Technical University of Crete
School of Electrical and Computer Engineering
Course: **Optimization**
Exercise 1 (80/1000)
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In this exercise, we will study simple concepts from Calculus of Several Variables (Taylor expansions), Convex Sets, and Convex Functions (Note: $\mathbb{R}_+ = \{x \in \mathbb{R} \,|\, x \geq 0\}$, $\mathbb{R}_{++} = \{x \in \mathbb{R} \,|\, x > 0\}$).

`You must prepare an electronic (LaTeX) report and upload the pdf file.`

1. (0) Let $f : \mathbb{R}_+ \to \mathbb{R}$, with $f(x) = \frac{1}{1+x}$. Let $x_0 \in \mathbb{R}_+$, and define the first- and second-order Taylor approximations of f at x_0 as

$$\begin{aligned} f_{(1)}(x) &= f(x_0) + f'(x_0)(x - x_0), \\ f_{(2)}(x) &= f(x_0) + f'(x_0)(x - x_0) + \frac{1}{2} f''(x_0)(x - x_0)^2. \end{aligned} \tag{1}$$

   (a) Find the analytic expressions for functions f' and f'';

   (b) (0) Draw in a common plot $f(x)$, $f_{(1)}(x)$ and $f_{(2)}(x)$ and, in order to understand the behavior of the approximations, consider various values of x_0.

2. (0) Let $f : \mathbb{R}_+^2 \to \mathbb{R}$, with $f(x_1, x_2) = \frac{1}{1+x_1+x_2}$.

   (a) (0) Compute and plot, via `mesh`, f for $x_1, x_2 \in [0, x_*]$, with $x_* > 0$.

   (b) (0) Plot the level sets of f, via `contour`. What do you observe? Can you explain the phenomenon?

   (c) (0) Compute the first- and second-order Taylor approximations of f at point $\mathbf{x}_0 = (x_{0,1}, x_{0,2})$.

   (d) (0) Draw on a common plot f and its first-order Taylor approximation.

   (e) (0) Draw on a common plot f and its second-order Taylor approximation.

3. (20) Let $\mathbb{S}_{\mathbf{a},b} = \{\mathbf{x} \in \mathbb{R}^n \,|\, \mathbf{a}^T\mathbf{x} \leq b\}$.

   (a) (5) Prove that $\mathbb{S}_{\mathbf{a},b}$ is convex.

   (b) (15) Prove that $\mathbb{S}_{\mathbf{a},b}$ is *not* affine (a counterexample is sufficient).

4. (10) Find the point $\mathbf{x}_*$ that is co-linear with $\mathbf{a}$ and lies on the hyperplane $\mathbb{H}_{\mathbf{a},b} := \{\mathbf{x} \in \mathbb{R}^n \,|\, \mathbf{a}^T\mathbf{x} = b\}$.

5. (20) Check whether the following functions are convex or not (using, for example, the second derivative rule).

   (a) (5) $f : \mathbb{R}_+ \to \mathbb{R}$, with $f(x) = \frac{1}{1+x}$;

   (b) (5) $f : \mathbb{R}^2_+ \to \mathbb{R}$, with $f(x_1, x_2) = \frac{1}{1+x_1+x_2}$;

   (c) (5) $f : \mathbb{R}_{++} \to \mathbb{R}$, with $f(x) = x^a$, for (to get a better feeling, plot function x^a, for various values of a)

      i. $a \geq 1$ and $a \leq 0$;

      ii. $0 \leq a \leq 1$.

   (d) (5) $f_1, f_2 : \mathbb{R}^n \to \mathbb{R}$, with $f_1(\mathbf{x}) = \|\mathbf{x}\|_2$ and $f_2(\mathbf{x}) = \|\mathbf{x}\|_2^2$ (plot the functions for $n = 2$).

6. (20) Let $\mathbf{A} \in \mathbb{R}^{m \times n}$, $\mathbf{b} \in \mathbb{R}^m$, and $f : \mathbb{R}^n \to \mathbb{R}$, with $f(\mathbf{x}) = \|\mathbf{A}\mathbf{x} - \mathbf{b}\|_2^2$.

   (a) (10) Assume that the columns of $\mathbf{A}$ are linearly independent and prove that f is strictly convex (Prove that the Hessian of $f(\mathbf{x})$ is positive definite).

   (b) (10) Plot f for $m = 3$ and $n = 2$. In order to generate the data, generate a random (3×2) matrix $\mathbf{A}$, a random (2×1) vector $\mathbf{x}$, and compute $\mathbf{b} = \mathbf{A}\mathbf{x}$. Then, plot, via `mesh`, function f in a square around the true value $\mathbf{x}$ - use also the `contour` statement. What do you observe?

      Repeat the above procedure by assuming that $\mathbf{b} = \mathbf{A}\mathbf{x} + \mathbf{e}$, where $\mathbf{e}$ is a "small noise" vector. What do you observe?