Turbocharging Monte Carlo pricing under rough volatility

Mikko Pakkanen

Department of Mathematics, Imperial College London, UK

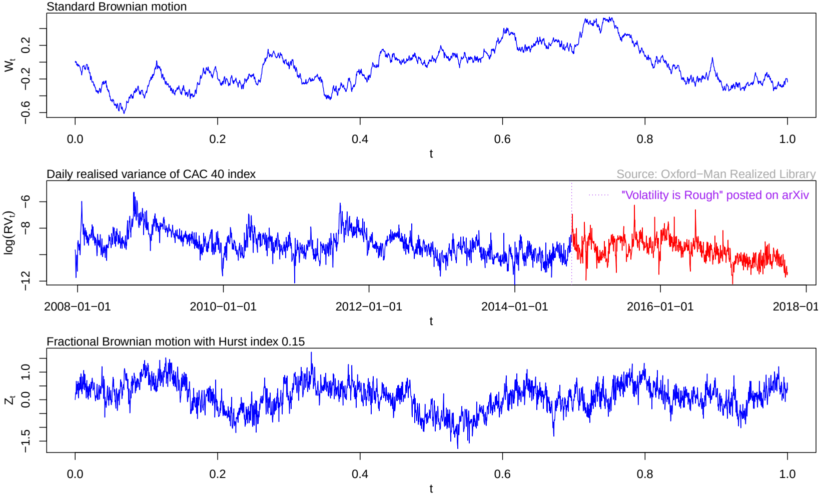
Jim Gatheral's 60th Birthday Conference
Courant Institute, New York, 14 October 2017

Joint work with Ryan McCrickerd

Imperial College London

Imperial Network of Excellence in Probabilistic Methods and Modelling

Volatility is (still) rough — 3rd anniversary!



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where W and $W^{\perp}$ are independent Brownians and $\rho \in [-1, 1]$. The spot variance V_t is a product $V_t = \xi_0(t)\mathcal{E}(\eta W^{\alpha})_t$ of

- the forward variance curve $t \mapsto \xi_0(t)$, known at time 0,
- the Wick exponential $\mathcal{E}(\eta W^{\alpha})_t = \exp\left(\eta W_t^{\alpha} - \frac{1}{2}\mathbf{Var}[\eta W_t^{\alpha}]\right)$ of a parameter $\eta > 0$ times a Gaussian random variable W_t^{α}.

The rough Bergomi model (cont.)

The random variable W_t^α follows the Gaussian Riemann–Liouville process

$$W_t^\alpha = \sqrt{2\alpha + 1} \int_0^t (t-s)^\alpha dW_s, \quad t \geq 0,$$

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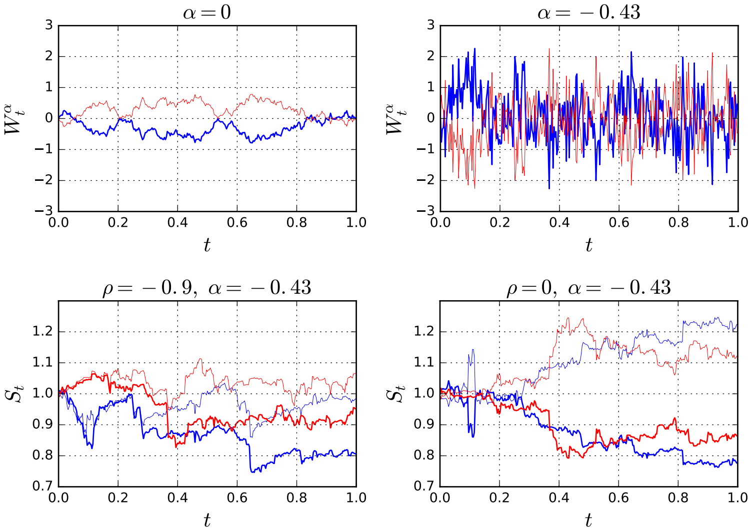
The paths of W^α have Hölder regularity $\alpha + \frac{1}{2}$ and locally look like the paths of a fractional Brownian motion with

