assumptions 1, 2, 4, and represent the system (Σ, Ω) in the linear coordinate transformation $z_1 = -x$, $z_2 = h$ as follows,

$$\begin{aligned}\dot{z}_1 &= Az_1 + Bz_2 - Bd_pu - \delta_x(t) \\ \dot{z}_2 &= \dot{u}(\alpha - |z_2|\Phi(u, \dot{u}, z_2)) + \delta_h(t).\end{aligned} \tag{22}$$

Then, the following system,

$$\begin{aligned}\dot{\hat{z}}_1 &= A\hat{z}_1 + B\hat{z}_2 - Bd_pu - l_1\eta_1(e_1) \\ \dot{\hat{z}}_2 &= \dot{u}\big(\alpha - q|\hat{z}_2|\big) - l_2\eta_2(e_1).\end{aligned} \tag{23}$$

with

$$\begin{aligned}\eta_1(e_1) &= \mu_1|e_1|^{1/2}\mathrm{sgn}(e_1) + \mu_2|e_1|^p\mathrm{sgn}(e_1) \\ \eta_2(e_1) &= \Big(\frac{\mu_1}{2}|e_1|^{-1/2} + \mu_2p|e_1|^{p-1}\Big)(\eta_1(e_1))\end{aligned} \tag{24}$$

where (μ_1, μ_2) are positive gains and p is a positive parameter to be chosen, is an observer for system (22). Furthermore, the zero equilibrium is defined as,

$$e_1 = \hat{z}_1 - z_1, \quad e_2 = \hat{z}_2 - z_2. \tag{25}$$

is finite-time globally stable with convergence time $t \le \frac{2}{\omega}\log\big(\frac{\omega}{\psi}V_\epsilon^{\frac{1}{2}}(\epsilon(0)) + 1\big)$ with positive constants (ψ, ω) defined by (46).

Proof: The error dynamics of (25) is given as:

$$\begin{aligned}\dot{e}_1 &= A\hat{z}_1 + B\hat{z}_2 - Bd_pu - l_1\eta_1(e_1) \\ &\quad -\big(Az_1 + Bz_2 - Bd_pu - \delta_x(t)\big) \\ \dot{e}_2 &= \dot{u}\big(\alpha - q|\hat{z}_2|\big) - l_2\eta_2(e_1) \\ &\quad -(\dot{u}(\alpha - |z_2|\Phi(u, \dot{u}, z_2)) + \delta_h(t))\end{aligned} \tag{26}$$

simplifying the last expression:

$$\begin{aligned}\dot{e}_1 &= Ae_1 + Be_2 - l_1\eta_1(e_1) + \delta_x(t) \\ \dot{e}_2 &= \dot{u}\big(\Phi(u, \dot{u}, z_2)|z_2| - q|\hat{z}_2|\big) - l_2\eta_2(e_1) - \delta_h(t).\end{aligned} \tag{27}$$

Notice that,

$$\begin{aligned}\Big|\Phi(\cdot)|z_2| - q|\hat{z}_2|\Big| &\le \Big|\big(\Phi(\cdot)|z_2| - q|\hat{z}_2|\big) + \big(q|z_2| - \Phi(\cdot)|\hat{z}_2|\big)\Big| \\ &= \Big|\big(\Phi(\cdot) + q\big)\big(|z_2| - |\hat{z}_2|\big),\Big|\end{aligned} \tag{28}$$

where when $\mathrm{sgn}\Phi(\cdot) = 1$ (the worst scenario in this case) the above inequality is verified as long as $\frac{|z_2|}{|\hat{z}_2|} \ge \underbrace{(\frac{\Phi(\cdot)}{q})}_{\epsilon}$, and from (3), $\epsilon \ll 1$ is verified for bounded states. Thus, the last equation in (27) can be simplified as,

$$\dot{e}_2 \le |\dot{u}|\big|(\Phi(u, \dot{u}, z_2) + q)\big(|z_2| - |\hat{z}_2|\big)\big| - l_2\eta_2(e_1) + \varrho|e_2|. \tag{29}$$

Then, from (3), applying the triangle inequality in the first term of the second inequation (29), and since the absolute value function is a globally Lipschitz function, it follows that $\dot{e}_2 \le (2q\vartheta + \varrho)|e_2| - l_2\eta_2(e_1)$, where $\vartheta \in \mathbb{R}_{>0}$ is a bound of controller (6).

