Recursivity and the Estimation of Dynamic Games with Continuous Controls

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Abstract

We study the estimation of dynamic games with continuous control variables, such as investments in R&D, quality, and capacity. We use the recursive characterization of Markov Perfect Equilibria (MPE) to develop estimators that exploit the structure of optimal policies. In particular, we derive a pseudo maximum likelihood estimator for models with shocks to firms' marginal costs of investment. We evaluate the performance of these estimators in two Monte Carlo exercises, including a version of the Hashmi and van Biesebroeck (2016) model of innovation in the automobile industry extended to allow for entry and exit. We find that estimators based on recursive equilibrium conditions perform well and outperform the inequality estimator of Bajari, Benkard, and Levin (2007).

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1 Introduction

Many questions of interest to Industrial Organization economists involve firm choices that have persistent effects on market conditions. Such choices include investments in research and development, the choice of productive capacity, and the choice of product characteristics. Many other examples can be given. Decisions of this type are inherently dynamic and are often taken in industries with few firms. Therefore, their study necessitates the use of dynamic oligopoly models. Furthermore, many of these choices, such as the ones above, are naturally modeled as continuous variables.

This paper studies the estimation of dynamic oligopoly models with continuous controls. We propose to estimate such models using estimators that exploit the recursive optimality conditions that characterize equilibrium behavior. In particular, we derive a pseudo maximum likelihood estimator based on those conditions. Estimators based on optimality conditions for firm behavior stand in contrast to the estimator proposed by Bajari et al. (2007), based on value function inequalities. That seminal contribution made the estimation of dynamic games with continuous controls feasible, and the methods therein have been widely applied in empirical work. Because the estimators we propose make fuller use of the model structure, we expect them to exhibit improved econometric performance. We evaluate the performance of our proposed estimators relative to multiple implementations of the Bajari et al. (2007) estimator in a set of Monte Carlo exercises.

We base our Monte Carlo exercises on two models. First, a model closely inspired by the quality ladder game in Bajari et al. (2007). Second, an extension of the Hashmi and van Biesebroeck (2016) model of innovation in the automobile industry, here adapted to allow for firm entry and exit. The latter model is the framework of an actual empirical application, and thus an accurate representation of models used in empirical work by Industrial Organization economists. Estimating this model serves to illustrate that the methods propose here are computationally feasible for empirically relevant models.

The first step in our proposed estimation routine consists of estimating policy functions and state transitions from the data. This step is similar to the Bajari et al. (2007) estimator, as well as estimators of dynamic games with discrete controls such as the ones proposed by Aguirregabiria and Mira (2007), Pakes, Ostrovsky, and Berry (2007), and Pesendorfer and Schmidt-Dengler (2008). We depart from prior literature in the second step. We use the estimated policy functions and state transitions to form an empirical analogue of the maximization objective in firms' Bellman equations. We use that object to derive contributions to a pseudo likelihood function. That derivation rests on the monotonicity of firms' policy functions, a property for which we provide two sets of sufficient conditions.[1] This pseudo MLE estimator applies to models with private information shocks to firms' marginal costs of setting their continuous control, for instance investment or capacity adjustment costs. When those shocks are not present, as in the Bajari et al. (2007) and Ryan (2012) models, one can still use recursive optimality conditions to solve for firms' optimal behavior, and use those to form an objective function for estimation. We follow this approach to estimate the parameters of the Bajari et al. (2007) model in our first Monte Carlo exercise.

In our Monte Carlo exercises, we evaluate the performance of estimators based on recursive equilibrium conditions and the Bajari et al. (2007) estimator (henceforth, BBL estimator). We consider three implementations of the BBL estimator, each using different forms of policy deviations: additive, multiplicative, and what we term asymptotic. The first one uses additive perturbations to the estimated policy function, as in Bajari et al. (2007) and Ryan (2012). The second one uses multiplicative perturbations, as in Hashmi and van Biesebroeck (2016) and recommended by Srisuma (2013). The third uses the asymptotic distribution of the empirical policies to construct deviations.

We find that estimators based on recursive equilibrium conditions perform well in estimating both the Bajari et al. (2007) and the Hashmi and van Biesebroeck (2016) models. In estimating the Bajari et al. (2007) model, the BBL estimator with additive deviations performs well, but the implementations with multiplicative and asymptotic deviations perform poorly. In estimating the Hashmi and van Biesebroeck (2016) model, all three implementations of the BBL estimator perform poorly. Taken together, these results show, first, that the performance of the BBL estimator depends on the choice of deviations. Second, that estimators based on recursive equilibrium conditions perform at least as well as the BBL estimator and can significantly outperform it in empirically relevant settings.

A number of empirical papers apply the BBL estimator. These include Ryan (2012), Hashmi and van Biesebroeck (2016), Fowlie, Reguant, and Ryan (2016),

[1] The first of these sets of sufficient conditions builds closely on Doraszelski and Satterthwaite (2010). That paper also establishes the existence of Markov Perfect Equilibria in dynamic games with continuous controls without shocks to firms' marginal costs. We extend the existence result to allow for such shocks.

Liu and Siebert (2022), and Egan, Hortaçsu, Kaplan, Sunderam, and Yao (2025). Other papers have used firms' optimality conditions to estimate dynamic games with continuous controls. Jofre-Bonet and Pesendorfer (2003) show, in a dynamic auction model, that firms' first-order conditions and the observed distribution of bids identify the distribution of firms' costs. Lim and Yurukoglu (2018) estimate a model of regulation of a monopolist electricity distributor who engages in a dynamic investment choice. They estimate their model matching policy functions estimated in a first stage to those implied by solving an empirical Bellman equation. This bears similarity to our approach, in particular to our estimation of the Bajari et al. (2007) model. Our pseudo MLE estimator, which applies to models with private information shocks to the cost of setting the continuous control, differs significantly from these prior contributions, performs well in our Monte Carlo experiment, and is computationally cheaper as it avoids solving for optimal behavior.

A paper closely related to ours is Srisuma (2013). He observes that the BBL inequalities may fail to identify structural parameters and proposes an estimator that makes use of agents' optimization problems in a two-step procedure. Srisuma's estimator is based on minimizing a distance between the observed conditional distributions of agents' actions and the one implied by agents' (pseudo) maximization problems. This is an expensive objective to compute for the models that practitioners take to data. Indeed, the Monte Carlo simulations in Srisuma (2013) are based on simple static models. Finally, Wang and Zhai (2025) propose a pseudo MLE estimator to estimate a dynamic model of institutional landlords' investments in the single-family housing market. That contribution is complementary to ours in that it supports the notion that the ideas discussed here are widely applicable to empirically-relevant models. Our derivation of the pseudo MLE estimator differs from theirs.

The rest of the paper is organized as follows. In Section 2 we discuss a general model of dynamic competition in an oligopolistic industry. In Section 3 we introduce the pseudo-MLE estimator based on firms' recursive optimality conditions and briefly review the BBL inequality estimator. In Section 4 we report the results from our Monte Carlo exercises. Section 5 concludes.

2 The Economic Model

We model the dynamic interaction between oligopolistic competitors. There are $\bar{N}$ firms in the market, including N incumbents and $\bar{N} - N$ potential entrants. $\bar{N}$ is a parameter of the model whereas N is an endogenous variable. Each firm has characteristics $\xi_i \in \Xi$. The set of possible firm characteristics Ξ satisfies $\Xi \subset \mathbb{R} \cup \{-\infty\}$ and $-\infty \in \Xi$, where $-\infty$ represents the firm being inactive. Time is discrete and the horizon is infinite. The state of the industry at time t is $\boldsymbol{\xi}_t = (\xi_{1t}, \ldots, \xi_{\bar{N}t})$.[2]

At the beginning of the period firm i earns flow profit $\pi_i(\boldsymbol{\xi}_t)$. If $\xi_i = -\infty$, then $\pi_i(\boldsymbol{\xi}_t) = 0$. Flow profits are typically modeled as the outcome of competition in static variables such as prices or quantities. We do not need to specify the underlying model that generates π_i. Rather, we treat these functions as parameters of the dynamic game. We assume that the functions π_i are symmetric, i.e., that

$$\pi_i(\xi_i, \xi_2, \ldots, \xi_{i-1}, \xi_1, \xi_{i+1}, \ldots, \xi_{\bar{N}}) = \pi_1(\boldsymbol{\xi}) = \pi(\boldsymbol{\xi}) \quad \text{for all } i = 2, \ldots, \bar{N} \tag{1}$$

and

$$\pi(\boldsymbol{\xi}_1, \boldsymbol{\xi}_{-1}) = \pi(\boldsymbol{\xi}_1, \boldsymbol{\xi}_{p(-1)}) \tag{2}$$

for any permutation $p(\cdot)$ of the indices $2, \ldots, \bar{N}$ – see, e.g., Doraszelski and Satterthwaite (2010).[3]

After firms earn profits, incumbents privately observe scrap values $\rho_{it} \in \mathbb{R}_+$ and potential entrants privately observe entry costs $\phi_{it} \in \mathbb{R}_+$. Scrap values and entry costs are iid draws from the distributions F_ρ and F_ϕ, respectively. Upon observing these random variables, firms simultaneously decide whether or not to be active in period $t+1$. We denote the decision to be active by $\alpha_{it} = 1$; choosing not to be active is represented by $\alpha_{it} = 0$. Firms who decide not to be active in the next period perish and are replaced by new potential entrants.

Besides entry and exit decisions, firms also invest to affect the evolution of their characteristics ξ_{it}. Investment choices are denoted by $x_{it} \in \mathbb{R}_+$. After entry and exit decisions are made, all firms that chose to be active in $t+1$ privately observe investment cost shocks $\nu_{it} \in \mathcal{S} \subseteq \mathbb{R}$. Investment cost shocks are iid

[2] It is straightforward to accommodate exogenous states that capture, e.g., changing demand and/or cost conditions.

[3] These conditions are sometimes called, respectively, symmetry and anonymity – see, e.g., Doraszelski and Pakes (2007). Doraszelski and Satterthwaite (2010) call a set of functions symmetric if they satisfy both conditions. We adopt their terminology.

draws from the distribution F_ν. Those firms then simultaneously choose their levels of investment and incur investment costs $c(x_{it}, \nu_{it})$.[4]

Assumption 1 (Investment Cost (IC))**.** The investment cost function $c(x, \nu)$ is strictly increasing and convex in x, and the marginal cost of investment is increasing in the cost shock ν, i.e.

$$\frac{\partial c(x,\nu)}{\partial x} > 0, \quad \frac{\partial^2 c(x,\nu)}{\partial x^2} \geq 0, \quad \text{and} \quad \frac{\partial^2 c(x,\nu)}{\partial \nu \partial x} > 0 \; .$$

Assumption 2 (Conditional Independence and No Spillovers (CINS))**.** The distribution of $\boldsymbol{\xi}_{t+1}$ conditional on $\boldsymbol{\xi}_t$ and firms' action profile $\boldsymbol{a}_t = (a_{1t}, \ldots, a_{\bar{N}t})$, where $a_{it} = (\alpha_{it}, x_{it})$, satisfies

$$F_{\boldsymbol{\xi}}(\boldsymbol{\xi}_{t+1} \mid \boldsymbol{\xi}_t, \boldsymbol{a}_t) = \int \prod_{i=1}^{\bar{N}} F_\xi(\xi_{t+1} \mid \xi_t, a_{it}, \eta_t) \, \mathrm{d}F_\eta \; . \tag{3}$$

In equation (3), $\eta \in \mathbb{R}$ is an aggregate shock that affects the transition of all firms' characteristics and is independent and identically distributed over time, with distribution F_η. It is realized after firms make their entry, exit, and investment decisions. Equation (3) rules out spillover effects in the evolution of firms' characteristics. Such dependence would not raise conceptual difficulties, but we rule it out to simplify the exposition and align with the literature. Finally, we restrict the transition function F_ξ so that firms move to state $\xi = -\infty$ only if they choose not to be active in the following period: for all $\xi \in \Xi$ and $x_t \in \mathbb{R}_+$, $F_\xi(-\infty \mid \xi, (\alpha, x), \eta) = 1 - \alpha$. In what follows, we will write $F_\xi(\xi_{t+1} \mid \xi_t, x_t, \eta)$ instead of $F_\xi(\xi_{t+1} \mid \xi_t, (1, x_t), \eta)$. We assume that the map $x \mapsto F_\xi(\cdot \mid \xi, x, \eta)$ is continuous for all $\xi \in \Xi$ and $\eta \in \text{supp}(F_\eta)$.[5]

2.1 Equilibrium Concept

Given the assumptions we have made imply *ex-ante* firm symmetry, our attention will be directed towards Symmetric Markov Perfect Equilibria (SMPE). The literature has largely focused on symmetric environments and SMPEs due

[4]We restrict attention to scalar firm characteristics ξ_{it} and investment x_{it}. Accommodating multidimensional characteristics and actions is conceptually straightforward but may generate computational challenges. Perhaps for this reason, we are unaware of papers estimating or studying dynamic games with multiple continuous choices. We thus restrict attention to the scalar case as that covers most (perhaps all) of the literature and saves on notation.

[5]As we will soon restrict ourselves to finite Ξ and $\text{supp}(F_\eta)$, continuity is taken to mean continuity of a map from the real line to a higher dimensional Euclidean space in the usual sense.

to their computational convenience. The Markov restriction constrains firm behavior to only depend on payoff relevant variables: publicly observed firm characteristics $\boldsymbol{\xi}_t$ and private information $\varepsilon_{it} = (\rho_{it}, \phi_{it}, \nu_{it})$. The symmetry restriction imposes that value and policy functions satisfy conditions analogous to (1) and (2). Under condition (1), it suffices to compute policy and value functions from the perspective of firm 1. We thus focus on firm 1's dynamic programming problems without loss of generality. Moreover, under condition (2) we can compute value and policy functions on a reduced state space. Instead of considering the original state space $\boldsymbol{\Xi} := \Xi^{\bar{N}}$, we can map states that are equivalent from firm 1's perspective onto an arbitrary member of that equivalence class. For instance, suppose $\bar{N} = 3$ and $\Xi = \{-\infty, 1, 2\}$. Then $\boldsymbol{\xi} = (1, 1, 2)$ and $\tilde{\boldsymbol{\xi}} = (1, 2, 1)$ are equivalent from firm 1's perspective. It suffices to compute value and policy functions for one of these two states. We focus on the reduced state space $\boldsymbol{\Xi}^R := \{\boldsymbol{\xi} \in \boldsymbol{\Xi} : \xi_2 \leq \xi_3 \leq \ldots \leq \xi_{\bar{N}}\}$. Given a state $\boldsymbol{\xi} \in \boldsymbol{\Xi}^R$, we denote by $\boldsymbol{\xi}_j$ the corresponding state in $\boldsymbol{\Xi}^R$ from the perspective of firm j, i.e., $\boldsymbol{\xi}_j = (\xi_j, s(\boldsymbol{\xi}_{-j}))$, where ξ_j is the j-th coordinate of $\boldsymbol{\xi}$ and $s(\boldsymbol{\xi}_{-j})$ denotes $\boldsymbol{\xi}_{-j}$ sorted in an increasing order.

In what follows, we will denote a strategy by

$$\sigma(\boldsymbol{\xi}, \varepsilon) = \left(\alpha^I(\boldsymbol{\xi}, \rho), \alpha^E(\boldsymbol{\xi}, \phi), \sigma^x(\boldsymbol{\xi}, \nu)\right) ,$$

where $\varepsilon = (\rho, \phi, \nu)$ and α^I and α^E denote, respectively, the incumbent's and entrant's decision to be active in $t + 1$.

2.2 The Incumbent's Problem

In what follows, we denote by $V_I(\boldsymbol{\xi}, \rho)$ the expected net present value (ENPV) of an incumbent faced with public state $\boldsymbol{\xi}$ and scrap value ρ and by $V_I^A(\boldsymbol{\xi}, \nu)$ the ENPV of an incumbent that has chosen to be active and has observed investment cost shock ν.

2.2.1 An Active Incumbent's Investment Problem

The objects $V_I^A(\boldsymbol{\xi}, \nu)$ and $V_I(\boldsymbol{\xi}, \rho)$ are related through the Bellman equation

$$V_I^A(\boldsymbol{\xi}, \nu) = \max_{x \in \mathbb{R}_+} \left\{ \pi(\boldsymbol{\xi}) - c(x, \nu) + \beta \mathbb{E}\left[V_I(\boldsymbol{\xi}', \rho) \mid \boldsymbol{\xi}, x, \sigma\right] \right\} \tag{4}$$

where

$$\mathbb{E}[V_I(\boldsymbol{\xi}', \rho) \mid \boldsymbol{\xi}, x, \sigma] = \int_{\boldsymbol{\varepsilon}_{-1}} \int_{\boldsymbol{\xi}'} \int_{\rho} V_I(\boldsymbol{\xi}', \rho) \, \mathrm{d}F_\rho \, \mathrm{d}F(\boldsymbol{\xi}' \mid \boldsymbol{\xi}, (1, x), \sigma_{-1}(\boldsymbol{\xi}, \boldsymbol{\varepsilon}_{-1})) \, \mathrm{d}G_{\boldsymbol{\varepsilon}_{-1}} \tag{5}$$

and $\sigma_{-1}(\boldsymbol{\xi}, \boldsymbol{\varepsilon}_{-1}) = (\sigma(\boldsymbol{\xi}_2, \varepsilon_2), \ldots, \sigma(\boldsymbol{\xi}_{\bar{N}}, \varepsilon_{\bar{N}}))$.

