Recursivity and the Estimation of Dynamic Games with Continuous Controls

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Abstract

We revisit the estimation of dynamic games with continuous control variables, such as investments in R&D, quality, and capacity. We show how to use the recursive characterization of Markov Perfect Equilibria to develop estimators that make full use of the structure of the model. Our estimator resembles an indirect inference estimator, albeit in a two-step procedure that is common in the estimation of dynamic games. We use Monte Carlo experiments based on an empirically-relevant model of investment in R&D to compare the performance of our estimator with alternatives. We find that our estimator outperforms the commonly-used inequality estimator of Bajari, Benkard, and Levin (2007) and a naive implementation of an estimator based on recursive equilibrium conditions.

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1 Introduction

Many questions of interest to Industrial Organization economists involve firm choices that have persistent effects on market conditions. Such choices include investments in research and development, the choice of productive capacity, and the choice of product characteristics. Many other examples can be given. Decisions of this type are inherently dynamic and are often taken in industries with few firms. Therefore, their study necessitates the use of dynamic oligopoly models. Furthermore, many of these choices, such as the ones above, are naturally modeled as continuous variables.

This paper revisits the estimation of dynamic oligopoly models with continuous controls. The estimation of such models was made feasible by the seminal contribution of Bajari et al. (2007), henceforth BBL. Here we make the observation that the main estimator proposed by BBL does not use the full structure of the model, in that it does not exploit the structure of equilibrium policies. Exploiting this structure should lead to efficiency gains. We propose an estimator that does use the structure of equilibrium policies and conduct Monte Carlo exercises that compare its performance to different implementations of BBL and one other natural alternative in an empirically relevant model.

Our Monte Carlo exercises are based on Hashmi and van Biesebroeck (2016) – henceforth HvB. HvB propose and estimate an equilibrium model of innovation in the automobile industry. They are interested in the equilibrium relationship between market structure and innovation. In their model, firms engage in R&D – measured by firms' patenting activity – because it boosts the quality of their products. The cost of R&D depends on the number of patents a firm files for and on an investment cost shock. We base our simulation exercises on the Hashmi and van Biesebroeck (2016) model because it underpins an actual empirical application, and thus accurately represents models used in practice.

The first step in our estimation routine consists of estimating policy functions and state transitions. This is just as in BBL and estimators of dynamic games with discrete controls, such as the ones proposed by Aguirregabiria and Mira (2007), Pakes, Ostrovsky, and Berry (2007), and Pesendorfer and Schmidt-Dengler (2008). We depart from BBL in the second step. We use the estimated policy functions and state transitions to form and solve the maximization problem in the right-hand-side of firms' Bellman equations. We do so at the states observed in the data and randomly drawn investment cost shocks. We then project these predicted investment levels onto the space spanned by the basis functions in the empirical policy. Finally, we minimize a measure of the distance between the estimated policy function coefficients and those implied by the projection step. Therefore, our estimator combines elements of the two-step estimators that sprung from Hotz and Miller (1993) with Indirect Inference estimators à la Gourieroux, Monfort, and Renault (1993).

The intuition behind our estimator is simple. Under the maintained assumption that the observed policies constitute an equilibrium, solving the right-hand-side of firms' Bellman equations must return the observed policy. It is therefore also natural to consider an estimator that directly compares the estimated policy and the one implied by firms' Bellman equations. This would be the continuous-control analog of the estimator proposed by Pesendorfer and Schmidt-Dengler (2008). We thus include this estimator in our Monte Carlo experiments.

We find that our estimator has desirable properties. In our Monte Carlo exercise, its finite-sample bias is small and the estimator is precise. We compare it to two implementations of BBL. The first one uses the policy deviations employed by HvB′′. The second one forms deviations at random based on the estimated policy function. In contrast to our estimator, we find that both implementations of BBL have very substantial finite sample bias. In fact, the estimates are orders of magnitude away from the true parameters. We relate these findings, naturally, to the shape of the objective functions that define each estimator. While our estimator's objective function attains its minimum close to the truth and has large curvature, we find the BBL objective to be flat around the true parameters. When we consider the estimator that directly compares the estimated and model-implied policies, we find it to have poor properties.

Our estimator enjoys other benefits relative to BBL. First, it does not require the econometrician to choose policy deviations. This is an advantage, as the performance of the BBL estimator may very well depend on the deviations chosen by the analyst. This concern is substantiated by the different performance of the two BBL alternatives we consider. Second, we find that our estimator is faster to compute. Though our estimator adds the cost of solving (static) optimization problems multiple times, it does away with the simulation in BBL – by far the most expensive part of the algorithm. We find that the savings trump the costs. Our estimator takes about one third of the BBL run time.

It would be remiss of us not to remind the reader that Bajari et al. (2007) do propose a second estimator based on solving for firms' optimal policies. Nevertheless, the empirical literature seems to have converged to using the estimator that they discuss at greater length, based on value function inequalities. A number of applications, including very recent ones, use the inequality estimator. These include Ryan (2012), Hashmi and van Biesebroeck (2016), Fowlie, Reguant, and Ryan (2016), and Liu and Siebert (2022). We are not aware of any empirical paper estimating a dynamic game with continuous controls using an estimator based on firms' Bellman equations.

We therefore view our contributions as threefold. First, we show that estimators based on recursive equilibrium conditions can outperform the inequality-based estimators applied in the empirical literature by a large margin. Second, we provide implementation details for an estimator based on recursive equilibrium conditions in an empirically relevant setting, including shocks to firms' marginal costs of investment. Third, we show that a natural alternative estimator known to perform well in discrete control settings can have poor performance in continuous-control environments. Along the way we also show how one can do away with value function simulation and can instead solve for these objects, thus reducing simulation error.

The rest of the paper is organized as follows. In section 2 we discuss the Hashmi and van Biesebroeck (2016) model. In section 3 we provide a brief review of the main BBL estimator. In section 4 we present the estimator we propose. In section 5 we discuss the results of our Monte Carlo exercises. Finally, in section 6 we offer concluding remarks.

2 The Economic Model

We base our comparison of estimators on the model of innovation in the automobile industry of Hashmi and van Biesebroeck (2016). As in all dynamic games following the Ericson and Pakes (1995) framework, there is a static part to the model and a dynamic one. In the static part, firms play a Nash-Bertrand game, given demand and marginal cost functions. In the dynamic part, firms invest in R&D. Investments in R&D affect the quality of firms' products, which in turn affect their demand and production costs. There is no entry or exit.[1]

The timing of the model is as follows. Firms start the period with given quality levels. They simultaneously set prices, which determine the quantity sold and their flow profits. They then privately observe a shock to their invest-

[1]Entry and exit are not observed in the Hashmi and van Biesebroeck (2016) data, which is why those decisions are not included in their model. For our purposes, this simplifies the exposition and allows us to focus on the continuous control, which is our main interest.

ment cost. Next, firms simultaneously decide how much to invest and pay the associated costs. Firm investments affect the distribution of the quality of their products in the beginning of the following period. Firms' random quality levels realize before the start of the new period.

We now discuss the static and dynamic parts of the model in turn.

2.1 Price Competition Given Demand and Costs

There are N single-product firms in the market, indexed by $j = 1, 2, \dots, N$.[2] There is a measure M of consumers in the market, and consumers are indexed by i. If consumer i buys firm j's product, she obtains conditional indirect utility

$$u_{ij} = \alpha p_j + \xi_j + \varepsilon_{ij} \ ,$$

where α is the marginal utility of income, p_j is the price of good j, ξ_j is a quality index and ε_{ij} is a random shock that follows a Type 1 Extreme Value distribution. There is also an outside option, indexed by $j = 0$. Its non-random utility is normalized to zero, $u_{i0} = \varepsilon_{i0}$.

As is well-known, this specification of utility yields the market share function

$$\sigma_j(\boldsymbol{p}; \boldsymbol{\xi}) = \frac{\exp(\alpha p_j + \xi_j)}{1 + \sum_k \exp(\alpha p_k + \xi_k)}, \quad j = 1, \dots, N$$

where $\boldsymbol{p} = (p_1, \dots, p_N)$ and $\boldsymbol{\xi} = (\xi_1, \dots, \xi_N)$. Prices are modeled as the outcome of a Bertrand game. Firm j solves

$$\max_{p_j} \ \pi_j(p_j; \boldsymbol{p}_{-j}, \boldsymbol{\xi}) := (p_j - \mu(\xi_j))\sigma_j(\boldsymbol{p}; \boldsymbol{\xi}) \ ,$$

where $\mu_j(\xi_j)$ is firm j's constant marginal cost of production, which depends on its quality level, and firm j takes $\boldsymbol{p}_{-j} = (p_1, \dots, p_{j-1}, p_{j+1}, \dots, p_N)$ as given. The marginal cost function is specified as

$$\mu(\xi_j) = \exp(\theta_{c1} + \theta_{c2}\xi_j) \ .$$

We know from Caplin and Nalebuff (1991) that this pricing game has a unique

[2] Hashmi and van Biesebroeck (2016) estimate a demand model that accounts for the many products sold by automakers. When setting up their dynamic game of innovation, they (heuristically) aggregate that model to the firm level. This aggregation makes the dynamic model tractable. Here, as our focus is on methods for estimating the dynamic parameters of the model, we start from a model of single-product firms.

equilibrium. Moreover, the equilibrium price vector, $\boldsymbol{p}^*(\boldsymbol{\xi}) = (p_1^*(\boldsymbol{\xi}), \ldots, p_N^*(\boldsymbol{\xi}))$, must satisfy the system of first-order conditions:

$$\alpha(1 - \sigma_j(\boldsymbol{p}^*; \boldsymbol{\xi}))(p_j^* - \mu_j(\xi_j)) + 1 = 0, \quad j = 1, \ldots, J \tag{1}$$

The equilibrium $\boldsymbol{p}^*(\boldsymbol{\xi})$ induces profits $\pi_j(\boldsymbol{\xi}) := [p_j^*(\boldsymbol{\xi}) - \mu(\xi_j)]\sigma_j(\boldsymbol{p}^*(\boldsymbol{\xi}), \boldsymbol{\xi})$.

