Notes on: angular spectrum with attenuation and dispersion and flux-conservative shock wave propagation

Gianmarco Pinton[1]

[1] Joint Department of Biomedical Engineering at the University of North Carolina at Chapel Hill and North Carolina State University, Chapel Hill, North Carolina, USA.

Objective: to introduce the equations, numerical methods, and code usage for the angular spectrum code with frequency-dependent attenuation and dispersion and flux-conservative shock wave propagation capabilities.

1 Description of code

The sample code is based around the propagation of an imaging pulse that is propagating in homogeneous tissue. Here we scanned a clinical imaging transducer with a hydrophone with a distance of 2mm between the hydrophone and the trasducer face. The scan plane was parallel to the transducer face and the recording was performed as a function of time (Fig.1)

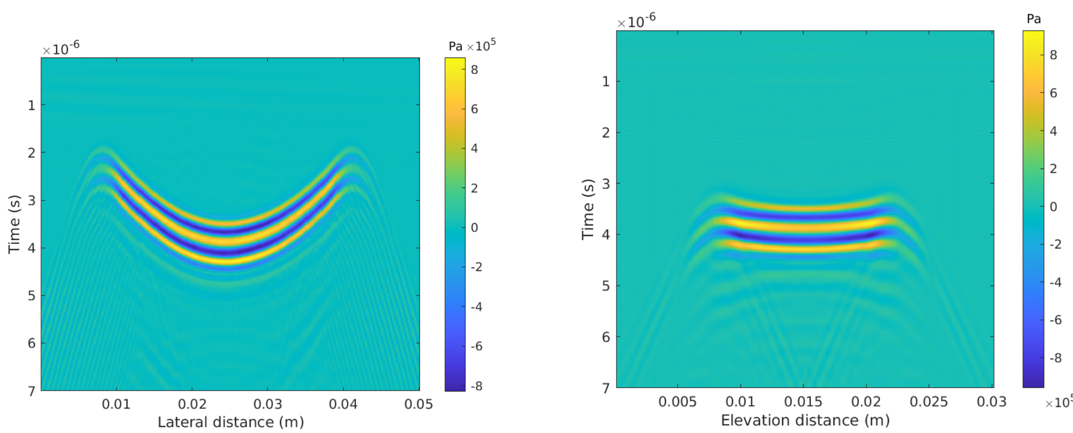

Figure 1: Sections of a 2D space, 1D time hydrophone scan of an imaging pulse emitted by a clinical transducer. Lateral plane (left) and elevational plane (right).

The sampling grid of the hydrophone measurements generally differs from the native simulation grid sampling in both space and time. The hydrophone measurement is thus interpolated to fit the numerical grid. first in time, using a 1D interpolation function, `interp1easy.m`, and then using a 2D interpolation function in space, `interp2easy.m`. A hydrophone scan is not required to run the code. You can, for example, use analytical representations of different acoustical fields.

The grid spacing is defined in terms of the characteristic wavelength, λ, or period, T. By default the simulation `launch_asr3.m` has a grid size of $\lambda/5$ in the $x-y$ plane and 1.6λ in z, the propagation dimension. The time step is by default $T/25$. Smaller grid sizes in space and time will have broader spectral supports that allows the resolution of higher nonlinearities by the simulation. It will also lengthen the computational time.

The propagation, attenuation, and absorbing boundary layer operators are pre-calculated for a given grid size. The propagation operator is pre-calculated in the `precalculate_mas` and it is based on the angular spectrum method. Its output is a propagation matrix `HH` which is then used in the Fourier domain multiplication.

The attenuation can be calculated using two functions. In the first, simpler case, `precalculate_ad` the attenuation is proportional to the f^2, as you would expect in thermoviscous fluids such as water. In this case there is no dispersion. The attenuation coefficient has units of dB/MHz^2/cm. In the second case, `precalculate_ad_pow2`, the attenuation law is assumed to be a power law which has an attenuation/dispersion relationship determined by the Kramers-Kronig relation. The definition of this power law is meant to be conformal to the ultrasound literature. The power in the power law is defined by the variable `pow` and the attenuation is determined by the coefficient α_0, which has units dB/MHz$^{\text{pow}}$/cm. For example,$\alpha_0 = 0.3$ and `pow`=1 is equivalant to a 0.3 dB/MHz/cm attenuation law. The function will automatically calculate the Kramers Kronig dispersion based on the definition of attenuation and there is no need to calculate it explicitly. In both cases the output of the function is a complex-valued filter where the real part corresponds to attenuation as a function of frequency and the complex part corresponds the the dispersion as a function of freqeuncy.

Absorbing boundary layers are calculated in `precalculate_abl` using a simple approach. The attenuation with the outer 1/5 of the simulation domain has an attenuation that increases as a quadratic function (Fig. 2). This attenuation boundary layered is multiplied with all three dimensions of the pressure field for each propagation step.