Let $\bar{V}_I(\boldsymbol{\xi}) := \int_\rho V_I(\boldsymbol{\xi}, \rho) \, \mathrm{d}F_\rho$. By conditional independence

$$\int_{\boldsymbol{\xi}'} \bar{V}_I(\boldsymbol{\xi}') \, \mathrm{d}F(\boldsymbol{\xi}' \mid \boldsymbol{\xi}, (1, x), \sigma_{-1}(\boldsymbol{\xi}, \boldsymbol{\varepsilon}_{-1})) = \int_\eta \int_{\xi_1'} W(\xi_1' \mid \boldsymbol{\xi}, \boldsymbol{\varepsilon}_{-1}, \sigma, \eta) \, \mathrm{d}F(\xi_1' \mid \xi_1, x, \eta) \, \mathrm{d}F_\eta$$

where $W(\xi_1' \mid \boldsymbol{\xi}, \boldsymbol{\varepsilon}_{-1}, \sigma, \eta)$ is given by

$$\int_{\xi_2'} \cdots \int_{\xi_{\bar{N}}'} \bar{V}_I(\xi_1', \boldsymbol{\xi}_{-1}') \, \mathrm{d}F(\xi_{\bar{N}}' \mid \xi_{\bar{N}}, \sigma(\boldsymbol{\xi}_{\bar{N}}, \varepsilon_{\bar{N}}), \eta) \ldots \mathrm{d}F(\xi_2' \mid \xi_2, \sigma(\boldsymbol{\xi}_2, \varepsilon_2), \eta)$$

is the incumbent's ENPV of starting a period with characteristic ξ_1' given $\boldsymbol{\xi}$, $\boldsymbol{\varepsilon}_{-1}$, η, and that its competitors behave according to σ.

Under the additional assumption of independence of the ε_i, the integral with respect to $\boldsymbol{\varepsilon}_{-1}$ can be written as a multiple integral. Changing the order of integration in (5), we have multiple terms of the form

$$\int_{\varepsilon_j} \int_{\xi_j'} \bar{V}_I(\boldsymbol{\xi}') \, \mathrm{d}F(\xi_j' \mid \xi_j, \sigma(\boldsymbol{\xi}_j, \varepsilon_j), \eta) \, \mathrm{d}G_{\varepsilon_j} = \int_{\xi_j'} \bar{V}_I(\boldsymbol{\xi}') \, \mathrm{d}F^\sigma(\xi_j' \mid \boldsymbol{\xi}_j) \tag{6}$$

where

$$F^\sigma(\xi_j' \mid \boldsymbol{\xi}_j, \eta) := \int_{\varepsilon_j} F(\xi_j' \mid \xi_j, \sigma(\boldsymbol{\xi}_j, \varepsilon_j), \eta) \, \mathrm{d}G_{\varepsilon_j} \, . \tag{7}$$

It is clear that that equation (6) holds when $F(\xi' \mid \xi, x, \eta)$ has a density for all (ξ, x, η) or Ξ is finite. It holds much more generally. Appendix A.1 states the relevant technical results. This implies that the ensuing discussion holds for continuous, discrete, and discrete-continuous public state transition processes.

In summary, we have

$$\mathbb{E}[V_I(\boldsymbol{\xi}', \rho) \mid \boldsymbol{\xi}, x, \sigma] = \int_\eta \int_{\xi_1'} W(\xi_1' \mid \boldsymbol{\xi}, F^\sigma, \eta) \, \mathrm{d}F(\xi_1' \mid \xi_1, x, \eta) \, \mathrm{d}F_\eta \, ,$$

where

$$W(\xi_1' \mid \boldsymbol{\xi}, F^\sigma, \eta) = \int_{\xi_2'} \cdots \int_{\xi_{\bar{N}}'} \bar{V}_I(\xi_1', \boldsymbol{\xi}_{-1}') \, \mathrm{d}F^\sigma(\xi_{\bar{N}}' \mid \boldsymbol{\xi}_{\bar{N}}, \eta) \ldots \mathrm{d}F^\sigma(\xi_2' \mid \boldsymbol{\xi}_2, \eta) \tag{8}$$

and F^σ is given by (7).[6]

Uniqueness of the Investment Decision. The first-order condition of the maximization problem on the right-hand side of (4) is

$$-\frac{\partial c(x,\nu)}{\partial x} + \beta \, \frac{\partial}{\partial x} \left(\int_\eta \int_{\xi_1'} W(\xi_1' \mid \boldsymbol{\xi}, F^\sigma, \eta) \, \mathrm{d}F(\xi_1' \mid \xi_1, x, \eta) \, \mathrm{d}F_\eta \right) \leq 0 \, ,$$

with equality if the solution is interior. It is desirable for the investment first-order condition to be sufficient for an optimum. Sufficiency is useful both in equilibrium computation and in estimation based on firms' optimal policies. To establish sufficiency of the investment first-order condition, we adapt the concept of UIC-admissible transitions functions by Doraszelski and Satterthwaite (2010) to the transition process outlined above.

Definition 1 (Doraszelski and Satterthwaite (2010))**.** The distribution $F(\xi' \mid \xi, x, \eta)$ is UIC-admissible if, for all $\xi', \xi \in \Xi$, $x \in \mathbb{R}_+$, and $\eta \in \mathrm{supp}(F_\eta)$,

$$F(\xi' \mid \xi, x, \eta) = L(\xi', \xi, \eta) + K(\xi', \xi, \eta) Q(\xi, x) \, ,$$

where $Q(\xi, x, \eta)$ is twice-continuously differentiable in x for all $\xi \in \Xi$ and $\eta \in \mathrm{supp}(F_\eta)$, and satisfies

$$\frac{\partial Q(\xi, x)}{\partial x} > 0 \quad \text{and} \quad \frac{\partial^2 Q(\xi, x)}{\partial x^2} < 0 \, .$$

Proposition 1. If Assumption 1 holds and $F(\xi' \mid \xi, x, \eta)$ is UIC-admissible, then the problem

$$\max_{x \in [0, \bar{x}]} \pi(\boldsymbol{\xi}) - c(x, \nu) + \beta \int_\eta \int_{\xi_i'} W(\xi_i' \mid \boldsymbol{\xi}, F^\sigma, \eta) \, \mathrm{d}F(\xi_i' \mid \xi_i, x, \eta) \, \mathrm{d}F_\eta$$

has a unique solution for all $\boldsymbol{\xi}$ and ν. Moreover, the maximizer is increasing in ν, and strictly so over the range of ν where it is interior.

Proof. See Appendix A.2. The proof of a unique solution is a rewriting of the proof of Proposition 3 in Doraszelski and Satterthwaite (2010) specialized to the transition process described above. □

[6] We abuse notation slightly by denoting the integrand in (8) by $W(\xi_1' \mid \boldsymbol{\xi}, F^\sigma, \eta)$ when we have already defined $W(\xi_1' \mid \boldsymbol{\xi}, \boldsymbol{\varepsilon}_{-1}, \sigma, \eta)$ above. We will, however, have no further need for the latter, and we thus retain the former.

2.2.2 The Incumbent's Exit Decision

The incumbent commits to an exit decision before observing investment cost shock ν. Therefore, it must make its decision on the basis of its expected continuation value conditional on it being active:

$$\bar{V}_I^A(\boldsymbol{\xi}) := \int V_I^A(\boldsymbol{\xi}, \nu)\, \mathrm{d}F_\nu\ . \tag{9}$$

Recall $V_I(\boldsymbol{\xi}, \rho)$ denotes the ENPV of an incumbent with scrap value ρ. Then

$$V_I(\boldsymbol{\xi}, \rho) = \max\left\{\pi(\boldsymbol{\xi}) + \rho, \bar{V}_I^A(\boldsymbol{\xi})\right\} = \max_{\chi\in\{0,1\}} \chi\bar{V}_I^A(\boldsymbol{\xi}) + (1-\chi)[\pi(\boldsymbol{\xi}) + \rho]\ . \tag{10}$$

The implied conditional probability of an incumbent remaining active is

$$\mathbb{P}\big(\alpha^I(\boldsymbol{\xi}, \rho) = 1 \mid \boldsymbol{\xi}\big) = F_\rho\big(\bar{V}_I^A(\boldsymbol{\xi}) - \pi(\boldsymbol{\xi})\big) \tag{11}$$

2.3 The Entrant's Problem

Denote by $V_E^A(\boldsymbol{\xi}_{-1}, \nu)$ the ENPV of a potential entrant that enters under public state $\boldsymbol{\xi}_{-1}$ and draws investment cost shock ν. This function is characterized by

$$V_E^A(\boldsymbol{\xi}_{-1}, \nu) = \max_{x\in\mathbb{R}_+}\left\{-c(x,\nu)+\beta\int\int_{\xi_1'} W(\xi_1' \mid (-\infty, \boldsymbol{\xi}_{-1}), F^\sigma, \eta)\, \mathrm{d}F(\xi_1' \mid \xi_e, x, \eta)\, \mathrm{d}F_\eta\right\}\ , \tag{12}$$

where $\xi_e \in \Xi$ is an exogenously specified initial quality level for potential entrants. In our simulations below we assume that $\xi_e = \min(\Xi \setminus -\infty)$.

Potential entrants either enter the market or perish. Therefore, their ENPV given entry cost ϕ is

$$V_E(\boldsymbol{\xi}_{-1}, \phi) = \max\left\{0, \bar{V}_E^A(\boldsymbol{\xi}_{-1}) - \phi\right\} = \max_{\chi\in\{0,1\}} \chi\big[\bar{V}_E^A(\boldsymbol{\xi}_{-1}) - \phi\big] \tag{13}$$

where we have normalized the value of entrants' outside option to zero and

$$\bar{V}_E^A(\boldsymbol{\xi}_{-1}) := \int V_E^A(\boldsymbol{\xi}_{-1}, \nu)\, \mathrm{d}F_\nu\ . \tag{14}$$

The conditional probability of entry is

$$\mathbb{P}\big(\alpha^E(\boldsymbol{\xi}_{-1}, \phi) = 1 \mid \boldsymbol{\xi}_{-1}\big) = F_\phi\big(\bar{V}_E^A(\boldsymbol{\xi}_{-1})\big) \tag{15}$$

2.4 Equilibrium

Definition 2. Let $\Xi_I := (\Xi \setminus \{-\infty\}) \times \Xi^{\bar{N}-1}$. A Symmetric Markov Perfect Equilibrium (SMPE) is a pair $(\bar{V}_I, \sigma)$ where $\bar{V}_I : \Xi_I \to \mathbb{R}$ and $\sigma = (\sigma^x, \alpha^E, \alpha^I)$ are such that

1. $\sigma^x(\boldsymbol{\xi}, \nu)$ solves the right-hand side of

$$V^A(\boldsymbol{\xi}, \nu) = \max_{x \in \mathbb{R}_+} \left\{ \pi(\boldsymbol{\xi}) - c(x, \nu) + \beta \int_\eta \int_{\xi_1'} W(\xi_1' \mid \boldsymbol{\xi}, F^\sigma, \eta) \, \mathrm{d}F(\xi_1' \mid \xi_1, x, \eta) \, \mathrm{d}F_\eta \right\} \tag{16}$$

for all $\boldsymbol{\xi} \in \Xi^N$ and ν in the support of F_ν, where $W(\xi' \mid \boldsymbol{\xi}, F^\sigma, \eta)$ is given by equation (8).

2. $\alpha^I(\boldsymbol{\xi}, \rho)$ solves problem (10) subject to (9) and (16), for all $\boldsymbol{\xi} \in \Xi_I$ and ρ in the support of F_ρ.

3. $\alpha^E(\boldsymbol{\xi}_{-1}, \phi)$ solves problem (13) subject to (14) and (12), for all $\boldsymbol{\xi}_{-1} \in \times \Xi^{\bar{N}-1}$ and ϕ in the support of F_ϕ.

4. For all $\boldsymbol{\xi} \in \Xi_I$, $\bar{V}_I(\boldsymbol{\xi}) = \int \{\alpha^I(\boldsymbol{\xi}, \rho) \bar{V}_I^A(\boldsymbol{\xi}) + (1 - \alpha^I(\boldsymbol{\xi}, \rho))[\pi(\boldsymbol{\xi}) + \rho]\} \, \mathrm{d}F_\rho$ where $\bar{V}_I^A(\boldsymbol{\xi})$ is given by (9).

We define an equilibrium only in terms of incumbents' integrated value functions. This is due to the assumption, common in this literature, that firms that choose to be inactive in the following period perish. As a result, when a firm's quality becomes $\xi' = -\infty$ (as a consequence of exit or no entry), that firm's continuation value is zero. Therefore, $\bar{V}_I$ alone is sufficient to determine firm behavior.[7]

To the best of our knowledge, the existing proofs of existence of a Markov Perfect Equilibrium in Ericson and Pakes (1995) type models – notably, Doraszelski and Satterthwaite (2010) – do not allow for shocks to firms' costs of investment. We thus provide an existence result.

Proposition 2. Suppose $|\Xi| < \infty$ and $|\mathrm{supp}(F_\eta)| < \infty$. Then, a Symmetric Markov Perfect Equilibrium exists.

Proof. See appendix A.3. □

[7]Note also that we do not define $\bar{V}_I$ on Ξ^R but rather on Ξ_I. Similarly we do not define the policy functions on the reduced state space. This is for the sake of precision, as strategies must be complete contingent plans. However, the discussion in Section 2.1 applies. In particular, when computing equilibria we do exploit symmetry, as discussed in this section.

The added complication brought about by investment cost shocks is that the investment policy is infinite-dimensional. Random scrap values and setup costs without investment cost shocks – as in Doraszelski and Satterthwaite (2010) – do not raise this difficulty. This is due to the fact that entry and exit policies have a cutoff structure, where the cutoffs depend only on the public state $\boldsymbol{\xi}$. This structure implies that entry and exit probabilities at a given public state or the cutoffs fully characterize entry and exit policy functions. Therefore, to establish the existence of an equilibrium in an environment with random scrap values and setup costs but deterministic cost of investment, one may invoke the Brouwer fixed-point theorem applied in the space of policy functions. That is not the case with investment cost shocks.

To establish Proposition 2, one potential strategy is to invoke a fixed-point theorem applicable to infinite-dimensional spaces. However, under the finiteness assumptions of that Proposition, the more elementary Brouwer fixed-point theorem suffices. The idea is to establish the existence of a fixed point of a certain map from the set of collections of probability distributions of the form $\{F^\sigma(\xi' \mid \boldsymbol{\xi}, \eta) : \xi' \in \Xi, \boldsymbol{\xi} \in \boldsymbol{\Xi}^R, \eta \in \mathrm{supp}(F_\eta)\}$ into itself. These collections of probability distributions are finite-dimensional objects under the finiteness assumption of Proposition 2, which enables the application of Brouwer's theorem. A symmetric Markov Perfect Equilibrium is then constructed using the strategy profile in which each firm plays the unique optimal policy subject to competitors' qualities having the transition described by the fixed-point. We provide further details in Appendix A.3.

This proof strategy underpins the method we employ to compute Symmetric Markov Perfect Equilibria. We start with guesses for $\bar{V}_I(\boldsymbol{\xi})$ and $F^\sigma(\xi' \mid \boldsymbol{\xi}, \eta)$. With these two objects we can compute $W(\xi' \mid \boldsymbol{\xi}, F^\sigma, \eta)$. We then solve for firms' optimal investment and entry and exit decisions, i.e., we perform the computations associated with conditions 1 to 3 in definition 2. With these we can update F^σ.[8] We compute the integral in condition 4 exploiting the fact that

$$\bar{V}_I(\boldsymbol{\xi}) = F_\rho(\bar{V}_I^A(\boldsymbol{\xi}) - \pi(\boldsymbol{\xi}))[\bar{V}_I^A(\boldsymbol{\xi}) - \pi(\boldsymbol{\xi})] + \pi(\boldsymbol{\xi}) + \int_{\bar{V}_I^A(\boldsymbol{\xi}) - \pi(\boldsymbol{\xi})}^{\infty} \rho \, \mathrm{d}F_\rho \,. \tag{17}$$

For suitable distributions, e.g. the exponential and lognormal, the integral in this equation can be written in closed form. We iterate on these steps until

[8]Updating F^σ involves an integral with respect to F_ν – see equation (42) in Appendix A.3. The choice of approximant to that integral will determine the values of ν for which we compute firms' optimal investment choices.

both $\bar{V}_I$ and F^σ converge. As the firms' objective function depends solely and continuously on these objects, Berge (1963)'s Maximum Theorem implies that the policy functions associated with the firms' problem also converge. By the symmetry assumption, it is sufficient to compute $\bar{V}_I(\boldsymbol{\xi})$ and $F^\sigma(\xi' \mid \boldsymbol{\xi}, \eta)$ for $\boldsymbol{\xi} \in \Xi^R$. As shown in Pakes and McGuire (1994), the reduced state space Ξ^R grows in the number of firms as a polynomial of order $|\Xi|$ rather than exponentially. For instance, symmetry reduces the state space cardinality of the Hashmi and van Biesebroeck (2016) model we consider from 1,048,576 to 62,016.

3 Estimators

This section introduces a pseudo-MLE estimator based on firms' optimal behavior in subsection 3.1 and reviews the BBL estimator in 3.2. Subsection 3.3 discusses the estimation of firms' integrated value functions, which is a key input to both estimators.

3.1 A Pseudo MLE Estimator Based on Recursive Equilibrium Conditions

In this section we discuss estimators based on firms' optimal behavior, i.e. behavior consistent with conditions 1 to 3 in Definition 2. Let the investment cost function depend on parameters $\boldsymbol{\theta}_x$, the distribution of scrap values depend on parameters $\boldsymbol{\theta}_\rho$, and the distribution of entry costs depend on parameters $\boldsymbol{\theta}_\phi$. We denote these parameters collectively by $\boldsymbol{\theta}$. We also use the notation $\boldsymbol{\theta}_{-\phi} := (\boldsymbol{\theta}_x, \boldsymbol{\theta}_\rho)$. Our goal is to estimate $\boldsymbol{\theta}$. To simplify the notation and discussion, we assume that there is no aggregate shock.