2.2 Investment in R&D

2.2.1 Quality Transitions

Firms invest to affect the quality of their product in the next period. Firms' qualities belong to the set $\Xi = \{\xi_{\min}, \xi_{\min}+\delta, \ldots, \xi_{\max}-\delta, \xi_{\max}\}$. If $\xi_{\min} < \xi < \xi_{\max}$, then ξ can decrease by δ, stay unchanged, or increase by δ. These transition probabilities are parameterized as

$$P(\xi'|\xi, x) = \begin{cases} \theta_{t1} * [1 - \text{up}(\xi, x)] & \text{if} \quad \xi' = \xi - \delta \\ 1 + \theta_{t1} - \text{up}(\xi, x)(1 - 2\theta_{t1}) & \text{if} \quad \xi' = \xi \\ (1 - \theta_{t1}) * \text{up}(\xi, x) & \text{if} \quad \xi' = \xi + \delta \\ 0 & \text{otherwise} \end{cases} \tag{2}$$

where

$$\text{up}(\xi, x) = e^{-e^{-\theta_{t2}\log(x+1) - \theta_{t3}\xi - \theta_{t4}\xi^2}}.$$

In the expressions above x denotes investment in R&D and $\boldsymbol{\theta}_t = (\theta_{t1}, \theta_{t2}, \theta_{t3}, \theta_{t4})$ is a vector of parameters governing the transition probabilities. To respect the finiteness of Ξ, we adjust these probabilities at the maximum and minimum values of Ξ. When $\xi = \xi_{min}$ we set $P(\xi|\xi, x)$ equal to the sum of the first two cases in equation 2. When $\xi = \xi_{max}$ we set $P(\xi|\xi, x)$ equal to the sum of the second and third cases in equation 2.

The interpretation of this specification is that the quality of a product is subject to a positive and a negative shock. One can think of the negative shock as capturing improvements in the outside option. The positive shock captures successes in the R&D process. The negative shock occurs with probability θ_{t1}, whereas the positive shock occurs with probability $\text{up}(\xi_j, x)$. Quality falls when the product's quality experiences a negative shock without an offsetting positive shock. It grows when quality experiences a positive shock unmatched by a negative shock. In all other cases, quality remains the same. Positive and negative shocks are independent across firms, conditional on $\boldsymbol{\xi}$ and $\boldsymbol{x}$.[3]

The probability of an improvement in the outside option is independent of firms' qualities and investments. The reasoning is that the outside option is outside of the focal market, and the process driving its quality is unresponsive to the conditions of the market being analyzed. The probability of an R&D success, however, does depend on a firm's investment and quality, but not on its competitors'. In particular, if $\theta_{t2} > 0$ and $\theta_{t3} < 0$, the specification of $\text{up}(\xi_j, x)$ implies that the probability of success is increasing in investment and decreasing in firm quality. This last effect captures the notion that it is harder to improve on a high-quality product. The sign of θ_{t4} determines whether an increase in quality reduces the probability of success at an increasing or decreasing rate.

2.2.2 The Firm's Problem

Firms choose their investment to maximize their present-discounted stream of profits. They trade-off better future prospects for the quality of their product against the immediate cost of investment, $c(x, \nu)$. This cost depends on the level of investment x and on a privately observed investment cost shock, ν. The cost shock is assumed to be iid across firms and follows a standard normal distribution. We focus throughout on Markov Perfect Equilibria. Firms' state variables are the vector of qualities in the market, $\boldsymbol{\xi}$, and their privately observed investment cost shock ν. Firm behaviour depends only on the state variables. Denote firm j's policy function by $\sigma_j(\boldsymbol{\xi}, \nu)$ and let $\boldsymbol{\sigma} = (\sigma_1, \ldots, \sigma_N)$. Firm j's problem can then be recursively represented as

$$V_j(\xi_j, \boldsymbol{\xi}_{-j}, \nu) = \max_{x \in \mathbb{R}_+} \left\{ \pi(\xi_j, \boldsymbol{\xi}_{-j}) - c(x, \nu) + \beta \mathbb{E}_{\boldsymbol{\sigma}_{-j}} \left[V_j(\xi_j', \boldsymbol{\xi'}_{-j}, \nu' | \xi_j, \boldsymbol{\xi}_{-j}, x) \right] \right\} \quad (3)$$

where

$$\mathbb{E}_{\boldsymbol{\sigma}_{-j}} \left[V_j(\xi_j', \boldsymbol{\xi}'_{-j}, \nu') | \xi_j, \boldsymbol{\xi}_{-j}, x \right] = \sum_{\xi_j'} \sum_{\boldsymbol{\xi}'_{-j}} \int_{\nu} V_j(\xi_j', \boldsymbol{\xi}'_{-j}, \nu') dF(\nu') P_{-j}(\boldsymbol{\xi}'_{-j} | \boldsymbol{\xi}; \boldsymbol{\sigma}_{-j}) P_j(\xi_j' | \xi_j, x) \quad (4)$$

[3]The independence of negative shocks across firms is at odds with its interpretation as capturing improvements in the outside option. We make this assumption for two reasons. First, it simplifies the exposition somewhat. Second, despite stating otherwise (p. 200), HvB seem to treat the negative shocks as independent at times (see their discussion of simulation in p. 201).

and

$$P_{-j}(\boldsymbol{\xi}'_{-j}|\boldsymbol{\xi};\boldsymbol{\sigma}_{-j}) = \prod_{k\neq j} P(\xi'_k|\boldsymbol{\xi};\sigma_k) = \prod_{k\neq j}\int_{\nu_k} P(\xi'_k|\xi_k,\sigma_k(\boldsymbol{\xi},\nu_k))dF(\nu_k)\ . \tag{5}$$

In this last expression, the terms $P(\xi'_k|\xi_k,\sigma_k(\boldsymbol{\xi},\nu_k))$ are derived from the quality transition model discussed in section 2.2.1.

We proceed as in Doraszelski and Pakes (2007). Define

$$EV_j(\boldsymbol{\xi}) := \int_{\nu_j} V_j(\xi_j,\boldsymbol{\xi}_{-j},\nu)dF(\nu)$$

and

$$W(\xi'_j|\boldsymbol{\xi};\boldsymbol{\sigma}_{-j}) := \sum_{\xi'_{-j}} EV(\xi'_j,\boldsymbol{\xi}'_{-j})P_{-j}(\boldsymbol{\xi}'_{-j}|\boldsymbol{\xi};\boldsymbol{\sigma}_{-j})\ .$$

$EV(\boldsymbol{\xi})$ is the expected present-discounted stream of profits when the vector of qualities is $\boldsymbol{\xi}$. The object $W(\xi'_j|\boldsymbol{\xi};\boldsymbol{\sigma}_{-j})$ is the expected value of landing on quality ξ'_j starting from the vector of qualities $\boldsymbol{\xi}$. With these definitions we have that

$$\mathbb{E}_{\boldsymbol{\sigma}_{-j}}\left[V_j(\xi'_j,\boldsymbol{\xi}'_{-j},\nu')|\xi_j,\boldsymbol{\xi}_{-j},x\right] = \sum_{\xi'_j} W(\xi'_j|\boldsymbol{\xi};\boldsymbol{\sigma}_{-j})P_j(\xi'_j|\xi_j,x)$$

The first-order condition of the maximization problem on the right hand side of the Bellman equation is then

$$\frac{\partial c(x,\nu)}{\partial x} = \beta\sum_{\xi'_j} W(\xi'_j|\boldsymbol{\xi};\boldsymbol{\sigma}_{-j})\frac{\partial P_j(\xi'_j|\xi_j,x)}{\partial x} \tag{6}$$

At the optimum level of investment, given $\boldsymbol{\xi}$ and ν, the firm equates the marginal cost of investment to its marginal benefit. The marginal cost of investment is an exogenous object. The marginal benefit of investment depends on how investment changes the distribution over quality levels in the following period. It also depends on the value of starting the following period with different levels of quality, ξ'_j. Moreover, these future values take into account the current quality-state. For instance, the gain in present-discounted profits from an increase in quality may be smaller if a firm's competitors all have substantially lower quality, relative to a case in which their qualities are similar to the focal firm's.

In our numerical exercise, we parameterize the investment cost function as

$$c(x,\nu) = \theta_{x1}x + \theta_{x2}x^2 + \theta_{x3}\nu x \ .$$

This is as in HvB, except that they also include a cubic term.

