Notes on:

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Symbol	Definition

1 Brownian Motion

Let $B_t \in (-\infty, \infty)$ be an ordinary Brownian motion. This has the explicit exact solution as a Gaussian of variance t from the initial condition:

$$B_t \stackrel{d}{=} \mathcal{N}(B_0, t)$$
$$\mathbf{E}[B_t] = B_0$$
$$\mathbf{Var}[B_t] = t$$

(and we can also calculate sample paths, or the joint distribution for any collection of times $t_1, \ldots, t_n$ etc for B_t)

2 Ornstein-Unlenbeck Process with parameter θ

Define the OU process $O_t \in (-\infty, \infty)$ with a parameter θ by the SDE:

$$\mathrm{d}O_t = -\theta \mathrm{d}t + \mathrm{d}B_t$$

This has the explicit exact solution

$$O_t \stackrel{d}{=} \mathcal{N}\left(O_0 e^{-\theta t}, \frac{1}{2\theta}\left(1 - e^{-2\theta t}\right)\right)$$
$$\mathbf{E}[O_t] = O_0 e^{-\theta t}$$
$$\mathbf{Var}[O_t] = \frac{1}{2\theta}\left(1 - e^{-2\theta t}\right)$$

(and we can also calculate sample paths, or the joint distribution for any collection of times $t_1, \ldots, t_n$ etc for O_t)

Note that in the limit $t \to \infty$, the OU process approaches the steady state distribution

$$\lim_{t\to\infty} O_t \stackrel{d}{=} \mathcal{N}(0, \frac{1}{2\theta})$$

So by tuning θ, we can choose how narrow or how wide the steady state distribution is

3 Sigmoid-OU

See a plot of how this thing evolves here https://www.desmos.com/calculator/aihfujprtw

Define the process $S_t \in (0,1)$ by applying the sigmoid function to O_t. i.e. with $\sigma(x) = \frac{e^x}{1+e^x} \in (0,1)$

$$S_t = \sigma(O_t)$$

By Ito's lemma, the SDE for S_t is:

$$\begin{aligned}
\mathrm{d}S_t &= \sigma'(O_t)\mathrm{d}O_t + \frac{1}{2}\sigma''(O_t)\left(\mathrm{d}O_t\right)^2 \\
\text{(where we have used } \mathrm{d}B_t^2 = \mathrm{d}t) &= \sigma'(O_t)\left(-\theta O_t \mathrm{d}t + \mathrm{d}B_t\right) + \frac{1}{2}\sigma''(O_t)\mathrm{d}t \\
&= \left(-\theta O_t \sigma'(O_t) + \frac{1}{2}\sigma''(O_t)\right)\mathrm{d}t + \sigma'(O_t)\mathrm{d}B_t
\end{aligned}$$

This can be simplified by using $\sigma'(x) = \sigma(x)(1-\sigma(x))$ and $\sigma''(x) = \sigma(x)(1-\sigma(x))(1-2\sigma(x))$ to see that:

$$\mathrm{d}S_t = \left(-\theta\sigma^{-1}(S_t)S_t(1-S_t) + \frac{1}{2}S_t(1-S_t)(1-2S_t)\right)\mathrm{d}t + S_t(1-S_t)\mathrm{d}B_t$$

where $\sigma^{-1}(x) = \ln(\frac{x}{1-x})$ is the logit function.

Note also that we have the exact solution for S_t at any time t by pushing forward the exact solution of O_t , namely:

$$S_t \sim \sigma\left[\mathcal{N}\left(O_0 e^{-\theta t}, \frac{1}{2\theta}\left(1-e^{-2\theta t}\right)\right)\right]$$

Note that in the limit $t \to \infty$, S_t will settle to a distribution which is centered around $\frac{1}{2}$. This can be seen by using the taylor series expansion for small x, that

$$\begin{aligned}
\sigma(x) &\approx \sigma(0) + \sigma'(0)x + \frac{1}{2}\sigma''(0)x^2 + \frac{1}{6}\sigma^{(3)}(0)x^3 + \frac{1}{24}\sigma^{(4)}(0)x^4 + O(x^5) \\
&= \frac{1}{2} + \frac{1}{4}x + 0 + \frac{1}{6\cdot 8}x^3 + 0 + O(x^5)
\end{aligned}$$

And so:

$$\begin{aligned}
\lim_{t\to\infty} S_t &\sim \sigma\left[\mathcal{N}\left(0, \frac{1}{2\theta}\right)\right] \\
&= \sigma\left[\frac{1}{\sqrt{2\theta}}Z\right] \\
&\approx \frac{1}{2} + \frac{1}{4\sqrt{2\theta}}Z + \frac{1}{48\left(2\theta\right)^{3/2}}Z^3 + \dots
\end{aligned}$$

4 Notes on the Sigmoid-Gaussian distribution

where $G \sim \mathcal{N}(\mu, v)$,$G = \mu + \sqrt{v}Z$, $Z \sim \mathcal{N}(0,1)$ be a Gaussian of mean μ and variance v. Lets examine the law of $\sigma(Z)$. The CDF is:

$$\begin{aligned}
CDF(x) = \mathbf{P}\left(\sigma\left[G\right] \le x\right) &= \mathbf{P}\left(Z \le \frac{\sigma^{-1}(x)-\mu}{\sqrt{v}}\right) \\
&= \frac{1}{2} + \frac{1}{2}erf\left(\frac{\sigma^{-1}(x)-\mu}{\sqrt{2v}}\right)
\end{aligned}$$

And the pdf is obtained by taking the derivative:

$$\begin{aligned}
PDF(x) &= CDF(x)' \\
&= \rho\left(Z = \frac{\sigma^{-1}(x)-\mu}{\sqrt{v}}\right)\frac{1}{\sqrt{v}}\left(\sigma^{-1}(x)\right)' \\
&= \frac{1}{\sqrt{2\pi v}x(1-x)}\exp\left(-\frac{\left(\sigma^{-1}(x)-\mu\right)^2}{2v}\right)
\end{aligned}$$

where we used $\sigma^{-1}(x) = \ln\left(\frac{x}{1-x}\right) = \ln(x) - \ln(1-x)$ so its derivative is $\left[\sigma^{-1}(x)\right]' = \frac{1}{x} + \frac{1}{1-x} = \frac{1}{x(1-x)}$
see https://www.desmos.com/calculator/aihfujprtw for plotting this thing.