Note: Identification of Dynamic Auctions with Incomplete Data

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Consider the dynamic first-price, IPV auctions model presented in Athey and Haile 2007, Section 9.

The ex-ante value function is there shown to solve

$$V_i(\mathbf{c}) = \int_{\underline{b}_i(\mathbf{c})}^{\bar{b}_i(\mathbf{c})} \frac{G_{M_i}(b_i|\mathbf{c})}{g_{M_i}(b_i|\mathbf{c})} G_{M_i}(b_i|\mathbf{c}) dG_{B_i}(b_i|\mathbf{c}) + \\ + \delta \sum_{j \neq i} V_i(\omega(\mathbf{c}, j)) \times \\ \left\{ \int_{b_i}^{\bar{b}_i(\mathbf{c})} \prod_{k \neq i,j} G_{B_k}(b_j|\mathbf{c}) g_{B_j}(b_j|\mathbf{c}) db_j + \int_{\underline{b}_i(\mathbf{c})}^{\bar{b}_i(\mathbf{c})} \frac{G_{M_i}(b_i|\mathbf{c})}{g_{M_i}(b_i|\mathbf{c})} \frac{g_{B_j}(b_i|\mathbf{c})}{G_{B_j}(b_i|\mathbf{c})} G_{M_i}(b_i|\mathbf{c}) dG_{B_i}(b_i|\mathbf{c}) \right\} \tag{1}$$

where $G_{M_i}(b_i|\mathbf{c}) = \prod_{j \neq i} G_{B_j}(b_i|\mathbf{c})$ is the CDF of the highest bid faced by i, with pdf $g_{M_i}(b_i|\mathbf{c}) = \left[\prod_{j \neq i} G_{B_j}(b_i|\mathbf{c})\right] \sum_{j \neq i} \frac{b_{B_j}(b_i|\mathbf{c})}{G_{B_j}(b_i|\mathbf{c})}$. Hence the above shows that the ex-ante value functions can be determined by solving a linear system in which the only unknowns are (fixing the discount factor δ, which is not identified) equilibrium conditional bid distributions $G_{B_j}(b_i|\mathbf{c})$ for each bidder j.

While identification of this dynamic model is typically discussed assuming the econometrician observes all the bids as well as the identity of all the bidders, the purpose of this note is to clarify what is identified upon observing the winning bid and the identity of the winner in each auction. This follows from Athey and Haile 2002; Prakasa Rao 1992.

Suppose in fact the econometrician observes, for any state $\mathbf{c}$, $H_i(b|\mathbf{c}) \equiv Pr(B^{n:n} < b, I^{n:n} = i)$, where $B^{n:n}$ is the highest bid and $I^{n:n}$ the identity of the highest bidder in an auction with n bidders. Then, conditional on n,

$$H_i(b|\mathbf{c}) = \int_b^{\infty} \prod_{j \neq i} G_{B_j}(x|\mathbf{c}) g_{B_i}(x|\mathbf{c}) dx = \int_b^{\infty} \frac{\prod_j G_{B_j}(x|\mathbf{c})}{G_{B_i}(x|\mathbf{c})} g_{B_i}(x|\mathbf{c}) dx = \int_b^{\infty} \prod_j G_{B_j}(x|\mathbf{c}) d \ln G_{B_i}(x|\mathbf{c}) \tag{2}$$

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note $\prod_j G_{B_j}(b|\mathbf{c}) = Pr(B^{n:n} < b) = \sum_i H_i(b|\mathbf{c})$, whence $H_i(b|\mathbf{c}) = \int_b^\infty \sum_j H_j(x|\mathbf{c}) d\ln G_{B_i}(x|\mathbf{c})$. Differentiating,

$$
\begin{aligned}
dH_i(b|\mathbf{c}) &= -\sum_j H_j(b|\mathbf{c}) d\ln G_{B_i}(b|\mathbf{c}) \\
d\ln G_{B_i}(b|\mathbf{c}) &= -\big[\sum_j H_j(b|\mathbf{c})\big]^{-1} dH_i(b|\mathbf{c}) \\
G_{B_i}(b|\mathbf{c}) &= \exp\bigg(-\int_b^\infty \big[\sum_j H_j(x|\mathbf{c})\big]^{-1} dH_i(x|\mathbf{c})\bigg)
\end{aligned}
\tag{3}
$$

Hence knowledge of $H_i(b|\mathbf{c})$ is *not sufficient* to identify $G_{B_i}(b|\mathbf{c})$ at any state $\mathbf{c}$ and for any number of bidders n – the identity of bidders matters. In particular, identification is restored either by observing the identity of bidders in each auction (though not their bids), or with complete participation (which however is typically far from what one empirically observes). Knowledge of $G_{B_i}(b|\mathbf{c})$ in turn identifies $V_i(\mathbf{c})$ from Equation (1) (assuming knowledge of the support of bids). The features of the distribution of bidder utility identified from knowledge of $G_{B_i}(b|\mathbf{c})$ and of $V_i(\mathbf{c})$ depend on the assumptions made on the form of bidder utility (Athey and Haile 2002, Theorem 6).

References

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