2.3 Equilibrium

We focus on symmetric Markov Perfect Equilibria, i.e., $\sigma_j(\xi,\nu) = \sigma(\xi,\nu)$ for $j = 1,\ldots,N$. As shown by equations (4) and (5), the expectation in the Bellman equation (3) depends on $\sigma(\cdot,\cdot)$. Therefore, one can think of equation (3) as defining an operator that maps policy and value functions into themselves: $T : \Sigma\times\mathcal{V} \to \Sigma\times\mathcal{V}$, where Σ denotes the set of feasible policy functions and $\mathcal{V}$ the set of feasible value functions. This operators returns a firm's best-response to its competitors all playing a given strategy σ (given V). A strategy profile $(\sigma,\ldots,\sigma)$ is a symmetric Markov Perfect Equilibrium if and only if (σ, V_σ) is a fixed-point of T, where V_σ is the expected present-discounted stream of profits when all firms play the strategy σ.

This representation underpins the method we employ to solve for Markov Perfect Equilibria. We start with a guess for value and policy functions. With the policy functions we can compute the implied transition probabilities for firms' qualities. Having transition probabilities and the guess for the value function, we can compute the terms $W(\xi_j'|\boldsymbol{\xi};\boldsymbol{\sigma}_{-j})$. This allows us to solve firms' first-order conditions, which yields new guesses for policy and value functions. We iterate on these steps until both value and policy functions converge. This is, of course, simply the Pakes and McGuire (1994) algorithm.[4]

3 A Brief Review of BBL

In this section we briefly review the main estimator proposed by Bajari et al. (2007), which is the main point of comparison for the estimator we propose in

[4]Alternatively, one can view equation 3 as an operator $\tilde{T} : \Sigma \to \Sigma$ defined in two steps. First, compute the value function implied by σ, i.e.,

$$V(\boldsymbol{\xi},\nu;\sigma) = \pi(\boldsymbol{\xi}) - c(\sigma(\boldsymbol{\xi},\nu),\nu) + \beta\mathbb{E}_\sigma\left[V(\boldsymbol{\xi}',\nu')|\boldsymbol{\xi},\boldsymbol{\sigma}(\boldsymbol{\xi},\nu)\right] \ .$$

Next, solve the right-hand-side of equation 3 to obtain $\tilde{T}\sigma$. A symmetric strategy profile $(\sigma,\ldots,\sigma)$ is a symmetric MPE if and only if it is a fixed point of $\tilde{T}$. One can thus solve for an MPE by iterating on the two steps above.

section 4. The expected present-discounted stream of profits of a firm when it plays a strategy σ_j and all its competitors play the strategy σ is given by

$$V(\boldsymbol{\xi}, \nu; \sigma_j, \sigma) = \mathbb{E}\left\{ \sum_{t=0}^{\infty} \beta^t \left[\pi(\xi_{jt}, \boldsymbol{\xi}_{-j,t}) - c(\sigma_j(\boldsymbol{\xi}_t, \nu_{jt}), \nu_{jt}) \right] \middle| \boldsymbol{\xi}_0 = \boldsymbol{\xi}, \nu_{j0} = \nu \right\}$$

where, letting $P(\xi, x)$ denote the probability mass function defined in equation 2, $\xi_{j,t+1} \sim P(\xi_{j,t}, \sigma_j(\boldsymbol{\xi}_t, \nu_{jt}))$ and for $k \neq j$, $\xi_{k,t+1} \sim P(\xi_{k,t}, \sigma(\boldsymbol{\xi}_t, \nu_{kt}))$.

By definition, a symmetric strategy profile $(\sigma, \ldots, \sigma)$ is a Markov Perfect Equilibrium if

$$V(\boldsymbol{\xi}, \nu; \sigma, \sigma) \geq V(\boldsymbol{\xi}, \nu; \sigma', \sigma) \quad \forall\, \boldsymbol{\xi}, \nu, \sigma' \tag{7}$$

Bajari et al. (2007) base their estimator on the equilibrium conditions (7). Define

$$g(\boldsymbol{\xi}, \nu, \sigma'; \sigma, \theta) := V(\boldsymbol{\xi}, \nu; \sigma, \sigma, \theta) - V(\boldsymbol{\xi}, \nu; \sigma', \sigma, \theta)$$

where we have made the dependence on the structural parameters θ explicit. Let H be a distribution over the space of tuples of the form $(\boldsymbol{\xi}, \nu, \sigma')$. Define

$$Q(\theta, \sigma) := \int \Big(\min\left\{ g(\boldsymbol{\xi}, \nu, \sigma'; \sigma, \theta), 0 \right\} \Big)^2 dH(\boldsymbol{\xi}, \nu, \sigma') \tag{8}$$

Let $\mathcal{E}(\theta)$ be the set of MPE when the parameters of the model are given by θ and let θ_0 denote the true parameter value. If $\sigma \in \mathcal{E}(\theta_0)$, the equilibrium conditions above imply that $Q(\theta_0, \sigma) = 0$.

Assumption 1 (ID). For any $\theta, \theta' \in \Theta$, $\mathcal{E}(\theta) \cap \mathcal{E}(\theta') = \emptyset$.

If assumption 1 holds, $\sigma \in \mathcal{E}(\theta_0)$, $\theta' \neq \theta_0$, and H is chosen judiciously, then $Q(\theta', \sigma) > 0$: assumption 1 implies that $\sigma \notin \mathcal{E}(\theta')$, so that $g(\boldsymbol{\xi}, \nu, \sigma'; \sigma, \theta') < 0$ for some $(\boldsymbol{\xi}, \nu, \sigma', \sigma)$.[5]

Bajari et al. (2007) propose estimating the structural parameters of the model by minimizing a sample analog of (8). In particular, for some set of $(\boldsymbol{\xi}, \nu, \sigma')$

[5]To be more precise, we require that $g(\boldsymbol{\xi}, \nu, \sigma'; \sigma, \theta') < 0$ on a set of positive H-measure for all $\theta' \neq \theta_0$. We can attach this condition to our definition of MPE. Given a measure μ on the set of tuples $(\boldsymbol{\xi}, \nu, \sigma')$, say that $(\sigma, \ldots, \sigma)$ is a symmetric MPE if $g(\boldsymbol{\xi}, \nu, \sigma'; \sigma, \theta_0) < 0$ with zero μ-measure. Then choose H such that μ is absolutely continuous with respect to H. If $\theta' \neq \theta_0$, assumption 1 implies that $g(\boldsymbol{\xi}, \nu, \sigma'; \sigma, \theta') < 0$ with positive μ-measure. This implies that $g(\boldsymbol{\xi}, \nu, \sigma'; \sigma, \theta') < 0$ with positive H-measure, otherwise absolute continuity of μ with respect to H would be violated.

tuples, indexed by $i = 1, \ldots, n_I$, they propose minimizing

$$\hat{Q}(\theta, \hat{\sigma}) := \frac{1}{n_I} \sum_{i=1}^{n_I} \left(\min \left\{ g(\boldsymbol{\xi}_i, \nu_i, \sigma_i'; \hat{\sigma}, \theta), 0 \right\} \right)^2$$

where $\hat{\sigma}$ is the strategy profile estimated in a first step. Evaluating this objective requires estimates of the value function V under the estimated policies and under the deviations. Bajari et al. (2007) propose obtaining these estimates by forward simulation. As they note, linearity of flow profits significantly reduces the computational burden of forward simulation. Linearity implies that there exists a function $W(\boldsymbol{\xi}, \nu; \sigma', \sigma)$ such that $V(\boldsymbol{\xi}, \nu; \sigma', \sigma, \theta) = W(\boldsymbol{\xi}, \nu; \sigma', \sigma)\theta$. Therefore, the forward simulation needs to be performed only once for each pair (σ', σ) to obtain $W(\boldsymbol{\xi}, \nu; \sigma', \sigma)$. Note that linearity of flow profits does apply to the Hashmi and van Biesebroeck (2016) model.

4 The Recursive Estimator

In this section we introduce our proposed estimator. The Bajari et al. (2007) estimator is appealing because of its simplicity. However, it fails to use the model-implied structure of optimal policies. This is clear in how the estimator is setup: it requires picking deviations σ', but the choice of these deviations is not tied to the model in any way. The presence of the deviations σ' in the estimator has practical drawbacks. The researcher has to choose these deviations somehow, and different choices of deviations may affect the performance of the estimator, as we show in our Monte Carlo experiments.

We instead propose an estimator that does exploit the structure of optimal policies. It is based on the Bellman Equation (3). This will not only yield a more efficient estimator, as it makes full use of the model structure. It will also free the researcher from having to experiment with and choose alternative choices of deviations. Our estimator will therefore have econometric and practical advantages over the one proposed by Bajari et al. (2007).[6]

[6] Another important point of comparison is computational cost. As will become clear as we discuss our estimator, there is a trade-off. Our estimator will require simulating or solving for value functions only once. The Bajari et al. (2007) estimator requires doing so for all pairs (σ', σ) that enter the objective function. However, our estimator requires solving at least as many static optimization problems as observations in the data. In the Monte Carlo experiments we perform in section 5 this trade-off is resolved in our favor, with the recursive estimator taking approximately a third of the time of BBL estimators. However, it need not always be the case that our estimator will be faster to compute than BBL.