Suppose we can obtain an estimate (up to parameters) $\widehat{\bar{V}}_I(\boldsymbol{\xi}; \boldsymbol{\theta}_{-\phi})$ of the ex-ante value function $\bar{V}_I(\boldsymbol{\xi})$.[9] Suppose we also have estimates of $F(\xi' \mid \xi, x)$ and $F^\sigma(\xi' \mid \boldsymbol{\xi})$, defined in equation (7). These estimates allow us to set up an empirical analog of firms' investment problem:

$$\max_{x \in \mathbb{R}_+} \left\{ \pi(\boldsymbol{\xi}) - c(x, \nu; \boldsymbol{\theta}_x) + \beta \int_{\xi_1'} \widehat{W}(\xi_1' \mid \boldsymbol{\xi}, \boldsymbol{\theta}_{-\phi}) \, \mathrm{d}\hat{F}(\xi_1' \mid \xi_1, x) \right\}, \tag{18}$$

where $\widehat{W}(\xi_1' \mid \boldsymbol{\xi}, \boldsymbol{\theta}_{-\phi})$ is given by (8) substituting $\widehat{\bar{V}}_I(\boldsymbol{\xi}; \boldsymbol{\theta}_{-\phi})$ for $\bar{V}_I(\boldsymbol{\xi})$ and $\hat{F}^\sigma$ for

[9] We discuss alternative estimators of $\bar{V}_I(\boldsymbol{\xi})$ in Section 3.3. Observe that our notation indicates that these value-function estimates do not depend on $\boldsymbol{\theta}_\phi$. We discuss why below.

F^σ. We base estimation on necessary conditions for observed investment levels to solve (18) and on empirical analogs to equations (11) and (15). We sketch the main ideas below, and save full details for Appendix A.5.

Let x be an observed investment level. The conditional distribution of investment given $\boldsymbol{\xi}$ is given by

$$\begin{aligned} F_X(x \mid \boldsymbol{\xi}) &= \mathbb{P}(\sigma^x(\boldsymbol{\xi}, \nu) \leq x \mid \boldsymbol{\xi}) \\ &= \mathbb{P}\left(\nu \geq (\sigma^x)^{-1}(x; \boldsymbol{\xi}) \mid \boldsymbol{\xi}\right) \\ &= 1 - F_\nu\left((\sigma^x)^{-1}(x; \boldsymbol{\xi})\right) , \end{aligned} \tag{19}$$

where $(\sigma^x)^{-1}(x;\boldsymbol{\xi})$ is the inverse of the investment policy with respect to ν.[10] The second equality uses the monotonicity of the investment policy in ν, sufficient conditions for which are given in Proposition 1 and Proposition 3. When $x = 0$, equation (19) is the contribution to the likelihood. When $x > 0$, we differentiate (19) to obtain the conditional density $f_X(x \mid \boldsymbol{\xi})$. We obtain

$$f_X(x \mid \boldsymbol{\xi}) = -f_\nu\left((\sigma^x)^{-1}(x;\boldsymbol{\xi})\right) \cdot \frac{\partial}{\partial x}(\sigma^x)^{-1}(x;\boldsymbol{\xi}) \tag{20}$$

$$= -f_\nu\left((\sigma^x)^{-1}(x;\boldsymbol{\xi})\right) \times \frac{-\partial_x^2 c(x,(\sigma^x)^{-1}(x,\boldsymbol{\xi})) + \partial_x MB(\boldsymbol{\xi},x)}{\partial_{\nu x}^2 c(x,(\sigma^x)^{-1}(x,\boldsymbol{\xi}))} , \tag{21}$$

where $MB(\boldsymbol{\xi}, x) := \beta \int_{\xi_1'} W(\xi_1' \mid \boldsymbol{\xi}, \boldsymbol{\theta}_{-\phi}) \, \mathrm{d}F(\xi_1' \mid \xi_1, x)$. See Appendix A.5 for details.

We now turn to exit decisions. If a is an indicator for whether an incumbent chooses to be active, then the contribution to the likelihood is given by

$$aF_\rho\left(\widehat{\widehat{V}}_I^A(\boldsymbol{\xi};\boldsymbol{\theta}_{-\phi}) - \pi(\boldsymbol{\xi}); \boldsymbol{\theta}_\rho\right) + (1-a)\left[1 - F_\rho\left(\widehat{\widehat{V}}_I^A(\boldsymbol{\xi};\boldsymbol{\theta}_{-\phi}) - \pi(\boldsymbol{\xi}); \boldsymbol{\theta}_\rho\right)\right] ,$$

where $\widehat{\widehat{V}}_I^A(\boldsymbol{\xi};\boldsymbol{\theta}_{-\phi})$ is an estimate of the value of being active.

There are two approaches to computing the estimate of the value of being active. They trade-off computational cost and efficiency. The first approach makes use of an estimate of the policy function, $\hat{\sigma}^x(\boldsymbol{\xi}, \nu)$. With such an estimate in hand, we can estimate an incumbent's value of being active as

$$\widehat{\widehat{V}}_I^A(\boldsymbol{\xi};\boldsymbol{\theta}_{-\phi}) = \int V_I^A(\boldsymbol{\xi}, \nu; \hat{\sigma}^x, \boldsymbol{\theta}_{-\phi}) \, \mathrm{d}F_\nu ,$$

where $V_I^A(\boldsymbol{\xi}, \nu; \hat{\sigma}^x, \boldsymbol{\theta}_{-\phi})$ substitutes $\hat{\sigma}^x(\boldsymbol{\xi}, \nu)$ for x in the objective function of

[10] When $x = 0$ we define $(\sigma^x)^{-1}(0;\boldsymbol{\xi}) := \inf\{\nu \in \mathrm{supp}(F_\nu) : \sigma^x(\boldsymbol{\xi}, \nu) = 0\}$.

problem (18). This approach avoids solving that maximization problem altogether and is thus computationally cheap. The second approach consists in estimating $V_I^A(\boldsymbol{\xi}, \nu)$ by solving the maximization problem in (18) at pre-specified values of ν to approximate the integral over the shock. This approach is computationally more demanding, but uses information regarding how parameters affect optimal investment decisions, and is thus more efficient. In our Monte Carlo experiments in Section 4, we implement the first approach and report satisfactory results. The likelihood contribution of entry decisions is analogous to the contribution of exit decisions.

Let $L_X(\boldsymbol{\theta}; x)$ be the investment contribution to the likelihood and let $L_A(\boldsymbol{\theta}; a)$ be the contribution to the likelihood of firms' decisions to be active. The likelihood is then

$$L(\boldsymbol{\theta}) = \prod_{f,m} L_X(\boldsymbol{\theta}; x_{fm}) \times L_A(\boldsymbol{\theta}; a_{fm}) \ . \tag{22}$$

Alternative estimators based on recursive optimality conditions There are alternatives as to how to use firms' optimality conditions for estimation. For instance, in Section 4.1 we use a non-linear least squares estimator to estimate the Bajari et al. (2007) model. This estimator is based on minimizing squared deviations between observed investment levels and investment levels predicted by the model, as well as squared deviations between observed decisions to be active and predicted probabilities of being active. Model-predicted investment levels are obtained by solving the investment problem in (18). As above, predicted probabilities of being active can be obtained either by using an estimate of the investment policy, or by using the model-predicted optimal level of investment. We use the former approach.

The results in Section 4.1 show that this estimator works well. We use it to estimate the BBL model because that model does not have shocks to the cost of investment, making the PMLE estimator above inapplicable. We have also experimented with an indirect inference estimator based on solving for firms' optimal behavior. That estimator also performs well, but is also computationally more demanding than the PMLE estimator. For models with investment cost shocks, we favor the PMLE estimator, as it is significantly cheaper computationally.

3.2 A Review of the BBL Inequality Estimator

In this section we review the Bajari et al. (2007) estimator, the main point of comparison for the estimators proposed in Section 3.1.

The expected discounted stream of profits of an incumbent playing strategy $\tilde{\sigma}^I = (\tilde{\sigma}^x, \tilde{\alpha}^I)$ when all its competitors play strategy $\sigma = (\sigma^x, \alpha^I, \alpha^E)$ is given by

$$\bar{V}_I(\boldsymbol{\xi}; \tilde{\sigma}^I, \sigma, \boldsymbol{\theta}_{-\phi}) = \mathbb{E}\left\{ \sum_{t=0}^{\tau_e} \beta^t \left[\pi(\boldsymbol{\xi}_t) - c(\tilde{\sigma}^x(\boldsymbol{\xi}_t, \nu_t), \nu_t; \boldsymbol{\theta}_x)\right] + \beta^{\tau_e} \rho_{\tau_e} \,\middle|\, \boldsymbol{\xi}_0 = \boldsymbol{\xi} \right\} \tag{23}$$

where $\tau_e := \min\{t : \tilde{\sigma}^I(\boldsymbol{\xi}_t, \rho_t) = 0\}$ is the incumbent's potentially infinite exit date and the public state evolves according to the probability distribution induced by the policy functions $(\tilde{\sigma}^I, \sigma)$ and F_ν, F_ρ, F_ϕ. The expected discounted stream of profits of a potential entrant playing strategy $\tilde{\sigma}^E = (\tilde{\sigma}^x, \tilde{\alpha}^I, \tilde{\alpha}^E)$ when all its competitors play strategy σ is

$$\bar{V}_E(\boldsymbol{\xi}; \tilde{\sigma}^E, \sigma, \boldsymbol{\theta}) = \int\int \tilde{\alpha}^E(\boldsymbol{\xi}, \phi) v(\phi, \nu, \boldsymbol{\xi}; \boldsymbol{\theta}_{-\phi}) \, \mathrm{d}F_\nu \, \mathrm{d}F_\phi(\boldsymbol{\theta}_\phi)$$

where

$$\begin{aligned} v(\phi, \nu, \boldsymbol{\xi}; \boldsymbol{\theta}_{-\phi}) &:= -\phi - c(\tilde{\sigma}^x(\boldsymbol{\xi}, \nu), \nu; \boldsymbol{\theta}_x) \\ &\quad + \beta \int\int \bar{V}_I(\boldsymbol{\xi}'; \tilde{\sigma}^E, \sigma, \boldsymbol{\theta}_{-\phi}) \, \mathrm{d}F^\sigma(\boldsymbol{\xi}'_{-1} \mid \boldsymbol{\xi}) \, \mathrm{d}F(\xi'_1 \mid \xi_1, \tilde{\sigma}^x(\boldsymbol{\xi}, \nu)) \,. \end{aligned}$$

A symmetric strategy profile $(\sigma, \ldots, \sigma)$ is a Symmetric Markov Perfect Equilibrium only if, for all $\boldsymbol{\xi}$ and σ',

$$\bar{V}_I(\boldsymbol{\xi}; \sigma, \sigma, \boldsymbol{\theta}_{-\phi}) \geq \bar{V}_I(\boldsymbol{\xi}; \sigma', \sigma, \boldsymbol{\theta}_{-\phi}) \quad \text{and} \quad \bar{V}_E(\boldsymbol{\xi}; \sigma, \sigma, \boldsymbol{\theta}) \geq \bar{V}_E(\boldsymbol{\xi}; \sigma', \sigma, \boldsymbol{\theta}) \,. \tag{24}$$

Bajari et al. (2007) base their estimator on the equilibrium conditions (24).[11] Though their estimator can in principle be used to estimate set-identified models, it has not, to our knowledge, been applied as such. We thus focus on point-identified models, which we now define. Let $\mathcal{E}(\boldsymbol{\theta})$ be the set of SMPEs when the parameters of the model are given by $\boldsymbol{\theta}$.

Assumption 3 (Identification)**.** For any $\boldsymbol{\theta}, \boldsymbol{\theta}' \in \Theta$, $\mathcal{E}(\boldsymbol{\theta}) \cap \mathcal{E}(\boldsymbol{\theta}') = \emptyset$.

[11] Condition (24) is slightly weaker than Markov Perfect Equilibrium as it allows violations of optimality at seats of measure zero (according F_ρ and F_ϕ).

Given a state $\boldsymbol{\xi}$ and policy functions σ, σ', define

$$g(\boldsymbol{\xi}, \sigma', \sigma; \boldsymbol{\theta}) := \bar{V}(\boldsymbol{\xi}; \sigma, \sigma, \boldsymbol{\theta}) - \bar{V}(\boldsymbol{\xi}; \sigma', \sigma, \boldsymbol{\theta}) ,$$

where $\bar{V}$ ought to be interpreted as either $\bar{V}_I$ or $\bar{V}_E$ depending on the first coordinate of $\boldsymbol{\xi}$. Let H be a distribution over the space of pairs $(\boldsymbol{\xi}, \sigma')$. Define

$$Q(\boldsymbol{\theta}, \sigma) := \int \left(\min \left\{ g(\boldsymbol{\xi}, \sigma', \sigma; \boldsymbol{\theta}), 0 \right\} \right)^2 \mathrm{d}H(\boldsymbol{\xi}, \sigma') . \tag{25}$$

Let $\boldsymbol{\theta}_0$ denote the true parameters of the model. If $\sigma \in \mathcal{E}(\boldsymbol{\theta}_0)$, the equilibrium conditions above imply that $Q(\boldsymbol{\theta}_0, \sigma) = 0$. Under assumption 3, $\sigma \in \mathcal{E}(\boldsymbol{\theta}_0) \Rightarrow \sigma \notin \mathcal{E}(\boldsymbol{\theta}')$ if $\boldsymbol{\theta}' \neq \boldsymbol{\theta}_0$. Therefore, if $\boldsymbol{\theta}' \neq \boldsymbol{\theta}_0$, then there must exist $(\boldsymbol{\xi}, \sigma')$ for which $g(\boldsymbol{\xi}, \sigma', \sigma; \boldsymbol{\theta}') < 0$. It follows that, for an appropriate choice of H, $Q(\boldsymbol{\theta}', \sigma) > 0$.[12]

Bajari et al. (2007) propose estimating the structural parameters of the model by minimizing a sample analog of (25). In particular, given a set of $\{(\boldsymbol{\xi}_i, \sigma'_i)\}_{i=1}^{n_I}$ pairs and an estimate of the strategy profile $\hat{\sigma}$, they propose minimizing

$$\hat{Q}(\boldsymbol{\theta}, \hat{\sigma}) := \frac{1}{n_I} \sum_{i=1}^{n_I} \left(\min \left\{ g(\boldsymbol{\xi}_i, \sigma'_i, \hat{\sigma}; \boldsymbol{\theta}), 0 \right\} \right)^2 \quad i = 1, \ldots, n_I . \tag{26}$$

Evaluating this objective requires estimates of $\bar{V}(\boldsymbol{\xi}; \sigma', \hat{\sigma}, \boldsymbol{\theta})$. We review Bajari et al. (2007)'s proposal to obtain these estimates and alternatives in the next subsection.

3.3 Estimating Integrated Value Functions

Integrated value functions are a key input to all estimators discussed above. This subsection discusses methods to estimate them up to parameters.

Bajari et al. (2007) propose estimating integrated value functions by forward simulation. As they note, linearity of the value function with respect to $\boldsymbol{\theta}$, which is a feature of many models including those in Section 4, significantly reduces

[12] To be more precise, $Q(\boldsymbol{\theta}', \sigma) > 0$ requires that $g(\boldsymbol{\xi}, \sigma'; \sigma, \boldsymbol{\theta}') < 0$ on a set of positive H-measure for all $\boldsymbol{\theta}' \neq \boldsymbol{\theta}_0$. We can attach this condition to our definition of MPE. Given a measure μ on the set of tuples $(\boldsymbol{\xi}, \sigma')$, say that $(\sigma, \ldots, \sigma)$ is a symmetric MPE if $g(\boldsymbol{\xi}, \sigma'; \sigma, \theta_0) < 0$ with zero μ-measure. Then choose H such that μ is absolutely continuous with respect to H. If $\theta' \neq \theta_0$, assumption 3 implies that $g(\boldsymbol{\xi}, \sigma'; \sigma, \theta') < 0$ with positive μ-measure. This implies that $g(\boldsymbol{\xi}, \sigma'; \sigma, \theta') < 0$ with positive H-measure, otherwise absolute continuity of μ with respect to H would be violated. Thus, $Q(\theta', \sigma) > 0$. This hints at difficulties with the BBL approach: the measure H has to be rich, in the sense of $\mu \ll H$, where μ is itself rich enough that we are willing to define MPE on its basis. If H is not sufficiently rich, the equilibrium conditions may be violated at a set of positive μ-measure that is neglected by H. In this case $Q(\boldsymbol{\theta}', \sigma) = 0$.

the computational burden of forward simulation. Under linearity there exists a function $\bar{\Lambda}(\boldsymbol{\xi}, \sigma', \sigma)$ such that $\bar{V}(\boldsymbol{\xi}; \sigma', \sigma, \boldsymbol{\theta}) = \bar{\Lambda}(\boldsymbol{\xi}, \sigma', \sigma) \cdot \boldsymbol{\theta}$ and forward simulation need not be repeated as $\boldsymbol{\theta}$ varies.