Now, we come back to the error dynamics (27) by taking into account Assumption 4,

$$\begin{aligned}\dot{e}_1 &\leq Ae_1 + Be_2 - l_1\eta_1(e_1) + \varpi|e_1| \\ \dot{e}_2 &\leq (2q\vartheta + \varrho)|e_2| - l_2\eta_2(e_1),\end{aligned} \tag{30}$$

where $\beta = 2q\vartheta + \varrho$. Now, let us consider the change of coordinates $\epsilon = \begin{pmatrix}\eta_1(e_1)\\ e_2\end{pmatrix}$ with $\eta_1(e_1)$ defined as in (24), where $\frac{\partial\eta_1}{\partial e_1} = \frac{\mu_1}{2}|e_1|^{-1/2} + \mu_2 p|e_1|^{p-1} \geq 0$. Then, the system (30) can be represented as,

$$\dot{\epsilon} \leq \frac{\partial\eta_1}{\partial e_1}A_\epsilon\epsilon + \begin{pmatrix}\frac{\partial\eta_1}{\partial e_1}(Ae_1 + \varpi|e_1|)\\ \beta|e_2|\end{pmatrix} \tag{31}$$

where $A_\epsilon = \begin{pmatrix}-l_1 & B\\ -l_2 & 0\end{pmatrix}$ is Hurwitz.[2] The latter is true since $\eta_2(e_1) = \frac{\partial\eta_1}{\partial e_1}\eta_1(e_1)$. Now, let us consider the following candidate Lyapunov function $V_\epsilon = \epsilon^\mathsf{T}P\epsilon$ with $P = \begin{pmatrix}p_1 & p_3\\ p_3 & p_2\end{pmatrix} = P^\mathsf{T} \succ 0$ that fulfill the Lyapunov equation $A_\epsilon^\mathsf{T}P + PA_\epsilon = -Q$, where $Q = Q^\mathsf{T} \succ 0$. The time-derivative of V_ϵ along the error solutions (31) is computed as,

$$\begin{aligned}\dot{V}_\epsilon &= \dot{\epsilon}^\mathsf{T}P\epsilon + \epsilon^\mathsf{T}P\dot{\epsilon}\\ &= \left(\frac{\partial\eta_1}{\partial e_1}\epsilon^\mathsf{T}A_\epsilon^\mathsf{T} + \frac{\partial\eta_1}{\partial e_1}[Ae_1 + \varpi|e_1|,\ 0] + [0,\ \beta|e_2|]\right)P\epsilon\\ &\quad + \epsilon^\mathsf{T}P\left(\frac{\partial\eta_1}{\partial e_1}A_\epsilon\epsilon + \frac{\partial\eta_1}{\partial e_1}[Ae_1 + \varpi|e_1|,\ 0]^\mathsf{T} + [0,\ \beta|e_2|]^\mathsf{T}\right)\\ &= \underbrace{-\frac{\partial\eta_1}{\partial e_1}\epsilon^\mathsf{T}Q\epsilon}_{\dot{V}_{\epsilon1}} + \underbrace{2\frac{\partial\eta_1}{\partial e_1}\epsilon^\mathsf{T}P[Ae_1 + \varpi|e_1|,\ 0]^\mathsf{T}}_{\dot{V}_{\epsilon2}}\\ &\quad + \underbrace{2\epsilon^\mathsf{T}P[0,\ \beta|e_2|]^\mathsf{T}}_{\dot{V}_{\epsilon3}}.\end{aligned} \tag{32}$$