As Bajari et al. (2007) and other estimators in the dynamic games, the first step in our estimation procedure is to estimate policy functions and state transitions. Given the model for state transitions set up in section 2.2.1, we can estimate the parameters θ_t by maximum likelihood. Next, letting $i = 1, \ldots, N$ index the observations in the data, we specify the empirical policy function as

$$x_i = \sum_{k=1}^{B} \gamma_k \Phi_k(\boldsymbol{\xi_i}) + \lambda \nu_i \tag{9}$$

for some choice of $\{\Phi_k\}_{k=1}^{B}$.[7] For example, Hashmi and van Biesebroeck (2016) use the firm's own quality, its square and cube, its rank in the vector $\boldsymbol{\xi}$, and the mean, standard deviation, skewness and kurtosis of the vector $\boldsymbol{\xi}$. Estimating 9 yields predicted investment levels at each state: $\hat{\sigma}(\boldsymbol{\xi}, \nu) = \sum_{k=1}^{B} \hat{\gamma}_k \Phi_k(\boldsymbol{\xi_i}) + \hat{\lambda}\nu$.

Next, suppose that we have estimates of $EV(\boldsymbol{\xi}; \sigma, \theta) := \int V(\boldsymbol{\xi}, \nu; \sigma, \sigma, \theta) dF(\nu)$. Denote these estimates by $\widehat{EV}(\boldsymbol{\xi}; \hat{\sigma}, \theta)$. These estimates could be obtained by forward simulation, as in Bajari et al. (2007). We discuss a more efficient alternative below. For now, however, let us focus on the main point of difference between our estimator and BBL. Using $\widehat{EV}(\boldsymbol{\xi}; \sigma)$, we can setup and solve the right-hand side of the Bellman Equation (3):

$$\max_{x \in \mathbb{R}_+} \left\{ \pi(\xi_j, \boldsymbol{\xi}_{-j}) - c(x, \nu) + \beta \mathbb{E}_{\hat{\sigma}} \left[\widehat{EV}(\xi_j', \boldsymbol{\xi'}_{-j}; \hat{\sigma}, \theta) | \xi_j, \boldsymbol{\xi}_{-j}, x \right] \right\}. \tag{10}$$

We solve the problem (10) for each $\boldsymbol{\xi}$ observed in the data and a randomly drawn $\nu \sim N(0, 1)$. This yields a new predicted level of investment at these states, which depends on the parameters θ. We denote these predicted investment levels by $\{T_{\theta,\hat{\sigma}}(\boldsymbol{\xi}_i, \nu_i)\}_{i=1}^{N}$.

Next, we re-estimate the model (9), but using $T_{\theta,\hat{\sigma}}(\boldsymbol{\xi}_i, \nu_i)$ as the dependent variable, rather than the observed levels of investment. This yields new estimates of $\boldsymbol{\gamma}$ and λ, which we denote by $\tilde{\boldsymbol{\gamma}}(\theta, \hat{\sigma})$ and $\tilde{\lambda}(\theta, \hat{\sigma})$. Let $\hat{\psi} := (\hat{\boldsymbol{\gamma}}', \hat{\lambda})'$ and $\tilde{\boldsymbol{\psi}}(\boldsymbol{\theta}, \hat{\sigma}) := (\tilde{\boldsymbol{\gamma}}(\theta, \hat{\sigma})', \tilde{\lambda}(\theta, \hat{\sigma}))'$. For some positive-definite weight matrix $\boldsymbol{W}$, we solve

$$\min_{\theta} Q(\theta, \boldsymbol{W}) := \left[\tilde{\boldsymbol{\psi}}(\theta, \hat{\sigma}) - \hat{\psi}\right]' \boldsymbol{W} \left[\tilde{\boldsymbol{\psi}}(\theta, \hat{\sigma}) - \hat{\psi}\right] \tag{11}$$

The intuition for the procedure above is the following. If the policy observed in the data, $\hat{\sigma}$, is a Markov Perfect Equilibrium, it must satisfy the re-

[7]This specification is inconsistent with the model, as ν enters the policy function nonlinearly. We adopt it here to follow Hashmi and van Biesebroeck (2016). Despite this misspecification, our estimator performs well. Note that Bajari et al. (2007) propose an estimator for the policy function that is consistent with the model and could be applied here.

cursive equilibrium conditions (3). Therefore, if we solve the right-hand side of the Bellman Equation using the transitions and value functions implied by $\hat{\sigma}$, we must obtain $\hat{\sigma}$ back. This is the continuous-control analog of the minimum distance estimator by Pesendorfer and Schmidt-Dengler (2008). As we show in section 5, an estimator based only on this intuition does not have good properties. Our intuition for this result is that the empirical policy and policies implied by the Bellman equation belong to different spaces. The best approximation of the former by the latter may not occur at the true parameters.

This observation motivates the final step in our estimator. We project the predicted levels of investment onto the same basis that generates the empirical policy function, and minimize a measure of the distance between the estimated coefficients. This estimator turns out to perform very well, as we show in section 5.

4.1 Estimating EV

Let us now return to the estimation of $EV(\boldsymbol{\xi};\sigma,\theta)$. As observed above, one could estimate this object by forward simulation as in BBL. Instead, we note that it is possible to solve for the value function instead, thus doing away with simulation error. If firms play the Markovian strategy $\hat{\sigma}$, the value function must satisfy

$$V(\boldsymbol{\xi},\nu;\hat{\sigma},\theta) = \pi(\xi_j,\boldsymbol{\xi}_{-j}) - c(\hat{\sigma}(\boldsymbol{\xi},\nu),\nu) + \beta\mathbb{E}_{\hat{\sigma}}\left[EV(\xi_j',\boldsymbol{\xi}'_{-j};\hat{\sigma},\theta)|\xi_j,\boldsymbol{\xi}_{-j},\hat{\sigma}(\boldsymbol{\xi},\nu)\right] \tag{12}$$

which implies, integrating over ν,

$$EV(\boldsymbol{\xi};\hat{\sigma},\theta) = \pi(\xi_j,\boldsymbol{\xi}_{-j}) - \int_\nu c(\hat{\sigma}(\boldsymbol{\xi},\nu),\nu_j)dF(\nu) + \beta\mathbb{E}_{\hat{\sigma}}\left[EV(\xi_j',\boldsymbol{\xi}'_{-j};\hat{\sigma},\theta)|\xi_j,\boldsymbol{\xi}_{-j},\hat{\sigma}(\boldsymbol{\xi},\nu)\right] \tag{13}$$

One can view the right-hand side of equation as an operator mapping EV values into themselves. Our object of interest, $EV(\boldsymbol{\xi};\sigma,\theta)$, is a fixed point of this operator. Moreover, by Blackwell's sufficient conditions, this operator is a contraction. We can thus solve for $EV(\boldsymbol{\xi};\sigma,\theta)$ by iterating on the right-hand side of (13).

Just as in BBL, linearity of flow profits reduces the computational burden associated with computing EV. That is because, letting flow profits be given by $w(\boldsymbol{\xi}, \nu, x) \cdot \theta$, we have that

$$EV(\boldsymbol{\xi}; \sigma, \theta) = \mathbb{E}_\sigma \left\{ \sum_{t=0}^{\infty} \beta^t w(\boldsymbol{\xi}_t, \nu_t, \sigma(\boldsymbol{\xi}_t, \nu_t)) \cdot \theta \middle| \boldsymbol{\xi}_0 = \boldsymbol{\xi} \right\} = \boldsymbol{W}(\boldsymbol{\xi}; \sigma) \cdot \theta$$

where $\boldsymbol{W}(\boldsymbol{\xi}; \sigma) = \mathbb{E}_\sigma \left\{ \sum_{t=0}^{\infty} \beta^t w(\boldsymbol{\xi}_t, \nu_t, \sigma(\boldsymbol{\xi}_t, \nu_t)) \middle| \boldsymbol{\xi}_0 = \boldsymbol{\xi} \right\}$. Moreover, the components of $\boldsymbol{W}$ themselves satisfy recursions like (13). For example, the model in Section 2 has $c(x, \nu, \boldsymbol{\theta}_x) = w_x(x, \nu) \cdot \boldsymbol{\theta}_x$, where $\boldsymbol{\theta}_x$ is the vector of structural parameters entering the cost of investment. Therefore, flow payoffs have the form $(\pi(\boldsymbol{\xi}), w_x(x, \nu)) \cdot (1, \boldsymbol{\theta}_x)$. It follows that $W_1(\boldsymbol{\xi})$ – the first component of $\boldsymbol{W}(\xi)$ – satisfies

$$W_1(\xi) = \pi(\xi) + \beta \mathbb{E}_\sigma[W_1(\xi')|\xi] \ . \tag{14}$$

Similar recursions hold for the other terms, with the caveat that some of those equations involve an integration with respect to ν as in equation (13). In short, linearity implies that the researcher only needs to solve for the $\boldsymbol{W}(\boldsymbol{\xi}; \sigma)$, and doing so can easily be done using (14). Note that this needs to be done only once, rather than for all guesses of θ, and only for the estimated policy $\hat{\sigma}$.