$$H = \alpha + \frac{1}{2}.$$

Example: rough Bergomi paths

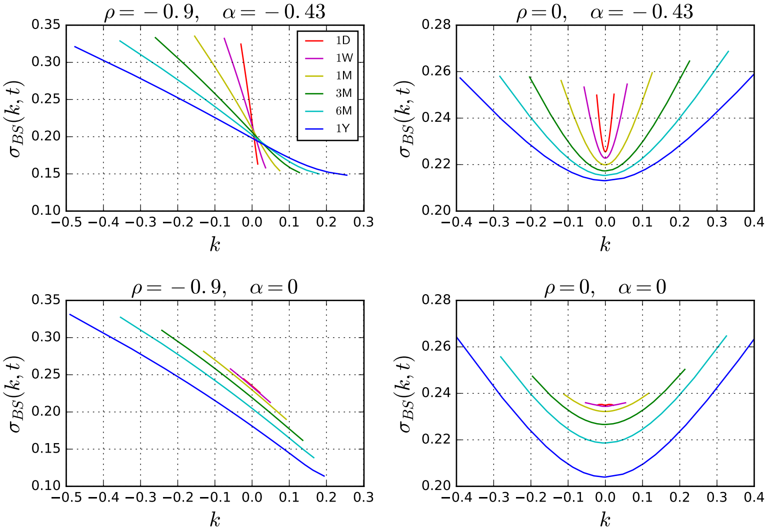


Intuition on the parameters

The rough Bergomi model has three time-homogeneous parameters, α, η, and ρ, with the following interpretations in terms of the implied volatility surface:

- η — smile,
- ρ — skew,
- α — near-maturity explosion (of smile and skew).

Example: rough Bergomi smiles



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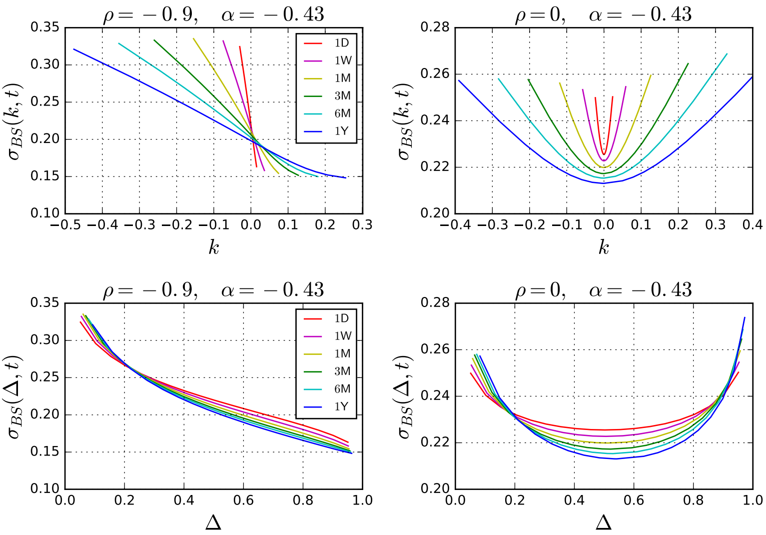

Pricing under rough volatility

Simulating rough volatility

Variance reduction methods

Pricing by Monte Carlo

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Thus it is worthwhile to try to optimise, “turbocharge”, Monte Carlo pricing as much as possible.

In general, a good Monte Carlo pricer should have:

- low bias, to avoid systematic error,
- low variance, such that good accuracy can be achieved in reasonable runtime.

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The covariance structure of (B, W^α) is not difficult to work out, so we could simulate exactly

$$\boldsymbol{X} := ((B_0, W^\alpha_0), (B_{1/n}, W^\alpha_{1/n}), (B_{2/n}, W^\alpha_{2/n}), \ldots, (B_{\lfloor nt \rfloor/n}, W^\alpha_{\lfloor nt \rfloor/n}))$$

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The Cholesky factor needs to be computed only once, but subsequent realisations of $\boldsymbol{X}$ still take $\mathcal{O}(n^2)$ flops.

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$$W^\alpha_{i/n} = \sum_{k=1}^{i} \int_{\frac{i}{n}-\frac{k}{n}}^{\frac{i}{n}-\frac{k-1}{n}} \left(\tfrac{i}{n}-s\right)^\alpha dW_s \approx \sum_{k=1}^{i} \left(\tfrac{k}{n}\right)^\alpha \left(W_{\frac{i}{n}-\frac{k-1}{n}} - W_{\frac{i}{n}-\frac{k}{n}}\right) =: \widehat{W}^{\alpha,n}_{i/n}.$$

Since $\widehat{W}^{\alpha,n}_{i/n}$ is a discrete convolution, $\widehat{W}^{\alpha,n}_{0}, \widehat{W}^{\alpha,n}_{1/n}, \ldots, \widehat{W}^{\alpha,n}_{\lfloor nt \rfloor/n}$ can be generated (using FFT) in $\mathcal{O}(n \log n)$ flops.

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However:

- Forward Riemann sums are inaccurate since the integrand $s \mapsto \left(\frac{i}{n} - s\right)^\alpha$ has a singularity.
- This leads to biased estimates of implied volatility.