We next develop each of the terms $(\dot{V}_{\epsilon1}, \dot{V}_{\epsilon2}, \dot{V}_{\epsilon3})$ above. Notice that,

$$\lambda_{\min}\{P\}\|\epsilon\|_2^2 \leq V_\epsilon \leq \lambda_{\max}\{P\}\|\epsilon\|_2^2 \tag{33}$$

where $(\lambda_{\min}\{P\}, \lambda_{\max}\{P\})$ are the minimum and maximum eigenvalues of the matrix P; and $\|\cdot\|_2$ is the Euclidean norm. By using (33) for $\dot{V}_\epsilon$ and Q, it follows that,

$$\dot{V}_{\epsilon1} = -\frac{\partial\eta_1}{\partial e_1}\epsilon^\mathsf{T}Q\epsilon \leq -\frac{\partial\eta_1}{\partial e_1}\lambda_{\min}\{Q\}\|\epsilon\|_2^2, \tag{34}$$

and using the right-hand-side inequity of (33) and the definition of $\frac{\partial\eta_1}{\partial e_1}$ above, it results in

$$\begin{aligned}\dot{V}_{\epsilon1} &\leq -\frac{\partial\eta_1}{\partial e_1}\frac{\lambda_{\min}\{Q\}}{\lambda_{\max}\{P\}}\epsilon^\mathsf{T}P\epsilon\\ &\leq -\frac{\mu_1}{2}|e_1|^{-1/2}\frac{\lambda_{\min}\{Q\}}{\lambda_{\max}\{P\}}V_\epsilon - \mu_2 p|e_1|^{p-1}\frac{\lambda_{\min}\{Q\}}{\lambda_{\max}\{P\}}V_\epsilon.\end{aligned} \tag{35}$$

By computing the square root of the left-hand-side of (33) we get, $\|\epsilon\|_2 \leq \frac{V^{1/2}}{\lambda_{\min}^{1/2}\{P\}}$. Also, it is easy to verify that

$$\begin{aligned}|e_1|^{1/2} &\leq \|\epsilon\|_2\\ &= \sqrt{|\mu_1|e_1|^{1/2}\mathrm{sgn}(e_1) + \mu_2|e_1|^p\mathrm{sgn}(e_1)|^2 + |e_2|^2}\end{aligned} \tag{36}$$

holds. Then, $|e_1|^{1/2} \leq \|\epsilon\|_2 \leq \frac{V^{1/2}}{\lambda_{\min}^{1/2}\{P\}}$, which can be represented as $-|e_1|^{-1/2} \leq -\frac{\lambda_{\min}^{1/2}\{P\}}{V^{1/2}}$ by using the inequalities properties. From that, we can rewrite (35) as,

$$\dot{V}_{\epsilon1} \leq -\frac{\mu_1}{2}\frac{\lambda_{\min}^{1/2}\{P\}\lambda_{\min}\{Q\}}{\lambda_{\max}\{P\}}V_\epsilon^{1/2} - \mu_2 p|e_1|^{p-1}\frac{\lambda_{\min}\{Q\}}{\lambda_{\max}\{P\}}V_\epsilon. \tag{37}$$

We continue by computing $\dot{V}_{\epsilon2}$ in (32) as follows,

$$\begin{aligned}\dot{V}_{\epsilon2} &= 2\frac{\partial\eta_1}{\partial e_1}\epsilon^\mathsf{T}P[Ae_1 + \varpi|e_1|,\ 0]^\mathsf{T} = \left(2\frac{\partial\eta_1}{\partial e_1}\right)\\ &\quad\times(p_1Ae_1\eta_1 + p_1\varpi|e_1|\eta_1 + p_3Ae_1e_2 + p_3\varpi|e_1|e_2).\end{aligned} \tag{38}$$

then, we add and subtract the term e_2^2, and after arranging the common terms, it follows that,