Equation (14) gives an alternative way of computing $\boldsymbol{W}(\xi)$. Let $\boldsymbol{M}(\boldsymbol{\xi}, \sigma)$ be the vector of probabilities over states $\boldsymbol{\xi}'$ induced by σ when the current state is $\boldsymbol{\xi}$. Then (14) is equivalent to

$$W_1(\xi) = \pi(\xi) + \beta \boldsymbol{M}(\xi, \sigma) \cdot \boldsymbol{W}_1$$

where $\boldsymbol{W}_1$ is the vector $(W_1(\xi_1), \ldots, W_1(\xi_{|\Omega|}))$. Stacking these equations we have that

$$(I - \beta \boldsymbol{M}(\sigma)) \boldsymbol{W}_1 = \boldsymbol{\pi}$$

where $\boldsymbol{M}(\sigma)$ is a matrix whose rows are the vectors $\boldsymbol{M}(\boldsymbol{\xi}, \sigma)$. Therefore, solving for $\boldsymbol{W}$ amounts to solving a few systems of linear equations – as many as the dimension of $\boldsymbol{\theta}_x$. The number of linear equations may be large, but the matrix $\boldsymbol{M}(\sigma)$ will be sparse in many applications.[8]

[8]For instance, in the economic model in our Monte Carlo experiments, the number of equations is equal to 45,900. However, only about 0.5% of the entries of $\boldsymbol{M}(\sigma)$ are nonzero.

Table 1: Exercise Parameters

Parameter	Value
BBL: Number of Inequalities	1000
BBL: Number of Simulated Paths	500
BBL: Simulation Horizon	80
BBL: Vector of Scalar Deviations	[0.9 0.95 1.05 1.1]
Discount Factor	0.925
Investment Cost Parameters	[2.625 1.624 0.5096]
Marginal Cost Parameters	[2.47 0.0]
Number of Firms	5
Number of Households	1.0e8
Number of Markets	100
Number of Periods	40
Price Sensitivity	-0.222
State Space	-1.4:0.2:1.4
Transition Probability Parameters	[0.547 0.062 -0.884 -0.285]

5 Monte Carlo Simulations

In this section we discuss the performance of the estimators discussed in Sections 3 and 4 for the model presented in Section 2.[9] We consider a rather data-rich environment, with data about 100 separate markets recorded for 40 periods. 5 firms compete in each market.[10] Firms have marginal cost parameters $\theta_{c1} = 2.47, \theta_{c2} = 0$ – i.e. constant marginal cost $mc = \exp(2.47)$; unobserved product quality goes from -1.4 to 1.4 in increments of .2. There is a substantial probability of quality downgrade shocks ($\theta_{t1} = 0.547$), and the probability of a quality upgrading shock is increasing in own investment ($\theta_{t2} = 0.062$) but decreasing (at a decreasing rate) in own quality level ($\theta_{t3}, \theta_{t4} < 0$). Investment costs are convex ($\theta_{x1}, \theta_{x2} > 0$) in investment and increasing in the 'shock' term ($\theta_{x3} > 0$). We outline all parameters in Table 1.

[9]While we eventually intend to consider multiple parameterisations of the model, for the moment we only discuss one such parameterisation.

[10]For comparison, Hashmi and van Biesebroeck (2016) aggregate data to have a single market with 14 firms for the 1982-2006 period. Ryan (2012) collected a total of 517 observations (an unbalanced panel of 27 markets over 19 years), with the number of firms per observation ranging from 1 to 20.

We compare the performance of four estimators:

1. the estimator proposed in Section 4 (recursive estimator);

2. Hashmi and van Biesebroeck (2016)'s BBL estimator, in which policy deviations result from multiplicative shifts of the policy function: $\sigma'(\boldsymbol{\xi}, \nu) = \iota\sigma(\boldsymbol{\xi}, \nu)$ for $\iota \in \{.90, .95, 1.05, 1.10\}$ (HvB estimator);

3. a BBL estimator using $\sigma'(\boldsymbol{\xi}, \nu) = \sum_{k=1}^{B} \gamma'_k \Phi_k(\boldsymbol{\xi_i}) + \nu$, with γ'_k drawn from the asymptotic distribution of $\hat{\gamma}_k$ (BBL estimator);

4. an estimator that directly compares the estimated policy function and the one implied by firms' Bellman equations (naïve estimator). Specifically, this estimator is defined by

$$\min_{\theta \in \Theta} \sum_{\xi} \int_{\mathbb{R}} [T_{\theta,\hat{\sigma}}(\xi, \nu) - \hat{\sigma}(\xi, \nu)]^2 \phi(\nu) \mathrm{d}\nu$$

Table 2 lists true parameter values along with estimate average and standard deviation for the four estimators over 250 simulations. Of the four estimators, the recursive estimator evidently performs the best: averages are close to the true parameters, with small standard deviations. BBL estimators and the naïve estimator display substantial bias and, at times, large standard deviations. Figures 1, 2, 3, and 4 present the distribution of parameter estimates for each algorithm; vertical dotted lines represent true parameter values. Recursive estimates are correctly centered around the true values.[11] HvB and BBL estimators are centered around wrong values, displaying a bias that can be orders of magnitude larger than the parameter to be estimated. The finite sample bias of the naïve estimator is smaller than that of the BBL implementations, but it is still substantial.

[11]There is a small mass at zero in the distribution of θ_{x3}. This is likely due to the non-convexity close to zero shown in figure 5. Due to this non-convexity, the gradient-based minimization routine we are currently employing can converge to local minimum that has $\theta_{x3} = 0$. This is simple to resolve. It will likely suffice to run a global minimization routine with very loose termination criteria to roughly identify the location of the global minimum, and then move onto a local gradient-based minimization routine.