Alternatively, as we now show, one can solve for $\bar{\boldsymbol{V}}_I(\boldsymbol{\xi}; \boldsymbol{\theta}_{-\phi})$ in closed form when the state space is finite. Let $P(\xi' \mid \xi, a)$ denote the probability that a firm's quality in $t+1$ is ξ' conditional on its current quality being ξ and its action being $a = (\alpha, x)$, where the notation is analogous to that in Section 2. Moreover, let Ξ_I^R denote the set of states in the reduced state space in which firm 1 is active, i.e., $\Xi_I^R := \{\boldsymbol{\xi} \in \Xi^R : \xi_1 > -\infty\}$. Let $\bar{\boldsymbol{V}}_I = [\bar{V}_I(\boldsymbol{\xi}) : \boldsymbol{\xi} \in \Xi_I^R]$ be a vector stacking incumbents' integrated values across states in Ξ_I^R. We show in appendix A.4 that $\bar{\boldsymbol{V}}_I$ satisfies

$$[I - \beta \boldsymbol{M}(\boldsymbol{P})] \bar{\boldsymbol{V}}_I = \boldsymbol{\pi} - \boldsymbol{K}(\boldsymbol{\theta}_x) + \boldsymbol{\Sigma}(F_\rho) \tag{27}$$

where

$$\boldsymbol{K}(\boldsymbol{\theta}_x) = \left[\mathbb{P}_I^A(\boldsymbol{\xi}) \int c(\sigma^x(\boldsymbol{\xi}, \nu), \nu; \boldsymbol{\theta}_x) \, \mathrm{d}F_\nu \ : \ \boldsymbol{\xi} \in \Xi_I^R \right] \tag{28}$$

$$\boldsymbol{\Sigma}(F_\rho) = \left[[1 - \mathbb{P}_I^A(\boldsymbol{\xi})] \mathbb{E}[\rho \mid \rho > F_\rho^{-1}(\mathbb{P}_I^A(\boldsymbol{\xi})); \boldsymbol{\theta}_\rho] \ : \ \boldsymbol{\xi} \in \Xi_I^R \right] \tag{29}$$

where $\mathbb{P}_I^A(\boldsymbol{\xi}) = \mathbb{P}(\alpha^I(\boldsymbol{\xi}, \rho) = 1 \mid \boldsymbol{\xi})$ is the probability that the incumbent chooses to be active in state $\boldsymbol{\xi}$ and $\boldsymbol{M}(\boldsymbol{P})$ is the transition matrix implied by the policy function σ, i.e.,[13]

$$\boldsymbol{M}(\boldsymbol{P}) = \left[\mathbb{P}^\sigma(\boldsymbol{\xi}_l \mid \boldsymbol{\xi}_k) \ : \ 1 \leq l, k \leq |\Xi_I^R| \right] \tag{30}$$

where

$$\mathbb{P}^\sigma(\boldsymbol{\xi}' | \boldsymbol{\xi}) = \prod_{j=1}^N P^\sigma(\xi_j' \mid \boldsymbol{\xi}) = \prod_{j=1}^N \int P(\xi_j' \mid \xi_j, \sigma(\boldsymbol{\xi}_j, \varepsilon)) \, \mathrm{d}G_\varepsilon \,. \tag{31}$$

Equations (27) to (31) imply that we can estimate $\bar{\boldsymbol{V}}_I$ up to parameters by estimating the probabilities $\mathbb{P}_I^A(\boldsymbol{\xi})$ and $P^\sigma(\xi' \mid \boldsymbol{\xi})$ and the investment policy function $\sigma^x(\boldsymbol{\xi}, \nu)$. The probabilities are directly estimable from the data on firms' characteristics. We discuss estimation of the investment policy function below.

Equation (27) allows us to solve efficiently for $\bar{\boldsymbol{V}}_I(\boldsymbol{\xi}; \boldsymbol{\theta}_{-\phi})$. That system of equations can be essentially solved only once, as we can compute a decomposition of $I - \beta \boldsymbol{M}(\boldsymbol{P})$ and store it in memory. Then, as we vary $\boldsymbol{\theta}_{-\phi}$, we only need

[13]We order states in Ξ^R and Ξ_I^R lexicographically, where firm 1 takes precedence over firm 2, who takes precedence over firm 3, so on and so forth. We do so by interpreting $\boldsymbol{\xi} = (\xi_1, \ldots, \xi_{\bar{N}})$ as a number in base $|\Xi|$.

to recompute the expected flow profits and solve the resulting system using the stored matrix decomposition.

Much like with forward simulation, linearity of flow payoffs is computationally helpful, as can be readily seen from equation (27). If the cost of investment is linear in parameters and the scrap value is either non-existent (as in Hashmi and van Biesebroeck (2016)), deterministic (as in Bajari et al. (2007)) or follows an exponential distribution, equation (27) implies that $\bar{V}_I$ is linear in parameters, i.e., $\bar{V}_I = [I - \beta \boldsymbol{M}(\boldsymbol{P})]^{-1} \boldsymbol{X} \boldsymbol{\theta}_{-\phi}$ for some matrix $\boldsymbol{X}$. In this case there are additional computational savings, as one can store the solution A to $[I - \beta \boldsymbol{M}(\boldsymbol{P})]A = \boldsymbol{X}$ in memory and only compute $A\boldsymbol{\theta}_{-\phi}$ as the parameters change.[14]

The trade-off between forward simulation and solving equation (27) is one between computational cost and simulation error. Forward simulation can be computationally cheaper when the state space is large, but involves simulation error that the closed-form solution avoids. Note that value function simulation is useful in conjunction with both estimators discussed above. If solving equation (27) is computationally infeasible, one can obtain the $\hat{W}(\xi_1' \mid \boldsymbol{\xi}, \boldsymbol{\theta}_{-\phi})$ estimates in equation (18) by forward simulation.

Estimation of the investment policy function. To estimate $\sigma^x(\boldsymbol{\xi}, \nu)$ we build on Bajari et al. (2007). Their argument implies, in the case in which $\sigma^x(\boldsymbol{\xi}, \nu)$ is decreasing in ν, that

$$\sigma^x(\boldsymbol{\xi}, \nu) = F_X^{-1}(1 - F_\nu(\nu) \mid \boldsymbol{\xi}) \ , \tag{32}$$

where $F_X(x \mid \boldsymbol{\xi})$ is the distribution of investment conditional on $\boldsymbol{\xi}$, which is identified. That is, the policy function is identified by the quantiles of the conditional distribution of investment.[15] Using this, the integral in equation (28)

[14]When these conditions on scrap values fail, equation (27) shows that $\bar{V}_I$ is no longer linear in parameters, even when the cost of investment is. However, the non-linearity arising from exit behavior does not add substantive computational burden to either approach to estimating integrated value functions. In the closed-form approach, one needs only to recompute the right-hand side of equation (27) as parameters change. When performing forward simulation, one can take $\tau \sim U[0,1]$ draws and use those to simulate exit decisions: incumbents remain active if and only if $\tau \leq P_I^A(\boldsymbol{\xi})$. As we vary structural parameters, these simulations do not need to be repeated. All that needs to be recomputed is the scrap value that accrues to firms when they do decide to exit, which is $F_\rho^{-1}(\tau; \boldsymbol{\theta}_\rho)$, when $\tau > P_I^A(\boldsymbol{\xi})$. The cost of repeatedly calling F_ρ^{-1} is typically dwarfed by the cost of repeating the simulation.

[15] This argument rests on the maintained assumption that F_ν is known. If F_ν is known up to parameters θ_ν, (32) must account for that dependence. This in turn implies that $\boldsymbol{K}(\boldsymbol{\theta}_x)$ in equation (28) also depends on $\boldsymbol{\theta}_\nu$. It can then be seen from equation (27) that the integrated

can be approximated by $\sum_{i=1}^{N} \omega_i c(F_X^{-1}(1 - F_\nu(\nu_i) \mid \boldsymbol{\xi}), \nu_i; \boldsymbol{\theta}_x)$, for judiciously chosen weights ω_i and nodes ν_i. Note that in principle this argument requires estimating investment quantiles at each element of Ξ_I^R, which is infeasible in practice. We estimate $F_X^{-1}(1 - F_\nu(\nu_i) \mid \boldsymbol{\xi})$ as the predicted values from quantile regressions of investment on features of $\boldsymbol{\xi}$.

4 Monte Carlo Simulations

We use two models to compare the performance of estimators based on recursive equilibrium conditions with that of the BBL estimator. First, we consider a model similar to the one simulated and estimated in Bajari et al. (2007). Then, we consider an extension of the model in Hashmi and van Biesebroeck (2016) that allows for entry and exit. Both models are private cases of the model presented in Section 2 and feature UIC-admissible transitions.

4.1 A Bajari et al. (2007) Inspired Model

There are $\bar{N}$ single-product firms in the market, N of which are active. We index a firm and its product by f. The quality of product f, denote ξ_f, is an element of the set $\Xi = \{-\infty, -20, -19, \ldots, 19, 20\}$. When consumer i purchases product f, she derives utility

$$u_{if} = \gamma_0 \ln(\xi_f) + \alpha p_f + \varepsilon_{if} \ , \tag{33}$$

where p_f is the price of product f and ε_{if} are independent and identically distributed Type 1 Extreme Value random variables. At the beginning of each period, incumbent firms compete in prices à la Nash-Bertrand and earn flow profits $\pi_f(\boldsymbol{\xi})$.

Firms invest to affect the quality of their product in the following period. If $-20 < \xi_f < 20$, ξ can increase by one step in Ξ, remain unchanged, or decrease by one step. Quality only transitions to $-\infty$ as a result of exit. Each firm's quality is affected by two shocks, one positive and one negative, which are independent from one another and across firms. The negative shock lowers quality with exogenous probability $\delta \in (0, 1)$. The positive shock instead increases quality by one step with probability $u(x) = \frac{\psi x}{1+\psi x}$. This can be interpreted as the probability of R&D success, which increases with investment.

value function ceases to be linear in the structural parameters even when the investment cost function is linear in $\boldsymbol{\theta}_x$ and the scrap value is deterministic (as in Bajari et al. (2007)).

Given independence of the shocks, quality transition probabilities satisfy

$$P(\xi' \mid \xi, x) = \begin{cases} \delta[1 - u(x)] & \text{if } \xi' = \xi - 1 \\ 1 - \delta - u(x)(1 - 2\delta) & \text{if } \xi' = \xi \\ (1 - \delta)u(x) & \text{if } \xi' = \xi + 1 \\ 0 & \text{otherwise} \end{cases} \tag{34}$$

when ξ is interior.[16] Given this structure, firms choose investment to maximise the present-discounted stream of profits. They balance expected higher product quality with an immediate cost of investment $c(x) = \kappa x$. There is no shock to the cost of investment. It is easy to see that the transitions in (34) are UIC-admissible.[17]

At the beginning of the period, incumbents observe a private scrap value $\rho \sim F_\rho$ and potential entrants privately observe an entry cost $\phi \sim F_\phi$, and simulataneously make entry and exit decisions. Those decisions are implemented at the end of the period. We set the maximum number of firms to $\bar{N} = 3$.

Our model differs from the Bajari et al. (2007) model in five aspects. First, we use the utility specification in (33), whereas the Bajari et al. (2007) specification is logarithmic in net income. We do so to apply our own existing code for computing Nash-Bertrand equilibria for pricing games with logit demands and linear utility. We set the values of γ_0 and α to approximate their specification. Second, in the Bajari et al. (2007) model all firms experience the same negative shock to their qualities, interpreted as a stochastic improvement to the outside option, whereas we model negative shocks as firm-specific and independent. This also leads us to define the set of possible qualities differently from Bajari et al. (2007). There, both the outside option and firms' products qualities are integers between zero and 20. Rather than treating the outside option explic-

[16] The probability that $\xi' = \xi$ at maximum (minimum) quality is defined to be the complement of the probability of ξ decreasing (increasing).

[17] The local structure of the transitions (34) can be exploited for computational gains. It implies that the transition matrix $\boldsymbol{M}(\boldsymbol{P})$ in equation (30) is banded. A banded matrix is a sparse matrix whose non-zero entries are confined to a band around the main diagonal. Importantly, the LU decomposition of a banded matrix has banded components. This implies substantial computational savings in both the computation of the LU decomposition and the substitutions used to compute the solution to the linear system (27). It is important for this observation that equation (27) refers only to incumbent states. The matrix of transitions over all states includes non-zero entries in its first few columns (i.e., those that pertain to transitions to $\xi = -\infty$), making its lower bandwidth large, and reducing the computational savings. The full computational savings are thus a consequence of both the local nature of transitions and the assumption that exiting firms perish.

Table 1: BBL Model Parameters

Parameter	**Value**
Data Structure	
Number of Quality Levels	39
Minimum Quality Level	-2.99573
Maximum Quality Level	2.99573
Maximum Number of Players	3
Number of Markets	100
Number of Periods	40
Model Parameters	
Discount Factor (β)	0.925
Demand System (γ_0, α)	[0.1, -0.25]
Marginal Cost (θ_c)	[1.09861, 0.0]
Quality Transition (δ, ψ)	[0.7, 7.0]
Investment Cost (θ_x)	1.0
Investment Cost Shock Distribution (F_ν)	Dirac(0.0)
Scrap Value Distribution (F_ρ)	Uniform(22.0, 23.0)
Entry Costs Distribution (F_ϕ)	Uniform(22.0, 30.0)
Bajari Benkard Levin Estimator Settings	
Number of Simulated Paths	250
Simulation Horizon	150

Parameterization of the BBL model used in Monte Carlo simulations.

itly, we allow for one with utility normalized to zero. Therefore, we interpret product qualities as relative to the outside option, and thus define Ξ to be the set of all possible quality differences relative to the outside option in the BBL specification.

Third, we use a stochastic scrap value upon exit, whereas Bajari et al. (2007) assume a deterministic scrap value. We do so because we found it challenging to compute an equilibrium to the model with a deterministic scrap value to a satisfactory degree of accuracy. This is intuitive. When the scrap value is deterministic, small changes in the value function and endogenous transition probabilities lead firms to change their exit decisions, leading to failure in convergence. Fourth, we allow all potential entrants to enter, whereas Bajari et al. (2007) allow only one entrant per period. Fifth, our parameter values differ from those in Bajari et al. (2007). This is because under their parameters we computed equilibria in which potential entrants always enter, and incumbents never exit. That is perhaps due to our different utility specification. We choose parameter values that approximate the entry and exit probabilities reported in the supplementary materials to Bajari et al. (2007).

4.1.1 Estimators

As discussed in Section 3.1, we estimate the BBL model using a nonlinear least squares estimator. That is because of the lack of a shock to investment costs, which makes the pseudo maximum likelihood estimator inapplicable. Explicitly, we solve

$$\min_{\boldsymbol{\theta}} \sum_{f,m,t} \left\{ \left(x_{fmt} - \sigma^x(\boldsymbol{\xi}_{fmt}; \boldsymbol{\theta}, \hat{\Phi}) \right)^2 + \left(a_{fmt} - \mathbb{P}\left(\alpha(\boldsymbol{\xi}, \eta) = 1 \mid \boldsymbol{\xi}_{fmt}, \boldsymbol{\theta}, \hat{\Phi} \right) \right)^2 \right\} . \tag{35}$$

In this expression, x_{fmt} is firm f' investment choice in market m and period t, $\sigma^x(\boldsymbol{\xi}_{fmt}; \boldsymbol{\theta}, \hat{\Phi})$ denotes the optimal investment when the firm's state is $\boldsymbol{\xi}_{fmt}$, given parameters $\boldsymbol{\theta}$ and first-stage estimates $\hat{\Phi}$, a_{fmt} is the observed decision to be active, and $\mathbb{P}(\alpha(\boldsymbol{\xi}, \eta) = 1 \mid \boldsymbol{\xi}_{fmt}, \boldsymbol{\theta}, \hat{\Phi})$ is the model-predicted probability of being active. The variable η should be interpreted as either the scrap value or the entry cost, depending on whether firm f is an incumbent or a potential entrant in market m and period t, as determined by $\boldsymbol{\xi}_{fmt}$.

Our implementation of the BBL estimator follows Bajari et al. (2007) and Pakes et al. (2007) in that we first estimate exit and cost parameters, and then estimate entry parameters holding exit and cost parameters at their estimated values. Exit and cost parameters are estimated by minimizing (26), which requires defining deviation policies. We implement two variants that have appeared in the literature, which we term *additive* and *multiplicative* deviations, as well as an *asymptotic* variant we devised. These classes of deviations perturb estimated policies additively, multiplicatively, or by drawing alternative coefficients for policy functions from the asymptotic distribution of estimated policy function parameters. We discuss implementation of each in detail in Appendix A.6. We set the number of criterion inequalities to the number of unique states observed in the data. As shown in Tables 1 and 3, we forward simulate 250 independent histories for 150 periods to approximate value functions. Having estimated exit and cost parameters, we estimate entry parameters by minimizing the average squared deviation between a_{fmt} and Equation (15), where we approximate $\bar{V}_E$ by forward simulation. That is, similarly to Equation (35) we solve

$$\min_{\boldsymbol{\theta}_\phi} \sum_{f,m,t} \left\{ a_{fmt} - F_\phi \left(\bar{V}_E(\boldsymbol{\xi}_{fmt}; \hat{\Phi}, \hat{\boldsymbol{\theta}}_{-\phi}); \boldsymbol{\theta}_\phi \right) \right\}^2$$

4.1.2 Monte Carlo Results

We report the results in Figure 1 and Table 2. Figure 1 shows histograms of parameter estimates over 100 Monte Carlo simulations for three of the four estimators discussed above.[18] The vertical dashed lines indicate the true parameter values. Table 2 reports the true values of each parameter along with the mean and standard deviation of parameter estimates across Monte Carlo simulations. The column labeled "Recursive" refers to the NLLS estimator defined in equation (35), which makes use of recursive equilibrium conditions. The columns labeled "Additive", "Multiplicative", and "Asymptotic" refer to the three implementations of the BBL estimator discussed above.

