Note: Identification of Dynamic Auctions with Incomplete Data

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Consider the dynamic first-price, IPV auctions model presented in Athey and Haile 2007, Section 9.

The ex-ante value function is there shown to solve

$$
\begin{aligned}
&V_i(\mathbf{c}) = \int_{\underline{b}_i(\mathbf{c})}^{\bar{b}_i(\mathbf{c})} \frac{G_{M_i}(b_i|\mathbf{c})}{g_{M_i}(b_i|\mathbf{c})} G_{M_i}(b_i|\mathbf{c}) dG_{B_i}(b_i|\mathbf{c}) + \\
&+ \delta \sum_{j\neq i} V_i(\omega(\mathbf{c}, j)) \times \\
&\left\{ \int_{b_i}^{\bar{b}_i(\mathbf{c})} \prod_{k\neq i,j} G_{B_k}(b_j|\mathbf{c}) g_{B_j}(b_j|\mathbf{c}) db_j + \int_{\underline{b}_i(\mathbf{c})}^{\bar{b}_i(\mathbf{c})} \frac{G_{M_i}(b_i|\mathbf{c})}{g_{M_i}(b_i|\mathbf{c})} \frac{g_{B_j}(b_i|\mathbf{c})}{G_{B_j}(b_i|\mathbf{c})} G_{M_i}(b_i|\mathbf{c}) dG_{B_i}(b_i|\mathbf{c}) \right\}
\end{aligned} \tag{1}
$$

where $G_{M_i}(b_i|\mathbf{c}) = \prod_{j\neq i} G_{B_j}(b_i|\mathbf{c})$ is the CDF of the highest bid faced by i, with pdf $g_{M_i}(b_i|\mathbf{c}) = \left[\prod_{j\neq i} G_{B_j}(b_i|\mathbf{c})\right] \sum_{j\neq i} \frac{b_{B_j}(b_i|\mathbf{c})}{G_{B_j}(b_i|\mathbf{c})}$. Hence the above shows that the ex-ante value functions can be determined by solving a linear system in which the only unknowns are (fixing the discount factor δ, which is not identified) equilibrium conditional bid distributions $G_{B_j}(b_i|\mathbf{c})$ for each bidder j.

While identification of this dynamic model is typically discussed assuming the econometrician observes all the bids as well as the identity of all the bidders, the purpose of this note is to clarify what is identified upon observing the winning bid and the identity of the winner in each auction. This follows from Athey and Haile 2002; Prakasa Rao 1992.

Suppose in fact the econometrician observes, for any state $\mathbf{c}$, $H_i(b|\mathbf{c}) \equiv Pr(B^{n:n} < b, I^{n:n} = i|\mathbf{c})$, where $B^{n:n}$ is the highest bid and $I^{n:n}$ the identity of the highest bidder in an auction with n bidders. The econometrician does not observe $\mathcal{P}$, the set of auction participants, but only its cardinality $|\mathcal{P}| = n$. Agents are assumed to observe $\mathcal{P}$. Then,

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$$
\begin{aligned}
H_i(b|\mathbf{c}) &= \int_b^\infty \prod_{j\neq i, j\in\mathcal{P}} G_{B_j}(x|\mathbf{c}) g_{B_i}(x|\mathbf{c}) dx = \int_b^\infty \frac{\prod_{j\in\mathcal{P}} G_{B_j}(x|\mathbf{c})}{G_{B_i}(x|\mathbf{c})} g_{B_i}(x|\mathbf{c}) dx = \\
&= \int_b^\infty \prod_{j\in\mathcal{P}} G_{B_j}(x|\mathbf{c}) d\ln G_{B_i}(x|\mathbf{c})
\end{aligned} \tag{2}
$$

Note $\prod_{j\in\mathcal{P}} G_{B_j}(b|\mathbf{c}) = Pr(B^{n:n} < x|\mathbf{c}, \mathcal{P}) = \sum_{j\in\mathcal{P}} H_j(b|\mathbf{c})$, whence $H_i(b|\mathbf{c}) = \int_b^\infty \sum_{j\in\mathcal{P}} H_j(x|\mathbf{c}) d\ln G_{B_i}(x|\mathbf{c})$. Differentiating,

$$
\begin{aligned}
dH_i(b|\mathbf{c}) &= -\sum_{j\in\mathcal{P}} H_j(b|\mathbf{c}) d\ln G_{B_i}(b|\mathbf{c}) \\
d\ln G_{B_i}(b|\mathbf{c}) &= -\Big[\sum_{j\in\mathcal{P}} H_j(b|\mathbf{c})\Big]^{-1} dH_i(b|\mathbf{c}) \\
G_{B_i}(b|\mathbf{c}) &= \exp\Big(-\int_0^b \Big[\sum_{j\in\mathcal{P}} H_j(x|\mathbf{c})\Big]^{-1} dH_i(x|\mathbf{c})\Big)
\end{aligned} \tag{3}
$$

Hence knowledge of $H_i(x|\mathbf{c})$ is *not sufficient* to identify $G_{B_i}(b|\mathbf{c})$ at any state $\mathbf{c}$ and for any number of bidders n – the identity of bidders matters. This is not particularly surprising, given how little we have assumed about bidder heterogeneity. In particular, identification is restored either by observing the identity of bidders, or with complete participation (which however is typically far from what one empirically observes).

On the other hand, $G_{B_i}(b|\mathbf{c})$ is trivially set identified without additional information. Because $H_i(x|\mathbf{c})$ is observed for all potential participants $\mathcal{N}$, one can define

$$
\Gamma_n(x|\mathbf{c}) \equiv \arg \max_{\Gamma\subset\mathcal{N}, |\Gamma|=n} \sum_{i\in\Gamma} H_i(x|\mathbf{c}) \qquad \text{and} \qquad L_n(x|\mathbf{c}) \equiv \arg \min_{L\subset\mathcal{N}, |L|=n} \sum_{i\in L} H_i(x|\mathbf{c})
$$

Evidently, whatever $\mathcal{P}$,

$$
\big[\sum_{j\in\Gamma_n} H_j(x|\mathbf{c})\big]^{-1} \leq \big[\sum_{j\in\mathcal{P}} H_j(x|\mathbf{c})\big]^{-1} \leq \big[\sum_{j\in L_n} H_j(x|\mathbf{c})\big]^{-1} \tag{4}
$$

which means the bounds can be plugged in the equation for $G_{B_i}(b|\mathbf{c})$ to construct upper and lower bound for any $x, \mathbf{c}$. Of course, the sets Γ_n, L_n could change (infinitely often) as x changes, complicating the construction of the bounds. But if, for any $\mathcal{P}$ (including Γ_n, L_n), $\sum_{j\in\mathcal{P}} H_j(x|\mathbf{c}) = Pr(B^{n:n} < x|\mathbf{c}, \mathcal{P})$ is finite, the bounds are integrable. The cases in which the bounds are not finite seem economically uninteresting.

References

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