Figure 1: Recursive Estimator Parameter Estimates

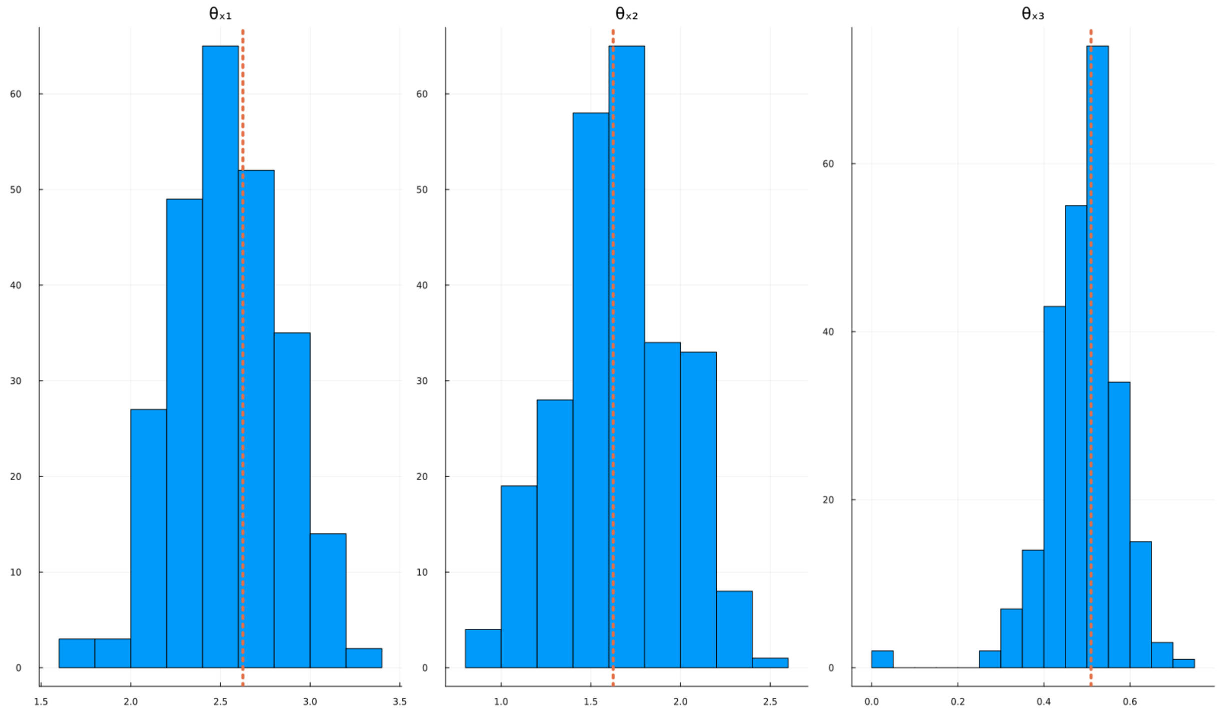

Figure 2: HvB Estimator Parameter Estimates

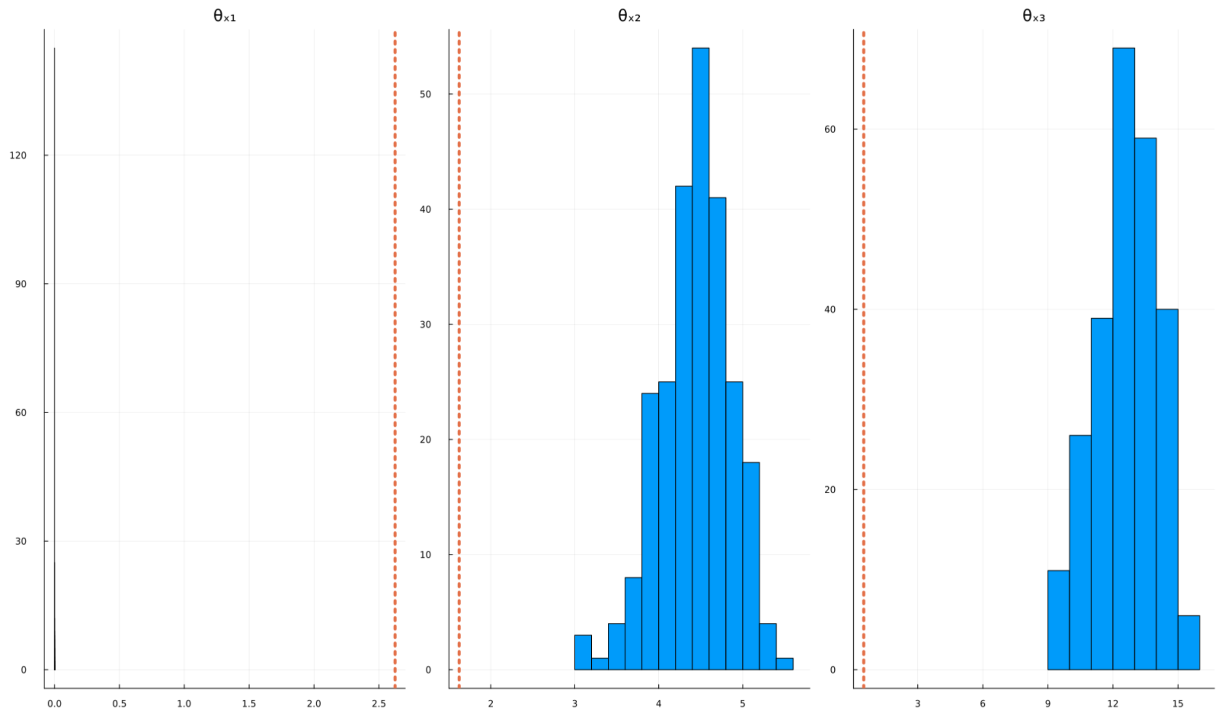

Figure 3: BBL Estimator Parameter Estimates

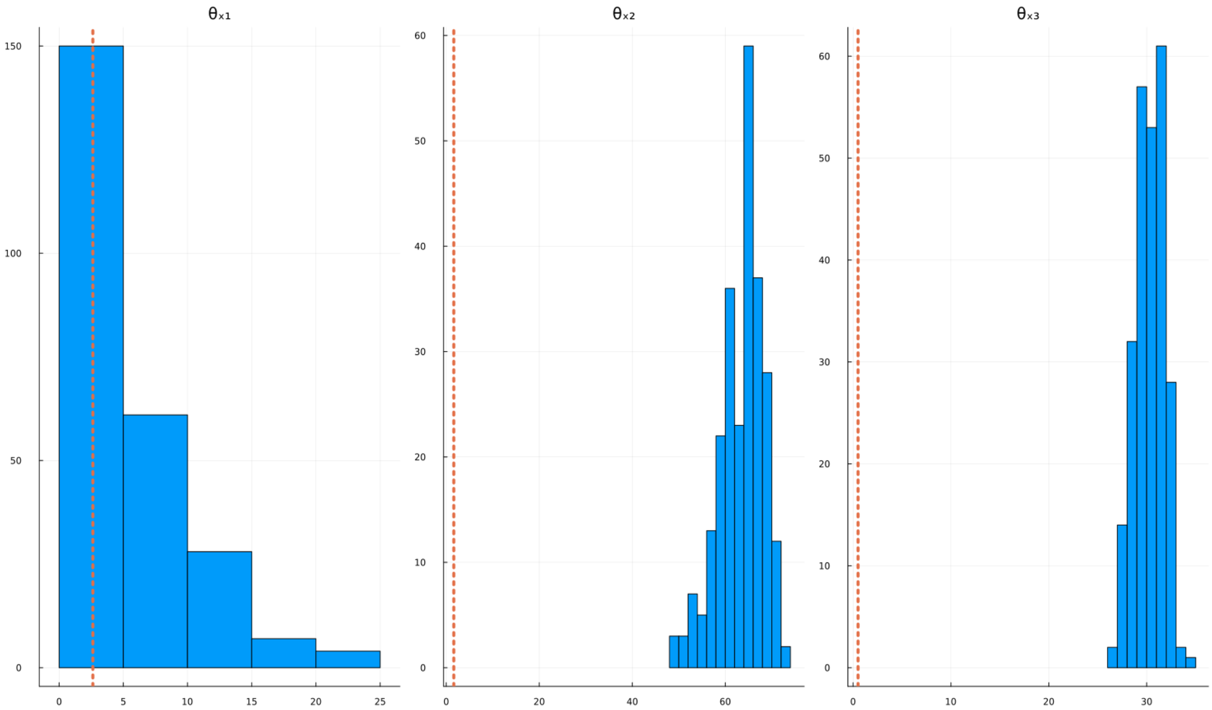

Figure 4: Naïve Estimator Parameter Estimates

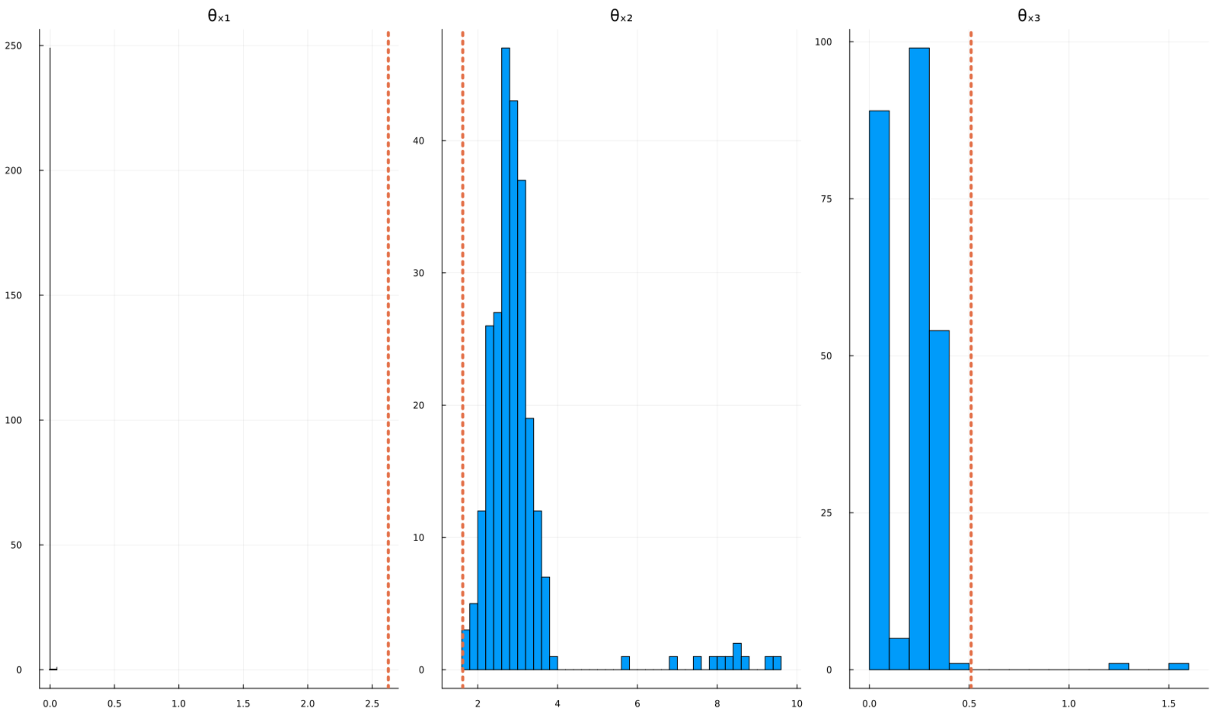

Table 2: Summary of Parameter Estimates

	True	Recursive	BBL	HvB	Recursive Naive
θ_1	2.625	2.537	4.850	1.490e-08	2.124e-04
		(0.298)	(4.993)	(2.107e-14)	(0.003)
θ_2	1.624	1.651	63.439	4.436	3.028
		(0.321)	(4.810)	(0.419)	(1.184)
θ_3	0.510	0.492	30.346	12.614	0.187
		(0.086)	(1.448)	(1.412)	(0.177)

The relative performance of the different estimators is a consequence of the differences in the objective functions that define them. We illustrate this by analysing the shape of each objective function in a neighbourhood of the true parameters for a particular simulated dataset. To do so, we plot two-dimensional slices of the objective function by varying one parameter at a time while holding the others fixed at their true values. To render the shape of different objective functions comparable, we normalise objective values on the parameter grid by dividing each by the objective value at the true parameter.
Figures 5 to 8 display the results. Vertical lines represent true parameter values.

Figure 5 shows that the objective function of the recursive estimator features pronounced local convexity around each true parameter, with minimal distance between the local objective minimum and the objective value at the true parameter. This is reflected in the good performance of the estimator. The picture is very different for the HvB and BBL estimators: they are approximately flat around the true parameter, and zooming in reveals that they display no convexity at all for the considered grid – they are almost linear around the correct parameter values. This is consistent with the large bias displayed in Figures 2 and 3: the objective is convex around parameter values that are wrong by orders of magnitude.