We find that the NLLS estimator performs well in this data-generating process, as does the additive BBL estimator. Both exhibit small finite sample bias and variance. The NLLS estimator has slightly lower variance for the marginal cost and scrap value parameters, whereas the BBL estimator has better precision for entry cost parameters.[19] The multiplicative BBL estimator performs faily

[18] As shown in Table 2, asymptotic deviations performed very poorly. We thus removed them to improve figure legibility.

[19] We conjecture that the better performance of the BBL estimator for entry cost parameters is a consequence of how our differing implementations affect the performance of numerical optimization with respect to those parameters. Our implementation of the BBL estimator estimates entry cost parameters separately from exit and cost parameters, whereas the NLLS es-

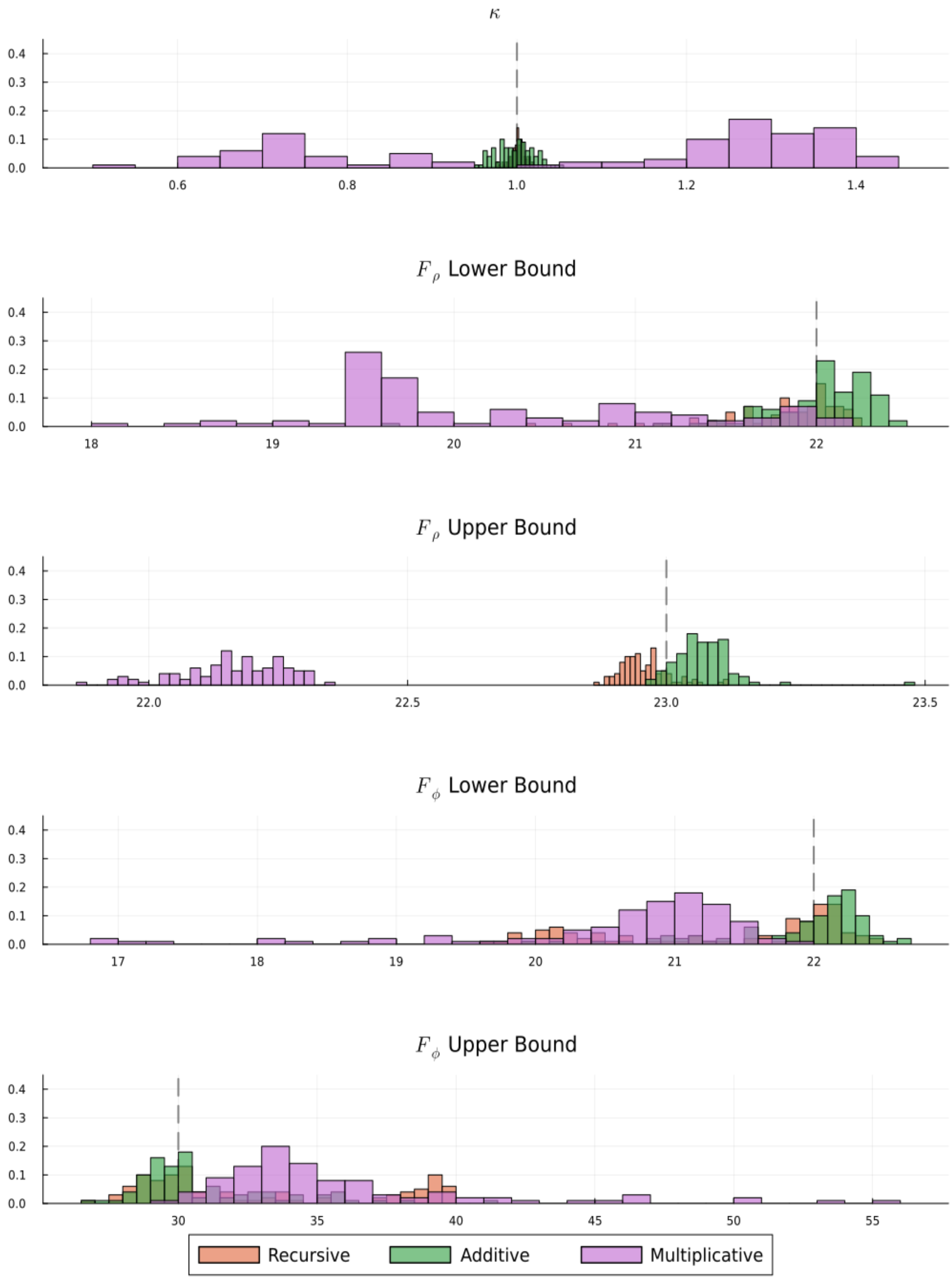


Figure 1: Parameter Estimates – BBL Model

Histograms of parameter estimates for the BBL model over 100 Monte Carlo simulations. The true parameter values are indicated by the vertical dashed lines. "Optimality" refers to the NLLS estimator defined in equation (35). "Additive", "Multiplicative", and "Asymptotic" refer, respectively, to the BBL estimator with additive, multiplicative, and asymptotic deviations.

Table 2: Estimation Results – BBL Model

Parameter	Value	Recursive	Additive	Multiplicative	Asymptotic
κ	1.0	1.002	0.998	1.096	-3.097
		(0.009)	(0.022)	(0.277)	(3.162)
F_ρ Lower Bound	22.0	21.820	22.019	20.226	24.250
		(0.334)	(0.344)	(0.949)	(1.987)
F_ρ Upper Bound	23.0	22.959	23.072	22.162	26.269
		(0.049)	(0.060)	(0.102)	(1.977)
F_ϕ Lower Bound	22.0	21.368	21.908	20.581	4.949
		(0.878)	(0.535)	(1.060)	(9.120)
F_ϕ Upper Bound	30.0	32.813	30.938	35.822	122.083
		(4.301)	(2.581)	(5.039)	(49.612)

Parameter estimates for the BBL model. The "Parameter" column describes the parameters being estimated. θ_x parameterizes investment costs $c(x) = \kappa x$. F_ρ and F_ϕ respectively denote scrap value and entry cost distributions. The "Value" column reports the true value of those parameters in the data-generating process. The remaining columns report the mean and standard deviation of parameter estimates obtained from different estimators across 100 Monte Carlo simulations. The "Recursive" estimator is the nonlinear least squares estimator defined in equation (35). "Additive", "Multiplicative", and "Asymptotic" refer, respectively, to the BBL estimator with additive, multiplicative, and asymptotic deviations, as discussed in the main text.

well, but exhibits larger finite sample bias and variance than both the NLLS estimator and the additive BBL estimator. Finally, the asymptotic BBL estimator performs poorly.

Table 2 contains at least two noteworthy findings. First, the performance of the BBL estimator depends meaningfully on the choice of deviation policies. This is a drawback, as it may not be clear which class of deviations is most appropriate in a given application. Second, the estimator based on recursive equilibrium conditions, which does away with the need to define deviation policies, performs at least as well as the best-performing BBL estimator.

4.2 An Extension of Hashmi and van Biesebroeck (2016)

In this section we present an extension of the Hashmi and van Biesebroeck (2016) model of R&D in the automobile industry that allows for entry and exit. As before, each period firms compete in prices à la Nash-Bertrand and invest to affect the quality of their product in the following period. Importantly, this

timator estimates all parameters jointly. Since entry decisions are relatively infrequent in this data-generating process, the joint objective function is likely relatively insensitive to changes in entry cost parameters. This may lead numerical optimizers to declare convergence somewhat prematurely. We have yet to investigate this conjecture thoroughly.

model features shocks to the marginal cost of investment.

4.2.1 Static Price Competition

Suppose there are N single-product firms active in the market, indexed by $j = 1, \ldots, N$. As firms are single-product, we let j denote interchangeably a firm and its product. Consumer i derives conditional indirect utility u_{ij} from purchasing firm j's product, where

$$u_{ij} = \begin{cases} \epsilon_i^{\text{out}} + (1-\varsigma)\varepsilon_{i0} & \text{if } j = 0 \\ \alpha p_j + \xi_j + \epsilon_i^{\text{in}} + (1-\varsigma)\varepsilon_{ij} & \text{if } j = 1, \ldots, J \end{cases} . \tag{36}$$

In (36), $j = 0$ denotes the outside good and p_j denotes good j's price. Goods are grouped into two nests, one containing all inside goods (i.e, those produced by one of the N firms) and one containing the outside good. The ε_{ij}'s are independent and identically distributed Type 1 Extreme Value random variables. The nest-level disturbances ϵ_i^{out} and ϵ_i^{in} follow the unique distribution such that $\epsilon_i^{\text{g}} + (1-\varsigma)\varepsilon_{ij}$, for g $=$ {in, out}, is also Type 1 Extreme Value distributed – see Cardell (1997). The parameter $\varsigma \in [0, 1]$ is the nesting parameter. This is a standard nested-logit demand specification, leading to a well-known functional form for market shares.

A firm selling a good of quality ξ_j has constant marginal cost

$$mc(\xi_j) = \exp(\theta_{c1} + \theta_{c2}\xi_j) .$$

Firms compete à la Nash-Bertrand. There is a unique equilibrium to the pricing game (Caplin and Nalebuff (1991)), so that profits $\pi(\boldsymbol{\xi})$ as a function of firms' product qualities are well-defined.

4.2.2 Quality Transitions and Investment Decision

As before, firms invest to affect the quality of their product in the following period. Product qualities are elements of $\Xi = \{-\infty, \xi_m, \xi_m + \delta, \ldots, \xi_M - \delta, \xi_M\}$. Transitions are as in the BBL model of Section 4.1, except that the probability of a positive quality shock is now allowed to depend not only on investment but also on current quality. Specifically,

$$u(\xi, x) = 1 - (1 + x)^{-\alpha(\xi)} , \tag{37}$$

where $\alpha(\xi) = \exp(\theta_{t2} + \theta_{t3}\xi + \theta_{t4}\xi^2)$. With parameters θ_t such that $\alpha(\xi)$ is a decreasing function of ξ over Ξ, this specification implies that the probability of quality improvement is decreasing in current quality, for any given level of investment. This captures the notion that it is harder to improve on a higher-quality product.[20]

Firms face investment cost

$$c(x, \nu) = \theta_{x1}x + \theta_{x2}x^2 + \theta_{x3}x\nu \ , \tag{38}$$

and ν follows a distribution F_ν that is known to the econometrician. The cost shock interacts with the investment level and therefore affects the optimal investment choice. This rationalizes that two firms facing the same quality vector $\boldsymbol{\xi}$ may optimally choose different levels of investment. At the beginning of the period, prior to observing the cost shock ν, potential entrants privately observe their entry cost $\phi \sim F_\phi$, incumbents privately observe their scrap value $\rho \sim F_\rho$, and all firms make entry and exit decisions simultaneously. Firms' entry and exit decisions determine the endogenous number of active firms $N \leq \bar{N}$, where $\bar{N}$ is the maximum number of firms in the market.

Other than including entry and exit, this model differs from the one in Hashmi and van Biesebroeck (2016) in two ways. First, our specification of upgrade probabilities in (37) differs from that in Hashmi and van Biesebroeck (2016). This is because their specification is not globally concave in investment, and thus does not meet the conditions of Proposition 1 Second, our investment cost specification in (38) omits a cubic term present in Hashmi and van Biesebroeck (2016).

4.2.3 Parameterization

Table 3 reports parameters governing the data generating process. We fix $\bar{N} = 5$ in each simulated market, in line with simulations in Hashmi and van Biesebroeck (2016).[21] As in their model, incumbent product quality can take on fifteen values, from -1.4 to 1.4 in increments of 0.2. Firms' marginal cost of pro-

[20] Our specification of quality transitions differs from HvB. HvB base their upgrade probabilities on the CDF of a Gumbel distribution. That specification violates the curvature assumption in the definition of UIC-admissibility and in the alternative condition for uniqueness of optimal investment discussed in Appendix A.2.

[21] In their empirical analysis, Hashmi and van Biesebroeck (2016) aggregate data to have a single market with 14 firms for the 1982-2006 period. When computing the equilibrium of the dynamic game, however, they restrict the number of firms to 5 to reduce computational burden. See Hashmi and van Biesebroeck (2016, Footnote 30).

Table 3: HvB Model Parameters

Parameter	**Value**
Data Structure	
Number of Quality Levels	15
Minimum Quality Level	-1.4
Maximum Quality Level	1.4
Distance Across Quality Levels (δ)	0.2
Maximum Number of Players	5
Number of Markets	100
Number of Periods	40
Model Parameters	
Discount Factor (β)	0.95
Demand System (γ_0, α)	[1.0, -0.222]
Marginal Cost (θ_c)	[2.47, 0.0]
Quality Transition (θ_t)	[0.347, -0.75, -0.3, -0.1]
Investment Cost (θ_x)	[2.625, 1.624, 0.5096]
Investment Cost Shock Distribution (F_ν)	Normal(0.0, 1.0)
Scrap Value Distribution (F_ρ)	Exponential(0.8)
Entry Costs Distribution (F_ϕ)	Exponential(11.0)
Bajari Benkard Levin Estimator Settings	
Number of Simulated Paths	250
Simulation Horizon	150

Parameterization of the Hashmi and van Biesebroeck (2016) model used in Monte Carlo simulations.

duction is constant and equal to $mc_j = \exp(2.47)$, i.e., we set $\theta_{c1} = 2.47$ and $\theta_{c2} = 0$. The probability of a quality downgrade shock is $\theta_{t1} = 0.347$, and the remaining transition parameters are such that the probability of a quality upgrade shock is decreasing in own quality level at an increasing rate. Investment costs are convex ($\theta_{x2} > 0$) in investment and the marginal cost of investment is increasing in the shock ($\theta_{x3} > 0$). We follow Hashmi and van Biesebroeck (2016) in assuming that the investment cost shock ν follows a standard normal distribution.[22] Our parameterization differs somewhat from that in Hashmi and van Biesebroeck (2016). For instance, we assume a lower probability of a quality downgrade shock. We also consider different values for investment cost parameters. For instance, HvB report a negative θ_{x2}. We impose a positive value to ensure convexity of the investment cost function. The changes in parameter values are not instrumental for the qualitative results we report in Section 4.2.4.

Finally, we have to take a stance on entry cost and scrap value distributions, which are not present in the Hashmi and van Biesebroeck (2016) model. We assume both are exponential. The mean of the scrap value distribution is much smaller than that of entry costs. This is so that the DGP generates somewhat low entry and exit probabilities, as is commonly observed in empirical applications.[23]

4.2.4 Monte Carlo Results

We report results in Figure 2 and Table 4. The results are now markedly different from those in Section 4.1. All implementations of the BBL estimator perform poorly in the estimation of the investment cost parameters, showing substantial finite sample bias and variance. The distributions in Figure 2 show no tendency to be centered around the true parameter values. Indeed, sometimes those distributions are close to uniform, and sometimes they exhibit mass points at the bounds used in optimization. The BBL estimation conditions fail to identify the investment cost parameters. By contrast, the pseudo MLE estimator which uses recursive equilibrium conditions performs well, with the average estimates being close to the true parameter values and with small variances.

[22] This implies that the marginal cost of investment can be negative, counter to Assumption 1. However, at the parameter values reported in Table 3, the probability that ν is sufficiently small to make the marginal cost of investment negative is less than 10^{-6}. We can straightforwardly accomodate different distributions.

[23] Table 3 details the distribution of scrap values and entry costs before they are scaled to have the same order of magnitude as firm profits. Scaling is implemented by multiplying these shocks by the average of $\pi(\boldsymbol{\xi})/(1-\beta)$ across states.