$$\begin{aligned}\dot{V}_{\epsilon2} \leq &-\underbrace{\left((2p_1)(\bar{A} - \varpi)(\frac{\mu_1^2}{2}) - (p_3\varpi\mu_1)\right)}_{\tilde{r}}|e_1|\\ &-\underbrace{\left((2p_1)(\bar{A} - \varpi)(\mu_1\mu_2)(\frac{1}{2} + p)\right)}_{\tilde{s}}|e_1|^{p+\frac{1}{2}}\\ &-\underbrace{\left((2p_1)(\bar{A} - \varpi)(\mu_2^2p) - (p_3\varpi\mu_2 p)\right)}_{\tilde{t}}|e_1|^{2p} - e_2^2\\ &\underbrace{-(2p_3\bar{A})\left(\frac{\mu_1}{2}|e_1|^{\frac{1}{2}}\mathrm{sgn}e_1 + \mu_2 p|e_1|^p\mathrm{sgn}e_1\right)e_2}_{X\text{: cross terms}}\\ &+\underbrace{((p_3\varpi)(\mu_1 + \mu_2 p) + 1)e_2^2}_{Y\text{: positive terms}},\end{aligned} \tag{39}$$

where $\bar{A} \in \mathbb{R}_{>0}$ is the positive part of A, since A is Hurwitz. Now, let us arrange the negative terms of (39):

$$\begin{aligned}&-\tilde{r}|e_1| - \tilde{s}|e_1|^{p+\frac{1}{2}} + \tilde{t}|e_1|^{2p} - e_2^2\\ &\leq -\mu_1^2|e_1| - 2\mu_1\mu_2|e_1|^{p+\frac{1}{2}} - \mu_2^2|e_1|^{2p} - e_2^2 \leq -\frac{V_\epsilon}{\lambda_{\max}\{P\}},\end{aligned} \tag{40}$$

since it is clear that $\tilde{r} > \mu_1^2$, $\tilde{s} > 2\mu_1\mu_2$, and $\tilde{t} > \mu_2^2$. We take into account the cross terms X in (39) in the following analysis. Notice that X in (39) can be decomposed using the Young inequality as follows:

$$\begin{aligned}X &\leq p_3\bar{A}\mu_1|e_1|^{\frac{1}{2}}|e_2| + 2p_3\bar{A}\mu_2 p|e_1|^p|e_2|\\ &\leq (\frac{p_3}{2}\bar{A}\mu_1)|e_1| + (p_3\bar{A}\mu_2 p)|e_1|^{2p} + (p_3\bar{A})(\frac{\mu_1}{2} + \mu_2 p)e_2^2.\end{aligned} \tag{41}$$

Let us continue by computing $\dot{V}_{\epsilon3}$ in (32),

$$\begin{aligned}\dot{V}_{\epsilon3} &= 2\beta p_3|e_2|\eta_1 + 2\beta p_2|e_2|e_2\\ &\leq 2\beta p_3\mu_1|e_2||e_1|^{\frac{1}{2}} + 2\beta p_3\mu_2|e_2||e_1|^p + 2\beta p_2e_2^2\\ &\leq (2\beta p_3\mu_1)\left(\frac{e_2^2}{2} + \frac{|e_1|}{2}\right) + (2\beta p_3\mu_2)\left(\frac{e_2^2}{2} + \frac{|e_1|^{2p}}{2}\right)\\ &\quad + 2\beta p_2e_2^2\\ &\leq \beta((p_3)(\mu_1 + \mu_2) + 2p_2)e_2^2 + \beta p_3\mu_1|e_1| + \beta p_3\mu_2|e_1|^{2p}\end{aligned} \tag{42}$$

[2]By tuning the positive observer gains (l_1, l_2), the matrix A_ϵ is Hurwitz for all the possible values of B according to model Ω.

where we have applied the Young inequality to the above cross terms. Now, let us arrange the rest of the terms: Y in (39), the cross-terms X in (41) and the terms in (42):

$$X + Y + \dot{V}_{\epsilon 3} \leq \underbrace{(p_3\mu_1)(\beta + \frac{\bar{A}}{2})}_{\tilde{a}} |e_1| + \underbrace{(p_3\mu_2)(\beta + \bar{A}p)}_{\tilde{b}} |e_1|^{2p} + \underbrace{\begin{bmatrix}(\beta)((p_3)(\mu_1 + \mu_2) + 2p_2) + (p_3\bar{A})(\frac{\mu_1}{2} + \mu_2 p) \\ +((p_3\varpi)(\mu_1 + \mu_2 p) + 1)\end{bmatrix}}_{\tilde{c}} e_2^2. \quad (43)$$

