A recursive parameterisation of unitary matrices

C. Jarlskog

Division of Mathematical Physics
LTH, Lund University
Box 118, S-22100 Lund, Sweden

Abstract

A simple recursive scheme for parameterisation of n-by-n unitary matrices is presented. The n-by-n matrix is expressed as a product containing the (n-1)-by-(n-1) matrix and a unitary matrix that contains the additional parameters needed to go from n-1 to n. The procedure is repeated to obtain recursion formulas for n-by-n unitary matrices.

1 The parameterisation

It has been known since a long time that unitary transformations play a central role in physics. An excellent example is Wigner's paper of 1939 [1] which has had a great impact in the development of physics and is still important.

It is also a known fact that a general n-by-n unitary matrix $X^{(n)}$ may be expressed as a product of three unitary matrices,

$$X^{(n)} = \Phi^{(n)}(\vec{\alpha}) V^{(n)} \Phi^{(n)}(\vec{\beta}) \tag{1}$$

where the matrices Φ are diagonal unitary matrices,

$$\Phi^{(n)}(\vec{\alpha}) = \begin{pmatrix} e^{i\alpha_1} & & & & \\ & e^{i\alpha_2} & & & \\ & & . & & \\ & & & . & \\ & & & & e^{i\alpha_n} \end{pmatrix} \tag{2}$$

$\Phi(\vec{\beta})$ is defined analogously; the α's and β's being real. The matrix $X^{(n)}$ has n^2 real parameters. In the following, for simplicity, the word parameter stands for real parameter. The quantities $\vec{\alpha}$ and $\vec{\beta}$ take care of $2n-1$ parameters of $X^{(n)}$ because only the sums $\alpha_i + \beta_j$ enter, where i and j run from 1 to n. The remaining $(n-1)^2$ parameters reside in the non-trivial matrix $V^{(n)}$ which is the subject of this study.

We start by putting $V^{(1)} = 1$, whereby $X^{(1)} = e^{i(\alpha_1+\beta_1)}$ is the most general one-by-one unitary "matrix". For $n \geq 2$, we write the matrix $V^{(n)}$ in the form

$$V^{(n)} = \begin{pmatrix} V^{(n-1)} + (1-c_n)|A^{(n-1)}><B^{(n-1)}| & s_n|A^{(n-1)}> \\ s_n<B^{(n-1)}| & c_n \end{pmatrix} \tag{3}$$

Here we have introduced an angle denoted by θ_n and have used the common notation $s_n = sin\theta_n$, $c_n = cos\theta_n$. The complex vectors "$A^{(n-1)}$" and "$B^{(n-1)}$" have each n-1 components, i.e.,

$$|A^{(n-1)}>= \begin{pmatrix} a_1^{(n-1)} \\ a_2^{(n-1)} \\ . \\ . \\ a_{n-1}^{(n-1)} \end{pmatrix}, \quad |B^{(n-1)}>= \begin{pmatrix} b_1^{(n-1)} \\ b_2^{(n-1)} \\ . \\ . \\ b_{n-1}^{(n-1)} \end{pmatrix} \tag{4}$$

Furthermore,

$$< B^{(n-1)}| = (b_1^{(n-1)\star}, b_2^{(n-1)\star}, \ . \ ., b_{n-1}^{(n-1)\star}) \tag{5}$$

and

$$(|A^{(n-1)} >< B^{(n-1)}|)_{ij} \equiv a_i^{(n-1)} b_j^{(n-1)\star} \tag{6}$$

$A^{(n-1)}$ and $B^{(n-1)}$ are not arbitrary but are required to satisfy the conditions

$$< A^{(n-1)}|A^{(n-1)} >= 1, \quad |B^{(n-1)} >= -V^{(n-1)\dagger}|A^{(n-1)} > \tag{7}$$

whereby

$$< B^{(n-1)}|B^{(n-1)} >= 1, \quad |A^{(n-1)} >= -V^{(n-1)}|B^{(n-1)} > \tag{8}$$