The hybrid scheme

The hybrid scheme of Bennedsen, Lunde, and Pakkanen (2017) fixes the deficiencies of forward Riemann sums by using:

$$\widetilde{W}^{\alpha,n}_{i/n} := \underbrace{\sum_{k=1}^{\kappa} \int_{\frac{i}{n}-\frac{k}{n}}^{\frac{i}{n}-\frac{k-1}{n}} \left(\tfrac{i}{n}-s\right)^{\alpha} dW_s}_{\text{exact for } \kappa \text{ slices}} + \underbrace{\sum_{k=\kappa+1}^{i} \left(\tfrac{b_k}{n}\right)^{\alpha} \left(W_{\frac{i}{n}-\frac{k-1}{n}} - W_{\frac{i}{n}-\frac{k}{n}}\right)}_{\text{Riemann sum for the rest}},$$

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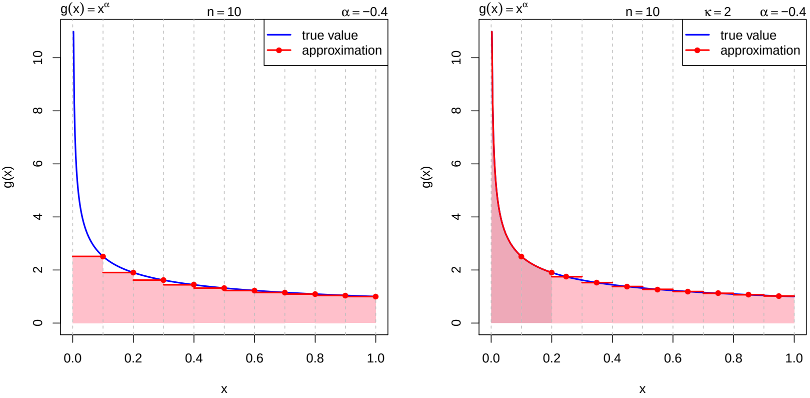
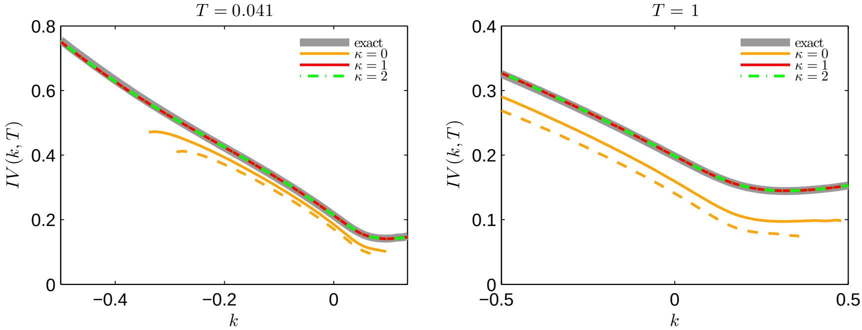
Again, this requires only $\mathcal{O}(n \log n)$ flops.

Approximating $x \mapsto x^{\alpha}$



Forward Riemann sums vs. the hybrid scheme

Numerical results: implied volatility smiles



Solid/patterned line: optimal b_k
Dashed line: $b_k = k$.

S_0	$\xi_0(t)$	η	α	ρ	n	paths
1	0.235^2	1.9	−0.43	−0.9	500	10^6

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To this end, we work with price estimators of the form

$$\hat{P}_n(k,t) := \frac{1}{n}\sum_{i=1}^{n}(X_i - \hat{\alpha}_n Y_i) - \hat{\alpha}_n \mathbf{E}[Y],$$

where $(X_1, Y_1), \ldots, (X_n, Y_n)$ are identical copies of a random vector (X, Y) and $\hat{\alpha}_n$ is a free parameter, to be defined shortly.

Base estimator

Our reference estimator, which we call the Base estimator, uses

$$X := f(S_t) := \begin{cases} (S_t - S_0 e^k)^+, & k \geq 0, \\ (S_0 e^k - S_t)^+, & k < 0, \end{cases}$$

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Without loss of generality, assume $S_0 = 1$.

Mixing formula

The well-known result of Romano and Touzi (1997) implies that

$$\mathbf{E}[f(S_t)] = \mathbf{E}\left[\mathrm{BS}\left((1-\rho^2)\int_0^t V_s ds, S_t^W, k\right)\right],$$

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This suggests that we could use

$$X := \mathrm{BS}\left((1-\rho^2)\int_0^t V_s ds, S_t^W, k\right).$$

This method alone is rather effective in reducing variance when $\rho \approx 0$, but its benefits evaporate as $\rho \to -1$.