Table 4: Parameter Estimates – HvB Model

Parameter	Value	Recursive	Additive	Multiplicative	Asymptotic
θ_{x1}	2.625	2.921	-2.845	-1.698	5.000
		(0.360)	(3.862)	(4.102)	(0.000)
θ_{x2}	1.624	1.601	-1.610	-3.196	5.000
		(0.076)	(3.216)	(2.530)	(0.000)
θ_{x3}	0.5096	0.552	5.417	5.890	0.000
		(0.065)	(4.892)	(3.295)	(0.000)
F_ρ Scale Parameter	0.8	0.754	0.810	0.452	0.599
		(0.061)	(0.029)	(0.044)	(0.017)
F_ϕ Scale Parameter	11.0	10.422	11.550	7.334	8.523
		(0.799)	(0.542)	(0.687)	(0.268)

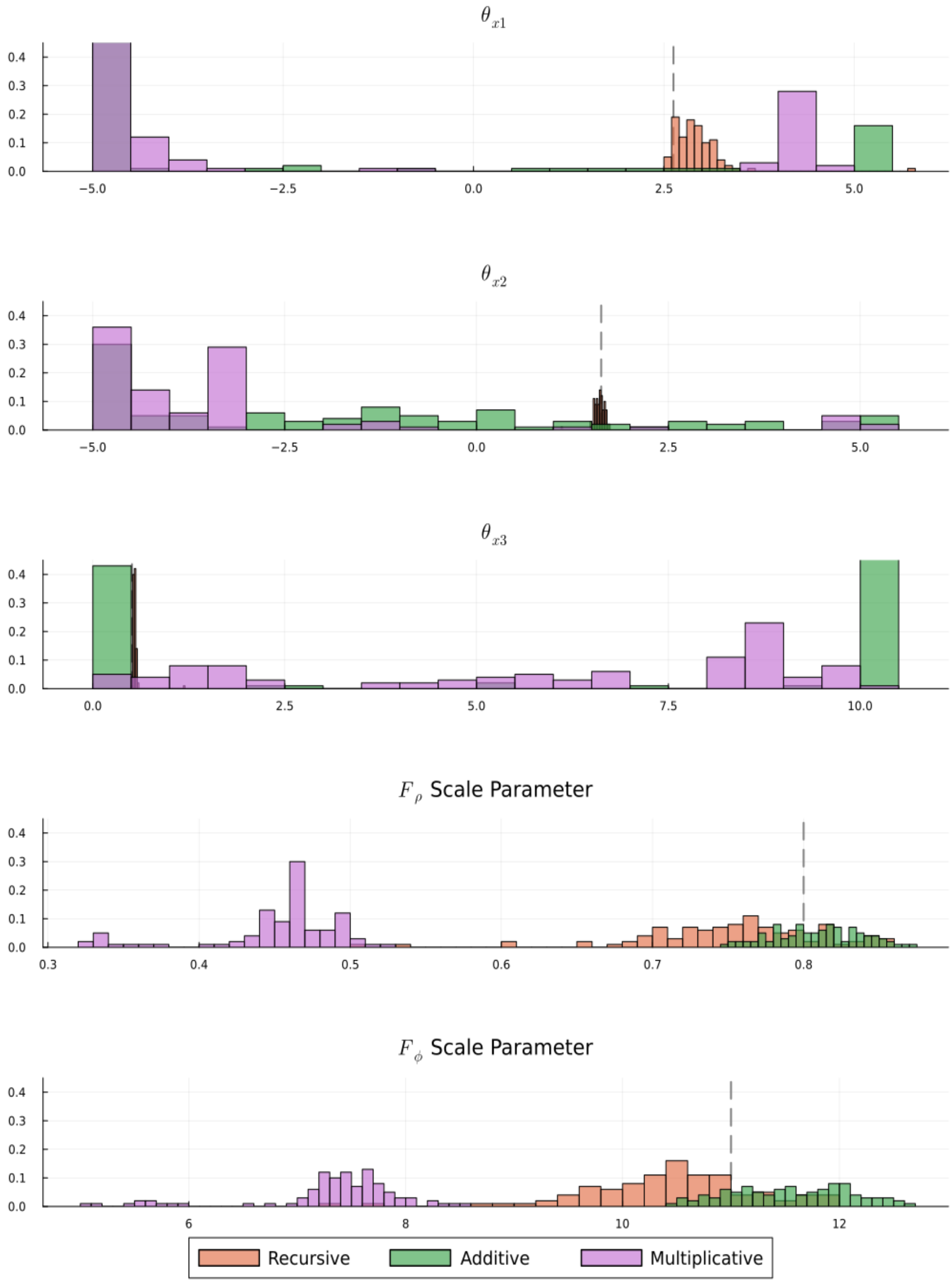
Parameter estimates for the HvB model. The "Parameter" column describes the parameters being estimated. $\theta_{x1}, \theta_{x2}, \theta_{x3}$ parameterize investment costs: see (38). F_ρ and F_ϕ respectively denote scrap value and entry cost distributions. The "Value" column reports the true value of those parameters in the data-generating process. The remaining columns report the mean and standard deviation of parameter estimates obtained from different estimators across 100 Monte Carlo simulations. The "Recursive" estimator is the pseudo-MLE estimator described in Section 3.1. "Additive", "Multiplicative", and "Asymptotic" refer, respectively, to the BBL estimator with additive, multiplicative, and asymptotic deviations, as discussed in the main text.

The BBL estimators do a better job of estimating the scrap value and entry cost parameters. Again, the implementation with additive deviations performs best. The pseudo MLE estimator also performs well in estimating these parameters. Overall, the PMLE estimator significantly outperforms all three implementations of the BBL estimator in this more complex model.

5 Conclusion

We have proposed estimators for dynamic games with continuous controls that exploit the optimality conditions characterizing firm behavior. In particular, we have derived a pseudo-MLE estimator for models with private information shocks to firms' marginal costs of setting their continuous controls, such as investment or capacity adjustment. We have evaluated the performance of those estimators vis-à-vis multiple implementations of the commonly used Bajari et al. (2007) estimator in a set of Monte Carlo exercises. The estimators we propose are sufficiently cheap to compute in empirically-relevant models, and exhibit superior econometric performance. We hope that these results will steer empirical researchers towards exploiting optimality conditions when possible.

Figure 2: Parameter Estimates – HvB Model



Histograms of parameter estimates for the BBL model over 100 Monte Carlo simulations. The true parameter values are indicated by the vertical dashed lines. "Optimality" refers to the pseudo MLE estimator defined in Section 3.1. "Additive", "Multiplicative", and "Asymptotic" refer, respectively, to the BBL estimator with additive, multiplicative, and asymptotic deviations.

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Appendices

Appendix A Proofs and Derivations

A.1 Validity of equations (6) and (7)

We repeat the two equations here for the reader's convenience:

$$\underbrace{\int_{\varepsilon_j}\int_{\xi_j'} \bar{V}_I(\boldsymbol{\xi}')\,\mathrm{d}F(\xi_j' \mid \xi_j, \sigma(\boldsymbol{\xi}_j, \varepsilon_j))\,\mathrm{d}G_{\varepsilon_j}}_{A} = \underbrace{\int_{\xi_j'} \bar{V}_I(\boldsymbol{\xi}')\,\mathrm{d}F^{\sigma}(\xi_j' \mid \boldsymbol{\xi}_j)}_{B} \quad (6 \text{ – repeated})$$

where

$$F^{\sigma}(\xi_j' \mid \boldsymbol{\xi}_j) := \int_{\varepsilon_j} F(\xi_j' \mid \xi_j, \sigma(\boldsymbol{\xi}_j, \varepsilon_j))\,\mathrm{d}G_{\varepsilon_j}\ . \quad (7 \text{ – repeated})$$

This equality states that the mixture of integrals with respect to the transition kernels $F(\xi_j' \mid \xi_j, \sigma(\boldsymbol{\xi}_j, \varepsilon_j))$ is equal to the integral with respect to the mixture distribution F^{σ}. This follows from Fubini-Tonelli type theorems for product measures constructed from a probability measure and a transition kernel, such as Theorem 6.3 in Chapter 1 of Çinlar (2011) or Theorem 14.29 of Klenke (2013).

For instance, the result in Çinlar (2011) is as follows.

Definition 3 (Transition kernel (Çınlar))**.** Let $(E, \mathcal{E})$ and $(F, \mathcal{F})$ be measurable spaces. A mapping $K : E \times \mathcal{F} \to \overline{\mathbb{R}}_+$ is called a *transition kernel* from $(E, \mathcal{E})$ into $(F, \mathcal{F})$ if

1. for every $B \in \mathcal{F}$, the mapping $x \mapsto K(x, B)$ is $\mathcal{E}$-measurable;
2. for every $x \in E$, the mapping $B \mapsto K(x, B)$ is a measure on $(F, \mathcal{F})$.

Theorem 1 (Measure–kernel–function (Çınlar, Thm. 6.3))**.** Let K be a transition kernel from $(E, \mathcal{E})$ into $(F, \mathcal{F})$. Then

$$Kf(x) := \int_F K(x, dy)\, f(y), \quad x \in E,$$

defines a function $Kf \in \mathcal{E}_+$ for every $f \in \mathcal{F}_+$. Moreover, for each measure μ on $(E, \mathcal{E})$,

$$\mu K(B) := \int_E \mu(dx)\, K(x, B), \quad B \in \mathcal{F},$$

defines a measure μK on $(F, \mathcal{F})$; and, for every measure μ on $(E, \mathcal{E})$ and $f \in \mathcal{F}_+$,

$$(\mu K)f \;=\; \mu(Kf) \;=\; \int_E \mu(dx) \int_F K(x, dy)\, f(y).$$

The last stament is precisely what we need. It says that the integral of f with respect to the mixture measure μK is equal to the mixture of integrals of f with respect to the kernels $K(x, \cdot)$, weighted by the measure μ. In the statement above, $\mathcal{E}_+$ and $\mathcal{F}_+$ denote the sets of non-negative measurable functions on $(E, \mathcal{E})$ and $(F, \mathcal{F})$, respectively. Our integrand in equation (6) is $\bar{V}_I(\xi_j, \boldsymbol{\xi}_{-j})$, which is non-negative in equilibrium. Therefore, the Theorem applies. Theorem 14.29 of Klenke (2013) applies to integrable f.

A.2 Uniqueness of the Investment Decision: Proof of Proposition 1

We start with the proof of Proposition 1, restated below for convenience.

Proposition (1)**.** If Assumption 1 holds and $F(\xi' \mid \xi, x, \eta)$ is UIC-admissible, then the problem

$$\max_{x \in [0, \bar{x}]} \pi(\boldsymbol{\xi}) - c(x, \nu) + \beta \int_\eta \int_{\xi_i'} W(\xi_i' \mid \boldsymbol{\xi}, F^\sigma, \eta)\, \mathrm{d}F(\xi_i' \mid \xi_i, x, \eta)\, \mathrm{d}F_\eta$$

has a unique solution for all $\boldsymbol{\xi}$. Moreover, the maximizer is increasing in ν, and strictly so over the range of ν where it is interior.

Proof. Under UIC-admissibility, the linearity of the Riemann-Stieltjes integral with respect to the integrator implies

$$\begin{aligned}
\int_{\xi_i'} W(\xi_i' \mid \boldsymbol{\xi}, F^\sigma, \eta)\, \mathrm{d}F(\xi_i' \mid \xi_i, x, \eta) &= \int_{\xi_i'} W(\xi_i' \mid \boldsymbol{\xi}, F^\sigma, \eta)\, \mathrm{d}\left[L(\xi', \xi, \eta) + K(\xi', \xi, \eta) Q(\xi, x)\right] \\
&= \int_{\xi_i'} W(\xi_i' \mid \boldsymbol{\xi}, F^\sigma, \eta)\, \mathrm{d}L(\xi', \xi, \eta) \\
&\quad + Q(\xi, x) \underbrace{\int_{\xi_i'} W(\xi_i' \mid \boldsymbol{\xi}, F^\sigma, \eta)\, \mathrm{d}K(\xi', \xi, \eta)}_{A(\boldsymbol{\xi}, F^\sigma, \eta)}
\end{aligned}$$

Let $A(\boldsymbol{\xi}, F^\sigma) := \int_\eta \int_{\xi_i'} W(\xi_i' \mid \boldsymbol{\xi}, F^\sigma, \eta)\, \mathrm{d}K(\xi', \xi, \eta)\, \mathrm{d}F_\eta$. The investment first-order condition is then

$$-\frac{\partial c(x, \nu)}{\partial x} + \beta A(\boldsymbol{\xi}, F^\sigma) \frac{\partial Q(\xi, x)}{\partial x} = 0\,,$$

Suppose $A(\boldsymbol{\xi}, F^\sigma) \leq 0$. Then the objective function is strictly decreasing in x and the solution is $x^* = 0$. Suppose instead that $A(\boldsymbol{\xi}, F^\sigma) > 0$. Then the objective function is strictly concave in x:

$$-\frac{\partial^2 c(x,\nu)}{\partial x^2} + \beta A(\boldsymbol{\xi}, F^\sigma) \frac{\partial^2 Q(\xi, x)}{\partial x^2} < 0 \ .$$

Let $v(x, \boldsymbol{\xi}, \nu)$ denote the objective function. If $\partial_x v(0, \boldsymbol{\xi}, \nu) < 0$, then the solution is again $x^* = 0$. If $\partial_x v(\bar{x}, \boldsymbol{\xi}, \nu) > 0$, then $x^* = \bar{x}$. Otherwise, the solution is the unique solution to the first-order condition.

To establish the monotonicity statement, suppose first that $A(\boldsymbol{\xi}, F^\sigma) \leq 0$. Then $x^* = 0$ for all ν. Next suppose $A(\boldsymbol{\xi}, F^\sigma) > 0$. If $\partial_x v(0, \boldsymbol{\xi}, \nu_1) < 0$, and $\nu_1 < \nu_2$, then $\partial_x v(0, \boldsymbol{\xi}, \nu_2) < 0$ and in both cases $x^* = 0$ is optimal. If $\partial_x v(\bar{x}, \boldsymbol{\xi}, \nu_1) > 0$ and $\nu_1 < \nu_2$, then either $\partial_x v(\bar{x}, \boldsymbol{\xi}, \nu_2) \geq 0$, and $\bar{x}$ is optimal in both cases, or $\partial_x v(\bar{x}, \boldsymbol{\xi}, \nu_2) < 0$ and the solution is interior and thus less than $\bar{x}$. If $(\boldsymbol{\xi}, \nu)$ are such that the solution is interior, an application of the Implicit Function Theorem implies $\frac{\partial x^*(\boldsymbol{\xi}, \nu)}{\partial \nu} < 0$. □

We note here that under the assumption that the $W(\cdot)$ function is increasing, it is possible to give an alternative, and perhaps more intuitive, condition for strict concavity of the local income function. The $W(\cdot)$ function is an endogenous object, and imposing restrictions on it is a limitation of the following result. However, if the analyst is willing to make this monotonicity assumption – i.e., that the equilibrium is such that an increase in own characteristics is good in this precise sense – than the result below is useful for the estimation and solution of games whose transitions do not satisfy UIC-admissibility. Capacity accumulation games are a class of such games.

Assumption 4. Let Ξ be a compact subset of the real line, and denote its minimum and maximum by ξ_m and ξ_M. Let Ξ° be the interior of Ξ (understood in the discrete case as $\Xi \setminus \{\xi_m, \xi_M\}$). The family of distributions $F(\cdot \mid \xi, x, \eta)$ is such that, for all $\xi \in \Xi$ and $\xi' \in \Xi^\circ$,

(a) $F(\xi' \mid \xi, x, \eta)$ is twice-continuously differentiable in x.

(b) $F(\xi' \mid \xi, x, \eta)$ is strictly decreasing and strictly convex in x;

(c) $F(\xi_m \mid \xi, x, \eta)$ is decreasing and convex in x.

Assumption 4(a) is a technical condition required in the proof, but has little economic content. It is also nonrestrictive in practice, as in applications the econometrician typically imposes a parametric restriction on $F(\xi' \mid \xi, x, \eta)$ that satisfy this condition. Assumption 4(b) states that for any current and future characteristics (other than the endpoints of Ξ), an increase in investment x causes the cumulative distribution function to decrease; i.e., an increase in x increases a firm's distribution of future quality in the first-order stochastic dominance sense. Moreover, investment has decreasing marginal returns in the sense that the reduction in the CDF is decreasing in investment. Assumption 4(c) allows for the continuous case, where $F(\xi_m \mid \xi, x, \eta) = 0$ for all ξ and x, and for the discrete case in which $F(\xi_m \mid \xi, x, \eta)$ may be strictly positive and depend on x.

Proposition 3. Suppose assumptions 1 and 4 hold. Moreover, suppose $W(\xi' \mid \boldsymbol{\xi}, F^\sigma)$ is increasing in ξ' for all $\boldsymbol{\xi}$. Then

$$v(x; \boldsymbol{\xi}, \nu) = \pi(\boldsymbol{\xi}) - c(x, \nu) + \beta \int_{\xi_1'} W(\xi_1' \mid \boldsymbol{\xi}, F^\sigma) \, \mathrm{d}F(\xi_1' \mid \xi_1, x, \eta)$$

is strictly concave in x.

Proof. By Integration by Parts,

$$\begin{aligned}\int_{\xi_i'} W(\xi_i' \mid \boldsymbol{\xi}, \sigma) \, \mathrm{d}F(\xi_i' \mid \xi_i, x) = &- \int F(\xi' \mid \xi, x) \, \mathrm{d}W(\xi' \mid \boldsymbol{\xi}, \sigma) \\ &+ W(\xi_M \mid \boldsymbol{\xi}, \sigma) \underbrace{F(\xi_M \mid \xi, x)}_{=1} - W(\xi_m \mid \boldsymbol{\xi}, \sigma) F(\xi_m \mid \xi, x)\end{aligned}$$

Therefore,

$$\begin{aligned}\frac{\partial^2}{\partial x^2} \left(\int_{\xi_i'} W(\xi_i' \mid \boldsymbol{\xi}, \sigma) \, \mathrm{d}F(\xi_i' \mid \xi_i, x) \right) &= - \int \frac{\partial^2}{\partial x^2} F(\xi' \mid \xi, x) \, \mathrm{d}W(\xi' \mid \boldsymbol{\xi}, \sigma) \\ &\quad - W(\xi_m \mid \boldsymbol{\xi}, \sigma) \frac{\partial^2}{\partial x^2} F(\xi_m \mid \xi, x) \\ &< 0\end{aligned}$$

The differentiation under the integral sign is valid due to the twice continuous differentiability of $F(\xi' \mid \xi, x)$ in x. The inequality is due to Assumption 4 and the monotonicity of W. This inequality, coupled with the strict convexity of $c(x, \nu)$, implies the desired property. □

In addition to Assumption 4, we impose the standard restrictions on the investment cost function summarized in Assumption 1 and assume that $W(\xi' \mid \boldsymbol{\xi}, F^\sigma)$ is increasing. This says that the ENPV of starting the following period at ξ' conditional on $\boldsymbol{\xi}$ is increasing in ξ': that is, firms would rather start the following period from a higher rather than lower ξ, for all current states $\boldsymbol{\xi}$. This is a restriction on an equilibrium object. In theory, it may be violated. It could happen, for instance, that a higher quality in $t+1$ induces higher investment by competitors, and so much so that this reduces a firm's ENPV of profits.[24] That being said, we have not computed any equilibrium that violates this monotonicity condition of the models considered in this paper.