Finally, objective plots for the naïve estimator are informative about its imprecision. Objective function minima are not far from the minima at the true parameter value (low bias) for θ_{x1}, θ_{x2}, but the naïve objective function is much less convex around local minima than the recursive objective function is. On the other hand, the objective function is not centered around the true parameter objective value for θ_{x3}.

Figure 5: Recursive Estimator Objective

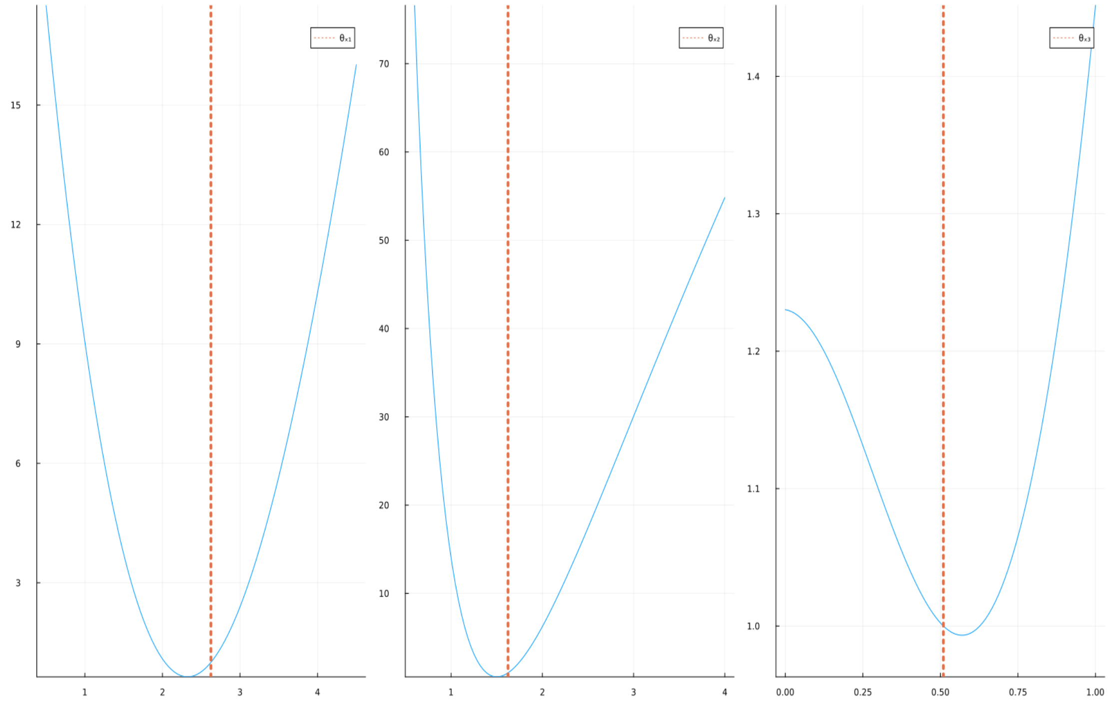

Figure 6: HvB Estimator Objective

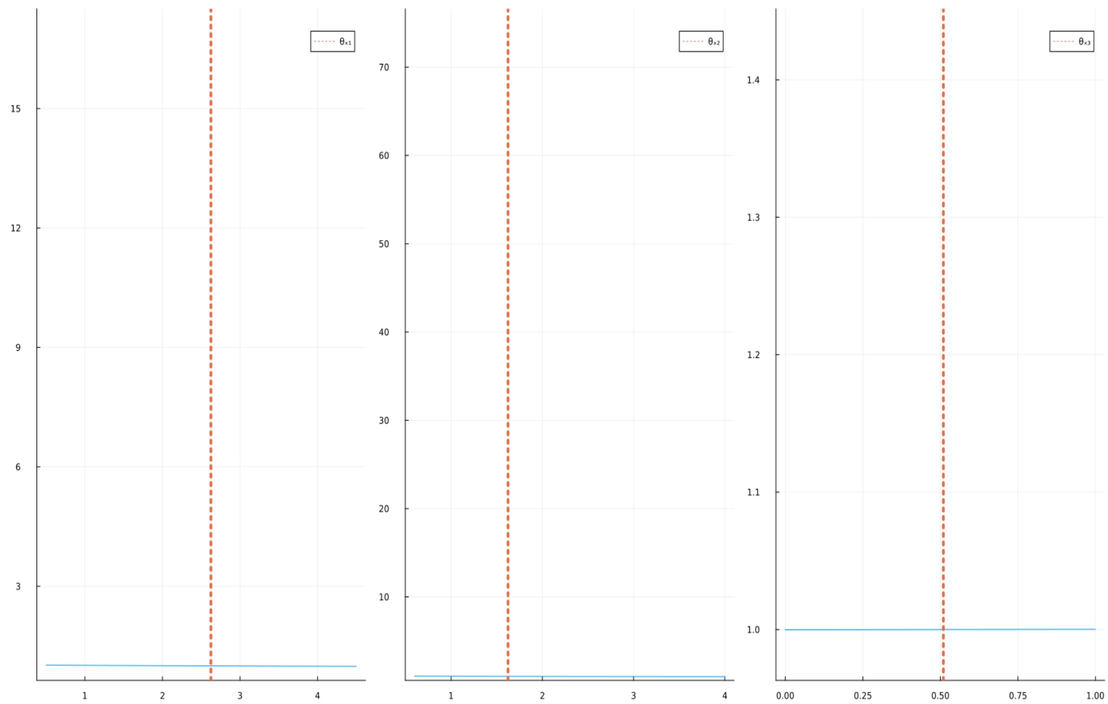

Figure 7: BBL Estimator Objective

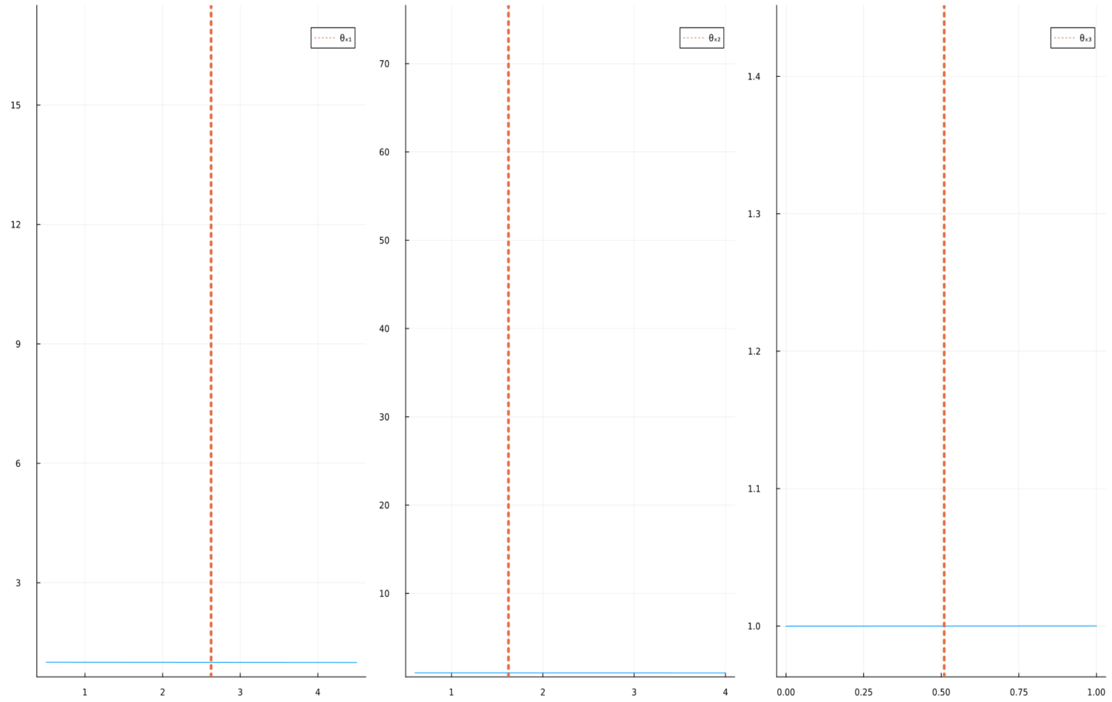

Figure 8: Naïve Estimator Objective

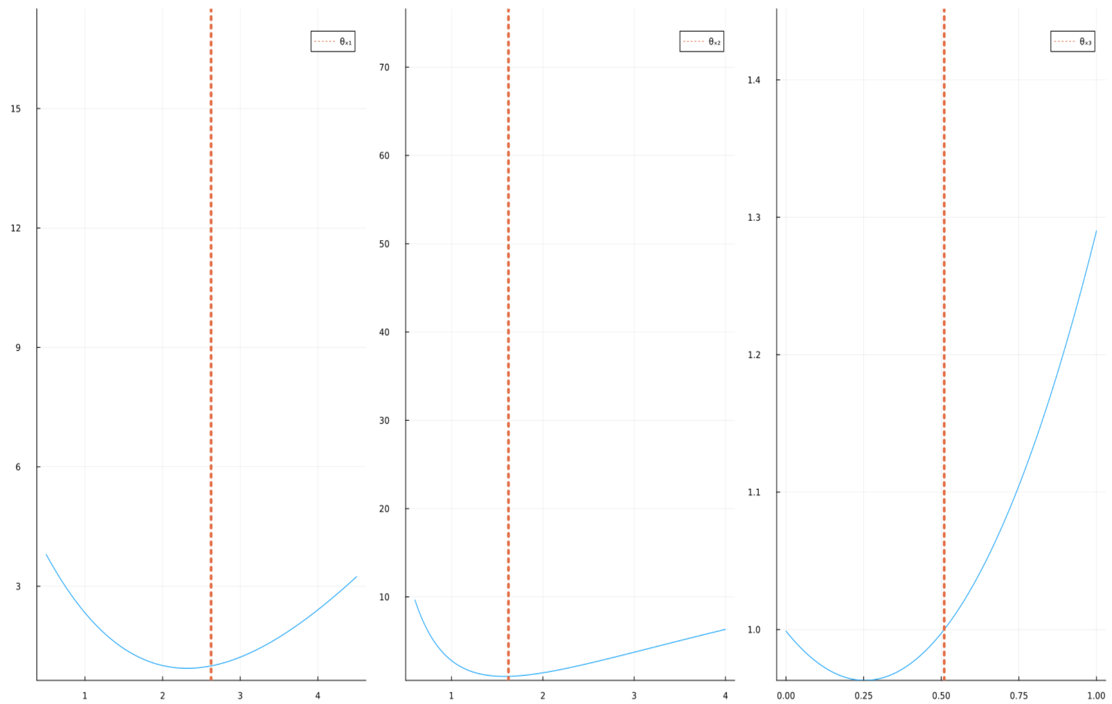

Table 3: Exercise Timings (25 Simulations)

Call	Average Time	Average Allocation
Overall		
Preliminaries	66.8s	134GiB
Simulation	541ms	715MiB
Step 1: Estimating Transition Parameters	563ms	582MiB
Step 1: Estimating Policy Functions	274ms	54.8MiB
Computing $\hat{\sigma}(\boldsymbol{\xi}, \nu)$ and $P(\xi' \vert \xi, \hat{\sigma}(\boldsymbol{\xi}, \nu))$	272ms	274MiB
Computing VF Covariates	38.7s	15.1GiB
Computing Projection Objects	873ms	748MiB
Step 2: Recursive Estimator	45.7s	148GiB
Step 2: Naive Estimator	577s	2.19TiB
Step 2: BBL Estimator	250s	349GiB
Simulation	246s	334GiB
Optimisation	3.52s	15.1GiB
Step 2: HvB Estimator	244s	349GiB
Simulation	240s	334GiB
Optimisation	4.14s	15.7GiB

We conclude this section with an overview of the computational costs of each estimator. Table 3 displays run time and memory allocation for the different steps of our simulation, averaged over 25 runs.[12] As expected Step 1 – shared by all the algorithms considered – is fast, as it only entails estimating transition parameters by maximum likelihood and policy function parameters by linear regression.

On the other hand, there is substantial variation in the time and memory costs of various Step 2 algorithms. The recursive estimator is the fastest algorithm of the four: between computing the objects needed to form the objective (39.5s) and estimation (45.7s), the estimator takes approximately 1m25s to return an estimate. BBL-type estimators take substantially longer, with both the HvB and BBL estimator taking more than 4 minutes. Of these 4 minutes, only 4 seconds are spent in optimisation, with continuation value simulation making up the rest of the time. Finally, the naïve estimator takes the most time of

[12]'Preliminaries' are objects that only need to be computed once to set up multiple simulations.

all, clocking at more than 10 minutes. The reason for this substantial cost is that evaluation of its objective and grandient requires solving a large number of static optimisation problems.

6 Conclusion

We have revisited the estimation of dynamic games with continuous controls. We note that the commonly applied inequality estimator of Bajari et al. (2007) does not fully exploit the structure of optimal policies and propose an estimator that does so. Our estimator combines two-step methods that are common in the estimation of dynamic models with indirect inference ideas. We conduct a Monte Carlo experiment based on an empirically-relevant model and find that the estimator we propose significaly outperforms available alternatives. Bajari et al. (2007) themselves propose an estimator that does use the structure of optimal policies. However, the empirical literature has converged to applying exclusively their inequality estimator. We hope that by providing clear implementation details and documenting the substantial advantages of fully exploiting the structure of the model, our contribution will steer the literature towards doing so whenever feasible.