Control variate

Inspired by the idea of Bergomi (2016) of using a timer option as a control variate, we choose

$$Y := \mathrm{BS}\left(\rho^2\left(\hat{Q}_n - \int_0^t V_s ds\right), S_t^W, k\right),$$

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By a martingale argument,

$$\mathbf{E}[Y] = \mathrm{BS}(\rho^2 \hat{Q}_n, 1, k)$$

Mixed estimator

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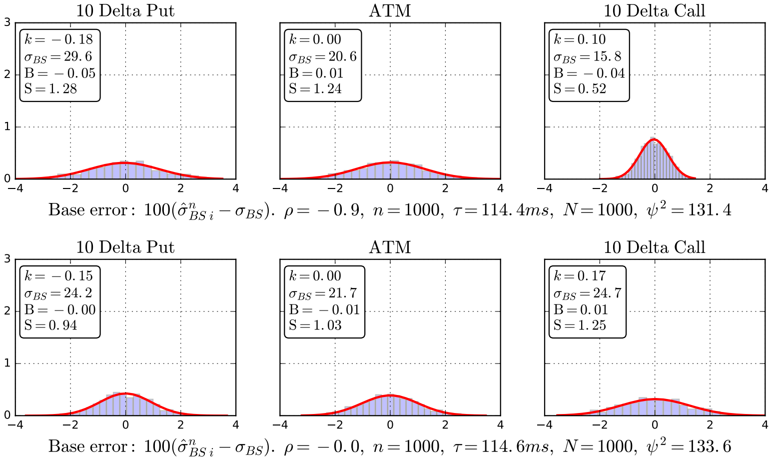
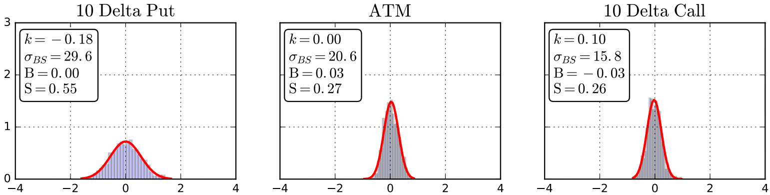
Additionally, we couple (X_{2i-1}, Y_{2i-1}) and (X_{2i}, Y_{2i}) for any $i \geq 1$ by using antithetic pairs (B, W) and $(-B, -W)$ as drivers.

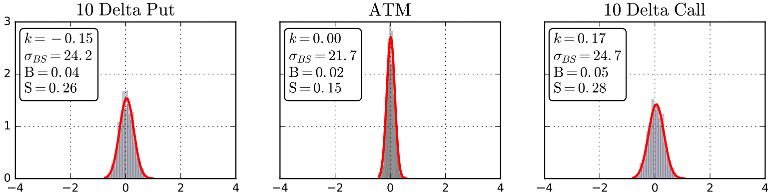
Numerical results: Base estimator



Numerical results: Mixed estimator



Mixed error: $100(\hat{\sigma}^n_{BS\,i}-\sigma_{BS})$. $\rho=-0.9$, $n=1000$, $\tau=70.6ms$, $N=1000$, $\psi^2=10.4$



Mixed error: $100(\hat{\sigma}^n_{BS\,i}-\sigma_{BS})$. $\rho=-0.0$, $n=1000$, $\tau=69.8ms$, $N=1000$, $\psi^2=3.9$

Calibration experiment

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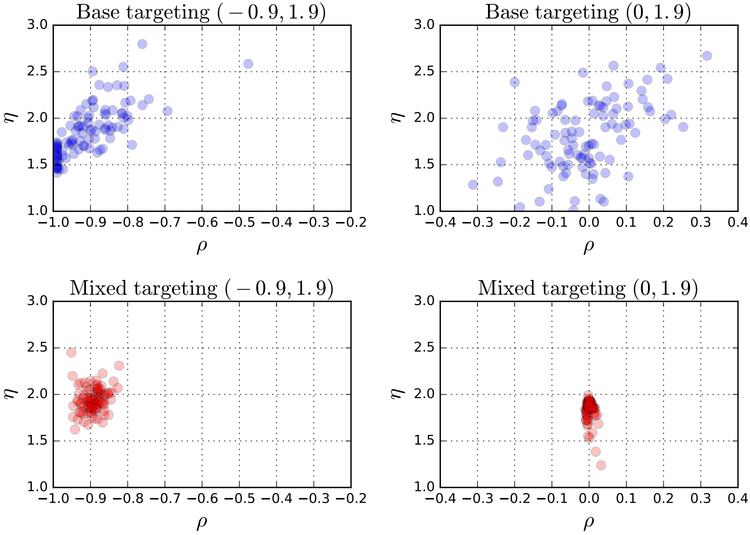
Throughout the experiment, we use $n = 1\,000$.

Numerical results: calibration experiment



Some alternative variance reduction methods

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- Importance sampling — seems unattractive as it would need to be tuned strike by strike.
- Quasi Monte Carlo (Sobol sequences etc) — applicable and useful here, albeit the speed-up appears not to be dramatic.

References

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Implementation

Python implementation of turbocharged pricing along with a Jupyter notebook are available from:

https://github.com/ryanmccrickerd/roughbergomi

Finally...

Happy birthday Jim!