In the above expression, one can choose some parameters to get $\max\{\tilde{a}, \tilde{b}, \tilde{c}\} = \tilde{c} > 1$, and, therefore, it holds that

$$\begin{aligned} X + Y + \dot{V}_{\epsilon 3} &\leq \tilde{a}|e_1| + \tilde{b}|e_1|^{2p} + \tilde{c}e_2^2 \\ &\leq \tilde{c}\Big(\mu_1^2|e_1| + 2\mu_1\mu_2|e_1|^{p+\frac{1}{2}} + \mu_2^2|e_1|^{2p} + e_2^2\Big) \\ &\leq \tilde{c}\|\epsilon\|_2^2 \leq \frac{\tilde{c}}{\lambda_{\min}\{P\}} V_\epsilon, \end{aligned} \quad (44)$$

as long as $\tilde{a} \leq \tilde{c}\mu_1$, $(p_3)(\beta + \bar{A}/2) \leq \tilde{c}\mu_1$, $\tilde{b} \leq \tilde{c}\mu_2$, and $(p_3)(\beta + \bar{A}p) \leq \tilde{c}\mu_2$, which can be achieved by tuning the observer gains and Q.

Finally, we put together the expressions (37), (40), and (44):

$$\begin{aligned} \dot{V}_\epsilon = &-\frac{\mu_1}{2}\frac{\lambda_{\min}^{1/2}\{P\}\lambda_{\min}\{Q\}}{\lambda_{\max}\{P\}} V_\epsilon^{1/2} - \mu_2 p|e_1|^{p-1}\frac{\lambda_{\min}\{Q\}}{\lambda_{\max}\{P\}} V_\epsilon \\ &- \frac{V_\epsilon}{\lambda_{\max}\{P\}} + \frac{\tilde{c}}{\lambda_{\min}\{P\}} V_\epsilon \end{aligned} \quad (45)$$

and if we choose the free parameter $p = 1$, then

$$\begin{aligned} \dot{V}_\epsilon = &-\underbrace{\left(\frac{\mu_1}{2}\frac{\lambda_{\min}^{1/2}\{P\}\lambda_{\min}\{Q\}}{\lambda_{\max}\{P\}}\right)}_{\psi} V_\epsilon^{1/2} \\ &- \underbrace{\left(\mu_2\frac{\lambda_{\min}\{Q\}}{\lambda_{\max}\{P\}} + \frac{1}{\lambda_{\max}\{P\}} - \frac{\tilde{c}}{\lambda_{\min}\{P\}}\right)}_{\omega} V_\epsilon \end{aligned} \quad (46)$$

where

$$\frac{\mu_2\lambda_{\min}\{Q\} + 1}{\lambda_{\max}\{P\}} \geq \frac{\tilde{c}}{\lambda_{\min}\{P\}}, \quad (47)$$

which can be easily achieved by tuning Q and (l_1, l_2) in A_ϵ. The convergence time can be computed following the same procedure as the previous Propositions. Thus, the convergence time results as given in the statement of Proposition 3. ■

V. Conclusion

In this letter, we proposed a novel control strategy that achieves to stabilize, in finite time, the error equilibrium of the disturbed generalized Bouc-Wen model. Besides, we assumed that we only know the parameters α, d_p in the hysteresis model. The rest of the hysteresis parameters are considered unknown.

Future work includes applying the proposed controller to a piezoelectric actuator given by a robotic hand.

Supplementary Material

Some simulation experiments that complement the work together with the MATLAB Simulink files can be found on: https://github.com/gfloresc/L-CSS-Bouc-Wen.

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