A.3 Proof of Proposition 2

Consider an incumbent's Bellman Equation after observing its scrap value:

$$V_I(\boldsymbol{\xi}, \rho) = \max \left\{ \pi(\boldsymbol{\xi}) + \rho, \int \max_{x \in \mathbb{R}_+} h(x, \boldsymbol{\xi}, \nu, \bar{V}_I; F^\sigma) \, \mathrm{d}F_\nu \right\} , \tag{39}$$

where

$$\begin{aligned} h(x, \boldsymbol{\xi}, \nu, \bar{V}_I; F^\sigma) := \pi(\boldsymbol{\xi}) - c(x, \nu) + \\ + \beta \int_\eta \int_{\xi_1'} W(\xi_1' \mid \bar{V}_I, \boldsymbol{\xi}, F^\sigma, \eta) \, \mathrm{d}F(\xi_1' \mid \xi_1, x, \eta) \, \mathrm{d}F_\eta , \end{aligned}$$

and

$$W(\xi_1' \mid \bar{V}_I, \boldsymbol{\xi}, F^\sigma, \eta) = \int_{\xi_2'} \cdots \int_{\xi_{\bar{N}}'} \bar{V}_I(\xi_1', \boldsymbol{\xi}_{-1}') \, \mathrm{d}F^\sigma(\xi_{\bar{N}}' \mid \boldsymbol{\xi}_{\bar{N}}, \eta) \ldots \mathrm{d}F^\sigma(\xi_2' \mid \boldsymbol{\xi}_2, \eta) . \tag{8 – repeated}$$

Here we have chosen to make the dependence of W on $\bar{V}_I$ explicit to make the arguments that follow clearer.

Integrating (39) with respect to ρ establishes that, in equilibrium, $\bar{V}_I$ must be a fixed point of the operator $T_{F^\sigma} : \mathbb{R}^{|\Xi|^R} \to \mathbb{R}^{|\Xi|^R}$ defined by

$$T_{F^\sigma} \bar{V}_I(\boldsymbol{\xi}) := \int \max \left\{ \pi(\boldsymbol{\xi}) + \rho, \int \max_{x \in \mathbb{R}_+} h(x, \boldsymbol{\xi}, \nu, \bar{V}_I; F^\sigma) \, \mathrm{d}F_\nu \right\} \, \mathrm{d}F_\rho . \tag{40}$$

Lemma 1. T_{F^σ} is a contraction.

Proof. T_{F^σ} satisfies Blackwell's sufficient conditions for a contraction (e.g., Stokey, Lucas, and Prescott (1989, Theorem 3.3)). Indeed,

[24] For a model where competitor investment can increase in a firm's quality, see Besanko and Doraszelski (2004).

1. *Monotonicity*. If $\bar{V}_I' \geq \bar{V}_I$, then

$$h(x^*(\boldsymbol{\xi}, \nu, \bar{V}_I; F^\sigma), \boldsymbol{\xi}, \nu, \bar{V}_I; F^\sigma) \leq h(x^*(\boldsymbol{\xi}, \nu, \bar{V}_I; F^\sigma), \boldsymbol{\xi}, \nu, \bar{V}_I'; F^\sigma)$$
$$\leq \max_{x \in \mathbb{R}_+} h(x, \boldsymbol{\xi}, \nu, \bar{V}_I'; F^\sigma)$$

where x^* is the solution to the inner maximization problem in Equation (40). This implies that $T_{F^\sigma}\bar{V}_I(\boldsymbol{\xi}) \leq T_{F^\sigma}\bar{V}_I'(\boldsymbol{\xi})$ for all $\boldsymbol{\xi}$.

2. *Discounting*. Note that

$$T_{F^\sigma}(\bar{V}_I + a)(\boldsymbol{\xi}) = \int \max\left\{\pi(\boldsymbol{\xi}) + \rho, \int \max_{x \in \mathbb{R}_+} h(x, \boldsymbol{\xi}, \nu, \bar{V}_I; F^\sigma)\, \mathrm{d}F_\nu + \beta a\right\} \mathrm{d}F_\rho$$
$$\leq T_{F^\sigma}(\bar{V}_I)(\boldsymbol{\xi}) + \beta a$$

The boundedness condition in Blackwell's Theorem is satisfied because the functions $\bar{V}_I$ are simply vectors in an Euclidean space. □

It follows from Lemma 1 and the Contraction Mapping Theorem that to each F^σ there corresponds a unique integrated incumbent value function $\bar{V}_I(F^\sigma)$. We will need to establish that the map $\bar{V}_I(F^\sigma)$ is continuous. The next result will be useful in that regard. Let $\mathcal{F}$ be the set collections of conditional distributions of firm quality conditional on the industry state and the aggregate shock. The elements of $\mathcal{F}$ are vectors of the form

$$\left[F(\xi' \mid \boldsymbol{\xi}, \eta) : \xi' \in \Xi, \boldsymbol{\xi} \in \boldsymbol{\Xi}^R, \eta \in \mathrm{supp}(F_\eta)\right] ,$$

where $0 \leq F(\xi' \mid \boldsymbol{\xi}, \eta)$ and $\sum_{\xi' \in \Xi} F(\xi' \mid \boldsymbol{\xi}, \eta) = 1$ for all $\xi \in \Xi$, $\boldsymbol{\xi} \in \boldsymbol{\Xi}^R$, and $\eta \in \mathrm{supp}(F_\eta)$. Let $\mathcal{V} \subseteq R^{|\boldsymbol{\Xi}^R|}$ be the set of possible integrated incumbent value functions $\bar{V}_I$.

Lemma 2. The map $T : \mathcal{F} \times \mathcal{V} \to \mathcal{V}$ with values $T_{F^\sigma}\bar{V}_I$ is continuous in F^σ.

Proof. The terms $W(\xi_1' \mid \bar{V}_I, \boldsymbol{\xi}, F^\sigma, \eta)$ are continuous in F^σ.[25] Therefore, the function h is continuous in F^σ and x.[26] The Maximum Theorem then implies that $\max_{x \in \mathbb{R}_+} h(x, \boldsymbol{\xi}, \nu, \bar{V}_I; F^\sigma)$ is continuous in F^σ. It then follows from equation (40)

[25]This is clear in the finite case we are restricting ourselves to. It also holds in greater generality. To see this, note that in the general case one can define T_{F^σ} in the space of bounded continuous functions on $\boldsymbol{\Xi}^R$ with the sup norm. For a continuous $\bar{V}_I$, continuity of $W(\xi_1' \mid \bar{V}_I, \boldsymbol{\xi}, F^\sigma, \eta)$ in F^σ follows from Helly's Second Theorem.

[26]Again, continuity in x is clear in the finite case. In the general case, one can again refer to Helly's Second Theorem.

that $T_{F^\sigma}\bar{V}_I(\boldsymbol{\xi})$ is continuous in F^σ and thus, since $|\Xi^R| < \infty$, that $T_{F^\sigma}\bar{V}_I$ is continuous in F^σ. $\square$

Denardo (1967) states the result below in terms of policy functions (δ in his notation) and their associated return functions v_δ, which are the expected net present value of payoffs when following policy δ. The result holds in our setting. Denardo (1967)'s proof goes through unchanged.

Lemma 3 (Theorem 1, Denardo (1967))**.** For all $F \in \mathcal{F}$ and $\bar{V}_I \in \mathcal{V}$,

$$\|\bar{V}_I(F^\sigma) - \bar{V}_I\| \leq \frac{1}{1-\beta}\|T_{F^\sigma}\bar{V}_I - \bar{V}_I\| \ .$$

Lemma 4. The map $\bar{V}_I : \mathcal{F} \to \mathcal{V}$ with values $\bar{V}_I(F^\sigma)$ is continuous.

Proof. This is very similar to Lemma 3.2 of Whitt (1980). Let $F^\sigma \in \mathcal{F}$ and $\{F^\sigma_n\}_{n\in\mathbb{N}}$ be a sequence in $\mathcal{F}$ such that $\lim_{n\to\infty} F^\sigma_n = F^\sigma$. Then

$$\begin{aligned}\|\bar{V}_I(F^\sigma_n) - \bar{V}_I(F^\sigma)\| &\leq \frac{1}{1-\beta}\|T_{F^\sigma_n}\bar{V}_I(F^\sigma) - \bar{V}_I(F^\sigma)\| \\ &= \frac{1}{1-\beta}\|T_{F^\sigma_n}\bar{V}_I(F^\sigma) - T_{F^\sigma}\bar{V}_I(F^\sigma)\| \to 0\end{aligned}$$

as $n \to \infty$. The inequality is due to Lemma 3. The equality is due to $\bar{V}_I(F^\sigma)$ being a fixed point of T_{F^σ}. The limit is due to Lemma 2. $\square$

Now consider the problem

$$\max_{x\in\mathbb{R}_+} h(x, \boldsymbol{\xi}, \nu, \bar{V}_I(F^\sigma); F^\sigma) \ . \tag{41}$$

Because of Proposition 1, this problem has a unique solution for each $\boldsymbol{\xi}$ and ν. Collect those in the policy function $\sigma^x(\boldsymbol{\xi}, \nu; F^\sigma)$. The value of problem (41) gives the ENPV of profits conditional on being active when competitor transitions are given by F^σ, $V_I^A(\boldsymbol{\xi}, \nu; F^\sigma)$, as in equation (16). By Berge's Maximum Theorem and the uniqueness of the optimal investment, $\sigma^x(\boldsymbol{\xi}, \nu; F^\sigma)$ and $V_I^A(\boldsymbol{\xi}, \nu; F^\sigma)$ are continuous in F^σ. The same is true of $V_E^A(\boldsymbol{\xi}_{-1}, \nu; F^\sigma)$.

It follows that

$$\bar{V}_s^A(\boldsymbol{\xi}; F^\sigma) := \int V_s^A(\boldsymbol{\xi}, \nu; F^\sigma)\, \mathrm{d}F_\nu \ , \ s \in \{E, I\}$$

are also continuous in F^σ. Therefore, the transitions implied by F^σ, given by

$$F(\xi' \mid \boldsymbol{\xi}, \eta; F^\sigma) = \begin{cases} 1 - F_\rho(\bar{V}_I^A(\boldsymbol{\xi}; F^\sigma) - \pi(\boldsymbol{\xi})) & \text{if } \xi > -\infty \\ & \text{and } \xi' = -\infty \\ F_\rho(\bar{V}_I^A(\boldsymbol{\xi}; F^\sigma) - \pi(\boldsymbol{\xi})) \int F(\xi' \mid \xi, \sigma^x(\boldsymbol{\xi}, \nu; F^\sigma), \eta)\, \mathrm{d}F_\nu & \text{if } \xi > -\infty \\ & \xi' > -\infty \\ 1 - F_\phi(\bar{V}_E^A(\boldsymbol{\xi}_{-1}; F^\sigma)) & \text{if } \xi = \xi' = -\infty \\ F_\phi(\bar{V}_E^A(\boldsymbol{\xi}_{-1}; F^\sigma)) \int F(\xi' \mid \xi_e, \sigma^x(\boldsymbol{\xi}_{-1}, \nu; F^\sigma), \eta)\, \mathrm{d}F_\nu & \text{if } \xi = -\infty \\ & \xi' > -\infty \end{cases} \quad (42)$$

are continuous in F^σ for all ξ', $\boldsymbol{\xi}$, and η.

We have thus established that the map $H : \mathcal{F} \to \mathcal{F}$ defined by (42) is continuous. It is clear that $\mathcal{F}$ is a compact and convex subset of $\mathbb{R}^{|\Xi| \times |\boldsymbol{\Xi}| \times |\mathrm{supp}(F_\eta)|}$. Therefore, Brouwer's Fixed-Point Theorem applies: there exists a $\bar{F}^\sigma \in \mathcal{F}$ such that $H(\bar{F}^\sigma) = \bar{F}^\sigma$.

To conclude the argument, consider the policy functions that obtain under $\bar{V}_I(\bar{F}^\sigma)$: the investment policy $\sigma^x(\boldsymbol{\xi}, \nu; \bar{F}^\sigma)$, the exit policy $\alpha^I(\boldsymbol{\xi}, \rho; \bar{F}^\sigma)$ that solves the outer maximization problem in (39), and the entry policy $\alpha^E(\boldsymbol{\xi}_{-1}, \phi; \bar{F}^\sigma)$ that solves problem (13). By definition, they are optimal given $\bar{F}^\sigma$. Moreover, because $\bar{F}^\sigma$ is a fixed point of H, $\sigma^x(\boldsymbol{\xi}, \nu; \bar{F}^\sigma)$ and $\sigma^A(\boldsymbol{\xi}, \rho; \bar{F}^\sigma)$ give rise to $\bar{F}^\sigma$. It follows that $\sigma^x(\boldsymbol{\xi}, \nu; \bar{F}^\sigma)$ and $\sigma^A(\boldsymbol{\xi}, \rho; \bar{F}^\sigma)$ are optimal against themselves, i.e., the strategy profile in which each firm plays these strategies is a symmetric Markov Perfect Equilibrium.[27]

A.4 Characterizing EV

This section derives equation (27).[28] We start from $V_I(\boldsymbol{\xi}, \rho)$:

$$V_I(\boldsymbol{\xi}, \rho) = \max\left\{\pi(\boldsymbol{\xi}) + \rho, \bar{V}_I^A(\boldsymbol{\xi})\right\} = \max_{\chi \in \{0,1\}} \left\{\chi \bar{V}_I^A(\boldsymbol{\xi}) + (1 - \chi)(\pi(\boldsymbol{\xi}) + \rho)\right\} \quad (43)$$

[27] Strictly speaking, we construct a candidate equilibrium on a reduced state-space. One still has to check that the strategies induced on the original state space constitute an equilibrium of the game. See Doraszelski and Satterthwaite (2010) section 6, in particular p. 239 (though the preceding pages give required preliminaries).

[28] Related calculations appear e.g. in Jofre-Bonet and Pesendorfer (2003); Pakes et al. (2007).

Letting $\alpha^I(\boldsymbol{\xi}, \rho)$ denote the optimal policy, we have

$$V_I(\boldsymbol{\xi}, \rho) = \alpha^I(\boldsymbol{\xi}, \rho)\bar{V}_I^A(\boldsymbol{\xi}) + (1 - \alpha^I(\boldsymbol{\xi}, \rho))(\pi(\boldsymbol{\xi}) + \rho)$$

and, integrating over ρ,

$$\begin{aligned}\bar{V}_I(\boldsymbol{\xi}) &:= \int V_I(\boldsymbol{\xi}, \rho)\, \mathrm{d}F_\rho \\ &= \mathbb{P}_I^A(\boldsymbol{\xi})\bar{V}_I^A(\boldsymbol{\xi}) + [1 - \mathbb{P}_I^A(\boldsymbol{\xi})]\pi(\boldsymbol{\xi}) \\ &+ [1 - \mathbb{P}_I^A(\boldsymbol{\xi})]\mathbb{E}[\rho \mid \rho > \bar{V}_I^A(\boldsymbol{\xi}) - \pi(\boldsymbol{\xi})] \end{aligned} \tag{44}$$

where $\mathbb{P}_I^A(\boldsymbol{\xi}) := \mathbb{P}(\alpha^I(\boldsymbol{\xi}, \rho) = 1)$ is the probability that an incumbent chooses to be active when its initial state is $\boldsymbol{\xi}$.

Next, letting $\sigma^x(\boldsymbol{\xi}, \nu)$ denote the optimal investment policy, we have that

$$V^A(\boldsymbol{\xi}, \nu) = \pi(\boldsymbol{\xi}) - c(\sigma^x(\boldsymbol{\xi}, \nu), \nu; \theta_x) + \beta \sum_{\xi_1' \in \Xi} W(\xi_1' \mid \boldsymbol{\xi}, \theta_{-\phi}) P(\xi_1' \mid \xi_1, \sigma^x(\boldsymbol{\xi}, \nu)) \tag{45}$$

where, with slight abuse of notation, $P(\xi_1' \mid \xi, x)$ denotes the probability of the firms own characteristic evolving from ξ_1 to ξ_1' when the firm invests x. Integrating (45) we get

$$\bar{V}_I^A(\boldsymbol{\xi}) = \pi(\boldsymbol{\xi}) - \int c(\sigma^x(\boldsymbol{\xi}, \nu), \nu; \theta_x)\, \mathrm{d}F_\nu + \beta \sum_{\xi_1' \in \Xi} W(\xi_1' \mid \boldsymbol{\xi}, \theta_{-\phi}) \mathbb{P}^A(\xi_1' \mid \boldsymbol{\xi}) \tag{46}$$

where $\mathbb{P}^A(\xi_1' \mid \boldsymbol{\xi}) := \int P(\xi_1' \mid \xi_1, \sigma^x(\boldsymbol{\xi}, \nu))\, \mathrm{d}F_\nu$ is the ex-ante probability of ξ_1' given that the firm chooses to be active and investments optimally in state $\boldsymbol{\xi}$. Then, using the definition of $W(\xi_1' \mid \boldsymbol{\xi}, \theta_{-\phi})$, we have that

$$\begin{aligned}\bar{V}_I^A(\boldsymbol{\xi}) &= \pi(\boldsymbol{\xi}) - \int c(\sigma^x(\boldsymbol{\xi}, \nu), \nu; \theta_x)\, \mathrm{d}F_\nu \\ &+ \beta \sum_{\xi_1' \in \Xi} \left(\sum_{\boldsymbol{\xi}_{-1}'} \bar{V}_I(\xi_1', \boldsymbol{\xi}_{-1}') \prod_{k>1} \mathbb{P}^\sigma(\xi_k' \mid \boldsymbol{\xi}) \right) \mathbb{P}^A(\xi_1' \mid \boldsymbol{\xi})\ . , \end{aligned} \tag{47}$$

where $P^\sigma(\xi_k' \mid \boldsymbol{\xi})$ is defined in equation (31).