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Appendices

Appendix A The Gradient of $Q(\theta)$

We have

$$\nabla Q(\theta) = 2\underset{1\times(k+1)}{[\hat{\gamma}(\theta) - \hat{\gamma}]'}\underset{(k+1)\times(k+1)}{W}\underset{(k+1)\times J}{D_\theta \hat{\gamma}(\theta)}$$

where $D_\theta \hat{\gamma}(\theta)$ is the derivative (i.e., the Jacobian) of $\hat{\gamma}(\theta)$ and we have made the dimensions explicit for clarity. The symbols k and J represent, respectively, the number of covariates in the empirical policy function and the number of structural parameters to be estimated. These objects have dimension $(k + 1)$ rather than k because we include the estimated standard deviation in our objective function.

From the above, all that is left to calculate is $D_\theta \hat{\gamma}(\theta)$. Remember that $\hat{\gamma}(\theta) = (\hat{\beta}(\theta), S(\theta))$. Therefore

$$\underset{(k+1)\times J}{D_\theta \hat{\gamma}(\theta)} = \begin{pmatrix} \underset{k\times J}{D_\theta \hat{\beta}(\theta)} \\ \underset{1\times J}{\nabla_\theta S(\theta)} \end{pmatrix}$$

The object $\hat{\beta}(\theta)$ is an OLS estimate, and thus satisfies

$$(X'X)\hat{\beta}(\theta) = X'T_\theta(\hat{\sigma})$$

where X is the matrix of features

$$X = \begin{pmatrix} \phi_1(\xi_1) & \dots & \phi_k(\xi_1) \\ \vdots & \ddots & \vdots \\ \phi_1(\xi_N) & \dots & \phi_k(\xi_N) \end{pmatrix}$$

and[13]

$$T_\theta(\hat{\sigma}) = \begin{pmatrix} T_\theta(\hat{\sigma})(\xi_1, \nu_1) \\ \vdots \\ T_\theta(\hat{\sigma})(\xi_N, \nu_N) \end{pmatrix},$$

where $T_\theta(\hat{\sigma})(\xi_i, \nu_i)$ is the optimal level of investment when the parameters are

[13] We draw the ν_i shocks once and store them in memory so that all parameter values use the same ν_i shocks.

θ, future behavior is given by $\hat{\sigma}$, and the state is (ξ_i, ν_i). Therefore,

$$(X'X)D_\theta\hat{\beta}(\theta) = X'D_\theta T_\theta(\hat{\sigma}) \tag{15}$$

where

$$D_\theta T_\theta(\hat{\sigma}) = \begin{pmatrix} \nabla_\theta T_\theta(\hat{\sigma})(\xi_1, \nu_i) \\ \vdots \\ \nabla_\theta T_\theta(\hat{\sigma})(\xi_N, \nu_N) \end{pmatrix}$$

By the Implicit Function Theorem, the gradients in this matrix are given by

$$\nabla_\theta T_\theta(\hat{\sigma})(\xi_i, \nu_i) = -\left[\frac{\partial f}{\partial x}(x^*, \xi_i, \nu_i; \theta, \hat{\sigma})\right]^{-1} \nabla_\theta f(x^*, \xi_i, \nu_i; \theta, , \hat{\sigma})$$

where $x^* = T_\theta(\hat{\sigma})(\xi, \nu)$ and $f(x, \xi, \nu; \theta, \hat{\sigma})$ is the investment first-order condition. We can then solve for $D_\theta\hat{\beta}$ from equation 15.

Next, we need $\nabla_\theta S(\theta)$. We have

$$S(\theta) = \left\{\frac{1}{n-k}\sum_{i=1}^{n}[T_\theta(\hat{\sigma})(\xi_i, \nu_i) - x(\xi_i)'\hat{\beta}(\theta)]^2\right\}^{\frac{1}{2}}$$

where $x(\xi_i) := (\phi_1(\xi_i) \ldots \phi_k(\xi_i))'$. Therefore,

$$\begin{aligned} \nabla_\theta S(\theta) &= \frac{1}{2}\{\cdot\}^{-\frac{1}{2}} \times \frac{2}{n-k}\sum_{i=1}^{n}[T_\theta(\hat{\sigma})(\xi_i, \nu_i) - x(\xi_i)'\hat{\beta}(\theta)]\big\{\nabla_\theta T_\theta(\hat{\sigma})(\xi_i, \nu_i) - \nabla_\theta[x(\xi_i)'\hat{\beta}(\theta)]\big\} \\ &= \frac{1}{S(\theta)(n-k)}\sum_{i=1}^{n}[T_\theta(\hat{\sigma})(\xi_i, \nu_i) - x(\xi_i)'\hat{\beta}(\theta)]\big\{\nabla_\theta T_\theta(\hat{\sigma})(\xi_i, \nu_i) - \nabla_\theta[x(\xi_i)'\hat{\beta}(\theta)]\big\} \end{aligned}$$

The first gradient in these expressions has already been characterized. The second gradient is

$$\nabla_\theta[x(\xi_i)'\hat{\beta}(\theta)] = x(\xi_i)'D_\theta\hat{\beta}(\theta)$$

and $D_\theta\hat{\beta}(\theta)$ has just been characterized.