We can easily check that if the matrix $V^{(n-1)}$, in Eq.(3), is unitary so is $V^{(n)}$. In order for $V^{(n)}$ to be the most general n-by-n unitary matrix, modulus the phase matrices Φ, it must have the required number of parameters. The vector $A^{(n-1)}$, having $n-1$ complex components, would seem to represent $2(n-1)$ parameters. But it has only $2(n-2)$ because it is normalised and its overall phase can be absorbed into the matrices Φ, i.e., the transformation

$$|A^{(n-1)} > \ \ \rightarrow e^{i\eta}|A^{(n-1)} > \tag{9}$$

yields

$$|B^{(n-1)} > \ \ \rightarrow e^{i\eta}|B^{(n-1)} > \tag{10}$$

and

$$V^{(n)} \rightarrow \Phi(0, 0, ..., e^{-i\eta})V^{(n)}\Phi(0, 0, ..., e^{i\eta}) \tag{11}$$

The parameter counting, therefore, goes as follows. On the LHS of Eq.(3) we need to have $(n-1)^2$ parameters. On the RHS, we have $(n-2)^2$ from $V^{(n-1)}$ and $2(n-2)$ from the vector $A^{(n-1)}$ Thus, together with the angle θ_n, the number of parameters is $(n-2)^2 + 2(n-2) + 1$ which equals $(n-1)^2$ as required.

We may use the relation (7) between $A^{(n-1)}$ and $B^{(n-1)}$ to rewrite the matrix $V^{(n)}$ in terms of either $A^{(n-1)}$ or $B^{(n-1)}$. In terms of $A^{(n-1)}$, we have

$$\begin{aligned} V^{(n)} &= \begin{pmatrix} 1-(1-c_n))|A^{(n-1)} >< A^{(n-1)}| & s_n|A^{(n-1)} > \\ -s_n < A^{(n-1)}| & c_n \end{pmatrix} \begin{pmatrix} V^{(n-1)} & 0 \\ 0 & 1 \end{pmatrix} \\ &\equiv A_{n,n-1} \begin{pmatrix} V^{(n-1)} & 0 \\ 0 & 1 \end{pmatrix} \end{aligned} \tag{12}$$

While writing the matrix in terms of $B^{(n-1)}$ yields

$$\begin{aligned} V^{(n)} &= \begin{pmatrix} V^{(n-1)} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1-(1-c_n))|B^{(n-1)}><B^{(n-1)}| & -s_n|B^{(n-1)}> \\ s_n<B^{(n-1)}| & c_n \end{pmatrix} \\ &\equiv \begin{pmatrix} V^{(n-1)} & 0 \\ 0 & 1 \end{pmatrix} B_{n,n-1} \end{aligned} \tag{13}$$

These relations allow for a systematic construction of unitary matrices order by order. By repeating the above procedure for the matrix $V^{(n-1)}$ in terms of $A^{(n-2)}$ and $B^{(n-2)}$, and following down the chain we find the recursion formulas that we are looking for,

$$V^{(n)} = A_{n,n-1}A_{n,n-2}...A_{n,2}A_{n,1} \tag{14}$$

$$V^{(n)} = B_{n,1}B_{n,2}...B_{n,n-2}B_{n,n-1} \tag{15}$$

The matrices $A_{n,n-1}$ and $B_{n,n-1}$ were previously defined in Eqs.(12) and (13). For $j < n-1$ we have

$$A_{n,j} = \begin{pmatrix} \begin{pmatrix} 1-(1-c_{j+1}))|A^{(j)}><A^{(j)}| & s_{j+1}|A^{(j)}> \\ -s_{j+1}<A^{(j)}| & c_{j+1} \end{pmatrix} & 0 \\ 0 & I_{n-j-1} \end{pmatrix} \tag{16}$$

$$B_{n,j} = \begin{pmatrix} \begin{pmatrix} 1-(1-c_{j+1}))|B^{(j)}><B^{(j)}| & -s_{j+1}|B^{(j)}> \\ s_{j+1}<B^{(j)}| & c_{j+1} \end{pmatrix} & 0 \\ 0 & I_{n-j-1} \end{pmatrix} \tag{17}$$