We now plug (47) into (44) to obtain

$$\begin{aligned}\bar{V}_I(\boldsymbol{\xi}) = \pi(\boldsymbol{\xi}) - \mathbb{P}_I^A(\boldsymbol{\xi}) \int c(\sigma^x(\boldsymbol{\xi}, \nu), \nu; \theta_x)\, \mathrm{d}F_\nu \\ + [1 - \mathbb{P}_I^A(\boldsymbol{\xi})]\mathbb{E}[\rho \mid \rho > \bar{V}_I^A(\boldsymbol{\xi}) - \pi(\boldsymbol{\xi})] \\ + \beta \sum_{\{\boldsymbol{\xi}' : \xi_1' > -\infty\}} \bar{V}_I(\boldsymbol{\xi}') \prod_{k=1}^{N} \mathbb{P}^\sigma(\xi_k' \mid \boldsymbol{\xi})\end{aligned} \tag{48}$$

where the last line uses the fact that $\mathbb{P}^\sigma(\xi_1' \mid \boldsymbol{\xi}) = \mathbb{P}^A(\xi_1' \mid \boldsymbol{\xi})\mathbb{P}_I^A(\boldsymbol{\xi})$. Note that we can also make the sum in the last line over all $\boldsymbol{\xi}'$ if we define $\bar{V}_I(\boldsymbol{\xi}') = 0$ when $\xi_1' = -\infty$. With this convention it is more accurate to interpret $\bar{V}_I(\boldsymbol{\xi})$ as the ENPV of landing in state $\boldsymbol{\xi}$ (given our assumption that firms that exit perish), rather than starting a period from state $\boldsymbol{\xi}$.

Observe that $\bar{\boldsymbol{V}}_I := [\bar{V}_I(\boldsymbol{\xi}) : \boldsymbol{\xi} \in \boldsymbol{\Xi}^R, \xi_1 > -\infty]$ enters equation (48) in a non-linear fashion through the $\mathbb{E}[\rho \mid \rho > \bar{V}^A(\boldsymbol{\xi}) - \pi(\boldsymbol{\xi})]$ term. Fortunately, assuming that F_ρ is strictly increasing, we can deal with that term as follows:

$$\begin{aligned}\mathbb{E}[\rho \mid \rho > \bar{V}^A(\boldsymbol{\xi}) - \pi(\boldsymbol{\xi})] &= \mathbb{E}[\rho \mid F_\rho(\rho) > F_\rho(\bar{V}^A(\boldsymbol{\xi}) - \pi(\boldsymbol{\xi}))] \\ &= \mathbb{E}[\rho \mid F_\rho(\rho) > \mathbb{P}_I^A(\boldsymbol{\xi})] \\ &= \mathbb{E}[\rho \mid \rho > F_\rho^{-1}(\mathbb{P}_I^A(\boldsymbol{\xi}))]\ ,\end{aligned} \tag{49}$$

which does away with $\bar{V}^A(\boldsymbol{\xi})$. We can now plug (49) into (48) and stack across states with $\boldsymbol{\xi}_1 > -\infty$:

$$\bar{\boldsymbol{V}}_I = \boldsymbol{\pi} - \boldsymbol{K}(\boldsymbol{\theta}_x) + \boldsymbol{\Sigma}(F_\rho) + \beta \boldsymbol{M}(\boldsymbol{P})\bar{\boldsymbol{V}}_I \qquad \text{(27 – Repeated)}$$

where the terms of this equation are defined in (28) to (31).

Finally, note that (27) does uniquely define $\bar{V}_I$ because the matrix $I - \beta\boldsymbol{M}(\boldsymbol{P})$ is invertible. Indeed, assume otherwise. Then there exists $x \in \boldsymbol{\Xi}_I^R \setminus \{\boldsymbol{0}\}$ such that $[I - \beta\boldsymbol{M}(\boldsymbol{P})]x = 0$, or $x = \beta\boldsymbol{M}x$. This implies $\|x\|_\infty = \|\beta\boldsymbol{M}x\|_\infty$. However, letting $(\boldsymbol{M}x)_i$ denote the i-th coordinate of $\boldsymbol{M}x$, we have

$$|(\boldsymbol{M}x)_i| = \left|\sum_{j=1}^{|\boldsymbol{\Xi}_I^R|} M_{ij}x_j\right| \leq \sum_{j=1}^{|\boldsymbol{\Xi}_I^R|} M_{ij}|x_j| \leq \|x\|_\infty \sum_{j=1}^{|\boldsymbol{\Xi}_I^R|} M_{ij} \leq \|x\|_\infty\ ,$$

where we have used that $\boldsymbol{M}$ is a sub-stochastic matrix, i.e. its rows sum to at most one.[29] Therefore $\|x\|_\infty = \|\beta\boldsymbol{M}x\|_\infty = \beta\|\boldsymbol{M}x\|_\infty \leq \beta\|x\|_\infty$, a contradiction.

[29] The rows of $\boldsymbol{M}(\boldsymbol{P})$ need not sum to one because $\boldsymbol{M}(\boldsymbol{P})$ is the matrix of transitions between

A.5 The Investment Contribution to the Likelihood

As shown in the main text, the conditional distribution of investment given $\boldsymbol{\xi}$ is given by

$$F_X(x \mid \boldsymbol{\xi}) = 1 - F_\nu\left((\sigma^x)^{-1}(x; \boldsymbol{\xi})\right) \ . \tag{19}$$

Here we build on the derivations in Appendix A.2. As shown there, the investment first-order condition can be written as

$$-\frac{\partial c(x, \nu; \boldsymbol{\theta}_x)}{\partial x} + \beta A(\boldsymbol{\xi}, F^\sigma; \boldsymbol{\theta}_{-\phi}) \frac{\partial Q(\xi, x)}{\partial x} = 0 \ ,$$

where $A(\boldsymbol{\xi}, F^\sigma; \boldsymbol{\theta}_{-\phi}) = \int_{\xi'} W(\xi' \mid \boldsymbol{\xi}, F^\sigma, \eta; \boldsymbol{\theta}_{-\phi}) \ \mathrm{d}K(\xi', \xi, \eta)$ and we have made explicit the dependence on the dynamic parameters to be estimated.

To characterize the likelihood at a parameter vector $\boldsymbol{\theta} = (\boldsymbol{\theta}_{-\phi}, \boldsymbol{\theta}_\phi)$, there are two cases to consider. First, suppose $A(\boldsymbol{\xi}, F^\sigma; \boldsymbol{\theta}_{-\phi}) \leq 0$. Then, per the proof of Proposition 1, the optimal investment level is $x^* = 0$ for all ν. Therefore, the investment contribution to the likelihood is 1 if $x = 0$ and 0 otherwise.

Next, suppose $A(\boldsymbol{\xi}, F^\sigma; \boldsymbol{\theta}_{-\phi}) > 0$. Then, again following the proof of Proposition 1, if $x = 0$ its contribution to the likelihood is given by equation (19). If $x > 0$, we need the conditional density. Differentiating (19) with respect to x gives the conditional density

$$f_X(x \mid \boldsymbol{\xi}) = -f_\nu\left((\sigma^x)^{-1}(x; \boldsymbol{\xi})\right) \cdot \frac{\partial}{\partial x}(\sigma^x)^{-1}(x; \boldsymbol{\xi}) \ . \tag{50}$$

The inverse $(\sigma^x)^{-1}(x; \boldsymbol{\xi})$ is characterized implicitly by the identity

$$\frac{\partial}{\partial x} c(x, (\sigma^x)^{-1}(x; \boldsymbol{\xi})) = \underbrace{\beta A(\boldsymbol{\xi}, F^\sigma; \boldsymbol{\theta}_{-\phi}) \frac{\partial Q(\xi, x)}{\partial x}}_{MB(\boldsymbol{\xi}, x)} \ . \tag{51}$$

Therefore,

$$(\sigma^x)^{-1}(x; \boldsymbol{\xi}) = (\partial_x c)^{-1}(MB(\boldsymbol{\xi}, x), x) \ , \tag{52}$$

where $(\partial_x c)^{-1}(\cdot, x)$ is the inverse of $\partial_x c(x, \nu)$ with respect to ν. We use (52) to compute the inverse policy. Moreover, applying the Implicit Function Theorem to (51) gives

$$\frac{\partial}{\partial x}(\sigma^x)^{-1}(x; \boldsymbol{\xi}) = \frac{-\partial_x^2 c(x, (\sigma^x)^{-1}(x, \boldsymbol{\xi})) + \partial_x MB(\boldsymbol{\xi}, x)}{\partial_{\nu x}^2 c(x, (\sigma^x)^{-1}(x, \boldsymbol{\xi}))} \ .$$

states in $\Xi_I^R = \{\boldsymbol{\xi} \in \Xi^R : \xi_1 > -\infty\}$ rather than Ξ^R.

As an illustration, let us apply the expression above to the specification in Section 4.2. We have

$$\begin{aligned}
-\partial_x^2 c(x, (\sigma^x)^{-1}(x; \boldsymbol{\xi})) &= -2\theta_{x2} \\
\partial_x MB(\boldsymbol{\xi}, x) &= \beta A(\boldsymbol{\xi}) \partial_x^2 Q(\boldsymbol{\xi}, x) \\
\partial_{\nu x}^2 c(x, (\sigma^x)^{-1}(x; \boldsymbol{\xi})) &= \theta_{x3} \ ,
\end{aligned}$$

so that

$$\frac{\partial}{\partial x}(\sigma^x)^{-1}(x; \boldsymbol{\xi}) = \frac{-2\theta_{x2} + \beta A(\boldsymbol{\xi}) \partial_x^2 Q(\boldsymbol{\xi}, x)}{\theta_{x3}} \ .$$

A.6 BBL Implementation Details

As explained in Section 3.3, we estimate the investment policy function by estimating a set of quantile regressions of investment on functions of $\boldsymbol{\xi}$. Formally, let $\tau \in (0, 1)$ and let ν_τ be the τ-th quantile of F_ν. Denote by $\hat{\chi}_\tau^x$ the quantile regression coefficient estimate associated with the τ-th conditional quantile. Our estimate of the investment policy function is

$$\hat{\sigma}^x(\boldsymbol{\xi}, \nu_\tau) = f^x(\boldsymbol{\xi})' \hat{\chi}_\tau^x \ ,$$

where, for example, the $f^x(\boldsymbol{\xi})$ used to estimate the model in Section 4.1 includes functions of $\boldsymbol{\xi}$ such as dummies for the firm's own quality, the number of active firms in the market, the rank of the firm's quality in the market, the mean quality in the market, and the maximum quality in the market.[30] With regards to entry and exit decisions, we estimate conditional probabilities of firms choosing to be active. We do so by estimating logistic regressions of those decisions on functions of $\boldsymbol{\xi}$. Let $\mathbb{P}^E(\boldsymbol{\xi}_{-1}) := \mathbb{P}(\alpha^E(\boldsymbol{\xi}_{-1}, \phi) = 1)$ and $\mathbb{P}^I(\boldsymbol{\xi}) := \mathbb{P}(\alpha^I(\boldsymbol{\xi}, \rho) = 1)$ denote the conditional probabilities that entrants and incumbents choose to be active, respectively. Our estimates of these probabilities are

$$\hat{\mathbb{P}}^E(\boldsymbol{\xi}_{-1}) = \Lambda\big(f^E(\boldsymbol{\xi})' \hat{\chi}^E\big) \qquad \hat{\mathbb{P}}^I(\boldsymbol{\xi}) = \Lambda\big(f^I(\boldsymbol{\xi})' \hat{\chi}^I\big) \ .$$

where Λ denotes the logistic CDF and $\hat{\chi}^E$ and $\hat{\chi}^E$ denote the logistic regression coefficient estimates. The vector $f^E(\boldsymbol{\xi})$ includes, for example, dummies for the number of firms and the average quality in the market, whereas $f^I(\boldsymbol{\xi})$ includes

[30] This is similar to the specification that HvB adopt to estimate the investment policy function. We adopt a specification that is more flexible with respect to own quality and omits higher order moments of the distribution of quality in a market, as we simulate a dataset with fewer firms per market.

dummies for the number of firms and the firm's own quality.

The BBL variants we consider differ in the construction of the deviations from these policies, which we denote as $\{\tilde{\sigma}^x(\boldsymbol{\xi},\nu_\tau), \tilde{\mathbb{P}}^E(\boldsymbol{\xi}_{-1}), \tilde{\mathbb{P}}^I(\boldsymbol{\xi})\}$. Let i index a deviation. In what we term Additive BBL we draw $o_i^x \sim N(0, 0.3)$, $o_i^E \sim N(0, 0.5)$, and $o_i^I \sim N(0, 0.5)$ for each inequality i, as in Bajari et al. (2007). We then form deviations as

$$
\begin{aligned}
\tilde{\sigma}_i^x(\boldsymbol{\xi},\nu_\tau) &= \max\{0, \hat{\sigma}^x(\boldsymbol{\xi},\nu_\tau) + o_i^x\} \\
\tilde{\mathbb{P}}_i^E(\boldsymbol{\xi}_{-1}) &= \min\{\max\{0, \hat{\mathbb{P}}^E(\boldsymbol{\xi}_{-1}) + o_i^E\}, 1\} \\
\tilde{\mathbb{P}}_i^I(\boldsymbol{\xi}) &= \min\{\max\{0, \hat{\mathbb{P}}^I(\boldsymbol{\xi}) + o_i^E\}, 1\} \ .
\end{aligned}
$$

In what we term Multiplicative BBL we form deviations as

$$
\begin{aligned}
\tilde{\sigma}_i^x(\boldsymbol{\xi},\nu_\tau) &= \iota_i^x \hat{\sigma}^x(\boldsymbol{\xi},\nu_\tau) \\
\tilde{\mathbb{P}}_i^E(\boldsymbol{\xi}_{-1}) &= \iota_i^E \hat{\mathbb{P}}^E(\boldsymbol{\xi}_{-1}) \\
\tilde{\mathbb{P}}_i^I(\boldsymbol{\xi}) &= \iota_i^I \hat{\mathbb{P}}^I(\boldsymbol{\xi}) \ ,
\end{aligned}
$$

where $\iota_i^x, \iota_i^E, \iota_i^I$ are sampled uniformly and independently from the vector $\{.90, .95, 1.05, 1.10\}$, as in Hashmi and van Biesebroeck (2016). Lastly, in what we term Asymptotic BBL we draw $\tilde{\boldsymbol{\chi}}_i \sim \mathrm{N}(\hat{\boldsymbol{\chi}}, \varsigma\hat{\boldsymbol{\Sigma}}_{\boldsymbol{\chi}})$ for each i, where ς is a scaling constant we set equal to 1. We then form deviations as

$$
\begin{aligned}
\tilde{\sigma}_i^x(\boldsymbol{\xi},\nu_\tau) &= f^x(\boldsymbol{\xi})'\tilde{\chi}_{i\tau}^x \\
\tilde{\mathbb{P}}_i^E(\boldsymbol{\xi}_{-1}) &= \Lambda\big(f^E(\boldsymbol{\xi})\tilde{\chi}_i^E\big) \\
\tilde{\mathbb{P}}_i^I(\boldsymbol{\xi}) &= \Lambda\big(f^I(\boldsymbol{\xi})\tilde{\chi}_i^I\big).
\end{aligned}
$$

Note that all three sets of BBL deviations require the econometrician to choose tuning parameters: the distributions of additive deviations, the vector of constants constituting the multiplicative deviations, and the scaling constant for asymptotic deviations. The estimators we present in this paper dispense with this requirement.

Forward simulation of potentially non-linear value functions. Given estimated policy functions and deviations, typical implementations of the BBL inequality estimator compute value functions by forward simulation. As shown by equation (27), the incumbent value function depends on the conditional expectation of scrap values which may cause the value function to be non-linear in model parameters. This nonlinearity increases the computational cost of the forward simulation routine, as the discussion in Section 3.3 no longer applies.

However, it is still the case that the forward simulation routine can be performed essentially once. To see this, note that we can define the exit policy as a function of a $U[0,1]$ random variable by means of a change of variable: $\breve{\sigma}^I(\boldsymbol{\xi},\tau) := \sigma^I(\boldsymbol{\xi}, F_\rho^{-1}(\tau))$. It follows that $\breve{\sigma}^I(\boldsymbol{\xi},\tau) = \mathbf{1}\{\tau \le \mathbb{P}^I(\boldsymbol{\xi})\}$.[31] We thus take $\tau \sim U[0,1]$ draws and use those to simulate exit decisions: incumbents remain active if and only if $\tau \le P^I(\boldsymbol{\xi})$. As we vary structural parameters, these simulations do not need to be repeated. All that needs to be recomputed is the scrap value that accrues to firms when they do decide to exit, which is $F_\rho^{-1}(\tau;\boldsymbol{\theta}_\rho)$, when $\tau > \mathbb{P}^I(\boldsymbol{\xi})$. The cost of repeatedly calling F_ρ^{-1} is typically dwarfed by the cost of repeating the simulation.

[31]Indeed, $\breve{\sigma}^I(\boldsymbol{\xi},\tau) = \sigma^I(\boldsymbol{\xi}, F_\rho^{-1}(\tau)) = \mathbb{1}\{F_\rho^{-1}(\tau) \le \bar{V}_I^A(\boldsymbol{\xi}) - \pi(\boldsymbol{\xi})\} = \mathbb{1}\{\tau \le F_\rho(\bar{V}_I^A(\boldsymbol{\xi}) - \pi(\boldsymbol{\xi}))\} = \mathbb{1}\{\tau \le \mathbb{P}^I(\boldsymbol{\xi})\}$.