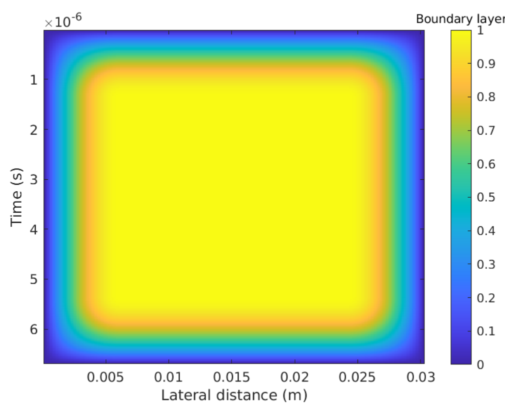

Figure 2: Absorbing boundary layer.

2 Description of equations

Acoustic propagation was described in 3D and included the effects of nonlinearity and attenuation. The propagation equations were solved numerically using the angular spectrum approach in combination with a Rusanov method for flux-conservative shock propagation.

The acoustic pressure can be defined by a pseudo-potential ϕ, which has advantages for the continuity through shocks [1, 2]

$$p(\vec{x}) = \frac{\partial \phi}{\partial t}(\vec{x}) \tag{1}$$

where p is the acoustical pressure. Wave propagation is described in retarded time, $\tau = t - x/\bar{c_0}$, which represents a time frame moving with the mean sound speed, $\bar{c_0}$, in the x-direction. Wave propagation equation can then be written as

$$\frac{\partial \phi}{\partial x}(\vec{x}, \tau) = D\phi(\vec{x}, \tau) + H\phi(\vec{x}, \tau) + N\phi(\vec{x}, \tau) + A\phi(\vec{x}, \tau) \tag{2}$$

The operator D describes diffraction:

$$D\phi(\vec{x}, \tau) = \frac{\bar{c_0}}{2} \int_{-\infty}^{\tau} \left(\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} \right) dt \tag{3}$$

and equation $\frac{\partial \phi}{\partial x}(\vec{x}, \tau) = D\phi(\vec{x}, \tau)$ is thus the 3D wave equation in a constant speed of sound medium with no attenuation in a reference frame that moves with the wave in the x-direction. The nonlinear operator N describes quadratic nonlinearity

$$N\phi(\vec{x}, \tau) = \frac{\beta}{2\bar{\rho}\bar{c_0}^3} \left(\frac{\partial \phi}{\partial t} \right)^2 \tag{4}$$

The attenuation operator A can describes empirical attenuation and dispersion laws in the frequency domain that follow the Kramers Kronig causality relations [3], which allows for a more flexible and accurate representation of the varyious power law attenuations that are observed in different soft tissues. The implementation of this attenuation and dispersion is performed numerically in the Fourier domain using a filtering approach [4].

The numerical solutions of these equations are based on a second order Strang splitting [5]

$$\phi(x + \Delta x, y, z, \tau) = \phi^N_{\frac{\Delta x}{2}} \circ \phi^D_{\frac{\Delta x}{2}} \circ \phi^{H+A}_{\Delta x} \circ \phi^D_{\frac{\Delta x}{2}} \circ \phi^N_{\frac{\Delta x}{2}} \circ (x, y, z, \tau) + \mathcal{O}(\Delta x^2) \tag{5}$$

which allows each term to be solved using largely independent numerical methods. The diffraction term is solved using the angular spectrum, the nonlinear term is solved using a flux-conservative Rusano approach [6], and the attenuation and dipsersion is solved empirically as a Fourier domain filter [4].

References

[1] Wallace D Hayes and Harry L Runyan Jr. Sonic-boom propagation through a stratified atmosphere. *The Journal of the Acoustical Society of America*, 51(2C):695–701, 1972.

[2] David Luquet, Régis Marchiano, and François Coulouvrat. Long range numerical simulation of acoustical shock waves in a 3d moving heterogeneous and absorbing medium. *Journal of Computational Physics*, 379:237–261, 2019.

[3] Matthew O'Donnell, ET Jaynes, and JG Miller. Kramers–kronig relationship between ultrasonic attenuation and phase velocity. *The Journal of the Acoustical Society of America*, 69(3):696–701, 1981.

[4] Bruno Giammarinaro, François Coulouvrat, and Gianmarco Pinton. Numerical simulation of focused shock shear waves in soft solids and a two-dimensional nonlinear homogeneous model of the brain. *Journal of biomechanical engineering*, 138(4), 2016.

[5] Gilbert Strang. On the construction and comparison of difference schemes. *SIAM journal on numerical analysis*, 5(3):506–517, 1968.

[6] David Espíndola, Stephen Lee, and Gianmarco Pinton. Shear shock waves observed in the brain. *Physical Review Applied*, 8(4):044024, 2017.