Here I_{n-j-1} is the unit matrix of order $n-j-1$. The two unitary matrices $A_{n,j}$ and $B_{n,j}$ are related by

$$A_{n,j} = \begin{pmatrix} V^{(j)} & 0 \\ 0 & I_{n-j} \end{pmatrix} B_{n,j} \begin{pmatrix} V^{(j)\dagger} & 0 \\ 0 & I_{n-j} \end{pmatrix} \tag{18}$$

2 Simple examples

The simplest case is $n = 2$ for which we take $|A^{(1)}>= 1$ whereby $|B^{(1)}>= -1$. Using $V^{(1)} = 1$ we obtain, from Eqs.(12) and (13)

$$V^{(2)} = A_{2,1} = B_{2,1} = \begin{pmatrix} c_2 & s_2 \\ -s_2 & c_2 \end{pmatrix} \tag{19}$$

This is the familiar rotation matrix $R_2(\theta_2)$, θ_2 being the rotation angle in two dimensions.

The next simplest case is $n = 3$ for which we may either use the "mixed form" or the pure forms. For the mixed form we have

$$V^{(3)} = \begin{pmatrix} R_2(\theta) + (1-c_3)|A^{(2)}><B^{(2)}| & s_3|A^{(2)}> \\ s_3<B^{(2)}| & c_3 \end{pmatrix} \tag{20}$$

Here $R_2(\theta)$ is again the rotation matrix in Eq.(19) and we may put

$$|A^{(2)}>= \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}, \quad |B^{(2)}>= \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} \tag{21}$$

From Eq.(7) follows that

$$|A^{(2)}>= -R_2(\theta)|B^{(2)}>, \quad |B^{(2)}>= -R_2(-\theta)|A^{(2)}> \tag{22}$$

Hence these vectors represent two parameters, for example

$$|A^{(2)}>= \begin{pmatrix} cos\gamma \\ sin\gamma e^{i\delta} \end{pmatrix} \tag{23}$$

where γ and δ are real. The pure forms are also obtained very simply, for example

$$V^{(3)} = \begin{pmatrix} 1-(1-c_3)|A^{(2)}>< A^{(2)}| & s_3|A^{(2)}> \\ -s_3 < A^{(2)}| & c_3 \end{pmatrix} \begin{pmatrix} R_2(\theta_2) & 0 \\ 0 & 1 \end{pmatrix} \tag{24}$$

where $A^{(2)}$ is as defined in Eq.(23). Evidently, depending on the application one has in mind some choices may be more convenient than others. This is demonstrated in Refs.[2] and [3] which deal with the so called quark and lepton mixing matrices. The essential point is that $V^{(3)}$ is described in a rather simple fashion by four parameters as it should be. For the case of $n = 4$ we could, for example, take $A^{(3)}$ to be

$$|A^{(3)}>= \begin{pmatrix} cos\rho \\ sin\rho cos\sigma e^{i\delta_1} \\ sin\rho sin\sigma e^{i\delta_2} \end{pmatrix} \tag{25}$$

where ρ, σ, δ_1 and δ_2 are the four parameters needed to define the most general $A^{(3)}$.

In principle, the above recursive procedure may "easily" be extended to much larger n with the help of computers.

References

[1] E. P. Wigner, Ann. Math. **40** (1939) 149; see also E. P. Wigner, "Symmetries and Reflections" (MIT Press 1970)

[2] C. Jarlskog, hep-ph/0503199, to appear in Phys. Lett. B

[3] C. Jarlskog, hep-ph/0504012